The Cortex of Nilpotent Lie Algebras of Dimensions Less or Equal to 7 and Semi-Direct Product of Vector Groups: Nilpotent Case

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Abstract.   The paper deals with the cortex of real nilpotent Lie algebras. We first show that for any real nilpotent Lie algebra $\mathfrak{g}$ of dimension less or equal to 6, its cortex coincides with the set of the common zeros of the G-invariant polynomials on $\mathfrak{g}^\star$ namely the I-cortex, where G is the corresponding connected and simply connected Lie group and $\mathfrak{g}^\star$ is its dual. Next we give an example of 7-dimensional (real) nilpotent Lie algebra for which the cortex is a proper semi-algebraic set in the I-cortex. Finally we study the cortex of a class of nilpotent Lie groups given by a semi-direct product of abelian groups $G := \mathbb{R}^m \rtimes_\pi V$ where π is the continuous representation of $\mathbb{R}^n$ on the m-dimensional (real) vector space V defined by $\pi(t_1, \dots, t_n) = \exp\left(\sum_{i=1}^n t_i A_i\right)$ with $\{A_1, \dots, A_n\}$ is a set of pairwise commuting nilpotent matrices in $\mathbb{R}^{m \times m}$.

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1.   Introduction

Let G be a locally compact, the set $\widehat{G}$ of the equivalence classes of the irreducible unitary representations of G is naturally equipped with the Fell topology. It is well known that this topology is generally not Hausdorff, especially, there are classes $[\pi]$ of unitary representations which are not separated to the class $[\mathbf{1}_G]$ of the trivial representation. By definition, the cortex of the group G is the set of all such classes of representations.

Suppose now that $G = \exp \mathfrak{g}$ is an exponential Lie group, with Lie algebra $\mathfrak{g}$. Then $\widehat{G}$ is homeomorphic to the set $\mathfrak{g}^\star / \mathrm{Ad}^\star G$ of coadjoint orbits in the dual $\mathfrak{g}^\star$ of $\mathfrak{g}$, equipped with the quotient topology. Using this identification, we can see the cortex of G as the set of orbits which are not separated to the trivial orbit $\{0\}$. For simplicity, in [6], the authors define the cortex of $\mathfrak{g}^\star$ as the union of these orbits. In other words, the cortex of $\mathfrak{g}^\star$ is the set of points y of $\mathfrak{g}^\star$ which are limit of a sequence $x^{(p)} = \mathrm{Ad}^\star s_p \ell^{(p)}$, where, for each p, s_p belongs to $G = \exp \mathfrak{g}$, $\ell^{(p)}$ to $\mathfrak{g}^\star$ and $\lim_p \ell^{(p)} = 0$:

$$\mathrm{Cor}(\mathfrak{g}^\star) = \left\{ y \in \mathfrak{g}^\star : \ y = \lim_p \mathrm{Ad}^\star_{s_p} \ell^{(p)}, \ \lim_p \ell^{(p)} = 0 \right\}.$$

Motivated by this situation, the authors in [8] generalize this notion and define the cortex $\mathrm{Cor}(V) = C_V(G)$ of a representation of a locally compact group G on a finite-dimensional vector space V as the set of all $v \in V$ for which $G.v$ and $\{0\}$ cannot be Hausdorff-separated in the orbit-space V/G. They give a precise description of $C_V(G)$ in the case $G = \mathbb{R}$.

In fact the cortex of V (or $\mathfrak{g}^\star$) is generally not easy to determine and describe, even if $G = V \rtimes H$ is a nilpotent, connected and simply connected Lie group. From now one, we restrict ourselves to this nilpotent situation.

For this reason, in [6] and [8], the authors consider another subset of V (or $\mathfrak{g}^\star$), called the I-Cortex. By definition, the I-Cortex of V is the set of zero of all the H-invariant polynomial functions on V:

$$\mathrm{ICor}(V) = \big\{y \in V : \ P(y) = 0 \text{ for any invariant polynomial function } P\big\}.$$

It is easy to see that $\mathrm{Cor}(V)$ is a closed subset of $\mathrm{ICor}(V)$, and if $G = V \rtimes H$ is nilpotent, and this last set is a projective subvariety of V.

Moreover, when G is nilpotent, connected and simply connected, $\mathrm{ICor}(V)$ is really easier to describe. However, in [6] the authors presented an example of a 8-dimensional, simply connected nilpotent Lie group G for which $\mathrm{Cor}(\mathfrak{g}^\star) \subsetneq \mathrm{ICor}(\mathfrak{g}^\star)$. In this example $\mathrm{Cor}(\mathfrak{g}^\star)$ is a projective variety in $\mathfrak{g}^\star$.

On the other hand, in [5], by a proof based on a computation of the Pukànszky parametrization of the generic coadjoint orbits, the author proves that the equality $\mathrm{Cor}(\mathfrak{g}^\star) = \mathrm{ICor}(\mathfrak{g}^\star)$ holds for any nilpotent Lie algebra $\mathfrak{g}$ such that $\dim \mathfrak{g} \le 6$.

These results bring up two natural questions

1. Is the equality $\mathrm{Cor}(\mathfrak{g}^\star) = \mathrm{ICor}(\mathfrak{g}^\star)$ holding for every nilpotent 7-dimensional Lie algebra?

2. If this equality does not hold, is it however true that the cortex of a nilpotent Lie algebra is an algebraic set?

Recall that the dimension 7 is the minimal dimension for which there are continuous series of non-equivalent nilpotent Lie algebras.

In a previous work (see [13]) we investigate the cortex of a class of Lie groups $G = V \rtimes_\pi \mathbb{R}^n$ where $\mathbb{R}^n$ is identified with $\exp\left(\sum_{i=1}^n \mathbb{R}A_i\right)$ with $\{A_1, \ldots, A_n\}$ is a set of pairwise commuting non-singular matrices in $\mathbb{R}^{m \times m}$ (in [4] G is called the semi direct product of the vector groups V and $\mathbb{R}^n$).

In this paper, we first concentrate ourselves to the case of the dual of the 7-dimensional nilpotent Lie algebras. We prove that the equality $\mathrm{Cor}(\mathfrak{g}^\star) = \mathrm{ICor}(\mathfrak{g}^\star)$ holds if the generic coadjoint are of dimension $0, 2$ or 6, or the dimension of the center of $\mathfrak{g}$ is greater than 1. Then we compute explicitly the sets $\mathrm{ICor}(\mathfrak{g}^\star)$ for all the remaining cases.

Then we answer by the negative to the two above questions, by computing an example of a 7-dimensional nilpotent Lie algebra $\mathfrak{g}$ for which $\mathrm{Cor}(\mathfrak{g}^\star)$ is not an algebraic set, and especially $\mathrm{Cor}(\mathfrak{g}^\star) \subsetneq \mathrm{ICor}(\mathfrak{g}^\star)$.

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In this article, we study the cortex of such a semi-direct product group $G = V \rtimes_\pi \mathbb{R}^n$ where $\mathbb{R}^n$ is identified with $\exp\left(\sum_{i=1}^n \mathbb{R}A_i\right)$ with $\{A_1, \ldots, A_n\}$ is a set of pairwise commuting nilpotent matrices in $\mathbb{R}^{m \times m}$ and π the continuous representation of $\mathbb{R}^n$ given by $\pi(t_1, \ldots, t_n) = \exp\left(\sum_{i=1}^n t_i A_i\right)$.

The paper is organized as follows: in Section 2, we recall the definition and general properties of cortex and I-cortex of Lie algebra or a vector space on which a matrix group H acts. We recall also some basic tools which will be useful for the rest of the text. In Section 3, we investigate the cortex of the nilpotent Lie algebras $\mathfrak{g}$ with $\dim \mathfrak{g} \leq 6$, we show, by a different way from [5], that the equality $\mathrm{Cor}(\mathfrak{g}^\star) = \mathrm{ICor}(\mathfrak{g}^\star)$ holds for any nilpotent (real) Lie algebra $\mathfrak{g}$ such that $\dim \mathfrak{g} \leq 6$. In Section 4, we give the list of the "interesting" 7-dimensional (real) nilpotent Lie algebras, namely the nilpotent Lie algebras with 1-dimensional center and 4-dimensional generic coadjoint orbits. Then we present the complete computation of the cortex of two examples among these Lie algebras and proving it is not algebraic for one of these two examples. Finally and in Section 5, we consider the class of nilpotent Lie groups $G = V \rtimes_\pi \mathbb{R}^n$ where V is a real vector space of dimension m and π is the continuous representation of $\mathbb{R}^n \equiv \exp\left(\sum_{i=1}^n \mathbb{R}A_i\right)$ given by $\pi(t_1, \ldots, t_n) = \exp\left(\sum_{i=1}^n t_i A_i\right)$ with $\{A_1, \ldots, A_n\}$ is a set of pairwise commuting nilpotent matrices in $\mathbb{R}^{m \times m}$.

2. Background material and notations

If G is a locally compact group, Vershik and Karpushev [26], introduce the notion of cortex of G as the set of all unitary irreducible representations of G that cannot be Hausdorff separated from the trivial representation. If G a Lie group with Lie algebra $\mathfrak{g}$, it's known that G acts on $\mathfrak{g}$ by the adjoint action denoted by Ad and on $\mathfrak{g}^\star$ by the coadjoint action denoted by $\mathrm{Ad}^\star$. More generally, if π is a finite dimensional representation of G on V, then G acts on the space $\mathbb{R}[V^\star]$ of the polynomial functions on V by:

$$(s \cdot P)(v) = P(\pi(s^{-1})v) \qquad (s \in G, \ P \in \mathbb{R}[V^\star]).$$

Following [8], we recall the following:

Definition 2.1. Let π be a continuous representation of a locally compact Lie group G on a finite (real) space W we define

$$\mathrm{C}_W(\pi) = \{w = \lim_{p \to \infty} \pi(s_p)w^{(p)} : \ \{w^{(p)}\}_p \in W, \lim_{p \to \infty} w^{(p)} = 0, \{s_p\}_p \subset G\},$$

and the cortex of invariants of π as

$$\mathrm{IC}_W(\pi) = \{w \in W : \ P(w) = P(0) \text{ for all } G - \text{invariant polynomials } P \text{ on } W\}.$$

In particular when π is the contragredient representation of G on the dual space $\mathfrak{g}^\star$ of the Lie algebra $\mathfrak{g}$ of G, we define the cortex of $\mathfrak{g}^\star$ as

$$\mathrm{Cor}(\mathfrak{g}^\star) = \{\lim_{p \to \infty} \mathrm{Ad}_{s_p}^\star(\ell^{(p)}) : \ \{s_p\}_p \subset G, \ \{\ell^{(p)}\}_p \subset \mathfrak{g}^\star \text{ with } \lim_{p \to \infty} \ell^{(p)} = 0\},$$

and the cortex of invariants

$$\mathrm{ICor}(\mathfrak{g}^\star) = \{\ell \in \mathfrak{g}^\star : P(\ell) = P(0), \quad \text{for all } G - \text{invariant polynomial } P \text{ on } \mathfrak{g}^\star\}.$$

Note that one has the inclusion $C_W(\pi) \subset IC_W(\pi)$ (see [8]), and if G is exponential non nilpotent Lie group, then it can happen that the only invariant polynomial functions on $\mathfrak{g}^\star$ are the constant functions. If σ_1, σ_2 are two continuous representations of G on W such that it exists a non-singular matrix P in $\mathbb{R}^{m \times m}$ ($\dim W = m$)

with
$$\sigma_1(g) = P\sigma_2(g)P^{-1}, \quad \forall g \in G,$$

then
$$C_W(\sigma_1) = PC_W(\sigma_2).$$

Now if U is a Zariski open set in W then by density of U one has

$$C_W(\pi) = \{\lim_{p \to \infty} \pi^\star(g_p)w^{(p)} : \{g_p\}_p \subset G, \{w^{(p)}\}_p \subset U, \text{ with } \lim_{p \to \infty} w^{(p)} = 0\}.$$

When G is nilpotent Lie group and $\mathfrak{g}^\star/\mathrm{Ad}^\star(G)$ denotes the orbit space of the coadjoint representation of G on $\mathfrak{g}^\star$, Kirillov's theory establishes a bijection between $\mathfrak{g}^\star/\mathrm{Ad}^\star(G)$ and $\widehat{G}$ (the unitary dual of G). More precisely, let $\ell \in \mathfrak{g}^\star$ and $\mathfrak{p}_\ell$ be a Pukànszky polarization at ℓ, we define the representation $\pi_{\ell,\mathfrak{p}_\ell}$ by

$$\pi_{\ell,\mathfrak{p}_\ell} := ind_{\mathcal{P}_\ell}^G \chi_\ell,$$

where $\mathcal{P}_\ell = \exp \mathfrak{p}_\ell$ and χ_ℓ is the unitary character associated with $\mathcal{P}_\ell$ given by

$$\chi_\ell(\exp X) = e^{-i<\ell,X>}, \quad X \in \mathfrak{p}_\ell,$$

then one has

Theorem 2.1. *The representation $\pi_{\ell,\mathfrak{p}_\ell}$ is an irreducible representation of G, its equivalence class $[\pi_{\ell,\mathfrak{p}_\ell}]$ does not really depend on ℓ or the choice of $\mathfrak{p}_\ell$, it depends only upon the coadjoint orbit $\mathcal{O}_\ell$ of ℓ, and each unitary irreducible representation π of G is equivalent to a representation $\pi_{\ell,\mathfrak{p}(\ell)}$.*
Moreover the mapping, called the Kirillov map:

$$K : \mathfrak{g}^\star/\mathrm{Ad}_G^\star \longrightarrow \widehat{G}, \qquad \mathcal{O}_\ell \longmapsto [\pi_{\ell,\mathfrak{p}(\ell)}] := \pi_{\mathcal{O}_\ell}$$

is a homeomorphism between $\mathfrak{g}^\star/\mathrm{Ad}_G^\star$, equipped with the quotient topology and $\widehat{G}$, with the Fell topology.

Auslander and Kostant extended the Kirillov map to the case of an exponential Lie group (see [7, 3, 17] for instance). In [17], the authors show that the Kirillov mapping is still a homeomorphism. With this in mind, we see that if G is an exponential Lie group, then the following assertions are equivalent

1. $\pi := \pi_{\ell,\mathfrak{p}_\ell}$ is not separated to the trivial representation,

2. ℓ belongs to $\mathrm{Cor}(\mathfrak{g}^\star)$.

Throughout the paper, G will always denote a connected and simply connected nilpotent Lie group with (real) Lie algebra $\mathfrak{g}$. We denote by $\mathfrak{z}$ the center of $\mathfrak{g}$ and $\mathfrak{g}^\star$ its dual space.

For any $\mathfrak{a} \subset \mathfrak{g}$, we write $\mathfrak{a}^\perp = \{\ell \in \mathfrak{g}^\star : \ell(A) = 0 \text{ for all } A \in \mathfrak{a}\}$, which is a subspace in $\mathfrak{g}^\star$.

2.1. Stratification of $\mathfrak{g}^\star$

We let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ such that $\dim \mathfrak{g} = 7$ and let

$$\mathfrak{g} = \mathfrak{g}_n \supset \mathfrak{g}_{n-1} \supset \cdots \supset \mathfrak{g}_1 \supset \mathfrak{g}_0 = \{0\}$$

be a Jordan-Hölder sequence for $\mathfrak{g}$, i.e. a decreasing sequence of ideals such that $\dim \mathfrak{g}_j = j$ and $[\mathfrak{g}, \mathfrak{g}_j] \subset \mathfrak{g}_{j-1}$ for $j = 1, \ldots, n$. For $\ell \in \mathfrak{g}^\star$ we have the skew-symmetric bilinear form B_ℓ defined on $\mathfrak{g}$ by

$$B_\ell(X, Y) = \ell([X, Y]), \quad X, Y \in \mathfrak{g}.$$

The radical of B_ℓ is equal to the Lie algebra $\mathfrak{g}^\ell$ of the stabilizer G_ℓ of ℓ:

$$\mathfrak{g}^\ell = \{X \in \mathfrak{g} : \ B_\ell(X, Y) = 0, \quad \forall Y \in \mathfrak{g}\}.$$

For $\ell \in \mathfrak{g}^\star$ we define $\mathbf{e}_\ell$ to be the set

$$\mathbf{e}_\ell = \{j = 1, \ldots, n : \ \mathfrak{g}_j \subsetneqq \mathfrak{g}_{j-1} + \mathfrak{g}^\ell\}.$$

It is easily seen that, if $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$ for $j = 1, \ldots, n$, then

$$\mathbf{e}_\ell = \{j = 1, \ldots, n : \ X_j \notin \mathfrak{g}_{j-1} + \mathfrak{g}^\ell\}.$$

The set $\mathbf{e}_\ell$ (may be empty) is called the set of jump indices of ℓ (see [1, 21]) and we know that $|\mathbf{e}_\ell| = \dim O_\ell = rank(B_\ell)$, which must be an even number; here $\mathcal{O}_\ell = \mathrm{Ad}^\star(G)\ell$ is the coadjoint orbit of ℓ.

The set $\mathcal{E} = \{\mathbf{e}_\ell, \ \ell \in \mathfrak{g}^\star\}$ has a total ordering: if $\mathbf{e}, \mathbf{e}' \in \mathcal{E}$ with $\mathbf{e} = \{e_1 < \cdots < e_{2d}\}$, $\mathbf{e}' = \{e'_1 < \cdots < e'_{2d'}\}$, we say that $\mathbf{e} \prec \mathbf{e}'$ if:

$$\begin{cases} d > d', \ \ \text{or} \\ d = d' \ \ \text{and} \ \ e_r < e'_r, \ \text{where} \ r = \min\{s \ : \ e_s \neq e'_s\}. \end{cases}$$

For each $\mathbf{e} \in \mathcal{E}$, let $\Omega_\mathbf{e}$ be the set $\ \Omega_\mathbf{e} = \{\ell \in \mathfrak{g}^\star \ : \ \mathbf{e}_\ell = \mathbf{e}\}$.

Each $\Omega_\mathbf{e}$ is a G-invariant algebraic set, the collection $\{\Omega_\mathbf{e}\}$ constitutes a partition of $\mathfrak{g}^\star$, and for each $\mathbf{e}$, the set $\bigcup_{\mathbf{e}' \succeq \mathbf{e}} \Omega_{\mathbf{e}'}$ is a Zariski-open subset of $\mathfrak{g}^\star$. Let $\mathbf{e} \in \mathcal{E}$ with $\mathbf{e} \neq \emptyset$, then set $\mathbf{e} = \{e_1 < i_2 < \cdots < e_{2d}\}$, and let $\ell \in \Omega_\mathbf{e}$. We define the skew-symmetric matrix $M_\mathbf{e}(\ell)$ by

$$M_\mathbf{e}(\ell) = \Big(B_\ell\big([X_{e_r}, X_{e_s}]\big)\Big)_{1 \leq r, s \leq 2d},$$

and let $P_\mathbf{e}(\ell)$ denotes its Pfaffian. Note that if $\mathbf{e} = \emptyset$ we set $P_\mathbf{e}(\ell) = 1$. Each layer $\Omega_\mathbf{e}$ in $\mathfrak{g}$ is given by (see [21])

$$\Omega_\mathbf{e} = \{\ell \in \mathfrak{g}^\star; \ P_\mathbf{e}(\ell) \neq 0 \ \text{and} \ P_{\mathbf{e}'}(\ell) = 0 \ \text{for} \ \mathbf{e}' \prec \mathbf{e}\}.$$

For $\mathbf{e} \in \mathcal{E}$, let $T_\mathbf{e} = \{\ell \in \mathfrak{g}^\star : \ell(X_j) = 0 \ \ \forall j \in \mathbf{e}\}$, as it is well known, the set

$$\Sigma_\mathbf{e} = \Omega_\mathbf{e} \cap T_\mathbf{e} = \{\ell \in \Omega_\mathbf{e} : \ \ell(X_j) = 0, \quad j \notin \mathbf{e}\},$$

is a cross-section to the coadjoint orbits in $\Omega_\mathbf{e}$. Each coadjoint orbit $\mathcal{O}_\ell$ in $\Omega_\mathbf{e}$ meets $\sum_\mathbf{e}$ in a unique and single point called the fundamental element.

In [25], M. Vergne constructs canonical coordinates for generic coadjoint orbits in the dual of a nilpotent Lie algebra. Let $\Omega_{\mathbf{e}}$ be the minimal coarse Zariski-open. As it is well-known, the projection onto the cross-section is given by rational functions $(\lambda_i), i \notin \mathbf{e}$, which are regular on $\Omega_{\mathbf{e}}$ and each λ_i is of the form

$$\lambda_i(\ell) = \ell_i + f_i(\ell_1, \ldots, \ell_{i-1}), \quad (\ell_i = \ell(X_i)).$$

Each coadjoint orbit $\mathcal{O}_\ell \in \Omega_{\mathbf{e}}$ can be characterized as follows

$$\mathcal{O}_\ell = \{\xi = (x_1, \ldots, x_n) \in \mathfrak{g}^\star : \quad \lambda_i(\xi) = \lambda_i(\ell)\}$$

Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. In all that follows we assume that $\dim \mathfrak{g} \leq 7$, and hence if $\mathcal{O}$ is a coadjoint orbit in $\mathfrak{g}^\star$, then $\dim \mathcal{O} \in \{0, 2, 4, 6\}$.

It is obvious that when the coadjoint orbits in the minimal layer of $\mathfrak{g}^\star$ are 0-dimensional then $\mathrm{Cor}(\mathfrak{g}^\star) = \mathrm{ICor}(\mathfrak{g}^\star)$ since $\mathfrak{g}$ is abelian. We denote by Ω the minimal layer in $\mathfrak{g}^\star$. Then, since it is Zariski open, due to the density one has

$$\mathrm{Cor}(\mathfrak{g}^\star) = \{\lim_{p \to \infty} \mathrm{Ad}^\star_{s_p}(\ell^{(p)}) : \quad \{s_p\}_p \subset G, \ \{\ell^{(p)}\}_p \subset \Omega \text{ with } \lim_{p \to \infty} \ell^{(p)} = 0\}.$$

Our study of the cortex of nilpotent Lie algebras $\mathfrak{g}$ satisfying $\dim \mathfrak{g} \leq 7$ is based on the dimension of its generic coadjoint orbits.

3. The cortex of nilpotent Lie algebras of dimensions less or equal to 6

In this section we assume that $\mathfrak{g}$ is a nilpotent Lie algebra with $\dim \mathfrak{g} \leq 6$, then it is known that $\mathrm{Cor}(\mathfrak{g}^\star) = \mathrm{ICor}(\mathfrak{g}^\star)$ (see [5]). The technics used in [5] are based on the computation of the Pukànszky's polynomials appearing in the parametrization of the generic coadjoint orbits of any nilpotent real Lie algebras $\mathfrak{g}$ satisfying $\dim \mathfrak{g} \leq 6$ and hence the proof turns out to be very long. Below we give an alternate short proof of this result. To this end, let Ω denotes the minimal layer in $\mathfrak{g}^\star$ then each coadjoint orbit $\mathcal{O} \subset \Omega$ satisfies $\dim \mathcal{O} \in \{0, 2, 4\}$. It is obvious that if $\dim \mathcal{O} = 0$ then $\mathfrak{g}$ is abelian and $\mathrm{Cor}(\mathfrak{g}^\star) = \mathrm{ICor}(\mathfrak{g}^\star) = \mathfrak{g}^\star$.

3.1. The generic orbits are 2-dimensional

Proposition 3.1. ([2]) *Let G be a connected, simply connected solvable Lie group whose coadjoint orbits have dimension smaller or equal to two. Let $\mathfrak{g}$ be the corresponding Lie algebra. Then, up to a direct central factor, $\mathfrak{g}$ belongs to the following list of algebras:*

(i) $\mathbb{R}T \oplus \mathfrak{a}$, *where $\mathfrak{a}$ is an abelian ideal and $\mathrm{ad}_T \in End(\mathfrak{a})$.*

(ii) $\mathbb{R}T \oplus \mathfrak{h}_3$, *where $\mathfrak{h}_3$ is the 3-dimensional Heisenberg Lie algebra spanned by (Z, Y, X) with $[X, Y] = Z$ and*
  − *either $[T, X] = X, [T, Y] = Y, [T, Z] = 0$ (the 4-dimensional Diamond algebra).*
  − *or $[T, X] = Y, [T, Y] = -X, [T, Z] = 0$*

(iii) $\mathfrak{g}$ *is 5-dimensional with basis $(X_1, X_2, X_3, Y_1, Y_2)$ with*
$$[X_1, X_2] = X_3, [X_1, X_3] = Y_1, [X_2, X_3] = Y_2.$$

(iv) $\mathfrak{g}$ *is 6-dimensional with basis $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ with*
$$[X_1, X_2] = Y_3, [X_2, X_3] = Y_1, [X_3, X_1] = Y_2.$$

Proposition 3.2.   *Let $\mathfrak{g}$ be a real nilpotent Lie algebra with $\dim \mathfrak{g} \leq 6$ and such that the dimension of its generic coadjoint orbits are smaller or equal to 2, then*

$$\mathrm{Cor}(\mathfrak{g}^\star) = \mathrm{ICor}(\mathfrak{g}^\star).$$

Proof.   The proof is based on the results given in Proposition 3.1. Let G be a connected, simply connected nilpotent Lie group whose coadjoint orbits have dimension smaller or equal to two and let $\mathfrak{g}$ be the corresponding nilpotent Lie algebra.

(i): $\mathfrak{g} = \mathbb{R}T \oplus \mathfrak{a}$, where $\mathfrak{a}$ is an abelian ideal and $\mathrm{ad}_T \in End(\mathfrak{a})$. In this case the corresponding Lie group, is up to a direct central factor, $G = \mathbb{R}^n \rtimes \mathbb{R}$ (with $1 \leq n \leq 5$) and by [8], one has $\mathrm{Cor}(\mathfrak{g}^\star) = \mathrm{ICor}(\mathfrak{g}^\star)$.

(ii): $\mathfrak{g}$ is 5-dimensional with basis $(X_1, X_2, X_3, Y_1, Y_2)$ with

$$[X_1, X_2] = X_3, [X_1, X_3] = Y_1, [X_2, X_3] = Y_2.$$

Let $(U_1 = Y_2, U_2 = Y_1, U_3 = X_3, U_4 = X_2, U_5 = X_1)$ then the minimal layer is

$$\Omega := \Omega_\mathbf{e} = \{\ell = \sum_{i=1}^5 \ell_i U_i^\star \in \mathfrak{g}^\star : \ell_1 \neq 0\}, \quad \mathbf{e} = \{3, 4\}.$$

The corresponding cross-section mapping is given by

$$\Psi : \Omega \to \Omega, \quad \ell \mapsto \Psi(\ell) = (\ell_1, \ell_2, 0, 0, f(\ell)),$$

where $f(\ell) = \ell_5 + (\ell_3^2 - 2\ell_2\ell_4)/2\ell_1$. Hence we get three invariant rational functions generating algebraically the field of invariant rational functions:

$$\lambda_1(x) = x_1, \quad \lambda_2(x) = x_2, \quad \lambda_3(x) = x_5 + \frac{x_3^2 - 2x_2x_4}{2x_1},$$

and any invariant polynomial function is a rational function of these three polynomials. Therefore we have the fundamental invariant polynomials:

$$x_1, \quad x_2, \quad P(x) = x_3^2 + 2x_1x_5 - 2x_2x_4.$$

This immediately gives: $\mathrm{ICor}(\mathfrak{g}^\star) = \{\ell \in \mathfrak{g}^\star : \ell_1 = \ell_2 = \ell_3 = 0\}$.

On the other hand, the coadjoint orbit of $\ell \in \Omega$ is given by

$$\mathcal{O} = \{x = (x_1, \ldots, x_5) \in \mathfrak{g}^\star : \quad x_1 = \ell_1, \quad x_2 = \ell_2, \quad P(x) = P(\ell)\}.$$

Let us show that $\mathrm{ICor}(\mathfrak{g}^\star) = \mathrm{Cor}(\mathfrak{g}^\star)$, to this end let $y = (0, 0, 0, y_4, y_5) \in \mathrm{ICor}(\mathfrak{g}^\star)$ with $y_4 \neq 0$ and consider

$$\ell_\varepsilon = (\varepsilon, \frac{\varepsilon y_5}{y_4}, 0, 0, 0), \quad y_\varepsilon = (\varepsilon, \frac{\varepsilon y_5}{y_4}, 0, y_4, y_5) \quad \text{with } \varepsilon \neq 0.$$

We can easily check that ℓ_ε and y_ε are in the same coadjoint orbit in Ω, and $y = \lim_{\varepsilon \to 0} y_\varepsilon$. Thus any generic element in the I-cortex is in the cortex and hence

$$\mathrm{Cor}(\mathfrak{g}^\star) = \mathrm{ICor}(\mathfrak{g}^\star).$$

(iii): $\mathfrak{g}$ is 6-dimensional with basis $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ with

$$[X_1, X_2] = Y_3, [X_2, X_3] = Y_1, [X_3, X_1] = Y_2.$$

Let $(U_1 = Y_3, U_2 = Y_2, U_3 = Y_1, U_4 = X_3, U_5 = X_2, U_6 = X_1)$. The minimal layer is

$$\Omega := \Omega_{\mathbf{e}} = \{\ell = \sum_{i=1}^{6} \ell_i U_i^{\star} \in \mathfrak{g}^{\star} : \ell_3 \neq 0\}, \quad \mathbf{e} = \{4, 5\}.$$

The corresponding cross-section mapping is given by

$$\Psi : \Omega \to \Omega, \quad \ell \mapsto \Psi(\ell) = (\ell_1, \ell_2, \ell_3, 0, 0, \ell_6 + \frac{\ell_2 \ell_5 + \ell_1 \ell_4}{\ell_3}).$$

Hence the algebra of G-invariant polynomials is

$$Pol(\mathfrak{g}^{\star})^G = \mathbb{R}[\ell_1, \ell_2, \ell_3, \ell_3 \ell_6 + \ell_2 \ell_5 + \ell_1 \ell_4],$$

and $$\mathrm{ICor}(\mathfrak{g}^{\star}) = \{\ell \in \mathfrak{g}^{\star} : \ell_1 = \ell_2 = \ell_3 = 0\}.$$

Now if $y = (0, 0, 0, y_4, y_5, y_6) \in \mathrm{ICor}(\mathfrak{g}^{\star})$ with $y_5 + y_6 \neq 0$ and $\varepsilon \neq 0$ let

$$\ell_{\varepsilon} = (\varepsilon, -\frac{\varepsilon y_4}{y_5 + y_6}, -\frac{\varepsilon y_4}{y_5 + y_6}, 0, 0, 0), \quad y_{\varepsilon} = (\varepsilon, -\frac{\varepsilon y_4}{y_5 + y_6}, -\frac{\varepsilon y_4}{y_5 + y_6}, y_4, y_5, y_6).$$

We can easily check that ℓ_{ε} and y_{ε} are in the same coadjoint orbit in Ω, and $y = \lim_{\varepsilon \to 0} y_{\varepsilon}$. Hence any generic element in the I-cortex is in the cortex and thus

$$\mathrm{Cor}(\mathfrak{g}^{\star}) = \mathrm{ICor}(\mathfrak{g}^{\star}) = \mathfrak{z}^{\perp}. \qquad \blacksquare$$

3.2. The generic coadjoint orbits are 4-dimensional

Let $\mathfrak{g}$ be a nilpotent Lie algebra with $\dim \mathfrak{g} \leq 6$ and such that the coadjoint orbit in the minimal layer is 4-dimensional.

3.2.1. $\dim \mathfrak{g} = 5$
Since $\mathfrak{g}$ is nilpotent, then the orbit of ℓ is $\mathcal{O}_{\ell} = \{\ell\} + \mathfrak{z}^{\perp}$, thus $\mathrm{Cor}(\mathfrak{g}^{\star}) = \mathfrak{z}^{\perp}$.
On the other hand, since $Pol(\mathfrak{g}^{\star})^G = Pol(\mathfrak{z}^{\star})$, therefore $\mathrm{Cor}(\mathfrak{g}^{\star}) = \mathrm{ICor}(\mathfrak{g}^{\star}) = \mathfrak{z}^{\perp}$.

3.2.2. $\dim \mathfrak{g} = 6$
(i): $\dim \mathfrak{z} = 2$. In this case, if $\mathcal{O}_{\ell}$ is the coadjoint orbit of a generic ℓ then

$$\mathcal{O}_{\ell} = \{\ell\} + \mathfrak{z}^{\perp} \quad \text{and} \quad Pol(\mathfrak{g}^{\star})^G = Pol(\mathfrak{z}^{\star}).$$

Thus $$\mathrm{Cor}(\mathfrak{g}^{\star}) = \mathrm{ICor}(\mathfrak{g}^{\star}) = \mathfrak{z}^{\perp}.$$

(ii): $\dim \mathfrak{z} = 1$. There are 13 nilpotent Lie algebras corresponding to this case (see [20]). For all these Lie algebras, the minimal layer Ω is given by

$$\Omega = \{\ell \in \mathfrak{g}^{\star} : \ell_1 \neq 0\}.$$

The field of invariant rational functions is generated by two invariant rational functions x_1 and P. Below we give the list of these nilpotent Lie algebras $(\mathfrak{g}_t)_{1 \leq t \leq 13}$ with the non trivial invariant polynomial function P and the corresponding I-cortex.

N°	Brackets	Invariant polynomial P	I - Cortex
1	$[X_6, X_5] = X_4,\ [X_6, X_4] = X_1,$ $[X_3, X_1] = X_1$	$2x_1x_5 - x_4^2$	$y_1 = y_4 = 0$
2	$[X_6, X_5] = X_4,\ [X_6, X_4] = X_3,$ $[X_6, X_3] = X_1,\ [X_5, X_4] = X_1$	$2x_1x_4 - x_3^2$	$y_1 = y_3 = 0$
3	$[X_6, X_5] = X_4,\ [X_6, X_4] = X_3,$ $[X_6, X_3] = X_1,\ [X_5, X_4] = X_1,$ $[X_5, X_2] = X_1$	$2x_1(x_4 - x_2) - x_3^2$	$y_1 = y_3 = 0$
4	$[X_6, X_5] = X_3,\ [X_6, X_4] = X_2,$ $[X_5, X_2] = X_1,\ [X_4, X_3] = X_1$	$1x_6 + x_2x_3$	$y_1 = y_2y_3 = 0$
5	$[X_6, X_5] = X_3,\ [X_6, X_4] = X_2,$ $[X_6, X_3] = X_1,\ [X_4, X_2] = X_1$	$2x_1x_5 - x_3^2$	$y_1 = y_3 = 0$
6	$[X_6, X_5] = X_3,\ [X_6, X_4] = X_2,$ $[X_5, X_3] = X_1,\ [X_4, X_2] = X_1$	$x_1x_6 + x_2^2 + x_3^2$	$y_1 = y_2 = y_3 = 0$
7	$[X_6, X_5] = X_3,\ [X_6, X_3] = X_2,$ $[X_6, X_2] = X_1,\ [X_5, X_4] = X_1,$ $[X_4, X_3] = X_1$	$3x_1^2x_5 - 3x_1x_2x_3 + x_2^3$	$y_1 = y_2 = 0$
8	$[X_6, X_5] = X_4,\ [X_6, X_4] = X_3,$ $[X_6, X_2] = X_1,\ [X_5, X_4] = X_2,$ $[X_5, X_3] = X_1$	$x_1x_4 - x_2x_3$	$y_1 = y_2y_3 = 0$
9	$[X_6, X_5] = X_4,\ [X_6, X_4] = X_3,$ $[X_6, X_3] = X_1,\ [X_5, X_4] = X_2,$ $[X_5, X_2] = X_1$	$2x_1x_4 - x_2^2 - x_3^2$	$y_1 = y_2 = y_3 = 0$
10	$[X_6, X_5] = X_4,\ [X_6, X_4] = X_3,$ $[X_6, X_3] = X_2,\ [X_6, X_2] = X_1,$ $[X_5, X_4] = X_1$	$2x_1x_3 - x_2^2$	$y_1 = y_2 = 0$
11	$[X_6, X_5] = X_4,\ [X_6, X_4] = X_3,$ $[X_6, X_3] = X_2,\ [X_5, X_2] = X_1,$ $[X_4, X_3] = X_1$	$2x_1x_6 - 2x_2x_4 + x_3^2$	$y_1 = y_3^2 - 2y_2y_4 = 0$
12	$[X_6, X_5] = X_4,\ [X_6, X_4] = X_3,$ $[X_6, X_3] = X_2,\ [X_6, X_2] = X_1,$ $[X_5, X_4] = X_2,\ [X_5, X_3] = X_1$	$3x_1^2x_4 - 3x_1x_2x_3 + x_2^3$	$y_1 = y_2 = 0$
13	$[X_6, X_5] = X_4,\ [X_6, X_4] = X_3,$ $[X_6, X_3] = X_2,\ [X_6, X_2] = X_1,$ $[X_5, X_4] = X_2,\ [X_4, X_3] = X_1$	$6x_1^2x_6 + 3x_1x_3^2 - 6x_1x_2x_4 - x_2^3$	$y_1 = y_2 = 0$

Proposition 3.3.   *For each Lie algebra of the above list, one has*

$$\mathrm{Cor}(\mathfrak{g}^\star) = \mathrm{ICor}(\mathfrak{g}^\star).$$

Proof.   Since $\mathrm{Cor}(\mathfrak{g}^\star) \subset \mathrm{ICor}(\mathfrak{g}^\star)$, then we shall prove the converse inclusion. To this end, and for any generic $y \in \mathrm{ICor}(\mathfrak{g}_t^\star)$ $(t = 1, \ldots, 13)$, we give below a sequence $(\ell^{(p)})_p \subset \Omega$ converging to zero and a sequence $(y^{(p)})_p$ such that $y^{(p)} \in \mathrm{Ad}^\star(G)\ell^{(p)}$ (for any positive integer p) and $\lim_{p\to\infty} y^{(p)} = y$. On the other hand, recall that the coadjoint orbit $\mathcal{O}_\ell$ (for $\ell \in \Omega$) is given by

$$\mathcal{O}_\ell = \{x = (x_1, \ldots, x_6) \in \Omega : \ x_1 = \ell_1 \text{ and } P(x) = P(\ell)\}.$$

3.3. Case of the Lie algebras $\mathfrak{g}_t$, $t = 1, 2, 5, 10$

For these Lie algebras the G-invariant polynomial P is of the form $P = 2x_1 x_{k_t} - x_{j_t}^2$ where $2 < k_t \notin \mathbf{e}$ and $j_t = k_t - 1$ or $k_t - 2$. On other hand

$$\mathrm{ICor}(\mathfrak{g}_t^\star) = \{y = (y_1, \ldots, y_6) : \quad y_1 = y_j = 0\}.$$

For $y = (y_1, \ldots, y_6) \in \mathrm{ICor}(\mathfrak{g}_t^\star)$ such $y_{k_t} \neq 0$ we let $\varepsilon_t = \frac{y_{k_t}}{|y_{k_t}|}$, $\quad y^{(p)} = (y_1^{(p)}, \ldots, y_6^{(p)})$ and $\ell^{(p)} = (\ell_1^{(p)}, 0, \ldots, 0)$ with

$$y_i^{(p)} = \begin{cases} \ell_1^{(p)} = \frac{\varepsilon_t}{p}, & \text{if } i = 1 \\ \sqrt{\frac{2|y_k|}{p}}, & \text{if } i = k_t \\ y_i, & \text{otherwise.} \end{cases}$$

3.4. Case of the Lie algebra $\mathfrak{g}_3$

In this case, one has $P = 2x_1(x_4 - x_2) - x_3^2$ and

$$\mathrm{ICor}(\mathfrak{g}_3^\star) = \{y = (y_1, \ldots, y_6) : \quad y_1 = y_3 = 0\}.$$

If $y = (0, y_2, 0, y_4, y_5, y_6) \in \mathrm{ICor}(\mathfrak{g}_k^\star)$ has the property that $y_4 - y_2 \neq 0$, we let $y^{(p)} = (y_1^{(p)}, \ldots, y_6^{(p)})$, $\ell^{(p)} = (\ell_1^{(p)}, 0, \ldots, 0)$ and $\varepsilon = \frac{y_4 - y_2}{|y_4 - y_2|}$ with

$$y_i^{(p)} = \begin{cases} \ell_1^{(p)} = \frac{\varepsilon}{p}, & \text{if } i = 1 \\ \sqrt{\frac{2|y_4 - y_2|}{p}}, & \text{if } i = 3 \\ y_i, & \text{otherwise.} \end{cases}$$

3.5. Case of the Lie algebras $\mathfrak{g}_t$, $t = 4, 8$

For both of these Lie algebras, one has

$$\mathrm{ICor}(\mathfrak{g}_t^\star) = \{y = (y_1, \ldots, y_6) : \quad y_1 = y_2 y_3 = 0\}.$$

For any $y \in \mathrm{ICor}(\mathfrak{g}_t^\star)$ we let

$$y^{(p)} = (\frac{1}{p}, y_2, y_3, y_4, y_5, y_6) \quad \text{and} \quad \ell^{(p)} = \begin{cases} (\frac{1}{p}, \frac{1}{\sqrt{p}}, \frac{y_6}{\sqrt{p}}, 0, 0, 0), & \text{if } t = 4 \\ (\frac{1}{p}, \frac{1}{\sqrt{p}}, -\frac{y_4}{\sqrt{p}}, 0, 0, 0), & \text{if } t = 8. \end{cases}$$

3.6. Case of the Lie algebras $\mathfrak{g}_t$, $t = 6, 9$

For both of these Lie algebras, one has

$$\mathrm{ICor}(\mathfrak{g}_t^\star) = \{y = (y_1, \ldots, y_6) : \quad y_1 = y_2 = y_3 = 0\}.$$

On other hand, the G-invariant polynomial P takes the form $P = 2x_1 x_{k_t} \pm (x_2^2 + x_3^3)$ (for some $2 < k_t$). Now if $y \in \mathrm{ICor}(\mathfrak{g}_t^\star)$ with $y_{k_t} \neq 0$ we let $\varepsilon_t = \frac{y_k}{|y_k|}$ and

$$y^{(p)} = \begin{cases} (-\frac{\varepsilon_t}{p}, \sqrt{\frac{2|y_6|}{p}}, 0, y_4, y_5, y_6), & \text{if } t = 6 \\ (\frac{\varepsilon_t}{p}, -\sqrt{\frac{2|y_4|}{p}}, 0, y_4, y_5, y_6), & \text{if } t = 9. \end{cases} \quad \text{and} \quad \ell^{(p)} = \begin{cases} (-\frac{\varepsilon_t}{p}, 0, 0, 0, 0, 0), & \text{if } t = 6 \\ (\frac{\varepsilon_t}{p}, 0, 0, 0, 0, 0), & \text{if } t = 9. \end{cases}$$

3.7. Case of the Lie algebras $\mathfrak{g}_t$, $t = 7, 12, 13$

For these Lie algebras, one has

$$\mathrm{ICor}(\mathfrak{g}_t^\star) = \{y = (y_1, \dots, y_6) : \quad y_1 = y_2 = 0\}, \quad t = 7, 12, 13$$

On other hand, the G-invariant polynomial P is homogeneous of degree 3. For $t = 7, 12, 13$, we define

$$k_t = \begin{cases} 5, & \text{if } t = 7 \\ 4, & \text{if } t = 12 \\ 6, & \text{if } t = 13 \end{cases}$$

and

$$U_t = \{y = (0, 0, y_3, y_4, y_5, y_6) : \quad y_{k_t} \neq 0\}.$$

For $y \in U_t$ we let $\ell^{(p)} = (\ell_1^{(p)}, 0, 0, 0, 0, 0)$, $(\alpha_p)_p$ a real sequence converging to zero and $y^{(p)} \in \Omega$ as follows

N°	$y^{(p)}$	Condition on $(\alpha_p)_p$
7	$(\ell_1^{(p)} = \dfrac{3\alpha_p y_3 \pm \sqrt{9\alpha_p^2 y_3^2 - 12\alpha_p^3 y_5}}{6 y_5}, \alpha_p, y_3, y_4, y_5, y_6)$	$\alpha_p y_5 < 0$
12	$(\ell_1^{(p)} = \dfrac{3\alpha_p y_3 \pm \sqrt{9\alpha_p^2 y_3^2 - 12\alpha_p^3 y_4}}{6 y_4}, \alpha_p, y_3, y_4, y_5, y_6)$	$\alpha_p y_4 < 0$
13	$(\ell_1^{(p)} = \dfrac{6\alpha_p y_4 - 3\alpha_p^3 \pm \sqrt{(3\alpha_p^3 - 6\alpha_p y_4)^2 + 24\alpha_p^3 y_6}}{12 y_6}, \alpha_p, y_3, y_4, y_5, y_6)$	$\alpha_p y_6 > 0$

3.8. Case of the Lie algebra $\mathfrak{g}_{11}$

For this Lie algebra the G-invariant polynomial P is $2x_1 x_6 - 2x_2 x_4 + x_3^2$ and

$$\mathrm{ICor}(\mathfrak{g}_{11}^\star) = \{y = (y_1, \dots, y_6) \in \mathfrak{g}_{11}^\star : \quad y_1 = y_3^2 - 2y_2 y_4 = 0\}.$$

Then we let $\quad y^{(p)} = (\dfrac{1}{p}, y_2, y_3, y_4, y_5, y_6) \quad$ and $\quad \ell^{(p)} = (\dfrac{1}{p}, -\dfrac{1}{\sqrt{p}}, 0, \dfrac{y_6}{\sqrt{p}}, 0, 0)$. ∎

4. The cortex of 7-dimensional nilpotent Lie algebras

This section is devoted to the study of the cortex of the 7-dimensional nilpotent Lie algebras over the field of real numbers, and we are essentially concerned with the postulate if the cortex and the I-cortex of a given nilpotent Lie algebra coincide as in the case of nilpotent Lie algebras of dimension smaller or equal to 6. As it is known there are infinite non isomorphic 7-dimensional nilpotent Lie algebras. Historically, there has been a good deal of activity in the subject of nilpotent Lie algebras. Since Safiullina's first attempt to classify all 7-dimensional nilpotent Lie algebras there have been a number of works in that direction, for instance see [15, 18, 19, 23, 24]. In [24] the author gave a classification of all 7-dimensional nilpotent Lie algebras over the filed of complex numbers, however in [15] one also gives a classification on the field of real numbers. His classification was based on the upper central series of the Lie algebras and knowledge of all lower dimensional nilpotent Lie algebras.

We continue with same reasoning as is Section 3, that is, we focus on the dimension of the generic coadjoint orbits. For this purpose, if $\mathfrak{g}$ is a 7-dimensional (real) nilpotent Lie algebra and if $\mathcal{O}$ is a generic coadjoint orbit then $\dim \mathcal{O} \in \{0, 2, 4, 6\}$. It is easy to conclude that the cortex and I-cortex coincide when $\dim \mathcal{O} \in \{0, 6\}$, and if $\dim \mathcal{O} = 2$ then by Proposition (3.1), the corresponding Lie group is, up to a central factor, $G = \mathbb{R}^n \rtimes \mathbb{R}$ ($n \leq 6$) and due to [8] one has again $\mathrm{Cor}(\mathfrak{g}^\star) = \mathrm{ICor}(\mathfrak{g}^\star)$.

Finally it remains to dissect the class of 7-dimensional (real) nilpotent Lie algebras in which the generic coadjoint orbits are 4-dimensional. In this situation the center $\mathfrak{z}$ of $\mathfrak{g}$ is either of dimension 1 or 2. The case $\dim \mathfrak{z} = 2$ can be treated as in Proposition 3.3, and therefore it seems that the 'interesting' case is when $\mathfrak{g}$ is a 7-dimensional (real) nilpotent Lie algebras with 1-dimensional center and 4-dimensional generic coadjoint orbits.

Let $\mathfrak{g}$ be a nilpotent Lie algebra with a Jordan-Hölder basis $(X_1, \ldots, X_7)$ with dual basis $(X_1^\star, \ldots, X_7^\star)$ and let $\Omega_{\mathbf{e}}$ be the minimal layer (Zariski open set) in $\mathfrak{g}^\star$. As it is well known the set

$$\Omega_{\mathbf{e}} \cap \{\ell \in \mathfrak{g}^\star : \ell(X_j) = 0, \ j \notin \mathbf{e}\}$$

is a cross-section for the coadjoint orbits in $\Omega_{\mathbf{e}}$ and the projection onto the cross-section is given by rational functions $\lambda_i, i \notin \mathbf{e}$, which are regular on $\Omega_{\mathbf{e}}$ and each λ_i is of the form

$$\lambda_i(\ell) = \ell_{k_i} + f_i(\ell_1, \ldots, \ell_{k_i-1}), \quad \ell_j = \ell(X_j), \ i = 1, 2,$$

and all G-invariant rational functions are rational combinations of the λ_i, $i \notin \mathbf{e}$. Note that, since $\dim \mathfrak{z} = 1$ then 2 must be a jump index ($2 \in \mathbf{e}$) and the corresponding cross-section mapping has the form

$$\Psi : \Omega_{\mathbf{e}} \to \Omega_{\mathbf{e}} \ell = (\ell_1, \ldots, \ell_7) \mapsto \ell_1 X_1^\star + \lambda_1(\ell) X_{k_1}^\star + \lambda_2(\ell) X_{k_2}^\star,$$

where $\{k_1 < k_2\} = \{2, \ldots, 7\} \setminus \mathbf{e}$. We use the methods of [1] for finding the set of jump indices of the minimal layer, then we compute explicitly the G-invariant rational functions λ_1, λ_2 associated to the cross-section mapping and hence we deduce the $\mathrm{ICor}(\mathfrak{g}_i^\star)$.

Below the list of all these Lie algebras (we refer the reader to [15, 24]), and for any Lie algebra of this list, we give the we give the non zero brackets, the non trivial fundamental invariants λ_1, λ_2, the set of jump indices and the I-cortex. We see that for any Lie algebra of this list, one has

$$\Omega := \Omega_{\mathbf{e}} = \{\ell \in \mathfrak{g}^\star : \ \ell_1 \neq 0\}.$$

	$[X_7, X_6] = X_5, [X_7, X_5] = X_4, [X_7, X_4] = X_1, [X_3, X_2] = X_1$
$\mathfrak{g}_1$	$\lambda_1 = x_5 - \dfrac{x_4^2}{2x_1}, \quad \lambda_2 = x_6 - \dfrac{x_5 x_4}{x_1} + \dfrac{x_4^3}{3x_1^2}$
	$\mathrm{ICor}(\mathfrak{g}_1^\star) = \{\ell : \ \ell_1 = \ell_4 = 0\}, \quad \mathbf{e} = \{2, 3, 4, 7\}$
	$[X_7, X_6] = X_3, [X_7, X_3] = X_1, [X_5, X_4] = X_2, [X_5, X_2] = X_1$
$\mathfrak{g}_2$	$\lambda_1 = x_4 - \dfrac{x_2^2}{2x_1}, \quad \lambda_2 = x_6 - \dfrac{x_3^2}{2x_1}$
	$\mathrm{ICor}(\mathfrak{g}_2^\star) = \{\ell : \ \ell_1 = \ell_2 = \ell_3 = 0\}, \quad \mathbf{e} = \{2, 3, 5, 7\}$

$\mathfrak{g}_3$	$[X_7, X_6] = X_3,\ [X_7, X_3] = X_1,\ [X_6, X_5] = X_1,\ [X_5, X_4] = X_2,$ $[X_5, X_2] = X_1$
	$\lambda_1 = x_4 - \dfrac{x_2^2}{2x_1}, \quad \lambda_2 = x_6 + x_2 - \dfrac{x_3^2}{2x_1}$
	$\mathrm{ICor}(\mathfrak{g}_3^\star) = \{\ell :\ \ell_1 = \ell_2 = \ell_3 = 0\}, \quad \mathbf{e} = \{2,3,5,7\}$
$\mathfrak{g}_4$	$[X_7, X_6] = X_3,\ [X_7, X_5] = X_2,\ [X_7, X_3] = X_1,\ [X_5, X_4] = X_3,$ $[X_4, X_2] = -X_1$
	$\lambda_1 = x_5 - \dfrac{x_2 x_3}{x_1}, \quad \lambda_2 = x_6 - \dfrac{x_3^2}{2x_1}$
	$\mathrm{ICor}(\mathfrak{g}_4^\star) = \{\ell :\ \ell_1 = \ell_3 = 0\}, \quad \mathbf{e} = \{2,3,4,7\}$
$\mathfrak{g}_5$	$[X_7, X_6] = X_5,\ [X_7, X_5] = X_3,\ [X_7, X_3] = X_1,\ [X_6, X_4] = X_3,$ $[X_5, X_4] = X_1, [X_4, X_2] = X_1$
	$\lambda_1 = x_5 + x_2 - \dfrac{x_3}{2x_1^2}, \quad \lambda_2 = x_6 - \dfrac{x_5 x_3}{x_1} + \dfrac{x_3^3}{3x_1^2}$
	$\mathrm{ICor}(\mathfrak{g}_5^\star) = \{\ell :\ \ell_1 = \ell_3 = 0\}, \quad \mathbf{e} = \{2,3,4,7\}$
$\mathfrak{g}_6$	$[X_7, X_6] = X_5,\ [X_7, X_5] = X_3,\ [X_7, X_4] = X_2,\ [X_7, X_3] = X_1,$ $[X_4, X_2] = X_1$
	$\lambda_1 = x_5 - \dfrac{x_3^2}{2x_1}, \quad \lambda_2 = x_6 - \dfrac{x_5 x_3}{x_1} + \dfrac{x_3^3}{3x_1^2}$
	$\mathrm{ICor}(\mathfrak{g}_6^\star) = \{\ell :\ \ell_1 = \ell_3 = 0\}, \quad \mathbf{e} = \{2,3,4,7\}$
$\mathfrak{g}_7$	$[X_7, X_6] = X_5,\ [X_7, X_5] = X_3,\ [X_7, X_3] = X_1,\ [X_6, X_4] = X_2,$ $[X_6, X_2] = X_1$
	$\lambda_1 = x_4 - \dfrac{x_2^2}{2x_1}, \quad \lambda_2 = x_5 - \dfrac{x_3^2}{2x_1}$
	$\mathrm{ICor}(\mathfrak{g}_7^\star) = \{\ell :\ \ell_1 = \ell_2 = \ell_3 = 0\}, \quad \mathbf{e} = \{2,3,6,7\}$
$\mathfrak{g}_8$	$[X_7, X_6] = X_5,\ [X_7, X_5] = X_3,\ [X_7, X_3] = X_1,\ [X_6, X_4] = X_2,$ $[X_4, X_2] = X_1$
	$\lambda_1 = x_5 - \dfrac{x_3^2}{2x_1}, \quad \lambda_2 = x_6 + \dfrac{x_2^2 - 2x_5 x_3}{2x_1} + \dfrac{x_3^3}{3x_1^2}$
	$\mathrm{ICor}(\mathfrak{g}_8^\star) = \{\ell :\ \ell_1 = \ell_3 = 0\}, \quad \mathbf{e} = \{2,3,4,7\}$
$\mathfrak{g}_9$	$[X_7, X_6] = X_5,\ [X_7, X_5] = X_3,\ [X_7, X_4] = X_2,\ [X_7, X_3] = X_1,$ $[X_6, X_4] = X_3,\ [X_6, X_2] = X_1$
	$\lambda_1 = x_4 - \dfrac{x_3 x_2}{x_1}, \quad \lambda_2 = x_5 - \dfrac{x_3^2}{2x_1}$
	$\mathrm{ICor}(\mathfrak{g}_9^\star) = \{\ell :\ \ell_1 = \ell_3 = 0\}, \quad \mathbf{e} = \{2,3,6,7\}$
$\mathfrak{g}_{10}$	$[X_7, X_6] = X_5,\ [X_7, X_5] = X_3,\ [X_7, X_4] = X_2,\ [X_7, X_3] = X_1,$ $[X_6, X_4] = X_3,\ [X_5, X_4] = X_1, [X_4, X_2] = X_1$
	$\lambda_1 = \lambda_1 = x_5 + x_2 - \dfrac{x_3^2}{2x_1}, \quad \lambda_2 = x_6 - \dfrac{x_5 x_3}{x_1} + \dfrac{x_3^3}{3x_1^2}$
	$\mathrm{ICor}(\mathfrak{g}_{10}^\star) = \{\ell :\ \ell_1 = \ell_3 = 0\}, \quad \mathbf{e} = \{2,3,6,7\}$

$\mathfrak{g}_{11}$	$[X_7,X_6]=X_5,\ [X_7,X_5]=X_3,\ [X_7,X_4]=X_2,$ $[X_7,X_2]=X_1,\ [X_6,X_5]=X_2,\ [X_6,X_3]=X_1$
	$\lambda_1 = x_4 - \dfrac{x_2^2}{2x_1},\quad \lambda_2 = x_5 - \dfrac{x_3 x_2}{2x_1}$
	$\mathrm{ICor}(\mathfrak{g}_{11}^\star)=\{\ell:\ \ell_1=\ell_2=0\},\quad \mathbf{e}=\{2,3,4,7\}$
$\mathfrak{g}_{12}$	$[X_7,X_6]=X_5+X_4,\ [X_7,X_4]=X_3,$ $[X_7,X_3]=X_1,\ [X_6,X_5]=X_2,\ [X_6,X_2]=X_1$
	$\lambda_1 = x_4 - \dfrac{x_3^2}{2x_1},\quad \lambda_2 = x_5 - \dfrac{x_2^2}{2x_1}$
	$\mathrm{ICor}(\mathfrak{g}_{12}^\star)=\{\ell:\ \ell_1=\ell_2=\ell_3=0\},\quad \mathbf{e}=\{2,3,6,7\}$
$\mathfrak{g}_{13}$	$[X_7,X_6]=X_5,\ [X_7,X_5]=X_4,\ [X_7,X_4]=X_3,$ $[X_7,X_3]=X_1,\ [X_6,X_2]=X_1$
	$\lambda_1 = x_4 - \dfrac{x_3^2}{2x_1},\quad \lambda_2 = x_5 - \dfrac{x_4 x_3}{x_1} + \dfrac{x_3^3}{3x_1^2}$
	$\mathrm{ICor}(\mathfrak{g}_{13}^\star)=\{\ell:\ \ell_1=\ell_3=0\},\quad \mathbf{e}=\{2,3,6,7\}$
$\mathfrak{g}_{14}$	$[X_7,X_6]=X_5,\ [X_7,X_5]=X_4,\ [X_7,X_4]=X_3,$ $[X_7,X_3]=X_1,\ [X_6,X_2]=X_1,\ [X_6,X_5]=X_1$
	$\lambda_1 = x_4 - \dfrac{x_3^2}{2x_1},\quad \lambda_2 = x_5 - x_2 - \dfrac{x_4 x_3}{x_1} + \dfrac{x_3^3}{3x_1^2}$
	$\mathrm{ICor}(\mathfrak{g}_{14}^\star)=\{\ell:\ \ell_1=\ell_3=0\},\quad \mathbf{e}=\{2,3,6,7\}$
$\mathfrak{g}_{15}$	$[X_7,X_6]=X_5,\ [X_7,X_5]=X_4,\ [X_7,X_4]=X_3,\ [X_7,X_3]=X_1,$ $[X_6,X_5]=X_3,\ [X_6,X_2]=X_1,\ [X_6,X_4]=X_1$
	$\lambda_1 = x_4 - x_2 - \dfrac{x_3^2}{2x_1},\quad \lambda_2 = x_5 - \dfrac{x_4 x_3}{x_1} + \dfrac{x_3^3}{3x_1^2}$
	$\mathrm{ICor}(\mathfrak{g}_{15}^\star)=\{\ell:\ \ell_1=\ell_3=0\},\quad \mathbf{e}=\{2,3,6,7\}$
$\mathfrak{g}_{16}$	$[X_7,X_6]=X_5,\ [X_7,X_5]=X_4,\ [X_7,X_4]=X_3,$ $[X_7,X_3]=X_1,\ [X_6,X_5]=X_2,\ [X_6,X_2]=X_1$
	$\lambda_1 = x_4 - \dfrac{x_3^2}{2x_1},\quad \lambda_2 = x_5 - \dfrac{x_4 x_3}{x_1} - \dfrac{x_2^2}{2x_1} + \dfrac{x_3^3}{3x_1^2}$
	$\mathrm{ICor}(\mathfrak{g}_{16}^\star)=\{\ell:\ \ell_1=\ell_3=0\},\quad \mathbf{e}=\{2,3,6,7\}$
$\mathfrak{g}_{17}$	$[X_7,X_6]=X_5,\ [X_7,X_5]=X_4,\ [X_7,X_4]=X_2,$ $[X_7,X_2]=X_1,\ [X_6,X_3]=X_2,\ [X_5,X_3]=X_1$
	$\lambda_1 = x_4 - \dfrac{x_2^2}{2x_1},\quad \lambda_2 = x_6 - \dfrac{x_2 x_5}{x_1} + \dfrac{x_2^2 x_4}{2x_1^2} - \dfrac{x_2^4}{8x_1^3}$
	$\mathrm{ICor}(\mathfrak{g}_{17}^\star)=\{\ell:\ \ell_1=\ell_2=0\},\quad \mathbf{e}=\{2,3,5,7\}$
$\mathfrak{g}_{18}$	$[X_7,X_6]=X_5,\ [X_7,X_5]=X_4,\ [X_7,X_4]=X_2,$ $[X_7,X_2]=X_1,\ [X_6,X_3]=X_2+X_1,\ [X_5,X_3]=X_1$
	$\lambda_1 = x_4 - \dfrac{x_2^2}{2x_1},\quad \lambda_2 = x_6 - x_5 + \dfrac{x_4 x_2 - x_2 x_5}{x_1} + \dfrac{3x_2^2 x_4 - 2x_2^3}{6x_1^2} - \dfrac{x_2^4}{8x_1^3}$
	$\mathrm{ICor}(\mathfrak{g}_{18}^\star)=\{\ell:\ \ell_1=\ell_2=0\},\quad \mathbf{e}=\{2,3,5,7\}$

$\mathfrak{g}_{19}$	$[X_7, X_6] = X_5,\ [X_7, X_5] = X_4,\ [X_7, X_4] = X_2,$ $[X_6, X_3] = X_2,\ [X_6, X_2] = X_1,\ [X_5, X_4] = -X_1$
	$\lambda_1 = x_3 - \dfrac{x_2^2}{2x_1},\quad \lambda_2 = x_7 + \dfrac{2x_2 x_5 - x_4^2}{x_1}$
	$\mathrm{ICor}(\mathfrak{g}_{19}^\star) = \{\ell:\ \ell_1 = \ell_2 = \ell_4 = 0\},\quad \mathbf{e} = \{2, 4, 5, 6\}$
$\mathfrak{g}_{20}$	$[X_7, X_6] = X_5,\ [X_7, X_5] = X_4,\ [X_7, X_3] = X_2,\ [X_7, X_2] = X_1,$ $[X_6, X_5] = X_3,\ [X_6, X_4] = X_2,\ [X_5, X_4] = X_1$
	$\lambda_1 = x_3 - \dfrac{x_2^2}{2x_1},\quad \lambda_2 = x_6 + \dfrac{x_4 x_3 - x_2 x_5}{x_1}$
	$\mathrm{ICor}(\mathfrak{g}_{20}^\star) = \{\ell:\ \ell_1 = \ell_2 = \ell_3 \ell_4 = 0\},\quad \mathbf{e} = \{2, 4, 5, 7\}$
$\mathfrak{g}_{21}$	$[X_7, X_6] = X_5,\ [X_7, X_5] = X_4,\ [X_7, X_4] = X_1,\ [X_7, X_3] = X_2,$ $[X_7, X_2] = X_1,\ [X_6, X_5] = X_3,\ [X_6, X_4] = X_2,\ [X_5, X_4] = X_1$
	$\lambda_1 = x_3 - \dfrac{x_2^2}{2x_1},\quad \lambda_2 = x_6 + \dfrac{x_3 x_4 - x_2 x_5 - x_2 x_3}{x_1} + \dfrac{x_2^3}{3x_1^2}$
	$\mathrm{ICor}(\mathfrak{g}_{21}^\star) = \{\ell:\ \ell_1 = \ell_2 = 0\},\quad \mathbf{e} = \{2, 3, 5, 7\}$
$\mathfrak{g}_{22}$	$[X_7, X_6] = X_5,\ [X_7, X_5] = X_3,\ [X_7, X_3] = X_2,\ [X_7, X_2] = X_1,$ $[X_6, X_4] = X_3,\ [X_5, X_4] = X_2,\ [X_4, X_3] = -X_1$
	$\lambda_1 = x_5 - \dfrac{x_2 x_3}{x_1} + \dfrac{x_2^3}{3x_1^2},\quad \lambda_2 = x_6 - \dfrac{x_3^2 + 2x_2 x_5}{2x_1} + \dfrac{x_2^2 x_3}{x_1^2} - \dfrac{x_2^4}{4x_1^3}$
	$\mathrm{ICor}(\mathfrak{g}_{22}^\star) = \{\ell:\ \ell_1 = \ell_2 = 0\},\quad \mathbf{e} = \{2, 3, 4, 7\}$
$\mathfrak{g}_{23}$	$[X_7, X_6] = X_5,\ [X_7, X_5] = X_3,\ [X_7, X_3] = X_2,\ [X_7, X_2] = X_1,$ $[X_6, X_4] = X_3 + X_1,\ [X_5, X_4] = X_2,\ [X_4, X_3] = -X_1$
	$\lambda_1 = x_5 - \dfrac{x_3 x_2}{x_1} + \dfrac{x_2^3}{3x_1^2},\quad \lambda_2 = x_6 - x_3 - \dfrac{x_3^2 - x_2^2 + 2x_5 x_2}{2x_1} + \dfrac{x_3 x_2^2}{x_1^2} - \dfrac{x_2^4}{4x_1^3}$
	$\mathrm{ICor}(\mathfrak{g}_{23}^\star) = \{\ell:\ \ell_1 = \ell_2 = 0\},\quad \mathbf{e} = \{2, 3, 4, 7\}$
$\mathfrak{g}_{24}$	$[X_7, X_6] = X_5,\ [X_7, X_5] = X_4,\ [X_7, X_4] = X_3,$ $[X_7, X_3] = X_2,\ [X_7, X_2] = X_1,\ [X_6, X_5] = X_1$
	$\lambda_1 = x_3 - \dfrac{x_2^2}{2x_1},\quad \lambda_2 = x_4 - \dfrac{x_3 x_2}{x_1} + \dfrac{x_2^3}{3x_1^2}$
	$\mathrm{ICor}(\mathfrak{g}_{24}^\star) = \{\ell:\ \ell_1 = \ell_2 = 0\},\quad \mathbf{e} = \{2, 5, 6, 7\}$
$\mathfrak{g}_{25}$	$[X_7, X_6] = X_5,\ [X_7, X_5] = X_4,\ [X_7, X_4] = X_3,\ [X_7, X_3] = X_2,$ $[X_7, X_2] = X_1,\ [X_6, X_5] = X_2,\ [X_6, X_4] = X_1$
	$\lambda_1 = x_3 - \dfrac{x_2^2}{2x_1},\quad \lambda_2 = x_5 - \dfrac{x_4 x_2}{x_1} + \dfrac{x_3 x_2^2}{2x_1^2} - \dfrac{x_2^4}{8x_1^3}$
	$\mathrm{ICor}(\mathfrak{g}_{25}^\star) = \{\ell:\ \ell_1 = \ell_2 = 0\},\quad \mathbf{e} = \{2, 4, 6, 7\}$
$\mathfrak{g}_{26}$	$[X_7, X_6] = X_5,\ [X_7, X_5] = X_4,\ [X_7, X_4] = X_3,\ [X_7, X_3] = X_2,$ $[X_7, X_2] = X_1,\ [X_6, X_5] = X_2 + X_1,\ [X_6, X_4] = X_1$
	$\lambda_1 = x_3 - \dfrac{x_2^2}{2x_1},\quad \lambda_2 = x_5 - x_4 + \dfrac{x_3 x_2 - x_4 x_2}{x_1} + \dfrac{x_3 x_2^2}{2x_1^2} - \dfrac{x_2^3}{3x_1^2} - \dfrac{x_2^4}{8x_1^3}$
	$\mathrm{ICor}(\mathfrak{g}_{26}^\star) = \{\ell:\ \ell_1 = \ell_2 = 0\},\quad \mathbf{e} = \{2, 4, 6, 7\}$

$\mathfrak{g}_{27}$	$[X_7, X_6] = X_5, \ [X_7, X_5] = X_4, \ [X_7, X_4] = X_3, \ [X_7, X_3] = X_2,$ $[X_7, X_2] = X_1, \ [X_6, X_5] = X_3, \ [X_6, X_4] = X_2, \ [X_6, X_3] = X_1$
	$\lambda_1 = x_4 - \dfrac{x_2 x_3}{x_1} + \dfrac{x_2^3}{3x_1^2}, \quad \lambda_2 = x_5 - \dfrac{x_3^2 + 2x_4 x_2}{2x_1} + \dfrac{x_3 x_2^2}{x_1^2} - \dfrac{x_2^4}{4x_1^3}$
	$\mathrm{ICor}(\mathfrak{g}_{27}^\star) = \{\ell : \ \ell_1 = \ell_2 = 0\}, \quad \mathbf{e} = \{2, 3, 6, 7\}$
$\mathfrak{g}_{28}$	$[X_7, X_6] = X_5, \ [X_7, X_5] = X_4, \ [X_7, X_4] = X_3, \ [X_7, X_3] = X_2,$ $[X_7, X_2] = X_1, \ [X_6, X_5] = X_3 + X_1, \ [X_6, X_4] = X_2, \ [X_6, X_3] = X_1$
	$\lambda_1 = x_4 - \dfrac{x_2 x_3}{x_1} + \dfrac{x_2^3}{3x_1^2}, \quad \lambda_2 = x_5 - x_3 + \dfrac{x_2^2 - x_3^2 - 2x_2 x_4}{x_1} + \dfrac{x_2^2 x_3}{x_1^2} - \dfrac{x_2^4}{4x_1^3}$
	$\mathrm{ICor}(\mathfrak{g}_{28}^\star) = \{\ell : \ \ell_1 = \ell_2 = 0\}, \quad \mathbf{e} = \{2, 3, 6, 7\}$
$\mathfrak{g}_{29}$	$[X_7, X_6] = X_5, \ [X_7, X_5] = X_4, \ [X_7, X_4] = X_3, \ [X_7, X_3] = X_2,$ $[X_7, X_2] = X_1, \ [X_6, X_5] = X_3, \ [X_6, X_4] = X_2, \ [X_5, X_4] = X_1$
	$\lambda_1 = x_3 - \dfrac{x_2^2}{2x_1}, \quad \lambda_2 = x_6 + \dfrac{x_3 x_4 - x_2 x_5}{x_1} - \dfrac{x_2 x_3^2}{x_1^2} + \dfrac{2x_2^3 x_3}{3x_1^3} - \dfrac{2x_2^5}{15x_1^4}$
	$\mathrm{ICor}(\mathfrak{g}_{29}^\star) = \{\ell : \ \ell_1 = \ell_2 = 0\}, \quad \mathbf{e} = \{2, 4, 5, 7\}$
$\mathfrak{g}_{30}$	$[X_7, X_6] = X_5, \ [X_7, X_5] = X_3, \ [X_7, X_3] = X_1,$ $[X_6, X_5] = X_2, \ [X_5, X_4] = X_2, \ [X_6, X_2] = -X_1$
	$\lambda_1 = x_4 + \dfrac{x_2^2}{2x_1}, \quad \lambda_2 = x_5 - \dfrac{x_3^2}{2x_1} + \dfrac{x_3^3}{3x_1^2}$
	$\mathrm{ICor}(\mathfrak{g}_{30}^\star) = \{\ell : \ \ell_1 = \ell_2 = \ell_3 = 0\}, \quad \mathbf{e} = \{2, 3, 6, 7\}$
$\mathfrak{g}_{31}$	$[X_7, X_6] = X_5, \ [X_7, X_5] = X_3, \ [X_7, X_4] = X_2, \ [X_7, X_3] = X_1,$ $[X_6, X_5] = -X_2, \ [X_6, X_4] = X_3, \ [X_6, X_2] = X_1$
	$\lambda_1 = x_4 - \dfrac{x_3 x_2}{x_1}, \quad \lambda_2 = x_5 - \dfrac{x_2^2 + x_3^2}{2x_1}$
	$\mathrm{ICor}(\mathfrak{g}_{31}^\star) = \{\ell : \ \ell_1 = \ell_2 = \ell_3 = 0\}, \quad \mathbf{e} = \{2, 3, 6, 7\}$
$\mathfrak{g}_{32}$	$[X_7, X_6] = X_5, \ [X_7, X_5] = X_4, \ [X_7, X_4] = -X_2, \ [X_7, X_2] = X_1,$ $[X_6, X_5] = X_3, \ [X_6, X_3] = -X_2, \ [X_5, X_3] = -X_1$
	$\lambda_1 = x_4 + \dfrac{x_2^2}{2x_1}, \quad \lambda_2 = x_6 - \dfrac{x_3^2 + x_2 x_5}{2x_1} + \dfrac{x_2^2 x_4}{2x_1^2} + \dfrac{x_2^4}{8x_1^3}$
	$\mathrm{ICor}(\mathfrak{g}_{32}^\star) = \{\ell : \ \ell_1 = \ell_2 = 0\}, \quad \mathbf{e} = \{2, 3, 5, 7\}$
$\mathfrak{g}_{33}$	$[X_7, X_6] = X_4, \ [X_7, X_4] = X_3, \ [X_7, X_3] = X_2, \ [X_7, X_2] = X_1,$ $[X_6, X_5] = X_3 - X_1, \ [X_5, X_4] = -X_2, \ [X_5, X_3] = -X_1$
	$\lambda_1 = x_4 - \dfrac{x_3 x_2}{2x_1} + \dfrac{x_2^3}{3x_1^2}, \quad \lambda_2 = x_6 + x_3 + \dfrac{2x_3 x_2^2 - 2x_4 x_2 - x_3^2 - x_2^2}{2x_1} - \dfrac{x_2^4}{4x_1^3}$
	$\mathrm{ICor}(\mathfrak{g}_{33}^\star) = \{\ell : \ \ell_1 = \ell_2 = 0\}, \quad \mathbf{e} = \{2, 3, 5, 7\}$
$\mathfrak{g}_{34}$	$[X_7, X_6] = X_5, \ [X_7, X_5] = X_4, \ [X_7, X_4] = X_3, \ [X_7, X_3] = X_2,$ $[X_7, X_2] = -X_1, \ [X_6, X_5] = X_3 + X_1, \ [X_6, X_4] = X_2, \ [X_6, X_3] = -X_1$
	$\lambda_1 = x_4 + \dfrac{x_2 x_3}{x_1} - \dfrac{x_2^3}{3x_1^2}, \quad \lambda_2 = x_5 + x_3 + \dfrac{2x_2 x_4 + x_2^2 + x_3^2}{2x_1} - \dfrac{x_2^4}{4x_1^3}$
	$\mathrm{ICor}(\mathfrak{g}_{34}^\star) = \{\ell : \ \ell_1 = \ell_2 = 0\}, \quad \mathbf{e} = \{2, 3, 6, 7\}$

Most, but not all, of the Lie algebras of the above list verify the equality between the cortex and the I-cortex. Below we present two interesting examples.

Example 4.1. The Lie algebra $\mathfrak{g} := \mathfrak{g}_5$ has a basis $(X_1, \ldots, X_7)$ with:

$$[X_7, X_6] = X_5, \quad [X_7, X_5] = X_3, \quad [X_7, X_3] = X_1,$$

$$[X_6, X_4] = X_3, \quad [X_5, X_4] = X_1, \quad [X_4, X_2] = X_1.$$

The minimal layer is $\Omega = \{x : x_1 \neq 0\}$ and there are three invariant polynomial functions, x_1, $P_1(x) = 2x_1(x_2 + x_5) - x_3^2$, and $P_2(x) = 3x_1^2 x_6 - 3x_1 x_5 x_3 + x_3^3$, and any invariant polynomial function is a rational function of these three polynomials. This immediately gives the set:

$$\mathrm{ICor}(\mathfrak{g}^\star) = \{y \in \mathfrak{g}^\star : y_1 = y_3 = 0\}.$$

On other hand, the coadjoint orbit $\mathcal{O}_\ell$ of any $\ell \in \Omega$ is given by

$$\mathcal{O}_\ell = \{x = (x_1, \ldots, x_7) \in \mathfrak{g}^\star : x_1 = \ell_1, P_i(x) = P_i(\ell), \quad i = 1, 2\}.$$

Proposition 4.2. *The cortex of the Lie algebra $\mathfrak{g}$ is the semi-algebraic set given by*

$$\mathrm{Cor}(\mathfrak{g}^\star) = \{y \in \mathfrak{g}^\star : y_1 = y_3 = 0, \ (2y_2 - y_5)(2y_2 + 3y_5) \leq 0\} \subsetneq \mathrm{ICor}(\mathfrak{g}^\star).$$

Proof. First we put:

$$Q_1(x, \ell) = P_1(x) - P_1(\ell)|_{\ell_1 = x_1} = 2x_1(x_2 + x_5 - \ell_2 - \ell_5) - (x_3^2 - \ell_3^2),$$

$$Q_2(x, \ell) = P_2(x) - P_2(\ell)|_{\ell_1 = x_1} = 3x_1^2(x_6 - \ell_6) - 3x_1(x_5 x_3 - \ell_5 \ell_3) + (x_3^3 - \ell_3^3).$$

The points x and ℓ such that $x_1 = \ell_1 \neq 0$ are in the same orbit if and only if

$$Q_1(x, \ell) = Q_2(x, \ell) = 0.$$

We consider these functions as polynomial functions in x_1 and compute their resultant (or, in this case, express x_1 by $Q_1(x, \ell) = 0$ in term of the other variables, then put it in $Q_2(x, \ell)$). We get the polynomial:

$$R(x, \ell) = \begin{vmatrix} 2(x_2 + x_5 - \ell_2 - \ell_5) & 0 & 3(x_6 - \ell_6) \\ -(x_3^2 - \ell_3^2) & 2(x_2 + x_5 - \ell_2 - \ell_5) & -3(x_5 x_3 - \ell_5 \ell_3) \\ 0 & -(x_3^2 - \ell_3^2) & x_3^3 - \ell_3^3 \end{vmatrix}$$

$$= 4(x_2 + x_5 - \ell_2 - \ell_5)^2(x_3^3 - \ell_3^3) + 3(x_6 - \ell_6)(x_3^2 - \ell_3^2)^2$$

$$\quad - 6(x_2 + x_5 - \ell_2 - \ell_5)(x_3^2 - \ell_3^2)(x_5 x_3 - \ell_5 \ell_3).$$

Recall that $R(x, \ell) = 0$ is equivalent to say that Q_1 and Q_2 have a common root in x_1. Thus if $(x^{(p)}, \ell^{(p)}) \to (y, 0)$, and $x^{(p)}$ and $\ell^{(p)}$ are in the same generic orbit in Ω, we have $R(x^{(p)}, \ell^{(p)}) = 0$ for any p.

Suppose now that y is in $\mathrm{ICor}(\mathfrak{g}^\star)$ and that $y_2 + y_5 \neq 0$. In consequence we have $(x_3^{(p)})^2 - (\ell_3^{(p)})^2 \neq 0$ for any p, thus we can suppose that at least one of the sequences $(x_3^{(p)})$ and $(\ell_3^{(p)})$ does not vanish.

We claim that $x_3^{(p)} = o(\ell_3^{(p)})$ is impossible. In fact, if it was the case, we could write for any p $x_3^{(p)} = u^{(p)} \ell_3^{(p)}$, with $u_3^{(p)} \to 0$.

But we can write, forgetting the superscripts:

$$R(x, \ell) = \ell_3^3 (1 - u) \mathcal{R}(x, \ell),$$

with

$$\mathcal{R}(x, \ell) = -4(x_2 + x_5 - \ell_2 - \ell_5)^2 (1 + u + u^2) + 3(x_6 - \ell_6)\ell_3 (1 + u^2)^2 (1 - u)$$
$$- 6(x_2 + x_5 - \ell_2 - \ell_5)(1 + u)(u x_5 - \ell_5).$$

Recall that $\ell_3^3(1 - u)(1 + u + u^2) = \ell_3^3 - x_3^3 \neq 0$, thus we have $\mathcal{R}(x, \ell) = 0$ and at the limit:

$$\lim \mathcal{R}(x, \ell) = -4(y_2 + y_5)^2 \neq 0.$$

This is impossible proving our claim.

We now suppose that $x_3^{(p)} \neq 0$ for each p, that $\ell_3^{(p)} = t^{(p)} x_3^{(p)}$ and $t^{(p)} \to s$, where s is a real number.

We replace in $R(x, \ell)$, getting (without superscript) $R(x, \ell) = x_3^3 (1 - t) \mathcal{R}(x, \ell)$

$$\mathcal{R}(x, \ell) = 4(x_2 + x_5 - \ell_2 - \ell_5)^2 (1 + t + t^2) + 3(x_6 - \ell_6) x_3 (1 + t)^2 (1 - t)$$
$$- 6(x_2 + x_5 - \ell_2 - \ell_5)(1 + t)(x_5 - t\ell_5),$$

passing to the limit, we get that s is a root of the equation:

$$0 = 4(y_2 + y_5)^2 (1 + s + s^2) - 6y_5(y_2 + y_5)(1 + s)$$
$$= 2(y_2 + y_5)\left[2(y_2 + y_5)s^2 + (2y_2 - y_5)s + (2y_2 - y_5)\right].$$

The second degree polynomial in s which is in the bracket has a real root, its discriminant is positive, thus:

$$\Delta = (2y_2 - y_5)^2 - 8(y_2 + y_5)(2y_2 - y_5) = -3(2y_2 - y_5)(2y_2 + 3y_5) \geq 0.$$

We thus have

$$\{y \in \mathrm{Cor}(\mathfrak{g}^\star) : \; y_2 + y_5 \neq 0\} \subset \{y : \; y_1 = y_3 = 0, \; (2y_2 - y_5)(2y_2 + 3y_5) \leq 0\},$$

and

$$\mathrm{Cor}(\mathfrak{g}^\star) = \overline{\{y \in C : \; y_2 + y_5 \neq 0\}} \subset \{y : \; y_1 = y_3 = 0, \; (2y_2 - y_5)(2y_2 + 3y_5) \leq 0\}$$
$$\subsetneq \mathrm{ICor}(\mathfrak{g}^\star).$$

We proved that $\mathrm{Cor}(\mathfrak{g}^\star)$ does not coincide with $\mathrm{ICor}(\mathfrak{g}^\star)$.

It remains to prove that we have the equality

$$\mathrm{Cor}(\mathfrak{g}^\star) = \{y : \; y_1 = y_3 = 0, \; (2y_2 - y_5)(2y_2 + 3y_5) \leq 0\}.$$

For this purpose we fix y such that the conditions $y_2 y_6 (y_2 + y_5) \neq 0$, $y_1 = y_3 = 0$ and $(2y_2 - y_5)(2y_2 + 3y_5) \leq 0$ are fulfilled. We solve the above equation, and find s:

$$s = \frac{1}{4(y_2 + y_5)}\left(-(2y_2 - y_5) \pm \sqrt{-3(2y_2 - y_5)(2y_2 + 3y_5)}\right).$$

Remark that $s \neq \pm 1$. Indeed, replacing s by 1 in our equation gives:

$$4(y_2 + y_5)^2(1 + s + s^2) - 6y_5(y_2 + y_5)(1 + s) = 12(y_2 + y_5)y_2 \neq 0,$$

similarly, replacing s by -1 gives:

$$4(y_2 + y_5)^2(1 + s + s^2) - 6y_5(y_2 + y_5)(1 + s) = 4(y_2 + y_5)^2 \neq 0.$$

We now use the same method than in the preceding example, we fix ε 'small', put $t(y, \varepsilon) = s(y) + \varepsilon$, where $s(y)$ is one of this solution, and choose

$$x(y, \varepsilon) = (x_1(y, \varepsilon), y_2, x_3(y, \varepsilon), y_4, y_5, y_6, y_7)$$

and $\qquad \ell(y, \varepsilon) = (x_1(y, \varepsilon), 0, t(y, \varepsilon)x_3(y, \varepsilon), 0, 0, 0, 0).$

It remains only $x_3(y, \varepsilon)$ and $x_1(y, \varepsilon)$ to determine.

Now $x_3(y, \varepsilon)$ is solution of the equation:

$$\begin{aligned}
0 &= 4(y_2 + y_5)^2(1 + (s + \varepsilon) + (s + \varepsilon)^2) + 3y_6 x_3(1 + s + \varepsilon)^2(1 - s - \varepsilon) \\
&\quad - 6(y_2 + y_5)(1 + s + \varepsilon)y_5 \\
&= 4(y_2 + y_5)^2 \varepsilon(1 + 2s + \varepsilon) + 3y_6 x_3(1 + s + \varepsilon)^2(1 - s - \varepsilon) - 6(y_2 + y_5)y_5\varepsilon.
\end{aligned}$$

Therefore: $\quad x_3(y, \varepsilon) = 2\varepsilon(y_2 + y_5)\dfrac{6y_5 - 4(y_2 + y_5)(1 + 2s + \varepsilon)}{3y_6(1 + s + \varepsilon)^2(1 - s - \varepsilon)}.$

Since $y_6(1 + s + \varepsilon)^2(1 - s - \varepsilon) \to y_6(1 + s)^2(1 - s) \neq 0$, we have $x_3(y, \varepsilon) \to 0$. Moreover we have

$$\frac{\partial}{\partial \varepsilon}(6y_5 - 4(y_2 + y_5)(1 + 2s + \varepsilon)) = -8(y_2 + y_5) \neq 0,$$

thus we can choose ε sufficiently small such that $x_3(y, \varepsilon) \neq 0$ is $\varepsilon \neq 0$.

Finally, we come back to the first equation $Q_1(x, \ell) = 0$, getting $x_1(y, \varepsilon)$:

$$x_1(y, \varepsilon) = \frac{1}{2(y_2 + y_5)}x_3(y, \varepsilon)^2(1 - (s + \varepsilon)^2).$$

Clearly $x_1(y, \varepsilon) \to 0$ and $x_1(y, \varepsilon) \neq 0$ for $\varepsilon \neq 0$ 'small'.

By our construction, we have $Q_1(x(y, \varepsilon), \ell(y, \varepsilon)) = 0$ and $R(x(y, \varepsilon), \ell(y, \varepsilon)) = 0$, thus as polynomial in x_1, Q_1 and Q_2 have a common root, namely $x_1(\ell, \varepsilon)$, and we have also, $Q_2(x(y, \varepsilon), \ell(y, \varepsilon)) = 0$, since $x_1(y, \varepsilon) = \ell_1(y, \varepsilon) \neq 0$, this proves that $x(y, \varepsilon)$ and $\ell(y, \varepsilon)$ belong to the same orbit in Ω. Moreover:

$$\lim_{\varepsilon \to 0}(x(y, \varepsilon), \ell(y, \varepsilon)) = (y, 0).$$

This proves that:

$$\{y \in \mathfrak{g}^\star : \ y_2 y_6(y_2 + y_5) \neq 0, \ y_1 = y_3 = 0 \text{ and } (2y_2 - y_5)(2y_2 + 3y_5) \leq 0\} \subset \mathrm{Cor}(\mathfrak{g}^\star),$$

thus:

$$\mathrm{Cor}(\mathfrak{g}^\star) \supset \{y \in \mathfrak{g}^\star : \ y_1 = y_3 = 0, \ (2y_2 - y_5)(2y_2 + 3y_5) \leq 0\}$$

$$= \overline{\{y \in \mathfrak{g}^\star : \ y_2 y_6(y_2 + y_5) \neq 0, \ y_1 = y_3 = 0, \ (2y_2 - y_5)(2y_2 + 3y_5) \leq 0\}}.$$

We finally conclude that, the cortex is the semi-algebraic set:

$$\mathrm{Cor}(\mathfrak{g}^\star) = \{y \in \mathfrak{g}^\star : y_1 = y_3 = 0 \text{ and } (2y_2 - y_5)(2y_2 + 3y_5) \leq 0\} \subsetneq \mathrm{ICor}(\mathfrak{g}^\star). \qquad \blacksquare$$

Remark 4.3. The cortex of the Lie algebra $\mathfrak{g}_{15}$ is also a semi-algebraic set and we can show that

$$\mathrm{Cor}(\mathfrak{g}_{15}^{\star}) = \{y \in \mathfrak{g}_{15}^{\star} : \quad y_1 = y_3 = 0, \ (2y_2 + y_4)(-2y_2 + 3y_4) \geq 0\} \subsetneq \mathrm{ICor}(\mathfrak{g}_{15}^{\star}).$$

Example 4.4. Let $\mathfrak{h} := \mathfrak{g}_{29}$ be the 7-dimensional (real) nilpotent Lie algebra with basis $(X_1, \ldots, X_7)$ and the non zero brackets

$$[X_7, X_6] = X_5, \quad [X_7, X_5] = X_4, \quad [X_7, X_4] = X_3, \quad [X_7, X_3] = X_2,$$
$$[X_7, X_2] = X_1, \quad [X_6, X_5] = X_3, \quad [X_6, X_4] = X_2, \quad [X_5, X_4] = X_1.$$

The minimal layer is $\Omega = \{x : \quad x_1 \neq 0\}$ and the generic coadjoint orbits are 4-dimensional. The invariant polynomials are x_1 and

$$P_1(x) = 2x_1 x_3 - x_2^2,$$
$$P_2(x) = 15x_1^4 x_6 + 15x_1^3(x_3 x_4 - x_2 x_5) - 15x_1^2 x_2 x_3^2 + 10x_1 x_2^3 x_3 - 2x_2^5.$$

Therefore $\mathrm{ICor}(\mathfrak{h}^{\star}) = \{y \in \mathfrak{h}^{\star} : y_1 = y_2 = 0\}$.

Now we give the following

Proposition 4.5. *The cortex of the Lie algebra $\mathfrak{h}$ is a proper algebraic set in* $\mathrm{ICor}(\mathfrak{h}^{\star})$ *given by*

$$\mathrm{Cor}(\mathfrak{h}^{\star}) = \{y \in \mathfrak{h}^{\star} : \quad y_1 = y_2 = y_3 = 0\} \subsetneq \mathrm{ICor}(\mathfrak{h}^{\star}).$$

Proof. As in Example 4.1., consider:

$$Q_1(x, \ell) = P_1(x) - P_1(\ell)|_{\ell_1 = x_1} = 2x_1(x_3 - \ell_3) - (x_2^2 - \ell_2^2),$$
$$Q_2(x, \ell) = P_2(x) - P_2(\ell)|_{\ell_1 = x_1} = 15x_1^4(x_6 - \ell_6) + 15x_1^3\big((x_3 x_4 - \ell_3 \ell_4) - (x_2 x_5 - \ell_2 \ell_5)\big)$$
$$- 15x_1^2(x_2 x_3^2 - \ell_2 \ell_3^2) + 10x_1(x_2^3 x_3 - \ell_2^3 \ell_3) - 2(x_2^5 - \ell_2^5).$$

We compute the resultant or we substitute x_1 from the first equation and we report it in the second one. Let $y \in \mathrm{Cor}(\mathfrak{h}^{\star})$ and suppose that $\ell_2 = tx_2$, then we get:

$$0 = 2x_1(x_3 - \ell_3) - x_2^2(1 - t^2),$$
$$0 = \frac{15}{16}x_2^8(x_6 - \ell_6)(1 - t^2)^4 + \frac{15}{8}x_2^6(1 - t^2)^3\big((x_3 x_4 - \ell_3 \ell_4) - x_2(x_5 - t\ell_5)\big)$$
$$- \frac{15}{4}x_2^5(1 - t^2)^2(x_3^2 - t\ell_3^2) + 5x_2^5(1 - t^2)(x_3 - t^3\ell_3) - 2x_2^5(1 - t^5).$$

Dividing by x_2^5 and assume that $t \to s \in \mathbb{R}$, then when ℓ tends to zero we get,

$$0 = -\frac{15}{4}(1 - s^2)^2 y_3^2 + 5(1 - s^2)y_3 - 2(1 - s^5).$$

On the other hand, since $x_1 \neq 0$, then $1 - t^2 \neq 0$, and by taking the limit:

$$0 = -\frac{15}{4}(1 - s)(1 + s)^2 y_3^2 + 5(1 + s)y_3 - 2(1 + s + s^2 + s^3 + s^4). \qquad (1)$$

Let a be a non zero real number. It is clear that if $(x, \ell) \to (y, 0)$, then we have $(ax, a\ell) \to (ay, 0)$; if ℓ is in the orbit of x, $a\ell$ is in the orbit of ax, $ax_1 \neq 0$ and if $x_2 = t\ell_2$, $ax_2 = at\ell_2$, thus if s is finite, we may substitute y_3 by ay_3, and we get:

$$0 = -\frac{15}{4}(1-s)(1+s)^2 a^2 y_3^2 + 5(1+s)ay_3 - 2(1+s+s^2+s^3+s^4).$$

Therefore: $0 = (a-1)y_3\left[(a+1)15(1-s)(1+s)^2 y_3 - 20(1+s)\right]$,   and if $y_3 \neq 0$,

we obtain $15(a+1)(1-s)(1+s)^2 y_3 - 20(1+s), \quad a \in \mathbb{R}$

Taking the derivative with respect to a, we get $15(1-s)(1+s)^2 y_3 = 0$, that is, $1 - s^2 = 0$. If $s = 1$ we obtain from eq. (1):

$$10y_3 = 10, \quad y_3 = 1.$$

But this is impossible since we get $ay_3 = 1$ (for any non zero real number a). Hence $s = -1$ and eq. (1) lead to

$$0 = -2(1-1+1-1+1) = -2.$$

This is also impossible. Therefore either $t \to \pm\infty$ or $y_3 = 0$.

In the first situation, we let $x_2 = u\ell_2$, with $u \to 0$ and the resultant is:

$$0 = \frac{15}{16}\ell_2^8(x_6 - \ell_6)(u^2 - 1)^4 + \frac{15}{8}\ell_2^6(u^2 - 1)^3\left((x_3 x_4 - \ell_3 \ell_4) - \ell_2(ux_5 - \ell_5)\right)$$

$$- \frac{15}{4}\ell_2^5(u_2^2 - 1)^2(ux_3^2 - \ell_3^2) + \ell_2^5(u^2 - 1)(u^3 x_3 - \ell_3) - 2\ell_2^5(u^5 - 1).$$

Taking the limit, after dividing by ℓ_2^5, we obtain $0 = 2$.

This is impossible, thus $y_3 = 0$ and

$$\mathrm{Cor}(\mathfrak{h}^\star) \subset \{y \in \mathfrak{h}^\star : y_1 = y_2 = y_3 = 0\} \subsetneq \mathrm{ICor}(\mathfrak{h}^\star).$$

Conversely let $y = (0, 0, 0, y_4, y_5, y_6, y_7) \in \mathfrak{h}^\star$ such that $y_5 \neq 0$, we can check that

$$x = (\varepsilon, \frac{y_6}{y_5}\varepsilon, 0, y_4, y_5, y_6, y_7) \text{ and } \ell = (\varepsilon, \frac{y_6}{y_5}\varepsilon, 0, 0, 0, 0, 0), \text{ with } \varepsilon \neq 0, \ \varepsilon \to 0$$

are in the same orbit since $Q_1(x, \ell) = Q_2(x, \ell) = 0$. Finally since $\mathrm{Cor}(\mathfrak{h}^\star)$ is closed in $\mathrm{ICor}(\mathfrak{h}^\star)$, we get:

$$\mathrm{Cor}(\mathfrak{h}^\star) = \{y \in \mathfrak{h}^\star : \ y_1 = y_2 = y_3 = 0\} \subsetneq \mathrm{ICor}(\mathfrak{h}^\star). \qquad \blacksquare$$

5.   The cortex of semi-direct product of vector groups: nilpotent case

5.1. The cortex of a unipotent representation

Let V be a real vector space or dimension m and $\{A_1, \ldots, A_n\}$ be a set of pairwise commuting matrices in $\mathbb{R}^{m \times m}$. Then $\mathfrak{h} = \sum_{j=1}^n \mathbb{R}A_j$ is an abelian sub-Lie algebra in $\mathfrak{gl}(m, \mathbb{R})$, and let $H = \exp \mathfrak{h}$, here $\exp \ : \mathfrak{gl}(m, \mathbb{R}) \to GL(m, \mathbb{R})$ denotes the exponential matrix mapping.

The group H is acting on V via

$$H \times V \to V, \quad (e^A, v) \mapsto e^A v.$$

In equivalent way, we have a continuous finite dimensional representation of the topological additive (abelian) group $\mathbb{R}^n$:

$$\pi : \mathbb{R}^n \to GL(V), \quad \mathbf{t} = (t_1, \ldots, t_n) \mapsto \pi(\mathbf{t}) = e^{\mathbf{tA}},$$

where $\qquad \mathbf{tA} := \sum_{j=1}^{n} t_j A_j, \quad \mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n, \quad \mathbf{A} = (A_1, \ldots, A_n).$

The orbit of $v \in V$ under π is denoted by $\mathcal{O}_v^\pi$ and is given by

$$\mathcal{O}_v^\pi = \{e^{\mathbf{tA}} v, \quad \mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n\}.$$

On other hand, the representation $\pi^\star$ on the dual space $V^\star$ of V derives from π as:

$$\pi^\star(\mathbf{t}) = {}^t(\pi(-\mathbf{t})), \quad \mathbf{t} \in \mathbb{R}^n.$$

The orbit, under $\pi^\star$, of $x \in V^\star$ is given

$$\mathcal{O}_\xi^{\pi^\star} = \{\pi^\star(\mathbf{t})x, \quad \mathbf{t} \in \mathbb{R}^n\}.$$

By Engel's theorem $H = \exp \mathfrak{h}$ may be viewed as a closed connected group of upper triangular matrices with 1 on the diagonal. To this end, let $(e_1, \ldots, e_m)$ be a Jordan-Hölder basis in $V^\star$ and fix $(N_1, \ldots, N_n)$ a basis in $\mathfrak{h}$ such that each N_j $(1 \le j \le n)$ is strictly lower triangular matrix. That is, it exists a non-singular matrix $P \in \mathbb{R}^{m \times m}$ such that

$$A_j = P N_j P^{-1}, \quad j = 1, \ldots, n.$$

In this paper we are concerned with the nilpotent case, that is, $\{A_1, \ldots, A_n\}$ is a set of pairwise commuting nilpotent matrices in $\mathbb{R}^{m \times m}$ and hence the representation π (respectively $\pi^\star$) is unipotent.

5.1.1. Parametrization of the orbits of $V^\star$

By Engel's theorem $H = \exp \mathfrak{h}$ may be viewed as a closed connected group of upper triangular matrices with 1 on the diagonal. To this end, let $(v_1, \ldots, v_m)$ be a Jordan-Hölder basis in V in which any matrix $A \in \mathfrak{h}$ is written in strictly upper triangular form and let $(e_1, \ldots, e_m)$ be the corresponding dual basis in $V^\star$. Following [9] (see also [7]), one has

Theorem 5.1. (Chevalley-Rosenlicht) *For $x \in V^\star$, there are $X_1, \ldots, X_k \in \mathfrak{h}$ such that the orbit of x is*

$$\mathcal{O}_x^{\pi^\star} := {}^t H \cdot \xi = \{\exp(t_1 {}^t X_1) \cdot \cdots \cdot \exp(t_k {}^t X_k)x, \quad t_1, \ldots, t_k \in \mathbb{R}\}.$$

The map $\varphi(t_1, \ldots, t_k) = (\exp(t_1 {}^t X_1) \cdots \exp(t_k {}^t X_k))x$ is a diffeomorphism between $\mathbb{R}^k$ and the orbit of x, which is a closed submanifold in $V^\star$.

Let $(e_1, \ldots, e_m)$ be a Jordan-Hölder basis in $V^\star$ and define a map Q and polynomials $Q_1, \ldots, Q_m$ such that

$$Q(t_1, \ldots, t_k) = (\exp(t_1{}^tX_1) \cdot \ldots \cdot \exp(t_k{}^tX_k))x = \sum_{i=1}^m Q_j(t_1, \ldots, t_k)e_j.$$

Then there are disjoint sets of indices $S \cup T = \{1, \ldots, m\}$ with $S = \{j_1 < \cdots < j_k\}$ such that Q_j depends only on the variables t_i with $j_i \le j$. Moreover

$$Q_{j_i} = t_i + (a\ polynomial\ in)\ t_1, \ldots, t_{i-1}, \qquad 1 \le i \le k.$$

Remark 5.2. Note that, since the action of H is unipotent, then for $x \in V^\star$ such that $S \ne \emptyset$, one has $S = \{j_1 < \cdots < j_k\} \subset \{2, \ldots, m\}$.

The space $V^\star$ is systematically split into disjoint ${}^tH$-invariant and disjoint layers and a similar parametrization exists for each layer (see [3, 9, 1]). We are concerned with the layer of generic orbits, and under the same assumptions we have

Theorem 5.3. (Theorem 3.1.6 [9]) *Let* $(e_1, \ldots, e_m)$ *be a Jordan-Hölder basis in* $V^\star$ *as in Theorem* (5.1). *Then there are disjoint sets of indices* $S \cup T = \{1, \ldots, m\}$, *a Zariski open set* $\mathcal{U} \subset V^\star$, *and rational functions* $Q_1(x, t), \ldots, Q_m(x, t)$ *of the variables* $(x, t) = (x_1, \ldots, x_m, t_1, \ldots, t_k)$ *where* $k = Card(S)$, *with the following properties: if* $S = \{j_1 < \cdots < j_k\}$ *and if we identify* $x \in V^\star$ *with* $(x_1, \ldots, x_m) \in \mathbb{R}^m$, *then*

- *The functions* $Q_1(x, t), \ldots, Q_m(x, t)$ *are rational non singular on* $\mathcal{U} \times \mathbb{R}^k$. *For fixed* x, *they are polynomial in* t.

- *For each* $x \in \mathcal{U}$, $Q(x, t) = \sum_{i=1}^m Q_i(x, t)e_i$ *maps* $\mathbb{R}^k$ *diffeomorphically onto the orbit* ${}^tH \cdot x$.

- *For fixed* $x \in \mathcal{U}$, $Q_j(x, t)$ *depends only on those* t_i *such that* $j_i \le j$.

- *If* $j \notin S$ *the* $Q_j(x, t) = x_j + R_j(x_1, \ldots, x_{j-1}, t_1, \ldots, t_i)$ *where* i *is the largest index such that* $j_i \le j$ *and* R_j *rational. Moreover* $Q_1(x, t) = x_1$.

- $Q_{j_i}(x, t) = t_i + x_{j_i} + R(x_1, \ldots, x_{j_i-1}, t_1, \ldots, t_{i-1})$.

With the same notations as in Theorem 5.3, we let

$$V_T^\star = span_{\mathbb{R}}\{e_j, \quad j \in T\}, \qquad V_S^\star = span_{\mathbb{R}}\{e_j, \quad j \in S\}.$$

Then every ${}^tH$-orbit in U meets $V_T^\star$ in a unique point, in particular $V_T^\star \cap \mathcal{U}$ is a Zariski open set in $V_T^\star$. On other hand the cross-section mapping

$$\Psi : U \to V_T^\star, \quad x \mapsto \Psi(x_1, \ldots, x_m) = \sum_{j \in T} \Psi_j(x)e_j, \tag{2}$$

is rational and ${}^tH$-invariant. Note that every Ψ_j has the form

$$\Psi_j(x_1, \ldots, x_m) = x_j + \varphi_j(x_1, \ldots, x_{j-1}), \quad j \in T, \tag{3}$$

where φ_j is rational invariant non-singular function on $\mathcal{U}$ $(j \in T)$.

Proposition 5.4. *With use the same notations as in Theorem* (5.3) *and we assume that* $\dim {}^t H \cdot x + j_1 = \dim V + 1$. *Then* $C_{V^\star}(\pi^\star)$ *is the vector space given by*

$$C_{V^\star}(\pi^\star) = \sum_{j=j_1}^{m} \mathbb{R} e_j.$$

Proof.   Let $x = (x_1, \ldots, x_m) \in \mathcal{U}$ then

$${}^t H \cdot x = \{y = (y_1, \ldots, y_m) \in \mathcal{U} : y_j = x_j \text{ for } j = 1, \ldots, j_1 - 1\}$$

$$= \{(x_1, \ldots, x_{j_1-1})\} \times \sum_{j=j_1}^{m} \mathbb{R} e_j.$$

Therefore $C_{V^\star}(\pi^\star) = \sum_{j=j_1}^{m} \mathbb{R} e_j$, completing the proof. ∎

Now let us consider the general settings, to this end recall that orbit of each $x \in U$ is given by

$$\mathcal{O}_x^{\pi^\star} = \{y = (y_1, \ldots, y_m) \in \mathcal{U} : \Psi(y) = \Psi(x)\}, \quad x = (x_1, \ldots, x_m) \in \mathcal{U},$$

where Ψ is the cross-section mapping and which can be written as

$$\Psi(x_1, \ldots, x_m) = \sum_{j=1}^{j_1-1} x_j e_j + \sum_{j \in T, j > j_1} \psi_j(x) e_j, \quad x = (x_1, \ldots, x_m) \in \mathcal{U}.$$

Each $\psi_j (j \in T, j > j_1)$ is a rational non singular function on U and $\pi^\star$-invariant. On other hand, since the representation $\pi^\star$ is unipotent, each coordinate function ψ_j can be written as:

$$\psi_j = \frac{U_j}{V_j}, \quad g.c.d(U_j, V_j) = 1, \quad U_j, V_j \in Pol(V^\star)^{\pi^\star}.$$

Now let $F : V^\star \times V^\star \to \mathbb{R}^{m+1-k-j_1}$ be the mapping given by

$$F(x, y) = ((x_j - y_j)_{1 \leq j \leq j_1-1}, (U_j(x) - U_j(y))_{j \in T, j > j_1}, (V_j(x) - V_j(y))_{j \in T, j > j_1}). \quad (4)$$

We see that $x, y \in U$ are in the same orbit if and only if $F(x, y) = 0$ with $x, y \in U$. Therefore we deduce that, if $x \in U$, its orbit is characterized by

$$\mathcal{O}_x^{\pi^\star} = \{y = (y_1, \ldots, y_m) \in \mathcal{U} : \quad F(x, y) = 0\}.$$

Due to the density of the Zariski open set $\mathcal{U}$ we deduce that

$$C_{V^\star}(\pi^\star) \subset Z_{V^\star}(\pi^\star), \quad (5)$$

where

$$Z_{V^\star}(\pi^\star) = \{x \in V^\star : x_i = U_j(x) = V_j(x) = 0, \ i = 1, \ldots, j_1 - 1, j \in T, \ j_1 < j\}.$$

Theorem 5.5. *Let* S, U *as in Theorem* (5.3). *If for any* $x = (x_1, \ldots, x_m) \in Z_{V^\star}(\pi^\star)$ *the map*

$$(x_i, y_i)_{i \in S} \longmapsto$$

$$\left(\left((U_j(x) - U_j(y)) \big|_{y_i = x_i, \ 1 \leq i \leq j_1-1} \right)_{j \in T, j_1 < j}, \left((V_j(x) - V_j(y)) \big|_{y_i = x_i, \ 1 \leq i \leq j_1-1} \right)_{j \in T, j_1 < j} \right).$$

is of maximal rank in $((x_{j_1}, \ldots, x_m), 0)$, *then* $C_{V^\star}(\pi^\star) = Z_{V^\star}(\pi^\star)$.

Proof. Two elements $x, y \in \mathcal{U}$, are in the same orbit if and only if $F(x, y) = 0$. Fix $x \in \mathcal{U}$ and by substituting each of y_k by x_k (for $k = 1, \ldots, j_1 - 1$) in (5.1.1), we consider the map $f : V^\star \times V^\star \longrightarrow \mathbb{R}^{2s}$ given by

$$f(x, y) =$$
$$\left(\big((U_j(x) - U_j(y)) |_{y_i = x_i, \ 1 \le i \le j_1 - 1} \big)_{j \in T, j_1 < j}, \ \big((V_j(x) - V_j(y)) |_{y_i = x_i, \ 1 \le i \le j_1 - 1} \big)_{j \in T, j_1 < j} \right).$$

If for any $x = (x_1, \ldots, x_m) \in Z_{V^\star}(\pi^\star)$ (note that $x_1 = \cdots = x_{j_1 - 1} = 0$), the rank of the map

$$(x_i, y_i)_{i \in S} \mapsto f(x, y)$$

is maximum in $((x_{j_1}, \ldots, x_m), 0)$, then, by implicit function Theorem, there is a sequence $(x^{(p)}, y^{(p)})$ of generic elements, converging to $(x, 0)$ and such that

$$f(x^{(p)}, y^{(p)}) = 0 \quad \text{for all } p \in \mathbb{N}.$$

This, together with the relation (5) we conclude $Z_{V^\star}(\pi^\star) = C_{V^\star}(\pi^\star)$, completing the proof. ∎

5.2. The cortex of $G = V \rtimes_\pi \mathbb{R}^n$: nilpotent case

First let's recall some of the results of [3, 10], if $G = V \rtimes_\pi \mathbb{R}^n$ be the group endowed with the law

$$(v, \mathbf{t}) \cdot (w, \mathbf{s}) = (v + \pi(\mathbf{t})w, \mathbf{t} + \mathbf{s}) = (v + e^{\mathbf{tA}}w, \mathbf{t} + \mathbf{s}),$$

where $v, w \in V$, $\mathbf{tA} = \sum_{i=1}^n t_i A_i$, $\mathbf{s} = \sum_{i=1}^n s_i A_i$, as in [4]. G is called the semidirect product of the vector groups V and $\mathbb{R}^n$. The Lie algebra of G is $\mathfrak{g} = V \times \mathfrak{h} \equiv V \times \mathbb{R}^n$ is equipped with the Lie bracket

$$[(v, \mathbf{t}), (w, \mathbf{s}) = (\mathbf{t} \cdot w - \mathbf{s} \cdot v, 0), \quad v, w \in V, \quad \mathbf{t}, \mathbf{s} \in \mathbb{R}^n,$$

where
$$\mathbf{t} \cdot v = \sum_{i=1}^n t_i A_i v,$$

so that
$$[(v, \mathbf{t}), (w, \mathbf{s})] = (\sum_{i=1}^n A_i (t_i w - s_i v), 0), \quad v, w \in V.$$

Since $\mathfrak{g} \equiv V \times \mathbb{R}^n$, $\mathrm{ad}_v := \mathrm{ad}_{(v,0)}$ and $\mathrm{ad}_{\mathbf{t}} := \mathrm{ad}_{(0,\mathbf{t})}$ can be written in 2×2 matrix form:

$$\mathrm{ad}_v = \begin{bmatrix} 0 & N_v \\ 0 & 0 \end{bmatrix}, \quad \mathrm{ad}_{\mathbf{t}} = \begin{bmatrix} \mathbf{tA} & 0 \\ 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} \mathbf{t} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\mathbf{tA} = \sum_{i=1}^n t_i A_i$ and $N_v : \mathfrak{h} \to V$ is the linear mapping (which we identify with its matrix) given by $N_v(\mathbf{s}) = -\mathbf{s} \cdot v$. Since $\mathrm{ad}_v^2 = 0$ thus

$$\mathrm{Ad}_{(v,\mathbf{t})} = \begin{bmatrix} e^{\mathbf{tA}} & N_v \\ 0 & I_n \end{bmatrix}.$$

Similarly, if $\mathfrak{g}^\star$ denotes the dual space of $\mathfrak{g}$, then $\mathfrak{g}^\star = V^\star \times \mathfrak{h}^\star$ and the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^\star$ is given by

$$\mathrm{ad}^\star_{(v,\mathbf{t})} = \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} -{}^t(\mathbf{tA}) & 0 \\ -{}^t N_v & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix},$$

where $\mathbf{t} = \sum_{i=1}^n t_i {}^t A_i$ and ${}^t N_{-v}$ is the map dual (or transpose) to N_v.

We next turn to the coadjoint action of G on $\mathfrak{g}^\star$, we get

$$\mathrm{Ad}^\star_{(v,\mathbf{t})} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} {}^t e^{-\mathbf{t}\mathbf{A}} & 0 \\ -{}^t N_v & I_n \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix},$$

From these formulae, we derive that, one has $spec(\mathrm{ad}_{(v,\mathbf{t})}) \subset \{0\} \cup spec\{\mathbf{t}\}$ with $\mathbf{t} = \sum_{i=1}^n t_i A_i$ (for more details see [10]). For $x \in V^\star$ let

$$\mathfrak{h}_x = \{\mathfrak{h} \ni A = \sum_{i=1}^n \mathbb{R}A_i : \quad {}^t Ax = 0\} := \ker \left[A \mapsto {}^t Ax \right],$$

and
$$\mathfrak{h}_x^\perp = \{\lambda \in \mathfrak{h}^\star : \ <\lambda, \mathfrak{h}_x> = 0\},$$

where $< , >$ is the duality pairing between $\mathfrak{h}^\star$ and $\mathfrak{h}$. By Lemma 15 ([10]), on has

$$\mathrm{Ad}^\star(G)(x, \lambda) = ((\mathrm{Ad}^\star H)x) \times (\lambda + \mathfrak{h}_x^\perp),$$

where
$$H = \{e^{\sum_{i=1}^n t_i A_i}, \quad t_1, \ldots, t_n \in \mathbb{R}\}.$$

Note that $\mathfrak{h}_x^\perp$ is a subspace in $\mathfrak{h}^\star$. Now we give the following

Theorem 5.6. *Let G be the connected and simply connected nilpotent Lie group given by $G = V \rtimes_\pi \mathbb{R}^n$ with Lie algebra $\mathfrak{g} = V \times \mathfrak{h}$ where $\mathfrak{h} = \sum_{j=1}^n \mathbb{R}A_j$ with $\{A_1, \ldots, A_n\}$ is a set of pairwise commuting nilpotent matrices in $\mathbb{R}^{m \times m}$ and V is a m-dimensional (real) vector space. The cortex of the Lie algebra $\mathfrak{g}^\star$ is given by*

$$\mathrm{Cor}(\mathfrak{g}^\star) \subset C_{V^\star}(\pi^\star) \times \mathfrak{h}_0^\perp, \quad \text{where} \quad \mathfrak{h}_0^\perp = \{\lim_{x \to 0} \lambda_x, \quad \lambda_x \in \mathfrak{h}_x^\perp\}.$$

Proof. This a direct consequence of the definition of $\mathrm{Cor}(\mathfrak{g}^\star)$ and the fact that

$$\mathrm{Ad}^\star(G)(x, \lambda) = \mathrm{Ad}^\star(H)x \times (\lambda + \mathfrak{h}_x^\perp). \qquad \blacksquare$$

Finally we conclude the paper by the following

Corollary 5.1. *Under the same assumptions as in Theorem (5.6) one has*

$$\mathfrak{h}_0^\perp = \overline{\bigcup_{x \in \mathcal{U}} \mathfrak{h}_x^\perp}.$$

Proof. First note that for any $x \in \mathcal{U}$ and $\varepsilon \neq 0$ ($\varepsilon \in \mathbb{R}$) one has $\mathfrak{h}_x = \mathfrak{h}_{\varepsilon x}$. This gives that if $\lambda \in \mathfrak{h}_x^\perp$, then $\lambda := \lambda_x = \lim_{\varepsilon \to 0} \lambda_{\varepsilon x}$ which is in $\mathfrak{h}_0^\star$, and so

$$\bigcup_{x \in \mathcal{U}} \mathfrak{h}_x^\perp \subset \mathfrak{h}_0^\perp.$$

On other hand, the cortex is closed, and thus

$$\overline{\bigcup_{x \in \mathcal{U}} \mathfrak{h}_x^\perp} \subset \overline{\mathfrak{h}_0^\perp} = \mathfrak{h}_0^\perp.$$

Conversely, if $\lambda \in \mathfrak{h}_0^\perp$, then for any norm on $\mathfrak{h}^\star$ and $\varepsilon > 0$, there exists $x \in \mathcal{U}$ and $\lambda_x \in \mathfrak{h}_x^\perp \subset \bigcup_{y \in \mathcal{U}} \mathfrak{h}_y^\perp$ such that $||\lambda - \lambda_x|| < \varepsilon$, which is equivalent to

$$\mathfrak{h}_0^\perp \subset \overline{\bigcup_{x \in \mathcal{U}} \mathfrak{h}_x^\perp}. \qquad \blacksquare$$

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