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A Correspondence Between Boundary Coefficients of Real Flag Manifolds and Height of Roots

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Abstract. We prove a new formula for the cellular homology coefficients of real flag manifolds in terms of the height of certain roots. For real flag manifolds of type A, we get simple expressions for the coefficients that allow us to compute the first and second integral homology groups exhibiting their generators.

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1. Introduction

In this work, we revisit the determination of the boundary map coefficients for the cellular homology of real flag manifolds, a problem equivalent to finding the incidence coefficients of the differential map for the cohomology. It follows a series of previous work on this subject like Kocherlakota [6] with its Morse Theory approach, Casian-Stanton [2] by a representation theoretical view, Rabelo-San Martin [14] within the frame of the cellular homology and, lastly, Matszangosz [9] through the cohomology of the Vassiliev complex. We prove a new formula for these coefficients with respect to the height of certain roots.

A generalized flag manifold $\mathbb{F}$ is a homogeneous space G/P, where G is a real noncompact semisimple Lie Group and P is a parabolic subgroup. It admits a cellular decomposition called Bruhat decomposition, where the cells are the Schubert cells and parametrized by the Weyl group $\mathcal{W}$. Consider a pair of Schubert cells $\mathcal{S}_w, \mathcal{S}_{w'}$ such that w covers w', i.e., $w' \leq w$ by the Bruhat-Chevalley order and $\ell(w') = \ell(w) - 1$. In this case, there is a root β such that $w = s_\beta \cdot w'$. According to both [6] and [14], we may summarize how to compute the coefficient $c(w, w')$ as it follows: Consider the set Π_w of positive roots sent to negative by w^{-1} and denote by $\phi(w)$ the sum of all such roots. Then, $c(w, w') = \pm(1 + (-1)^{\kappa(w,w')})$, where κ is obtained from the equation $\phi(w) - \phi(w') = \kappa(w, w') \cdot \beta$. The papers [13] and [7] apply this procedure in the context of the Isotropic Grassmannians and the results obtained (for instance, see [8], Theorem 3.12) suggest a formula for the coefficients in terms of the height of some root.

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Theorem 1.1. *If we write $w = w' \cdot s_\gamma$, it follows that $\kappa(w, w') = \mathrm{ht}(\gamma^\vee)$, where $\mathrm{ht}(\gamma^\vee)$ is the height of the corresponding dual coroot $\gamma^\vee$ with respect to the dual system of roots.*

In other words, the parity of $\mathrm{ht}(\gamma^\vee)$ determines whether the coefficient $c(w, w')$ is zero or ± 2. Notice the role developed by the roots β and γ or, equivalently, by the right and left actions in the understanding of this topic.

In the context of flag manifolds of type A_{n-1}, this theorem simplifies the task of computing the coefficient. Schubert cells are parametrized by the symmetric group S_n. If we denote a permutation by the one-line notation as $w = w_1 \cdots w_n$ then w covers w' if and only if there is $i < j$ such that $w = w' \cdot (i, j)$, $w'_i < w'_j$ and there is no $i < k < j$ such that $w'_i < w_k < w'_j$. Since the simple system of roots is equal to its dual, it is immediate to conclude that $\kappa(w, w') = j - i$ (see Proposition 5.2).

This provides a very nice link between combinatorics and topology. As a consequence, we retrieve the orientability condition for this class of flags according to Patrão-San Martin-Santos-Seco [11] (Proposition 5.3).

We may also go further and derive an explicit formula for the first and second homology groups of any partial flag manifold of type A.

Theorem 1.2. *Consider a partial flag manifold $\mathbb{F}_\Theta$ of $G = \mathrm{Sl}(n, \mathbb{R})$, where $\Theta \subset \Sigma = \{a_1, \ldots, a_{n-1}\}$, and the simple reflections $s_i = s_{a_i}$, for $a_i \in \Sigma$. Denote by r_Θ the number of connected components of the Dynkin diagram of Θ.*

(1) *For $n \geqslant 3$, the first homology group is*

$$H_1(\mathbb{F}_\Theta, \mathbb{Z}) \cong (\mathbb{Z}_2)^{n-|\Theta|-1}$$

and it is generated by the set of Schubert cells $\mathcal{S}_{s_i}$ such that $a_i \in \Sigma - \Theta$.

(2) *For $n \geqslant 4$, the second homology group is*

$$H_2(\mathbb{F}_\Theta, \mathbb{Z}) \cong (\mathbb{Z}_2)^{N_\Theta}, \text{ where } N_\Theta = \binom{n - |\Theta| - 1}{2} + r_\Theta - 1$$

and it is generated by the set

- $X_{i,j} = \mathcal{S}_{s_i s_j}$, *for any $i, j \in [n-1]$ such that $j - i \geqslant 2$ and $a_i, a_j \in \Sigma - \Theta$;*
- $X_{i,i+1} = \mathcal{S}_{s_i s_{i+1}} - \mathcal{S}_{s_{i+2} s_{i+1}}$, *for every $i \in [n-3]$ such that $a_{i+1} \in \Sigma - \Theta$.*

Concerning the first homology group, being the abelianization of the fundamental group, our results match those of Wiggerman [17] which gives a presentation of $\pi_1(\mathbb{F}_\Theta)$ with generators in the set $\Sigma \setminus \Theta$ subject to some relations. In this sense, although it is not a kind of new result, our combinatorial approach presents a direct and simple computation for such abelianization. A similar result that should be mentioned is that in the context of the 2-homotopy of the complex flag manifolds, Grama-Seco ([5], Theorem 2.4) obtained by geometric methods that it is generated by 2-spheres which are also enumerated by the complement of Θ inside Σ. For the second homology group, Del Barco-San Martin ([3], Theorem 4.1) has shown that it is a torsion group $\mathbb{Z}_2^N$ without providing any description for N.

The article is organized as follows. In Section 2, we introduce the main definitions about flag manifolds, root and coroot systems, Bruhat decomposition, and combinatorics of the symmetric group. In Section 3, we describe a formula for the elements of Π_w. In Section 4, we prove the height formula for the coefficients. In Section 5, we study the flag manifolds of type A and present their respective first and second homology groups. Finally, in Section 6 we point out some further directions.

2. Preliminaries

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}$ be the set of integers. For $n, m \in \mathbb{Z}$, where $n \leqslant m$, denote the set $[n, m] = \{n, n+1, \dots, m\}$. For $n \in \mathbb{N}$, denote $[n] = [1, n]$.

We define the flag manifolds as homogeneous spaces G/P where G is a non-compact semi-simple Lie group and P is a parabolic subgroup of G. The flag manifolds for the several groups G with non-compact real semi-simple Lie algebra $\mathfrak{g}$ are the same.

If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is a Cartan decomposition, let $\mathfrak{a}$ be a maximal abelian sub-algebra contained in $\mathfrak{s}$. A sub-algebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be a *Cartan sub-algebra* if $\mathfrak{h}_{\mathbb{C}}$ is a Cartan sub-algebra of $\mathfrak{g}_{\mathbb{C}}$. If $\mathfrak{h} = \mathfrak{a}$ is a Cartan sub-algebra of $\mathfrak{g}$, we say that $\mathfrak{g}$ is a split real form of $\mathfrak{g}_{\mathbb{C}}$.

Let Π be the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$ and fix a simple system of roots $\Sigma \subset \Pi$. Denote by $\Pi^{\pm}$ respectively the set of positive and negative roots and by $\mathfrak{a}^+$ the Weyl chamber $\mathfrak{a}^+ = \{H \in \mathfrak{a} \colon \alpha(H) > 0 \text{ for all } \alpha \in \Sigma\}$. The direct sum of root spaces corresponding to the positive roots is denoted by $\mathfrak{n} = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$. The Iwasawa decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. The notations K and N refer respectively to the connected subgroups whose Lie algebras are $\mathfrak{k}$ and $\mathfrak{n}$.

A minimal parabolic sub-algebra of $\mathfrak{g}$ is given by $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Let P be the minimal parabolic subgroup with Lie algebra $\mathfrak{p}$. Note that P is the normalizer of $\mathfrak{p}$ in G. We call $\mathbb{F} = G/P$ the maximal flag manifold of G and denote by b_0 the base point $1 \cdot P$ in G/P.

Now, assume that $\Theta \subset \Sigma$ is any subset of simple roots. Such a choice provides a very interesting way to obtain several flag manifolds, called partial flag manifolds as we now explain. Denote by $\mathfrak{g}(\Theta)$ the semi-simple Lie algebra generated by $\mathfrak{g}_{\pm\alpha}$, $\alpha \in \Theta$. Let $G(\Theta)$ be the connected group with Lie algebra $\mathfrak{g}(\Theta)$. Let $\mathfrak{n}_\Theta$ be the sub-algebra generated by the roots spaces $\mathfrak{g}_{-\alpha}$, $\alpha \in \Theta$ and consider $\mathfrak{p}_\Theta = \mathfrak{n}_\Theta \oplus \mathfrak{p}$. The normalizer P_Θ of $\mathfrak{p}_\Theta$ in G is a standard parabolic subgroup which contains P. Finally, the corresponding flag manifold $\mathbb{F}_\Theta = G/P_\Theta$ is called a partial flag manifold of G of type Θ. We denote by b_Θ the base point $1 \cdot P_\Theta$ in G/P_Θ.

Now, we highlight some results about the dual system of a root system which will play a key role in the sequel. We follow closely the book of Perrin [12].

Let E be a finite dimensional vector space. A set $\Pi \subset E$ is an abstract system of roots if it is finite, spans E and does not contain 0 and satisfies

(1) for every $\alpha \in \Pi$, there exists a reflection s_α with respect to α such that $s_\alpha(\Pi) = \Pi$;

(2) for every $\alpha, \beta \in \Pi$, $s_\alpha(\beta) - \beta$ is an integer multiple of α.

The Weyl group $\mathcal{W}$ of the root system Π is the group generated by reflections s_α, $\alpha \in \Pi$. It is possible to show that there exists an inner product $\langle \cdot, \cdot \rangle$ in E which is invariant by $\mathcal{W}$.

It follows that s_α is the corresponding orthogonal reflection

$$s_\alpha(\beta) = \beta - 2\frac{\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

From the identification of E with E^* by $\langle\cdot,\cdot\rangle$, for each root $\alpha \in \Pi$, we define $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$. We call $\alpha^\vee$ the *coroot* of α. It follows that if $\beta\in\Pi$ is another root,

$$s_\alpha(\beta) = \beta - \langle \alpha^\vee, \beta\rangle\, \alpha. \tag{1}$$

We know that if we choose $\langle\cdot,\cdot\rangle$ to be the Killing form, then the set of roots Π associated to the Lie algebra $\mathfrak{g}$ is an abstract root system. Furthermore, the set of coroots Π^* is also a root system in E^* which is called the dual root system.

Proposition 2.1 ([12], Proposition 11.4.1)**.** *Let $\alpha, \beta\in\Pi$. Then, $(s_\alpha\beta)^\vee = s_{\alpha^\vee}(\beta^\vee)$.*

Hence, we derive the following corollary that will be very useful.

Corollary 2.2. *Suppose that $\alpha \in \Pi$ is given by $\alpha = s_1\cdots s_{m-1}(\delta_m)$ such that $s_i = s_{\delta_i}$ is the simple reflection associated to $\delta_i \in \Pi$. Then, the coroot $\alpha^\vee$ can be written as follows*

$$\alpha^\vee = s_{\delta_1^\vee}\cdots s_{\delta_{m-1}^\vee}(\delta_m^\vee). \tag{2}$$

A system of simple roots $\Sigma \subset \Pi$ is a basis of E such that every root $\alpha \in \Pi$ is written as a linear combination with integer coefficients with the same sign, i.e., all of them either non-negative or non-positive. A root system is called reduced when for $\alpha \in \Pi$, $\mathbb{R}\alpha \cap \Pi = \{-\alpha, \alpha\}$. Such a reduced root system is always the set of roots of a Cartan subalgebra over an algebraically closed field. Non-reduced root systems appear in the study of real semi-simple Lie algebras. However, split real forms correspond to reduced root systems.

Proposition 2.3 ([12], Proposition 11.6.13)**.**
If Π is reduced then $\Sigma^* = \{\alpha^\vee \colon \alpha \in \Sigma\}$ is a simple root system of Π^*.

By this Proposition, if $\alpha \in \Pi$ is given in terms of the system Σ of simple roots as

$$\alpha = \sum_{\delta\in\Sigma} d_\delta \delta \tag{3}$$

we should obtain an analogous expression for $\alpha^\vee$ with respect to the simple root system Σ^*. Indeed, it follows that

$$\alpha^\vee = \sum_{\delta\in\Sigma} d_\delta \frac{\langle\delta,\delta\rangle}{\langle\alpha,\alpha\rangle}\delta^\vee. \tag{4}$$

As a consequence, if $\alpha^\vee = \sum_{\delta^\vee\in\Sigma^*} d_\delta^*\delta^\vee$ is the coroot of α given by (3), the relationship between its coefficients is given by

$$d_\delta^* = d_\delta \frac{\langle\delta,\delta\rangle}{\langle\alpha,\alpha\rangle}.$$

By Equation (4), we also conclude that if $\mathfrak{g}$ is of type A, D, E then its dual root system is isomorphic to itself while the dual root system of a lie algebra of type B is isomorphic to the root system of type C (and vice-versa).

The height of the root α, denoted by $\mathrm{ht}(\alpha)$ is the sum of the coefficients that appear in the decomposition of α in Equation (3)

$$\mathrm{ht}(\alpha) = \sum_{\delta \in \Sigma} d_\delta .$$

If we consider the elements of $\mathcal{W}$ as a product of simple reflections $s_i = s_{\alpha_i}$, $\alpha_i \in \Sigma$, the length $\ell(w)$ of $w \in \mathcal{W}$ is defined as the number of simple reflections in any reduced decomposition of w.

There is a partial order $\leq$ in the Weyl group called the Bruhat-Chevalley order: we say that $w_1 \leqslant w_2$ if given a reduced decomposition $w_2 = s_{j_1} \cdots s_{j_r}$ then $w_1 = s_{j_{i_1}} \cdots s_{j_{i_k}}$ for some $1 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant r$. When there exists $w, w' \in \mathcal{W}$ such that $w' \leqslant w$ and $\ell(w) = \ell(w') + 1$ we say that w covers w' (alternatively, w, w' is a covering pair). If w covers w' and given a reduced decomposition $w = s_1 \cdots s_\ell$ then we will denote by I the integer in $[\ell]$ such that $w' = s_1 \cdots \widehat{s_I} \cdots s_\ell$, where the integer I depends on w' and the choice of the reduced decomposition of w. For convenience, we will sometimes refer to this decomposition of w' as $\widehat{w}_I$.

For the subset $\Theta \subset \Sigma$, we define the subgroup $\mathcal{W}_\Theta$ generated by the reflections with respect to the roots $\alpha \in \Theta$. We denote by $\mathcal{W}^\Theta$ the subset of minimal representatives of the cosets of $\mathcal{W}_\Theta$ in $\mathcal{W}$.

The Bruhat decomposition presents flag manifolds as a union of N-orbits, namely,

$$\mathbb{F}_\Theta = \coprod_{w \in \mathcal{W}^\Theta} N \cdot w b_\Theta ,$$

where $b_\Theta = 1 \cdot P_\Theta$ is the base point. Each orbit $N \cdot w b_\Theta, w \in \mathcal{W}$, is called a Bruhat cell. It is diffeomorphic to a Euclidean space and, in the case of a split real form, its dimension coincides with the length of w, i.e., $\dim (N \cdot w b_\Theta) = \ell(w)$ (for details, see Corollary 3.8 of Duistermaat-Kolk-Varadarajan [4] and Corollary 2.6 of [17]). A Schubert cell $\mathcal{S}_w$ is the closure of a Bruhat cell. The Bruhat-Chevalley order also characterizes a partial order between the corresponding Schubert varieties. It also endows the flag manifolds with a cellular structure where $\mathcal{S}_w^\Theta = \bigcup_{u \leq w} N \cdot u b_\Theta$.

3. A formula for the roots of Π_w

A very important role is developed by the set $\Pi_w = \Pi^+ \cap w\Pi^-$ composed of the positive roots sent to negative roots by w^{-1}. If $w = s_1 \cdots s_m$ is a reduced decomposition of w then $\Pi_w = \{\beta_1, \ldots, \beta_m\}$ where

$$\beta_k = s_1 \cdots s_{k-1}(\delta_k) \ , \ \text{for } 1 \leq k \leq m. \tag{5}$$

In particular, we have that $\ell(w)$ equals the cardinality of Π_w. In this section, we consider a slightly more general setting in which we derive a formula for a root that is obtained after a finite sequence of compositions of reflections over a simple root. As a consequence, we obtain a formula for those roots of Π_w.

Consider a finite ordered sequence of simple roots $(\delta_1, \ldots, \delta_m)$, possibly with repetition. We are interested in writing $s_1 \cdots s_{m-1}(\delta_m)$ in terms of $\delta_1, \ldots, \delta_m$. When $m = 1$, it is trivial. However, as m increases, we may observe the occurrence of a pattern.

We show what happens for $m = 2,3,4$ after successive applications of Equation (1).

$$
\begin{aligned}
s_1(\delta_2) &= \delta_2 - \langle \delta_1^\vee, \delta_2\rangle\, \delta_1 \\
s_1 s_2(\delta_3) &= \delta_3 - \langle \delta_2^\vee, \delta_3\rangle\, \delta_2 - \langle \delta_1^\vee, \delta_3\rangle\, \delta_1 + \langle \delta_1^\vee, \delta_2\rangle \langle \delta_2^\vee, \delta_3\rangle\, \delta_1 \\
s_1 s_2 s_3(\delta_4) &= \delta_4 - \langle \delta_3^\vee, \delta_4\rangle\, \delta_3 - \langle \delta_2^\vee, \delta_4\rangle\, \delta_2 + \langle \delta_2^\vee, \delta_3\rangle \langle \delta_3^\vee, \delta_4\rangle\, \delta_2 - \langle \delta_1^\vee, \delta_4\rangle\, \delta_1 + \\
&\quad + \langle \delta_1^\vee, \delta_2\rangle \langle \delta_2^\vee, \delta_4\rangle\, \delta_1 + \langle \delta_1^\vee, \delta_3\rangle \langle \delta_3^\vee, \delta_4\rangle\, \delta_1 - \langle \delta_1^\vee, \delta_2\rangle \langle \delta_2^\vee, \delta_3\rangle \langle \delta_3^\vee, \delta_4\rangle\, \delta_1 .
\end{aligned}
$$

By these equations, it turns out that $s_1 \cdots s_{m-1}(\delta_m)$ is an integer coefficient combination of the roots $\delta_1, \ldots, \delta_m$. The coefficients are given as alternating sums of products of the Killing numbers of the roots that belong to a specific interval. Besides, as m increases, the sum is given by products of a greater number of factors. We now proceed to obtain a general formula for this expression.

Given a subset $J = (i_1 < i_2 < \cdots < i_r)$, let us define

$$
\Delta_J = \prod_{i=1}^{r-1} \langle \delta_{j_i}^\vee, \delta_{j_{i+1}} \rangle .
$$

In particular, $\Delta_{\{j\}} = 1$ for any $j \geqslant 1$.

Proposition 3.1. *Let $\delta_1, \ldots, \delta_m$ be an ordered sequence of simple roots whose simple reflections are, respectively, $s_1, \ldots, s_m$. Then, for every $k \in [m]$,*

$$
s_k s_{k+1} \cdots s_{m-1}(\delta_m) = \sum_{\substack{J \subset [k,m] \\ m \in J}} (-1)^{|J|-1} \Delta_J \cdot \delta_{\min J}, \tag{6}
$$

where the sum runs among all subsets J of $[k,m]$ that contains m.

Proof. We will prove by induction in the number k of roots. If $k = m$ then Equation (6) is trivially satisfied. Now, suppose the formula is valid for $k+1$. Then,

$$
\begin{aligned}
s_k(s_{k+1} \cdots s_{m-1}(\delta_m)) &= \sum_{\substack{J \subset [k+1,m] \\ m \in J}} (-1)^{|J|-1} \Delta_J \cdot s_k(\delta_{\min J}) \\
&= \sum_{\substack{J \subset [k+1,m] \\ m \in J}} (-1)^{|J|-1} \Delta_J \cdot \delta_{\min J} + \sum_{\substack{J \subset [k+1,m] \\ m \in J}} (-1)^{|J|} \Delta_J \cdot \langle \delta_k^\vee, \delta_{\min J}\rangle\, \delta_k \\
&= \sum_{\substack{J \subset [k+1,m] \\ m \in J}} (-1)^{|J|-1} \Delta_J \cdot \delta_{\min J} + \sum_{\substack{J \subset [k,m] \\ k,m \in J}} (-1)^{|J|-1} \Delta_J \cdot \delta_k \\
&= \sum_{\substack{J \subset [k,m] \\ m \in J}} (-1)^{|J|-1} \Delta_J \cdot \delta_{\min J} .
\end{aligned}
$$

■

4. A new formula for the coefficient of the boundary map

In this section we present a formula for the coefficients of the boundary map of real flag manifolds by means of the height of some roots in the Lie algebra. Our main applications will be given alone in the context of split real forms.

Let us begin by reviewing some main results about the determination of the cellular homology coefficients following [14]. We start in the context of the maximal flag manifold $\mathbb{F}$.

Given a Schubert cell $\mathcal{S}_w$, we fix once and for all reduced decompositions

$$w = s_1 \cdots s_\ell$$

as a product of simple reflections, for each $w \in \mathcal{W}$, with $\ell = \ell(w)$. Let $\mathcal{C}$ be the $\mathbb{Z}$-module freely generated by $\mathcal{S}_w$, $w \in \mathcal{W}$. The boundary map ∂ defined over $\mathcal{C}$ is given by $\partial \mathcal{S}_w = \sum_{w'} c(w,w')\mathcal{S}_{w'}$, where $c(w,w') \in \mathbb{Z}$ in such way that non-trivial coefficients may occur when w covers w'. Furthermore, the non-trivial coefficients must be equal to ± 2 ([14], Theorem 2.2).

Notice also that, by [14] Proposition 1.10, the condition for w, w' to be a covering pair is equivalent to say that if $w = s_1 \cdots s_\ell$ is a reduced decomposition of $w \in \mathcal{W}$ as a product of simple reflections, then $w' = s_1 \cdots \widehat{s_I} \cdots s_\ell$ is a uniquely defined reduced decomposition with $\mathfrak{g}(\alpha_I) \cong \mathfrak{sl}(2,\mathbb{R})$.

It will also be useful to denote by $v = s_1 \cdots s_{I-1}$ and $u = s_{I+1} \cdots s_\ell$ such that $w' = v \cdot u$. There are roots not necessarily simple $\beta = v(\alpha_I)$ and $\gamma = u^{-1}(\alpha_I)$ with

$$w = s_\beta \cdot w' \quad \text{and} \quad w = w' \cdot s_\gamma.$$

Let us determine if $c(w,w')$ is either 0 or ± 2. We define

$$\sigma(w,w') = \sum_{\delta \in \Pi_u} \langle \alpha_I^\vee, \delta \rangle \cdot \dim \mathfrak{g}_\delta. \tag{7}$$

For $w \in \mathcal{W}$, let

$$\phi(w) = \sum_{\delta \in \Pi_w} \dim \mathfrak{g}_\delta \cdot \delta.$$

The following results show how we determine when $c(w,w')$ is either 0 or ± 2.

Proposition 4.1 ([14], Proposition 2.7)**.** *Let β the unique root such that $w = s_\beta \cdot w'$. Then*

$$\phi(w) - \phi(w') = \kappa(w,w') \cdot \beta,$$

where $\kappa(w,w') = 1 - \sigma(w,w')$.

Theorem 4.2 ([14], Theorem 2.8, [6], Theorem 1.1.4)**.** *Suppose that w covers w'. Then the coefficient $c(w,w')$ is given as follows:*

$$c(w,w') = \pm \left(1 + (-1)^{\kappa(w,w')}\right).$$

Now, we address the question about the determination of the sign. This method is developed in [14] whose argument is based on the reduced decompositions of the elements $w \in \mathcal{W}$.

For each w fix a reduced decomposition. There is a first $(-1)^I$ ingredient which is related with the deleted position. The second component appears when the fixed reduced decomposition for w' is not equal to that $\widehat{w}_I$. According to [14] Proposition 1.9, there are characteristic maps for $\mathcal{S}_{w'}$ given by $\Phi_{w'} \colon B_{w'} \to \mathcal{S}_{w'}$ and $\Phi_{\widehat{w}_I} \colon B_{\widehat{w}_I} \to \mathcal{S}_{w'}$, where $B_{w'}$ and $B_{\widehat{w}_I}$ are balls of dimension $\ell(w')$. The first map is obtained by the choice of reduced decomposition for w' whereas the latter follows from the deletion operation. In addition, there is a property where both maps $\Phi_{w'}$ and $\Phi_{\widehat{w}_I}$ are diffeomorphisms when restricted to the interior of the respective balls.

Theorem 4.3 ([14], Theorem 2.8)**.**

$$c(w,w') = (-1)^I \cdot \deg\left(\Phi_{w'}^{-1} \circ \Phi_{\widehat{w}_I}\right) \cdot \left(1 + (-1)^{\kappa(w,w')}\right).$$

Remark 4.4. When both reduced decompositions w' and $\widehat{w}_I$ are equal then $\deg\left(\Phi_{w'}^{-1} \circ \Phi_{\widehat{w}_I}\right) = 1$. We compute the degree of the composition $\Phi_w^{-1} \circ \Phi_{\widehat{w}_I}$ considered as map between spheres in which the boundaries of the balls are collapsed to points.

$$\begin{array}{ccccc} B_{w'} & \xrightarrow{\Phi_{w'}} & \mathcal{S}_{w'} & \xleftarrow{\Phi_{\widehat{w}_I}} & B_{\widehat{w}_I} \\ \downarrow & & \downarrow & & \downarrow \\ B_{w'}/\partial(B_{w'}) & \longrightarrow & \sigma_{w'} & \longleftarrow & B_{\widehat{w}_I}/\partial(B_{\widehat{w}_I}) \end{array}$$

We denote by $\sigma_{w'} = \mathcal{S}_{w'}/(\mathcal{S}_{w'} \setminus N \cdot w'b_0)$ the space obtained by identifying the complement of the Bruhat cell $N \cdot w'b_0$ to a point. ■

Let $\Theta \subset \Sigma$ and consider the partial flag manifold $\mathbb{F}_\Theta$. Recall that the Schubert varieties are $\mathcal{S}_w^\Theta$ are the closure of the Bruhat cells $N \cdot wb_\Theta$, for $w \in \mathcal{W}^\Theta$. Let $\mathcal{C}^\Theta$ be the $\mathbb{Z}$-module freely generated by $\mathcal{S}_w^\Theta$, for every element w of $\mathcal{W}^\Theta$. The boundary maps $\partial^\Theta : \mathcal{C}^\Theta \to \mathcal{C}^\Theta$ are defined by

$$\partial^\Theta \mathcal{S}_w^\Theta = \sum_{w' \in \mathcal{W}^\Theta} c^\Theta(w,w') \mathcal{S}_{w'}^\Theta$$

for some coefficients $c^\Theta(w,w') \in \mathbb{Z}$.

Theorem 4.5 ([14], Theorem 3.4)**.** *The integral homology of the flag manifold $\mathbb{F}_\Theta = G/P_\Theta$ is isomorphic to the homology of $(\mathcal{C}^\Theta, \partial^\Theta)$, where ∂^Θ obtained by restricting ∂ and projecting it onto $\mathcal{C}^\Theta$.*

Hence, $c^\Theta(w,w') = c(w,w')$, where $w, w' \in \mathcal{W}^\Theta$.

We now seek a formula for the cellular homology coefficients in terms of the height of roots.

This formula previously established for $\kappa(w,w')$ takes the relationship of w and w' by a left action ($w = s_\beta \cdot w'$) into account. We will show how to obtain an equivalent expression by exploring the right action, i.e, $w = w' \cdot s_\gamma$, which becomes impressively simple for split real forms.

Assume that $w = s_1 \cdots s_\ell$ and $w' = s_1 \cdots \widehat{s_I} \cdots s_\ell = v \cdot u$, with $v = s_1 \cdots s_{I-1}$ and $u = s_{I+1} \cdots s_\ell$, be reduced decompositions such that $(\delta_1, \ldots, \delta_\ell)$ is the corresponding sequence of simple roots associated with this decomposition. Our strategy consists on finding an explicit computation of $\sigma(w,w')$ as defined in Equation (7) in terms of the root u. When u is non-trivial, by Equation (5), recall that the roots of $\Pi_u = \Pi^+ \cap u\Pi^-$ are given by

$$\beta_j = s_{I+1} \cdots s_{j-1}(\delta_j), \quad j \in [I+1, \ell].$$

Proposition 4.6. *The coefficient $\kappa(w,w')$ is*

$$\kappa(w,w') = 1 + \sum_{\substack{J\subset[I,\ell]\\ I\in J,|J|>1}} (-1)^{|J|-1}\Delta_J \dim(\mathfrak{g}_{\beta_{\max J}}).$$

where the sum runs among all subsets J of $[I,\ell]$ that contains I with at least two elements.

Proof. Let $j\in[I+1,\ell]$. By Proposition 3.1, we have

$$\langle\delta_I^\vee,\beta_j\rangle = \sum_{\substack{J\subset[I+1,j]\\ j\in J}} (-1)^{|J|-1}\Delta_J\cdot\langle\delta_I^\vee,\delta_{\min J}\rangle = \sum_{\substack{J\subset[I+1,j]\\ j\in J}} (-1)^{|J|-1}\Delta_{\{I\}\cup J} = \sum_{\substack{J\subset[I,j]\\ I,j\in J}} (-1)^{|J|}\Delta_J.$$

Hence, by Equation (7),

$$\begin{aligned}\kappa(w,w') = 1-\sigma(w,w') &= 1-\sum_{j=I+1}^{\ell}\left(\sum_{\substack{J\subset[I,j]\\ I,j\in J}} (-1)^{|J|}\Delta_J\right)\dim(\mathfrak{g}_{\beta_j})\\ &= 1+\sum_{\substack{J\subset[I,\ell]\\ I\in J,|J|>1}} (-1)^{|J|-1}\Delta_J\dim(\mathfrak{g}_{\beta_{\max J}}).\end{aligned}$$ ∎

The expression of κ in Proposition 4.6 becomes apparent when the Lie algebra is a split real form.

Theorem 4.7. *Assume that $\mathfrak{g}$ is a split real form. Let $\gamma = u^{-1}(\delta_I)$ be the root for which $w = w'\cdot s_\gamma$. Then*

$$\kappa(w,w') = \mathrm{ht}(\gamma^\vee),$$

where $\mathrm{ht}(\gamma^\vee)$ is the height of the coroot $\gamma^\vee$ in the dual root system Π^*.

Proof. By hypothesis, $\mathfrak{g}$ is split real form, which means that $\dim\mathfrak{g}_\alpha = 1$, for every root $\alpha\in\Pi$. First of all, if $I=\ell$ then $\gamma=\delta_\ell$ is a simple root and $\gamma^\vee=\delta_\ell^\vee$. Hence, $\kappa(w,w') = 1 = \mathrm{ht}(\gamma^\vee)$.

Suppose that $I<\ell$, the root γ can be written as $\gamma = s_\ell s_{\ell-1}\cdots s_{I+1}(\delta_I)$. By Corollary 2.2, the coroot of γ is $\gamma^\vee = s_{\delta_\ell^\vee}s_{\delta_{\ell-1}^\vee}\cdots s_{\delta_{I+1}^\vee}(\delta_I^\vee)$. Consider the sequence of coroots $\delta'_I,\ldots,\delta'_\ell$ given by $\delta'_j = \delta^\vee_{\ell+I-j}$, for $j\in[I,\ell]$, and their simple reflections $s'_I = s_{\delta_\ell^\vee},\ldots,s'_\ell = s_{\delta_I^\vee}$. By Proposition 3.1,

$$\gamma^\vee = s'_I\ldots s'_{\ell-1}(\delta'_\ell) = \sum_{\substack{J\subset[I,\ell]\\ \ell\in J}} (-1)^{|J|-1}\Delta'_J\cdot\delta'_{\min J} = \sum_{\substack{J\subset[I,\ell]\\ \ell\in J}} (-1)^{|J|-1}\Delta'_J\cdot\delta^\vee_{\ell+I-\min J} \quad (8)$$

where

$$\Delta'_J = \prod_{i=1}^{r-1}\langle\delta'^\vee_{j_i},\delta'_{j_{i+1}}\rangle = \prod_{i=1}^{r-1}\langle(\delta^\vee_{\ell+I-j_i})^\vee,\delta^\vee_{\ell+I-j_{i+1}}\rangle = \prod_{i=1}^{r-1}\langle\delta^\vee_{\ell+I-j_{i+1}},\delta_{\ell+I-j_i}\rangle,$$

since $(\delta^\vee)^\vee = \delta$.

In the summation (8), change j to $\ell + I - j$ to get

$$\gamma^\vee = \sum_{\substack{J\subset[I,\ell]\\ I\in J}} (-1)^{|J|-1}\Delta_J \cdot \delta^\vee_{\max J}.$$

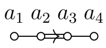
Therefore, by Proposition 4.6, the height of the coroot $\gamma^\vee$ is

$$\mathrm{ht}(\gamma^\vee) = \sum_{\substack{J\subset[I,\ell]\\ I\in J}} (-1)^{|J|-1}\Delta_J = 1 + \sum_{\substack{J\subset[I,\ell]\\ I\in J, |J|>1}} (-1)^{|J|-1}\Delta_J = \kappa(w,w'). \qquad \blacksquare$$

Corollary 4.8. *Let $\mathfrak{g}$ be a Lie algebra of type A_n, D_n, E_6, E_7 or E_8. Then*

$$\kappa(w,w') = \mathrm{ht}(\gamma).$$

Proof. In the context of a split real form of a Lie algebra of type A_n, D_n, E_6, E_7 or E_8, all roots have the same length. Then, $\mathrm{ht}(\alpha) = \mathrm{ht}(\alpha^\vee)$ for all $\alpha \in \Pi$. ■

Remark 4.9. We can use Theorem 4.7 to get the formula for F_4 and G_2 Lie algebras. If $\mathfrak{g}$ is a Lie algebra of type F_4, suppose that the simple roots are ordered canonically as follows 

. Then,

$$\kappa(w,w') = \mathrm{ht}(\widetilde{\gamma}),$$

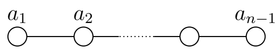
where $\widetilde{\gamma}$ is the root obtained from γ by reversing the simple roots $\Sigma = \{a_1, a_2, a_3, a_4\}$ to $\Sigma^\vee = \{a_4, a_3, a_2, a_1\}$, respectively, since the dual simple roots correspond to reverse roots in the Dynkin diagram, i.e., $a_1^\vee = a_4, a_2^\vee = a_3, a_3^\vee = a_2, a_4^\vee = a_1$. The same idea applies to the Lie algebra G_2. ■

Remark 4.10. We also can use Theorem 4.7 to get the formula for B_n and C_n Lie algebras from the isomorphism $\Pi_B^* \cong \Pi_C$. For instance, in the context of isotropic and odd orthogonal Grassmannians, it coincides with Theorem 3.12 of [8]. ■

5. Type A case

In this section, we present some immediate consequences of Theorem 4.7 for flags associated with type A Lie algebras. It emphasizes the convenience of the permutation model the symmetric group provides parametrizing the Schubert cells.

Let $G = \mathrm{Sl}(n,\mathbb{R})$ be a Lie group of type A and $\Sigma = \{a_1, \ldots, a_{n-1}\}$ the simple roots ordered as follows:



The respective Weyl group $\mathcal{W}$ is the symmetric group S_n. We denote a permutation $w \in S_n$ in the one-line notation by $w = w_1 w_2 \cdots w_\ell$ where $w_i = w(i)$ for all $i = 1, \ldots, \ell$.

The following lemma provides a characterization of the covering relation of two permutations using the one-line notation.

Lemma 5.1 ([1], Lemma 2.1.4). *Let $w, w' \in S_n$. Then, w covers w' in the Bruhat order if and only if $w = w' \cdot (i,j)$ for some transposition (i,j) with $i < j$ such that $w'(i) < w'(j)$ and there does not exist any k such that $i < k < j$, $w'(i) < w'(k) < w'(j)$.*

The lemma says that if $w = w_1 \cdots w_n$ is the one-line notation of $w \in S_n$ then w' is covered by w if and only if the one-line notation of w' is obtained from w by switching the values in position i and j, for some pair $i < j$ and such that no value between positions i and j lies in $[w(j), w(i)]$.

The next proposition follows from the covering relation stated in Lemma 5.1.

Proposition 5.2. *Let $w, w' \in S_n$ such that w' is covered by w, i.e., $w = w' \cdot (i,j)$ for some $i < j$. Then, the coefficient $\kappa(w, w')$ is given by*

$$\kappa(w, w') = j - i.$$

In particular, $c(w, w') = 0$ if, and only if, $j - i$ is odd.

Proof. Since γ is the root such that $w = w's_\gamma$, the transposition (i,j) is the reflection s_γ through γ. Considering the simple reflections $s_i = s_{a_i}$, for $a_i \in \Sigma$, a reduced decomposition for (i,j) is $s_{j-1} \cdots s_{i+1} s_i s_{i+1} \cdots s_{j-1}$. Using the fact that $s_{w(\alpha)} = w s_\alpha w^{-1}$ we have $(i,j) = s_{s_{j-1} \cdots s_{i+1}(a_i)}$ and, then, $\gamma = s_{j-1} \cdots s_{i+1}(a_i)$.

Applying Proposition 3.1, the root γ is the sum of simple roots $a_i + a_{i+1} + \cdots + a_{j-1}$. Therefore, by Corollary 4.8, $\kappa(w, w') = \mathrm{ht}(\gamma) = j - i$. ■

This proposition simplifies the task to compute the boundary coefficient. For instance, given $w = 1\,3\,7\,5\,8\,2\,9\,4\,6$ and $w' = 1\,3\,7\,\mathbf{2}\,8\,\mathbf{5}\,9\,4\,6$, we see that $w = w' \cdot (4,6)$. Hence, $\kappa(w, w') = 6 - 4 = 2$ and $c(w, w') = \pm(1 + (-1)^{\kappa(w,w')}) = \pm 2$.

Given $\Theta \subset \Sigma$, recall that $\mathcal{W}^\Theta = \{w \in \mathcal{W} \colon \ell(w) < \ell(w s_a), \forall a \in \Theta\}$ is the set of minimal representatives of the cosets of $\mathcal{W}_\Theta$ in $\mathcal{W}$. In type A, it can be described as follows (see Theorem A of Tan [16]): let the complement of Θ with respect to Σ be the set of roots $\{a_{d_1}, a_{d_2}, \ldots, a_{d_k}\}$ where $d_0 = 0 < d_1 < \cdots < d_k < n = d_{k+1}$. Then,

$$\mathcal{W}^\Theta = \{w_1 \cdots w_n \in S_n \colon w_1 < \cdots < w_{d_1}, w_{d_1+1} < \cdots < w_{d_2}, \ldots, w_{d_k+1} < \cdots < w_n\}. \quad (9)$$

In other words, a permutation $w \in \mathcal{W}^\Theta$ has descents only in entries d_i, for every $i \in [k]$.

Let us give a combinatorial proof of the condition for the orientability of any real flag manifold of type A ([11], Proposition 4.1) as a direct application of Proposition 5.2.

Proposition 5.3. *Let the complement of Θ with respect to Σ be the set of roots $\{a_{d_1}, a_{d_2}, \ldots, a_{d_k}\}$ where $d_0 = 0 < d_1 < \cdots < d_k < n = d_{k+1}$. Then, the flag manifold $\mathbb{F}_\Theta$ is orientable if and only if $d_{j+1} - d_j$ have the same parity, for every $j \in [k]$.*

Proof. We will establish a criterion for orientability by seeking in which condition the top dimensional homology group is $\mathbb{Z}$. Let us begin with the (unique) Schubert top cell $\mathcal{S}_{w_\Theta}$.

The associated permutation w_Θ is the longest permutation (with respect to the Bruhat order) with descents at positions $d_1, \ldots, d_k$. In one-line notation,

$$w_\Theta = (d_k + 1) \cdots n | (d_{k-1} + 1) \cdots d_k | \cdots | 1 \cdots d_1.$$

There are k Schubert cells $\mathcal{S}_{(w_\Theta)'_j}$, $j \in [k]$, covered by $\mathcal{S}_{w_\Theta}$. For each $j \in [k]$, the corresponding permutations of w_Θ and $(w_\Theta)'_j$ differ only by the values at positions $d_{j-1}+1$ and d_{j+1}, where we consider $d_0 = 0$ and $d_{k+1} = n$, i.e.,

$$w_\Theta = (w_\Theta)'_j \cdot (d_{j-1}+1, d_{j+1}).$$

By Proposition 5.2, it follows that

$$\kappa(w_\Theta, (w_\Theta)'_j) = d_{j+1} - (d_{j-1}+1),$$

i.e., $c(w_\Theta, (w_\Theta)'_j) = 0$ if and only if $(d_{j+1} - d_{j-1})$ is even. Since

$$d_{j+1} - d_{j-1} = (d_{j+1} - d_j) + (d_j - d_{j-1}),$$

$\partial \mathcal{S}_{w_\Theta} = 0$ if and only if $d_{j+1} - d_j$ have the same parity, for every j. ■

We now seek a formula for the first and second homology groups for partial flag manifolds of type A. This require introducing a few combinatorial notations.

The code (also called Lehmer code) of a permutation $w \in S_n$ is an integer sequence α with $\alpha_i = \#\{k > i \mid w_k < w_i\}$ and it will be denoted by $\mathsf{code}(w)$. In other words, each entry of the code corresponds to the number of inversions to the right of w_i. It is clear that $0 \leqslant \alpha_i \leqslant n - i$. The code provides a bijection between S_n and the set $[0, n-1] \times [0, n-2] \times \cdots \times [0, 1]$.

Given $w \in S_n$, the *code spectrum* of w is the unique sequence

$$0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_l < n$$

such that the code α of w is given by $\alpha_i = \#\{j \colon b_j = i\}$. We will denote by $[\![b_1, \cdots, b_\ell]\!]$ the permutation w given by such code spectrum to distinguish it from the other notations. For instance, the code spectrum of $w \in S_5$ with $\mathsf{code}(w) = (0, 2, 0, 1)$ is $[\![2, 2, 4]\!]$.

This notation allows us to easily describe permutations with some chosen length. Let us describe all permutations $w \in S_n$ with length up to three:

- $s_i = [\![i]\!]$ for $i \in [n-1]$;
- $s_i s_j = [\![i, j]\!]$ for $i < j$ and $i, j \in [n-1]$;
- $s_{i+1} s_i = [\![i, i]\!]$ for $i \in [n-2]$;
- $s_i s_j s_k = [\![i, j, k]\!]$ for $i < j < k$ and $i, j, k \in [n-1]$;
- $s_{i+1} s_i s_k = [\![i, i, k]\!]$ for $i < k$ and $i, k \in [n-1]$;
- $s_i s_{j+1} s_j = [\![i, j, j]\!]$ for $i < j$ and $i, j \in [n-2]$;
- $s_{i+2} s_{i+1} s_i = [\![i, i, i]\!]$ for $i \in [n-3]$.

When we need to fix a reduced decomposition, we will choose the ones as above.

The following lemma provides us all the boundary maps required to compute the first and second homology groups of any partial flag manifold of type A.

Lemma 5.4.

(1) $\partial \mathcal{S}_{[\![i]\!]} = 0$, *for* $i \in [n-1]$;

(2) $\partial \mathcal{S}_{[\![i,i]\!]} = -2\mathcal{S}_{[\![i]\!]}$, *for* $i \in [n-2]$;

(3) $\partial \mathcal{S}_{[\![i,i+1]\!]} = -2\mathcal{S}_{[\![i+1]\!]}$, *for* $i \in [n-2]$;

(4) $\partial \mathcal{S}_{[\![i,j]\!]} = 0$, *for* $i \in [n-3]$ *and* $j \in [i+2, n-1]$;

(5) $\partial \mathcal{S}_{[\![i,j-1,j]\!]} = 2\mathcal{S}_{[\![i,j]\!]}$, *for* $i \in [n-3]$ *and* $j \in [i+2, n-1]$;

(6) $\partial \mathcal{S}_{[\![i,i+1,i+1]\!]} = 2\mathcal{S}_{[\![i,i+1]\!]} - 2\mathcal{S}_{[\![i+1,i+1]\!]}$ *for* $i \in [n-3]$.

Proof. The general formula for the coefficients is given by Theorem 4.3 whereas Proposition 5.2 gives us when it is ± 2. Recall that $\hat{w}_I$ is the reduced decomposition of w' obtained by removing the I-th simple reflection of w. To get the sign we can observe that if we choose to fix the reduced decomposition for w and w' as given above, both reduced decompositions w' and $\hat{w}_I$ are exactly the same. Then, $\Phi_{w'} = \Phi_{\hat{w}_I}$ and $\deg\left(\Phi_{w'}^{-1} \circ \Phi_{\hat{w}_I}\right) = 1$. Then, the sign of the coefficient is given by $(-1)^I$.

Let us proof the expression (6) for $w = [\![i, i+1, i+1]\!] \in S_3$, with $i \in [n-3]$. This w covers the following permutations of length two:

$$\hat{w}_1 = [\![i+1, i+1]\!],\ \hat{w}_2 = [\![i, i+1]\!], \text{ and } \hat{w}_3 = [\![i, i+2]\!].$$

In one-line notation, $w = \cdots (i+1)\,(i+3)\,i\,(i+2) \cdots$ and the covering relations are given as follows:

$$\begin{aligned} \hat{w}_1 &= \cdots i\,(i+3)\,(i+1)\,(i+2) \cdots \quad , \quad w = \hat{w}_1 \cdot (i, i+2);\\ \hat{w}_2 &= \cdots (i+1)\,(i+2)\,i\,(i+3) \cdots \quad , \quad w = \hat{w}_2 \cdot (i+1, i+3);\\ \hat{w}_3 &= \cdots (i+1)\,i\,(i+3)\,(i+2) \cdots \quad , \quad w = \hat{w}_3 \cdot (i+1, i+2). \end{aligned}$$

By Proposition 5.2, $c(w, \hat{w}_1) = (-1)^1 \cdot 2 = -2, c(w, \hat{w}_2) = (-1)^2 \cdot 2 = 2$ and $c(w, \hat{w}_3) = (-1)^3 \cdot 0 = 0$. The remaining expressions follow analogously. ■

Remark 5.5. The list of boundary maps in Lemma 5.4 does not present a formula for every three dimensional Schubert cell since it intends to provide only the necessary to compute the first and second homology groups. In this cases, Theorem 5.7 will show that $\mathrm{im}(\partial)$ coincides with $2 \cdot \ker(\partial)$. ■

The next lemma provides necessary and sufficient conditions for permutations of length one and two in $\mathcal{W}^\Theta$.

Lemma 5.6.

(1) *Given* $i \in [n-1]$, *we have that* $[\![i]\!] \in \mathcal{W}^\Theta$ *if, and only if,* $a_i \in \Sigma - \Theta$;

(2) *Given* $i \in [n-2]$ *and* $j \in [n-1]$ *such that* $i \leqslant j$, *we have that* $[\![i,j]\!] \in \mathcal{W}^\Theta$ *if, and only if, one of the following happens:*

- $a_i, a_j \in \Sigma - \Theta$;
- $j = i+1$, $a_i \in \Theta$, *and* $a_{i+1} \in \Sigma - \Theta$.

Proof. The permutation $w = [\![i]\!]$ is given by $w = 1 \cdots (i+1)\, i \cdots n$, i.e., it has a descent only at the i-th entry. Then, by (9), $w \in \mathcal{W}^{\Theta}$ if, and only if, $a_i \in \Sigma - \Theta$. The permutation $w = [\![i,j]\!]$ may be written as it follows:

- If $j - i \geqslant 2$ then $w = 1 \cdots (i+1)\, i \cdots (j+1)\, j\ \cdots n$. By (9), $w \in \mathcal{W}^{\Theta}$ if, and only if, $a_i, a_j \in \Sigma - \Theta$;
- If $j = i+1$ then $w = 1 \cdots (i+1)\,(i+2)\, i \cdots n$. By (9), $w \in \mathcal{W}^{\Theta}$ if, and only if, $a_{i+1} \in \Sigma - \Theta$;
- If $j = i$ then $w = 1 \cdots (i+2)\, i\, (i+1) \cdots n$. By (9), $w \in \mathcal{W}^{\Theta}$ if, and only if, $a_i \in \Sigma - \Theta$. ■

Denote by r_{Θ} the number of connected components of the Dynkin diagram of Θ.

Theorem 5.7. *Consider a partial flag manifold $\mathbb{F}_{\Theta}$ of $G = \mathrm{Sl}(n, \mathbb{R})$, where $\Theta \subset \Sigma = \{a_1, \ldots, a_{n-1}\}$.*

(1) *For $n \geqslant 3$, the first homology group is*

$$H_1(\mathbb{F}_{\Theta}, \mathbb{Z}) \cong (\mathbb{Z}_2)^{n-|\Theta|-1}$$

and it is generated by the set of Schubert cells $\mathcal{S}_{[\![i]\!]}$ such that $a_i \in \Sigma - \Theta$.

(2) *For $n \geqslant 4$, the second homology group is*

$$H_2(\mathbb{F}_{\Theta}, \mathbb{Z}) \cong (\mathbb{Z}_2)^{N_{\Theta}}$$

where $N_{\Theta} = \binom{n-|\Theta|-1}{2} + r_{\Theta} - 1$ and it is generated by the set of Schubert cells

- *$X_{i,j} = \mathcal{S}_{[\![i,j]\!]}$, for any $i, j \in [n-1]$ such that $j - i \geqslant 2$ and $a_i, a_j \in \Sigma - \Theta$;*
- *$X_{i,i+1} = \mathcal{S}_{[\![i,i+1]\!]} - \mathcal{S}_{[\![i+1,i+1]\!]}$, for every $i \in [n-3]$ such that $a_{i+1} \in \Sigma - \Theta$.*

Proof. By Lemmas 5.6 and 5.4, the kernel of ∂^{Θ} is generated by $\mathcal{S}_{[\![i]\!]}$ such that $a_i \in \Sigma - \Theta$. For every $i \in [n-2]$ such that $a_i \in \Sigma - \Theta$, we have that $[\![i,i]\!] \in \mathcal{W}^{\Theta}$ and $\partial^{\Theta}\mathcal{S}_{[\![i,i]\!]} = -2\mathcal{S}_{[\![i]\!]}$. If $a_{n-1} \in \Sigma - \Theta$ then $\partial^{\Theta}\mathcal{S}_{[\![n-2,n-1]\!]} = -2\mathcal{S}_{[\![n-1]\!]}$.
We conclude that $H_1(\mathbb{F}_{\Theta}, \mathbb{Z})$ has no free part and

$$H_1(\mathbb{F}_{\Theta}, \mathbb{Z}) \cong (\mathbb{Z}_2)^{|\{\mathcal{S}_{[\![i]\!]}\,:\, a_i \in \Sigma - \Theta\}|} = (\mathbb{Z}_2)^{n-|\Theta|-1}.$$

To compute the second homology group, let us prove that it has no free part. By Lemmas 5.6 and 5.4, the kernel of ∂^{Θ} is generated by

- $X_{i,j} = \mathcal{S}_{[\![i,j]\!]}$, for any $i, j \in [n-1]$ such that $j - i \geqslant 2$ and $a_i, a_j \in \Sigma - \Theta$;
- $X_{i,i+1} = \mathcal{S}_{[\![i,i+1]\!]} - \mathcal{S}_{[\![i+1,i+1]\!]}$, for any $i \in [n-3]$ such that $a_{i+1} \in \Sigma - \Theta$.

Notice that we do not allow $X_{n-2,n-1}$ since $[\![n-1, n-1]\!] \notin S_n$. Also by Lemma 5.4, we have that each $X_{i,j}$ is image through ∂^{Θ} of the following Schubert cells:

- For any $i, j \in [n-1]$ such that $j - i \geqslant 2$ and $a_i, a_j \in \Sigma - \Theta$, we have $[\![i, j-1, j]\!] \in \mathcal{W}^{\Theta}$. Hence, $\partial^{\Theta}\mathcal{S}_{[\![i,j-1,j]\!]} = 2X_{i,j}$;
- For any $i \in [n-3]$ such that $a_{i+1} \in \Sigma - \Theta$, we have $[\![i, i+1, i+1]\!] \in \mathcal{W}^{\Theta}$. Hence, $\partial^{\Theta}\mathcal{S}_{[\![i,i+1,i+1]\!]} = 2X_{i,i+1}$.

Therefore, the second homology group has no free part and the torsion is generated by all $X_{i,j}$ such that $[\![i,j]\!] \in \mathcal{W}^{\Theta}$.

Notice that for $X_{i,i+1}$, either $a_i \in \Theta$ or $a_i \in \Sigma - \Theta$. Then, we can split the generators of $H_2(\mathbb{F}_\Theta, \mathbb{Z})$ in two sets

$$A_1 = \{X_{i,j} \colon 1 \leqslant i < j \leqslant n-1 \text{ and } a_i, a_j \in \Sigma - \Theta \text{ except } (i,j) = (n-2, n-1)\},$$

$$A_2 = \{X_{i,i+i} \colon i \in [n-3], a_i \in \Theta, \text{ and } a_{i+1} \in \Sigma - \Theta\}.$$

The cardinality of A_1 is given as follows:

- If $a_{n-1}, a_{n-2} \in \Sigma - \Theta$ then $|A_1|$ is the number of 2-combinations from $\Sigma - \Theta$ except the pair $(n-2, n-1)$. Hence, $|A_1| = \binom{n-|\Theta|-1}{2} - 1$;
- If $a_{n-1} \in \Theta$ or $a_{n-2} \in \Theta$ then then $|A_1|$ is the number of 2-combinations from $\Sigma - \Theta$. Hence, $|A_1| = \binom{n-|\Theta|-1}{2}$.

The cardinality of A_2 is given by:

- If $a_{n-1}, a_{n-2} \in \Sigma - \Theta$ then $|A_2|$ is the number of connected components r_Θ of Θ. Hence, $|A_2| = r_\Theta$;
- If $a_{n-1} \in \Theta$ or $a_{n-2} \in \Theta$ then $|A_2| = r_\Theta - 1$ since A_2 counts up to $i = n-3$ and there should be a connected component containing either a_{n-1} or a_{n-2}.

Therefore, $N_\Theta = |A_1| + |A_2| = \binom{n-|\Theta|-1}{2} + r_\Theta - 1$. ∎

6. Final comments and further directions

We would like to highlight that, although this is a classical theme – topology of real flag manifolds such as Projective spaces and Grassmannian manifolds – we have not found in the literature such simple formula to compute its homology groups. It is remarkable finding such closed and direct expressions for first and second homology groups of partial flag manifolds of type A.

With the results obtained in this paper, we can identify other directions to research as listed below.

- For type A flag manifolds, it seems possible to get a formula for third and fourth homology groups. It will require achieving a better understanding of the coefficient since the degree in Theorem 4.3 is not so easy to compute. Indeed, the computational evidences has shown us that in some cases the homology groups have free part. A forthcoming paper will deal with the combinatorics involved to explicitly compute the sign in the type A case.
- Theorem 4.7 provides a formula of the boundary coefficient for split real forms. It is reasonable to ask about low dimensional homology of other types of flag manifolds. This would require to get a nicer combinatorial model for the Weyl group.

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