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# Generic 1-Connectivity of Flag Domains in Hermitian Symmetric Spaces

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**Abstract.** A flag domain is an open real group orbit in a complex flag manifold. It has been shown that a flag domain is either pseudoconvex or pseudoconcave. Moreover, generically 1-connected flag domains are pseudoconcave. In this study, for flag domains contained in irreducible Hermitian symmetric spaces of type *AIII* or *CI*, we determine which pseudoconcave flag domain is generically 1-connected.

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*Key Words:* Flag domain, Hermitian symmetric space, Weyl group.

## 1. Introduction

Let $G$ be a connected complex semisimple Lie group, and let $G_0$ be a real form of $G$. An open $G_0$-orbit in a $G$-flag manifold is called a flag domain. For example, a Hermitian symmetric domain is a flag domain. According to [8, 6], a flag domain is either pseudoconvex or pseudoconcave. A pseudoconvex flag domain, such as a Hermitian symmetric domain, possesses plenty of global functions. In contrast, any global function on a pseudoconcave flag domain is constant. In this study, we investigate pseudoconcave flag domains, focusing on *generic* $1$*-connectivity.*

Let $K_0$ be a maximally compact subgroup of $G_0$. By the Matsuki duality [11], $G_0$-orbits correspond to $K$-orbits with complexification $K$ of $K_0$. Through this correspondence, a flag domain $D$ contains a compact submanifold called the base cycle of $D$. The base cycle and its $G$-transformations play an important role in the study of pseudoconcave flag domains. Let us choose a base point $z$ in the base cycle. The isotropy subgroup $Q_z$ at $z$ is a parabolic subgroup, and we have a unique open $Q_z$-orbit in the ambient flag manifold $G(z)$. We say that $D$ is generically 1-connected if the open $Q_z$-orbit intersects with the base cycle. Huckleberry [9] showed that a flag domain is pseudoconcave if it is generically 1-connected. He also showed that $D$ is generically 1-connected if $K$ is a simple Lie group. For example, all $SL(n,\mathbb{R})$-flag domains are generically 1-connected. However, the following problem is still open: *are all pseudoconcave flag domains generically* $1$*-connected?* In this study, we provide an answer for flag domains contained in irreducible compact Hermitian symmetric spaces of type *AIII* or *CI*.

All flag domains contained in the Hermitian symmetric space under consideration correspond to the signature, and our results indicate that generic 1-connectivity

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depends on the numerical condition of the signature. In the case of type *CI*, where $G_0 = Sp(2n, \mathbb{R})$, few flag domains are generically 1-connected: the Hermitian symmetric space contains $(n+1)$ flag domains, of which $(n-1)$ are pseudoconcave, and at most one of them is generically 1-connected. In contrast, in the case of type *AIII*, where $G_0 = SU(p,q)$, more than one flag domain can be generically 1-connected: almost half of the flag domains under consideration are generically 1-connected if $2p < q$. We prove these by using combinatorics of the Weyl groups and their action on the roots. Moreover, we consider the generic 1-connectivity of a certain type of flag domain fibered over the flag domains in the Hermitian symmetric spaces.

## 2. Cycle Connectivity of Flag Domains

In this section, we review pseudoconcavity, cycle connectivity, and generic 1-connectivity. Subsequently, we present combinatorial conditions that are equivalent to generic 1-connectivity.

### 2.1. Pseudoconcavity

Let $X$ be a connected complex manifold. Andreotti [1] defined pseudoconcavity as follows:

**Definition 2.1.** $X$ is pseudoconcave if we can find a relatively compact open subset $Y \subset X$ such that at every point $z \in \mathrm{bd}\,(Y)$, a holomorphic map $\rho$ on the unit disk $\mathbb{D}$ to $c\ell\,(Y)$ satisfying $\rho(0) = z$ and $\mathrm{bd}\,(\rho(\mathbb{D})) \subset Y$ exists.

This definition is weaker than the definition of $q$*-pseudoconcavity* in [2], where a smooth exhaustion is required for the definition. Similar to the finiteness theorem of [2] for higher cohomologies of $q$-pseudoconcave manifolds, we have a weaker version of the finiteness theorem:

**Proposition 2.2** ([1]). *If $X$ is pseudoconcave, then any global function on $X$ is constant, and $\dim_{\mathbb{C}} H^0(X, \mathcal{F}) < \infty$ for any coherent sheaf $\mathcal{F}$.*

To prove this finiteness theorem, the maximum principle works essentially.

**Remark 2.3.** Higher cohomologies of a pseudoconcave flag domain have a significant meaning in several aspects. In Hodge theory, Green et al. [5] studied them with specific Mumford-Tate domains in connection with automorphic cohomology. In representation theory, higher cohomologies give a geometric realization of Zuckerman derived functor modules, see [10] and references therein.

### 2.2. Cycle connectivity

Let $G$ be a connected complex Lie group. For a $G$-flag manifold $Z$, we fix a base point $z \in Z$. Then, $Z \cong G/Q_z \cong G(z)$, where $Q_z$ is the parabolic subgroup that stabilizes $z$. Let $\mathfrak{g}_0$ be a real form of the Lie algebra $\mathfrak{g}$ of $G$, and let $\tau$ be the associated complex conjugation. The $\tau$-invariant complex subspace $\mathfrak{q}_z \cap \tau\mathfrak{q}_z$ contains a $\tau$-stable Cartan subalgebra $\mathfrak{h}$, where $\mathfrak{q}_z$ is the Lie algebra of $Q_z$. For a $\mathfrak{h}$-root system $\Sigma$ of $\mathfrak{g}$, we choose a positive root system $\Sigma^+$ such that $\mathfrak{q}_z$ contains the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. Let $\Psi$ be the simple root system corresponding to $\Sigma^+$.

For a subset $\Phi \subset \Psi$, we define

$$\Phi_{\mathrm{r}} = \Big\{ \sum_{\psi \in \Psi} \epsilon_\psi \psi \in \Sigma \mid \epsilon_\psi = 0 \text{ whenever } \psi \notin \Phi \Big\}$$
$$\Phi_{\mathrm{n}}^{\pm} = \{\alpha \in \pm\Sigma^+ \mid \alpha \notin \Phi_{\mathrm{r}}\}.$$

We may choose $\Phi$ such that $\Sigma(\mathfrak{q}_z) = \{\alpha \in \Sigma \mid \mathfrak{g}_\alpha \subset \mathfrak{q}_z\} = \Phi_{\mathrm{r}} \cup \Phi_{\mathrm{n}}^+$. Here $\Phi_{\mathrm{r}}$ (resp. $\Phi_{\mathrm{n}}^+$) is reductive (resp. nilpotent) part of $\Sigma(\mathfrak{q}_z)$.

Let $G_0$ be the real form of $G$ corresponding to $\mathfrak{g}_0$. By [3, 12], $G_0$-orbits in $Z$ are finitely many, and there is an open orbit. An open $G_0$-orbit is called a flag domain. Suppose that $D = G_0(z)$ is a flag domain. Let $\theta$ be a Cartan involution that commutes with $\tau$. Then, we may assume that $\mathfrak{h}$ and $\Sigma^+$ satisfy the following conditions (see [12, Theorem 4.5]):

- $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$ is a $\theta$-stable maximally compact Cartan subalgebra of $\mathfrak{g}_0$;
- $\tau\Sigma^+ = -\Sigma^+$.

For the compact subgroup $K_0 = G_0^\theta$ and its complexification $K$, $K_0$-orbit $K_0(z)$ coincides with the $K$-orbit, and $C_0 = K_0(z) = K(z)$ is a complex compact manifold (see [4, Theorem 4.3.1]). Here $C_0$ is called the base cycle.

For any point $x, y \in D$, we write $x \sim y$ if there exists $C_i = g_i(C_0) \subset D$ with $g_i \in G$ such that $x \in C_1$ and $y \in C_N$, where the chain $C_1 \cup \cdots \cup C_N$ is connected. The relation $\sim$ is an equivalence relation, and $D/\sim$ is classified into two types:

**Proposition 2.4** ([9]). *$D/\sim$ is either a Hermitian symmetric domain or a point. In the former case, $D$ is pseudoconvex. In the latter case, we say $D$ is cycle connected.*

Because a holomorphic function on $D$ is factored as $D \to D/\sim \to \mathbb{C}$, the flag domain $D$ is cycle connected if and only if any global function on $D$ is constant. Moreover, by Proposition 2.2, pseudoconcave flag domains are cycle connected. Rather, the following theorem holds.

**Theorem 2.5** ([8, 6]). *A flag domain $D$ is cycle connected if, and only if, $D$ is pseudoconcave.*

### 2.3. Generic 1-connectivity

Let $W = W(G, H)$ be the Weyl group with the Cartan subgroup $H = \exp(\mathfrak{h})$, and let $W_\Phi$ be the subgroup generated by the simple reflections associated with $\Phi$. By the Bruhat decomposition, $Q_z$-orbits in $Z$ are parameterized by $W_\Phi \backslash W / W_\Phi$, and there is a unique open $Q_z$-orbit $\mathcal{O}$.

**Definition 2.6.** A flag domain $D$ is generically 1-connected if $C_0 \cap \mathcal{O} \neq \emptyset$.

The preimage of the base point under $D \to D/\sim$ contains an open subset if $D$ is generically 1-connected. Then, by Proposition 2.4, we have the following corollary:

**Corollary 2.7.** *Generically 1-connected flag domains are cycle connected (or equivalently pseudoconcave).*

The above corollary implies generic 1-connectivity is a kind of cycle connectivity. While cycle connectivity guarantees that any two points are connected by a chain of cycles of finite length, generic 1-connectivity ensures that any point in $\mathcal{O}$ is connected to the base point by a chain of length 1. In fact, if $D$ is generically 1-connected, any point in $\mathcal{O}$ is written as $g(z')$ with $z' \in C_0 \cap \mathcal{O}$ and $g \in Q_z$. Then both $z$ and $g(z')$ are contained in $g(C_0)$.

Now $K$ is a reductive subgroup of $G$. Because $C_0$ is a projective variety, $Q_z^K = K \cap Q_z$ is a parabolic subgroup. Then $C_0$ can be decomposed into the disjoint union of $Q_z^K$-orbits. Let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be the $\theta$-stable decomposition, where $\mathfrak{t}_0 = \mathfrak{h}_0 \cap \mathfrak{k}_0$ and $\mathfrak{a}_0 = \mathfrak{h}_0 \cap \mathfrak{p}_0$ with the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$. Because $\mathfrak{h}_0$ is maximally compact, $\mathfrak{t}_0$ is a maximal abelian subalgebra of $\mathfrak{k}_0$. Let $W_K = W(K_0, T_0)$ be the Weyl group with the maximal torus $T_0 = \exp(\mathfrak{t}_0)$. Then, each $Q_z^K$-orbit in $C_0$ is the orbit at $w(z)$ with some $w \in W_K$. Let $w_0^K$ be the longest element in $W_K$ with respect to the simple root system corresponding to the Borel subgroup contained in $Q_z^K$. Then the $Q_z^K$-orbit at $w_0^K(z)$ is open.

**Proposition 2.8.** *The following conditions are equivalent:*

(1) *$D$ is generically 1-connected;*

(2) $w_0^K(z) \in \mathcal{O}$*;*

(3) $w_0^K(\Phi_{\mathrm{n}}^-) \cap \Phi_{\mathrm{n}}^- = \emptyset$.

**Proof.** First, we show the equivalence between (1) and (2). If $D$ is not generically 1-connected, $\mathcal{O} \cap C_0 = \emptyset$, and thus $w(z) \notin \mathcal{O}$ for all $w \in W_K$. Contrastingly, if $D$ is generically 1-connected, $\mathcal{O} \cap C_0$ is a $Q_z^K$-invariant subset that must contain the open $Q_z^K$-orbit in $C_0$. Hence $w_0^K(z) \in \mathcal{O}$.

Next, we show the equivalence between (2) and (3). We write $z' = w_0^K(z)$. Since $\Sigma = w_0^K(\Sigma(\mathfrak{q}_z) \cup \Phi_{\mathrm{n}}^-) = \Sigma(\mathfrak{q}_{z'}) \cup w_0^K(\Phi_{\mathrm{n}}^-)$, we have

$$\begin{aligned}\dim(Q_z(z')) &= |\Sigma(\mathfrak{q}_z) \cap w_0^K(\Phi_{\mathrm{n}}^-)| = |w_0^K(\Phi_{\mathrm{n}}^-)| - |\Phi_{\mathrm{n}}^- \cap w_0^K(\Phi_{\mathrm{n}}^-)| \\ &= \dim D - |\Phi_{\mathrm{n}}^- \cap w_0^K(\Phi_{\mathrm{n}}^-)|.\end{aligned}$$

Then $Q_z(z')$ is open if, and only if, $w_0^K(\Phi_{\mathrm{n}}^-) \cap \Phi_{\mathrm{n}}^- = \emptyset$. ■

For the longest element $w_0 \in W$, the $Q_z$-orbit at $w_0(z)$ is open. Moreover, any element $w \in W$ such that $w(z)$ contained in $\mathcal{O}$ is written as $w = w_1 w_0 w_2$ with $w_1, w_2 \in W_\Phi$. If $\mathfrak{h}_0$ is compact, that is, $\mathfrak{h}_0 = \mathfrak{t}_0$, then $W_K$ is a subset of $W$. In this case, $w_0^K$ is written as $w_0^K = w_1 w_0 w_2$ if and only if $w_0^K(z) \in \mathcal{O}$. By Proposition 2.8, we have the following corollary:

**Corollary 2.9.** *In the case where $\mathfrak{h}_0$ is compact, $D$ is generically 1-connected if, and only if, there exists $w_1, w_2 \in W_\Phi$ such that $w_1 w_0^K w_2$ is the longest element in $W$.*

## 3. Flag domains in Hermitian symmetric spaces

In this section, we suppose that $Z$ is an irreducible Hermitian symmetric space of compact type. We then have the dual Hermitian symmetric domain $G_0/K_0$. To state our result, we review the root structure of $\mathfrak{g}$. Let $\mathfrak{h}_0$ be a maximal abelian subalgebra of $\mathfrak{k}_0$. We can choose a simple $\mathfrak{h}$-root system $\{\psi_1, \dots, \psi_n\}$ such that only one root is noncompact and compact otherwise. We suppose $\psi_m$ is noncompact.

Then, the set $\Sigma$ of roots can be decomposed into $\Sigma = \Sigma_{\mathrm{c}} \cup \Sigma_{\mathrm{nc}}^{+} \cup \Sigma_{\mathrm{nc}}^{-}$, where

$$\Sigma_{\mathrm{c}} = \Big\{ \sum \epsilon_i \psi_i \mid \epsilon_m = 0 \Big\}, \quad \Sigma_{\mathrm{nc}}^{\pm} = \Big\{ \sum \epsilon_i \psi_i \mid \epsilon_m = \pm 1 \Big\}.$$

Let $\mathfrak{p}^{\pm} = \sum_{\alpha \in \Sigma_{\mathrm{nc}}^{\pm}} \mathfrak{g}_\alpha$ and $P^{\pm} = \exp(\mathfrak{p}^{\pm})$. Then, we have $Z \cong G/KP^{+}$, and the Hermitian symmetric domain $G_0/K_0$ is regarded as the $G_0$-orbit at the identity coset $z_0 \in Z$.

In the Hermitian symmetric space $Z$, all $G_0$-orbits are related by the Cayley transforms. Choosing a maximal set $\Xi = \{\xi_1, \dots, \xi_r\} \subset \Sigma_{\mathrm{nc}}^{+}$ of strongly orthogonal roots with $r = \operatorname{rank}_{\mathbb{R}} \mathfrak{g}_0$, the partial Cayley transform $c_\xi$ and the product $c_\Gamma = \prod_{\xi \in \Gamma} c_\xi$ is constructed from $\xi \in \Gamma \subset \Xi$ (see [12, 13] for this construction). For disjoint subsets $\Gamma, \Delta \subset \Xi$, we define $z_{\Gamma,\Delta} = c_\Gamma c_\Delta^2(z_0)$. By [12, 13], the following properties hold:

- Every $G_0$-orbit on $Z$ is written as $G_0(z_{\Gamma,\Delta})$ with some $\Gamma, \Delta \subset \Xi$;
- $G_0(z_{\Gamma,\Delta}) = G_0(z_{\Gamma',\Delta'})$ if and only if $|\Gamma| = |\Gamma'|$ and $|\Delta| = |\Delta'|$;
- $G_0(z_{\Gamma,\Delta})$ is open if and only if $\Gamma = \emptyset$

Then, any flag domain in $Z$ is written as $G_0(z_{\emptyset,\Delta})$, and it depends on the cardinality of $\Delta$. We choose $\Delta$ as the set $\{\xi_1, \dots, \xi_a\}$ with $1 \le a \le r$. The square $c_\Delta^2$ of the partial Cayley transform is $s_\Delta = \prod_{1 \le i \le a} s_i$ with the reflection $s_i$ with respect to $\xi_i \in \Delta$. We set $z_a = s_\Delta(z_0)$ as a base point. Then the $G_0$-orbit $D_a = G_0(z_a)$ is a flag domain in $Z$. We denote by $\mathfrak{q}_a$ the parabolic subalgebra at $z_a$. Here, $\mathfrak{q}_a = s_\Delta(\mathfrak{k} + \mathfrak{p}_+)$, which contains the Borel subalgebra corresponding to the simple root system $\Psi = \{s_\Delta(\psi_1), \dots, s_\Delta(\psi_n)\}$. The set $\Sigma(\mathfrak{q}_a)$ of roots is decomposed as

$$\Sigma(\mathfrak{q}_a) = \Phi_{\mathrm{r}} \cup \Phi_{\mathrm{n}}^{+} \quad \text{with } \Phi = \Psi \setminus \{s_\Delta(\psi_m)\}.$$

Now $\Phi_{\mathrm{n}}^{-}$ is decomposed as $\Phi_{\mathrm{n}}^{-} = (\Phi_{\mathrm{n}}^{-} \cap \Sigma_{\mathrm{c}}) \cup (\Phi_{\mathrm{n}}^{-} \cap \Sigma_{\mathrm{nc}})$. Using the longest element $w_K^0$, we have $w_K^0(\Phi_{\mathrm{n}}^{-} \cap \Sigma_{\mathrm{c}}) \subset \Sigma_{\mathrm{c}}$ and $w_K^0(\Phi_{\mathrm{n}}^{-} \cap \Sigma_{\mathrm{c}}) \cap (\Phi_{\mathrm{n}}^{-} \cap \Sigma_{\mathrm{c}}) = \emptyset$. Then, to show generic 1-connectivity, Proposition 2.8 is simplified as follows:

**Corollary 3.1.** *$D_a$ is generically 1-connected if and only if the following condition is satisfied: $w_0^K(\Phi_{\mathrm{n}}^{-} \cap \Sigma_{\mathrm{nc}}) \cap (\Phi_{\mathrm{n}}^{-} \cap \Sigma_{\mathrm{nc}}) = \emptyset$.*

The Hermitian symmetric space with a classical group $G$ can be classified into four types (see [7, Chapter VII] for details): *AIII*, *DIII*, *BDI*, and *CI*. We consider the 1-connectivity of $D_a$ in the cases of type *AIII* and *CI*.

### 3.1. Case for type *CI*

We fix a symplectic form $\omega$ on $\mathbb{R}^{2n}$, and let $G_0 = Sp(2n, \mathbb{R})$ be the subgroup of $SL(2n, \mathbb{R})$, leaving invariant this form. We have a basis $\{f_i\}_i$ that satisfies $\sqrt{-1}\omega(v, \bar{w}) = \sum_{i \le n}(v_i w_i - v_{n+i} w_{i+i})$ for $v = \sum v_i f_i$ and $w = \sum w_i f_i$. Using this basis, we may regard $G_0$ as $U(n,n) \cap G$.

The Grassmannian $Z$ of $\omega$-isotropic $n$-planes in $\mathbb{C}^{2n}$ is a $G$-flag manifold, and all open $G_0$-orbits correspond to pairs of numbers of positive and negative signatures of the associated Hermitian form. Let

$$z_a = \operatorname{Span}\{f_i \mid a < i \le n + a\} \in Z$$

for $0 \le a \le n$, where $\operatorname{sgn}(z_a) = (n - a, a)$. Then, any flag domain can be written as

$D_a = G_0(z_a)$, and both $D_0$ and $D_n$ are Hermitian symmetric domains, that is, each one is the Siegel upper (or lower) half space.

**Theorem 3.2.** *An $Sp(2n,\mathbb{R})$-flag domain $D_a$ in the Hermitian symmetric space is generically 1-connected if and only if $2a = n$.*

For this proof, we consider the root structure and the Weyl group action. We choose $\mathfrak{k}_0 = \mathfrak{u}(2n) \cap \mathfrak{g}_0 \cong \mathfrak{u}(n)$, and let $\mathfrak{h}_0 \subset \mathfrak{u}(n)$ be the maximal torus consisting of diagonal matrices. We define $e_i \in \mathfrak{h}^*$ by $e_i(X) = a_i$ for $X = \operatorname{diag}(a_1, \ldots, a_n) \in \mathfrak{h}$. Then, we may choose the simple root system $\{\psi_1, \ldots, \psi_n\}$, where $\psi_i = e_i - e_{i+1}$ for $i < n$ and $\psi_n = 2e_n$, and we can write

$$\Sigma_{\mathrm{c}} = \Big\{ \sum \epsilon_i \psi_i \mid \epsilon_n = 0 \Big\} = \{ e_i - e_j \mid i \neq j \},$$
$$\Sigma_{\mathrm{nc}}^{\pm} = \Big\{ \sum \epsilon_i \psi_i \mid \epsilon_n = \pm 1 \Big\} = \{ \pm(e_i + e_j) \mid 1 \leq i \leq j \leq n \}$$

Now $G_0$ is the split real form, i.e. $\operatorname{rank}_{\mathbb{R}} \mathfrak{g}_0 = \dim \mathfrak{h}_0 = n$. The maximal set of noncompact orthogonal roots is $\{\xi_1, \ldots, \xi_n\}$ with $\xi_i = 2e_i$. Then $s_\Delta(e_i) = -e_i$ if $i < a$, and $s_\Delta(e_i) = e_i$ otherwise. Since $-s_\Delta(\psi_i) \in \Sigma_{\mathrm{c}} \cup \Sigma_{\mathrm{nc}}^+$, i.e. $-\psi_i \in s_\Delta(\Sigma_{\mathrm{c}} \cup \Sigma_{\mathrm{nc}}^+)$, for $1 \leq i \leq n-1$, $\Sigma(\mathfrak{q}_a) = s_\Delta(\Sigma_{\mathrm{c}} \cup \Sigma_{\mathrm{nc}}^+)$ contains the set $\{-\psi_i\}_{i \neq n}$ of simple roots of $\Sigma_{\mathrm{c}}$. By using these simple roots, $W_K$ is the symmetry group $S_n$ in terms of the permutations of the indices of $e_1, \ldots, e_n$, and the longest element in $W_K$ is

$$w_0^K = \left( \begin{smallmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{smallmatrix} \right). \tag{1}$$

**Proof of Theorem 3.2.** We apply Corollary 3.1. Since $\Phi_{\mathrm{n}}^- = s_\Delta(\Sigma_{\mathrm{nc}}^-)$, we have

$$\Phi_{\mathrm{n}}^- \cap \Sigma_{\mathrm{nc}} = \{ e_i + e_j \mid 1 \leq i \leq j \leq a \} \cup \{ -e_i - e_j \mid a+1 \leq i \leq j \leq n \}.$$

Then, $w_0^K(\Phi_{\mathrm{n}}^- \cap \Sigma_{\mathrm{nc}}) \cap (\Phi_{\mathrm{n}}^- \cap \Sigma_{\mathrm{nc}}) = \emptyset$ if and only if $2a = n$. ■

Next, we consider the generic 1-connectivity of a certain type of flag domain fibered over $D_a$ with $0 < a < n$. Let $Z_m$ be the flag manifold consisting of sequences $V_1 \subset V_2$ of $\omega$-isotropic subspaces with $0 < \dim V_1 = m \leq a < \dim V_2 = n$. We set

$$z_{m,0} = (F_m \subset F_n) \in Z_m \quad \text{where } F_r = \operatorname{Span}\{f_i\}_{i \leq r}. \tag{2}$$

We continue to assume $\Delta = \{\xi_1, \ldots, \xi_a\}$ and set $z_{m,a} = s_\Delta(z_{m,0})$ as the base point. Then, $D_{m,a} = G_0(z_{m,a})$ is the flag domain in $Z_m$, determined by

$$\operatorname{sgn}(V_1) = (0, m), \quad \operatorname{sgn}(V_2) = (n-a, a).$$

**Proposition 3.3.** *The flag domain $D_{m,a}$ is not generically 1-connected.*

**Proof.** We denote by $\mathfrak{q}_{m,a}$ the parabolic subalgebra at $z_{m,a}$.
Then, $\mathfrak{q}_{m,a}$ contains the Borel subalgebra corresponding to the simple root system $\Psi = \{s_\Delta(\psi_1), \ldots, s_\Delta(\psi_n)\}$, where $\Sigma(\mathfrak{q}_{m,a}) = \Phi_{\mathrm{r}} \cup \Phi_{\mathrm{n}}^+$ with $\Phi = \Psi \backslash \{s_\Delta(\psi_m), s_\Delta(\psi_n)\}$.

We have
$$e_1 + e_n = s_\Delta(-e_1 + e_n) = -\sum_{i=1}^{n-1} s_\Delta(\psi_i) \in \Phi_{\mathrm{n}}^-.$$

Moreover, $\Sigma(\mathfrak{q}_{m,a})$ contains the set $\{-\psi_i\}_{i \neq n}$ of simple roots of $\Sigma_{\mathrm{c}}$, and the longest element $w_0^K$ is defined in (1). Then, $w_0^K(e_1 + e_n) = e_1 + e_n$; hence, $D_{m,a}$ is not generically 1-connected by Proposition 2.8. ■

### 3.2. Case for type *AIII*

We fix a Hermitian form $\langle \bullet, \bullet \rangle$ on $\mathbb{C}^{p+q}$, and let $G_0 = SU(p,q)$ be the subgroup of $SL(p+q,\mathbb{C})$, leaving invariant this form. We may assume $p \le q$. We have a basis $\{f_i\}_i$ that satisfies $\langle v, w\rangle = \sum_{i\le p} v_i w_i - \sum_{j>p} v_j w_j$ for $v = \sum v_i f_i$ and $w = \sum w_i f_i$. The Grassmannian $Z$ of the $p$-planes in $\mathbb{C}^{p+q}$ is a $G$-flag manifold, and all open $G_0$-orbits correspond to pairs of numbers of positive and negative signatures. Let

$$z_a = \mathrm{Span}\,\{f_i \mid a < i \le p,\ p+q-a < i \le p+q\}$$

for $0 \le a \le p$, where $\mathrm{sgn}\,(z_a) = (p-a, a)$. Then, any flag domain is written as $D_a = G_0(z_a)$, and $D_0$ is the Hermitian symmetric domain $\{X \in M_{p,q}(\mathbb{C})|\ I - {}^t\bar{X}X > 0\}$.

**Theorem 3.4.** *An $SU(p,q)$-flag domain $D_a$ in the Hermitian symmetric space is generically 1-connected if and only if $p \le 2a \le q$.*

As in type *CI*, we consider the root structure and Weyl group action. We choose $\mathfrak{k}_0 = \mathfrak{su}(p,q) \cap \mathfrak{u}(p+q)$, and let $\mathfrak{h}_0 \subset \mathfrak{k}_0$ be the maximal torus consisting of diagonal matrices. Then, we may choose the simple root system $\{\psi_1, \dots, \psi_{p+q-1}\}$, where $\psi_i = e_i - e_{i+1}$. The simple root $\psi_i$ is compact if $i = p$ and is noncompact otherwise. Then we can write

$$\Sigma_{\mathrm{c}} = \left\{ \sum \epsilon_i \psi_i \mid \epsilon_p = 0 \right\} = \{e_i - e_j \mid i,j \le p \text{ or } p < i,j\},$$
$$\Sigma_{\mathrm{nc}}^{\pm} = \left\{ \sum \epsilon_i \psi_i \mid \epsilon_p = \pm 1 \right\} = \{\pm(e_i - e_j) \mid i \le p < j\}.$$

We set the maximal set $\{\xi_1, \dots, \xi_p\}$ of strongly orthogonal noncompact roots with $\xi_i = e_i - e_{p+q+1-i}$. The reflection $s_i$ with respect to $\xi_i$ is the permutation of the indices of $e_1, \dots, e_{p+q}$, which exchanges $i$ and $p+q+1-i$. Then $s_\Delta = \prod_{i=1}^a s_i$ is the permutation

$$s_\Delta = \begin{pmatrix} 1 & \cdots & a & a+1 & \cdots & p+q-a & p+q-a+1 & \cdots & p+q \\ p+q & \cdots & p+q-a+1 & a+1 & \cdots & p+q-a & a & \cdots & 1 \end{pmatrix}. \tag{3}$$

Therefore $\Sigma(\mathfrak{q}_a) = s_\Delta(\Sigma_{\mathrm{c}} \cup \Sigma_{\mathrm{nc}}^+)$ contains the set $\{-\psi_i\}_{i\ne p}$ of simple roots of $\Sigma_{\mathrm{c}}$. By using these simple roots, $W \cong S_{p+q}$ and $W_K \cong S_p \times S_q$. The longest element in $W_K$ is the permutation

$$w_0^K = \begin{pmatrix} 1 & \cdots & p & p+1 & \cdots & p+q \\ p & \cdots & 1 & p+q & \cdots & p+1 \end{pmatrix}. \tag{4}$$

**Proof of Theorem 3.4.** Similar to the proof of Theorem 3.2, we have

$$\begin{aligned} \Phi_{\mathrm{n}}^- \cap \Sigma_{\mathrm{nc}} =& \{e_i - e_j \mid 1 \le i \le a,\ p+q-a+1 \le j \le p+q\} \\ &\cup \{-e_i + e_j \mid a+1 \le i \le p,\ p+1 \le j \le p+q-a\}. \end{aligned} \tag{5}$$

Then, it is immediate to verify that $w_0^K(\Phi_{\mathrm{n}}^- \cap \Sigma_{\mathrm{nc}}) \cap (\Phi_{\mathrm{n}}^- \cap \Sigma_{\mathrm{nc}}) = \emptyset$ if and only if $p \le 2a \le q$. Hence Theorem 3.4 follows from Corollary 3.1. ■

Next, we consider the generic 1-connectivity of a certain type of flag domain fibered over $D_a$ with $p \le 2a \le q$. We define sequences $\mathbf{s} : s_1 \le \cdots \le s_{a+1}$ and $\mathbf{t} : t_1 \ge \cdots \ge t_{a+1}$ with

$$s_i = \begin{cases} i & \text{if } i \le a \\ p & \text{if } i = a+1, \end{cases} \qquad t_i = \begin{cases} p+q-i & \text{if } i \le a \\ p & \text{if } i = a+1. \end{cases}$$

Let $Z_{\mathbf{s}}$ (resp. $Z_{\mathbf{t}}$) be a flag manifold consisting of sequences $V_1 \subset \cdots \subset V_{a+1}$ with $\dim V_i = s_i$ (resp. $V_1 \supset \cdots \supset V_{a+1}$ with $\dim V_i = t_i$). We let

$$z_{\mathbf{s},0} = (F_1 \subset \cdots \subset F_a \subset F_p) \quad \text{and} \quad z_{\mathbf{t},0} = (F_{p+q-1} \supset \cdots F_{p+q-a} \supset F_p),$$

where $F_i$ is defined as in (2). We continue to assume $\Delta = \{\xi_1, \dots, \xi_a\}$ and set $z_{\mathbf{s}} = s_\Delta(z_{\mathbf{s},0})$ and $z_{\mathbf{t}} = s_\Delta(z_{\mathbf{t},0})$. Then, $D_{\mathbf{s}} = G_0(z_{\mathbf{s}})$ (resp. $D_{\mathbf{t}} = G_0(z_{\mathbf{t}})$) is the flag domain in $Z_{\mathbf{s}}$ (resp. $Z_{\mathbf{t}}$) determined by

$$\operatorname{sgn}(V_i) = \begin{cases} (0,i) & \text{if } i \le a \\ (p-a,a) & \text{if } i = a+1. \end{cases} \quad \left( \text{resp. } \operatorname{sgn}(V_i) = \begin{cases} (p-i,q) & \text{if } i \le a \\ (p-a,a) & \text{if } i = a+1. \end{cases} \right)$$

**Proposition 3.5.** *The flag domains $D_{\mathbf{s}}$ and $D_{\mathbf{t}}$ are generically 1-connected.*

**Proof.** We denote by $\mathfrak{q}_{\mathbf{s}}$ the parabolic subalgebra at $z_{\mathbf{s}}$. Then, $\mathfrak{q}_{\mathbf{s}}$ contains the Borel subalgebra corresponding to the simple root system $\Psi = \{s_\Delta(\psi_i)\}_{1 \le i \le p+q-1}$, and $\Sigma(\mathfrak{q}_{\mathbf{s}}) = \Phi_{\mathrm{r}} \cup \Phi_{\mathrm{n}}^+$ with $\Phi = \Psi \setminus \{s_\Delta(\psi_1), \dots, s_\Delta(\psi_a), s_\Delta(\psi_p)\}$. By the permutation (3), we have

$$\begin{aligned} \Phi_{\mathrm{n}}^- \cap \Sigma_{\mathrm{nc}} = {} & (\text{the right hand side of (5)}) \\ & \cup \{e_i - e_j \mid a+1 \le i \le p,\ p+q-a+1 \le j \le p+q\}. \end{aligned}$$

Moreover, $\mathfrak{q}_{\mathbf{s}}$ contains the set $\{-\psi_i\}_{i \ne p}$ of simple roots of $\Sigma_{\mathrm{c}}$, and the longest element is the permutation (4). Then, we have $w_0^K(\Phi_{\mathrm{n}}^- \cap \Sigma_{\mathrm{nc}}) \cap (\Phi_{\mathrm{n}}^- \cap \Sigma_{\mathrm{nc}}) = \emptyset$ and $D_{\mathbf{s}}$ is generically 1-connected.

For the proof of $D_{\mathbf{t}}$ the set $\Phi$ is replaced by

$$\Psi \setminus \{s_\Delta(\psi_{p+q-1}), \dots, s_\Delta(\psi_{p+q-a}), s_\Delta(\psi_p)\}.$$

Then, $\Phi_{\mathrm{n}}^- \cap \Sigma_{\mathrm{nc}}$ is written as

$$(\text{the right hand side of (5)}) \cup \{e_i - e_j \mid 1 \le i \le a,\ p+1 \le j \le p+q-a\}.$$

Therefore, $D_{\mathbf{t}}$ is generically 1-connected similarly as in the case for $D_{\mathbf{s}}$. ∎

Let $\mathbf{s}'$ (resp. $\mathbf{t}'$) be a subsequence of $\mathbf{s}$ (resp. $\mathbf{t}$), and we define the subsequences $z_{\mathbf{s}'}$ (resp. $z_{\mathbf{t}'}$) of $z_{\mathbf{s}}$ (resp. $z_{\mathbf{t}}$), as described above. Then, we have the flag domain $D_{\mathbf{s}'} = G_0(z_{\mathbf{s}'})$ and $D_{\mathbf{t}'} = G_0(z_{\mathbf{t}'})$, which compose the fibration

$$D_{\mathbf{s}} \to D_{\mathbf{s}'} \to D_a \leftarrow D_{\mathbf{t}'} \leftarrow D_{\mathbf{t}}$$

Because the parabolic subalgebra at $z_{\mathbf{s}'}$ (resp. $z_{\mathbf{t}'}$) contains $\mathfrak{q}_{\mathbf{s}}$ (resp. $\mathfrak{q}_{\mathbf{t}}$), the above proposition and Proposition 2.8 imply the following corollary:

**Corollary 3.6.** *The flag domains $D_{\mathbf{s}'}$ and $D_{\mathbf{t}'}$ are generically 1-connected.*

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