

A Generalization of Duflo’s Conjecture

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Abstract. We generalize Duflo’s conjecture to understand the branching laws of non-discrete series. We give a unified description on the geometric side about the restriction of an irreducible unitary representation π of $\mathrm{GL}_n(k)$, $k = \mathbb{R}$ or \mathbb{C} , to the mirabolic subgroup, where π is attached to a certain kind of coadjoint orbit.

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1. Introduction

One goal of the orbit method is to establish a correspondence between coadjoint orbits and irreducible unitary representations of Lie groups. One can benefit a lot from such a correspondence, for example, finding out irreducible unitary representations through coadjoint orbits; or predicting the decomposition of $\pi|_H$ by looking at the decomposition of $\mathrm{p}(\mathcal{O}_\pi)$ as H -orbit, where π is a unitary representation of the Lie group G and is attached to the coadjoint orbit \mathcal{O}_π , H is a closed subgroup of G , and p is the moment map from \mathcal{O}_π to \mathfrak{h}^* with \mathfrak{h}^* the dual of the Lie algebra of H ; etc. (see Kirillov [12], Introduction).

The orbit method is brought up by Kirillov, who attached an irreducible unitary representation to a coadjoint orbit for nilpotent groups in a perfect way. Later, Kostant (see Auslander-Kostant [3]) established the theory of quantization and built up the orbit method for solvable groups. Duflo [6] constructed all irreducible unitary representations of almost algebraic groups from certain coadjoint orbits, assuming one knew unitary dual of reductive groups. It suggested that to build up the orbit method for general Lie groups, one should concentrate on reductive groups. The quantization problem of coadjoint orbits of reductive groups can be reduced to quantizing nilpotent coadjoint orbits. It has not been solved completely, but there are some wonderful results (see Vogan [31], Chapter 10).

Through the orbit method, it’s possible to describe the branching laws in a geometric way. Heckman [11] and Guillemin-Sternberg [10] described it for compact groups, and Fujiwara [9] solved for exponential solvable groups. Later, Kobayashi [14],[15],[16],[17] obtained geometric conditions for ”admissibility” of the restriction of irreducible representations of reductive groups to their reductive subgroups. In spite by these results, Duflo (see Duflo-Vargas [8]) formulated a conjecture for the branching problem of discrete series of almost algebraic group (see also Liu [21]).

Let G be an (almost) algebraic group, let H be a closed algebraic subgroup of G . Let \mathfrak{g} (resp. \mathfrak{h}) be the Lie algebra of G (resp. of H). Let \mathfrak{g}^* (resp. \mathfrak{h}^*) be the dual of \mathfrak{g} (resp. of \mathfrak{h}). Let π be a discrete series representation of G . By Duflo [7], π is attached to a strongly regular G -coadjoint orbit \mathcal{O}_π . Consider the restriction of π to H , denoted by $\pi|_H$, Duflo's conjecture says that

- (i) $\pi|_H$ is H -admissible if and only if the moment map $p : \mathcal{O}_\pi \rightarrow \mathfrak{h}^*$ is weakly proper.
- (ii) If $\pi|_H$ is H -admissible, then each irreducible H -representation σ which appears in $\pi|_H$ is attached to a strongly regular H -coadjoint orbit Ω which is contained in $p(\mathcal{O}_\pi)$.
- (iii) If $\pi|_H$ is H -admissible, then the multiplicity of each such σ can be expressed geometrically in terms of the reduced space of Ω with respect to the moment map p .

Let us explain in detail. The notion “almost algebraic group” is explained in [7]. An element $f \in \mathfrak{g}^*$ is called strongly regular if f is regular (i.e. the coadjoint orbit containing f is of maximal dimension) and its “reductive factor”

$$\mathfrak{s}(f) := \{X \in \mathfrak{g}(f) : \text{ad}X \text{ is semisimple}\}$$

is of maximal dimension among the reductive factors of all the regular elements in \mathfrak{g}^* , where $\mathfrak{g}(f)$ denotes the centralizer of f in \mathfrak{g} .

Let \mathcal{O} be a G -coadjoint orbit in \mathfrak{g}^* . Then \mathcal{O} is equipped with the Kirillov-Kostant-Souriau symplectic form ω and becomes an H -Hamiltonian space. The corresponding moment map is the natural projection $p : \mathcal{O} \rightarrow \mathfrak{h}^*$.

In (i), the notion “ H -admissible” is due to Kobayashi (see [14]), which means that $\pi|_H$ is discretely decomposable (i.e. $\pi|_H$ is a direct sum of irreducible unitary representations of H) and all irreducible representations of H has only finite multiplicities in $\pi|_H$. The notion “weak properness” means that the preimage (for p) of each compact subset which is contained in $p(\mathcal{O}_\pi) \cap \Upsilon_{sr}$ is compact in \mathcal{O}_π , where Υ_{sr} denotes the set of strongly regular elements in \mathfrak{h}^* .

In (iii), the reduce space of Ω is equal to $p^{-1}(\Omega)/H$.

Duflo's conjecture has been proved for some cases including that Liu [21] proved it for the branching problem to the Iwasawa AN -subgroup of a semisimple Lie group, Kobayashi-Ørsted [20] proved it for branching problem of unitary representations attached to minimal nilpotent coadjoint orbits in $O(p, q)$, Kobayashi-Nasrin [18],[19] proved it for branching problem to the symmetric pairs, Paradan [25],[26] proved it for branching problem to the reductive subgroup of a real reductive group, Liu-Yu [23] generalized Duflo's conjecture to the restriction of tempered representations of $GL_n(k)$ (for $k = \mathbb{R}$ or \mathbb{C}) to its mirabolic subgroup and gave a geometric interpretation of Kirillov's conjecture, and Liu-Oshima-Yu [22] verified it for the restriction of irreducible unitary representations of $Spin(N, 1)$ to its parabolic subgroups.

This article generalizes Duflo's conjecture in the framework of Kirillov's conjecture in spirit by Liu-Yu [23]. We obtain a generalization for the irreducible unitary representations of $GL_n(k)$ (for $k = \mathbb{R}$ or \mathbb{C}) which are attached to some coadjoint orbits in section 5.

Let us say a few words about Kirillov's conjecture. The conjecture says that the restriction of any irreducible unitary representation of $GL_n(k)$ (for $k = \mathbb{R}, \mathbb{C}$ or Q_p)

to the mirabolic subgroup is also irreducible. It was proved by Bernstein for p -adic fields [5], Sahi [27] for tempered representations of $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$, and Sahi-Stein [29] for Speh representations of $GL_n(\mathbb{R})$, and Baruch [4] for the archimedean fields.

Here are the main results of this article. Let $k = \mathbb{R}$ or \mathbb{C} , let $G = GL_n(k)$, and let $P = P_n(k)$ be the mirabolic subgroup of G , consisting of the elements whose last row is $(0, \dots, 0, 1)$. Let \mathfrak{p} denote the Lie algebra of P and \mathfrak{p}^* denote the dual of \mathfrak{p} . Assume that π is an irreducible unitary representation of G and π is attached to the G -coadjoint orbit \mathcal{O}_π in section 5. It turns out that the image of the moment map $\mathfrak{p} : \mathcal{O}_\pi \rightarrow \mathfrak{p}^*$ contains finite P -coadjoint orbits, and there is a unique dense P -coadjoint orbit in $\mathfrak{p}(\mathcal{O}_\pi)$, denoted by Ω . The moment map \mathfrak{p} is proper over Ω . Moreover, $\pi|_P$ is attached to Ω in the sense of Duflo. It explores more about the geometry of Kirillov's conjecture.

The article is organized as follows. In Section 2, we present the classification of the $P_n(k)$ -coadjoint orbits and its proof, which comes from Liu-Yu [23]. In Section 3, we compute the moment map and obtain similar results as Liu-Yu [23] on the geometric side. In Section 4, we summarize some results of Kirillov's conjecture, from which we obtain the restrictions of all irreducible unitary representations of $GL_n(k)$ to $P_n(k)$. In Section 5, firstly, we summarize the orbit method for reductive groups, and obtain the correspondence between coadjoint orbits and irreducible unitary representations of $GL_n(k)$. Secondly, we show how to construct irreducible unitary representations from coadjoint orbits of $P_n(k)$ in the sense of Duflo. In Section 6, our generalization is presented and proved by comparing the results of the moment maps and the restrictions of irreducible unitary representations of $GL_n(k)$ to $P_n(k)$.

2. $P_n(k)$ -coadjoint orbit

This section is mainly adapted from Liu-Yu [23], section 3, which will be used in the rest of the article.

2.1. Coadjoint action, the dual map and the moment map

Let k be the field \mathbb{R} or \mathbb{C} , and let $n \in \mathbb{Z}_+$. Set $G_n(k) = GL_n(k)$ and

$$P_n(k) = \left\{ \begin{pmatrix} A & \alpha \\ 0 & 1 \end{pmatrix} : A \in GL_{n-1}(k), \alpha \in k^{n-1} \right\}$$

be the mirabolic subgroup of $G_n(k)$. $\mathfrak{g}_n(k)$ denotes the Lie algebra of $G_n(k)$ and

$$\mathfrak{p}_n(k) = \left\{ \begin{pmatrix} A & \alpha \\ 0 & 0 \end{pmatrix} : A \in \mathfrak{gl}_{n-1}(k), \alpha \in k^{n-1} \right\}$$

denotes the Lie algebra of $P_n(k)$. $\mathfrak{g}_n(k)^*$ and $\mathfrak{p}_n(k)^*$ denote the dual spaces of $\mathfrak{g}_n(k)$ and $\mathfrak{p}_n(k)$, respectively. $G_n(k)$ acts on $\mathfrak{g}_n(k)^*$ through

$$(g \cdot f)\xi = f(g^{-1} \cdot \xi), \forall g \in G_n(k), \forall f \in \mathfrak{g}_n(k)^*, \forall \xi \in \mathfrak{g}_n(k).$$

This is called the *coadjoint action*, and a $G_n(k)$ -orbit in $\mathfrak{g}_n(k)^*$ is called a *coadjoint orbit*. Similarly, one can define the coadjoint action of $P_n(k)$ on $\mathfrak{p}_n(k)^*$ and the corresponding coadjoint orbit.

By taking the trace, we can define a $G_n(k)$ -invariant nondegenerate bilinear form on $\mathfrak{g}_n(k) \times \mathfrak{g}_n(k)$,

$$(X, Y) \mapsto \text{tr}(XY), X, Y \in \mathfrak{g}_n(k).$$

We have a $G_n(k)$ -module isomorphism

$$\text{pr} : \mathfrak{g}_n(k) \rightarrow \mathfrak{g}_n(k)^*, X \mapsto (Y \mapsto \text{tr}(XY)), \forall Y \in \mathfrak{g}_n(k), \forall X \in \mathfrak{g}_n(k).$$

Let $\text{pr}' : \mathfrak{g}_n(k) \rightarrow \mathfrak{p}_n(k)^*$, $X \mapsto (Y \mapsto \text{tr}(XY)), \forall Y \in \mathfrak{p}_n(k), \forall X \in \mathfrak{g}_n(k)$, then

$$\ker \text{pr}' = \left\{ \begin{pmatrix} 0 & \alpha \\ 0 & t \end{pmatrix} : \alpha \in k^{n-1}, t \in k \right\}.$$

Let $\bar{\mathfrak{p}}_n(k) = \left\{ \begin{pmatrix} A & 0 \\ \alpha & 0 \end{pmatrix} \mid A \in \mathfrak{g}_{n-1}(k), \alpha \in k^{n-1} \right\}$, then $\mathfrak{g}_n(k) = \ker \text{pr}' \oplus \bar{\mathfrak{p}}_n(k)$, and $\text{pr}'|_{\bar{\mathfrak{p}}_n(k)}$ is an isomorphism. We use pr_n (resp. pr'_n) to denote pr (resp. pr') if there is ambiguity.

2.2. Classification of $P_n(k)$ -coadjoint orbit

Let
$$L_n(k) = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \mid A \in G_{n-1}(k) \right\}$$

and
$$N_n(k) = \left\{ \begin{pmatrix} I_{n-1} & \alpha \\ 0 & 1 \end{pmatrix} \mid \alpha \in k^{n-1} \right\}.$$

Let $\mathfrak{l}_n(k)$ and $\mathfrak{n}_n(k)$ denote the Lie algebra of $L_n(k)$ and $N_n(k)$, respectively. Then $L_n(k)$ is the Levi subgroup of $P_n(k)$, $N_n(k)$ is the unipotent radical of $P_n(k)$, and $P_n(k) = L_n(k) \ltimes N_n(k)$. We have following key theorem.

Theorem 2.1. *Every element of $\mathfrak{p}_n(k)^*$ is $P_n(k)$ -conjugated to an element of the form $\alpha = \text{pr}'(\xi)$, where $\xi = \begin{pmatrix} A & \\ & J_j \end{pmatrix}$, where $A \in \mathfrak{g}_{n-j}(k)$, and*

$$J_j = \begin{pmatrix} 0 & & & & \\ 1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \in \mathfrak{g}_j(k).$$

The stabilizer $\text{Stab}_{P_n(k)}(\alpha)$ is isomorphic to $\text{Stab}_{G_{n-j}(k)}(A) \ltimes N_{n-j+1}(k)$.

Proof. See Liu-Yu, [23], 3.2. ■

Definition 2.2. In Theorem 2.1, define the *depth* of $\text{pr}'(\xi)$ in $\mathfrak{p}_n(k)^*$ as j .

Corollary 2.3. *The elements with depth n form a unique dense open $P_n(k)$ -orbit in $\mathfrak{p}_n(k)^*$ and they are all strongly regular elements in $\mathfrak{p}_n(k)^*$.*

Proof. Applying Theorem 2.1, we have $\alpha \in \mathfrak{p}_n(k)^*$ has depth n if and only if $\text{Stab}_{P_n(k)}(\alpha)$ has minimal dimension, if and only if $\text{Stab}_{P_n(k)}(\alpha) = 1$. So we get the statement immediately. ■

Example 2.4. Set

$$B = \begin{pmatrix} a_1 & & & & \\ & \ddots & & & \\ & & a_{n-1} & & \\ b_1 & \cdots & b_{n-1} & 0 & \end{pmatrix} \in \mathfrak{g}_n(k),$$

where $a_i \neq a_j$ when $1 \leq i \neq j \leq n - 1$; $b_i \neq 0$, when $1 \leq i \leq n - 1$, and where $f = \text{pr}'(B) \in \mathfrak{p}_n(k)^*$. Then f has depth n .

Proof. It is sufficient to check that $\text{Stab}_{P_n(k)}(f) = \{1\}$. See Liu-Yu [23], 3.2. ■

3. Geometry of the moment map $\mathfrak{p} : \mathcal{O}_f \rightarrow \mathfrak{p}^*$

3.1. Moment map in the $\text{GL}_n(\mathbb{C})$ case

Let $n \in \mathbb{Z}_+$, let $G = G_n(\mathbb{C})$ and let $P = P_n(\mathbb{C})$ be the mirabolic subgroup of G . \mathfrak{g} (resp. \mathfrak{p}) denotes the Lie algebra of G (resp. of P). Let \mathfrak{g}^* (resp. \mathfrak{p}^*) denote the dual space of \mathfrak{g} (resp. of \mathfrak{p}). By the Lie theory, we have

$$n = \min_{x \in \mathfrak{g}^*} \dim(\text{Stab}_G(x)).$$

For any element $f \in \mathfrak{g}^*$, we call f a regular element in \mathfrak{g}^* , or say that f is regular, if $\dim(\text{Stab}_G(f)) = n$. And we call a G -coadjoint orbit \mathcal{O}_f a regular orbit if f is regular.

We will calculate the moment map $\mathfrak{p} : \mathcal{O}_f \rightarrow \mathfrak{p}^*$ for any $f \in \mathfrak{g}^*$. We find that the results about a regular element $f \in \mathfrak{g}^*$ is similar to the results about a semisimple regular element, for example, $\mathfrak{p}(\mathcal{O}_f)$ contains the unique strongly regular orbit of \mathfrak{p}^* (see Liu-Yu [23]). Using the results about the regular elements, we can settle the cases of the singular elements in \mathfrak{g}^* .

Furthermore, using the results of the case of $\mathfrak{g}_n(\mathbb{C})$, one can settle the case of $\mathfrak{g}_n(\mathbb{R})$ immediately.

3.1.1. Regular orbits

Write $J_s(a) = \begin{pmatrix} a & & & & \\ 1 & a & & & \\ & \ddots & \ddots & & \\ & & & 1 & a \end{pmatrix} \in \mathfrak{g}_s(\mathbb{C})$, $s \in \mathbb{Z}$ and $s \geq 2, a \in \mathbb{C}$,

and $J_1(a) = a_{1 \times 1}, a \in \mathbb{C}$. Set $J_s = J_s(0), \forall s \in \mathbb{Z}_+$. Let $a_i \in \mathbb{C}, s_i \in \mathbb{Z}_+, \forall 1 \leq i \leq m$, such that $a_i \neq a_j$ when $1 \leq i \neq j \leq m$, and $\sum_{i=1}^m s_i = n$. Set

$$\xi = \text{diag}(J_{s_1}(a_1), \dots, J_{s_m}(a_m)) \in \mathfrak{g},$$

and set $f = \text{pr}(\xi) \in \mathfrak{g}^*$. It is clear that every regular G -coadjoint orbit in \mathfrak{g}^* is of the form \mathcal{O}_f . Firstly, we describe all P -orbits in \mathcal{O}_f . For any $s \in \mathbb{Z}_+$, let

$$L_s = \text{Stab}_{G_s(\mathbb{C})}(J_s) = \left\{ \begin{pmatrix} x_1 & & & & \\ x_2 & x_1 & & & \\ \vdots & \ddots & \ddots & & \\ x_s & \cdots & x_2 & x_1 \end{pmatrix} \mid x_i \in \mathbb{C}, 1 \leq i \leq s, x_1 \neq 0 \right\},$$

Set $L = \text{Stab}_G(f) = \{\text{diag}(X_{s_1}, \dots, X_{s_m}) \mid X_{s_i} \in L_{s_i}, 1 \leq i \leq m\}$.

We have $\mathcal{O}_f \cong G/L$ and $P \setminus \mathcal{O}_f \cong P \setminus G/L$. Define the right action of G on \mathbb{C}^n by

$$v \cdot g = (x_1, \dots, x_n)A,$$

where $v = (x_1, \dots, x_n)^t \in \mathbb{C}^n$ and $g = A \in G_n(\mathbb{C})$. Set $v_0 = (0, \dots, 0, 1)^t$, we get $\text{Stab}_G(v_0) = P$, and $\mathbb{C}^n - \{0\} \cong P \setminus G$. To find the representatives of $P \setminus \mathcal{O}_f$, we need to find the representatives of $(\mathbb{C}^n - \{0\})/L$.

Proposition 3.1. *Set $I_0 = \{1, 2, \dots, n\}$, $K_0 = \{1, \dots, m\}$ and $s_0 = 0$. For any $\emptyset \neq K \subseteq K_0$ and $r_k \in \{1, 2, \dots, s_k\}$ for $k \in K$, define*

$$I = \left\{ \sum_{j=0}^{k-1} s_j + r_k \mid k \in K \right\} \subseteq I_0$$

attached to K and $\{r_k, k \in K\}$. Define $v_I = (x_1, \dots, x_n)$ by $x_i = 1$ when $i \in I$ and $x_i = 0$ when $i \notin I$. Let I_a be the set of all I 's constructed as above.

Then $\{v_I \mid I \in I_a\}$ form all different representatives of $(\mathbb{C}^n - \{0\})/L$.

Proof. The L -orbit of v_I is the set

$$\{(y_1, \dots, y_n)^t \mid y_i \in \mathbb{C}, 1 \leq i \leq n; y_i \neq 0 \text{ if } i \in I; y_i = 0 \text{ if } i \notin I\},$$

$$J = \bigsqcup_{k \in K} \left\{ \sum_{j=0}^{k-1} s_j + 1, \dots, \sum_{j=0}^{k-1} s_j + r_k \right\}.$$

It follows that $\{v_I \mid I \in I_a\}$ are all different representatives of $(\mathbb{C}^n - \{0\})/L$. \blacksquare

Let $g_I \in G$ be the element corresponding to the representative v_I under the isomorphism $\mathbb{C}^n - \{0\} \cong P \setminus G$, then

$$G = \bigsqcup_{I \in I_a} P g_I L.$$

So there are $\#I_a = \prod_{1 \leq i \leq m} (s_i + 1) - 1$ P -orbits in \mathcal{O}_f , $\{P \cdot (g_I \cdot f) \mid I \in I_a\}$.

Set $S = \{s_1, s_1 + s_2, \dots, n\}$, then $P g_S L$ is the unique dense open subset of G among $\{P g_I L, I \in I_a\}$, since $v_S \cdot L$ is the unique dense L -orbit in $\mathbb{C}^n - \{0\}$. As a consequence, $P \cdot (g_S \cdot f)$ is the unique dense open P -orbit in \mathcal{O}_f .

By definition, we can choose $\{g_I \mid I \in I_a\}$ as follows.

When $n \in I$, $g_I = \begin{pmatrix} I_{n-1} & 0 \\ \beta & 1 \end{pmatrix}$, $\beta = (x_1, \dots, x_{n-1})$ is defined by $x_i = 1$ when $i \in I - \{n\}$ and $x_i = 0$ when $i \notin I - \{n\}$.

When $n \notin I$, let $k = \max\{i : i \in I\}$, $g_I = \begin{pmatrix} I_{k-1} & & \\ & I_{n-k} & \\ \beta & & 1 \end{pmatrix}$, $\beta = (x_1, \dots, x_{k-1})$ is defined by $x_i = 1$ when $i \in I - \{k\}$ and $x_i = 0$ when $i \notin I - \{k\}$.

We go to calculate the image of each P -orbit of $g_I \cdot f$ under the moment map $p : \mathcal{O}_f \rightarrow \mathfrak{p}^*$ and find that $p(\mathcal{O}_f)$ contains the unique dense open P -orbit in \mathfrak{p}^* , more precisely, $p(g_S \cdot f)$ is a strongly regular element in \mathfrak{p}^* .

Lemma 3.2. *We have $\text{Stab}_P(g_S \cdot f) = 1$, so the dimension of P -orbit of $g_S \cdot f$ equals $\dim P = n^2 - n$.*

Proof. It is obvious that $\text{Stab}_P(g_S \cdot f) = P \cap g_S \text{Stab}_G(f) g_S^{-1}$

and $P \cap g_S \text{Stab}_G(f) g_S^{-1} = P \cap g_S L g_S^{-1} = g_S (g_S^{-1} P g_S \cap L) g_S^{-1}$.

Since $g_S^{-1} P g_S$ is the stabilizer of $v_0 \cdot g_S = v_S$, we have $g_S^{-1} P g_S \cap L = \text{Stab}_L(v_S)$.

By direct calculation, $\text{Stab}_L(v_S) = 1$, so $\text{Stab}_P(g_S \cdot f) = 1$. ■

For convenience, let $[U_1, \dots, U_k]$ denote an $n \times n$ matrix with k blocks, where the i -th block is an $n \times n_i$ ($1 \leq n_i \leq n$) submatrix U_i for $1 \leq i \leq k$, and $\sum_{i=1}^k n_i = n$.

Lemma 3.3. *If $x \in \mathfrak{p}^*$ is not strongly regular, then for any element $z \in \mathfrak{p}'^{-1}(x)$, the P -orbit of z has dimension $< n^2 - n$. Here $\mathfrak{p}' : \mathfrak{g}^* \rightarrow \mathfrak{p}^*$ is the natural projection.*

Proof. Take any element $x = \text{pr}'([X, 0_{n \times 1}]) \in \mathfrak{p}^*$ which is not strongly regular, by the definition, $\mathfrak{p}'^{-1}(x) = \{\text{pr}([X, Y]), \tilde{Y} \in \mathbb{C}^n\}$. Fix an element $z \in \mathfrak{p}'^{-1}(x)$, $z = \text{pr}([X, Y])$ for some $Y = (y_1, \dots, y_n)^t \in \mathbb{C}^n$.

Since the P -orbit of z has dimension

$$\dim P - \dim(\text{Stab}_P(z)) = n^2 - n - \dim(\text{Stab}_P(z)),$$

we only need to prove $\dim(\text{Stab}_P(z)) \geq 1$.

Since $\{\text{pr}([0, \tilde{Y}]), \tilde{Y} \in \mathbb{C}^n\}$ is a P -stable subspace, $\text{Stab}_P(z) \subseteq \text{Stab}_P(x)$.

Using the result of the classification of P -coadjoint orbits in \mathfrak{p}^* , we know that x is P -conjugated to an element of the form $\alpha = \text{pr}' \begin{pmatrix} A & \\ & J_k \end{pmatrix}$, where $A \in \mathfrak{g}_{n-k}(\mathbb{C})$,

$$\text{Stab}_P(\alpha) \cong \text{Stab}_{G_{n-k}(\mathbb{C})}(A) \times N_{n-k+1}(\mathbb{C})$$

with dimension $\geq 2(n - k)$ and $k \leq n - 1$ as x is not strongly regular. It doesn't matter to replace x by the elements in its P -orbit and we assume that $x = \alpha$. By direct calculation, we get

$$\begin{aligned} \text{Stab}_P(x) = & \left\{ \begin{pmatrix} C & \\ & I_k \end{pmatrix} \mid C \in G_{n-k}(\mathbb{C}), CAC^{-1} = A \right\} \times \left\{ \begin{pmatrix} I_{n-k} & N \\ & I_k \end{pmatrix} \mid \right. \\ & \left. N = [N_1, \dots, N_k], N_i \in \mathbb{C}^{n-k}, 1 \leq i \leq k, N_i = A^{i-1} N_1, 2 \leq i \leq k \right\}. \end{aligned}$$

Let $t = \begin{pmatrix} C & CN \\ & I_k \end{pmatrix} \in \text{Stab}_P(x)$, then $t \in \text{Stab}_P(z)$ if and only if

$$\begin{pmatrix} C & CN \\ & I_k \end{pmatrix} \left(\begin{pmatrix} A & \\ & J_k \end{pmatrix} + [0, Y] \right) \begin{pmatrix} C^{-1} & -N \\ & I_k \end{pmatrix} = \begin{pmatrix} A & \\ & J_k \end{pmatrix} + [0, Y],$$

So

$$t \in \text{Stab}_P(z) \Leftrightarrow -CA^k N_1 + C \begin{pmatrix} y_1 \\ \vdots \\ y_{n-k} \end{pmatrix} + CN \begin{pmatrix} y_{n-k+1} \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-k} \end{pmatrix}.$$

Since $1 \in \text{Stab}_P(z)$, we obtain a nonempty subvariety $\text{Stab}_P(z)$ of $\text{Stab}_P(x)$ given by $n - k$ algebraic equations.

Therefore $\dim(\text{Stab}_P(z)) \geq \dim(\text{Stab}_P(x)) - (n - k)$.

Since $\dim(\text{Stab}_P(x)) \geq 2(n - k)$ and $n - k \geq 1$, we get $\dim(\text{Stab}_P(z)) \geq 1$. ■

Proposition 3.4. $p(g_S \cdot f)$ is a strongly regular element in \mathfrak{p}^* .

Proof. By Lemma 3.2, the P -orbit of $g_S \cdot f$ has dimension $n^2 - n$. Applying Lemma 3.3, we see that $p(g_S \cdot f)$ is strongly regular. ■

For convenience, let $A(I_1, I_2)$ denote the submatrix of the matrix A with rows indexed by I_1 and columns indexed by I_2 , and let $a(I_1)$ denote the subvector of the vector a with index I_1 .

With the result above, we can calculate the P -orbit of $p(g_I \cdot f)$ for any $I \in I_a$. We find out that $\{p(g_I \cdot f) \mid I \in I_a\}$ represent all different P -orbits in $p(\mathcal{O}_f)$. More precisely, we have the following result.

Proposition 3.5. For any $\emptyset \neq K \subseteq K_0$ and $r_k \in \{1, 2, \dots, s_k\}$ for $k \in K$, define $I = \{\sum_{j=0}^{k-1} s_j + r_k \mid k \in K\}$ and g_I as above. Then $p(g_I \cdot f)$ has depth $d = \sum_{k \in K} r_k$. Moreover, if we set

$$t_k = \begin{cases} r_k, & k \in K, \\ 0, & k \in K_0 - K, \end{cases}$$

then $p(g_I \cdot f)$ is P -conjugated to $\text{pr}'(\eta)$, where

$$\eta = \text{diag}(J_{s_1-t_1}(a_1), \dots, J_{s_m-t_m}(a_m), J_d).$$

Proof. Firstly, we change $p(g_I \cdot f) = \text{pr}'(g_I \cdot \xi)$ to a good form by a series of explicit P -conjugations, then we use the result of Proposition 3.4 to get the statement. Set $A_I = g_I \cdot \xi$.

In the case of $n \in I$, by direct calculation, we have

$$A_I = \text{diag}(J_{s_1}(a_1), \dots, J_{s_m}(a_m)) + \begin{pmatrix} 0_{(n-1) \times n} \\ \alpha \end{pmatrix},$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$, $\alpha_k = (\alpha_{k,1}, \dots, \alpha_{k,s_k})$ is defined by

$$\alpha_k = \begin{cases} (0, \dots, 0), & k \notin K - \{n\}, \\ (a_k - a_m, \underbrace{0, \dots, 0}_{s_k-1}), & k \in K - \{n\}, r_k = 1 \\ (\underbrace{0, \dots, 0}_{r_k-2}, 1, a_k - a_m, \underbrace{0, \dots, 0}_{s_k-r_k}), & k \in K - \{n\}, r_k > 1. \end{cases}$$

We claim that $\text{pr}'(A_I)$ is P -conjugated to $\text{pr}'(\zeta)$, where

$$\zeta = \begin{pmatrix} J_{t_1}(a_1) & & & & \\ & J_{s_1-t_1}(a_1) & & & \\ & & \ddots & & \\ & & & J_{t_m}(a_m) & \\ & & & & J_{s_m-t_m}(a_m) \end{pmatrix} + \begin{pmatrix} 0_{(n-1) \times n} \\ \alpha \end{pmatrix}.$$

Step 2. Let

$$P_2 = \begin{pmatrix} I_{s_1+\dots+s_{k-1}} & 0 & 0 & 0 & 0 \\ 0 & I_{r_k} & 0 & 0 & 0 \\ M_1 & 0 & I_{s_k-r_k} & M_2 & 0 \\ 0 & 0 & 0 & I_{s_{k+1}+\dots+s_{m-1}} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

M_1 is an $(s_k-r_k) \times (s_1+\dots+s_{k-1})$ matrix, and M_2 is an $(s_k-r_k) \times (s_{k+1}+\dots+s_{m-1})$ matrix, then

$$P_2(P_1\zeta_{k-1}P_1^{-1})P_2^{-1} = \begin{pmatrix} B_1 & 0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 & 0 \\ N_2v_1 + (M_1B_1 - C_2M_1) & 0 & C_2 & N_2v_2 + (M_2B_2 - C_2M_2) & a_mN_2 - C_2N_2 \\ 0 & 0 & 0 & B_2 & 0 \\ v_1 & w_1 & 0 & v_2 & a_m \end{pmatrix}.$$

By solving the linear equations, one see that there exist M_1 and M_2 , such that

$$N_2v_1 + (M_1B_1 - C_2M_1) = 0, \text{ and } N_2v_2 + (M_2B_2 - C_2M_2) = 0.$$

Actually, we can write $B_1 = D' + N'$ and $C_2 = D'' + N''$ as Jordan decomposition, where D', D'' are semisimple matrices and N', N'' are nilpotent matrices. It is easy to see that the linear map $M_1 \mapsto M_1N' - N''M_1$ is nilpotent, that the linear map $M_1 \mapsto M_1D' - D''M_1$ is semisimple, and that they commute with each other.

Moreover, the linear map $M_1 \mapsto M_1D' - D''M_1$ is nondegenerate since B_1 and C_2 have no common eigenvalues. Therefore, we can easily see that the equation $N_2v_1 + (M_1B_1 - C_2M_1) = 0$ has a (unique) solution M_1 . Similarly we can get M_2 . Using such P_2 defined as above, we have $\text{pr}'(P_2(P_1\zeta_{k-1}P_1^{-1})P_2^{-1}) = \text{pr}'(\zeta_k)$. By induction, one can finish the proof of the claim.

Let ζ' be the submatrix of ζ with the rows and columns indexed by

$$\bigsqcup_{h \in K} \left\{ \sum_{j=0}^{h-1} s_j + 1, \dots, \sum_{j=0}^{h-1} s_j + r_h \right\}.$$

Applying Proposition 3.4, we see that $\text{pr}'_d(\zeta')$ is $P_d(\mathbb{C})$ -conjugated to $\text{pr}'_d(J_d)$, where $\text{pr}'_d : \mathfrak{g}_d(\mathbb{C}) \rightarrow \mathfrak{p}_d(\mathbb{C})^*$. So $\text{pr}'(\zeta)$ is P -conjugated to the element $\text{pr}'(\eta)$ in the statement, and so is $\text{p}(g_I \cdot \xi)$.

In the case of $n \notin I$, $m \in K$, one can prove the statement similarly. We have

$$A_I = \begin{pmatrix} B_1 & 0 & 0 \\ C & B_2 & v_1 \\ v_2 & 0 & a_m \end{pmatrix},$$

where $B_1 = \text{diag}(J_{s_1}(a_1), \dots, J_{s_{m-1}}(a_{m-1}), J_{r_m-1}(a_m)), B_2 = J_{s_m-r_m}(a_m),$

$$C = \begin{pmatrix} -v_I(1, \dots, n - (s_m - r_m) - 1) \\ 0_{(s_m-r_m) \times (n-(s_m-r_m)-1)} \end{pmatrix},$$

$$v_1 = (1, \underbrace{0, \dots, 0}_{s_m-r_{m-1}})^t, v_2 = \begin{cases} (\alpha(1, \dots, n - s_m), \underbrace{0, \dots, 0}_{r_m-2}, 1), & r_m \geq 2 \\ (\alpha(1, \dots, n - s_m), & r_m = 1. \end{cases}$$

We can use the method of step 2 to show that $p(A_I)$ is P -conjugated to $\text{pr}(\zeta')$, where

$$\zeta' = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & B_2 & v_1 \\ v_2 & 0 & a_m \end{pmatrix}.$$

Then the submatrix of ζ' with rows and columns indexed by $\{1, \dots, n\} - \{n - (s_k - r_k), \dots, n - 1\}$

is
$$\begin{pmatrix} B_1 & 0 \\ v_2 & a_m \end{pmatrix},$$

which is of the form in the case of $n \in I$. Applying the result of the case of $n \in I$, we can get the statement in the case of $n \notin I, m \in K$.

In the case of $n \notin I, m \notin K$, assume that $m_0 = \max\{k \in K\}$. By direct calculation we derive that the submatrix of A_I with rows and columns indexed by the set $\{1, 2, \dots, s_1 + \dots + s_{m_0} - 1\} \cup \{n\}$ is of the same form as in the above two cases. Applying the results of the above two cases, we can get the statement in the case of $n \notin I, m \notin K$. This completes the proof of the statement. ■

3.1.2. General orbits

We go to calculate the moment map $p : \mathcal{O}_f \rightarrow \mathfrak{p}^*$ for any $f \in \mathfrak{g}^*$. Write

$$J_k^l(a) = \text{diag}(\underbrace{J_k(a), \dots, J_k(a)}_{l \text{ blocks}}),$$

for any $a \in \mathbb{C}, k, l \in \mathbb{Z}_+$. Let $m \in \mathbb{Z}_+, m \leq n$. Let $a_1, \dots, a_m \in \mathbb{C}, a_i \neq a_j$ when $1 \leq i \neq j \leq m$. Set $J_{k,a_i}^l = J_k^l(a_i)$ for $1 \leq i \leq m, k, l \in \mathbb{Z}_+$. Let

$$\xi = \text{diag}(J_{k_{11},a_1}^{l_{11}}, J_{k_{12},a_1}^{l_{12}}, \dots, J_{k_{1r_1},a_1}^{l_{1r_1}}, \dots, J_{k_{m1},a_m}^{l_{m1}}, J_{k_{m2},a_m}^{l_{m2}}, \dots, J_{k_{mr_m},a_m}^{l_{mr_m}}),$$

where $r_j \in \mathbb{Z}_+, 1 \leq j \leq m$; and $l_{ji}, k_{ji} \in \mathbb{Z}_+, \forall 1 \leq j \leq m, 1 \leq i \leq r_j; k_{ji_1} < k_{ji_2}$ when $1 \leq j \leq m, 1 \leq i_1 < i_2 \leq r_j$; and

$$\sum_{1 \leq j \leq m, 1 \leq i \leq r_j} k_{ji} \cdot l_{ji} = n.$$

So ξ has m different eigenvalues, denoted by $a_j, 1 \leq j \leq m$, and has r_j different kinds of Jordan blocks for the eigenvalue a_j , with size k_{ji} and l_{ji} pieces of such size for $1 \leq i \leq r_j$.

Set $f = \text{pr}(\xi)$. It is clear that every orbit in \mathfrak{g}^* is of the form \mathcal{O}_f . Now we fix a ξ and set

$$\xi_j = \text{diag}(J_{k_{j1},a_j}^{l_{j1}}, J_{k_{j2},a_j}^{l_{j2}}, \dots, J_{k_{jr_j},a_j}^{l_{jr_j}}), \tag{1}$$

and $n_j = \sum_{r=1}^{r_j} k_{jr} \cdot l_{jr}$ for $1 \leq j \leq m$. For convenience, set $n_0 = 0, k_{j0} = 0, l_{j0} = 0$ for $1 \leq j \leq m$. Let $M_{s \times s}(\mathbb{C})$ be the set of all $s \times s$ matrices over \mathbb{C} . Write

$$K(x_1, \dots, x_s) = \begin{pmatrix} x_1 & & & \\ x_2 & x_1 & & \\ \vdots & \ddots & \ddots & \\ x_s & \cdots & x_2 & x_1 \end{pmatrix}, x_i \in \mathbb{C}, 1 \leq i \leq s,$$

then $\{X \in M_{s \times s}(\mathbb{C}) \mid XJ_s = J_sX\} = \{K(x_1, \dots, x_s) \mid x_i \in \mathbb{C}, 1 \leq i \leq s\}$, and we have the following Proposition.

Proposition 3.6. For any $a \in \mathbb{C}$, $k, l \in \mathbb{Z}_+$,

$$L_{k,l} = \text{Stab}_{\text{GL}_{k \cdot l}(\mathbb{C})}(J_k^l(a)) = \left\{ \left(\begin{array}{ccc} K(x_{11,1}, \dots, x_{11,k}) & \cdots & K(x_{1l,1}, \dots, x_{1l,k}) \\ \vdots & \ddots & \vdots \\ K(x_{l1,1}, \dots, x_{l1,k}) & \cdots & K(x_{ll,1}, \dots, x_{ll,k}) \end{array} \right) \left| \begin{array}{l} x_{ij,h} \in \mathbb{C}, \forall 1 \leq i, j \leq l, \\ 1 \leq h \leq k, \\ \det(x_{ij,1})_{1 \leq i, j \leq l} \neq 0 \end{array} \right. \right\}.$$

Proof. By direct calculation, we see that the determinate of the element on the right-hand side equals $(\det(x_{ij,1})_{1 \leq i, j \leq l})^k$ and get the statement immediately. ■

For any $k_1, k_2 \in \mathbb{Z}_+$ such that $k_1 \leq k_2$, and $x_1, \dots, x_{k_1} \in \mathbb{C}$, define $H_1(x_1, \dots, x_{k_1})$ as the $k_1 \times k_2$ matrix

$$[K(x_1, \dots, x_{k_1}), 0_{k_1 \times (k_2 - k_1)}],$$

and define $H_2(x_1, \dots, x_{k_1})$ as the $k_2 \times k_1$ matrix

$$\begin{pmatrix} 0_{(k_2 - k_1) \times k_1} \\ K(x_1, \dots, x_{k_1}) \end{pmatrix}.$$

Proposition 3.7. For any $a \in \mathbb{C}$, $k_1, k_2, l_1, l_2 \in \mathbb{Z}_+$, $k_1 \leq k_2$,

$$\begin{aligned} M_{k_1, k_2}^{l_1, l_2} &= \{X \in M_{k_1 l_1 \times k_2 l_2}(\mathbb{C}) \mid X(J_{k_2}^{l_2}(a)) = (J_{k_1}^{l_1}(a))X\} \\ &= \left\{ \left(\begin{array}{ccc} H_1(x_{11,1}, \dots, x_{11, k_1}) & \cdots & H_1(x_{1l_2,1}, \dots, x_{1l_2, k_1}) \\ \vdots & \ddots & \vdots \\ H_1(x_{l_1,1,1}, \dots, x_{l_1,1, k_1}) & \cdots & H_1(x_{l_1, l_2, 1}, \dots, x_{l_1, l_2, k_1}) \end{array} \right) \right. \\ &\quad \left. \mid x_{ij,h} \in \mathbb{C}, \forall 1 \leq i \leq l_1, 1 \leq j \leq l_2, 1 \leq h \leq k_1 \right\}, \end{aligned}$$

$$\begin{aligned} N_{k_1, k_2}^{l_1, l_2} &= \{X \in M_{k_2 l_2 \times k_1 l_1}(\mathbb{C}) \mid X(J_{k_1}^{l_1}(a)) = (J_{k_2}^{l_2}(a))X\} \\ &= \left\{ \left(\begin{array}{ccc} H_2(x_{11,1}, \dots, x_{11, k_1}) & \cdots & H_2(x_{1l_1,1}, \dots, x_{1l_1, k_1}) \\ \vdots & \ddots & \vdots \\ H_2(x_{l_2,1,1}, \dots, x_{l_2,1, k_1}) & \cdots & H_2(x_{l_2, l_1, 1}, \dots, x_{l_2, l_1, k_1}) \end{array} \right) \right. \\ &\quad \left. \mid x_{ij,h} \in \mathbb{C}, \forall 1 \leq i \leq l_2, 1 \leq j \leq l_1, 1 \leq h \leq k_1 \right\}. \end{aligned}$$

Proof. This is easy to check out. ■

For convenience, we fix j and set $r = r_j$, $k_u = k_{ju}$, $l_u = l_{ju}$, for $1 \leq u \leq r_j$, in the Proposition 3.8 and its proof.

Proposition 3.8. The stabilizer is

$$L_j = \text{Stab}_{\text{GL}_{n_j}(\mathbb{C})}(\xi_j) = \left\{ \left(\begin{array}{cccc} X_{11} & X_{12} & \cdots & X_{1r} \\ X_{21} & X_{22} & \cdots & X_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ X_{r1} & X_{r2} & \cdots & X_{rr} \end{array} \right) \left| \begin{array}{l} X_{uu} \in L_{k_u, l_u}, \forall 1 \leq u \leq r, \\ X_{st} \in M_{k_s, k_t}^{l_s, l_t}, \forall 1 \leq s < t \leq r, \\ X_{st} \in N_{k_s, k_t}^{l_s, l_t}, \forall 1 \leq t < s \leq r \end{array} \right. \right\}.$$

Proof. It is clear that

$$\text{Stab}_{\text{GL}_{n_j}(\mathbb{C})}(\xi_j) \subseteq \left\{ \left(\begin{array}{cccc} X_{11} & X_{12} & \cdots & X_{1r} \\ X_{21} & X_{22} & \cdots & X_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ X_{r1} & X_{r2} & \cdots & X_{rr} \end{array} \right) \left| \begin{array}{l} X_{st} \in M_{k_s, k_t}^{l_s, l_t}, \forall 1 \leq s \leq t \leq r, \\ X_{st} \in N_{k_s, k_t}^{l_s, l_t}, \forall 1 \leq t < s \leq r \end{array} \right. \right\},$$

so we only need to prove the element on the right-hand side is invertible if and only if X_{uu} is invertible for all $1 \leq u \leq r$. It follows from the fact that the determinant of the element on the right hand side is equal to $\prod_{1 \leq u \leq r} \det X_{uu}$. ■

It is clear that

$$\text{Stab}_G(f) = \text{Stab}_G(\xi) = \{\text{diag}(X_1, \dots, X_m) \mid X_i \in L_i, 1 \leq i \leq m\}.$$

We have $\mathcal{O}_f \cong G/L$ and $P \setminus \mathcal{O}_f \cong P \setminus G/L$. Define the right action of G on \mathbb{C}^n by

$$v \cdot g = (x_1, \dots, x_n)A,$$

where $v = (x_1, \dots, x_n)^t \in \mathbb{C}^n$ and $g = A \in G_n(\mathbb{C})$. Set $v_0 = (0, \dots, 0, 1)^t$, we get $\text{Stab}_G(v_0) = P$, and $\mathbb{C}^n - \{0\} \cong P \setminus G$.

To calculate $P \setminus G/L$, we need to calculate $(\mathbb{C}^n - \{0\})/L$. Actually, we only need to calculate $(\mathbb{C}^{n_j} - \{0\})/L_j$. Once we obtain all the different representatives of $(\mathbb{C}^{n_j} - \{0\})/L_j$, denoted by V_j , we can get all the different representatives of $(\mathbb{C}^n - \{0\})/L$ as follows,

$$\{v = (v_1, \dots, v_m) \neq 0 \mid v_i \in \mathbb{C}^{n_i}, v_i \in V_i \text{ or } v_i = 0, 1 \leq i \leq m\}.$$

We fix j and adopt the same notation as Proposition 3.8. We have following proposition about $(\mathbb{C}^{n_j} - \{0\})/L_j$, see Baruch [4], Chapter 5.

Proposition 3.9. Write $R_0 = \{1, \dots, r\}$. For any $\emptyset \neq R \subset R_0$ and $1 \leq x_i \leq k_i$ for $i \in R$, such that

- (1) $x_i < x_{i'}$ for any $i, i' \in R$ such that $i < i'$,
- (2) $k_i - x_i < k_{i'} - x_{i'}$ for any $i, i' \in R$ such that $i < i'$,

we define the set $I = \left\{ \sum_{u=0}^{i-1} k_u l_u + k_i(l_i - 1) + x_i, i \in R \right\} \subseteq \{1, 2, \dots, n_j\}$, and

$v_I = (y_1, y_2, \dots, y_{n_j})$, $y_i = 1$ when $i \in I$ and $y_i = 0$ when $i \notin I$. Let I_a be the set of all I 's constructed as above. Then $\{v_I \mid I \in I_a\}$ represent all different L_j -orbits in $\mathbb{C}^{n_j} - \{0\}$.

Remark 3.10. Let $R = \{r\}$ and $x_r = k_r$, define $I^o = \{n_j\}$, and the corresponding v_{I^o} . From the structure of L_j , we can see that the L_j -orbit of v_{I^o} is the unique dense open orbit in $\mathbb{C}^{n_j} - \{0\}$.

Remark 3.11. If we just assume $[k_{ji} \neq k_{ji'} \text{ for any } 1 \leq i \neq i' \leq r_j]$ rather than $[k_{ji} < k_{ji'} \text{ for any } 1 \leq i < i' \leq r_j]$, the conditions (1) and (2) in Proposition 3.9 should be replaced by (1)' $x_i < x_{i'}$ for any $i, i' \in R$ such that $k_{ji} < k_{ji'}$, and (2)' $k_i - x_i < k_{i'} - x_{i'}$ for any $i, i' \in R$ such that $k_{ji} < k_{ji'}$.

With the result above, we can get the representatives of $(\mathbb{C}^n - \{0\})/L$ immediately.

Proposition 3.12. *Set $M_0 = \{1, 2, \dots, m\}$ and $R_{j_0} = \{1, 2, \dots, r_j\}$, $1 \leq j \leq m$. For any $\emptyset \neq M \subseteq M_0$, and $\emptyset \neq R_j \subseteq R_{j_0}$ for each $j \in M$, and $1 \leq x_{ji} \leq k_{ji}$ for $i \in R_j$, $j \in M$ satisfying the following two conditions,*

- (1) $x_{ji_1} < x_{ji_2}$ for any $i_1, i_2 \in R_j$ such that $i_1 < i_2$,
- (2) $k_{ji_1} - x_{ji_1} < k_{ji_2} - x_{ji_2}$ for any $i_1, i_2 \in R_j$ such that $i_1 < i_2$,

set
$$I = \bigsqcup_{j \in M} \left\{ \sum_{v=0}^{j-1} n_v + \sum_{u=0}^{i-1} k_{ju} l_{ju} + k_{ji}(l_{ji} - 1) + x_{ji}, i \in R_j \right\} \subseteq \{1, 2, \dots, n\},$$

and $v_I = (y_1, y_2, \dots, y_n)$, $y_i = 1$ when $i \in I$ and $y_i = 0$ when $i \notin I$. Let I_a be the set of all such I 's. Then $\{v_I \mid I \in I_a\}$ form all different representatives of $(\mathbb{C}^n - \{0\})/L$.

Proof. This follows from the Proposition 3.9. ■

Remark 3.13. Set $M = M_0$, $R_j = \{r_j\}$ and $x_{jr_j} = k_{jr_j}$ for $1 \leq j \leq m$. Let I^o be the set corresponding to such M , R_j and x_{jr_j} , i.e. $I^o = \{n_1, n_1 + n_2, \dots, n\}$. Then v_{I^o} is the unique dense open L -orbit in $\mathbb{C}^n - \{0\}$. ■

We can define g_I in the same way as the contents about regular orbits, see the content above the Lemma 3.2, and such $g_I \cdot f$'s form all different representatives of $P \setminus \mathcal{O}_f$ and the orbit of $g_{I^o} \cdot f$ is the unique dense open orbit among them.

We go to calculate the moment map $p : \mathcal{O}_f \rightarrow \mathfrak{p}^*$. Let's fix some I corresponding to the given M , $R_j(j \in M)$ and $1 \leq x_{ji} \leq k_{ji}(i \in R_j, j \in M)$. For any j , let $q_j = \#R_j$, and $R_j = \{c_{j1}, \dots, c_{jq_j}\}$ with $c_{j1} < c_{j2} < \dots < c_{jq_j}$. For $j \in M$, set $\bar{x}_{jp} = x_{jc_{jp}}$, $\bar{k}_{jp} = k_{jc_{jp}}$ for $1 \leq p \leq q_j$, $\bar{x}_{j0} = 0$, and $d_j = \bar{x}_{jq_j}$.

Proposition 3.14. *Given I as above, then $p(g_I \cdot f)$ has depth $d = \sum_{j \in M} d_j$, and is P -conjugated to the element $\text{pr}'(\eta)$, where*

$$\eta = \text{diag}(A_1, \dots, A_m, J_d),$$

where $A_j = \xi_j$ if $j \notin M$ (ξ_j is defined by (1) after the definition of ξ and f); and $A_j = \text{diag}(A_{j1}, A_{j2}, \dots, A_{jq_j})$, if $j \in M$, where A_{ji} 's are defined as follows:

$$A_{ji} = \begin{cases} J_{k_{ji}, a_j}^{l_{ji}} & \text{if } i \notin R_j, \\ \text{diag}(J_{k_{ji}, a_j}^{l_{ji}-1}, J_{t, a_j}) & \text{with } t = \bar{k}_{jh} - \bar{x}_{jh} + \bar{x}_{j(h-1)}, \\ & \text{if } i = c_{jh} \in R_j \text{ for } 1 \leq h \leq q_j. \end{cases}$$

Proof. As in the proof of Proposition 3.5, by block calculation, we can assume $l_{ji} = 1, \forall 1 \leq i \leq r_j$ and $R_j = \{1, \dots, r_j\}$ for all $1 \leq j \leq m$ at the beginning. Hence, $d_j = x_{jr_j}$ for $1 \leq j \leq m$. For convenience, set $A_I = g_I \cdot \xi$.

In the case of $n \in I$, we have $r_m = 1$ since $\{x_{mi}, i \in R_m\}$ satisfies condition (1) and (2).

Similarly, we have

$$\begin{aligned} \text{Stab}_P(\mathfrak{p}(g_{I^o} \cdot f)) &\cong (G_{l_1} \times G_{l_2} \times \cdots \times G_{l_{r-1}}) \rtimes N \left(\sum_{i=1}^{r-1} k_i l_i \cdot (2(l_{i+1} + \cdots + l_r - 1)) \right. \\ &\quad \left. + \sum_{i=1}^{r-1} l_i^2 (k_i - 1) + (l_r - 1)^2 (k_r - 1) + \left(\sum_{i=1}^{r-1} k_i l_i + (l_r - 1) k_r \right) \right). \end{aligned}$$

By direct calculation, we have that the quotient

$$\text{Stab}_P(\mathfrak{p}(g_{I^o} \cdot f)) / \text{Stab}_P(g_{I^o} \cdot f)$$

is a single element. So the restriction of \mathfrak{p} on $P \cdot (g_{I^o} \cdot f)$ is proper.

In the case of $m > 1$, one can get the statement immediately. ■

3.2. Moment map in the $GL_n(\mathbb{R})$ case

Now, we can calculate the moment map in the $G_n(\mathbb{R})$ case and get the similar results as the case of $G_n(\mathbb{C})$.

Let $n \in \mathbb{Z}_+$, let $G = G_n(\mathbb{R})$ and let $P = P_n(\mathbb{R})$ be the mirabolic subgroup of G . \mathfrak{g} (resp. \mathfrak{p}) denotes the Lie algebra of G (resp. of P). Write

$$J_s(a) = \begin{pmatrix} a & & & & \\ 1 & a & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 & a \end{pmatrix} \in \mathfrak{g}_s(\mathbb{R}), \quad s \in \mathbb{Z} \text{ and } s \geq 2, \quad a \in \mathbb{R},$$

and $J_1(a) = a_{1 \times 1}, a \in \mathbb{R}$. Set $J_s = J_s(0), \forall s \in \mathbb{Z}_+$. Write

$$J_k^l(a) = \text{diag}(\underbrace{J_k(a), \dots, J_k(a)}_{l \text{ blocks}}) \quad \forall l \in \mathbb{Z}_+.$$

Write
$$J_k(a, b) = \begin{pmatrix} J_k(a) & bI_k \\ -bI_k & J_k(a) \end{pmatrix}, \quad \forall a, b \in \mathbb{R}, \quad k \in \mathbb{Z}_+,$$

and
$$J_k^l(a, b) = \text{diag}(\underbrace{J_k(a, b), \dots, J_k(a, b)}_{l \text{ blocks}}),$$

Let $u, v \in \mathbb{N}$ such that $2u + v \leq n$. Let $a_1, \dots, a_{u+v}, b_1, \dots, b_u \in \mathbb{R}$ such that $z_{2k-1} = a_k + ib_k, z_{2k} = a_k - ib_k$, for $k = 1, \dots, u$, and $z_{2u+k} = a_{u+k}$ for $1 \leq k \leq v$, are $2u + v$ distinct complex number, $\prod_{1 \leq i \leq u} b_i \neq 0$. Let $\xi = \text{diag}(\xi_1, \dots, \xi_{u+v})$ with

$$\xi_j = \begin{cases} \text{diag}(J_{k_{j1}}^{l_{j1}}(a_j, b_j), J_{k_{j2}}^{l_{j2}}(a_j, b_j), \dots, J_{k_{jr_j}}^{l_{jr_j}}(a_j, b_j)), & 1 \leq j \leq u, \\ \text{diag}(J_{k_{j1}}^{l_{j1}}(a_j), J_{k_{j2}}^{l_{j2}}(a_j), \dots, J_{k_{jr_j}}^{l_{jr_j}}(a_j)), & u + 1 \leq j \leq u + v, \end{cases}$$

where $r_j \in \mathbb{Z}_+, 1 \leq j \leq u + v, l_{ji}, k_{ji} \in \mathbb{Z}_+, \forall 1 \leq j \leq u + v, 1 \leq i \leq r_j$, and $k_{ji_1} < k_{ji_2}$ for $1 \leq j \leq u + v, 1 \leq i_1 < i_2 \leq r_j$, and

$$\sum_{1 \leq j \leq u, 1 \leq i \leq r_j} 2k_{ji}l_{ji} + \sum_{u+1 \leq j \leq u+v, 1 \leq i \leq r_j} k_{ji}l_{ji} = n.$$

Set $f = \text{pr}(\xi)$. It is clear that every orbit in \mathfrak{g}^* is of the form \mathcal{O}_f .

Now, we fix a ξ . Set

$$n_j = \begin{cases} \sum_{i=1}^{r_j} 2k_{ji} \cdot l_{ji}, & 1 \leq j \leq u, \\ \sum_{i=1}^{r_j} k_{ji} \cdot l_{ji}, & u + 1 \leq j \leq u + v, \end{cases}$$

and $n_0 = 0$. For convenience set $k_{j0} = 0, l_{j0} = 0$ for $1 \leq j \leq u + v$ and $L = \text{Stab}_G(f)$.

Then
$$P \setminus \mathcal{O}_f \cong P \setminus G/L \cong (\mathbb{R}^n - \{0\})/L.$$

By calculating the representatives of $(\mathbb{R}^n - \{0\})/L$, one can get the representatives of the P -orbits in \mathcal{O}_f . We have the following results about $(\mathbb{R}^n - \{0\})/L$.

Proposition 3.16. *Set $K_0 = \{1, \dots, u + v\}, R_{j0} = \{1, \dots, r_j\}, \forall 1 \leq j \leq u + v$. Let $\emptyset \neq K \subseteq K_0$, and $\emptyset \neq R_j \subseteq R_{j0}$ for any $j \in K$. Choose an integer $1 \leq x_{ji} \leq k_{ji}$ for $j \in K, i \in R_j$, such that*

- (1) $x_{ji_1} < x_{ji_2}$, for any $i_1, i_2 \in R_j$ such that $i_1 < i_2$,
- (2) $k_{ji_1} - x_{ji_1} < k_{ji_2} - x_{ji_2}$, for any $i_1, i_2 \in R_j$ such that $i_1 < i_2$.

$$\text{Set } I = \bigsqcup_{j \in K, 1 \leq j \leq u} \left\{ \sum_{l=0}^{j-1} n_l + \sum_{p=0}^{i-1} 2k_{jp}l_{jp} + k_{ji}(2l_{ji} - 1) + x_{ji}, i \in R_j \right\} \bigsqcup_{j \in K, u+1 \leq j \leq u+v} \left\{ \sum_{l=0}^{j-1} n_l + \sum_{p=0}^{i-1} k_{jp}l_{jp} + k_{ji}(l_{ji} - 1) + x_{ji}, i \in R_j \right\},$$

and $v_I = (v_1, \dots, v_n)$ with $v_i = 1$ when $i \in I$ and $v_i = 0$ when $i \notin I$. Let I_a be the set of all I 's constructed above, then $\{v_I \mid I \in I_a\}$ form all different representatives of $(\mathbb{R}^n - \{0\})/L$.

Proof. It can be proved similarly as Proposition 3.12. ■

We can define g_I in the same way as the contents about regular orbits, see the content above the Lemma 3.2, and such $g_I \cdot f$'s form all different representatives of $P \setminus \mathcal{O}_f$ and the orbit of $g_{I^o} \cdot f, I^o = \{n_1, n_1 + n_2, \dots, n\}$, is the unique dense open orbit among them.

We go to calculate the moment map $p : \mathcal{O}_f \rightarrow \mathfrak{p}^*$. Let's fix some I corresponding to the given $K, R_j (j \in K)$ and $1 \leq x_{ji} \leq r_j (i \in R_j, j \in K)$. For any $1 \leq j \leq u + v$, let $q_j = \#R_j$, and $R_j = \{c_{j1}, \dots, c_{jq_j}\}, c_{j1} < c_{j2} < \dots < c_{jq_j}$. Set $\bar{x}_{jh} = x_{jc_{jh}}, \bar{k}_{jh} = k_{jc_{jh}}, \bar{l}_{jh} = l_{jc_{jh}}$ for $1 \leq h \leq q_j$. Set $d_j = \bar{x}_{jq_j}$ and $\bar{x}_{j0} = 0$.

Proposition 3.17. *We have that $p(g_I \cdot f)$ has depth*

$$d = \sum_{j \in M, 1 \leq j \leq u} 2d_j + \sum_{j \in M, u+1 \leq j \leq u+v} d_j,$$

and is P -conjugated to the element $\text{pr}'(\eta)$, where $\eta = \text{diag}(A_1, \dots, A_m, J_d)$, where $A_j = \xi_j$ if $j \notin M$, and $A_j = \text{diag}(A_{j1}, A_{j2}, \dots, A_{jr_j})$, if $j \in M$, where A_{ji} 's are defined as follows, when $i \notin R_j$,

$$A_{ji} = \begin{cases} J_{k_{ji}}^{l_{ji}}(a_j, b_j), & 1 \leq j \leq u, \\ J_{k_{ji}}^{l_{ji}}(a_j), & u + 1 \leq j \leq u + v, \end{cases}$$

when $i = c_{jh} \in R_j$ for some $1 \leq h \leq q_j$,

$$A_{ji} = \begin{cases} \text{diag}(J_{k_{ji}}^{l_{ji}^{i-1}}(a_j, b_j), J_t(a_j, b_j)), & 1 \leq j \leq u, \\ \text{diag}(J_{k_{ji}}^{l_{ji}^{i-1}}(a_j), J_t(a_j)), & u + 1 \leq j \leq u + v, \end{cases}$$

here $t = k_{ji} - \bar{x}_{jh} + \bar{x}_{j(h-1)}$.

Proof. We regard $\mathfrak{p}_n(\mathbb{R})$ as the real form of $\mathfrak{p}_n(\mathbb{C})$. Use the result of $G_n(\mathbb{C})$, we get that $p(g_I \cdot f)$ is $P_n(\mathbb{C})$ -conjugated to $\text{pr}'(\eta)$. So $p(g_I \cdot f)$ is also $P_n(\mathbb{R})$ -conjugated to $\text{pr}'(\eta)$. ■

Theorem 3.18. $\mathfrak{p} : \mathcal{O}_f \rightarrow \mathfrak{p}^*$ sends different P -orbits of \mathcal{O}_f to different P -orbits of \mathfrak{p}^* , the image $p(\mathcal{O}_f)$ contains a unique dense orbit $P \cdot (g_I \circ \cdot f)$, and the restriction of \mathfrak{p} on $P \cdot (g_I \circ \cdot f)$ is proper. The reduce space of the unique open dense orbit is a singleton.

Proof. It follows from Proposition 3.17 and the similar result of the $G_n(\mathbb{C})$ case (see Theorem 3.15). ■

4. Results of Kirillov’s conjecture

Let k be the field \mathbb{R} or \mathbb{C} , let $P_n(k)$ be the mirabolic subgroup of $\text{GL}_n(k)$. Kirillov’s conjecture states that the irreducible unitary representations of $\text{GL}_n(k)$ remains irreducible upon restriction to $P_n(k)$ and was proved by Sahi ([27]) for tempered representations of $\text{GL}_n(\mathbb{R})$ or $\text{GL}_n(\mathbb{C})$, and Sahi-Stein ([29]) for Speh representations of $\text{GL}_n(\mathbb{R})$, and Baruch ([4]) in archimedean fields general. Later Aizenbud-Gourevitch-Sahi ([2]) calculated the adduced representations of the Speh complementary series.

Sahi’s and Sahi-Stein’s proofs are based on Mackey’s theory of the unitary representations of semi-direct products and Vogan’s result about the classification of the irreducible unitary representations of $\text{GL}_n(k)$, and their key method is constructing the intertwining operator by Fourier transform. Baruch proved by studying the $P_n(k)$ -invariant distribution. Aizenbud-Gourevitch-Sahi’s used annihilator variety and degenerate Whittaker models to get the adduced representations of the Speh complementary series. Here we summarize some results of Kirillov’s conjecture.

Firstly, we show how to get the unitary dual of $P_n(k)$, denoted by $\widehat{P_n(k)}$. The following two facts are easy to see.

- (i) $P_n(k) \cong \text{GL}_{n-1}(k) \ltimes k^{n-1}$.
- (ii) $\text{GL}_{n-1}(k)$ has two orbits in $(k^{n-1})^* : \{0\}$ and $(k^{n-1})^* - \{0\}$. Let $\xi \in (k^{n-1})^* - \{0\}$ be defined by $\xi(x_1, \dots, x_{n-1}) = x_{n-1}$. Then $\text{Stab}_{\text{GL}_{n-1}(k)}(\xi) \cong P_{n-1}(k)$.

Based on Mackey’s theory, we have the following:

Every irreducible unitary representation of $P_n(k)$ is obtained in one of the following two ways:

- (i) by trivially extending an irreducible unitary representation of $\text{GL}_{n-1}(k)$,
- (ii) by extending an irreducible unitary representation of $P_{n-1}(k)$ to $P_{n-1}(k) \ltimes k^{n-1}$ by the character ξ and then unitarily inducing to $P_n(k)$.

We use E and I to denote functors from the above constructions (i) and (ii) respectively. Then,

$$\widehat{P_n(k)} = E(\widehat{GL_{n-1}(k)}) \bigsqcup I(\widehat{P_{n-1}(k)}).$$

Moreover, we have the following fact: each irreducible unitary representation τ of $P_n(k)$ is of the form $\tau = I^{j-1}E\sigma$, where the integer $j \geq 1$ and $\sigma \in \widehat{GL_{n-j}(k)}$ are uniquely determined by τ .

Now we describe the unitary dual of $GL_n(k)$, which is finally obtained by Vogan [30]. Let π_1 and π_2 be the representations of $GL_{n_1}(k)$ and $GL_{n_2}(k)$, $\pi_1 \times \pi_2$ denotes the unitary parabolic induction from the representation $\pi_1 \otimes \pi_2$ of the Levi subgroup $GL_{n_1}(k) \times GL_{n_2}(k)$ to $GL_{n_1+n_2}(k)$. Then every irreducible unitary representation of $GL_n(\mathbb{R})$ is a \times -product of unitary characters, Stein representations, Speh representations, and Speh complementary series representations. And every irreducible unitary representation of $GL_n(\mathbb{C})$ is a \times -product of unitary characters and Stein representations.

For convenience, we give the descriptions of Speh representations, Stein representations, and Speh complementary series representations. They can be described as the (subrepresentations of) degenerated principle series.

(i): Speh representations. Let Q be the subgroup of $GL_{2n}(\mathbb{R})$ defined as

$$Q = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mid A, D \in GL_n(\mathbb{R}), B \in M_{n \times n}(\mathbb{R}) \right\}.$$

For any $m \in \mathbb{Z}_+$, we define a character $\chi_m : Q \rightarrow \mathbb{C}^\times$ as

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mapsto |\det(A)|^{m/2} \text{sgn}(\det(A))^{m+1} |\det(D)|^{-m/2}.$$

The Speh representation of $GL_{2n}(\mathbb{R})$, denoted by $\delta(n, m)$, is the unique nonzero irreducible subrepresentation of $\text{Ind}_Q^G(\chi_m)$, where the smooth vector in $\text{Ind}_Q^G(\chi_m)$ is

$$\{f \in C^\infty(G) \mid f(qg) = \lambda(q)\chi_m(q)f(g), \forall q \in Q, g \in G\},$$

here λ is the modular character defined as

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mapsto |\det(A)|^{n/2} |\det(D)|^{-n/2}.$$

Simply speaking, the Speh representation $\delta(n, m)$ is the unique nonzero irreducible subrepresentation of

$$(|\det|^{m/2} \cdot \text{sgn}(\det)^{m+1})|_{GL_n(\mathbb{R})} \times |\det|^{-m/2}|_{GL_n(\mathbb{R})}.$$

(ii): Stein representation. For any $s \in (0, 1/2)$, we can define the Stein representation of $GL_{2n}(\mathbb{R})$, denoted by $\sigma(n, s)$ (or $\sigma(n, s)|_{GL_{2n}(\mathbb{R})}$ when there is ambiguity), as the representation

$$|\det|^s|_{GL_n(\mathbb{R})} \times |\det|^{-s}|_{GL_n(\mathbb{R})}.$$

The Stein representation of $GL_{2n}(\mathbb{C})$, denoted by $\sigma(n, s)$ (or $\sigma(n, s)|_{GL_{2n}(\mathbb{C})}$ when there is ambiguity), can be defined the same as that of $GL_{2n}(\mathbb{R})$,

$$|\det|^s|_{GL_n(\mathbb{C})} \times |\det|^{-s}|_{GL_n(\mathbb{C})}.$$

(iii): Speh complementary series representation. For any $m \in \mathbb{Z}_+$, and $s \in (0, 1/2)$, we can define the Speh complementary series representation of $\mathrm{GL}_{4n}(\mathbb{R})$, denoted by $\Delta(n, m, s)$, as the representation

$$|\det|^s \delta(n, m) \times |\det|^{-s} \delta(n, m).$$

We now state some results on Kirillov's conjecture. Firstly, S.Sahi [27] established Kirillov's conjecture of $\mathrm{GL}_n(k)$ for the \times -product of unitary characters and Stein representations. Sahi [27] defined that

- (i) a unitary representation of $P_n(k)$, denoted by τ , is homogeneous of depth j if $\tau = I^{j-1}E\sigma$ for some unitary representation σ of $\mathrm{GL}_{n-j}(k)$,
- (ii) a unitary representation of $\mathrm{GL}_n(k)$, denoted by ρ , is adducible of depth j if $\rho|_{P_n(k)}$ is homogeneous of depth j , and if $\rho|_{P_n(k)} = I^{j-1}E\sigma$, we shall write $\sigma = A\rho$ and call it the adduced representation of ρ .

Then Sahi obtained the following key fact based on Mackey's theory and the partial Fourier transform, and got the result in that paper.

Theorem 4.1 ([27], Theorem 2.1). *If ρ and σ are adducible representation of $\mathrm{GL}_r(k)$ and $\mathrm{GL}_s(k)$ of depths l and m , then $\rho \times \sigma$ is adducible of depth $l + m$. Moreover, $A(\rho \times \sigma) = (A\rho) \times (A\sigma)$.*

We also have the following result.

Theorem 4.2 ([27], Lemma 3.1). *Let π be a unitary character of $\mathrm{GL}_n(k)$, then π is adducible of depth 1 and $A\pi = \pi|_{\mathrm{GL}_{n-1}(k)}$, where $\mathrm{GL}_{n-1}(k)$ is imbedded on the top left corner of $\mathrm{GL}_n(k)$.*

Later Sahi [28], Sahi-Stein [29] obtained the adduced representations of Stein representations and Speh representations. The key method is constructing the intertwining operator by partial Fourier transform.

Theorem 4.3 ([28], 2.4). *Let $\sigma(n+1, s)$ be the Stein representation of $\mathrm{GL}_{2n+2}(\mathbb{R})$, $s \in (0, 1/2)$, then $A\sigma(n+1, s) = \sigma(n, s)$.*

Theorem 4.4 ([29], Theorem 3). *Let $\delta(n+1, m)$ be the m -th Speh representation of $\mathrm{GL}_{2n+2}(\mathbb{R})$, $m \in \mathbb{Z}_+$, then $A\delta(n+1, m) = \delta(n, m)$.*

After Baruch [4] proved Kirillov's conjecture in archimedean fields, Aizenbud-Gourevitch-Sahi [2] calculated the adduced representations of the Speh complementary series.

Theorem 4.5 ([2], Theorem 4.2.4). *Let $\Delta(n+1, m, s)$ be the Speh complementary series representation of $\mathrm{GL}_{4n+4}(\mathbb{R})$, $m \in \mathbb{Z}_+$ and $s \in (0, 1/2)$. Then*

$$A\Delta(n+1, m, s) = \Delta(n, m, s).$$

Example 4.6. In the case of $\mathrm{GL}_n(\mathbb{C})$, let $a_i \in \mathbb{R}$, $s_i \in \mathbb{Z}_+$, $1 \leq i \leq m$, $\sum_{i=1}^m s_i = n$, let $\pi = (\det)^{ia_1}|_{\mathrm{GL}_{s_1}(\mathbb{C})} \times \cdots \times (\det)^{ia_m}|_{\mathrm{GL}_{s_m}(\mathbb{C})}$ be the irreducible unitary representation of $\mathrm{GL}_n(\mathbb{C})$, then

$$\pi|_{P_n} = I^{m-1}E((\det)^{ia_1}|_{\mathrm{GL}_{s_1-1}(\mathbb{C})} \times \cdots \times (\det)^{ia_m}|_{\mathrm{GL}_{s_m-1}(\mathbb{C})}).$$

5. Orbit method

The orbit method tries to establish a close connection existed between irreducible unitary representations of a Lie group and its orbits in the coadjoint representation, and to provide a clear geometric picture of irreducible unitary representations. In the 1950s, before the orbit method developed, Mackey and others had obtained some wonderful results on the irreducible unitary representations of Lie groups, showing how to use induced representation to obtain irreducible unitary representations, see Mackey [24]. In the 1960s, Kirillov found a way to think about the set of all irreducible unitary representations of a simply connected nilpotent Lie group, establishing a bijection between the coadjoint orbits and the irreducible unitary representations (see Kirillov [12]), and brought up the orbit method. Later, Kostant developed a quantization theory to obtain irreducible unitary representations, and generalized the results of Kirillov to solvable Lie groups (see Auslander-Kostant [3]). In the 1970s-1980s, Duflo got all the irreducible unitary representations of (almost) algebraic groups assuming the unitary dual of reductive Lie groups, see Duflo [6].

The orbit method (or the quantization problem) of reductive groups is not completely solved until now. Roughly speaking, the quantization of coadjoint orbits can be reduced to the case of nilpotent coadjoint orbits (defined below), and such irreducible unitary representations attached to nilpotent coadjoint orbits, called unipotent representations, are not easy to understand in general, see Vogan [33]. We will summarize some results about the orbit method of reductive Lie groups and algebraic groups in the following two parts. The method of constructing representations is using induced representation (includes parabolic induction and cohomological induction).

5.1. Representations attached to coadjoint orbits of $GL_n(k)$

5.1.1. Jordan decomposition

To begin with, we use Jordan decomposition to describe the coadjoint orbits of reductive groups (see Vogan [33], Lecture 2).

Let G be a reductive Lie group and let \mathfrak{g} be its Lie algebra. Let θ be a Cartan involution of \mathfrak{g} and let

$$G = K \exp \mathfrak{s}$$

be the Cartan decomposition, where $K = G^\theta$ and $\mathfrak{s} = -1$ eigenspace of θ on \mathfrak{g} .

Let B be a nondegenerate G -invariant and θ -invariant bilinear form on \mathfrak{g} such that the quadratic form

$$\mathfrak{g} \ni X \mapsto -B(X, \theta X)$$

on \mathfrak{g} is positive definite. Let \mathfrak{g}^* be the dual of \mathfrak{g} . Then we have a G -isomorphism

$$\phi : \mathfrak{g} \rightarrow \mathfrak{g}^*, X \mapsto (Y \mapsto B(X, Y), \forall Y \in \mathfrak{g}), \forall X \in \mathfrak{g}.$$

For any $\lambda \in \mathfrak{g}^*$, we call it semisimple (resp. nilpotent, hyperbolic, elliptic) if $\phi^{-1}(\lambda)$ is a semisimple (resp. nilpotent, hyperbolic, elliptic) element in \mathfrak{g} . By the Jordan decomposition, we can decompose any $\lambda \in \mathfrak{g}^*$ uniquely as

$$\lambda = \lambda_h + \lambda_e + \lambda_n,$$

where λ_h is hyperbolic, λ_e is elliptic, λ_n is nilpotent, and $\phi^{-1}(\lambda_h)$, $\phi^{-1}(\lambda_e)$, $\phi^{-1}(\lambda_n)$ commute with each other.

Set $X_h = \phi^{-1}(\lambda_h)$, $X_e = \phi^{-1}(\lambda_e)$, and $X_n = \phi^{-1}(\lambda_n)$. Let $G(X)$ (resp. $G(\lambda)$) be the stabilizer of X (resp. of λ), and $\mathfrak{g}(X)$ (resp. $\mathfrak{g}(\lambda)$) be the Lie algebra of $G(X)$ (resp. of $G(\lambda)$). Then we have the following.

Proposition 5.1 (Vogan [30], Proposition 2.10, 2.11, and 2.12).

- (i) *Any hyperbolic element in \mathfrak{g} is conjugated into \mathfrak{s} . If $X \in \mathfrak{s}$ is hyperbolic, then $G(X)$ is reductive group with Cartan involution $\theta|_{G(X)}$.*
- (ii) *There is a bijection between the G -orbits of elements in \mathfrak{g} , which hyperbolic part are G -conjugated to X_h , and the $G(X_h)$ -orbits of elements in $\mathfrak{g}(X_h)$, which hyperbolic part is zero.*
- (iii) *Any elliptic element in \mathfrak{g} is conjugated into \mathfrak{k} . If $X \in \mathfrak{k}$ is elliptic, then $G(X)$ is a reductive group with Cartan involution $\theta|_{G(X)}$.*
- (iv) *There is a bijection between the G -orbits of elements in \mathfrak{g} , which semisimple part are G -conjugated to $X_h + X_e$, and the $G(X_h + X_e)$ -orbits of elements in $\mathfrak{g}(X_h + X_e)$, which semisimple part is zero.*

Therefore we can assume $X_h \in \mathfrak{s}$ and $X_e \in \mathfrak{k}$.

An irreducible unitary representation which is attached to a coadjoint orbit has a kind of ‘‘Jordan decomposition’’. To attach an irreducible unitary representation to the orbit of λ , we firstly attach an irreducible representation π_n of $G(\lambda_h + \lambda_e)$ to the $G(\lambda_h + \lambda_e)$ -orbit of $\lambda_n|_{\mathfrak{g}(\lambda_h + \lambda_e)^*}$, which is called as nilpotent step; then using λ_e and applying cohomological induction to π_n , we get an irreducible unitary representation π_h of $G(\lambda_h)$, which is called as elliptic step; finally, using λ_h and applying parabolic induction to π_h , we get an irreducible unitary representation of G , which is called as hyperbolic step.

5.1.2. Hyperbolic step

The easiest step is the hyperbolic step, which uses parabolic induction, see Vogan [33], Lecture 2. Set $L = G(\lambda_h)$ with its Lie algebra \mathfrak{l} . Set

$$\mathfrak{g}_r = \{Y \in \mathfrak{g} | [X_h, Y] = rY\}, r \in \mathbb{R}$$

and $\mathfrak{u} = \bigoplus_{r>0} \mathfrak{g}_r$. Set $U = \exp \mathfrak{u}$ and $Q = LU$. We define the unitary character $\chi(\lambda_h)$ of L as follows:

$$\chi(\lambda_h)(k \cdot \exp(Z)) = \exp(i\lambda_h(Z)), k \in L \cap K, Z \in \mathfrak{l} \cap \mathfrak{s}.$$

Suppose that π_L is any unitary representation of $G(\lambda_h)$ attached to the $G(\lambda_h)$ -coadjoint orbit of $(\lambda_e + \lambda_n)|_{\mathfrak{l}}$, then the unitary representation of G (this unitary representation may be reducible)

$$\pi_G = \text{Ind}_Q^G(\pi_L \otimes \chi(\lambda_h))$$

is attached to the coadjoint orbit of λ .

5.1.3. Elliptic step

The elliptic step is more complicated, which uses cohomological induction, see Vogan [32]. Set $\lambda_s = \lambda_h + \lambda_e$ be semisimple part of λ , and set $X_s = X_h + X_e$.

Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of \mathfrak{g} , and let \mathfrak{q} be the parabolic subalgebra having $\mathfrak{g}_{\mathbb{C}}(\lambda_s)$ as a Levi factor with nilpotent radical \mathfrak{u} such that the eigenvalues of X_s acting on \mathfrak{u} are in

$$\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0, \text{ or } \operatorname{Re}(z) = 0, \operatorname{Im}(z) > 0\}.$$

Let $e^{2\rho(\mathfrak{u})}$ be the character of the adjoint action of $G(\lambda_s)$ on the top exterior power of \mathfrak{u} . Assume that τ is any irreducible representation of $G(\lambda_s)$ such that

$$d\tau = i\lambda_s + \rho(\mathfrak{u}).$$

Then there is attached to λ a unitary representation π_λ such that the underlying $(\mathfrak{g}_{\mathbb{C}}, K)$ -module $\Pi(\lambda)$ is defined as follows,

$$\Pi(\lambda) = (\Gamma_{\mathfrak{g}_{\mathbb{C}}, G(\lambda_s) \cap K}^{\mathfrak{g}_{\mathbb{C}}, K})^d(\operatorname{pro}_{\mathfrak{q}, G(\lambda_s) \cap K}^{\mathfrak{g}_{\mathbb{C}}, G(\lambda_s) \cap K}(\tau \otimes \pi_n)), \tag{2}$$

where Γ is the Zuckerman functor, pro is a kind of Hom functor, d is the dimension of $\mathfrak{u} \cap \mathfrak{k}$, π_n is the unipotent representation of $G(\lambda_s)$ which is attached to $\lambda_n|_{(\mathfrak{g}(\lambda_s))^*}$. To define the Zuckerman functor (see Knapp-Vogan [13], Introduction), we need to define the Hecke algebra. Assume that \mathfrak{h} is a complex reductive Lie algebra of Lie group H , M is a compact subgroup of H , then the Hecke algebra $R(\mathfrak{h}, M)$ can be defined as the algebra of left M -finite distributions on H with support in M , with convolution as multiplication.

Let us fix some notations to define the Zuckerman functor and the functor pro . Let G be a linear connected reductive Lie group and $G_{\mathbb{C}}$ be the complexification of G . Let K be the maximal compact subgroup of G . Let \mathfrak{g} denote the Lie algebra of G and $\mathfrak{g}_{\mathbb{C}}$ be the complexification of \mathfrak{g} .

Assume that T is a torus in G , and $L = Z_G(T)$ with Lie algebra \mathfrak{l} . Let $L_{\mathbb{C}}$ be the analytic subgroup of $G_{\mathbb{C}}$ with Lie subalgebra $\mathfrak{l}_{\mathbb{C}}$. Let Q be a parabolic subgroup in $G_{\mathbb{C}}$ containing $L_{\mathbb{C}}$ as Levi subgroup and let \mathfrak{q} be the Lie algebra of Q .

Assume that V is a $(\mathfrak{g}_{\mathbb{C}}, L \cap K)$ -module, W is an $(\mathfrak{l}_{\mathbb{C}}, L \cap K)$ -module, hence is also a $(\mathfrak{q}, L \cap K)$ -module by trivial extension to the radical \mathfrak{u} of \mathfrak{q} .

Define the functor pro from $(\mathfrak{q}, L \cap K)$ -module to $(\mathfrak{g}_{\mathbb{C}}, L \cap K)$ -module as

$$\operatorname{pro}_{\mathfrak{q}, L \cap K}^{\mathfrak{g}_{\mathbb{C}}, L \cap K}(W) = \operatorname{Hom}_{\mathfrak{q}}(U(\mathfrak{g}_{\mathbb{C}}), W)_{L \cap K}.$$

Here the subscript $L \cap K$ means taking all the $L \cap K$ finite vectors.

Define the functor from $(\mathfrak{g}_{\mathbb{C}}, L \cap K)$ -module to $(\mathfrak{g}_{\mathbb{C}}, K)$ -module as

$$\Gamma_{\mathfrak{g}_{\mathbb{C}}, L \cap K}^{\mathfrak{g}_{\mathbb{C}}, K}(V) = \operatorname{Hom}_{R(\mathfrak{g}_{\mathbb{C}}, L \cap K)}(R(\mathfrak{g}_{\mathbb{C}}, K), V)_K.$$

It is left exact and has i -th right derivative functors, denoted by $(\Gamma_{\mathfrak{g}_{\mathbb{C}}, L \cap K}^{\mathfrak{g}_{\mathbb{C}}, K})^i$.

Let $({}^u\mathcal{R}_{\mathfrak{q}, L \cap K}^{\mathfrak{g}_{\mathbb{C}}, K})^i(W)$ denote $(\Gamma_{\mathfrak{g}_{\mathbb{C}}, K \cap L}^{\mathfrak{g}_{\mathbb{C}}, K})^i(\operatorname{pro}_{\mathfrak{q}, L \cap K}^{\mathfrak{g}_{\mathbb{C}}, L \cap K}(W))$, then equation (2) can be written as

$$({}^u\mathcal{R}_{\mathfrak{q}, G(\lambda_s) \cap K}^{\mathfrak{g}_{\mathbb{C}}, K})^d(\tau \otimes \pi_n).$$

5.1.4. Nilpotent step

We focus on the special unipotent representations which are defined by Arthur, Barbasch, and Vogan (see Adams-Barbasch-Vogan [1], Chapter 27).

Let's introduce notation about partition firstly. Let $n \in \mathbb{Z}_+$, let $r \in \mathbb{Z}_+$, $k_i, l_i \in \mathbb{Z}_+$ such that $\sum_{1 \leq i \leq r} k_i l_i = n$ and $k_i > k_j$ for any $1 \leq i < j \leq r$. We use $\{k_1^{l_1} \cdots k_r^{l_r}\}$ to denote the partition of n ,

$$\{\underbrace{k_1, \dots, k_1}_{l_1}, \dots, \underbrace{k_r, \dots, k_r}_{l_r}\}.$$

Let P denote this partition. We will use J_P to denote $\text{diag}(J_{k_1}^{l_1}, \dots, J_{k_r}^{l_r})$, $J_P(a)$ to denote $\text{diag}(J_{k_1}^{l_1}(a), \dots, J_{k_r}^{l_r}(a))$, and $J_P(a, b)$ to denote $\text{diag}(J_{k_1}^{l_1}(a, b), \dots, J_{k_r}^{l_r}(a, b))$. For the case of $G = G_n(\mathbb{R})$, let $f = \text{pr}(\xi)$ with $\xi = J_P$. The special unipotent representation attached to \mathcal{O}_f is

$$\pi = \text{sgn}(\det)^{w_1}|_{G_{t_1}(\mathbb{R})} \times \cdots \times \text{sgn}(\det)^{w_p}|_{G_{t_p}(\mathbb{R})},$$

where $\{t_1, \dots, t_p\}$ is the dual partition of $\{k_1^{l_1} \cdots k_r^{l_r}\}$ and $w_i = 0, 1$ for $1 \leq i \leq p$. For the case of $G = G_n(\mathbb{C})$, let $f = \text{pr}(\xi)$ with $\xi = J_P$. The special unipotent representation attached to \mathcal{O}_f is

$$\pi = 1|_{G_{t_1}(\mathbb{C})} \times \cdots \times 1|_{G_{t_p}(\mathbb{C})}$$

where $\{t_1, \dots, t_p\}$ is the dual partition of $\{k_1^{l_1} \cdots k_r^{l_r}\}$.

In general, we have the following correspondence between coadjoint orbits and irreducible unitary representations.

Let us consider the case of $G = G_n(\mathbb{R})$.

When $n = 2m$, assume that $P = \{k_1^{l_1} \cdots k_r^{l_r}\}$ is a partition of m . Let $f \in \mathfrak{g}^*$ and $f = \text{pr}(\xi)$ with $\xi = J_P(0, b/2)$, $b \in \mathbb{Z}_+$.

Firstly, we use the nilpotent step and attach the orbit $f_n = \text{pr}(\xi_n)$, $\xi_n = J_P(0, 0)$, with the irreducible unitary representation π_n of $G(f_s) \cong G_m(\mathbb{C})$

$$1|_{G_{t_1}(\mathbb{C})} \times \cdots \times 1|_{G_{t_p}(\mathbb{C})},$$

where $\{t_1, \dots, t_p\}$ is the dual partition of $\{k_1^{l_1} \cdots k_r^{l_r}\}$. By direct calculation, we get

$$\tau = \left(\frac{\det}{|\det|}\right)^{b+m}|_{G_m(\mathbb{C})} \quad \text{and} \quad e^{\rho(u)} = \left(\frac{\det}{|\det|}\right)^m|_{G_m(\mathbb{C})}.$$

Using Vogan [30], Theorem 17.6, we have

$$({}^u\mathcal{R}_{\mathfrak{q}, G(f_s) \cap K}^{\mathfrak{g}_\mathbb{C}, K})^d(\tau \otimes \pi_n) = \prod_{i=1}^p \delta(t_i, b).$$

So the irreducible unitary representation $\prod_{i=1}^p \delta(t_i, b)$ is attached to the coadjoint orbit of $f = \text{pr}(\xi)$ with $\xi = J_P(0, b/2)$. When $f = \text{pr}(\xi)$ with

$$\xi = \text{diag}(J_{P_1}(0, b_1/2), \dots, J_{P_r}(0, b_r/2), J_P),$$

where $r \in \mathbb{Z}_+$, $b_i \in \mathbb{Z}_+$, for $1 \leq i \leq r$, $b_1 > b_2 > \cdots > b_r$, P_i is a partition of n_i and P is a partition of n_0 with $\sum_{i=1}^r 2n_i + n_0 = n$.

Let $\{t_{i1}, \dots, t_{is_i}\}$ be the dual partition of P_i for $1 \leq i \leq r$. Let $\{t_1, \dots, t_s\}$ be the dual partition of P .

Let $H = G_{2n_1}(\mathbb{R}) \times \cdots \times G_{2n_p}(\mathbb{R}) \times G_{n_0}(\mathbb{R})$,

then $G(f_s) \subset H$. Let \mathfrak{h} be the Lie algebra of H and $\mathfrak{h}_{\mathbb{C}}$ be the complexification of \mathfrak{h} . Let $\mathfrak{q}_1 = \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{q}$, and let \mathfrak{u}_1 be the nilpotent radical of \mathfrak{q}_1 . Let Q be the parabolic subgroup containing H and the invertible upper triangular matrices. Set $d_1 = \dim(\mathfrak{u}_1 \cap \mathfrak{k})$. We use Vogan [30], Theorem 17.6, and get

$$\begin{aligned} ({}^u\mathcal{R}_{\mathfrak{q}, G(f_s) \cap K}^{\mathfrak{g}_{\mathbb{C}}, K})^d(\tau \otimes \pi_n) &= \text{Ind}_Q^G (({}^u\mathcal{R}_{\mathfrak{q}_1, G(f_s) \cap K}^{\mathfrak{h}_{\mathbb{C}}, K \cap H})^{d_1}(\tau \otimes \pi_n)) \\ &= \left(\prod_{j=1}^p \prod_{i=1}^{s_j} \delta(t_{ji}, b_j) \right) \times \left(\prod_{j=1}^s \text{sgn}(\det)^{w_j} |_{G_{t_j}(\mathbb{R})} \right), \end{aligned}$$

where $w_j = 0, 1$ for any $1 \leq j \leq s$.

5.1.5. Representations of $GL_n(k)$ attached to some kind of coadjoint orbits

In general, for the case of $G_n(\mathbb{R})$, let $f = \text{pr}(\xi)$ with $\xi = \text{diag}(\xi_1, \dots, \xi_m)$, $m \in \mathbb{Z}_+$,

and $\xi_i = \text{diag}(J_{P_{i1}}(a_i, b_{i1}/2), \dots, J_{P_{ip_i}}(a_i, b_{ip_i}/2), J_{P_i}(a_i)), 1 \leq i \leq m$,

where $p_i \in \mathbb{Z}_+$, $a_i \in \mathbb{R}$ for $1 \leq i \leq m$, $a_1 > \dots > a_m$, $b_{ij} \in \mathbb{Z}_+$, and $b_{i1} > \dots > b_{ip_i}$, P_{ij} is a partition of n_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq p_i$, P_i is a partition of n_i for $1 \leq i \leq m$, n_{ij}, n_i satisfy

$$\sum_{1 \leq i \leq m} \sum_{1 \leq j \leq p_i} 2n_{ij} + \sum_{1 \leq i \leq m} n_i = n.$$

Assume that $\{t_{ij,1}, \dots, t_{ij,s_{ij}}\}$ is the dual partition of P_{ij} , and that $\{t_{i,1}, \dots, t_{i,s_i}\}$ is the dual partition of P_i .

The representation attached to \mathcal{O}_f is $\pi = \pi_1 \times \dots \times \pi_m$, where

$$\pi_l = |\det|^{ia_l} \otimes \left(\left(\prod_{j=1}^{p_l} \prod_{k=1}^{s_{lj}} \delta(t_{lj,k}, b_{lj}) \right) \times \left(\prod_{k=1}^{s_l} \text{sgn}(\det)^{w_{l,k}} |_{G_{t_{l,k}}(\mathbb{R})} \right) \right),$$

with $w_{l,k} = 0, 1$ for any $1 \leq l \leq m$, $1 \leq k \leq s_l$.

For the case of $G = G_n(\mathbb{C})$, let $f = \text{pr}(\xi)$ with

$$\xi = \text{diag}(J_{P_1}(a_1), \dots, J_{P_m}(a_m)),$$

where $a_i \in \mathbb{R}$ for any $1 \leq i \leq m$, and $a_1 > \dots > a_m$, P_i is a partition of n_i for $1 \leq i \leq m$, and $n_i (1 \leq i \leq m)$ satisfy $\sum_{1 \leq i \leq m} n_i = n$.

The representation attached to \mathcal{O}_f is

$$\pi = \prod_{j=1}^m \prod_{k=1}^{s_j} |\det|^{ia_j} |_{G_{p_{jk}}(\mathbb{C})},$$

where $\{p_{j1}, \dots, p_{js_j}\}$ is the dual partition of P_j for $1 \leq j \leq m$.

5.2. Representations attached to coadjoint orbits of $P_n(k)$

Given an (almost) algebraic group G , Duflo [6] constructed all the irreducible unitary representations of G assuming the unitary dual of some reductive groups, basing on Mackey's theory. It suggests that the problem of attaching irreducible unitary representations with coadjoint orbits can be reduced to the problem in the

case of reductive groups. We summarize some results of Duflo [6] without proofs in the following.

5.2.1. Coadjoint orbits of algebraic groups

Duflo showed how to get all the coadjoint orbits of \mathfrak{g}^* assuming that one knew the coadjoint orbits of reductive groups.

Let \mathfrak{g} be the Lie algebra of G , and let \mathfrak{g}^* be the dual of \mathfrak{g} . For any $x \in \mathfrak{g}^*$, let $G(x)$ be the stabilizer of x in G and let $\mathfrak{g}(x)$ be the Lie algebra of $G(x)$, let ${}^uG(x)$ be the unipotent radical of $G(x)$ and let ${}^u\mathfrak{g}(x)$ be the Lie algebra of ${}^uG(x)$.

On $\mathfrak{g}/\mathfrak{g}(x)$, there is a natural symplectic form, denoted by ω_x ,

$$\omega_x(\bar{\alpha}, \bar{\beta}) = x([\alpha, \beta]),$$

where $\alpha, \beta \in \mathfrak{g}$ and $\bar{\alpha}, \bar{\beta}$ denote the image of α and β in $\mathfrak{g}/\mathfrak{g}(x)$.

Firstly, we need to define the linear forms of unipotent type (see [6], I). If G is unipotent, all of \mathfrak{g}^* are of unipotent type, and if G is reductive, 0 is the unique element of \mathfrak{g}^* of unipotent type. In general, its definition depends on the following things.

Definition 5.2. Let $x \in \mathfrak{g}^*$ and \mathfrak{b} be a subalgebra of \mathfrak{g} . We say that

- (i) \mathfrak{b} is *coisotropic relative to x* (or coisotropic) if the orthogonal of \mathfrak{b} in \mathfrak{g} with respect to ω_x , denoted by \mathfrak{b}^\perp , is contained in \mathfrak{b} .
- (ii) \mathfrak{b} is a *polarization of x* if $\mathfrak{b}^\perp = \mathfrak{b}$.
- (iii) An coisotropic subalgebra \mathfrak{b} is of strongly unipotent type if \mathfrak{b} is algebraic (i.e. there exists algebraic subgroup with Lie subalgebra \mathfrak{b}) and $\mathfrak{b} = \mathfrak{g}(x) + {}^u\mathfrak{b}$.

Definition 5.3. An element x in \mathfrak{g}^* is said to be of unipotent type if it satisfies

- (i) There exists a reductive factor of $\mathfrak{g}(x)$ contained in $\ker x$.
- (ii) There is a subalgebra of strongly unipotent type relative to x .

Let $x \in \mathfrak{g}^*$ be a form of unipotent type, then we can construct a subalgebra of strongly unipotent type relative to x by induction on $\dim \mathfrak{g}$ canonically (see [6], I.20). Let $u = x|_{\mathfrak{u}\mathfrak{g}}$ and let $\mathfrak{h} = \mathfrak{g}(u)$. If $\mathfrak{h} = \mathfrak{g}$, then set $\mathfrak{b} = \mathfrak{h}$. If $\mathfrak{h} \neq \mathfrak{g}$, then $y = x|_{\mathfrak{h}}$ is a form of unipotent type. Let $\mathfrak{l} \subset \mathfrak{h}$ be the canonical subalgebra of strongly unipotent type relative to y , set $\mathfrak{b} = {}^u\mathfrak{g} + \mathfrak{h}$. Then \mathfrak{b} is the canonical subalgebra of strongly unipotent type relative to x .

Let D be the set of pairs (x, λ) , where x is a linear form of unipotent type on \mathfrak{g} , and λ is an element in $L(x)$, $L(x)$ is the set of linear forms over $\mathfrak{g}(x)$ whose restriction to ${}^u\mathfrak{g}(x)$ is equal to $x|_{{}^u\mathfrak{g}(x)}$.

Let \mathfrak{r} be a reductive factor of $\mathfrak{g}(x)$, then the restriction map establish a bijection of $L(x)$ and \mathfrak{r}^* , sending $y \in L(x)$ to $y|_{\mathfrak{r}}$.

Let $(x, \lambda) \in D$. Let \mathfrak{b} be a subalgebra of \mathfrak{g} , which is of unipotent type relative to x (see [6], I.8 for definition, from [6], I.9, I.22, we have that when x is of unipotent type, a subalgebra is of unipotent type relative to x if and only if it is strongly unipotent type relative to x). Let f be an element of \mathfrak{g}^* whose restriction to ${}^u\mathfrak{b}$ is equal to $x|_{{}^u\mathfrak{b}}$, and whose restriction to $\mathfrak{g}(x)$ is equal to λ . Such an element f exists by the definition of $L(x)$.

Let G act on D naturally, then we have the following.

Proposition 5.4 ([6], II.5). (i) *The orbit $G \cdot f$ in \mathfrak{g}^* does not depend on the choices of \mathfrak{b} and f . We denote it by $O_{x,\lambda}$.*

(ii) *The map $(x, \lambda) \mapsto O_{x,\lambda}$ induces a bijection from D/G to \mathfrak{g}^*/G .*

5.2.2. Construction of unitary dual of algebraic groups

Duflo showed how to construct the irreducible unitary representations of G using the coadjoint orbits and the irreducible unitary representations of some reductive groups. The construction of representations used Metaplectic groups.

Since $G(x)$ acts on $\mathfrak{g}/\mathfrak{g}(x)$ and keeps the form ω_x , we get a morphism

$$\phi : G(x) \rightarrow \text{Sp}(\mathfrak{g}/\mathfrak{g}(x), \omega_x).$$

Let $\text{Mp}(\mathfrak{g}/\mathfrak{g}(x), \omega_x)$ be the metaplectic group, i.e., there is a nontrivial connected double cover

$$\varphi : \text{Mp}(\mathfrak{g}/\mathfrak{g}(x), \omega_x) \rightarrow \text{Sp}(\mathfrak{g}/\mathfrak{g}(x), \omega_x).$$

We define a double cover of $G(x)$ as

$$G(x)^\mathfrak{g} = \{(g, t) \mid g \in G(x), t \in \text{Mp}(\mathfrak{g}/\mathfrak{g}(x), \omega_x) \text{ and } \phi(g) = \varphi(t)\}.$$

Let $\psi : G(x)^\mathfrak{g} \rightarrow G(x)$ be the covering morphism, and let $(1, -1)$ denote the nontrivial element of $\ker \psi$.

Similarly, for any group H , which acts on a symplectic vector space \mathfrak{m} and keep the symplectic form, we have a morphism $\phi' : H \rightarrow \text{Sp}(\mathfrak{m})$ and a double covering morphism $\varphi' : \text{Mp}(\mathfrak{m}) \rightarrow \text{Sp}(\mathfrak{m})$. Define a double cover

$$\tilde{H} = \{(h, t) \mid h \in H, t \in \text{Mp}(\mathfrak{m}) \text{ and } \phi'(h) = \varphi'(t)\}.$$

By Segal-Shale-Weil representation (see Duflo [6], II.6), any $\hat{s} \in \text{Mp}(\mathfrak{m})$ which is mapped to $s \in \text{Sp}(\mathfrak{m})$ can be represented by $(s, \theta_{\hat{s}})$ (or just denoted by (s, θ)), where $\theta_{\hat{s}}$ is a function depended on \hat{s} , with complex value module 1 over the set of maximal isotropic subspaces in \mathfrak{m} . Therefore, any element $(h, t) \in \tilde{H}$ ($h \in H, t \in \text{Mp}(\mathfrak{m})$) can be represented by (h, θ_t) . We will also use $H^{\mathfrak{m}}$ to denote \tilde{H} .

Since ${}^uG(x)$ is unipotent, ${}^uG(x)$ can be embedded into $G(x)^\mathfrak{g}$. Let

$$Y^{irr}(x) = \{\text{irreducible unitary representation } \pi \text{ of } G(x)^\mathfrak{g}/{}^uG(x) \mid \pi((1, -1)) = -1\}.$$

Let C denote the set of couples (x, τ) , where x is a linear form of unipotent type, and $\tau \in Y^{irr}(x)$. Then the group G acts on C naturally.

Now, we can construct an irreducible unitary representation $T_{x,\tau}$ of G (see [6], III for details). Let \mathfrak{b} be a subalgebra of strongly unipotent type relative to x which is stable under $G(x)$, and let \mathfrak{v} denote the nilpotent radical of \mathfrak{b} . Let V be the subgroup corresponding to \mathfrak{v} . Set $B = G(x)V$. Let v denote the restriction of x to \mathfrak{v} , and $G(x)^\mathfrak{v}$ is defined.

Let $\mathfrak{l} \subset \mathfrak{v}$ be a polarization relative to v , and let L be the unipotent subgroup with Lie algebra \mathfrak{l} . Define the irreducible unitary representation of V attached to v as

$$T_v = \text{Ind}_L^V e^{iv|_{\mathfrak{l}}}.$$

Define $T_{x,\tau} = \text{Ind}_B^G(\tau' \otimes S_v T_v)$,

where τ' and S_v are defined as follows, and we have that $T_{x,\tau}$ is independent of the choice of B (see [6], III.16). Define τ' to be the representation of $G(x)^\mathfrak{v}$ as

$$\tau'(y, \psi) = (\varphi\psi^{-1})\bar{\tau}(y, \varphi),$$

here $\bar{\tau} = \tau \otimes \chi_x$ of $(G(x)^\mathfrak{g}/{}^uG(x)) \rtimes {}^uG(x) \cong G(x)^\mathfrak{g}$ with χ_x defined by $d\chi_x = ix|_{\mathfrak{u}\mathfrak{g}(x)}$, and

$$\varphi\psi^{-1} = \varphi(\mathfrak{l}' + \mathfrak{g}^\perp)\psi(\mathfrak{l}')^{-1}$$

is a constant which is independent on the choice of the maximal totally isotropic subspace \mathfrak{l}' of \mathfrak{v} (see [6], II.8). Define S_v to be the action of $G(x)^\mathfrak{v}$ on T_v by

$$S_v(y, \varphi) = \varphi(\mathfrak{l}) \cdot S'_{v,\mathfrak{l}}(y),$$

and $S'_{v,\mathfrak{l}}$ is the action of $G(x)$ on T_v defined by

$$S'_{v,\mathfrak{l}}(y) = \|A(y)\|^{-1} F_{\mathfrak{l},y\mathfrak{l}} A(y),$$

$F_{\mathfrak{l},y\mathfrak{l}}$ is the intertwining operator from $T_{y\mathfrak{v}}$ to T_v , and $A(y)$ is the operator from T_v to $T_{y\mathfrak{v}}$ with

$$A(y)\alpha(z) = \alpha(y^{-1}(z)), \alpha \in T_v, z \in V.$$

It can be checked that S_v is independent of the choice of \mathfrak{l} (see [6], II.10).

Proposition 5.5 ([6], III.12). *For any element $(x, \tau) \in C$, we get an irreducible unitary representation $T_{x,\tau}$ of G . Moreover, the map $(x, \tau) \mapsto T_{x,\tau}$ induces a bijection from C/G to \widehat{G} .*

Now, we can obtain the correspondence between coadjoint orbits and irreducible unitary representations of an algebraic group G (see [6], III.19, III.20).

Let $f \in \mathfrak{g}^*$, by Proposition 5.4, we get $\mathcal{O}_f = \mathcal{O}_{x,\lambda}$ for some $x \in \mathfrak{g}^*$ of unipotent type and $\lambda \in L(x) \cong (\mathfrak{g}(x)/{}^u\mathfrak{g}(x))^*$. Assume that τ is an irreducible unitary representation of $G(x)^\mathfrak{g}/{}^uG(x)$ attached to λ such that $\tau((1, -1)) = -1$, then $T_{x,\tau}$ is an irreducible unitary representation of G attached to \mathcal{O}_f .

5.2.3. Duflo's construction in the $P_n(k)$ case

Set $G = P_n(k)$, we show how to attach representations with coadjoint orbits by the above method (using the same notation).

For any element $f \in \mathfrak{g}^*$, applying the results in Section 2, we have that f is conjugate to an element $\text{pr}'(\text{diag}(A, J_m))$ for some $m \in \mathbb{Z}_+$ and $A \in \mathfrak{gl}_{n-m}(k)$. Let $x = \text{pr}'(\text{diag}(0, J_m))$, then x is of unipotent type. Actually, $\mathfrak{g}(x)$ has reductive factor

$$\mathfrak{g}' = \{\text{diag}(X, 0_{m \times m}) \mid X \in \mathfrak{gl}_{n-m}(k)\} \subset \ker(x).$$

Let $\mathfrak{b} = \left\{ \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \mid X \in \mathfrak{gl}_{n-m}(k), Y \in M_{(n-m) \times m}(k), Z \in \mathfrak{u}_m(k) \right\}$,

where $\mathfrak{u}_m(k)$ is the set of strictly upper triangular matrices over k , then \mathfrak{b} is a subalgebra of strongly unipotent type relative to x . Therefore, x is of unipotent type.

Let $\lambda' = \text{pr}'(\text{diag}(A, 0))$, then $\lambda := \lambda'|_{\mathfrak{g}(x)} \in L(x)$, since $x|_{\mathfrak{u}\mathfrak{g}(x)} = 0$ and $\lambda|_{\mathfrak{u}\mathfrak{g}(x)} = 0$.

Moreover, $L(x) \cong (\mathfrak{g}(x)/{}^u\mathfrak{g}(x))^* \cong \mathfrak{gl}_{n-m}(k)^*$.

Also it is easy to see that $\mathcal{O}_f = \mathcal{O}_{x,\lambda}$. Furthermore, we have that $G(x)^\mathfrak{g}$ is a trivial double cover of $G(x)$, that is $G(x)^\mathfrak{g} \cong G(x) \times \mathbb{Z}_2$. Actually,

$$G(x) = \left\{ \left(\begin{array}{ccc} Y & Z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{m-1} \end{array} \right) \mid Y \in \text{GL}_{n-m}(k), Z \in M_{(n-m) \times 1}(k) \right\}.$$

It is not hard to see that the image of the reductive factor

$$\{\text{diag}(Y, I_m) \mid Y \in \text{GL}_{n-m}(k)\}$$

in $\text{Sp}(\mathfrak{g}/\mathfrak{g}(x), \omega_x) \cong \text{Sp}((n - \frac{m}{2})(m - 1), k)$

has trivial double cover in the metaplectic group $\text{Mp}((n - \frac{m}{2})(m - 1), k)$. Therefore, $G(x)^\mathfrak{g}$ is a trivial double cover of $G(x)$.

To attach \mathcal{O}_f to the representation of $P_n(k)$, we assume that τ_0 is the irreducible unitary representation of

$$G(x)/{}^uG(x) \cong \text{GL}_{n-m}(k)$$

which is attached to the $\text{GL}_{n-m}(k)$ -orbit of $\lambda \in L(x) \cong \mathfrak{gl}_{n-m}(k)^*$. Define an irreducible unitary representation of $G(x)/{}^uG(x) \times \mathbb{Z}_2$, denoted by τ , such that $\tau|_{G(x)/{}^uG(x)} = \tau_0$ and $\tau((1, -1)) = -1$. Then the irreducible representation of the irreducible unitary representation attached to \mathcal{O}_f is

$$T_{x,\tau} = \text{Ind}_B^G(\tau' \otimes S_v T_v).$$

The subalgebra of unipotent type relative to x constructed above, \mathfrak{b} , is the canonical one. And the corresponding subgroup B is

$$B = \left\{ \left(\begin{array}{cc} Y & Z \\ 0 & U \end{array} \right) \mid Y \in \text{GL}_{n-m}(k), Z \in M_{(n-m) \times m}(k), U \in U_m(k) \right\}.$$

Here $U_m(k)$ is the set of upper triangular unipotent matrices over k . Moreover, $\mathfrak{v} = {}^u\mathfrak{b}$ itself is the polarization of $v = x|_{\mathfrak{v}}$, and we get a one dimensional unitary representation of V , $T_v = e^{ix|_{\mathfrak{v}}}$.

Since ω_x is trivial on \mathfrak{b} , we get $G(x)^\mathfrak{v} \cong G(x) \times \mathbb{Z}_2$ and $G(x)^\mathfrak{b} \cong G(x) \times \mathbb{Z}_2$, and therefore $S_v|_{G(x)}$ is the one dimensional trivial representation, so we have

$$\tau' \otimes S_v T_v = \tau_0 \otimes e^{ix}.$$

Therefore, the irreducible unitary representation of G attached to \mathcal{O}_f with $f = \text{pr}'(\text{diag}(A, J_m))$ is

$$T_{x,\tau} = \text{Ind}_B^G(\tau_0 \otimes e^{ix}) = \Gamma^{m-1} E \tau_0,$$

where τ_0 is an irreducible unitary representation of $\text{GL}_{n-m}(k)$ attached to the $\text{GL}_{n-m}(k)$ -orbit of $\text{pr}_{n-m}(A)$.

6. A generalization of Duflo’s conjecture

We establish a generalization of Duflo’s conjecture for the restriction of an irreducible unitary representation π of $G = \text{GL}_n(k)$ to the mirabolic subgroup $P = P_n(k)$, $k = \mathbb{R}$ or \mathbb{C} , where π is attached to a G -coadjoint orbit of $f \in \mathfrak{g}^*$ in Section 5. Set $\mathcal{O}_\pi = \mathcal{O}_f$. Let $\mathfrak{p} : \mathcal{O}_\pi \rightarrow \mathfrak{p}^*$ be the moment map. Then we have the following theorem.

Theorem 6.1. *There are only finitely many P -orbits in $\mathfrak{p}(\mathcal{O}_\pi)$, including a unique open P -orbit Ω in $\mathfrak{p}(\mathcal{O}_\pi)$. Moreover,*

- (1) *the moment map $\mathfrak{p} : \mathcal{O}_\pi \rightarrow \mathfrak{p}^*$ is proper over Ω ,*
- (2) *the restriction of π to P , $\pi|_P$ is irreducible, and is attached to Ω in the sense of Duflo,*
- (3) *the reduced space of Ω (with respect to the moment map \mathfrak{p}) is a single point.*

Proof. By Theorem 3.15, Theorem 3.18, we get (1),(3) immediately. To get (2), we firstly get $\pi|_P$ from section 4 and get $\Omega = \mathfrak{p}(g_{I^o} \cdot f)$ from the results of moment maps in Section 3. It remains to verify if $\pi|_P$ is attached to Ω by Duflo’s construction in Section 5. We verify one example and one can verify the other cases easily.

Let $1 \leq t_p \leq \dots \leq t_1 \in \mathbb{Z}_+$ such that $\sum_{i=1}^p t_i = n$. Let π be the unipotent representation

$$1|_{G_{t_1}(\mathbb{C})} \times \dots \times 1|_{G_{t_p}(\mathbb{C})}$$

of $G_n(\mathbb{C})$, which is attached to the coadjoint orbit of $f = \text{pr}(\xi)$ with

$$\xi = \text{diag}(J_{k_1}^{l_1}, \dots, J_{k_r}^{l_r}),$$

where $k_i, l_i \in \mathbb{Z}_+$ for $1 \leq i \leq r$, and $\{\underbrace{k_1, \dots, k_1}_{l_1}, \dots, \underbrace{k_r, \dots, k_r}_{l_r}\}$ is the dual partition of $\{t_1, \dots, t_p\}$ (so $k_1 \geq \dots \geq k_r$). Then

$$\pi|_P = I^p E(1|_{G_{t_1-1}(\mathbb{C})} \times \dots \times 1|_{G_{t_p-1}(\mathbb{C})}).$$

For obtaining the unique dense open P -orbit in $\mathfrak{p}(\mathcal{O}_f)$, we define $\tilde{f} = \text{pr}(\tilde{\zeta})$ with $\tilde{\zeta} = \text{diag}(J_{k_r}^{l_r}, \dots, J_{k_1}^{l_1})$. Then \tilde{f} is $G_n(\mathbb{C})$ -conjugated to f , so $\mathcal{O}_f = \mathcal{O}_{\tilde{f}}$. By Theorem 3.15, we have $\Omega = P \cdot \mathfrak{p}(g_{I^o} \cdot \tilde{f})$ is the unique dense open P -orbit in $\mathfrak{p}(\mathcal{O}_f)$. Since $I^o = \{n\}$ and $g_{I^o} = I_n$, we have $g_{I^o} \cdot \tilde{f} = \tilde{f}$. It is easy to see that $\mathfrak{p}(\tilde{f})$ is P -conjugated to the element $\text{pr}'(\zeta)$ with

$$\zeta = \text{diag}(J_{k_1}^{l_1-1}, J_{k_2}^{l_2}, \dots, J_{k_r}^{l_r}, J_{k_1}),$$

So we have $\Omega = P \cdot \text{pr}'(\zeta)$. Based on Duflo’s construction, we get that the irreducible unitary representation of P attached to Ω is

$$I^{k_1} E(1|_{G_{s_1}(\mathbb{C})} \times \dots \times 1|_{G_{s_q}(\mathbb{C})}),$$

where $\{s_1, \dots, s_q\}$ is the dual partition of $\{\underbrace{k_1, \dots, k_1}_{l_1-1}, \underbrace{k_2, \dots, k_2}_{l_2}, \dots, \underbrace{k_r, \dots, k_r}_{l_r}\}$.

By definition of dual partition, we have $p = q$, $s_i = t_i - 1$ for $1 \leq i \leq p$, and $k_1 = p$. Hence we get that $\pi|_P$ is attached to Ω . ■

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