

Some Results on Quasi-Trace Functions on (Pre-)Lie Algebras

Ming Chen, Qi Wang*, Senrong Xu

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Abstract. The aim of this paper is to compare the representations, symplectic structures, relative Rota-Baxter operators and the cohomology theory of Lie algebras and the 3-Lie algebras induced by Lie algebras with quasi-trace functions. We introduce a notion of a quasi-trace function on a pre-Lie algebra, which can be used to construct a 3-pre-Lie algebra. The relationships between symplectic structures, relative Rota-Baxter operators and quasi-trace functions on (pre-) Lie algebras are discussed.

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Key Words: 3-Lie algebra, quasi-trace function, relative Rota-Baxter operator, symplectic structure, cohomology.

1. Introduction

The notion of an n -Lie algebra was introduced and studied by V. T. Filippov in [17]. It is the algebraic structure corresponding to Nambu mechanics ([20, 21]). n -Lie algebras, or more generally, n -Leibniz algebras, have attracted attention from several fields of mathematics and physics due to their close connection with dynamics, geometries as well as string theory. For example, the structure of 3-Lie algebras are applied to the study of the supersymmetry and gauge symmetry transformations of the world-volume theory of multiple M2-branes and the generalized identity for 3-Lie algebras can be regarded as a generalized Plücker relation in the physics literature. See the review article [15] for more details.

The quantization problem of the Nambu bracket turned out to be very difficult. The authors in [16] gave an unusual version of deformation quantization called Zariski quantization. In order to realize the quantum Nambu bracket in terms of matrices, as posed in the original paper by Nambu, the authors in [6] proposed the realization of quantum Nambu bracket defined by

$$[A, B, C] = \operatorname{tr}(A)[B, C] + \operatorname{tr}(B)[C, A] + \operatorname{tr}(C)[A, B], \quad (1)$$

where A, B, C are either square matrices or cubic matrices, and proved that the general linear Lie algebra endowed with the triple quantum Nambu bracket becomes a 3-Lie algebra. Motivated by the above construction of 3-Lie algebras from general linear Lie algebras with trace forms, the authors in [2, 3, 9] provided a construction of 3-Lie algebras from Lie algebras with certain linear functions.

* Corresponding author.

More precisely, let $(\mathfrak{g}, [-, -]_{\mathfrak{g}})$ be a Lie algebra with an element $\tau \in \mathfrak{g}^*$. Define the bracket

$$[x, y, z]_{\tau} := \tau(x)[y, z] + \tau(y)[z, x] + \tau(z)[x, y], \quad \forall x, y, z \in \mathfrak{g}. \quad (2)$$

Then it is shown in [3, 9] that if τ satisfies $\tau([x, y]_{\mathfrak{g}}) = 0$, then $(\mathfrak{g}, [-, -, -]_{\tau})$ is a 3-Lie algebra. Furthermore, it is shown in [2] that $(\mathfrak{g}, [-, -, -]_{\tau})$ is a 3-Lie algebra if τ satisfies

$$\tau(x)\tau([y, z]) + \tau(y)\tau([z, x]) + \tau(z)\tau([x, y]) = 0, \quad \forall x, y, z \in \mathfrak{g}. \quad (3)$$

In this paper, we call a linear function τ a *trace function* on \mathfrak{g} if $\tau([x, y]_{\mathfrak{g}}) = 0$ and we call a linear function τ a *quasi-trace function* on \mathfrak{g} if τ satisfies (3). It is obvious that a trace function on a Lie algebra is a quasi-trace function. The trace functions on n -Lie algebras were studied in [5, 11]. The author in [6] used the Weil algebra of a Lie algebra to construct Weil algebra of the induced 3-Lie algebra. The authors in [4] studied the connections between the structural properties and the cohomology of a Lie algebra and the induced 3-Lie algebra. See [1, 2] for more applications of (quasi-)trace functions on Lie superalgebras.

The classical Yang-Baxter equation plays a significant role in many fields in mathematics and mathematical physics. In order to gain better understanding of the relationship between the classical Yang-Baxter equation and the related integrable systems, the more general notion of a relative Rota-Baxter operator (also called \mathcal{O} -operator) on a Lie algebra was introduced by Kupershmidt ([19]). To study solutions of 3-Lie classical Yang-Baxter equation, the notion of a relative Rota-Baxter operator on a 3-LieRep pair was introduced in [8]. A relative Rota-Baxter operator on a 3-LieRep pair $(\mathfrak{g}; \text{ad})$, where ad is the adjoint representation of the 3-Lie algebra \mathfrak{g} , is exactly the Rota-Baxter operator on the 3-Lie algebra \mathfrak{g} introduced in [10]. See [13, 18, 22] for more details and applications about Rota-Baxter operators on n -Lie algebras.

The notion of a 3-pre-Lie algebra was introduced in [8] and it is shown that the underlying structure of a symplectic structure, or more generally a relative Rota-Baxter operator, on a 3-Lie algebra is a 3-pre-Lie algebra. The importance of a 3-pre-Lie algebra is that it can be used to construct a solution of the 3-Lie classical Yang-Baxter equation and thus gives a local cocycle 3-Lie bialgebra. However, the structure of the 3-pre-Lie algebra is complicated, it is not easy to give examples directly. We introduce the notion of a quasi-trace function on a pre-Lie algebra and show that a pre-Lie algebra with a quasi-trace function can give a 3-pre-Lie algebra. This method is more easier to construct 3-pre-Lie algebras. See the survey [12] and the references therein for more details on pre-Lie algebras.

In this paper, we investigate the connections between representations, symplectic structures, relative Rota-Baxter operators and the cohomology of a Lie algebra and the induced 3-Lie algebra. Based on the close relationships between pre-Lie algebras (3-pre-Lie algebras) and symplectic structures or relative Rota-Baxter operators on Lie algebras (3-Lie algebras), we analyze relations between (pre-)Lie algebras with quasi-trace functions and 3-(pre-)Lie algebras.

The paper is organized as follows. In Section 2, we first use a quasi-trace function on a Lie algebra to establish relationships between representations of the Lie algebra and

the induced 3-Lie algebra. Then we introduce the notion of a quasi-trace function on a pre-Lie algebra and show that a pre-Lie algebra with a quasi-trace function gives a 3-pre-Lie algebra. Some examples are given. In Section 3, we establish connections between symplectic structures and relative Rota-Baxter operators on Lie algebras and those structures on the induced 3-Lie algebras. We analyze relations between quasi-trace functions on pre-Lie algebras and Lie algebras within the framework of symplectic structures and relative Rota-Baxter operators on Lie algebras and induced 3-Lie algebras. In Section 4, we recall the cohomology complexes for Lie algebras and 3-Lie algebras, then we study relations between the cohomology theory of a Lie algebra and the induced 3-Lie algebra.

In this paper, all the vector spaces are over an algebraically closed field \mathbb{K} of characteristic 0, and finite dimensional.

2. Some properties of quasi-trace functions on (pre)-Lie algebras

2.1. Quasi-trace functions on Lie algebras

Definition 2.1. A 3-Lie algebra is a vector space \mathfrak{g} together with a trilinear bracket $[-, -, -] : \wedge^3 \mathfrak{g} \rightarrow \mathfrak{g}$ such that the following *fundamental identity* is satisfied

$$[x, y, [z, w, t]] = [[x, y, z], w, t] + [z, [x, y, w], t] + [z, w, [x, y, t]], \quad \forall x, y, z, w, t \in \mathfrak{g}. \quad (4)$$

Definition 2.2. Let $(\mathfrak{g}, [-, -, -]_{\mathfrak{g}})$ and $(\mathfrak{h}, [-, -, -]_{\mathfrak{h}})$ be two 3-Lie algebras. A linear map $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ is said to be a 3-Lie algebra homomorphism if

$$\phi([x, y, z]_{\mathfrak{g}}) = [\phi(x), \phi(y), \phi(z)]_{\mathfrak{h}}, \quad \forall x, y, z \in \mathfrak{g}. \quad (5)$$

Definition 2.3. A Lie algebra $(\mathfrak{g}, [-, -])$ with a quasi-trace function $\tau \in \mathfrak{g}^*$ is called a *quasi-trace Lie algebra*. We denote it by $(\mathfrak{g}, [-, -]; \tau)$.

We denote the 3-Lie algebra induced by a quasi-trace Lie algebra $(\mathfrak{g}, [-, -]; \tau)$ by \mathfrak{g}_{τ} .

Definition 2.4. Let $(\mathfrak{g}, [-, -]_{\mathfrak{g}}; \tau_{\mathfrak{g}})$ and $(\mathfrak{h}, [-, -]_{\mathfrak{h}}; \tau_{\mathfrak{h}})$ be two quasi-trace Lie algebras. A linear map $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ is called a *homomorphism* of quasi-trace Lie algebras if ϕ is a Lie algebra homomorphism and $\tau_{\mathfrak{g}} = \tau_{\mathfrak{h}} \circ \phi$.

Proposition 2.5. If ϕ is a homomorphism from the quasi-trace Lie algebra $(\mathfrak{g}, [-, -]_{\mathfrak{g}}; \tau_{\mathfrak{g}})$ to the quasi-trace Lie algebra $(\mathfrak{h}, [-, -]_{\mathfrak{h}}; \tau_{\mathfrak{h}})$, then ϕ is a 3-Lie algebra homomorphism from $\mathfrak{g}_{\tau_{\mathfrak{g}}}$ to $\mathfrak{h}_{\tau_{\mathfrak{h}}}$.

Proof. Since ϕ is a homomorphism from the quasi-trace Lie algebra \mathfrak{g} to \mathfrak{h} , for $x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned} \phi([x, y, z]_{\tau_{\mathfrak{g}}}) &= \tau_{\mathfrak{g}}(x)[\phi(y), \phi(z)]_{\mathfrak{h}} + \tau_{\mathfrak{g}}(y)[\phi(z), \phi(x)]_{\mathfrak{h}} + \tau_{\mathfrak{g}}(z)[\phi(x), \phi(y)]_{\mathfrak{h}} \\ &= \tau_{\mathfrak{h}}(\phi(x))[\phi(y), \phi(z)]_{\mathfrak{h}} + \tau_{\mathfrak{h}}(\phi(y))[\phi(z), \phi(x)]_{\mathfrak{h}} + \tau_{\mathfrak{h}}(\phi(z))[\phi(x), \phi(y)]_{\mathfrak{h}} \\ &= [\phi(x), \phi(y), \phi(z)]_{\tau_{\mathfrak{h}}}. \end{aligned}$$

Thus ϕ is a 3-Lie algebra homomorphism from $\mathfrak{g}_{\tau_{\mathfrak{g}}}$ to $\mathfrak{h}_{\tau_{\mathfrak{h}}}$. ■

Definition 2.6. Let $(\mathfrak{g}, [-, -, -])$ be a 3-Lie algebra and V a vector space. A representation of \mathfrak{g} on V is a bilinear map $\rho : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ such that

- (i) $\rho(x, y)\rho(z, w) - \rho(z, w)\rho(x, y) = \rho([x, y, z], w) - \rho([x, y, w], z),$
- (ii) $\rho([x, y, z], w) = \rho(x, y)\rho(z, w) + \rho(y, z)\rho(x, w) + \rho(z, x)\rho(y, w), \quad \forall x, y, z, w \in \mathfrak{g}.$

Proposition 2.7. Let $(\mathfrak{g}, [-, -])$ be a Lie algebra and $(V; \rho)$ a representation of \mathfrak{g} on V . If $\tau \in \mathfrak{g}^*$ is a quasi-trace function on \mathfrak{g} , then the bilinear map $\varrho : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ defined by

$$\varrho(x, y) = \tau(x)\rho(y) - \tau(y)\rho(x), \quad \forall x, y \in \mathfrak{g} \quad (6)$$

is a representation of the 3-Lie algebra \mathfrak{g}_τ on V .

Proof. For $x_1, x_2, x_3, x_4 \in \mathfrak{g}$, we have

$$\begin{aligned} & \varrho(x_1, x_2)\varrho(x_3, x_4) - \varrho(x_3, x_4)\varrho(x_1, x_2) - \varrho([x_1, x_2, x_3]_\tau, x_4) + \varrho([x_1, x_2, x_4]_\tau, x_3) \\ = & \tau(x_1)\tau(x_3)\rho(x_2)\rho(x_4) - \tau(x_1)\tau(x_4)\rho(x_2)\rho(x_3) - \tau(x_2)\tau(x_3)\rho(x_1)\rho(x_4) \\ & + \tau(x_2)\tau(x_4)\rho(x_1)\rho(x_3) - \tau(x_3)\tau(x_1)\rho(x_4)\rho(x_2) + \tau(x_2)\tau(x_3)\rho(x_4)\rho(x_1) \\ & + \tau(x_1)\tau(x_4)\rho(x_3)\rho(x_2) - \tau(x_4)\tau(x_2)\rho(x_3)\rho(x_1) + \tau(x_4)\tau(x_1)\rho([x_2, x_3]) \\ & + \tau(x_4)\tau(x_2)\rho([x_3, x_1]) + \tau(x_4)\tau(x_3)\rho([x_1, x_2]) - \tau(x_3)\tau(x_1)\rho([x_2, x_4]) \\ & - \tau(x_3)\tau(x_2)\rho([x_4, x_1]) - \tau(x_4)\tau(x_3)\rho([x_1, x_2]) \\ = & \tau(x_1)\tau(x_3)([\rho(x_2), \rho(x_4)] - \rho([x_2, x_4])) - \tau(x_1)\tau(x_4)([\rho(x_2), \rho(x_3)] - \rho([x_2, x_3])) \\ & - \tau(x_2)\tau(x_3)([\rho(x_1), \rho(x_4)] - \rho([x_1, x_4])) + \tau(x_2)\tau(x_4)([\rho(x_1), \rho(x_3)] - \rho([x_1, x_3])) \\ & - \tau(x_3)\tau(x_4)(\rho([x_1, x_2]) - \rho([x_1, x_2])) = 0. \end{aligned}$$

Similarly, we can show

$$\varrho(x_1, x_2)\varrho(x_3, x_4) + \varrho(x_2, x_3)\varrho(x_1, x_4) + \varrho(x_3, x_1)\varrho(x_2, x_4) - \varrho([x_1, x_2, x_3]_\tau, x_4) = 0.$$

Thus ϱ is a representation of the 3-Lie algebra \mathfrak{g}_τ on V . ■

Remark 2.8. Let $(\mathfrak{g}, [-, -]; \tau)$ be a quasi-trace Lie algebra. Define the representations $\text{ad}, \mathfrak{ad} : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ by

$$\text{ad}_x y = [x, y], \quad \mathfrak{ad}_{x,y} z = [x, y, z]_\tau, \quad \forall x, y, z \in \mathfrak{g}.$$

Then $(\mathfrak{g}; \text{ad})$ and $(\mathfrak{g}; \mathfrak{ad})$ are the adjoint representations of the Lie algebra \mathfrak{g} and the 3-Lie algebra \mathfrak{g}_τ , respectively. By Proposition 2.7, we obtain a representation of the 3-Lie algebra \mathfrak{g}_τ on \mathfrak{g} given by

$$\text{ad}_{x,y}^\tau = \tau(x)\text{ad}_y - \tau(y)\text{ad}_x,$$

which is not the adjoint representation of the 3-Lie algebra \mathfrak{g}_τ . More precisely, we have

$$\mathfrak{ad}_{x,y} z = \text{ad}_{x,y}^\tau z + \tau(z)[x, y], \quad \forall x, y, z \in \mathfrak{g}.$$

2.2. Quasi-trace functions on pre-Lie algebras and 3-pre-Lie algebras

Definition 2.9. A pre-Lie algebra is a pair (A, \star) , where A is a vector space and $\star : A \otimes A \rightarrow A$ is a bilinear multiplication satisfying that for all $x, y, z \in A$, the associator $(x, y, z) = (x \star y) \star z - x \star (y \star z)$ is symmetric in x, y , i.e.

$$(x, y, z) = (y, x, z), \quad \text{or equivalently, } (x \star y) \star z - x \star (y \star z) = (y \star x) \star z - y \star (x \star z).$$

Let (A, \star) be a pre-Lie algebra. The commutator $[x, y]^c = x \star y - y \star x$ defines a Lie algebra structure on A , which is called the *sub-adjacent Lie algebra* of (A, \star) and denoted by A^c . The pre-Lie algebra (\mathfrak{g}, \star) is also called a *compatible pre-Lie algebra* on the Lie algebra A^c . Furthermore, $L : A \rightarrow \mathfrak{gl}(A)$ with $x \rightarrow L_x$, where $L_x y = x \star y$, for all $x, y \in A$, gives a representation of the Lie algebra A^c on A . See [12] for more details.

Definition 2.10. ([8]) Let A be a vector space with a trilinear map $\{-, -, -\} : \otimes^3 A \rightarrow A$. Then $(A, \{-, -, -\})$ is called a *3-pre-Lie algebra* if the following conditions are satisfied:

$$\{x, y, z\} = -\{y, x, z\}; \tag{7}$$

$$\begin{aligned} \{x_1, x_2, \{x_3, x_4, x_5\}\} &= \{[x_1, x_2, x_3]^c, x_4, x_5\} + \{x_3, [x_1, x_2, x_4]^c, x_5\} \\ &\quad + \{x_3, x_4, \{x_1, x_2, x_5\}\}; \end{aligned} \tag{8}$$

$$\begin{aligned} \{[x_1, x_2, x_3]^c, x_4, x_5\} &= \{x_1, x_2, \{x_3, x_4, x_5\}\} + \{x_2, x_3, \{x_1, x_4, x_5\}\} \\ &\quad + \{x_3, x_1, \{x_2, x_4, x_5\}\}, \end{aligned} \tag{9}$$

where $x, y, z, x_i \in A$, $1 \leq i \leq 5$ and $[-, -, -]^c$ is defined by

$$[x, y, z]^c = \{x, y, z\} + \{y, z, x\} + \{z, x, y\}, \quad \forall x, y, z \in A. \tag{10}$$

Proposition 2.11. ([8]) Let $(A, \{-, -, -\})$ be a 3-pre-Lie algebra. Then $(A, [-, -, -]^c)$ is a 3-Lie algebra, which is called the *sub-adjacent 3-Lie algebra* of A and denoted by A^c . $(A, \{-, -, -\})$ is called the *compatible 3-pre-Lie algebra structure* on the 3-Lie algebra A^c .

Definition 2.12. Let (A, \star) be a pre-Lie algebra. A linear function $\tau \in A^*$ is called a *quasi-trace function* on A if τ satisfies

$$\tau(x)\tau(y \star z - z \star y) + \tau(y)\tau(z \star x - x \star z) + \tau(z)\tau(x \star y - y \star x) = 0, \quad \forall x, y, z \in A. \tag{11}$$

It is obvious that $\tau \in A^*$ is a quasi-trace function on the sub-adjacent Lie algebra A^c of the pre-Lie algebra (A, \star) .

Proposition 2.13. Let (A, \star) be a pre-Lie algebra. If $\tau \in A^*$ is a quasi-trace function on A , then $(A, \{\cdot, \cdot, \cdot\}_\tau)$ is a 3-pre-Lie algebra, where the bracket $\{-, -, -\}_\tau$ is given by

$$\{x, y, z\}_\tau := \tau(x)(y \star z) - \tau(y)(x \star z), \quad \forall x, y, z \in A. \tag{12}$$

Furthermore, the corresponding 3-Lie algebra structure is given by

$$[x, y, z]^c = \tau(x)[y, z]^c + \tau(y)[z, x]^c + \tau(z)[x, y]^c.$$

Proof. For $x, y, z \in A$, we have

$$\begin{aligned} \{x, y, z\}_\tau &= \tau(x)(y \star z) - \tau(y)(x \star z) \\ &= -(\tau(y)(x \star z) - \tau(x)(y \star z)) \\ &= -\{y, x, z\}_\tau, \end{aligned}$$

which implies that (7) holds.

For $x_1, x_2, x_3, x_4, x_5 \in A$, by the fact (A, \star) is a pre-Lie algebra and $\tau \in A^*$ is a quasi-trace function on A , we have

$$\begin{aligned} & \{[x_1, x_2, x_3]^c, x_4, x_5\}_\tau + \{x_3, [x_1, x_2, x_4]^c, x_5\}_\tau + \{x_3, x_4, \{x_1, x_2, x_5\}_\tau\}_\tau \\ & \quad - \{x_1, x_2, \{x_3, x_4, x_5\}_\tau\}_\tau \\ & = -\tau(x_1)\tau(x_4)((x_2 \star x_3) \star x_5 - x_2 \star (x_3 \star x_5) - (x_3 \star x_2) \star x_5 + x_3 \star (x_2 \star x_5)) \\ & \quad + \tau(x_2)\tau(x_4)((x_1 \star x_3) \star x_5 - x_1 \star (x_3 \star x_5) - (x_3 \star x_1) \star x_5 + x_3 \star (x_1 \star x_5)) \\ & \quad + \tau(x_1)\tau(x_3)((x_2 \star x_4) \star x_5 - x_2 \star (x_4 \star x_5) - (x_4 \star x_2) \star x_5 + x_4 \star (x_2 \star x_5)) \\ & \quad - \tau(x_2)\tau(x_3)((x_1 \star x_4) \star x_5 - x_1 \star (x_4 \star x_5) - (x_4 \star x_1) \star x_5 + x_4 \star (x_1 \star x_5)) \\ & = 0, \end{aligned}$$

which implies that (8) holds. Similarly, we have

$$\begin{aligned} & \{x_1, x_2, \{x_3, x_4, x_5\}_\tau\}_\tau + \{x_2, x_3, \{x_1, x_4, x_5\}_\tau\}_\tau + \{x_3, x_1, \{x_2, x_4, x_5\}_\tau\}_\tau \\ & \quad - \{[x_1, x_2, x_3]^c, x_4, x_5\}_\tau \\ & = \tau(x_1)\tau(x_4)((x_2 \star x_3) \star x_5 - x_2 \star (x_3 \star x_5) - (x_3 \star x_2) \star x_5 + x_3 \star (x_2 \star x_5)) \\ & \quad - \tau(x_2)\tau(x_4)((x_1 \star x_3) \star x_5 - x_1 \star (x_3 \star x_5) - (x_3 \star x_1) \star x_5 + x_3 \star (x_1 \star x_5)) \\ & \quad + \tau(x_3)\tau(x_4)((x_1 \star x_2) \star x_5 - x_1 \star (x_2 \star x_5) - (x_2 \star x_1) \star x_5 + x_2 \star (x_1 \star x_5)) \\ & = 0, \end{aligned}$$

which implies that (9) holds. The rest result follows by a direct calculation. \blacksquare

Let (A, \star) be a pre-Lie algebra and $\tau \in A^*$ a quasi-trace function on A . Since the left multiplication operation L_x defined by $L_x y = x \star y$ for $x, y \in A$, gives a representation of the Lie algebra A^c on A , by Proposition 2.7, we have

Corollary 2.14. *Keeping the above notations, the bilinear map $\mathfrak{L} : \wedge^2 A \rightarrow \mathfrak{gl}(A)$ defined by*

$$\mathfrak{L}_{x,y} z = \tau(x)(y \star z) - \tau(y)(x \star z) = \{x, y, z\}_\tau, \quad \forall x, y, z \in A \quad (13)$$

gives a representation of the induced 3-Lie algebra $(A, [-, -, -]^c)$.

Example 2.15. Let (A, \star) be a 2-dimensional pre-Lie algebra with a basis $\{e_1, e_2\}$ and the nonzero multiplication is given by

$$e_2 \star e_1 = -e_1, \quad e_2 \star e_2 = e_2.$$

Define $\tau \in A^*$ by $\tau(e_1) = \tau(e_2) = 1$. It is straightforward to check that τ is a quasi-trace function on A . By Proposition 2.13, $(A, \{-, -, -\}_\tau)$ is a 3-pre-Lie algebra, where the non-zero bracket $\{-, -, -\}_\tau$ is given by

$$\{e_1, e_2, e_1\}_\tau = -e_1, \quad \{e_1, e_2, e_2\}_\tau = e_2.$$

Example 2.16. Let (A, \star) be a 3-dimensional pre-Lie algebra with a basis $\{e_1, e_2, e_3\}$ and the nonzero multiplication is given by

$$e_3 \star e_2 = e_2, \quad e_3 \star e_3 = -e_3.$$

Define $\tau \in A^*$ by $\tau(e_1) = \tau(e_3) = 0, \tau(e_2) = 1$. It is straightforward to check that τ is a quasi-trace function on A .

By Proposition 2.13, $(A, \{-, -, -\}_\tau)$ is a 3-pre-Lie algebra, where the non-zero bracket $\{-, -, -\}_\tau$ is given by

$$\{e_2, e_3, e_2\}_\tau = e_2, \quad \{e_2, e_3, e_3\}_\tau = -e_3.$$

3. Symplectic structures and relative Rota-Baxter operators on Lie algebras and induced 3-Lie algebras

In this section, we explore the connections between symplectic structures and relative Rota-Baxter operators on Lie algebras and those structures on the 3-Lie algebras induced by Lie algebras with quasi-trace functions. With the help of quasi-trace functions on pre-Lie algebras, we also analyze the relations between pre-Lie algebra structures and 3-pre-Lie algebra structures obtained by symplectic structures and relative Rota-Baxter operators on Lie algebras and induced 3-Lie algebras, respectively.

3.1. Symplectic structures on Lie algebras and induced 3-Lie algebras

Recall that a *symplectic structure* on a Lie algebra $(\mathfrak{g}, [-, -])$ is a nondegenerate skew-symmetric bilinear form $\omega \in \wedge^2 \mathfrak{g}^*$ satisfying

$$\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0, \quad \forall x, y, z \in \mathfrak{g}. \quad (14)$$

A Lie algebra $(\mathfrak{g}, [-, -])$ with a symplectic structure ω is called a *symplectic Lie algebra*. We denote it by $(\mathfrak{g}, [-, -], \omega)$.

Definition 3.1. ([8]) A *symplectic structure* on a 3-Lie algebra $(\mathfrak{g}, [-, -, -])$ is a nondegenerate skew-symmetric bilinear form $\omega \in \wedge^2 \mathfrak{g}^*$ satisfying

$$\omega([x_1, x_2, x_3], x_4) - \omega([x_2, x_3, x_4], x_1) + \omega([x_3, x_4, x_1], x_2) - \omega([x_4, x_1, x_2], x_3) = 0, \quad (15)$$

where $x_1, x_2, x_3, x_4 \in \mathfrak{g}$. A 3-Lie algebra $(\mathfrak{g}, [-, -, -])$ with a symplectic structure ω is called a *symplectic 3-Lie algebra*. We denote it by $(\mathfrak{g}, [-, -, -], \omega)$.

Proposition 3.2. Let $(\mathfrak{g}, [-, -], \omega)$ be a symplectic Lie algebra and $\tau \in \mathfrak{g}^*$ a quasi-trace function on the Lie algebra \mathfrak{g} . Then $(\mathfrak{g}, [-, -, -]_\tau, \omega)$ is a symplectic 3-Lie algebra, where $[-, -, -]_\tau$ is given by (2).

Proof. We only need to prove that (15) holds. By (14), for $x_1, x_2, x_3, x_4 \in \mathfrak{g}$, we have

$$\begin{aligned} & \omega([x_1, x_2, x_3]_\tau, x_4) - \omega([x_2, x_3, x_4]_\tau, x_1) + \omega([x_3, x_4, x_1]_\tau, x_2) - \omega([x_4, x_1, x_2]_\tau, x_3) \\ &= \tau(x_1)(\omega([x_2, x_3], x_4) + \omega([x_3, x_4], x_2) + \omega([x_4, x_2], x_3)) \\ & \quad - \tau(x_2)(\omega([x_1, x_3], x_4) + \omega([x_3, x_4], x_1) + \omega([x_4, x_1], x_3)) \\ & \quad + \tau(x_3)(\omega([x_1, x_2], x_4) + \omega([x_2, x_4], x_1) + \omega([x_4, x_1], x_2)) \\ & \quad - \tau(x_4)(\omega([x_1, x_2], x_3) + \omega([x_2, x_3], x_1) + \omega([x_3, x_1], x_2)) \\ &= 0. \end{aligned}$$

The conclusion follows. ■

Lemma 3.3. ([14]) Let $(\mathfrak{g}, [-, -], \omega)$ be a symplectic Lie algebra. Then there exists a compatible pre-Lie algebra structure \star on \mathfrak{g} given by

$$\omega(x \star y, z) = -\omega(y, [x, z]), \quad \forall x, y, z \in \mathfrak{g}. \quad (16)$$

Proposition 3.4. *Let $(\mathfrak{g}, [-, -], \omega)$ be a symplectic Lie algebra and $\tau \in \mathfrak{g}^*$ a quasi-trace function on the Lie algebra \mathfrak{g} . Then τ is a quasi-trace function on the pre-Lie algebra (\mathfrak{g}, \star) , where \star is given by (16). Moreover, the 3-pre Lie algebra structure $\{-, -, -\}_\tau$ on \mathfrak{g} given by Proposition 2.13 is determined by*

$$\omega(\{x, y, z\}_\tau, w) = -\omega(z, \tau(x)[y, w] - \tau(y)[x, w]), \quad \forall x, y, z, w \in \mathfrak{g}. \quad (17)$$

Proof. By (14), we have

$$-\omega([x, y], z) - \omega(y \star x, z) + \omega(x \star y, z) = 0.$$

By the nondegeneracy of ω , we have $[x, y] = x \star y - y \star x$.

Because τ is a quasi-trace function on the Lie algebra $(\mathfrak{g}, [-, -])$, τ is a quasi-trace function on the pre-Lie algebra (\mathfrak{g}, \star) . The second claim follows by a direct calculation. \blacksquare

Remark 3.5. Let $(\mathfrak{g}, [-, -, -], \omega)$ be a symplectic 3-Lie algebra. It is shown in [8] that there exists a 3-pre-Lie algebra structure $\{-, -, -\}$ on \mathfrak{g} given by

$$\omega(\{x, y, z\}, w) = -\omega(z, [x, y, w]), \quad \forall x, y, z, w \in \mathfrak{g}. \quad (18)$$

Now assume that $(\mathfrak{g}, [-, -, -]_\tau, \omega)$ is a symplectic 3-Lie algebra obtained by a symplectic Lie algebra $(\mathfrak{g}, [-, -], \omega)$ with a quasi-trace function τ . Then there exists a 3-pre-Lie algebra structure $\{-, -, -\}$ on \mathfrak{g} given by

$$\omega(\{x, y, z\}, w) = -\omega(z, [x, y, w]_\tau) = -\omega(z, \tau(x)[y, w] - \tau(y)[x, w] + \tau(w)[x, y]).$$

This 3-pre-Lie algebra structure is different from the one given by (17).

Example 3.6. Let $(\mathfrak{g}, [-, -], \omega)$ be a symplectic Lie algebra with the linear basis $\{e_1, e_2, e_3, e_4\}$ and the Lie bracket and symplectic structure ω are given by

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \quad \text{and} \quad \omega = e^2 \wedge e^3 + e^1 \wedge e^4$$

respectively, where $\{e^i | 1 \leq i \leq 4\}$ is the dual basis of $\{e_i | 1 \leq i \leq 4\}$. It is straightforward to check that $\tau \in \mathfrak{g}^*$ given by

$$\tau(e_1) = \tau(e_3) = \tau(e_4) = 0, \quad \tau(e_2) = -1$$

is a quasi-trace function on the Lie algebra \mathfrak{g} . Then by Proposition 3.2 the triple $(\mathfrak{g}, [-, -, -]_\tau, \omega)$ is a symplectic 3-Lie algebra, where the bracket $[-, -, -]_\tau$ is given by

$$[e_1, e_2, e_3]_\tau = e_4.$$

By Remark 3.5, there is a 3-pre-Lie algebra structure $\{-, -, -\}$ on \mathfrak{g} given by

$$\{e_1, e_2, e_1\} = -e_2, \quad \{e_1, e_3, e_1\} = -e_3, \quad \{e_2, e_3, e_1\} = e_4.$$

Then, by Lemma 3.3, there is a pre-Lie algebra structure on \mathfrak{g} given by

$$e_1 \star e_1 = -e_2, \quad e_1 \star e_2 = e_3, \quad e_2 \star e_2 = -e_4, \quad e_3 \star e_1 = -e_4.$$

Furthermore, by Proposition 3.4, τ is a quasi-trace function on the pre-Lie algebra (\mathfrak{g}, \star) . Thus we also have a 3-pre-Lie algebra structure on \mathfrak{g} given by

$$\{e_1, e_2, e_1\}_\tau = -e_2, \quad \{e_1, e_2, e_2\}_\tau = e_3, \quad \{e_2, e_3, e_1\}_\tau = e_4.$$

3.2. Relative Rota-Baxter operators on Lie algebras and induced 3-Lie algebras

Recall that a **LieRep pair** consists of a Lie algebra $(\mathfrak{g}, [-, -])$ together with a representation ρ on V . We denote a **LieRep pair** by $(\mathfrak{g}, [-, -]; \rho)$.

Definition 3.7. ([19]) A *relative Rota-Baxter operator* on a **LieRep pair** $(\mathfrak{g}, [-, -]; \rho)$ is a linear map $T : V \rightarrow \mathfrak{g}$ satisfying

$$[T(u), T(v)] = T(\rho(T(u))(v) - \rho(T(v))(u)), \quad \forall u, v \in V. \tag{19}$$

In particular, a relative Rota-Baxter operator on a **LieRep pair** $(\mathfrak{g}, [-, -]; \text{ad})$ is called a *Rota-Baxter operator* on \mathfrak{g} .

The following result establishes the connection between relative Rota-Baxter operators on **LieRep pairs** and pre-Lie algebras.

Lemma 3.8. ([7]) Let $T : V \rightarrow \mathfrak{g}$ be a relative Rota-Baxter operator on a **LieRep pair** $(\mathfrak{g}, [-, -]; \rho)$. Define a multiplication \star^T on V by

$$u \star^T v = \rho(Tu)(v), \quad \forall u, v \in V. \tag{20}$$

Then (V, \star^T) is a pre-Lie algebra. Moreover, $(V, [-, -]^T)$ is a Lie algebra, where $[-, -]^T$ is given by

$$[u, v]^T = \rho(Tu)(v) - \rho(Tv)(u), \quad \forall u, v \in V. \tag{21}$$

Proposition 3.9. Let $T : V \rightarrow \mathfrak{g}$ be a relative Rota-Baxter operator on a **LieRep pair** $(\mathfrak{g}, [-, -]; \rho)$ and $\tau \in \mathfrak{g}^*$ be a quasi-trace function on the Lie algebra \mathfrak{g} . Then $\tilde{\tau} = \tau \circ T$ is a quasi-trace function on the pre-Lie algebra (V, \star^T) , where \star^T is given by (20). Moreover, there exists a 3-pre-Lie algebra structure $\{-, -, -\}_\tau$ on V given by

$$\{u, v, w\}_\tau = \tau(Tu)(\rho(Tv)(w)) - \tau(Tv)(\rho(Tu)(w)), \quad \forall u, v, w \in V. \tag{22}$$

Proof. For $u, v, w \in V$, by the fact that T is a relative Rota-Baxter operator and τ is a quasi-trace function, we have

$$\begin{aligned} & \tilde{\tau}(u)\tilde{\tau}(v \star^T w - w \star^T v) + \tilde{\tau}(v)\tilde{\tau}(w \star^T u - u \star^T w) + \tilde{\tau}(w)\tilde{\tau}(u \star^T v - v \star^T u) \\ &= \tau(Tu)\tau(T[v, w]^T) + \tau(Tv)\tau(T[w, u]^T) + \tau(Tw)\tau(T[u, v]^T) \\ &= \tau(Tu)\tau([Tv, Tw]) + \tau(Tv)\tau([Tw, Tu]) + \tau(Tw)\tau([Tu, Tv]) \\ &= 0, \end{aligned}$$

which implies that $\tilde{\tau}$ is a quasi-trace function on the pre-Lie algebra (V, \star^T) .

By Proposition 2.13, the second conclusion follows. ■

Recall that a **3-LieRep pair** consists of a 3-Lie algebra $(\mathfrak{g}, [-, -, -])$ and a representation ρ of \mathfrak{g} on V . We denote a **3-LieRep pair** by $(\mathfrak{g}, [-, -, -]; \rho)$ or simply by $(\mathfrak{g}; \rho)$.

Definition 3.10. ([8]) Let $(\mathfrak{g}, [-, -, -]; \rho)$ be a 3-LieRep pair.

A linear map $T : V \rightarrow \mathfrak{g}$ is called a *relative Rota-Baxter operator* on a 3-LieRep pair $(\mathfrak{g}, [-, -, -]; \rho)$ if T satisfies:

$$[Tu, Tv, Tw] = T(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v), \quad \forall u, v, w \in V. \quad (23)$$

In particular, a relative Rota-Baxter operator on a 3-LieRep pair $(\mathfrak{g}, [-, -, -]; \mathbf{ad})$ is called a *Rota-Baxter operator* on \mathfrak{g} . See [10] for more details on Rota-Baxter operators on 3-Lie algebras.

Lemma 3.11. ([8]) *Let $(\mathfrak{g}, [-, -, -]; \rho)$ be a 3-LieRep pair. If the linear map $T : V \rightarrow \mathfrak{g}$ is a relative Rota-Baxter operator on $(\mathfrak{g}, [-, -, -]; \rho)$, then there exists a 3-pre-Lie algebra structure on V defined by*

$$\{u, v, w\}^T = \rho(Tu, Tv)w, \quad \forall u, v, w \in V. \quad (24)$$

Moreover, $(V, [-, -, -]^T)$ is a 3-Lie algebra, where $[-, -, -]^T$ is defined by

$$[u, v, w]^T = \rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v, \quad \forall u, v, w \in V. \quad (25)$$

Theorem 3.12. *Let $T : V \rightarrow \mathfrak{g}$ be a relative Rota-Baxter operator on a LieRep pair $(\mathfrak{g}, [-, -]; \rho)$. If $\tau \in \mathfrak{g}^*$ is a quasi-trace function on the Lie algebra \mathfrak{g} , then T is a relative Rota-Baxter operator on the 3-LieRep pair $(\mathfrak{g}, [-, -, -]_\tau; \varrho)$, where $[-, -, -]_\tau$ and ϱ are given by (2) and (6), respectively.*

Proof. Since $T : V \rightarrow \mathfrak{g}$ is a relative Rota-Baxter operator on the LieRep pair $(\mathfrak{g}, [-, -]; \rho)$, we have

$$\begin{aligned} [Tu, Tv, Tw]_\tau &= \tau(Tu)[Tv, Tw] + \tau(Tv)[Tw, Tu] + \tau(Tw)[Tu, Tv] \\ &= \tau(Tu)T(\rho(Tv)w - \rho(Tw)v) + \tau(Tv)T(\rho(Tw)u - \rho(Tu)w) \\ &\quad + \tau(Tw)T(\rho(Tu)v - \rho(Tv)u) \\ &= T((\tau(Tu)\rho(Tv) - \tau(Tv)\rho(Tu))w + (\tau(Tv)\rho(Tw) - \tau(Tw)\rho(Tv))u \\ &\quad + (\tau(Tw)\rho(Tu) - \tau(Tu)\rho(Tw))v) \\ &= T(\varrho(Tu, Tv)w + \varrho(Tv, Tw)u + \varrho(Tw, Tu)v), \end{aligned}$$

which implies that T is a relative Rota-Baxter operator on the 3-LieRep pair $(\mathfrak{g}, [-, -, -]_\tau; \varrho)$. ■

Remark 3.13. Let $T : \mathfrak{g} \rightarrow \mathfrak{g}$ be a Rota-Baxter operator on a Lie algebra $(\mathfrak{g}, [-, -])$. Assume that $\tau \in \mathfrak{g}^*$ is a quasi-trace function on the Lie algebra \mathfrak{g} . Generally, T is not a Rota-Baxter operator on the 3-Lie algebra \mathfrak{g}_τ . More precisely, T is a Rota-Baxter operator on the 3-Lie algebra \mathfrak{g}_τ if and only if

$$\tau(x)T[Ty, Tz] + \tau(y)T[Tz, Tx] + \tau(z)T[Tx, Ty] = 0, \quad \forall x, y, z \in \mathfrak{g}.$$

Corollary 3.14. *Let $T : V \rightarrow \mathfrak{g}$ be a relative Rota-Baxter operator on a LieRep pair $(\mathfrak{g}, [-, -]; \rho)$ and $\tau \in \mathfrak{g}^*$ a quasi-trace function on the Lie algebra \mathfrak{g} . Then $(V, [-, -, -]_\tau^T)$ is a 3-Lie algebra, where $[-, -, -]_\tau^T$ is given by*

$$\begin{aligned} [u, v, w]_\tau^T &= \varrho(Tu, Tv)w + \varrho(Tv, Tw)u + \varrho(Tw, Tu)v \\ &= \{u, v, w\}_\tau + \{w, u, v\}_\tau + \{v, w, u\}_\tau, \quad \forall u, v, w \in V, \end{aligned}$$

where ϱ is given by (6) and $\{-, -, -\}_\tau$ is given by (22).

Proof. By Theorem 3.12, T is a relative Rota-Baxter operator on the 3-LieRep pair $(\mathfrak{g}, [-, -, -]_\tau; \varrho)$, $(V, [-, -, -]_\tau^T)$ is a 3-Lie algebra, where

$$[u, v, w]_\tau^T = \varrho(Tu, Tv)w + \varrho(Tv, Tw)u + \varrho(Tw, Tu)v.$$

Note that $\varrho(Tu, Tv)w = \tau(Tu)\rho(Tv)w - \tau(Tv)\rho(Tu)w = \{u, v, w\}_\tau$, and the result follows. ■

Example 3.15. Let $(\mathfrak{g}, [-, -])$ be a 3-dimensional Lie algebra with a basis $\{e_1, e_2, e_3\}$ and the nonzero multiplication is given by

$$[e_1, e_2] = e_1, \quad [e_2, e_3] = -e_3.$$

Let $T : \mathfrak{g} \rightarrow \mathfrak{g}$ be a Rota-Baxter operator on \mathfrak{g} and $\tau \in \mathfrak{g}^*$ a quasi-trace function on the Lie algebra \mathfrak{g} given by

$$T \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 4 \\ -4 & 0 & -2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

and

$$\tau(e_i) = 1, \quad 1 \leq i \leq 3.$$

Therefore, $(\mathfrak{g}, [-, -, -]_\tau^T)$ is a 3-Lie algebra, thus $T : \mathfrak{g} \rightarrow \mathfrak{g}$ is a relative Rota-Baxter operator on 3-LieRep pair $(\mathfrak{g}, [-, -, -]_\tau^T; \varrho)$, where $[-, -, -]_\tau^T$ is given by

$$[e_1, e_2, e_3]_\tau^T = -18e_1 - 9e_3.$$

4. Cohomology of Lie algebras and induced 3-Lie algebras

The Chevalley-Eilenberg cohomology theory for a Lie algebra $(\mathfrak{g}, [-, -])$ with coefficients in a representation $(V; \rho)$ is given as follows. Denote by $C_{\text{CE}}^n(\mathfrak{g}, V) := \text{Hom}(\wedge^n \mathfrak{g}, V)$, the space of n -cochains. The corresponding Chevalley-Eilenberg coboundary operator $d_\rho : C_{\text{CE}}^n(\mathfrak{g}, V) \rightarrow C_{\text{CE}}^{n+1}(\mathfrak{g}, V)$ is given by

$$\begin{aligned} d_\rho f(x_1, \dots, x_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \rho(x_i) f(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \\ &\quad + \sum_{1 \leq i < j \leq n+1}^{n+1} (-1)^{i+j} f([x_i, x_j], \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}) \end{aligned}$$

for all $f \in C_{\text{CE}}^n(\mathfrak{g}, V)$ and $x_1, x_2, \dots, x_{n+1} \in \mathfrak{g}$.

The cohomology theory for a 3-Lie algebra $(\mathfrak{g}, [-, -, -])$ with coefficients in a representation $(V; \varrho)$ is given as follows. Denote by

$$C_{3\text{-Lie}}^n(\mathfrak{g}, V) := \text{Hom}(\otimes^{n-1}(\wedge^2 \mathfrak{g}) \wedge \mathfrak{g}, V),$$

the space of n -cochains. The corresponding coboundary operator

$$\partial_\varrho : C_{3\text{-Lie}}^n(\mathfrak{g}, V) \rightarrow C_{3\text{-Lie}}^{n+1}(\mathfrak{g}, V)$$

is defined by

$$\begin{aligned} & \partial_\varrho f(\mathfrak{X}_1, \dots, \mathfrak{X}_n, z) \\ &= \sum_{1 \leq j < k \leq n} (-1)^j f(\mathfrak{X}_1, \dots, \hat{\mathfrak{X}}_j, \dots, \mathfrak{X}_{k-1}, [x_j, y_j, x_k] \wedge y_k + x_k \wedge [x_j, y_j, y_k], \dots, \mathfrak{X}_n, z) \\ & \quad + \sum_{j=1}^n (-1)^j f(\mathfrak{X}_1, \dots, \hat{\mathfrak{X}}_j, \dots, \mathfrak{X}_n, [x_j, y_j, z]) \\ & \quad + \sum_{j=1}^n (-1)^{j+1} \varrho(\mathfrak{X}_j) f(\mathfrak{X}_1, \dots, \hat{\mathfrak{X}}_j, \dots, \mathfrak{X}_n, z) \\ & \quad + (-1)^{n+1} \varrho(y_n, z) f(\mathfrak{X}_1, \dots, \mathfrak{X}_{n-1}, x_n) + (-1)^n \varrho(x_n, z) f(\mathfrak{X}_1, \dots, \mathfrak{X}_{n-1}, y_n) \end{aligned} \quad (26)$$

for all $f \in C_{3\text{-Lie}}^n(\mathfrak{g}, V)$ and $\mathfrak{X}_i = x_i \wedge y_i \in \wedge^2 \mathfrak{g}$, $z \in \mathfrak{g}$, $i = 1, 2, \dots, n$.

Let $(\mathfrak{g}, [-, -])$ be a Lie algebra and $(V; \rho)$ a representation of \mathfrak{g} . Assume that $\tau \in \mathfrak{g}^*$ is a quasi-trace function on \mathfrak{g} . Denote the cochain complexes of 3-Lie algebra \mathfrak{g}_τ associated to the representation ϱ by $(\oplus_{m=1}^{+\infty} C_{3\text{-Lie}}^m(\mathfrak{g}_\tau, V), \partial_\varrho)$, where ϱ is given by (6).

Theorem 4.1. *With the above notation, for $P \in C_{\text{CE}}^2(\mathfrak{g}, V)$, define $\tilde{P}(x_1, x_2, x_3)$ for all $x_1, x_2, x_3 \in \mathfrak{g}$ by*

$$\tilde{P}(x_1, x_2, x_3) = \tau(x_1)P(x_2, x_3) + \tau(x_2)P(x_3, x_1) + \tau(x_3)P(x_1, x_2). \quad (27)$$

For $P \in C_{\text{CE}}^{m+1}(\mathfrak{g}, V)$ ($m \geq 2$) define $\tilde{P}(\mathfrak{X}_1, \dots, \mathfrak{X}_m, x)$ by

$$\begin{aligned} \tilde{P}(\mathfrak{X}_1, \dots, \mathfrak{X}_m, x) &:= \sum_{i_1, \dots, i_m=1}^2 (-1)^{i_1 + \dots + i_m - m} \tau(x_1^{i_1}) \cdots \tau(x_m^{i_m}) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, x) \\ & \quad + \sum_{i_1, \dots, i_{m-1}=1}^2 (-1)^{i_1 + \dots + i_{m-1} + 1 - m} \tau(x_1^{i_1}) \cdots \tau(x_{m-1}^{i_{m-1}}) \tau(x) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_{m-1}^{i_{m-1}}, \mathfrak{X}_m), \end{aligned} \quad (28)$$

where $\mathfrak{X}_j := x_j^1 \wedge x_j^2 \in \wedge^2 \mathfrak{g}$, $\widehat{\mathfrak{X}}_j^1 := x_j^2$ and $\widehat{\mathfrak{X}}_j^2 := x_j^1$.

Then $\tilde{P} \in C_{3\text{-Lie}}^{m+1}(\mathfrak{g}_\tau, V)$ ($m \geq 1$), i.e., \tilde{P} is an $(m+1)$ -cochain of the 3-Lie algebra \mathfrak{g}_τ . Thus we obtain a well-defined linear map

$$\Phi : \oplus_{m=1}^{+\infty} C_{\text{CE}}^m(\mathfrak{g}, V) \longrightarrow \oplus_{m=1}^{+\infty} C_{3\text{-Lie}}^m(\mathfrak{g}_\tau, V)$$

defined by

$$\Phi(P) = \begin{cases} \tilde{P}, & \forall P \in C_{\text{CE}}^m(\mathfrak{g}, V) \ (m \geq 2), \\ P, & \forall P \in C_{\text{CE}}^1(\mathfrak{g}, V). \end{cases}$$

Furthermore, we also have $\partial_\rho \circ \Phi = \Phi \circ d_\rho$, i.e., Φ is a chain map between $(\oplus_{m=1}^{+\infty} C_{\text{CE}}^m(\mathfrak{g}, V), d_\rho)$ and $(\oplus_{m=1}^{+\infty} C_{3\text{-Lie}}^m(\mathfrak{g}_\tau, V), \partial_\rho)$. Thus Φ induces a map

$$\Phi_* : \oplus_{m=1}^{+\infty} H_{\text{CE}}^m(\mathfrak{g}, V) \longrightarrow \oplus_{m=1}^{+\infty} H_{3\text{-Lie}}^m(\mathfrak{g}_\tau, V)$$

given by $\Phi_*([P]) = [\Phi(P)]$, $\forall [P] \in H_{\text{CE}}^m(\mathfrak{g}, V)$.

Proof. For $P \in C_{\text{CE}}^{m+1}(\mathfrak{g}; V)$, it is straightforward to check that \tilde{P} defined by (28) is an $(m+1)$ -cochain of the 3-Lie algebra \mathfrak{g}_τ . For $P \in C_{\text{CE}}^1(\mathfrak{g}, V)$, it is straightforward to check that $\partial_\rho \Phi(P) = \Phi d_\rho(P)$. For $P \in C_{\text{CE}}^2(\mathfrak{g}, V)$, we have

$$\begin{aligned} & \partial_\rho \Phi(P)(x_1 \wedge x_2, x_3 \wedge x_4, x_5) \\ &= \partial_\rho \tilde{P}(x_1 \wedge x_2, x_3 \wedge x_4, x_5) \\ &= -\tilde{P}([x_1, x_2, x_3]_\tau, x_4, x_5) - \tilde{P}(x_3, [x_1, x_2, x_4]_\tau, x_5) - \tilde{P}(x_3, x_4, [x_1, x_2, x_5]_\tau) \\ &\quad + \tilde{P}(x_1, x_2, [x_3, x_4, x_5]_\tau) + \varrho(x_1, x_2) \tilde{P}(x_3, x_4, x_5) - \varrho(x_3, x_4) \tilde{P}(x_1, x_2, x_5) \\ &\quad - \varrho(x_4, x_5) \tilde{P}(x_1, x_2, x_3) + \varrho(x_3, x_5) \tilde{P}(x_1, x_2, x_4) \\ &= \tau(x_1)\tau(x_3)(-P([x_2, x_4], x_5) - P(x_4, [x_2, x_5]) + P(x_2, [x_4, x_5]) + \rho(x_2)P(x_4, x_5) \\ &\quad - \rho(x_4)P(x_2, x_5) + \rho(x_5)P(x_2, x_4)) \\ &\quad + \tau(x_1)\tau(x_4)(P([x_2, x_3], x_5) + P(x_3, [x_2, x_5]) - P(x_2, [x_3, x_5]) + \rho(x_2)P(x_3, x_5) \\ &\quad + \rho(x_3)P(x_2, x_5) + \rho(x_5)P(x_2, x_3)) \\ &\quad + \tau(x_2)\tau(x_3)(P([x_1, x_4], x_5) + P(x_4, [x_1, x_5]) - P(x_1, [x_4, x_5]) + \rho(x_1)P(x_4, x_5) \\ &\quad + \rho(x_4)P(x_1, x_5) + \rho(x_5)P(x_1, x_4)) \\ &\quad + \tau(x_2)\tau(x_4)(-P([x_1, x_3], x_5) - P(x_3, [x_1, x_5]) + P(x_1, [x_3, x_5]) + \rho(x_1)P(x_3, x_5) \\ &\quad - \rho(x_3)P(x_1, x_5) + \rho(x_5)P(x_1, x_3)) \\ &\quad + \tau(x_1)\tau(x_5)(-P([x_2, x_3], x_4) - P(x_3, [x_2, x_4]) + P(x_2, [x_3, x_4]) + \rho(x_2)P(x_3, x_4) \\ &\quad - \rho(x_3)P(x_2, x_4) + \rho(x_5)P(x_2, x_3)) \\ &\quad + \tau(x_2)\tau(x_5)(P([x_1, x_3], x_4) + P(x_3, [x_1, x_4]) - P(x_1, [x_3, x_4]) + \rho(x_1)P(x_3, x_4) \\ &\quad + \rho(x_3)P(x_1, x_4) + \rho(x_4)P(x_1, x_3)) \\ &= \widetilde{d_\rho P}(x_1 \wedge x_2, x_3 \wedge x_4, x_5). \end{aligned}$$

Moreover, for $P \in C_{\text{CE}}^{m+1}(\mathfrak{g}, V)$ ($m \geq 2$) and $\mathfrak{X}_j := x_j^1 \wedge x_j^2 \in \wedge^2 \mathfrak{g}$ whenever $j = 1, 2, \dots, m+1$ and $x_{m+2} \in \mathfrak{g}$, we have

$$\begin{aligned} & \partial_\rho \Phi(P)(\mathfrak{X}_1, \dots, \mathfrak{X}_m, \mathfrak{X}_{m+1}, x_{m+2}) \\ &= \partial_\rho \tilde{P}(\mathfrak{X}_1, \dots, \mathfrak{X}_m, \mathfrak{X}_{m+1}, x_{m+2}) \\ &= \sum_{1 \leq j < k \leq m+1} (-1)^j \tilde{P}(\mathfrak{X}_1, \dots, \widehat{\mathfrak{X}}_j, \dots, \mathfrak{X}_{k-1}, [x_j^1, x_j^2, x_k^1]_\tau \wedge x_k^2 + x_k^1 \wedge [x_j^1, x_j^2, x_k^2]_\tau, \dots, \mathfrak{X}_{m+1}, x_{m+2}) \\ &\quad + \sum_{j=1}^{m+1} (-1)^j \tilde{P}(\mathfrak{X}_1, \dots, \widehat{\mathfrak{X}}_j, \dots, \mathfrak{X}_{m+1}, [x_j^1, x_j^2, x_{m+2}]_\tau) \\ &\quad + \sum_{j=1}^{m+1} (-1)^{j+1} \varrho(\mathfrak{X}_j) \tilde{P}(\mathfrak{X}_1, \dots, \widehat{\mathfrak{X}}_j, \dots, \mathfrak{X}_m, \mathfrak{X}_{m+1}, x_{m+2}) \\ &\quad + (-1)^m \varrho(x_{m+1}^2, x_{m+2}) \tilde{P}(\mathfrak{X}_1, \dots, \mathfrak{X}_m, x_{m+1}^1) + (-1)^{m+1} \varrho(x_{m+1}^1, x_{m+2}) \tilde{P}(\mathfrak{X}_1, \dots, \mathfrak{X}_m, x_{m+1}^2) \end{aligned}$$

$$\begin{aligned}
&= \sum_{1 \leq j < k \leq m+1} \sum_{\substack{i_1, \dots, \widehat{i_j}, \dots, \\ i_k, \dots, i_{m+1}=1}}^2 (-1)^{i_1 + \dots + \widehat{i_j} + \dots + i_{m+1} + j - m} \tau(x_1^{i_1}) \cdots \widehat{\tau(x_j^{i_j})} \cdots \tau(x_k^{i_k}) \cdots \tau(x_{m+1}^{i_{m+1}}) \\
&P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_{k-1}^{i_{k-1}}, [x_j^1, x_j^2, \widehat{\mathfrak{X}}_k], \widehat{\mathfrak{X}}_{k+1}^{i_{k+1}}, \dots, \widehat{\mathfrak{X}}_{m+1}^{i_{m+1}}, x_{m+2}) \\
&+ \sum_{1 \leq j < k \leq m+1} \sum_{\substack{i_1, \dots, \widehat{i_j}, \dots, \\ i_k, \dots, i_m=1}}^2 (-1)^{i_1 + \dots + \widehat{i_j} + \dots + i_m + 1 + j - m} \tau(x_1^{i_1}) \cdots \widehat{\tau(x_j^{i_j})} \cdots \tau(x_k^{i_k}) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \\
&P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_{k-1}^{i_{k-1}}, [x_j^1, x_j^2, \widehat{\mathfrak{X}}_k], \widehat{\mathfrak{X}}_{k+1}^{i_{k+1}}, \dots, \widehat{\mathfrak{X}}_{m+1}) \\
&+ \sum_{j=1}^{m+1} \sum_{\substack{i_1, \dots, \widehat{i_j}, \dots, \\ i_{m+1}=1}}^2 (-1)^{i_1 + \dots + \widehat{i_j} + \dots + i_{m+1} + j - m} \tau(x_1^{i_1}) \cdots \widehat{\tau(x_j^{i_j})} \cdots \tau(x_{m+1}^{i_{m+1}}) \\
&P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_{m+1}^{i_{m+1}}, [x_j^1, x_j^2, x_{m+2}] \tau) \\
&+ \sum_{j=1}^{m+1} \sum_{\substack{i_1, \dots, \widehat{i_j}, \dots, \\ i_{m+1}=1}}^2 (-1)^{i_1 + \dots + \widehat{i_j} + \dots + i_{m+1} + 1 + j - m} \tau(x_1^{i_1}) \cdots \widehat{\tau(x_j^{i_j})} \cdots \tau(x_{m+1}^{i_{m+1}}) \tau(x_j^1) \\
&\rho(x_j^2) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_{m+1}^{i_{m+1}}, x_{m+2}) \\
&+ \sum_{j=1}^{m+1} \sum_{\substack{i_1, \dots, \widehat{i_j}, \dots, \\ i_{m+1}=1}}^2 (-1)^{i_1 + \dots + \widehat{i_j} + \dots + i_{m+1} + j - m} \tau(x_1^{i_1}) \cdots \widehat{\tau(x_j^{i_j})} \cdots \tau(x_{m+1}^{i_{m+1}}) \tau(x_j^2) \\
&\rho(x_j^1) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_{m+1}^{i_{m+1}}, x_{m+2}) \\
&+ \sum_{j=1}^m \sum_{\substack{i_1, \dots, \widehat{i_j}, \dots, \\ i_m=1}}^2 (-1)^{i_1 + \dots + \widehat{i_j} + \dots + i_m + j - m} \tau(x_1^{i_1}) \cdots \widehat{\tau(x_j^{i_j})} \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \tau(x_j^1) \\
&\rho(x_j^2) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, \widehat{\mathfrak{X}}_{m+1}) \\
&+ \sum_{j=1}^m \sum_{\substack{i_1, \dots, \widehat{i_j}, \dots, \\ i_m=1}}^2 (-1)^{i_1 + \dots + \widehat{i_j} + \dots + i_m + j + 1 - m} \tau(x_1^{i_1}) \cdots \widehat{\tau(x_j^{i_j})} \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \tau(x_j^2) \\
&\rho(x_j^1) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, \widehat{\mathfrak{X}}_{m+1}) \\
&+ \sum_{i_1, \dots, i_{m-1}=1}^2 (-1)^{i_1 + \dots + i_{m-1} + 1} \tau(x_1^{i_1}) \cdots \tau(x_{m-1}^{i_{m-1}}) \tau(x_{m+2}) \tau(x_{m+1}^1) \rho(x_{m+1}^2) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_{m-1}^{i_{m-1}}, \widehat{\mathfrak{X}}_m) \\
&+ \sum_{i_1, \dots, i_{m-1}=1}^2 (-1)^{i_1 + \dots + i_{m-1}} \tau(x_1^{i_1}) \cdots \tau(x_{m-1}^{i_{m-1}}) \tau(x_{m+2}) \tau(x_{m+1}^2) \rho(x_{m+1}^1) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_{m-1}^{i_{m-1}}, \widehat{\mathfrak{X}}_m) \\
&+ \sum_{i_1, \dots, i_m=1}^2 (-1)^{i_1 + \dots + i_m} \tau(x_1^{i_1}) \cdots \tau(x_m^{i_m}) \tau(x_{m+1}^2) \rho(x_{m+2}) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, x_{m+1}^1) \\
&+ \sum_{i_1, \dots, i_m=1}^2 (-1)^{i_1 + \dots + i_m + 1} \tau(x_1^{i_1}) \cdots \tau(x_m^{i_m}) \tau(x_{m+1}^1) \rho(x_{m+2}) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, x_{m+1}^2) \\
&+ \sum_{i_1, \dots, i_{m-1}=1}^2 (-1)^{i_1 + \dots + i_{m-1} + 1} \tau(x_1^{i_1}) \cdots \tau(x_{m-1}^{i_{m-1}}) \tau(x_{m+1}^1) \tau(x_{m+1}^2) \rho(x_{m+2}) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_{m-1}^{i_{m-1}}, \widehat{\mathfrak{X}}_m) \\
&+ \sum_{i_1, \dots, i_{m-1}=1}^2 (-1)^{i_1 + \dots + i_{m-1}} \tau(x_1^{i_1}) \cdots \tau(x_{m-1}^{i_{m-1}}) \tau(x_{m+1}^2) \tau(x_{m+1}^1) \rho(x_{m+2}) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_{m-1}^{i_{m-1}}, \widehat{\mathfrak{X}}_m)
\end{aligned}$$

$$\begin{aligned}
 &+ \sum_{i_1, \dots, i_m=1}^2 (-1)^{i_1+\dots+i_m-1} \tau(x_1^{i_1}) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \rho(x_{m+1}^2) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, x_{m+1}^1) \\
 &+ \sum_{i_1, \dots, i_m=1}^2 (-1)^{i_1+\dots+i_m} \tau(x_1^{i_1}) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \rho(x_{m+1}^1) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, x_{m+1}^2) \\
 &+ \sum_{i_1, \dots, i_{m-1}=1}^2 (-1)^{i_1+\dots+i_{m-1}} \tau(x_1^{i_1}) \cdots \tau(x_{m-1}^{i_{m-1}}) \tau(x_{m+1}^1) \tau(x_{m+2}) \rho(x_{m+1}^2) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_{m-1}^{i_{m-1}}, \widehat{\mathfrak{X}}_m) \\
 &+ \sum_{i_1, \dots, i_{m-1}=1}^2 (-1)^{i_1+\dots+i_{m-1}+1} \tau(x_1^{i_1}) \cdots \tau(x_{m-1}^{i_{m-1}}) \tau(x_{m+1}^2) \tau(x_{m+2}) \rho(x_{m+1}^1) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_{m-1}^{i_{m-1}}, \widehat{\mathfrak{X}}_m) \\
 &= \sum_{1 \leq j < k \leq m+1} \sum_{\substack{i_1, \dots, i_j, \dots, \\ i_k, \dots, i_{m+1}=1}}^2 (-1)^{i_1+\dots+i_j+\dots+i_{m+1}+j-m-1} \tau(x_1^{i_1}) \cdots \tau(x_j^{i_j}) \cdots \tau(x_k^{i_k}) \cdots \tau(x_{m+1}^{i_{m+1}}) \\
 &P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_{k-1}^{i_{k-1}}, [\widehat{\mathfrak{X}}_j^{i_j}, \widehat{\mathfrak{X}}_k^{i_k}], \widehat{\mathfrak{X}}_{k+1}^{i_{k+1}}, \dots, \widehat{\mathfrak{X}}_{m+1}^{i_{m+1}}, x_{m+2}) \\
 &+ \sum_{1 \leq j < k \leq m+1} \sum_{\substack{i_1, \dots, i_j, \dots, \\ i_k, \dots, i_{m+1}=1}}^2 (-1)^{i_1+\dots+i_j+\dots+i_{m+1}+j-m} \tau(x_1^{i_1}) \cdots \tau(x_j^{i_j}) \cdots \tau(x_k^{i_k}) \cdots \tau(x_{m+1}^{i_{m+1}}) \tau(\widehat{\mathfrak{X}}_k^{i_k}) \\
 &P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_{k-1}^{i_{k-1}}, [x_j^1, x_j^2], \dots, \widehat{\mathfrak{X}}_{m+1}^{i_{m+1}}, x_{m+2}) \\
 &+ \sum_{1 \leq j < k \leq m+1} \sum_{\substack{i_1, \dots, i_j, \dots, \\ i_k, \dots, i_m=1}}^2 (-1)^{i_1+\dots+i_j+\dots+i_m+j-m} \tau(x_1^{i_1}) \cdots \tau(x_j^{i_j}) \cdots \tau(x_k^{i_k}) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \\
 &P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_{k-1}^{i_{k-1}}, [\widehat{\mathfrak{X}}_j^{i_j}, \widehat{\mathfrak{X}}_k^{i_k}], \dots, \widehat{\mathfrak{X}}_{m+1}) \\
 &+ \sum_{1 \leq j < k \leq m+1} \sum_{\substack{i_1, \dots, i_j, \\ \dots, i_m=1}}^2 (-1)^{i_1+\dots+i_j+\dots+i_m+j-m-1} \tau(x_1^{i_1}) \cdots \tau(x_j^{i_j}) \cdots \tau(x_k^{i_k}) \cdots \tau(x_m^{i_m}) \tau(\widehat{\mathfrak{X}}_k^{i_k}) \tau(x_{m+2}) \\
 &P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_{k-1}^{i_{k-1}}, [x_j^1, x_j^2], \dots, \widehat{\mathfrak{X}}_{m+1}) \\
 &+ \sum_{j=1}^{m+1} \sum_{i_1, \dots, i_j, \dots, i_{m+1}=1}^2 (-1)^{i_1+\dots+i_j+\dots+i_{m+1}+j-m-1} \tau(x_1^{i_1}) \cdots \tau(x_j^{i_j}) \cdots \tau(x_{m+1}^{i_{m+1}}) \\
 &P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, \widehat{\mathfrak{X}}_{m+1}^{i_{m+1}}, [\widehat{\mathfrak{X}}_j^{i_j}, x_{m+2}]) \\
 &+ \sum_{j=1}^{m+1} \sum_{i_1, \dots, i_j, \dots, i_{m+1}=1}^2 (-1)^{i_1+\dots+i_j+\dots+i_{m+1}+j-m} \tau(x_1^{i_1}) \cdots \tau(x_j^{i_j}) \cdots \tau(x_{m+1}^{i_{m+1}}) \tau(x_{m+2}) \\
 &P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_{m+1}^{i_{m+1}}, [x_j^1, x_j^2]) \\
 &+ \sum_{j=1}^{m+1} \sum_{i_1, \dots, i_j, \dots, i_{m+1}=1}^2 (-1)^{i_1+\dots+1+\dots+i_{m+1}+j-m} \tau(x_1^{i_1}) \cdots \tau(x_j^1) \cdots \tau(x_{m+1}^{i_{m+1}}) \\
 &\rho(x_j^2) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_{m+1}^{i_{m+1}}, x_{m+2}) \\
 &+ \sum_{j=1}^{m+1} \sum_{i_1, \dots, i_j, \dots, i_{m+1}=1}^2 (-1)^{i_1+\dots+2+\dots+i_{m+1}+j-m} \tau(x_1^{i_1}) \cdots \tau(x_j^2) \cdots \tau(x_{m+1}^{i_{m+1}}) \\
 &\rho(x_j^1) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_{m+1}^{i_{m+1}}, x_{m+2}) \\
 &+ \sum_{j=1}^m \sum_{i_1, \dots, i_j, \dots, i_m=1}^2 (-1)^{i_1+\dots+1+\dots+i_m-m+j+1} \tau(x_1^{i_1}) \cdots \tau(x_j^1) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \\
 &\rho(x_j^2) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, \widehat{\mathfrak{X}}_{m+1})
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^m \sum_{i_1, \dots, \widehat{i_j}, \dots, i_m=1}^2 (-1)^{i_1+\dots+2+\dots+i_m-m+j+1} \tau(x_1^{i_1}) \cdots \tau(x_j^2) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \\
& \rho(x_j^1) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, \mathfrak{X}_{m+1}) \\
& + \sum_{i_1, \dots, i_{m-1}=1}^2 (-1)^{i_1+\dots+i_{m-1}+1} \tau(x_1^{i_1}) \cdots \tau(x_{m-1}^{i_{m-1}}) \tau(x_{m+1}^1) \tau(x_{m+2}) \rho(x_{m+1}^2) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_{m-1}^{i_{m-1}}, \mathfrak{X}_m) \\
& + \sum_{i_1, \dots, i_{m-1}=1}^2 (-1)^{i_1+\dots+i_{m-1}} \tau(x_1^{i_1}) \cdots \tau(x_{m-1}^{i_{m-1}}) \tau(x_{m+1}^2) \tau(x_{m+2}) \rho(x_{m+1}^1) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_{m-1}^{i_{m-1}}, \mathfrak{X}_m) \\
& + \sum_{i_1, \dots, i_m=1}^2 (-1)^{i_1+\dots+i_m+1} \tau(x_1^{i_1}) \cdots \tau(x_{m-1}^{i_{m-1}}) \tau(x_m^{i_m}) \tau(x_{m+1}^2) \rho(x_{m+2}) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, x_{m+1}^1) \\
& + \sum_{i_1, \dots, i_m=1}^2 (-1)^{i_1+\dots+i_m} \tau(x_1^{i_1}) \cdots \tau(x_{m-1}^{i_{m-1}}) \tau(x_m^{i_m}) \tau(x_{m+1}^1) \rho(x_{m+2}) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, x_{m+1}^2) \\
& + \sum_{i_1, \dots, i_m=1}^2 (-1)^{i_1+\dots+i_m-1} \tau(x_1^{i_1}) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \rho(x_{m+1}^2) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, x_{m+1}^1) \\
& + \sum_{i_1, \dots, i_m=1}^2 (-1)^{i_1+\dots+i_m} \tau(x_1^{i_1}) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \rho(x_{m+1}^1) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, x_{m+1}^2) \\
& + \sum_{i_1, \dots, i_{m-1}=1}^2 (-1)^{i_1+\dots+i_{m-1}} \tau(x_1^{i_1}) \cdots \tau(x_{m-1}^{i_{m-1}}) \tau(x_{m+1}^1) \tau(x_{m+2}) \rho(x_{m+1}^2) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_{m-1}^{i_{m-1}}, \mathfrak{X}_m) \\
& + \sum_{i_1, \dots, i_{m-1}=1}^2 (-1)^{i_1+\dots+i_{m-1}} \tau(x_1^{i_1}) \cdots \tau(x_{m-1}^{i_{m-1}}) \tau(x_{m+1}^2) \tau(x_{m+2}) \rho(x_{m+1}^1) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_{m-1}^{i_{m-1}}, \mathfrak{X}_m) \\
& = \sum_{\substack{i_1, \dots, i_j, \dots, \\ i_{m+1}=1}}^2 (-1)^{i_1+\dots+i_{m+1}-m-1} \tau(x_1^{i_1}) \cdots \tau(x_j^{i_j}) \cdots \tau(x_{m+1}^{i_{m+1}}) d_\rho P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_j^{i_j}, \dots, \widehat{\mathfrak{X}}_{m+1}^{i_{m+1}}, x_{m+2}) \\
& + \sum_{1 \leq j < k \leq m} \sum_{\substack{i_1, \dots, i_j, \dots, \\ i_k, \dots, i_m=1}}^2 (-1)^{i_1+\dots+i_j+\dots+i_m+j-m} \tau(x_1^{i_1}) \cdots \tau(x_j^{i_j}) \cdots \tau(x_k^{i_k}) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \\
& P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_{k-1}^{i_{k-1}}, [\widehat{\mathfrak{X}}_j^{i_j}, \widehat{\mathfrak{X}}_k^{i_k}], \dots, \mathfrak{X}_{m+1}) \\
& + \sum_{1 \leq j < m+1} \sum_{i_1, \dots, i_j, \dots, i_m=1}^2 (-1)^{i_1+\dots+i_j+\dots+i_m+j-m} \tau(x_1^{i_1}) \cdots \tau(x_j^{i_j}) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \\
& P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, [\widehat{\mathfrak{X}}_j^{i_j}, x_{m+1}^1] \wedge x_{m+1}^2) \\
& + \sum_{1 \leq j < m+1} \sum_{i_1, \dots, i_j, \dots, i_m=1}^2 (-1)^{i_1+\dots+i_j+\dots+i_m+j-m} \tau(x_1^{i_1}) \cdots \tau(x_j^{i_j}) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \\
& P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, x_{m+1}^1 \wedge [\widehat{\mathfrak{X}}_j^{i_j}, x_{m+1}^2]) \\
& + \sum_{i_1, \dots, i_j, \dots, i_m=1}^2 (-1)^{i_1+\dots+i_j+\dots+i_m+1} \tau(x_1^{i_1}) \cdots \tau(x_j^{i_j}) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \\
& P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, [x_{m+1}^1, x_{m+1}^2]) \\
& + \sum_{j=1}^m \sum_{i_1, \dots, \widehat{i_j}, \dots, i_m=1}^2 (-1)^{i_1+\dots+1+\dots+i_m-m+j+1} \tau(x_1^{i_1}) \cdots \tau(x_j^1) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \rho(x_j^2) \\
& P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, \mathfrak{X}_{m+1})
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^m \sum_{i_1, \dots, \widehat{i_j}, \dots, i_m=1}^2 (-1)^{i_1+\dots+2+\dots+i_m-m+j+1} \tau(x_1^{i_1}) \cdots \tau(x_j^2) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \rho(x_j^1) \\
 & P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{j}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, \mathfrak{X}_{m+1}) \\
 & + \sum_{i_1, \dots, i_m=1}^2 (-1)^{i_1+\dots+i_m-1} \tau(x_1^{i_1}) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \rho(x_{m+1}^2) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, x_{m+1}^1) \\
 & + \sum_{i_1, \dots, i_m=1}^2 (-1)^{i_1+\dots+i_m} \tau(x_1^{i_1}) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \rho(x_{m+1}^1) P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, x_{m+1}^2) \\
 & = \sum_{i_1, \dots, i_j, \dots, i_{m+1}=1}^2 (-1)^{i_1+\dots+i_{m+1}-m-1} \tau(x_1^{i_1}) \cdots \tau(x_j^{i_j}) \cdots \tau(x_{m+1}^{i_{m+1}}) \\
 & d_\rho P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_j^{i_j}, \dots, \widehat{\mathfrak{X}}_{m+1}^{i_{m+1}}, x_{m+2}) \\
 & + \sum_{i_1, \dots, i_j, \dots, i_m=1}^2 (-1)^{i_1+\dots+i_m-m} \tau(x_1^{i_1}) \cdots \tau(x_j^{i_j}) \cdots \tau(x_m^{i_m}) \tau(x_{m+2}) \\
 & d_\rho P(\widehat{\mathfrak{X}}_1^{i_1}, \dots, \widehat{\mathfrak{X}}_j^{i_j}, \dots, \widehat{\mathfrak{X}}_m^{i_m}, \mathfrak{X}_{m+1}) \\
 & = \widetilde{d}_\rho P(\mathfrak{X}_1, \dots, \mathfrak{X}_m, \mathfrak{X}_{m+1}, x_{m+2}) \\
 & = \Phi d_\rho(P)(\mathfrak{X}_1, \dots, \mathfrak{X}_m, \mathfrak{X}_{m+1}, x_{m+2}).
 \end{aligned}$$

Thus Φ is a chain map between $(\bigoplus_{m=1}^{+\infty} C_{\text{CE}}^m(\mathfrak{g}, V), d_\rho)$ and $(\bigoplus_{m=1}^{+\infty} C_{3\text{-Lie}}^m(\mathfrak{g}_\tau, V), \partial_\rho)$. ■

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Ming Chen, School of Mathematics and Statistics, Northeast Normal University, Changchun, Jilin, P. R. China; chenm468@nenu.edu.cn.

Qi Wang, School of Mathematics and Statistics, Changchun University of Technology, Changchun, Jilin, P. R. China; wangqi2018@ccut.edu.cn.

Senrong Xu, Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu, P. R. China; senrongxu@ujs.edu.cn.

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