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The Boundness and Lowest Two-Sided Cell of Weighted Coxeter Groups of Rank 3

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Abstract. We consider weighted Coxeter groups of rank 3 and its Hecke algebras in this paper. In this case, we show the boundness (in the sense of G. Lusztig) and the existence of lowest two-sided cell c_0. Then we can describe c_0 and the left cells in it. At last, we show that the conjectures $P1 - P15$ and $\widetilde{P}$ hold for c_0.

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Key Words: Weighted Coxeter group, Hecke algebra, two-sided cell, left cell, A-function.

1. Introduction

Let (W, S, L) be a weighted Coxeter group. In [7, 13.4], G. Lusztig conjectured that the maximal weight value of the longest elements of the finite parabolic subgroups of W is a bound for (W, S, L). This property is referred as boundness of a weighted Coxeter group [7, 13.2]. When W is finite, this conjecture is clear. In [5, 7.2], G. Lusztig proved this conjecture when W is an affine Weyl group. In [10, 2.1], N. Xi proved this conjecture when W has complete Coxeter graph and $L = l$. In [8, 3.2], J. Shi and G. Yang proved this conjecture when W has complete Coxeter graph without the assumption $L = l$. In [13, 2.1], P. Zhou proved this conjecture when $rank(W) = 3$ and $L = l$. In this paper, we prove this conjecture when $rank(W) = 3$ without the assumption $L = l$, see Theorem 3.1. Then we give some interesting consequences in Section 7. In Theorem 8.1, we show W has a lowest two-sided cell c_0. Then we give precise description for c_0 and the left cells in it, see Theorem 8.3. At last, we prove that conjectures $P1 - P15$ and $\widetilde{P}$ hold for c_0, see Theorem 8.5.

2. Preliminaries

In this paper, for any Coxeter group (W, S), we assume the generating set S is finite. We call $|S|$ the *rank* of (W, S) and denote it by $rank(W)$. We use l for the *length function* and $\leq$ for the Bruhat order of W. The neutral element of W will be denoted by e. For $x \in W$, we set

$$\mathcal{L}(x) = \{s \in S | sx < x\},\ \mathcal{R}(x) = \{s \in S | xs < x\}.$$

For $s, t \in W$, let $m_{st} \in \mathbb{Z}_{\geq 1} \bigcup \{\infty\}$ be the order of st in W.

For any $I \subseteq S$, let $W_I = \langle I \rangle$. Then (W_I, I) is also a Coxeter group, called a *parabolic subgroup* of (W, S).

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Denote the longest element of W_I by w_I if $|W_I| < \infty$. For $s,t \in S$, $s \neq t$, we use W_{st} instead of $W_{\{s,t\}}$ and w_{st} instead of $w_{\{s,t\}}$.

For $w_1, w_2, \cdots, w_n \in W$, we often use the notation $w_1 \cdot w_2 \cdot \cdots \cdot w_n$ instead of $w_1 w_2 \cdots w_n$ if $l(w_1 w_2 \cdots w_n) = l(w_1) + l(w_2) + \cdots + l(w_n)$.

Let (W,S) be a Coxeter group. A map $L : W \longrightarrow \mathbb{Z}$ is called a weight function if $L(ww') = L(w) + L(w')$ for any $w, w' \in W$ with $l(ww') = l(w) + l(w')$. Then we call (W,S,L) a *weighted Coxeter group*. In this paper, the weight function L for any weighted Coxeter group (W,S,L) is assumed to be positive, that is, $L(s) > 0$ for any $s \in S$. For any $J \subseteq S$, it is obvious that the parabolic subgroup $(W_J, J, L|_{W_J})$ is also a weighted Coxeter group.

Let (W,S,L) be a weighted Coxeter group and $\mathbb{Z}[v,v^{-1}]$ be the ring of Laurent polynomials in an indeterminate v with integer coefficients. For

$$f = \sum_{n\in\mathbb{Z}} a_n v^n \in \mathbb{Z}[v,v^{-1}] \setminus \{0\}$$

we define $\deg f = \max\{n \in \mathbb{Z} \mid a_n \neq 0\}$ and $\deg 0 = -\infty$.

The *Hecke algebra* $\mathcal{H}$ of (W,S,L) is the unital associative $\mathbb{Z}[v,v^{-1}]$-algebra defined by the generators T_s $(s \in S)$ and the relations

$$\underbrace{T_s T_t T_s \cdots}_{m_{st}\ factors} = \underbrace{T_t T_s T_t \cdots}_{m_{st}\ factors}, \quad s,t \in S,\ m_{st} < \infty,$$

and $$T_s^2 = 1 + \xi_s T_s,\ s \in S,\ \text{where}\ \xi_s = v^{L(s)} - v^{-L(s)} \in \mathbb{Z}[v,v^{-1}].$$

Obviously, the element T_e is the multiplicative unit of $\mathcal{H}$. For any $w \in W$, we define $T_w = T_{s_1} T_{s_2} \cdots T_{s_n} \in \mathcal{H}$, where $w = s_1 s_2 \cdots s_n$ is a reduced expression of w in W. Then T_w is independent of the choice of reduced expression and $\{T_w \mid w \in W\}$ is a $\mathbb{Z}[v,v^{-1}]$-basis of $\mathcal{H}$, called the standard basis. We define $f_{x,y,z} \in \mathbb{Z}[v,v^{-1}]$ for any $x,y,z \in W$ by the identity

$$T_x T_y = \sum_{z\in W} f_{x,y,z} T_z.$$

We have the following morphism of $\mathbb{Z}[v,v^{-1}]-$modules:

$$\tau : \quad \mathcal{H} \longrightarrow \mathbb{Z}[v,v^{-1}], \quad \sum_{w\in W} f_w T_w \longrightarrow f_e.$$

The following involutive automorphism of rings is useful, called the bar involution:

$$^{-}\colon \mathcal{H} \longrightarrow \mathcal{H}, \quad v^n \longrightarrow v^{-n}, \quad T_s \longrightarrow T_s^{-1}.$$

We have $\overline{T_w} = T_{w^{-1}}^{-1}$ for any $w \in W$. We set

$$\mathcal{H}_{\leq 0} = \bigoplus_{w\in W} \mathbb{Z}[v^{-1}] T_w,\ \mathcal{H}_{<0} = \bigoplus_{w\in W} v^{-1}\mathbb{Z}[v^{-1}] T_w.$$

The following facts are easy to get, see also [7].

Lemma 2.1. (1) *For any* $x,y \in W$, *we have* $f_{x,y,e} = \delta_{x,y^{-1}}$.

(2) *For any* $x,y,z \in W$, *we have* $f_{x,y,z^{-1}} = f_{y,z,x^{-1}} = f_{z,x,y^{-1}}$.

(3) *For any* $x,y,z \in W$, *we have* $\deg f_{x,y,z} \leq \min\{L(x), L(y), L(z)\}$.

(4) *For any finite parabolic subgroup* W_I *of* W, $x \in W_I$, *we have*

$$\deg f_{w_I, w_I, x} = L(x).$$

(5) *Assume* $s, t \in S$, $m_{st} < \infty$, $x, y, z \in W_{st}$ *such that* $\deg f_{x,y,z} = L(z)$.
If $l(z) \geq 2$, *then* $x = y = w_{st}$. *If* $z = s$, *then* $y = x^{-1}$, $s \in \mathcal{L}(x)$, $s \in \mathcal{R}(y)$.

Define the degree map

$$\deg : \mathcal{H} \longrightarrow \mathbb{Z} \cup \{-\infty\}, \quad \sum_{w \in W} f_w T_w \longrightarrow \max\{\deg f_w | w \in W\}.$$

We set $N = \max_{\substack{I \subseteq S \\ |W_I| < \infty}} L(w_I)$. G. Lusztig gave the following conjecture in [7, 13.4].

Conjecture 2.2. Let (W, S, L) be a weighted Coxeter group, S is finite and L is positive, then N is a bound for (W, S, L). Namely, $\deg(T_x T_y) \leq N$ for all $x, y \in W$.

This conjecture is important in studying the a-function and cells of weighted Coxeter groups. For example, if this conjecture is true, then the a-function is also bounded by this N. Moreover, W has a lowest two-sided cell c_0, see Section 8.

When W is a finite Coxeter group, this conjecture can be proved using Lemma 2.1(3). In [5, 7.2], G. Lusztig proved this conjecture when W is an affine Weyl group. In [8, 3.2], J. Shi and G. Yang proved this conjecture when W has complete Coxeter graph. In [13, 2.1], P. Zhou proved this conjecture when $rank(W) = 3$ and $L = l$. In this paper, we will prove this conjecture when $rank(W) = 3$ and L is positive, see Theorem 3.1.

For any $w \in W$, there exists a unique element $C_w \in \mathcal{H}_{\leq 0}$ such that $\overline{C_w} = C_w$ and $C_w - T_w \in \mathcal{H}_{<0}$. The elements $\{C_w | w \in W\}$ form a $\mathbb{Z}[v, v^{-1}]-$basis of $\mathcal{H}$, called the Kazhdan-Lusztig basis. We define $h_{x,y,z}, p_{x,y} \in \mathbb{Z}[v, v^{-1}]$ for any $x, y, z \in W$ such that

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z \quad \text{and} \quad C_y = \sum_{x \in W} p_{x,y} T_x.$$

These polynomials $p_{x,y}$ are called *Kazhdan-Lusztig* polynomials. For any finite parabolic subgroup W_I of W, we set

$$\eta_{w_I} = \sum_{y \leq w_I} v^{2L(y) - L(w_I)}.$$

For $w \in W$, $I = \mathcal{R}(w)$, $J = \mathcal{L}(w)$, we have the factorization $w = x \cdot w_I = w_J \cdot y$ for some $x, y \in W$. Then we set

$$E_x = \sum_{\substack{x' \leq x \\ l(x' w_I) = l(x') + l(w_I)}} p_{x' w_I, w} T_{x'} \quad \text{and} \quad F_y = \sum_{\substack{y' \leq y \\ l(w_J y') = l(w_J) + l(y')}} p_{w_J y', w} T_{y'}.$$

Using the Kazhdan-Lusztig basis, we can define the preorders $\underset{L}{\leq}$, $\underset{R}{\leq}$, $\underset{LR}{\leq}$ on W. These preorders give rise to equivalence relations $\underset{L}{\sim}$, $\underset{R}{\sim}$, $\underset{LR}{\sim}$ on W respectively. The equicalence classes are called left cells, right cells and two-sided cells of W.

Then we have partial orders $\underset{L}{\leq}$, $\underset{R}{\leq}$, $\underset{LR}{\leq}$ on the sets of left cells, right cells and two-sided cells of W respectively. For $x, y \in W$, we have $x \underset{L}{\leq} y$ if and only if $x^{-1} \underset{R}{\leq} y^{-1}$.

Now we assume Conjecture 2.2 holds and W has a lowest two-sided cell c_0. It is easy to see that $\deg h_{x,y,z} \leq N$ for any $x, y, z \in W$, so we can define the a-function

$$a : W \longrightarrow \mathbb{N}, \quad w \longrightarrow \max_{x,y \in W} \deg h_{x,y,w}.$$

For any $x, y, z \in W$, we define $\beta_{x,y,z}, \gamma_{x,y,z} \in \mathbb{Z}$ such that

$$f_{x,y,z^{-1}} = \beta_{x,y,z} v^N + \text{lower degree terms}$$

and

$$h_{x,y,z^{-1}} = \gamma_{x,y,z} v^{a(z)} + \text{lower degree terms.}$$

Lemma 2.3. *Let $x, y, z \in W$.*

(1) *We have $\beta_{x,y,z} = \beta_{y,z,x} = \beta_{z,x,y}$.*

(2) *We have $\gamma_{x,y,z} = \gamma_{y^{-1},x^{-1},z^{-1}}$.*

(3) *If $\beta_{x,y,z} \neq 0$, then $x \underset{L}{\sim} y^{-1}$, $y \underset{L}{\sim} z^{-1}$, $z \underset{L}{\sim} x^{-1}$, $a(x) = a(y) = a(z) = N$, and $\beta_{x,y,z} = \gamma_{x,y,z} = \gamma_{y,z,x} = \gamma_{z,x,y}$.*

(4) *If $\gamma_{x,y,z} \neq 0$ and $a(z) = N$, then $\beta_{x,y,z} = \gamma_{x,y,z} \neq 0$.*

The proof of the lemma above can be found in [7]. Furthermore, G. Lusztig gave some conjectures about the weighted Coxeter groups and their Hecke algebras in [7, 14.2, 14.3], which are called $P1 - P15$ and $\widetilde{P}$. These conjectures mainly concern some properties of the a-function and cells. We will prove them for c_0 when $rank(W) = 3$, see Theorem 8.5. When these conjectures hold for c_0, we can study the based ring J_0 of c_0, see [7, Chapter 18].

3. Weighted Coxeter Groups of Rank 3

From now on, we assume (W, S, L) is a weighted Coxeter group of rank 3 and L is positive. We set

$$N = \max_{\substack{I \subseteq S \\ |W_I| < \infty}} L(w_I), \quad M = \{w_J | J \subseteq S,\ |W_J| < \infty,\ L(w_J) = N\},$$

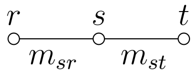
and

$$\Lambda = \{x \cdot u \cdot y | x, y \in W,\ u \in M\}.$$

We have the following result.

Theorem 3.1. *We have $\deg(T_x T_y) \leq N$ for any $x, y \in W$, and the equality holds only if $x, y \in \Lambda$. In particular, Conjecture 2.2 holds.*

When W is a finite Coxeter group, it is clear. When W is an affine Weyl group, see [5, 7.2] and [11, 3.1]. When W has complete Coxeter graph, see [8, 3.2] and [12, 3.4]. So we may assume $S = \{r, s, t\}$ with $m_{rt} = 2$. The Coxeter graph looks like



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When $m_{sr} = \infty$, $m_{st} = 2$, the Coxeter graph of (W, S) is not connected. In this case, (W, S) is the direct product of $(W_{sr}, \{s, r\})$ and $(\langle t\rangle, \{t\})$ as a Coxeter group, and the theorem is clear.

When $m_{sr} = m_{st} = \infty$, the case is also simple. For any $x \in W$, we have $\mathcal{L}(x) = \{s\}$ or $\mathcal{L}(x) \subseteq \{r, t\}$, and we also have $\mathcal{R}(x) = \{s\}$ or $\mathcal{R}(x) \subseteq \{r, t\}$. For any $x, y \in W$, if $\mathcal{R}(x) = \{s\}$ and $\mathcal{L}(y) \subseteq \{r, t\}$, or $\mathcal{R}(x) \subseteq \{r, t\}$ and $\mathcal{L}(y) = \{s\}$, then we have $T_xT_y = T_{xy}$. Therefore, it is easy to see $\deg(T_xT_y) \leq max\{L(rt), L(s)\}$ for any $x, y \in W$, and the equality holds only if $x, y \in \Lambda$.

Summarizing the discussions above, we only need to consider the following three cases. We will deal with them in Sections 4, 5, and 6.

(1) Case 1: $m_{sr} = \infty > m_{st} \geq 3$.

(2) Case 2: $\infty > m_{sr} \geq m_{st} \geq 4$, $m_{sr} \geq 5$.

(3) Case 3: $\infty > m_{sr} \geq 7$, $m_{st} = 3$.

4. Case 1

In this section, we prove Theorem 3.1 for the case $m_{sr} = \infty > m_{st} \geq 3$. In this case, $N = \max\{L(rt), L(w_{st})\}$. We can get the following lemma by easy computation.

Lemma 4.1. *Let $x \in W$.*

(1) *If $s \in \mathcal{R}(x)$, then $r \notin \mathcal{R}(x)$.*

(2) *If $s \in \mathcal{L}(x)$, then $r \notin \mathcal{L}(x)$.*

(3) *If $x = x_1 \cdot st$, then $r \notin \mathcal{R}(x)$.*

(4) *If $x = ts \cdot x_1$, then $r \notin \mathcal{L}(x)$.*

(5) *If $x = x_1 \cdot rs$, then $\mathcal{R}(x) = \{s\}$.*

(6) *If $x = sr \cdot x_1$, then $\mathcal{L}(x) = \{s\}$.*

Lemma 4.2. *Let $x, y \in W$.*

(1) *If $\mathcal{R}(x), \mathcal{L}(y) \subseteq \{s\}$, then $l(xtry) = l(xrty) = l(x) + l(y) + 2$.*

(2) *If $w \in W_{sr}$, $l(w) \geq 4$ or $w = rsr$, $\mathcal{R}(x)$, $\mathcal{L}(y) \subseteq \{t\}$, then $l(xwy) = l(x) + l(w) + l(y)$.*

(3) *If $w \in W_{st}$, $l(w) \geq 2$, $\mathcal{R}(x), \mathcal{L}(y) \subseteq \{r\}$, then $l(xwy) = l(x) + l(w) + l(y)$.*

(4) *If $\mathcal{R}(x), \mathcal{L}(y) \subseteq \{t\}$, and $\mathcal{R}(xs) = \{s\}$ or $\mathcal{L}(sy) = \{s\}$, then $l(xsrsy) = l(x) + l(y) + 3$.*

Proof. (1) See [13, 5.9].

(2) If $l(w) \geq 4$, see [13, 5.7]. If $w = rsr$, it is obvious if $x = e$. If $x \neq e$, we assume $x = x' \cdot t$ for some $x' \in W$ with $\mathcal{R}(x) \subseteq \{s\}$. By Lemma 4.1(6), we have $\mathcal{L}(sry) = \{s\}$, so

$$l(x) + l(rsr) + l(y) = l(x') + l(rt) + l(sry) = l(xrsry).$$

(3) If $l(w) \geq 3$, see [13, 5.8]. We only prove the case of $w = st$ because the case of $w = ts$ is similar.

If $l(x) \leq 1$ or $l(y) \leq 1$, by Lemma 4.1(2),(3), we have $l(xwy) = l(x) + l(w) + l(y)$.

If $l(x) \geq 2$ and $l(y) \geq 2$, we may assume $x = x_1 \cdot sr$, $y = rs \cdot y_1$ for some $x_1, y_1 \in W$. Since $\mathcal{L}(y) \subseteq \{r\}$, we have $\mathcal{L}(sy_1) = \{s\}$. By Lemma 4.1(5), we have $\mathcal{R}(x_1srs) = \{s\}$. Then by (2), we get

$$\begin{aligned} l(xwy) &= l(x_1srstrsy_1) = l(x_1srs) + l(tr) + l(sy_1) = l(x_1sr) + l(st) + l(rsy_1) \\ &= l(x) + l(w) + l(y). \end{aligned}$$

(4) We only prove the case of $\mathcal{R}(xs) = \{s\}$. If $y = e$, it is obvious. Otherwise, we may assume $y = y_1 \cdot y_2$ for some $y_1 \in W_{st} \setminus \{e\}$ and $y_2 \in W$ with $\mathcal{L}(y_2) \subseteq \{r\}$. Since $\mathcal{R}(xsr) = \{r\}$, $sy_1 \in W_{st}$ and $l(sy_1) \geq 2$, by (3), we have

$$l(xsrsy) = l(xsrsy_1y_2) = l(xsr) + l(sy_1) + l(y_2) = l(x) + l(y) + 3. \qquad \blacksquare$$

Proposition 4.3. *In case 1, we have* $\deg(T_xT_y) \leq N$ *for any* $x, y \in W$*, and the equality holds only if* $x, y \in \Lambda$*.*

Proof. We use induction on $l(y)$. When $l(y) = 0, 1$, the proposition is clear. Now we assume $l(y) \geq 2$ and the proposition is true for y' if $l(y') < l(y)$. We take $r_1 \in \mathcal{L}(y)$ and $r_2 \in \mathcal{L}(r_1y)$. Then we can write $x = x_1 \cdot u$, $y = u' \cdot y_1$ for some $u, u' \in W_{r_1r_2}$ and $x_1, y_1 \in W$ with $\mathcal{R}(x_1), \mathcal{L}(y_1) \subseteq S \setminus \{r_1, r_2\}$. Note that $l(y_1) \leq l(y) - 2$. We have

$$T_xT_y = T_{x_1}(T_uT_{u'})T_{y_1} = T_{x_1}(\sum_{w \in W_{r_1r_2}} f_{u,u',w}T_w)T_{y_1} = \sum_{w \in W_{r_1r_2}} f_{u,u',w}T_{x_1w}T_{y_1}.$$

We prove $\deg(f_{u,u',w}T_{x_1w}T_{y_1}) \leq N$ for all $w \in W_{r_1r_2}$, and the equality holds only if $x, y \in \Lambda$.

If we have $l(x_1wy_1) = l(x_1) + l(w) + l(y_1)$, then $\deg(f_{u,u',w}T_{x_1w}T_{y_1}) = \deg f_{u,u',w}$. If $m_{r_1r_2} < \infty$, we have $\deg f_{u,u',w} \leq L(w) \leq L(w_{r_1r_2}) \leq N$; the equality holds only if $u = u' = w = w_{r_1r_2}$ and $L(w_{r_1r_2}) = N$, which implies $x, y \in \Lambda$. If $m_{r_1r_2} = \infty$, we have $\deg f_{u,u',w} \leq max\{L(s), L(r)\} < N$.

If $w = e$, we use induction hypothesis on $T_{x_1}T_{y_1}$. If $l(w) = 1$, we have the equality

$$T_{x_1w}T_{wy_1} = \xi_w T_{x_1w}T_{y_1} + T_{x_1}T_{y_1}.$$

We use the induction hypothesis on $T_{x_1w}T_{wy_1}$ and $T_{x_1}T_{y_1}$ to obtain the relation $\deg(f_{u,u',w}T_{x_1w}T_{y_1}) \leq N$. If the equality holds, then $\deg f_{u,u',w} = L(w)$ and $\deg(T_{x_1w}T_{y_1}) = N - L(w)$. By Lemma 2.1(5), we have $w \in \mathcal{L}(u)$ and $w \in \mathcal{R}(u')$. On the other hand, by the equality, we have $\deg T_{x_1w}T_{wy_1} = N$ or $\deg T_{x_1}T_{y_1} = N$.

By using the induction hypothesis, we have $x = x_1 \cdot w = x_1w \cdot wu \in \Lambda$, and $y = u' \cdot y_1 = u'w \cdot wy_1 \in \Lambda$ for the former case, and $x = x_1 \cdot u \in \Lambda$, $y = u' \cdot y_1 \in \Lambda$ for the latter case.

From now on, we assume $l(x_1wy_1) < l(x_1) + l(w) + l(y_1)$ and $l(w) \geq 2$. By Lemma 4.2, we only need to consider the following 2 cases.

Case 1: $w = srs$.

By Lemma 4.2(4), we have $\mathcal{R}(x_1s) = \mathcal{L}(sy_1) = \{s, t\}$. So we assume $x_1 = x_2 \cdot w_{st}s$ and $y_1 = sw_{st} \cdot y_2$ for some $x_2, y_2 \in W$ with $\mathcal{R}(x_2), \mathcal{L}(y_2) \subseteq \{r\}$. Since we have $\mathcal{L}(r \cdot tw_{st} \cdot y_2) = \{r\}$, by Lemma 3.2(3), we obtain

$$\begin{aligned} T_{x_1srs}T_{y_1} &= T_{x_2 \cdot w_{st}}T_rT_{w_{st} \cdot y_2} = \xi_t T_{x_2 \cdot w_{st}}T_rT_{tw_{st} \cdot y_2} + T_{x_2 \cdot w_{st}t}T_rT_{tw_{st} \cdot y_2} \\ &= \xi_t T_{x_2 \cdot w_{st} \cdot r \cdot tw_{st} \cdot y_2} + T_{x_2 \cdot w_{st}t \cdot r \cdot tw_{st} \cdot y_2}. \end{aligned}$$

We obtain $\deg(f_{u,u',srs}T_{x_1srs}T_{y_1}) \leq max\{L(r), L(s)\} + L(t) \leq N$. When the equality holds, we must have $\deg f_{u,u',srs} = L(r) > L(s)$ and $L(rt) = N$. By Lemma 2.1(2), we have $f_{u,u',srs} = f_{srs,u,u'^{-1}} = f_{u',srs,u^{-1}}$. Thus $\mathcal{L}(u) = \mathcal{R}(u') = \{s\}$ and $l(u), l(u') \geq 2$, so rt appears in x and y, which implies $x, y \in \Lambda$.

Case 2: $w = rs$ (the case of $w = sr$ is similar).
We assume $x_1 = x_2 \cdot t$ for some x_2 with $\mathcal{R}(x_2) \subseteq \{s\}$. If $\mathcal{L}(sy_1) = \{s\}$, then $l(x_1rsy_1) = l(x_2) + l(rt) + l(sy_1) = l(x_1) + l(rs) + l(y_1)$ by Lemma 4.2(3). So we must have $\mathcal{L}(sy_1) = \{s, t\}$. We assume $y_1 = sw_{st} \cdot y_2$ for some y_2 with $\mathcal{L}(y_2) \subseteq \{r\}$. Since $\mathcal{R}(x_2r) \subseteq \{r\}$, by Lemma 4.2(3), we get

$$T_{x_1rs}T_{y_1} = T_{x_2}T_{rt}T_{w_{st}\cdot y_2} = \xi_t T_{x_2\cdot r}T_{w_{st}\cdot y_2} + T_{x_2\cdot r}T_{tw_{st}\cdot y_2} = \xi_t T_{x_2\cdot r\cdot w_{st}\cdot y_2} + T_{x_2\cdot r\cdot tw_{st}\cdot y_2}.$$

We obtain $\deg(f_{u,u',rs}T_{x_1rs}T_{y_1}) \leq max\{L(r), L(s)\} + L(t) \leq N$. When the equality holds, we must have $\deg f_{u,u',rs} = L(r) > L(s)$ and $L(rt) = N$. By Lemma 2.1(2), we have $f_{u,u',rs} = f_{sr,u,u'^{-1}} = f_{u',sr,u^{-1}}$. Thus $\mathcal{L}(u) = \{r\}$, $\mathcal{R}(u') = \{s\}$ and $l(u') \geq 2$, so rt appears in x and y, which implies $x, y \in \Lambda$. ■

5. Case 2

In this section, we prove Theorem 3.1 for the case $\infty > m_{sr} \geq m_{st} \geq 4$, $m_{sr} \geq 5$. In this case, $N = \max\{L(w_{sr}), L(w_{st})\} > L(rt)$. Note that $L(srst) < N$. First we have

Lemma 5.1. *Let $w \in W$.*

(1) *If $w = w_1 \cdot ts$, then $r \notin \mathcal{R}(w)$.*
(2) *If $w = w_1 \cdot rs$, then $t \notin \mathcal{R}(w)$.*
(3) *If $w = w_1 \cdot st$, $\mathcal{R}(w_1s) = \{s\}$, then $r \notin \mathcal{R}(w)$.*
(4) *If $w = w_1 \cdot sr$, $\mathcal{R}(w_1s) = \{s\}$, then $t \notin \mathcal{R}(w)$.*
(5) *If $w = w_1 \cdot tst$, then $r \notin \mathcal{R}(w)$.*
(6) *If $w = w_1 \cdot rsr$, then $t \notin \mathcal{R}(w)$.*
(7) *There is no $w_1, w_2 \in W$ such that $w = w_1 \cdot st = w_2 \cdot sr$.*
(8) *If $\mathcal{L}(w) \subseteq \{r\}$, then $\mathcal{L}(r \cdot tw_{st} \cdot w) = \{r\}$.*
(9) *If $\mathcal{L}(w) \subseteq \{t\}$, then $\mathcal{L}(t \cdot rw_{sr} \cdot w) = \{t\}$.*

Proof. We use induction on $l(w)$ to prove (1) and (2) simultaneously. When $l(w) = 0, 1, 2$, (1) and (2) are clear. Now assume $l(w) \geq 3$ and (1) and (2) are true for w' if $l(w') < l(w)$. If $w = w_2 \cdot r = w_1 \cdot ts$, we have $w = w_3 \cdot w_{sr}$, so $w_2 = w_3 \cdot w_{sr}r$ and $w_1 \cdot t = w_3 \cdot w_{sr}s$. We get $w_1 \cdot t = w_4 \cdot tr$. So $w_4 \cdot t = w_1tr = w_3 \cdot w_{sr}sr$, which contradicts with the induction hypothesis. If $w = w_2 \cdot t = w_1 \cdot rs$, we can find a contradiction similarly.

(3) If $r \in \mathcal{R}(w)$, then $\mathcal{R}(w) = \{r, t\}$. Thus $\mathcal{R}(w_1s) = \{s, r\}$, which contradicts with $\mathcal{R}(w_1s) = \{s\}$.

(4) Similar to (3).

(5) By (1), $\mathcal{R}(w_1 \cdot ts) = \{s\}$. By (3), $r \notin \mathcal{R}(w)$.

(6) Similar to (5).

(7) We use induction on $l(w)$. When $l(w)=0,1,2$, the lemma is clear. Now we assume $l(w)\geq 3$ and the lemma is true for w' if $l(w')<l(w)$. If $w=w_1{\cdot}st=w_2{\cdot}sr$, we have $w=w_3{\cdot}tr$. So $w_1s=w_3r,\ w_2s=w_3t$. We get $w_3r=w_4{\cdot}w_{sr},\ w_3t=w_5{\cdot}w_{st}$. Thus $w_3=w_4\cdot w_{sr}r=w_5\cdot w_{st}t$. So $w_4\cdot w_{sr}rs=w_5\cdot w_{st}ts$, which contradicts with the induction hypothesis.

(8) By (1), $\mathcal{L}(tw_{st}\cdot w)=\{s\}$, so $r\in\mathcal{L}(r\cdot tw_{st}\cdot w)$. By (4), $t\notin\mathcal{L}(r\cdot tw_{st}\cdot w)$. If $s\in\mathcal{L}(r\cdot tw_{st}\cdot w)$, we have $r\cdot tw_{st}\cdot w=w_{sr}\cdot w_1$, then $stw_{st}\cdot w=srw_{sr}\cdot w_1$, which contradicts with (7). So $s\notin\mathcal{L}(r\cdot tw_{st}\cdot w)$.

(9) Similar to (8). ■

Lemma 5.2. *Let $x,y\in W$.*

(1) *If $w\in W_{st}$, $l(w)\geq 4$, $\mathcal{R}(x),\mathcal{L}(y)\subseteq\{r\}$, then $l(xwy)=l(x)+l(w)+l(y)$.*

(2) *If $w\in W_{sr}$, $l(w)\geq 4$, $\mathcal{R}(x),\mathcal{L}(y)\subseteq\{t\}$, then $l(xwy)=l(x)+l(w)+l(y)$.*

(3) *If $\mathcal{R}(x),\mathcal{L}(y)\subseteq\{s\}$, $\mathcal{R}(xt)=\{t\}$, $\mathcal{R}(xr)=\{r\}$, then $l(xtry)=l(x)+l(y)+2$.*

(4) *If $\mathcal{R}(x),\mathcal{L}(y)\subseteq\{r\}$, $\mathcal{R}(xs)=\{s\}$, then $l(xstsy)=l(x)+l(y)+3$.*

(5) *If $\mathcal{R}(x),\mathcal{L}(y)\subseteq\{t\}$, $\mathcal{R}(xs)=\{s\}$, then $l(xsrsy)=l(x)+l(y)+3$.*

(6) *If $\mathcal{R}(x),\mathcal{L}(y)\subseteq\{r\}$, then $\deg(T_{xsts}T_y)\leq L(r)$.*

(7) *If $\mathcal{R}(x),\mathcal{L}(y)\subseteq\{r\}$, then $l(xtsty)=l(x)+l(y)+3$.*

Proof. (1)(2) See [13, 4.4].

(3) See the proof of [13, 4.5].

(4) Since $\mathcal{R}(x)\subseteq\{r\}$, $\mathcal{R}(xs)=\{s\}$, we have $\mathcal{R}(xst)=\{t\}$. If $y=(rsts)^i$, $(rsts)^ir$, $(rsts)^irs$ or $(rsts)^irst$ for some $i\in\mathbb{N}$, then $l(xstsy)=l(x)+l(y)+3$. Otherwise we use (1) or (2).

(5) Similar to (4).

(6) If $\mathcal{R}(xs)=\{s\}$ or $\mathcal{L}(sy)=\{s\}$, then by (4), $T_{xsts}T_y=T_{xstsy}$. If $\mathcal{R}(xs)=\mathcal{L}(sy)=\{r,s\}$, we assume $xs=x_1\cdot w_{sr}$, $sy=w_{sr}\cdot y_1$ for some $x_1,y_1\in W$ with $\mathcal{R}(x_1),\mathcal{L}(y_1)\subseteq\{t\}$. By Lemma 5.1(9) and Lemma 5.2(2),

$$\begin{aligned}T_{xsts}T_y&=T_{x_1w_{sr}t}T_{w_{sr}y_1}=\xi_rT_{x_1w_{sr}}T_{trw_{sr}y_1}+T_{x_1w_{sr}r}T_{trw_{sr}y_1}\\&=\xi_rT_{x_1w_{sr}trw_{sr}y_1}+T_{x_1w_{sr}rtrw_{sr}y_1}.\end{aligned}$$

We get $\deg(T_{xsts}T_y)\leq L(r)$.

(7) We may assume $y=ry'$ for some $y'\in W$ with $\mathcal{L}(y')\subseteq\{s\}$. By Lemma 5.1(5), $\mathcal{R}(xtst)=\{t\}$. By Lemma 5.1(1), $\mathcal{R}(xts)=\{s\}$. By Lemma 5.1(4), $t\notin\mathcal{R}(xtsr)$. By Lemma 5.1(5), $s\notin\mathcal{R}(xtsr)$. So $\mathcal{R}(xtsr)=\{r\}$. So by (3), we get $l(xtsty)=l(xtstry')=l(xts)+l(y')+2=l(x)+l(y)+3$. ■

Lemma 5.3. *Let $x,y\in W$, $\mathcal{R}(x),\mathcal{L}(y)\subseteq\{s\}$, then $\deg(T_{xtr}T_y)\leq L(s)$.*

Proof. We have 3 cases.

(1) $\mathcal{R}(xt)=\{t\}$, $\mathcal{R}(xr)=\{r\}$. By Lemma 5.2(3), $T_{xtr}T_y=T_{xtry}$.

(2) $\mathcal{R}(xt)=\{s,t\}$. We assume $xt=x'\cdot w_{st}$ for some $x'\in W$ with $\mathcal{R}(x')\subseteq\{r\}$.

If $\mathcal{L}(ry)=\{r\}$, by Lemma 5.2(1), $T_{xtr}T_y=T_{x'\cdot w_{st}\cdot ry}$. If $\mathcal{L}(ry)=\{s,r\}$, we assume $ry=w_{sr}\cdot y'$ for some $y'\in W$ with $\mathcal{L}(y')\subseteq\{t\}$. By Lemma 5.2(2),

$$\begin{aligned}T_{xtr}T_y&=T_{x'w_{st}}T_{w_{sr}y'}=\xi_s T_{x'w_{st}s}T_{w_{sr}y'}+T_{x'w_{st}s}T_{sw_{sr}y'}\\&=\xi_s T_{x'w_{st}s\cdot w_{sr}\cdot y'}+T_{x'w_{st}s\cdot sw_{sr}\cdot y'}.\end{aligned}$$

(3) $\mathcal{R}(xr)=\{s,r\}$. Similar to (2). ■

Lemma 5.4. *Let $x,y\in W$, $\mathcal{R}(x)\subseteq\{t\}$, and $\mathcal{L}(y)\subseteq\{s\}$. Then we have* $\deg(T_{xw_{sr}}T_{try})\le L(sr)$.

Proof. We have $T_{xw_{sr}}T_{try}=\xi_r T_{xw_{sr}r}T_{try}+T_{xw_{sr}r}T_{ty}$. Since $\mathcal{R}(xw_{sr}r)=\{s\}$, $\mathcal{L}(y)\subseteq\{s\}$, by Lemma 5.3, we get $\deg(\xi_r T_{xw_{sr}r}T_{try})\le L(sr)$.

We have 2 cases.

(1) $\mathcal{L}(ty)=\{t\}$. By Lemma 5.2(2), $T_{xw_{sr}r}T_{ty}=T_{xw_{sr}rty}$.

(2) $\mathcal{L}(ty)=\{s,t\}$. We have $ty=w_{st}\cdot y'$ for some $y'\in W$. Hence

$$\begin{aligned}T_{xw_{sr}r}T_{ty}&=T_{xw_{sr}r}T_{w_{st}y'}=\xi_s T_{xw_{sr}rs}T_{w_{st}y'}+T_{xw_{sr}rs}T_{sw_{st}y'}\\&=\xi_s T_{xw_{sr}rsw_{st}y'}+T_{xw_{sr}rsr}T_{rt}T_{tsw_{st}y'}.\end{aligned}$$

By Lemma 5.1(1)(2) we conclude $\mathcal{R}(xw_{sr}rsr)=\mathcal{L}(tsw_{st}y')=\{s\}$. By Lemma 5.3 we derive $\deg(T_{xw_{sr}rsr}\ T_{rt}T_{tsw_{st}y'})\le L(s)$. So we have $\deg(T_{xw_{sr}r}T_{ty})\le L(s)$ and $\deg(T_{xw_{sr}}T_{try})\le L(sr)$. ■

Lemma 5.5. *Let $x,y\in W$, $\mathcal{R}(x)\subseteq\{r\}$, and $\mathcal{L}(y)\subseteq\{s\}$. Then we have* $\deg(T_{xw_{st}}T_{try})\le Max\ \{L(st),L(sr)\}$. *Moreover, if $\mathcal{L}(ry)=\{r\}$, then we have* $\deg(T_{xw_{st}}T_{try})\le Max\{L(t),L(r)\}$.

Proof. First we have $T_{xw_{st}}T_{try}=\xi_t T_{xw_{st}}T_{ry}+T_{xw_{st}t}T_{ry}$.

Since $\mathcal{R}(xw_{st}t)=\{s\}$, $\mathcal{L}(y)\subseteq\{s\}$, by Lemma 5.3, we get $\deg(\xi_t T_{xw_{st}}T_{ry})\le L(st)$. If $\mathcal{L}(ry)=\{r\}$, by Lemma 5.2(1), $\deg(\xi_t T_{xw_{st}}T_{ry})\le L(t)$.

Now we consider $T_{xw_{st}t}T_{ry}$. We have 2 cases.

(1) $\mathcal{L}(ry)=\{r\}$.

(1.1) $m_{st}\ge 5$. By Lemma 5.2(1), $T_{xw_{st}t}T_{ry}=T_{xw_{st}try}$.

(1.2) $m_{st}=4$, $\mathcal{R}(xs)=\{s,r\}$, $\mathcal{L}(sry)=\{s\}$. We assume $xs=x_1\cdot w_{sr}$ and $\mathcal{R}(x_1)\subseteq\{t\}$. By Lemma 5.1(3), we have $\mathcal{L}(tsry)=\{t\}$. Then by Lemma 5.2(2), $T_{xw_{st}t}T_{ry}=T_{xsts}T_{ry}=T_{x_1w_{sr}}T_{tsry}=T_{x_1w_{sr}tsry}$.

(1.3) $m_{st}=4$, $\mathcal{R}(xs)=\{s,r\}$, $\mathcal{L}(sry)=\{s,r\}$. We assume $xs=x_1\cdot w_{sr}$ and $sry=w_{sr}\cdot y_1$ for some $x_1,y_1\in W$ with $\mathcal{R}(x_1),\mathcal{L}(y_1)\subseteq\{t\}$. Since $\mathcal{R}(x_1w_{sr}rt)=\{t\}$ and $\mathcal{L}(y_1)\subseteq\{t\}$, by Lemma 5.2(2),

$$\begin{aligned}T_{xw_{st}t}T_{ry}&=T_{xsts}T_{ry}=T_{x_1w_{sr}t}T_{w_{sr}y_1}=\xi_r T_{x_1w_{sr}rt}T_{w_{sr}y_1}+T_{x_1w_{sr}rt}T_{rw_{sr}y_1}\\&=\xi_r T_{x_1w_{sr}rtw_{sr}y_1}+T_{x_1w_{sr}rtrw_{sr}y_1}.\end{aligned}$$

(1.4) $m_{st}=4$, $\mathcal{R}(xs)=\{s\}$. By Lemma 5.2(4), $T_{xw_{st}t}T_{ry}=T_{xsts}T_{ry}=T_{xstsry}$.

(2) $\mathcal{L}(ry)=\{r,s\}$. We assume $ry=w_{sr}\cdot y'$, $\mathcal{L}(y')\subseteq\{t\}$. So

$$T_{xw_{st}t}T_{ry}=T_{xw_{st}t}T_{w_{sr}y'}=\xi_s T_{xw_{st}ts}T_{w_{sr}y'}+T_{xw_{st}ts}T_{sw_{sr}y'}.$$

(2.1) $\mathcal{R}(xw_{st}ts) = \{t\}$. By Lemma 5.2(2), $T_{xw_{st}ts}T_{w_{sr}y'} = T_{xw_{st}tsw_{sr}y'}$. On the other hand, $T_{xw_{st}ts}T_{sw_{sr}y'} = T_{xw_{st}tst}T_{tr}T_{rsw_{sr}y'}$. Since $\mathcal{R}(xw_{st}tst) = \mathcal{L}(rsw_{sr}y') = \{s\}$, by Lemma 5.3, we get $\deg(T_{xw_{st}ts}T_{sw_{sr}y'}) \le L(s)$ and $\deg(T_{xw_{st}t}T_{ry}) \le L(s)$.

(2.2) $\mathcal{R}(xw_{st}ts) = \{r, t\}$. Then we have $m_{st} = 4$ and $\mathcal{R}(xs) = \{r, s\}$. We conclude $xs = x' \cdot w_{sr}$ and $\mathcal{R}(x') \subseteq \{t\}$. So by Lemma 5.2(2),

$$\begin{aligned} T_{xw_{st}t}T_{ry} &= T_{xw_{st}t}T_{w_{sr}y'} = \xi_s T_{xst}T_{w_{sr}y'} + T_{xst}T_{sw_{sr}y'} = \xi_s T_{x'w_{sr}t}T_{w_{sr}y'} + T_{x'w_{sr}t}T_{sw_{sr}y'} \\ &= \xi_s\xi_r T_{x'w_{sr}rtw_{sr}y'} + \xi_r T_{x'w_{sr}rtrw_{sr}y'} + \xi_r T_{x'w_{sr}rtsw_{sr}y'} + T_{x'w_{sr}rtrsw_{sr}y'}. \end{aligned}$$

Summarizing the discussion above, we obtain $\deg(T_{xw_{st}}T_{try}) \le Max\{L(st), L(sr)\}$ for all cases. Moreover, $\deg(T_{xw_{st}}T_{try}) \le Max\{L(t), L(r)\}$ if $\mathcal{L}(ry) = \{r\}$. ■

Lemma 5.6. *Let $r_1, r_2 \in S$ with $r_1 \ne r_2$, $w \in W_{r_1 r_2}$ with $l(w) \ge 2$, $x, y \in W$ with $\mathcal{R}(x), \mathcal{L}(y) \subseteq S \setminus \{r_1, r_2\}$. Then we have $\deg(T_{xw}T_y) \le N - L(w)$, the equality holds only if $w \in \{w_{sr}, w_{st}\}$ and $L(w) = N$.*

Proof. We assume $\deg(T_{xw}T_y) > 0$ and $l(x), l(y) \ge 2$, otherwise the lemma is obvious. By Lemma 5.2, we need to consider the following cases.

(1) $w = sr$ (the case of $p = rs$ is similar).
We assume $y = ts \cdot y'$ for some $y' \in W$ with $\mathcal{L}(sy') = \{s\}$. Then we conclude $T_{xsr}T_y = T_{xs}T_{rt}T_{sy'}$. If $\mathcal{L}(xs) = \{s\}$, then Lemma 5.3 implies immediately $\deg(T_{xsr}T_y) \le L(s) < N - L(sr)$. If $\mathcal{L}(xs) = \{s, t\}$, then by Lemma 5.5, we get $\deg(T_{xsr}T_y) \le \max\{L(rs), L(st)\} < N - L(sr)$.

(2) $w = rsr$.
We assume $y = ts \cdot y'$ for some $y' \in W$ with $\mathcal{L}(sy') = \{s\}$. Then we conclude $T_{xrsr}T_y = T_{xrs}T_{rt}T_{sy'}$. Since $\mathcal{R}(xrs) = \mathcal{L}(sy') = \{s\}$, by Lemma 5.3, we get $\deg(T_{xrsr}T_y) \le L(s) < N - L(rsr)$.

(3) $w = srs$.
Since $\deg(T_{xsrs}T_y) > 0$, by Lemma 5.2(5), we have $\mathcal{R}(xs) = \mathcal{L}(sy) = \{s, t\}$. So we assume $x = x' \cdot w_{st}s$, $y = sw_{st} \cdot y'$ for some $x', y' \in W$ with $\mathcal{R}(x'), \mathcal{L}(y') \subseteq \{r\}$. Then $T_{xsrs}T_y = T_{x' \cdot w_{st}r}T_{w_{st} \cdot y'} = T_{x' \cdot w_{st}}T_{tr}T_{tw_{st} \cdot y'}$. Since $\mathcal{R}(x') \subseteq \{r\}$, $\mathcal{L}(tw_{st}y') = \{s\}$, $\mathcal{L}(rtw_{st}y') = \{r\}$, by Lemma 5.5, we obtain

$$\deg(T_{x' \cdot w_{st}}T_{tr}T_{tw_{st} \cdot y'}) \le Max\{L(t),\ L(r)\} < N - L(srs).$$

(4) $w = st$ (the case of $p = ts$ is similar).
We assume $y = rs \cdot y'$ for some $y' \in W$ with $\mathcal{L}(sy') = \{s\}$. Then we have $T_{xst}T_y = T_{xs}T_{rt}T_{sy'}$. If we have $\mathcal{L}(xs) = \{s\}$, then by Lemma 5.3, we obtain $\deg(T_{xst}T_y) \le L(s) < N - L(st)$. If $\mathcal{L}(xs) = \{s, r\}$, then by Lemma 5.4, we get $\deg(T_{xst}T_y) \le L(sr) < N - L(st)$.

(5) $w = sts$.
By Lemma 5.2(6), we have $\deg(T_{xsts}T_y) \le L(r) < N - L(sts)$.

(6) $w = rt$.
By Lemma 5.3, we have $\deg(T_{xrt}T_y) \le L(s) < N - L(rt)$. ■

Proposition 5.7. *In case 2, we have $\deg(T_xT_y) \le N$ for any $x, y \in W$, and the equality holds only if $x, y \in \Lambda$.*

Proof. We use induction on $l(y)$. When $l(y) = 0, 1$, the proposition is clear. Now we assume $l(y) \geq 2$ and the proposition is true for y' if $l(y') < l(y)$. We take $r_1 \in \mathcal{L}(y)$ and $r_2 \in \mathcal{L}(r_1 y)$. Then we can write $x = x_1 \cdot u$, $y = u' \cdot y_1$ for some $u, u' \in W_{r_1 r_2}$ and $x_1, y_1 \in W$ with $\mathcal{R}(x_1), \mathcal{L}(y_1) \subseteq S \setminus \{r_1, r_2\}$. Note that $l(y_1) \leq l(y) - 2$. We have

$$T_x T_y = T_{x_1}(T_u T_{u'})T_{y_1} = T_{x_1}\Big(\sum_{w \in W_{r_1 r_2}} f_{u,u',w} T_w\Big) T_{y_1} = \sum_{w \in W_{r_1 r_2}} f_{u,u',w} T_{x_1 w} T_{y_1}.$$

We prove $\deg(f_{u,u',w} T_{x_1 w} T_{y_1}) \leq N$ for all $w \in W_{r_1 r_2}$, and the equality holds only if $x, y \in \Lambda$. If $l(x_1 w y_1) = l(x_1) + l(w) + l(y_1)$ or $l(w) \leq 1$, we use the same proof in Proposition 4.3. If $l(x_1 w y_1) < l(x_1) + l(w) + l(y_1)$ and $l(w) \geq 2$, we use Lemma 5.6. ∎

6. Case 3

In this section, we prove Theorem 3.1 for the case $\infty > m_{sr} \geq 7$, $m_{st} = 3$. In this case, $N = L(w_{sr}) > \max\{L(w_{st}), L(rt)\}$.

Lemma 6.1. *Let $w \in W$.*

1. *There is no $w_1, w_2 \in W$ such that $w = w_1 \cdot st = w_2 \cdot sr$.*
2. *If $w = w_1 \cdot srs$, then $t \notin \mathcal{R}(w)$.*
3. *If $w = w_1 \cdot srsr$, then $t \notin \mathcal{R}(w)$.*
4. *If $w = w_1 \cdot ts$, then $r \notin \mathcal{R}(w)$.*
5. *If $w = w_1 \cdot tsr$, then $s \notin \mathcal{R}(w)$.*

Proof. (1) We use induction on $l(w)$. It is easy to check this lemma when $l(w) \leq 5$. Now assume $l(w) \geq 6$ and the lemma is true for w' if $l(w') < l(w)$. We assume $w = w_3 \cdot rt$. So $w_1 s = w_3 r$, $w_2 s = w_3 t$. We have $w_1 s = w_3 r = w_4 \cdot w_{sr}$ and $w_2 s = w_3 t = w_5 \cdot w_{st}$. Since $m_{sr} \geq 7$, we assume

$$w_3 = w_5 \cdot w_{st} t = w_5 \cdot ts = w_4 \cdot w_{sr} r = w_4' \cdot srsrs.$$

Since $\mathcal{R}(w_5) \subseteq \{r\}$ and $l(w_5) \geq 2$, we assume $w_5 = w_5' \cdot sr$. Thus $w_4' \cdot srs = w_5' \cdot st$. We get $w_4' \cdot srs = w_6 \cdot sts$, so $w_4' \cdot sr = w_6 \cdot st$, which contradicts with the induction hypothesis.

(2) If $w = w_1 \cdot srs = w_2 \cdot t$, we have $w = w_1 \cdot srs = w_3 \cdot sts$, so $w_1 \cdot sr = w_3 \cdot st$, which contradicts with (1).

(3) If $w = w_1 \cdot srsr = w_2 \cdot t$, we have $w = w_1 \cdot srsr = w_3 \cdot tr$, so $w_1 \cdot srs = w_3 \cdot t$, which contradicts with (2).

(4) If $w = w_1 \cdot ts = w_2 \cdot r$, we have $w = w_1 \cdot ts = w_3 \cdot w_{sr}$, so $w_1 \cdot t = w_3 \cdot w_{sr} s$, which contradicts with (3).

(5) If $s \in \mathcal{R}(w)$, then we have $w = w_2 \cdot srsrsr$ for some $w_2 \in W$, so $w_1 \cdot t = w_2 \cdot srsr$, which contradicts with (3). ∎

Lemma 6.2. *Let $x, y \in W$, $\mathcal{R}(x), \mathcal{L}(y) \subseteq \{t\}$, $w \in W_{sr}$, $l(w) \geq 6$ or $w = srsrs$, then $l(xwy) = l(x) + l(w) + l(y)$, $\mathcal{R}(xwy) = \mathcal{R}(wy)$, $\mathcal{L}(xwy) = \mathcal{L}(xw)$.*

Proof. See [13, 3.5]. ∎

Lemma 6.3. *Let $x, y \in W$, $\mathcal{R}(x), \mathcal{L}(y) \subseteq \{r\}$.*

(1) *If $\mathcal{R}(xs) = \{s\}$ or $\mathcal{L}(sy) = \{s\}$, then $T_{xsts}T_y = T_{xstsy}$.*

(2) *If $\mathcal{R}(xs) = \mathcal{L}(sy) = \{r, s\}$, then $\deg(T_{xsts}T_y) = L(r)$.*

Proof. (1) See the proof of [13, 3.6].

(2) Assume $x = x' \cdot w_{sr}s$, $y = sw_{sr} \cdot y'$ for some $x', y' \in W$ with $\mathcal{R}(x'), \mathcal{L}(y') \subseteq \{t\}$. By Lemma 6.2, we have

$$T_{xsts}T_y = T_{x' \cdot w_{sr}}T_tT_{w_{sr} \cdot y'} = \xi_r T_{x' \cdot w_{sr} \cdot t \cdot rw_{sr} \cdot y'} + T_{x' \cdot w_{sr}r \cdot t \cdot rw_{sr} \cdot y'}. \qquad \blacksquare$$

Lemma 6.4. *Let $x, y \in W$, $\mathcal{R}(x), \mathcal{L}(y) \subseteq \{s\}$.*

(1) *If $\mathcal{R}(xr) \neq \{s, r\}$, $\mathcal{R}(xt) \neq \{s, t\}$, $\mathcal{R}(xrs) \neq \{s, r\}$, then $T_{xtr}T_y = T_{xtry}$.*

(2) *If $\mathcal{R}(xr) = \{s, r\}$, then $\deg(T_{xtr}T_y) \leq L(sr)$.*

(3) *If $\mathcal{R}(xt) = \{s, t\}$, then $\deg(T_{xtr}T_y) \leq L(sr)$.*

(4) *If $\mathcal{R}(xrs) = \{s, r\}$, then $\deg(T_{xtr}T_y) \leq L(r)$.*

Proof. (1) See the proof of [13, 3.7].

(2) We assume $xr = x' \cdot w_{sr}$ for some $x' \in W$ with $\mathcal{R}(x') \subseteq \{t\}$, then $xtr = x' \cdot w_{sr} \cdot t$, so $T_{xtr}T_y = T_{x'w_{sr}}T_{ty}$. We assume $ty = u \cdot y'$ for some $u \in W_{sr}$ and $y' \in W$ with $\mathcal{L}(y') \subseteq \{t\}$. By Lemma 6.1(2), $u = e$ or $u = s$ or $u = sr$. If $u = e$ we conclude $T_{xtr}T_y = T_{x'w_{sr}}T_{y'} = T_{x'w_{sr}y'}$. If $u = s$, then $T_{xtr}T_y = T_{x'w_{sr}}T_{sy'} = \xi_s T_{x'w_{sr}y'} + T_{x'w_{sr}sy'}$. If $u = sr$, then

$$T_{xtr}T_y = T_{x'w_{sr}}T_{sry'} = \xi_s\xi_r T_{x'w_{sr}y'} + \xi_s T_{x'w_{sr}ry'} + \xi_r T_{x'w_{sr}sy'} + T_{x'w_{sr}sry'}.$$

(3) We assume $xt = x' \cdot w_{st}$ for some $x' \in W$, then $xtr = x' \cdot w_{st} \cdot r$, so we have $T_{xtr}T_y = T_{x'w_{st}}T_{ry}$. If $\mathcal{L}(ry) = \{r\}$, then by Lemma 6.3, $\deg(T_{xtr}T_y) \leq L(r)$. If $\mathcal{L}(ry) = \{s, r\}$, then we assume $ry = w_{sr} \cdot y'$ for some $y' \in W$ with $\mathcal{L}(y') \subseteq \{t\}$, so

$$T_{x'w_{st}}T_{ry} = T_{x'w_{st}}T_{w_{sr}y'} = \xi_s T_{x'w_{st}}T_{sw_{sr}y'} + T_{x'w_{st}s}T_{sw_{sr}y'}.$$

Since $\mathcal{R}(x') = \mathcal{L}(sw_{sr}y') = \{r\}$, by Lemma 6.3, we get $\deg(\xi_s T_{x'w_{st}}T_{sw_{sr}y'}) \leq L(sr)$. If $\mathcal{R}(x'w_{st}s) = \{t\}$, then by Lemma 6.2, $T_{x'w_{st}s}T_{sw_{sr}y'} = T_{x'stsw_{sr}y'}$. If we have $\mathcal{R}(x'w_{st}s) = \{r, t\}$, then $x's = x'' \cdot w_{sr}$ for some $x'' \in W$ with $\mathcal{R}(x'') \subseteq \{t\}$. So

$$\begin{aligned} T_{x'w_{st}s}T_{sw_{sr}u} &= T_{x''w_{sr}t}T_{sw_{sr}y'} = \xi_r T_{x''w_{sr}t}T_{rsw_{sr}y'} + T_{x''w_{sr}rt}T_{rsw_{sr}y'} \\ &= \xi_r T_{x''w_{sr}trsw_{sr}y'} + T_{x''w_{sr}rtrsw_{sr}y'}. \end{aligned}$$

(4) We assume $xrs = x' \cdot w_{sr}$ for some $x' \in W$, then $xtr = x' \cdot w_{sr}s \cdot t$, so $T_{xtr}T_y = T_{x'w_{sr}s}T_{ty}$. We have $ty = u \cdot y'$, $u \in W_{sr}$, $\mathcal{L}(y') \subseteq \{t\}$. By Lemma 6.1(2), $u = e$ or $u = s$ or $u = sr$. If $u = e$, then $T_{xtr}T_y = T_{x'w_{sr}s}T_{y'} = T_{x'w_{sr}sy'}$. If $u = s$, then $T_{xtr}T_y = T_{x'w_{sr}s}T_{sy'} = T_{x'w_{sr}y'}$. If $u = sr$, then

$$T_{xtr}T_y = T_{x'w_{sr}s}T_{sry'} = \xi_r T_{x'w_{sr}y'} + T_{x'w_{sr}ry'}. \qquad \blacksquare$$

Lemma 6.5. *Let $x, y \in W$, $\mathcal{R}(x) \subseteq \{r\}$, $\mathcal{L}(y) \subseteq \{t\}$, then $\deg(T_{xw_{st}}T_{w_{sr}y}) \leq L(sr)$.*

Proof. See the proof of Lemma 6.4(3). $\blacksquare$

Lemma 6.6. *Let* $x, y \in W$, $\mathcal{R}(x), \mathcal{L}(y) \subseteq \{t\}$, *then* $\deg(T_{xrsrsr}T_y) \leq L(s)$.

Proof. We suppose $l(xrsrsry) < l(x) + l(y) + 5$. We must have $l(x), l(y) \geq 1$, so we assume $x = x' \cdot t$ and $y = t \cdot y'$ for some $x', y' \in W$ with $\mathcal{R}(x'), \mathcal{L}(y') \subseteq \{s\}$. Then $T_x T_{rsrsr} T_y = T_{x'} T_{rt} T_{srsrty'}$. Since $\mathcal{L}(rsrsrty') = \{r\}$, $\mathcal{L}(srsrsrty') = \{s\}$, by Lemma 6.4, we must have $\mathcal{L}(tsrsrty') = \{s, t\}$. Thus we get $\mathcal{L}(ry') = \{r, s\}$. Similarly, we can prove $\mathcal{R}(x'r) = \{r, s\}$. Now we assume $x = x'' \cdot w_{sr} r \cdot t$, $y = t \cdot r w_{sr} \cdot y''$ for some $x'', y'' \in W$ with $\mathcal{R}(x''), \mathcal{L}(y'') \subseteq \{t\}$. By Lemma 6.2, we have $\mathcal{L}(tsrst \cdot sw_{sr} \cdot y'') = \mathcal{L}(tsrst \cdot sw_{sr}) = \{t\}$, thus

$$\begin{aligned} T_x T_{rsrsr} T_y &= T_{x'' \cdot w_{sr} r \cdot t} T_{rsrsr} T_{t \cdot r w_{sr} \cdot y''} = T_{x'' \cdot w_{sr} s \cdot tst} T_r T_{tst \cdot sw_{sr} \cdot y''} \\ &= \xi_t T_{x'' \cdot w_{sr} \cdot tsrst \cdot sw_{sr} \cdot y''} + T_{x'' \cdot w_{sr} s \cdot tsrst \cdot sw_{sr} \cdot y''}. \end{aligned}$$ ■

Lemma 6.7. *Let* $x, y \in W$. *If* $\mathcal{R}(x) \subseteq \{s\}$, $\mathcal{L}(y) \subseteq \{t\}$, *then* $\deg(T_{xtr}T_{w_{sr}y}) \leq L(rsr)$.

Proof. We have $T_{xtr} T_{w_{sr}y} = \xi_r T_{xtr} T_{rw_{sr}y} + T_{xt} T_{rw_{sr}y}$. Since $\mathcal{R}(x) \subseteq \{s\}$ and $\mathcal{L}(rw_{sr}y) \subseteq \{s\}$, by Lemma 6.4, we conclude $\deg(\xi_r T_{xtr} T_{rw_{sr}y}) \leq L(rsr)$.
If $\mathcal{R}(xt) = \{t\}$, by Lemma 6.2, we have $T_{xt} T_{rw_{sr}y} = T_{xtrw_{sr}y}$. If $\mathcal{R}(xt) = \{s, t\}$, we assume $xt = x' \cdot w_{st}$ for some $x' \in W$ with $\mathcal{R}(x') \subseteq \{r\}$. This leads us to

$$T_{xt} T_{rw_{sr}y} = \xi_s T_{x'sts} T_{srw_{sr}y} + T_{x'st} T_{srw_{sr}y}.$$

By Lemma 6.3, we have $\deg(\xi_s T_{x'sts} T_{srw_{sr}y}) \leq L(rs)$. If $\mathcal{R}(x'st) = \{t\}$, by Lemmas 6.2, 6.6, $\deg(T_{x'st} T_{srw_{sr}y}) \leq L(s)$. If $\mathcal{R}(x'st) = \{r, t\}$, we get $x'st = x'' \cdot w_{sr} \cdot t$, thus

$$\begin{aligned} T_{x'st} T_{srw_{sr}y} &= T_{x''w_{sr}t} T_{srw_{sr}y} = \xi_r T_{x''w_{sr}rt} T_{srw_{sr}y} + T_{x''w_{sr}rt} T_{rsrw_{sr}y} \\ &= \xi_r T_{x''w_{sr}rtsrw_{sr}y} + T_{x''w_{sr}rtrsrw_{sr}y}. \end{aligned}$$ ■

Lemma 6.8. *Let* $x, y \in W$. *If* $\mathcal{R}(x) \subseteq \{s\}$, $\mathcal{L}(y) \subseteq \{r\}$, *then* $\deg(T_{xtr}T_{w_{st}y}) \leq L(srsr)$.

Proof. We have $T_{xtr} T_{tsty} = \xi_s T_{xtr} T_{sty} + T_{xr} T_{sty}$. Since $\mathcal{L}(sty) = \{s\}$, by Lemma 6.4, we have $\deg(\xi_s T_{xtr} T_{sty}) \leq L(srs)$. Now we consider $T_{xr} T_{sty}$. If $y = e$, the lemma is clear, so we may assume $y = r \cdot y_1$ for some $y_1 \in W$ with $\mathcal{L}(y_1) \subseteq \{s\}$. We have 2 cases:

(1) $\mathcal{R}(xr) = \{r\}$.
If $\mathcal{R}(xrs) = \{s\}$, then $T_{xr} T_{sty} = T_{xrstr} T_{y_1}$. By Lemma 6.4, we get $\deg(T_{xr} T_{sty}) \leq L(sr)$. If $\mathcal{R}(xrs) = \{s, r\}$, then we assume $xrs = x' \cdot w_{sr}$ for some $x' \in W$ with $\mathcal{R}(x') \subseteq \{t\}$. So $T_{xr} T_{sty} = T_{x'w_{sr}} T_{try_1}$. By Lemma 6.7, $\deg(T_{xr} T_{sty}) \leq L(rsr)$.

(2) $\mathcal{R}(xr) = \{s, r\}$.
We assume $xr = x' \cdot w_{sr}$ for some $x' \in W$ with $\mathcal{R}(x') \subseteq \{t\}$, thus

$$\begin{aligned} T_{xr} T_{sty} &= T_{x'w_{sr}} T_{stry_1} = \xi_s T_{x'w_{sr}} T_{try_1} + T_{x'w_{sr}s} T_{try_1} \\ &= \xi_s T_{x'w_{sr}} T_{try_1} + \xi_r T_{x'w_{sr}sr} T_{try_1} + T_{x'w_{sr}sr} T_{ty_1}. \end{aligned}$$

By Lemma 6.7 we have $\deg(\xi_s T_{x'w_{sr}} T_{try_1}) \leq L(srsr)$. By Lemma 6.4 we conclude $\deg(\xi_r T_{x'w_{sr}sr} T_{try_1}) \leq L(rsr)$. If $\mathcal{L}(ty_1) = \{t\}$, then Lemma 6.2 leads to $T_{x'w_{sr}sr} T_{ty_1} = T_{x'w_{sr}srty_1}$. If $\mathcal{L}(ty_1) = \{s, t\}$, we assume $ty_1 = w_{st} \cdot y_2$ for some $y_2 \in W$ with $\mathcal{L}(y_2) \subseteq \{r\}$. Then

$$T_{x'w_{sr}sr} T_{ty_1} = T_{x'w_{sr}sr} T_{stsy_2} = \xi_s T_{x'w_{sr}sr} T_{tsy_2} + T_{x'w_{sr}srs} T_{tsy_2}.$$

If $\mathcal{L}(tsy_2) = \{t\}$, then by Lemma 6.2, $T_{x'w_{sr}sr} T_{tsy_2} = T_{x'w_{sr}srtsy_2}$. By Lemma 6.4, $\deg(T_{x'w_{sr}srs} T_{tsy_2}) = \deg(T_{x'w_{sr}srsr} T_{rtsy_2}) \leq L(sr)$.

If $\mathcal{L}(tsy_2) = \{r,t\}$, we assume $sy_2 = w_{sr}\cdot y_3$ for some $y_3 \in W$ with $\mathcal{L}(y_3) \subseteq \{t\}$. By Lemma 6.2, $T_{x'w_{sr}sr}T_{tsy_2} = T_{x'\cdot w_{sr}sr\cdot t}T_{w_{sr}\cdot y_3} = T_{x'\cdot w_{sr}sr\cdot t\cdot w_{sr}\cdot y_3}$. By Lemma 6.7, we get $\deg(T_{x'w_{sr}srs}T_{tsy_2}) = \deg(T_{x'\cdot w_{sr}srs}T_{t\cdot w_{sr}\cdot y_3}) = \deg(T_{x'\cdot w_{sr}srsr\cdot rt}T_{w_{sr}\cdot y_3}) \leq L(rsr)$. ■

Lemma 6.9. *Let $r_1, r_2 \in S$ with $r_1 \neq r_2$, $w \in W_{r_1r_2}$ with $l(w) \geq 2$, $x, y \in W$ with $\mathcal{R}(x), \mathcal{L}(y) \subseteq S \setminus \{r_1, r_2\}$. Then we have $\deg(T_{xw}T_y) \leq N - L(w)$, and the equality holds only if $w = w_{sr}$.*

Proof. We assume $\deg(T_{xw}T_y) > 0$ and $l(x), l(y) \geq 3$, otherwise the lemma is obvious. By Lemma 6.2, we need to consider the following cases.

(1) $w = rsrsr$. By Lemma 6.6, we have $\deg(T_{xrsrsr}T_y) \leq L(s) < N - L(rsrsr)$.

(2) $w = rsrs$ (the case of $p = srsr$ is similar). We assume $x = x' \cdot rst$ for some $x' \in W$ with $\mathcal{R}(x') \subseteq \{s\}$. Then $T_{xrsrs}T_y = T_{x'rs\cdot tr}T_{srs\cdot y}$. By Lemma 6.4, we have $\deg(T_{xrsrs}T_y) \leq L(sr) < N - L(rsrs)$.

(3) $w = srs$. We assume $x = x' \cdot rst$ and $y = tsr \cdot y'$ for some $x', y' \in W$ with $\mathcal{R}(x'), \mathcal{L}(y') \subseteq \{s\}$. Then we have $T_{xsrs}T_y = \xi_s T_{x'rt}T_{srstsry'} + T_{x'rt}T_{srstry'}$. Since $\mathcal{L}(srstsry') = \mathcal{L}(srstry') = \{s\}$, by Lemma 6.4, we immediately get the inequalities $\deg(T_{xsrs}T_y) \leq L(srs) < N - L(srs)$.

(4) $w = rsr$. We assume $x = x' \cdot rst$, $y = tsr \cdot y'$ for some $x', y' \in W$ with $\mathcal{R}(x'), \mathcal{L}(y') \subseteq \{s\}$. Then we have $T_{xrsr}T_y = T_{x'rsrts}T_{trsry'}$. If $\mathcal{R}(x'rsrts) = \{s\}$, then by Lemma 6.4, we have $\deg(T_{xrsr}T_y) \leq L(sr)$. If $\mathcal{R}(x'rsrts) = \{s,t\}$, we assume $x' \cdot rsr = x'' \cdot w_{sr}$ for some $x'' \in W$ with $\mathcal{R}(x'') \subseteq \{t\}$. Then

$$T_{x'rsrts}T_{trsry'} = \xi_s T_{x''\cdot w_{sr}s\cdot tst}T_{rsry'} + T_{x''\cdot w_{sr}s\cdot ts}T_{rsry'}.$$

Since $\mathcal{R}(x'' \cdot w_{sr}s) = \{r\}$, $\mathcal{L}(rsry') = \{r\}$ or $\{s,r\}$, by Lemmas 6.3, 6.5, we have $\deg(\xi_s T_{x''\cdot w_{sr}s\cdot tst}T_{rsry'}) \leq L(srs)$.

Now we consider $T_{x''\cdot w_{sr}s\cdot ts}T_{rsry'}$. First we assume $\mathcal{L}(rsry') = \{r\}$. Then we obtain $T_{x''\cdot w_{sr}s\cdot ts}T_{rsry'} = T_{x''\cdot w_{sr}sr\cdot rt}T_{srsry'}$. If $\mathcal{L}(srsry') = \{s\}$, we have $\deg(T_{x''\cdot w_{sr}sr\cdot rt}T_{srsry'}) \leq L(sr)$ by Lemma 6.4. If $\mathcal{L}(srsry') = \{s,r\}$, we have $\deg(T_{x''\cdot w_{sr}sr\cdot rt}T_{srsry'}) = L(r)$ by easy computation. Now we assume $\mathcal{L}(rsry') = \{s,r\}$, then $rsry' = w_{sr} \cdot y''$ for some $y'' \in W$ with $\mathcal{L}(y'') \subseteq \{t\}$. We have

$$\begin{aligned}
T_{x''\cdot w_{sr}s\cdot ts}T_{rsry'} &= T_{x''\cdot w_{sr}s\cdot ts}T_{w_{sr}\cdot y''} = \xi_s T_{x''\cdot w_{sr}s\cdot t}T_{w_{sr}\cdot y''} + T_{x''\cdot w_{sr}s\cdot t}T_{sw_{sr}\cdot y''} \\
&= \xi_s\xi_r T_{x''\cdot w_{sr}sr\cdot t}T_{w_{sr}\cdot y''} + \xi_s T_{x''\cdot w_{sr}sr\cdot t}T_{rw_{sr}\cdot y''} + \xi_r T_{x''\cdot w_{sr}sr\cdot t}T_{sw_{sr}\cdot y''} \\
&\quad + T_{x''\cdot w_{sr}sr\cdot t}T_{rsw_{sr}\cdot y''} \\
&= \xi_s\xi_r T_{x''\cdot w_{sr}sr\cdot t\cdot w_{sr}\cdot y''} + \xi_s T_{x''\cdot w_{sr}sr\cdot t\cdot rw_{sr}\cdot y''} + \xi_r T_{x''w_{sr}sr\cdot t\cdot sw_{sr}y''} + T_{x''w_{sr}sr\cdot t\cdot rsw_{sr}y''}.
\end{aligned}$$

Summarizing the discussions above, we get $\deg(T_{xrsr}T_y) \leq L(srs) < N - L(rsr)$.

(5) $w = rs$ (the case of $p = sr$ is similar). We assume $x = x' \cdot rst$, $y = tsr \cdot y'$ for some $x', y' \in W$ with $\mathcal{R}(x'), \mathcal{L}(y') \subseteq \{s\}$. Then $T_{xrs}T_y = T_{x'rs\cdot tr}T_{sts\cdot ry'}$. By Lemma 6.8, we have $\deg(T_{xrs}T_y) \leq L(srsr) < N - L(rs)$.

(6) $w = sts$. By Lemma 6.3, we have $\deg(T_{xsts}T_y) \leq L(r) < N - L(sts)$.

(7) $w = st$ (the case of $p = ts$ is similar). We assume $x = x' \cdot sr$, $y = rs \cdot y'$ for some $x', y' \in W$ with $\mathcal{R}(x' \cdot s) = \mathcal{L}(s \cdot y') = \{s\}$. Then $T_{xst}T_y = T_{x'\cdot srs}T_{tr\cdot sy'}$. By Lemma 6.4, 6.7, we have $\deg(T_{xst}T_y) \leq L(rsr) < N - L(st)$.

(8) $w = rt$. By Lemma 6.4, we have $\deg(T_{xrt}T_y) \leq L(sr) < N - L(rt)$. ■

Proposition 6.10. *In case 3, we have* $\deg(T_xT_y) \leq N$ *for any* $x, y \in W$*, and the equality holds only if* $x, y \in \Lambda$*.*

Proof. We use Lemma 6.9 and the method in the proof of Proposition 5.7. ∎

We have completed the proof of Theorem 3.1.

7. Some consequences

Proposition 7.1. *For any* $w_J \in M$*,* $q \leq w_J$*,* $x, y \in W$*,* $\mathcal{R}(x), \mathcal{L}(y) \subseteq S \setminus J$*, we have* $\deg(T_{xq}T_y) \leq N - L(q)$*. In particular,* $T_{xw_J}T_y = T_{xw_Jy}$*.*

Proof. We have

$$T_{xw_J}T_{w_Jy} = T_x(T_{w_J}T_{w_J})T_y = T_x\Big(\sum_{w\in W_J} f_{w_J,w_J,w}T_w\Big)T_y = \sum_{w\in W_J} f_{w_J,w_J,w}T_{xw}T_y.$$

Since $\deg(T_{xw_J}T_{w_Jy}) \leq N$ by Theorem 3.1, we get

$$\deg(T_{xq}T_y) \leq N - \deg f_{w_J,w_J,q} = N - L(q).$$ ∎

Proposition 7.2. *Let* $w_J \in M$*.*

(1) *The left cell of* W *containing* w_J *is* $\{x \cdot w_J | x \in W\} = \{y \in W | \mathcal{R}(y) = J\}$*.*

(2) *The right cell of* W *containing* w_J *is* $\{w_J \cdot x | x \in W\} = \{y \in W | \mathcal{L}(y) = J\}$*.*

Proof. (1) For $x \cdot w_J \in W$, we have $\deg f_{xw_J,w_J,xw_J} = N$ since $\deg f_{w_J,w_J,w_J} = N$. Therefore $\beta_{xw_J,w_J,w_Jx^{-1}} \neq 0$. By Lemma 2.3(3), we get $w_J \underset{L}{\sim} xw_J$. So we have $\{x \cdot w_J | x \in W\} \subseteq \{x | x \underset{L}{\sim} w_J\}$. On the other hand, for any $x \in W$ with $x \underset{L}{\sim} w_J$, we have $\mathcal{R}(x) = \mathcal{R}(w_J) = J$, so $\{x | x \underset{L}{\sim} w_J\} \subseteq \{x \cdot w_J | x \in W\} = \{y \in W | \mathcal{R}(y) = J\}$.

(2) The proof is similar to (1). ∎

For any $w_J \in M$, we define $B_J = \{x \in W | \mathcal{R}(x) \subseteq S \setminus J\}$ and

$$U_J = \{y \in W | \mathcal{L}(y) \subseteq S \setminus J \text{ and } sw_Jy \notin \Lambda \text{ for all } s \in J\}.$$

Then we have the following result.

Proposition 7.3. *Let* $w_J \in M$*,* $x \in B_J$*,* $q < w_J$*.*

(1) *If* $y \in B_J^{-1}$*,* $l(q) \geq 2$*, then* $\deg(T_{xq}T_y) < N - L(q)$ *for the three cases listed in Section* 3*.*

(2) *If* $y \in U_J$*, then* $\deg(T_{xq}T_y) < N - L(q)$ *for any weighted Coxeter group of rank* 3*.*

Proof. (1) See Lemmas 4.2(3), 5.6 and 6.9.

(2) It is obvious when W is a finite Coxeter group. We can get this inequality by [11, 3.1] when W is an affine Weyl group and by [12, 3.5] when W has complete Coxeter graph. Now we only consider other cases.
If $l(q) = 0$, by Theorem 3.1, we get $\deg(T_xT_y) < N$ since $y \notin \Lambda$. If $l(q) = 1$, by Theorem 3.1, we get $\deg(T_{xq}T_{qy}) < N$ since $qy \notin \Lambda$, so $\deg(T_{xq}T_y) < N - L(q)$.
If $l(q) \geq 2$, then we must be in the three cases listed in Section 3. Thus we get $\deg(T_{xq}T_y) < N - L(q)$ by (1). ∎

8. The lowest two-sided cell

Theorem 8.1. *We fix an element $w_J \in M$.*

(1) *The two-sided cell c_0 containing w_J is the lowest two-sided cell of W.*

(2) *We have $\Lambda = \{w \in W | a(w) = N\} \subseteq c_0$.*

Proof. (1) For $w \in W$, we may assume $w = y \cdot z$ for some $z \in W_J$ and $y \in W$ with $\mathcal{R}(y) \subseteq S \setminus J$. It is clear that $y \cdot w_J \underset{R}{\leq} w$. We have

$$T_{w_J y^{-1}} T_{y w_J} = T_{w_J}\Big(\sum_{z \in W} f_{y^{-1},y,z} T_z\Big) T_{w_J} = T_{w_J} T_{w_J} + T_{w_J}\Big(\sum_{z \in W\setminus\{e\}} f_{y^{-1},y,z} T_z\Big) T_{w_J}.$$

Since $\deg f_{w_J,w_J,w_J} = L(w_J)$, we get $\deg f_{w_J y^{-1}, y w_J, w_J} = L(w_J)$. So $\beta_{w_J y^{-1}, y w_J, w_J} \neq 0$ and $y w_J \underset{L}{\sim} w_J$. Thus $w_J \underset{L}{\sim} y w_J \underset{R}{\leq} w$. So $w_J \underset{LR}{\leq} w$ for all $w \in W$. We see c_0 is the lowest two-sided cell of W.

(2) First we prove $\Lambda = \{w \in W | a(w) = N\}$. For any $x \cdot u \cdot y \in \Lambda$, $x, y \in W$, $u \in M$, we have $\beta_{ux^{-1}, xuy, y^{-1}u} \neq 0$ since $\deg f_{ux^{-1}, xuy, uy} = N$. Using Lemma 2.3(3), we know $a(xuy) = N$, so $\Lambda \subseteq \{w \in W | a(w) = N\}$. On the other hand, if $a(w) = a(w^{-1}) = N$, take $x, y \in W$ such that $\deg h_{x,y,w^{-1}} = N$. Then $\gamma_{x,y,w} \neq 0$. By Lemma 2.3(1)(4), $\beta_{y,w,x} = \beta_{x,y,w} \neq 0$. So $\deg f_{y,w,x^{-1}} = N$. By Theorem 3.1, we get $w \in \Lambda$.

Now we prove $\{w \in W | a(w) = N\} \subseteq c_0$. For $w \in W$, $a(w) = a(w^{-1}) = N$, there exists $x, y \in W$ such that $\deg f_{x,y,w^{-1}} = N$. So $\beta_{x,y,w} = \gamma_{x,y,w} \neq 0$ and $w \underset{L}{\sim} x^{-1}$. We take $u \in W_J$ such that $l(yu) = l(y) + l(u)$ and $\mathcal{R}(yu) = J$, thus $yu \underset{LR}{\sim} w_J$. Since $T_x T_{yu} = (T_x T_y) T_u$ and N is a bound for (W, S, L), we have $\deg f_{x,yu,w^{-1}u} = N$. Thus $\beta_{x,yu,u^{-1}w} = \gamma_{x,yu,u^{-1}w} \neq 0$, $x \underset{L}{\sim} u^{-1}y^{-1}$. So $x^{-1} \underset{R}{\sim} yu$. We get $w \underset{L}{\sim} x^{-1} \underset{R}{\sim} yu \underset{LR}{\sim} w_J$. So we have $\{w \in W | a(w) = N\} \subseteq c_0$. ∎

Proposition 8.2. *Let $w_J \in M$.*

(1) *If $x \in B_J$, $y \in U_J$, then we have $C_{x w_J y} = E_x C_{w_J} F_y$.*

(2) *If $y \in U_J$, $x \in U_J^{-1}$, then the set $\Gamma_{J,y} = B_J w_J y$ is a left cell of W and the set $\Phi_{x,J} = x w_J B_J^{-1}$ is a right cell of W.*

Proof. Use Propositions 7.1, 7.2, 7.3 and the methods in the proofs of [12, 3.6, 3.7]. ∎

Now we can give a precise description for the lowest two-sided cell and the left cells, right cells in it.

Theorem 8.3. *Let c_0 be the lowest two-sided cell of W.*

(1) *We have $c_0 = \Lambda = \{w \in W | a(w) = N\}$.*

(2) *The two-sided cell c_0 can be decomposed into left cells as*

$$c_0 = \coprod_{w_J \in M, y \in U_J} \Gamma_{J,y},$$

and into right cells as

$$c_0 = \coprod_{w_J \in M, x \in U_J^{-1}} \Phi_{x,J}.$$

Proof. (1) It is clear that $\Lambda = \bigcup_{w_J \in M} B_J w_J U_J = \bigcup_{w_J \in M} U_J^{-1} w_J B_J^{-1} \subseteq c_0$.
We claim that for any $x \in W$, $y \in \Lambda$, if $x \underset{L}{\leftarrow} y$ or $x \underset{R}{\leftarrow} y$, then $x \in \Lambda$. We only prove the $x \underset{L}{\leftarrow} y$ case. First, we assume $y = z_1 w_J z_2$ for some $w_J \in M$, $z_1 \in B_J$, $z_2 \in U_J$, $h_{s,y,x} \neq 0$ for some $s \in S$. By Proposition 8.2(1), we have

$$C_s C_{z_1 w_J z_2} = C_s C_{z_2 w_J} F_{z_2} = \sum_{z_3 \in B_J} h_{s, z_1 w_J, z_3 w_J} C_{z_3 w_J} F_{z_2}$$
$$= \sum_{z_3 \in B_J} h_{s, z_1 w_J, z_3 w_J} C_{z_3 w_J z_2}.$$

We get $x = z_3 w_J z_2 \subseteq \Lambda$ for some $z_3 \in B_J$ and $z_2 \in U_J$, the claim is proved.
For $w \in c_0$, we have $w \underset{LR}{\leq} w_J$ and $w_J \in \Lambda$, so $w \in \Lambda$ by the claim. On the other hand, we have $\Lambda = \{w \in W | a(w) = N\} \subseteq c_0$ by Theorem 8.1(2). The first part of the theorem follows.
(2) We only prove the left cell decomposition of c_0. It is clear that

$$c_0 = \bigcup_{w_J \in M, y \in U_J} \Gamma_{J,y},$$

and we need to show it is a disjoint union. If $w_J, w_{J'} \in M$, $y \in U_J$, $y' \in U_{J'}$ such that $\Gamma_{J,y} = \Gamma_{J',y'}$, then there exists $x \in B_J$ such that $x \cdot w_J \cdot y = w_{J'} \cdot y'$. If $x \neq e$, we take $s \in \mathcal{L}(x)$, then $s \in \mathcal{L}(x \cdot w_J \cdot y) = \mathcal{L}(w_{J'} \cdot y') = J'$ and $s w_{J'} \cdot y' = sx \cdot w_J \cdot y \in \Lambda$, which contradicts with $y' \in U_{J'}$. So we have $x = e$. Thus, $J = \mathcal{L}(w_J \cdot y) = \mathcal{L}(w_{J'} \cdot y') = J'$, $y = y'$. Therefore, it is a disjoint union. ■

Proposition 8.4. *The number of left cells in c_0 is as follows:*

W	*the number of left cells in* c_0
finite Coxeter group	1
affine Weyl group	$\lvert W_0 \rvert$
$m_{sr} = \infty$, $m_{st} = m_{rt} = 2$	2
$m_{sr} = m_{st} = \infty$, $\infty > m_{rt} \geq 2$	∞ *or* $2m_{rt}$
$m_{sr} = m_{st} = m_{rt} = \infty$	∞ *or* 3 *or* 4
other cases	∞

When W is an affine Weyl group, W_0 denotes the finite Weyl group corresponding to W. When the number of left cells in c_0 has various possibilities, it depends on the weight function L.

Proof. It is clear when W is a finite Coxeter group. See [11, 3.7] when W is an affine Weyl group. We assume $\infty \geq m_{sr} \geq m_{st} \geq m_{rt} \geq 2$ and consider the following 11 cases.

(1) $m_{sr} = m_{st} = m_{rt} = \infty$. We have $N = Max\{L(r), L(s), L(t)\}$. We may assume $L(r) \geq L(s) \geq L(t)$. If $L(r) = L(s) = L(t)$, then $\Gamma_{\{r\},e}$, $\Gamma_{\{s\},e}$ and $\Gamma_{\{t\},e}$ are all the 3 left cells in c_0. If $L(r) = L(s) > L(t)$, then $\Gamma_{\{r\},e}$, $\Gamma_{\{s\},e}$, $\Gamma_{\{r\},t}$ and $\Gamma_{\{s\},t}$ are all the 4 left cells in c_0. If $L(r) > L(s) \geq L(t)$, then for different $k \in \mathbb{N}$, $\Gamma_{\{r\},(st)^k}$ are different left cells in c_0.

(2) $m_{sr} = m_{st} = \infty$, $\infty > m_{rt} \geq 2$. We have $N = Max\{L(w_{rt}), L(s)\}$. If $L(w_{rt}) > L(s)$, then for different $k \in \mathbb{N}$, $\Gamma_{\{r,t\},(sr)^k}$ are different left cells in c_0. If $L(w_{rt}) = L(s)$, then $\Gamma_{\{r,t\},e}$ and $\Gamma_{\{s\},w}, w \in W_{rt} \setminus \{w_{rt}\}$ are all the $2m_{rt}$ left cells in c_0. If $L(s) > L(w_{rt})$, then $\Gamma_{\{s\},w}, w \in W_{rt}$ are all the $2m_{rt}$ left cells in c_0.

(3) $m_{sr} = \infty$, $m_{st} = m_{rt} = 2$. Now we have $N = Max\{L(st), L(rt)\}$. If $L(s) > L(r)$, then $\Gamma_{\{s,t\},e}$ and $\Gamma_{\{s,t\},r}$ are all the 2 left cells in c_0. If $L(s) = L(r)$, then $\Gamma_{\{s,t\},e}$ and $\Gamma_{\{r,t\},e}$ are all the 2 left cells in c_0. If $L(r) > L(s)$, then $\Gamma_{\{r,t\},e}$ and $\Gamma_{\{r,t\},s}$ are all the 2 left cells in c_0.

(4) $m_{sr} = \infty$, $\infty > m_{st} \geq 3$, $m_{rt} = 2$. We have $N = Max\{L(w_{st}), L(rt)\}$. If $L(w_{st}) > L(rt)$, then for different $k \in \mathbb{N}$, $\Gamma_{\{s,t\},(rs)^k}$ are different left cells in c_0. If $L(rt) \geq L(w_{st})$, then for different $k \in \mathbb{N}$, $\Gamma_{\{r,t\},(sr)^k}$ are different left cells in c_0.

(5) $m_{sr} = \infty$, $\infty > m_{st}, m_{rt} \geq 3$. We have $N = Max\{L(w_{st}), L(w_{rt})\}$. We may assume $N = L(w_{st})$, then for different $k \in \mathbb{N}$, $\Gamma_{\{s,t\},(rst)^k}$ are different left cells in c_0.

(6) $\infty > m_{sr} \geq m_{st} \geq 4$, $m_{sr} \geq 5$, $m_{rt} = 2$. We have $N = Max\{L(w_{sr}), L(w_{st})\} > L(rt)$. If $L(w_{sr}) \geq L(w_{st})$, then for different $k \in \mathbb{N}$, $\Gamma_{\{s,r\},(tsrs)^k}$ are different left cells in c_0. If $L(w_{st}) > L(w_{sr})$, then for different $k \in \mathbb{N}$, $\Gamma_{\{s,t\},(rst)^k}$ are different left cells in c_0.

(7) $\infty > m_{sr} \geq 7$, $m_{st} = 3$, $m_{rt} = 2$. We have $N = L(w_{sr}) > Max\{L(w_{st}), L(rt)\}$. For different $k \in \mathbb{N}$, $\Gamma_{\{s,r\},(tsrsrs)^k}$ are different left cells in c_0.

(8) $\infty > m_{sr} \geq m_{st} \geq m_{rt} \geq 4$. If $L(w_{sr}) = N$, then for different $k \in \mathbb{N}$, $\Gamma_{\{s,r\},(tsr)^k}$ are different left cells in c_0. $L(w_{st}) = N$ and $L(w_{rt}) = N$ are similar.

(9) $\infty > m_{sr} \geq m_{st} \geq 4$, $m_{rt} = 3$, $L(w_{rt}) = N$. For different $k \in \mathbb{N}$, $\Gamma_{\{r,t\},(srt)^k}$ are different left cells in c_0.

(10) $\infty > m_{sr} \geq m_{st} \geq 4$, $m_{rt} = 3$, $L(w_{sr}) = N$. For different $k \in \mathbb{N}$, $\Gamma_{\{s,r\},(tsr)^k}$ are different left cells in c_0.

(11) $\infty > m_{sr} \geq 4$, $m_{st} = m_{rt} = 3$. Now we have $N = L(w_{sr}) > L(w_{st}) = L(w_{rt})$. For different $k \in \mathbb{N}$, $\Gamma_{\{s,r\},(tsr)^k}$ are different left cells in c_0. ∎

For $n \in \mathbb{Z}$, define $\pi_n : \mathbb{Z}[v, v^{-1}] \longrightarrow \mathbb{Z}, \quad \sum_{k\in\mathbb{Z}} a_k v^k \longrightarrow a_n$.

For $z \in W$, define $\Delta(z) \in \mathbb{N}$ and $n_z \in \mathbb{Z} \setminus \{0\}$, such that

$$p_{e,z} = n_z v^{-\Delta(z)} + \text{lower degree terms}.$$

Let
$$\mathcal{D} = \{z \in W | a(z) = \Delta(z)\}.$$

Theorem 8.5. *We have the following propositions $P1' - P15'$ and $\widetilde{P}'$. In particular, Conjectures $P1 - P15$ and $\widetilde{P}$ hold for c_0.*

$P1'$ *For $z \in c_0$, we have $a(z) \leq \Delta(z)$.*

$P2'$ *For $d \in \mathcal{D} \cap c_0$, $x, y \in c_0$, if $\gamma_{x,y,d} \neq 0$, then $x = y^{-1}$.*

$P3'$ *For $y \in c_0$, there exists unique $d \in \mathcal{D} \cap c_0$ such that $\gamma_{y^{-1},y,d} \neq 0$.*

$P4'$ *For $z, z' \in W$, if $z \in c_0$ or $z' \in c_0$, and $z' \underset{LR}{\leq} z$, then $a(z') \geq a(z)$.*

$P5'$ *For $d \in \mathcal{D} \cap c_0$, $y \in c_0$, if $\gamma_{y^{-1},y,d} \neq 0$, then $\gamma_{y^{-1},y,d} = n_d = 1$.*

$P6'$ *For $d \in \mathcal{D} \cap c_0$, we have $d^2 = e$.*

*P*7′   *For* $x, y, z \in W$, *if* $x \in c_0$ *or* $z \in c_0$, *then* $\gamma_{x,y,z} = \gamma_{y,z,x}$.

*P*8′   *For* $x, y \in W$, $z \in c_0$, *if* $\gamma_{x,y,z} \neq 0$, *then* $x \underset{L}{\sim} y^{-1}$, $y \underset{L}{\sim} z^{-1}$, $z \underset{L}{\sim} x^{-1}$.

*P*9′   *For* $z, z' \in W$, *if* $z \in c_0$ *or* $z' \in c_0$, $z' \underset{L}{\leq} z$, *and* $a(z') = a(z)$, *then* $z' \underset{L}{\sim} z$.

*P*10′   *For* $z, z' \in W$, *if* $z \in c_0$ *or* $z' \in c_0$, $z' \underset{R}{\leq} z$, *and* $a(z') = a(z)$, *then* $z' \underset{R}{\sim} z$.

*P*11′   *For* $z, z' \in W$, *if* $z \in c_0$ *or* $z' \in c_0$, $z' \underset{LR}{\leq} z$, *and* $a(z') = a(z)$, *then* $z' \underset{LR}{\sim} z$.

*P*12′   *If* $y \in c_0 \cap W_I$ *for some* $I \subseteq S$, *then* $a(y)$ *computed in terms of* W_I *is equal to* $a(y)$ *computed in terms of* W.

*P*13′   *For any left cell* Γ *in* c_0, *we have* $|\Gamma \cap \mathcal{D}| = 1$. *Assume* $\Gamma \cap \mathcal{D} = \{d\}$, *then* $\gamma_{x^{-1},x,d} \neq 0$ *for all* $x \in \Gamma$.

*P*14′   *For any* $z \in c_0$, *we have* $z \underset{LR}{\sim} z^{-1}$.

*P*15′   *For* $x, x' \in W$, $y, w \in c_0$, *we have the equality*

$$\sum_{y' \in c_0} h_{x,y',y} \otimes h_{w,x',y'} = \sum_{y' \in c_0} h_{x,w,y'} \otimes h_{y',x',y} \text{ in } \mathbb{Z}[v, v^{-1}] \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[v, v^{-1}].$$

$\widetilde{P}'$ *For* $x, y, z' \in W$, $z \in c_0$, *if* $\gamma_{x,y,z^{-1}} \neq 0$, $z' \underset{L}{\leftarrow} z$, *then there exists* $x' \in c_0$, *such that* $\pi_{a(z)}(h_{x',y,z'}) \neq 0$.

Proof. By above consequences and the methods in the proof of [11, 4.1], one can prove $P1' - P15'$. So we only give the proof of $\widetilde{P}'$.
Since $\gamma_{x,y,z^{-1}} \neq 0$, $z \in c_0$, by $P8'$, we have $y \underset{L}{\sim} z$. Since $z' \underset{L}{\leftarrow} z$, $z \in c_0$, we get $z' \underset{L}{\sim} z$ by $P9'$. So y, z, z' are in the same left cell in c_0. By Theorem 8.3(2), we assume $y = aw_Jb$, $z = a'w_Jb$, $z' = a''w_Jb$ for some $w_J \in M$, $b \in U_J$, $a, a', a'' \in B_J$. Then we have $\deg(T_yT_{z'^{-1}}) = \deg(T_{aw_Jb}T_{b^{-1}w_Ja''^{-1}}) = N$. Thus there exists $x' \in W$, such that $\deg f_{y,z'^{-1},x'^{-1}} = N$. By Lemma 2.3(3) and Theorem 8.3(1), we get $x' \in c_0$. We have $\deg f_{x',y,z'} = \deg f_{y,z'^{-1},x'^{-1}} = N$, so $\deg h_{x',y,z'} = N$, that is, $\pi_{a(z)}(h_{x',y,z'}) \neq 0$. ■

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