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Unified Products for Braided Lie Bialgebras with Applications

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Abstract. We construct unified products for braided Lie bialgebras. Some special cases of unified products such as crossed product and matched pair of braided Lie bialgebras are studied. It is proved that the extending problem for Lie bialgebras can be classified by some non-abelian cohomology theory of braided Lie bialgebras. As a byproduct, a non-abelian extension theory of Lie bialgebras is developed. Furthermore, one dimensional flag extending systems of Lie bialgebras are also investigated.

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Key Words: Lie bialgebra, braided Lie bialgebras, unified product, non-abelian cohomology, Yetter-Drinfeld modules.

1. Introduction

The notion of Lie bialgebras was introduced by Drinfeld in his remarkable work [12], where he also introduced the Drinfeld double Lie bialgebra $D(\mathfrak{g}) = \mathfrak{g} \bowtie \mathfrak{g}^{*op}$. Some years later, the theory of matched pairs of Lie algebras $(\mathfrak{g}, \mathfrak{m})$ was introduced by Majid in [16]. The double cross sum space $\mathfrak{g} \bowtie \mathfrak{m}$ is more general than Drinfeld classical double since $\mathfrak{m}$ need not to be the dual space of $\mathfrak{g}$. In [23], Sommerhäuser introduced the concept of braided Lie bialgebras (he call it Yetter-Drinfeld Lie algebra) to give a construction of symmetrizable Kac-Moody algebras. It is in fact an algebraic object in the module category ${}_{D(\mathfrak{g})}\mathcal{M}$. The theory of braided Lie bialgebras was also developed further by Majid in [18], where the bosonisation theorem for braided Lie bialgebras are proved. See also Grabowski's papers [13, 14, 15] for some concrete examples and [25, 26, 27] for ordinary Lie bialgebras. For the theory of Yetter-Drinfeld Hopf algebra, see [10, 11, 22, 24, 28].

There is a close relation between extension theory and cross product Lie bialgebras, see Masuoka [19]. For an abelian Lie bialgebra A, an abelian extension theory for Lie bialgebras was developed by Benayed in [7, 8, 9]. The general non-abelian Lie bialgebras case was not known since then. One of the motivations of this paper is to give a non-abelian extension theory for Lie bialgebras.

On the other hand, the theory of unified product and extending structure for many types of algebras were well developed by Agore and Militaru in [1, 2, 3, 4, 5, 6]. Let A be a Lie (associative, Leibniz, etc.) algebra and E a vector space containing A

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as a subspace. The extending problem is to describe and classify all Lie (associative, Leibniz, etc.) algebras structures on E such that A is a subalgebra of E. They show that associated to any extending structure of A by a complement space V, there is a unified product on the direct sum space $E \cong A \oplus V$. Recently, extending structures for 3-Lie algebras and Lie conformal superalgebras were studied in [31, 32].

It is a natural question whether can we develop a unified product theory to solve the extending problem for Lie bialgebras. In fact, this is more complicated than the theory founded in [4, 2, 6]. Thanks to the theory of braided Lie bialgebra developed by Sommerhäuser, we find that this problem can be solved affirmatively. In this paper, we give a most general construction of Lie bialgebras by using Sommerhäuser's braided Lie bialgebra, see Theorem 3.9 and Theorem 3.29. We will study unified products and extending systems for braided Lie bialgebras. In order to doing so, we have to introduced some new concepts of generalized Lie algebras and Lie coalgebras, such as σ-Lie algebras, Q-Lie coalgebras, see Definition 3.11. We also show how to classify extending systems for braided Lie bialgebras. As a by-product, a non-abelian extension theory for non-abelian Lie bialgebras is found, see Theorem 3.30, Theorem 4.1 and Theorem 4.2. Furthermore, one dimensional flag extending systems of Lie bialgebras are also investigated. The key observation of this paper is that we found Sommerhäuser's braided Lie bialgebra appeared naturally and play an important role in considering extending problem for Lie bialgebras.

The organization of this paper is as follows. In section 2, we review some basic facts and notations about Lie bialgebras and braided Lie bialgebras. In section 3, the definition of unified product for braided Lie bialgebras is introduced. We give the necessary and sufficient conditions for a unified product to form Lie bialgebras. In the last section 4, we study some applications of unified products, which include extending problem for braided Lie bialgebras and Lie bialgebras. We also study the flag extending systems.

Throughout this paper, all Lie algebras are assumed to be over an algebraically closed field k of characteristic different from 2 and 3. The space of linear maps from V to W is denoted by $\mathrm{Hom}(V, W)$. The identity map of a vector space V is denoted by $\mathrm{id}_V : V \to V$ or simply by $\mathrm{id} : V \to V$. The twisting maps $\tau : L \otimes L \to L \otimes L, \tau_{12}, \tau_{23} : L \otimes L \otimes L \to L \otimes L \otimes L$ are denoted by

$$\tau(a \otimes b) = b \otimes a, \quad \tau_{12}(a \otimes b \otimes c) = b \otimes a \otimes c, \quad \tau_{23}(a \otimes b \otimes c) = a \otimes c \otimes b.$$

2. Preliminaries

Definition 2.1. [12] A *Lie bialgebra H* is a vector space equipped simultaneously with a Lie algebra structure $(H, [\cdot,\cdot])$ and a Lie coalgebra (H, δ) structure such that the following compatibility condition is satisfied,

$$\text{(LB)} \quad \delta([a, b]) = \sum [a, b_1] \otimes b_2 + \sum b_1 \otimes [a, b_2] + \sum a_1 \otimes [a_2, b] + \sum [a_1, b] \otimes a_2,$$

where we use the sigma notation $\delta(a) = \sum a_1 \otimes a_2 = a_1 \otimes a_2$ and we denote it by $(H, [\cdot,\cdot], \delta)$. ■

We can also write the above equation as ad-actions on tensors by

$$\delta([a, b]) = [a, \delta(b)] + [\delta(a), b], \text{ where}$$

$$[a, \delta(b)] := \sum [a, b_1] \otimes b_2 + \sum b_1 \otimes [a, b_2], \ [\delta(a), b] := \sum a_1 \otimes [a_2, b] + \sum [a_1, b] \otimes a_2,.$$

A homomorphism of Lie bialgebras $\varphi : (H, [\cdot,\cdot], \delta) \to (H', [\cdot,\cdot]', \delta')$ is both a homomorphism of Lie algebras and a homomorphism of Lie coalgebras, i.e.,

$$\varphi([a,b]) = [\varphi(a), \varphi(b)]', \quad \delta'\varphi(a) = (\varphi\otimes\varphi)\delta(a) \quad \text{for all } a, b \in H,$$

Let A, H be both Lie algebras and Lie coalgebras. For $a, b \in A$, $x, y \in H$, we define the maps

$$\alpha : H\otimes A \to A, \quad \beta : H\otimes A \to H, \quad \phi : A \to H\otimes A, \quad \psi : H \to H\otimes A$$

by
$$\alpha(x\otimes a) = x \triangleright a, \qquad \beta(x\otimes a) = x \triangleleft a,$$
$$\phi(a) = \sum a_{(-1)} \otimes a_{(0)}, \qquad \psi(x) = \sum x_{(0)} \otimes x_{(1)}.$$

We now fix some notions. For a Lie algebra H and a linear map $\alpha : H\otimes A \to A$ with

$$[x,y] \triangleright a = x \triangleright (y \triangleright a) - y \triangleright (x \triangleright a), \quad \text{for all } x, y \in H,\ a \in A,$$

(A, α) is called a *left H-module.*
For a Lie coalgebra H and a linear map $\phi : A \to H\otimes A$ such that

$$\sum \delta_H(a_{(-1)}) \otimes a_{(0)} = \sum a_{(-1)} \otimes \phi(a_{(0)}) - \sum \tau_{12}\left(a_{(-1)} \otimes \phi(a_{(0)})\right),$$

(A, ϕ) is called a *left H-comodule.* If H and A are Lie algebras, A is a left H-module and

$$x \triangleright [a,b] = [x \triangleright a, b] + [a, x \triangleright b],$$

then $(A, [,], \alpha)$ is called a *left H-module Lie algebra.* If H is a Lie coalgebra and A is a Lie algebra, A is a left H-comodule and

$$\phi([a,b]) = \sum a_{(-1)} \otimes [a_{(0)}, b] + \sum b_{(-1)} \otimes [a, b_{(0)}],$$

then A is called a *left H-comodule Lie algebra.* Right Lie (co)module and Lie (co)module Lie (co)algebra can be defined similarly, see [17].

Definition 2.2. Let $(A, [\cdot,\cdot])$ be a given Lie algebra (Lie coalgebra, Lie bialgebra), E a vector space. An extending system of A through V is a Lie algebra (Lie coalgebra, Lie bialgebra) on E such that V a complement subspace of A in E, the canonical injection map $i : A \to E, a \mapsto (a, 0)$ or the canonical projection map $p : E \to A, (a, x) \mapsto a$ is a Lie algebra (Lie coalgebra, Lie bialgebra) homomorphism. The extending problem is to describe and classify up to an isomorphism the set of all Lie algebra (Lie coalgebra, Lie bialgebra) structures that can be defined on E. ■

We remark that our definition of extending system of A through V contains not only extending structure in [3, 1, 4] but also the global extension structure in [5]. The reason is that when we consider extending problem for Lie bialgebras, both of them are necessarily used, this will be clear in the context of next two sections. Note that in our extending system we do not demand $(A, [\cdot,\cdot])$ to be a subalgebra of E although A is always a Lie algebra. In fact, the canonical injection map $i : A \to E$ is a Lie (co)algebra homomorphism if and only if A is a Lie sub-(co)algebra of E.

Definition 2.3. Let A be a Lie algebra (Lie coalgebra, Lie bialgebra), E be a Lie algebra (Lie coalgebra, Lie bialgebra) such that A is a subspace of E and V a complement of A in E. Let $(E,[\cdot,\cdot])$ and $(E',[\cdot,\cdot]')$ be two Lie algebra (Lie coalgebra, Lie bialgebra) structures on E. For a linear map $\varphi : E \to E'$ we consider the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & E & \xrightarrow{\pi} & V & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}_A} & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \mathrm{id}_V} & & \\ 0 & \longrightarrow & A & \xrightarrow{i'} & E' & \xrightarrow{\pi'} & V & \longrightarrow & 0. \end{array} \tag{1}$$

where $\pi : E \to V$ is the canonical projection of $E = A \oplus V$ onto V and $i : A \to E$ is the inclusion map. We say that $\varphi : E \to E'$ *stabilizes* A if the left square of the diagram (1) is commutative. Two extending system $(E,[\cdot,\cdot])$ and $(E',[\cdot,\cdot]')$ are called *equivalent*, and we denote this by $(E,[\cdot,\cdot]) \equiv (E',[\cdot,\cdot]')$, if there exists a Lie algebra (Lie coalgebra, Lie bialgebra) isomorphism $\varphi : (E,[\cdot,\cdot]) \to (E',[\cdot,\cdot]')$ which stabilizes A. Denote by $Extd(E,A)$ ($CExtd(E,A)$, $BExtd(E,A)$) the set of equivalent classes of Lie algebra (Lie coalgebra, Lie bialgebra) structures on E. ■

3. Unified product for braided Lie bialgebras

In this section, we construct unified product for braided Lie bialgebras. First, we review the notion of matched pairs of Lie algebras and Lie coalgebras.

Definition 3.1. ([16]) Assume that A and H are Lie algebras. If (A,α) is a left H-module, (H,β) is a right A-module, and the following (M1) and (M2) hold, then (A,H,α,β) (or (A,H)) is called a *matched pair of Lie algebras.*

(BB1) $x \triangleright [a,b] = [x \triangleright a, b] + [a, x \triangleright b] + (x \triangleleft a) \triangleright b - (x \triangleleft b) \triangleright a$,

(BB2) $[x,y] \triangleleft a = [x, y \triangleleft a] + [x \triangleleft a, y] + x \triangleleft (y \triangleright a) - y \triangleleft (x \triangleright a)$. ■

Lemma 3.2. [16] *Let (A,H) be a matched pair of Lie algebras, then we get a new Lie algebra on the vector space $A \oplus H$ with bracket given by*

$$[(a,x),(b,y)] = ([a,b] + x \triangleright b - y \triangleright a, [x,y] + x \triangleleft b - y \triangleleft a).$$

We denoted it by $A \bowtie H$.

The dual version is the matched pair of Lie coalgebras which is introduced in [29, 30].

Definition 3.3. Two Lie coalgebras (A,H) form a *matched pair of Lie coalgebras* if (A,ϕ) is a left H-comodule and (H,ψ) is a right A-comodule, obeying the conditions

(BB3) $(\mathrm{id} \otimes \delta)\phi = (\phi \otimes \mathrm{id})\delta + (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\delta + (\psi \otimes \mathrm{id})\phi + (\mathrm{id} \otimes \tau)(\psi \otimes \mathrm{id})\phi$,

(BB4) $(\delta \otimes \mathrm{id})\psi = (\mathrm{id} \otimes \psi)\delta + (\mathrm{id} \otimes \tau)(\mathrm{id} \otimes \psi)\delta + (\mathrm{id} \otimes \phi)\psi + (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\psi$.

In sigma notation, the above conditions are

(BB3) $$\sum a_{(-1)} \otimes \delta_A(a_{(0)}) = \sum \phi(a_1) \otimes a_2 + \sum \tau_{12}\left(a_1 \otimes \phi(a_2)\right) + \sum \psi(a_{(-1)}) \otimes a_{(0)} - \sum \tau_{23}\left(\psi(a_{(-1)}) \otimes a_{(0)}\right),$$

(BB4) $$\sum \delta_H(x_{(0)}) \otimes x_{(1)} = \sum x_1 \otimes \psi(x_2) + \sum \tau_{23}\left(\psi(x_1) \otimes x_2\right) + \sum x_{(0)} \otimes \psi(x_{(1)}) - \sum \tau_{12}\left(x_{(0)} \otimes \psi(x_{(1)})\right).$$ ■

Lemma 3.4. [29, 30] *Let (A,H) be a matched pair of Lie coalgebras. We define $E = A \bowtie H$ as the vector space $A \oplus H$ with Lie cobracket*

$$\delta_E(a) = (\delta_A + \phi - \tau\phi)(a), \quad \delta_E(x) = (\delta_H + \psi - \tau\psi)(x),$$

that is

$$\delta_E(a) = \sum a_1 \otimes a_2 + \sum a_{(-1)} \otimes a_{(0)} - \sum a_{(0)} \otimes a_{(-1)},$$

$$\delta_E(x) = \sum x_1 \otimes x_2 + \sum x_{(0)} \otimes x_{(1)} - \sum x_{(1)} \otimes x_{(0)}.$$

Then $A \bowtie H$ is a Lie coalgebra.

Definition 3.5. [23] Let H be simultaneously a Lie algebra and a Lie coalgebra. If V is a left-H module and left H-comodule, satisfying

(YD1) $\quad \phi(x \triangleright v) = [x, v_{(-1)}] \otimes v_{(0)} + v_{(-1)} \otimes x \triangleright v_{(0)} + x_1 \otimes x_2 \triangleright v,$

then V is called a *left Yetter-Drinfeld module* over H. ■

We denote the category of Yetter-Drinfeld modules over H by ${}^H_H\mathcal{M}$. It can be shown that ${}^H_H\mathcal{M}$ form a monoidal category if H is Lie bialgebra [18].

Definition 3.6. Let A be simultaneously a Lie algebra and a Lie coalgebra. If V is a right A-module and right A-comodule, satisfying

(YD2) $\quad \psi(v \triangleleft a) = v_{(0)} \otimes [v_{(1)}, a] + v_{(0)} \triangleleft a \otimes v_{(1)} + v \triangleleft a_1 \otimes a_2,$

then V is called a *right Yetter-Drinfeld module over A*. ■

We denote the category of Yetter-Drinfeld modules over A by $\mathcal{M}^A_A$.

Definition 3.7. [23] If A be a Lie algebra and Lie coalgebra and H is a right Yetter-Drinfeld module over A, we call H a *braided Lie bialgebra* in $\mathcal{M}^A_A$, if the following condition is satisfied

(LBS) for H: $\quad \delta([x,y]) = [x, \delta(y)] + [\delta(x), y] - s(x \otimes y),$

where $s(x \otimes y) = x_{(0)} \otimes y \triangleleft x_{(1)} + x \triangleleft y_{(1)} \otimes y_{(0)} - y_{(0)} \otimes x \triangleleft y_{(1)} - y \triangleleft x_{(1)} \otimes x_{(0)}$. ■

In the rest of this section, we construct the double cross biproduct of braided Lie bialgebras. First, we give the condition for A to be a braided Lie bialgebra in ${}^H_H\mathcal{M}$:

(LBS) for A: $\quad \delta([a,b]) = [a, \delta(b)] + [\delta(a), b] - s(a \otimes b),$

where $s(a \otimes b) = b_{(-1)} \triangleright a \otimes b_{(0)} + a_{(0)} \otimes a_{(-1)} \triangleright b - a_{(-1)} \triangleright b \otimes a_{(0)} - b_{(0)} \otimes b_{(-1)} \triangleright a$.

Definition 3.8. Let A, H be two braided Lie bialgebras. If the following conditions hold:

(BB5) $\quad \delta_A(x \triangleright a) = x \triangleright a_1 \otimes a_2 + a_1 \otimes x \triangleright a_2 + x_{(0)} \triangleright a \otimes x_{(1)} - x_{(1)} \otimes x_{(0)} \triangleright a,$

(BB6) $\quad \delta_H(x \triangleleft a) = x_1 \otimes x_2 \triangleleft a + x_1 \triangleleft a \otimes x_2 + a_{(-1)} \otimes x \triangleleft a_{(0)} - x \triangleleft a_{(0)} \otimes a_{(-1)},$

(BB7) $\quad \phi([a,b]) = a_{(-1)} \otimes [a_{(0)}, b] + b_{(-1)} \otimes [a, b_{(0)}] + a_{(-1)} \triangleleft b \otimes a_{(0)} - b_{(-1)} \triangleleft a \otimes b_{(0)},$

(BB8) $\quad \psi([x,y]) = [x, y_{(0)}] \otimes y_{(1)} + [x_{(0)}, y] \otimes x_{(1)} + y_{(0)} \otimes x \triangleright y_{(1)} - x_{(0)} \otimes y \triangleright x_{(1)},$

(YDB) $\quad \phi(x \triangleright a) + \psi(x \triangleleft a) = [x, a_{(-1)}] \otimes a_{(0)} + a_{(-1)} \otimes x \triangleright a_{(-1)} + x_1 \otimes x_2 \triangleright a$
$\qquad + x_{(0)} \otimes [x_{(1)}, a] + x_{(0)} \triangleleft a \otimes x_{(1)} + x \triangleleft a_1 \otimes a_2,$

then (A, H) is called a *matched pair* of braided Lie bialgebras. ■

Theorem 3.9. [29, 30] *Let (A,H) be matched pair of Lie algebras and Lie coalgebras. If we define the double cross biproduct of A and H, denoted by $A \bowtie H$, $A \bowtie H = A \bowtie H$ as Lie algebra, $A \bowtie H = A \blacktriangleright\!\!\blacktriangleleft H$ as Lie coalgebra, then $A \bowtie H$ become a Lie bialgebra if and only if A is a braided Lie bialgebra in ${}^H_H\mathcal{M}$, H is a braided Lie bialgebra in $\mathcal{M}^A_A$, and (A,H) is a matched pair of braided Lie bialgebra.*

Example 3.10. Let $\mathfrak{g}$ be a finite-dimensional Lie bialgebra and $\mathfrak{g}^*$ its dual Lie bialgebras, then $(\mathfrak{g},\mathfrak{g}^*)$ is a matched pair of Lie bialgebras by the adjoint action $\alpha = \mathrm{ad}$ and coadjoint action $\beta = \mathrm{coad}$. Then the condition (YDB) is reduced to

$$\sum x_1 \otimes x_2 \triangleright a + \sum x \triangleleft a_1 \otimes a_2 = 0,$$

and $D(\mathfrak{g}) = \mathfrak{g} \bowtie \mathfrak{g}^{*op}$ becomes a Lie bialgebra which is called the classical Drinfeld double for Lie bialgebras. It was proved by Majid in [18] that if $\mathfrak{g}$ be a finite-dimensional quasitriangular Lie bialgebra and $\underline{\mathfrak{g}}^*$ the dual of its transmutation. Its bosonisation $\underline{\mathfrak{g}}^* \rtimes\!\!\!\triangleleft\, \mathfrak{g}$ is isomorphic as a Lie bialgebra to the Drinfeld double $D(\mathfrak{g})$. ■

Next, we will introduce the concept of unified product for braided Lie bialgebras. Define the maps

$$\sigma : H \otimes H \to A, \quad \theta : A \otimes A \to H, \quad P : A \to H \otimes H, \quad Q : H \to A \otimes A$$

by
$$\sigma(x,y) \in A, \quad \theta(a,b) \in H,$$
$$P(a) = \sum a_{[1]} \otimes a_{[2]} \in H \otimes H, \quad Q(x) = \sum x_{<1>} \otimes x_{<2>} \in A \otimes A.$$

An antisymmetric bilinear map $\sigma : H \otimes H \to A$ is called a *cocycle on H* if

(CC1) $x \triangleright \sigma(y,z) + y \triangleright \sigma(z,x) + z \triangleright \sigma(x,y) = \sigma([x,y],z) + \sigma([y,z],x) + \sigma([z,x],y)$.

An antisymmetric bilinear map $\theta : A \otimes A \to H$ is called a *cocycle on A* if

(CC2) $\theta(a,b) \triangleleft c + \theta(b,c) \triangleleft a + \theta(c,a) \triangleleft b = \theta(a,[b,c]) + \theta(b,[c,a]) + \theta(c,[a,b])$.

A co-antisymmetric linear map $P : A \to H \otimes H$ is called a cycle on A if

(CC3) $a_{(-1)} \otimes P(a_{(0)}) + \tau_{12}\tau_{23}\left(a_{(-1)} \otimes P(a_{(0)})\right) + \tau_{23}\tau_{12}\left(a_{(-1)} \otimes P(a_{(0)})\right)$
$= \delta(a_{[1]}) \otimes a_{[2]} + \tau_{12}\tau_{23}\left(\delta(a_{[1]}) \otimes a_{[2]}\right) + \tau_{23}\tau_{12}\left(\delta(a_{[1]}) \otimes a_{[2]}\right)$.

A co-antisymmetric linear map $Q : H \to A \otimes A$ is called a *cycle on H* if

(CC4) $Q(x_{(0)}) \otimes x_{(1)} + \tau_{12}\tau_{23}\left(Q(x_{(0)}) \otimes x_{(1)}\right) + \tau_{23}\tau_{12}\left(Q(x_{(0)}) \otimes x_{(1)}\right)$
$= x_{<1>} \otimes \delta(x_{<2>}) + \tau_{12}\tau_{23}\left(x_{<1>} \otimes \delta(x_{<2>})\right) + \tau_{12}\tau_{23}\left(x_{<1>} \otimes \delta(x_{<2>})\right)$.

In the following definitions, we introduce the new concept of cocycle Lie algebras and cycle Lie coalgebras, which are in fact not really ordinary Lie algebras and Lie coalgebras, but generalized ones.

Definition 3.11. (i): Let $\sigma : H \otimes H \to A$ be a cocycle on a vector space H equipped with an antisymmetric bilinear map $[\cdot,\cdot] : H \otimes H \to H$, satisfying the the following cocycle Jacobi identity:

(CC5) $[[x,y],z] + [[y,z],x] + [[z,x],y] = x \triangleleft \sigma(y,z) + y \triangleleft \sigma(z,x) + z \triangleleft \sigma(x,y)$.

Then H is called a *σ-Lie algebra.*

(ii) Let $\theta : A \otimes A \to H$ be a cocycle on a vector space A equipped with an anti-symmetric bilinear map $[\cdot,\cdot] : A \otimes A \to A$, satisfying the following cocycle Jacobi identity:

(CC6) $$[[a,b],c] + [[b,c],a] + [[c,a],b] = \theta(a,b) \triangleright c + \theta(b,c) \triangleright a + \theta(c,a) \triangleright b.$$

Then A is called a *θ-Lie algebra.*

(iii) Let $P : A \to H \otimes H$ be a cycle on a vector space H equipped with a co-antisymmetric linear map $\delta : H \to H \otimes H$, satisfying the the following cycle co-Jacobi identity:

(CC7) $$\delta(x_1) \otimes x_2 + \tau_{12}\tau_{23}\left(\delta(x_1) \otimes x_2\right) + \tau_{23}\tau_{12}\left(\delta(x_1) \otimes x_2\right) \\ = x_{(0)} \otimes P(x_{(1)}) + \tau_{12}\tau_{23}\left(x_{(0)} \otimes P(x_{(1)})\right) + \tau_{23}\tau_{12}\left(x_{(0)} \otimes P(x_{(1)})\right).$$

Then H is called a P-Lie coalgebra.

(iv) Let $Q : H \to A \otimes A$ be a cycle on a vector space A equipped with a co-antisymmetric linear map $\delta : A \to A \otimes A$, satisfying the following cycle co-Jacobi identity:

(CC8) $$\delta(a_1) \otimes a_2 + \tau_{12}\tau_{23}\left(\delta(a_1) \otimes a_2\right) + \tau_{23}\tau_{12}\left(\delta(a_1) \otimes a_2\right) \\ = Q(a_{(-1)}) \otimes a_{(0)} + \tau_{12}\tau_{23}\left(Q(a_{(-1)}) \otimes a_{(0)}\right) + \tau_{23}\tau_{12}\left(Q(a_{(-1)}) \otimes a_{(0)}\right).$$

Then A is called a *Q-Lie coalgebra.* ∎

Note that when $A = k$ or $H = k$, then our σ-Lie algebra or θ-Lie algebra is in fact the ω-Lie algebras introduced by Nurowski in [21] and studied by Zusmanovich in [33]. When σ, θ, P, Q are zero maps, then we obtain ordinary Lie algebras and Lie coalgebras.

Example 3.12. A Lie quasi-bialgebra is a triple $(\mathfrak{g}, \chi)$, where $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ is a finite-dimensional Lie algebra, $[\cdot,\cdot]_{\mathfrak{g}^*} : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$, and $\chi \in \wedge^3\mathfrak{g}$, satisfying compatibility conditions which are equivalent to the requirement that $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ is a Lie algebra with respect to the brackets:

$$[(u,0),(v,0)]_{\mathfrak{d}} = ([u,v]_{\mathfrak{g}}, 0), \quad [(v,0),(0,\mu)]_{\mathfrak{d}} = \left(-\operatorname{ad}^*_{\mu} v, \operatorname{ad}^*_{v} \mu\right),$$

$$[(0,\mu)(0,\nu)]_{\mathfrak{d}} = (\chi(\mu,\nu), [\mu,\nu]_{\mathfrak{g}^*}), \quad \text{for } u, v \in \mathfrak{g} \text{ and } \mu, \nu \in \mathfrak{g}^*.$$

The Lie algebra $(\mathfrak{d}, [\cdot,\cdot]_{\mathfrak{d}})$ is called the *Drinfeld double* of the Lie quasi-bialgebra $(\mathfrak{g}, \chi)$. If we define $\sigma(\mu,\nu) = \chi(\mu,\nu)$, then $(\mathfrak{g}^*, [\cdot,\cdot]_{\mathfrak{g}^*})$ is a σ-Lie algebra.

Theorem 3.13. *Let A be a θ-Lie algebra and H a σ-Lie algebra. Then $E = A \oplus H$ is a Lie algebra with bracket given by*

$$[(a,x),(b,y)] := ([a,b] + x \triangleright b - y \triangleright a + \sigma(x,y), [x,y] + x \triangleleft b - y \triangleleft a + \theta(a,b)) \quad (2)$$

if and only if the following compatibility conditions hold:

(TM1) $$[x,y] \triangleright a + [\sigma(x,y), a] = x \triangleright (y \triangleright a) - y \triangleright (x \triangleright a) + \sigma(x, y \triangleleft a) + \sigma(x \triangleleft a, y),$$

(TM2) $$x \triangleleft [a,b] + [x, \theta(a,b)] = (x \triangleleft a) \triangleleft b - (x \triangleleft b) \triangleleft a + \theta(x \triangleright a, b) + \theta(a, x \triangleright b),$$

(TBB1) $$x \triangleright [a,b] + \sigma(x, \theta(a,b)) = [x \triangleright a, b] + [a, x \triangleright b] + (x \triangleleft a) \triangleright b - (x \triangleleft b) \triangleright a,$$

(TBB2) $$[x,y] \triangleleft a + \theta(\sigma(x,y), a) = [x, y \triangleleft a] + [x \triangleleft a, y] + x \triangleleft (y \triangleright a) - y \triangleleft (x \triangleright a).$$

In this case, (A, H) is called a cocycle cross product system. *This Lie algebra will be denoted by $A_{\alpha,\theta}\#_{\beta,\sigma}H$.*

Proof. We have to check whether $[x,[a,b]_E]_E+[a,[b,x]_E]_E+[b,[x,a]_E]_E=0$.

In fact,
$$[x,[a,b]_E]_E = x\triangleright[a,b]+x\triangleleft[a,b]+[x,\theta(a,b)]+\sigma(x,\theta(a,b)),$$
$$[a,[b,x]_E]_E = -[a,x\triangleright b]-\theta(a,x\triangleright b)+(x\triangleleft b)\triangleright a+(x\triangleleft b)\triangleleft a,$$
$$[b,[x,a]_E]_E = [b,x\triangleright a]+\theta(b,x\triangleright a)-(x\triangleleft a)\triangleright b-(x\triangleleft a)\triangleleft b.$$

By (TM2) and (TBB1) we get the result. The other cases can easily be checked too. ■

There are two cases for $(A,[\cdot,\cdot]$ to be a Lie algebra. The first case is when $\alpha=0$, $\theta\neq 0$. From (TBB1) we get $\sigma(x,\theta(a,b))=0$, since $\theta\neq 0$ we assume $\sigma=0$ for simplicity, thus we obtain the following type (a1) unified product for Lie algebras.

Corollary 3.14. *Let $(A,[\cdot,\cdot])$ be a Lie algebra and V a vector space. An extending datum of A by V of type* (a1) *is $\Omega^{(1)}(A,V)=(\beta,\theta,[\cdot,\cdot]_V)$ consisting of the bilinear maps*
$$\theta: A\times A\to V,\qquad \beta: V\times A\to V,\qquad [\cdot,\cdot]_V: V\times V\to V.$$
Denote by $A_\theta\#_\beta V$ the vector space $E=A\oplus V$ with the bracket $[\cdot,\cdot]: E\times E\to E$ given by
$$[(a,x),(b,y)] := \left([a,b],\, x\triangleleft b-y\triangleleft a+[x,y]+\theta(a,b)\right), \tag{3}$$
for all a, $b\in A$, x, $y\in V$. Then $A_\theta\#_\beta V$ is a Lie algebra if and only if the following compatibility conditions hold for all a, $b\in A$, x, y, $z\in V$:

(A1) $\theta(a,a)=0,\quad [x,x]=0$,

(A2) $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$,

(A3) $[x,y]\triangleleft a=[x,y\triangleleft a]+[x\triangleleft a,y]$,

(A4) $x\triangleleft[a,b]+[x,\theta(a,b)]=(x\triangleleft a)\triangleleft b-(x\triangleleft b)\triangleleft a$,

(A5) $\theta(a,b)\triangleleft c+\theta(b,c)\triangleleft a+\theta(c,a)\triangleleft b=\theta(a,[b,c])+\theta(b,[c,a])+\theta(c,[a,b])$.

In the above Corollary 3.17, (A2) is derived from (CC5), (A3) is derived from (TBB2), (A4) is derived from (TM2), (A5) is derived from (CC2). Note that in this case by (A2) we obtain that V is a Lie algebra. Furthermore, V is in fact a subalgebra of $A_\theta\#_\beta V$ but A is not although $(A,[\cdot,\cdot])$ is itself a Lie algebra.

Denote the set of all Lie algebra extending datum of A by V of type (a1) by $\mathcal{A}^{(1)}(A,V)$.

Note that $A_\theta\#_\beta V$ is a Lie algebra containing V as a Lie sub-algebra. In fact any Lie algebraic structure on E containing A as subspace and V as Lie sub-algebra is isomorphic to such a unified product of this type.

In the following, we always assume that A is a subspace of a vector space E, there exists a projection map $p: E\to A$ such that $p(a)=a$, for all $a\in A$. Then the kernel space $V:=\ker(p)$ is also a subspace of E and a complement of A in E.

Lemma 3.15. *Let $(A,[\cdot,\cdot])$ be a Lie algebra and E a vector space containing A as a subspace. Suppose that there is a Lie algebraic structure $(E,[\cdot,\cdot]_E)$ on E such that V is a Lie subalgebra of E and the canonical projection map $p: E\to A$ is a Lie algebra homomorphism. Then there exists a Lie algebraic extending datum $\Omega^{(1)}(A,V)$ of A by V such that $(E,[\cdot,\cdot]_E)\cong A_\theta\#_\beta V$.*

Proof. Since V is a Lie subalgebra of E, we have $[x,y]_E \in V$. We define the extending datum of A through V by the following formulas:

$$\begin{aligned} \beta : V \times A \to V, &\qquad x \triangleleft a := [x,a]_E, \\ \theta : A \times A \to V, &\qquad \theta(a,b) := [a,b]_E - p\big([a,b]_E\big), \\ [\cdot,\cdot]_V : V \times V \to V, &\qquad [x,y]_V := [x,y]_E. \end{aligned}$$

for any $a, b \in A$ and $x, y \in V$. It is easy to see that the above maps are well defined and $\Omega^{(1)}(A,V) = \big(\theta, \beta, [\cdot,\cdot]_V\big)$ is an extending system of A through V and

$$\varphi : A_\theta \#_\beta V \to E, \qquad \varphi(a,x) := a + x$$

is an isomorphism of Lie algebras. ∎

Lemma 3.16. *Let $\Omega^{(1)}(A,V) = (\theta, \beta, [\cdot,\cdot]_V)$ and $\Omega'^{(1)}(A,V) = (\theta', \beta', [\cdot,\cdot]'_V)$ be two Lie algebraic extending datums of A by V of type (a1) and $A_\theta \#_\beta V$, $A_{\theta'} \#_{\beta'} V$ be the corresponding unified products. Then there exists a bijection between the set of all homomorphisms of Lie algebras $\varphi : A_\theta \#_\beta V \to A_{\theta'} \#_{\beta'} V$ whose restriction on A is the identity map and the set of pairs (r,s), where $r : V \to A$ and $s : V \to V$ are two linear maps satisfying*

$$r(x \triangleleft a) = [r(x), a], \tag{4}$$

$$[a,b]' = [a,b] + r\theta(a,b), \tag{5}$$

$$r([x,y]) = [r(x), r(y)]', \tag{6}$$

$$s(x) \triangleleft' a + \theta'(r(x), a) = s(x \triangleleft a), \tag{7}$$

$$\theta'(a,b) = s\theta(a,b), \tag{8}$$

$$s([x,y]) = [s(x), s(y)]' + s(x) \triangleleft' r(y) - s(y) \triangleleft' r(x) + \theta'(r(x), r(y)), \tag{9}$$

for all $a, b \in A$ and x, $y \in V$. Under the above bijection the homomorphism of Lie algebras $\varphi = \varphi_{r,s} : A_\theta \#_\beta V \to A_{\theta'} \#_{\beta'} V$ to (r,s) is given by $\varphi(a,x) = (a + r(x), s(x))$ for all $a \in A$ and $x \in V$. Moreover, $\varphi = \varphi_{r,s}$ is an isomorphism if and only if $s : V \to V$ is a linear isomorphism.

Proof. Let $\varphi : A_\theta \#_\beta V \to A_{\theta'} \#_{\beta'} V$ be a Lie algebra homomorphism whose restriction on A is the identity map. Then φ is determined by two linear maps $r : V \to A$ and $s : V \to V$ such that $\varphi(a,x) = (a + r(x), s(x))$ for all $a \in A$ and $x \in V$. In fact, we have to show

$$\varphi([(a,x),(b,y)]) = [\varphi(a,x), \varphi(b,y)]'.$$

The left hand side is equal to

$$\begin{aligned} \varphi([(a,x),(b,y)]) &= \varphi\left([a,b],\, x \triangleleft b - y \triangleleft a + [x,y] + \theta(a,b)\right) \\ &= \big([a,b] + r(x \triangleleft b) - r(y \triangleleft a) + r([x,y]) + r\theta(a,b), \\ &\qquad s(x \triangleleft b) - s(y \triangleleft a) + s([x,y]) + s\theta(a,b)\big), \end{aligned}$$

and the right hand side is equal to

$$\begin{aligned} [\varphi(a,x), \varphi(b,y)]' &= [(a + r(x), s(x)), (b + r(y), s(y))]' \\ &= \big([a + r(x), b + r(y)]', s(x) \triangleleft' (b + r(y)) - s(y) \triangleleft' (a + r(x)) \\ &\qquad + [s(x), s(y)]' + \theta'(a + r(x), b + r(y))\big). \end{aligned}$$

Thus φ is a homomorphism of Lie algebras if and only if the above conditions hold. ■

The second case is when $\theta = 0, \alpha \neq 0$, we obtain the following type (a2) unified product for Lie algebras which was developed in [4, Theorem 2.2].

Corollary 3.17. [4] *Let A be a Lie algebra and V a vector space. An extending datum of A by V of type* (a2) *is $\Omega^{(2)}(A,V) = (\alpha, \beta, \sigma, [\cdot,\cdot])$ consisting of four bilinear maps*

$$\alpha : V \times A \to A, \quad \beta : V \times A \to V, \quad \sigma : V \times V \to A, \quad [\cdot,\cdot] : V \times V \to V.$$

Denote by $A_\alpha \#_{\beta,\sigma} H$ the vector space $E = A \oplus V$ with the bilinear map $[\cdot,\cdot] : E \times E \to E$ given by

$$[(a,x),(b,y)] := ([a,b] + x \triangleright b - y \triangleright a + \sigma(x,y),\ x \triangleleft b - y \triangleleft a + [x,y]), \tag{10}$$

for all $a, b \in A$, $x, y \in V$. Then $A_\alpha \#_{\beta,\sigma} H$ is a Lie algebra if and only if the following compatibility conditions hold for all $a, b \in A$, $x, y, z \in V$:

(B1) $\sigma(x,x) = 0, \quad [x,x] = 0,$

(B2) $(V, \triangleleft)$ *is a right A-module,*

(B3) $x \triangleright [a,b] = [x \triangleright a, b] + [a, x \triangleright b] + (x \triangleleft a) \triangleright b - (x \triangleleft b) \triangleright a,$

(B4) $[x,y] \triangleleft a = [x, y \triangleleft a] + [x \triangleleft a, y] + x \triangleleft (y \triangleright a) - y \triangleleft (x \triangleright a),$

(B5) $[x,y] \triangleright a = x \triangleright (y \triangleright a) - y \triangleright (x \triangleright a) + [a, \sigma(x,y)] + \sigma(x, y \triangleleft a) + \sigma(x \triangleleft a, y),$

(B6) $\sigma(x,[y,z]) + \sigma(y,[z,x]) + \sigma(z,[x,y])$
$\quad + x \triangleright \sigma(y,z) + y \triangleright \sigma(z,x) + z \triangleright \sigma(x,y) = 0,$

(B7) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] + x \triangleleft \sigma(y,z) + y \triangleleft \sigma(z,x) + z \triangleleft \sigma(x,y) = 0.$

Note that in the above Corollary 3.17, (B2) is derived from (TM2), (B3) is derived from (TBB1), (B4) is derived from (TBB2), (B5) is derived from (TM1), (B6) is in fact (CC1) and (B7) is in fact (CC5). Thus V is in fact a σ-Lie algebra acting on A. Denote the set of all Lie algebra extending datum of A by V of type (a2) by $\mathcal{A}^{(2)}(A,V)$.

Note that $A_\alpha \#_{\beta,\sigma} H$ is a Lie algebra containing A as a Lie sub-algebra. In fact any Lie algebraic structure on E containing A as a Lie sub-algebra is isomorphic to such a unified product.

Lemma 3.18. [4] *Let $(A,[\cdot,\cdot])$ be a Lie algebra and E a vector space containing A as a subspace. Suppose that there is a Lie algebraic structure $(E,[\cdot,\cdot])$ on E such that $(A,[\cdot,\cdot])$ is a Lie subalgebra of E. Then there exists a Lie algebraic extending system $\Omega^{(2)}(A,V)$ of A by V such that $(E,[\cdot,\cdot],\delta_E) \cong A_\alpha \#_{\beta,\sigma} V$.*

Lemma 3.19. [4] *Let $\Omega^{(2)}(A,V) = (\alpha,\beta,\sigma,[\cdot,\cdot])$ and $\Omega'^{(2)}(A,V) = (\alpha',\beta',\sigma',[\cdot,\cdot]')$ be two Lie algebraic extending datums of A by V of type (a2) and $A_\alpha \#_{\beta,\sigma} V$, $A_{\alpha'} \#_{\beta',\sigma'} V$ be the corresponding unified products. Then there exists a bijection between the set of all homomorphisms of Lie algebras $\varphi : A_\alpha \#_{\beta,\sigma} V \to A_{\alpha'} \#_{\beta',\sigma'} V$ whose restriction on A is the identity map and the set of pairs (r,s), where $r : V \to A$ and $s : V \to V$ are two linear maps satisfying*

$$s(x) \triangleleft' a = s(x \triangleleft a), \tag{11}$$

$$r(x \triangleleft a) = [r(x), a] - x \triangleright a + s(x) \triangleright' a, \tag{12}$$

$$s([x,y]) = [s(x), s(y)]' + s(x) \triangleleft' r(y) - s(y) \triangleleft' r(x), \tag{13}$$

$$r([x,y]) = [r(x), r(y)] + s(x) \triangleright' r(y) - s(y) \triangleright' r(x) + \sigma'(s(x), s(y)) - \sigma(x,y) \tag{14}$$

for all $a \in A$ and x, $y \in V$.

Under the above bijection the homomorphism of Lie algebras

$$\varphi = \varphi_{r,s} : A_{\alpha}\#_{\beta,\sigma}V \to A_{\alpha'}\#_{\beta',\sigma'}V$$

to (r,s) is given by $\varphi(a,x) = (a + r(x), s(x))$ for all $a \in A$ and $x \in V$. Moreover, $\varphi = \varphi_{r,s}$ is an isomorphism if and only if $s : V \to V$ is a linear isomorphism.

Let $(A, [\cdot,\cdot])$ be a Lie algebra and V a vector space. Two Lie algebra extending systems $\Omega^{(i)}(A,V)$ and $\Omega'^{(i)}(A,V)$ are called equivalent if $\varphi_{r,s}$ is an isomorphism. We denote it by $\Omega^{(i)}(A,V) \equiv \Omega'^{(i)}(A,V)$. From the above lemmas, we obtain the following result.

Theorem 3.20. *Let $(A, [\cdot,\cdot])$ be a Lie algebra, E a vector space containing A as a subspace and V be a complement of A in E. Denote $\mathcal{HA}(V,A) := \mathcal{A}^{(1)}(A,V) \sqcup \mathcal{A}^{(2)}(A,V)/\equiv$. Then the map*

$$\Psi : \mathcal{HA}(V,A) \to Extd(E,A), \tag{15}$$

$$\overline{\Omega^{(1)}(A,V)} \mapsto A_{\theta}\#_{\beta}V, \quad \overline{\Omega^{(2)}(A,V)} \mapsto A_{\alpha}\#_{\beta,\sigma}V \tag{16}$$

is bijective, where $\overline{\Omega^{(i)}(A,V)}$ is the equivalence class of $\Omega^{(i)}(A,V)$ under $\equiv$.

Next we consider the coalgebra structures on $E = A^{\phi,P}\#^{\psi,Q}H$.

Theorem 3.21. [30] *Let A be a Q-Lie coalgebra and H be a P-Lie coalgebra. If we define $E = A^{\phi,P}\#^{\psi,Q}H$ as the vector space $A \oplus H$ with the Lie cobracket*

$$\delta_E(a) = \delta_A(a) + \phi(a) - \tau\phi(a) + P(a), \quad \delta_E(x) = \delta_H(x) + \psi(x) - \tau\psi(x) + Q(x), \tag{17}$$

then $A^{\phi,P}\#^{\psi,Q}H$ is a Lie coalgebra if and only if the following compatibility conditions hold:

(TM3) $\delta_H(a_{(-1)}) \otimes a_{(0)} + P(a_1) \otimes a_2$
$= a_{(-1)} \otimes \phi(a_{(0)}) - \tau_{12}\left(a_{(-1)} \otimes \phi(a_{(0)})\right) + a_{[1]} \otimes \psi\left(a_{[2]}\right) + \tau_{23}\left(\psi(a_{[1]}) \otimes a_{[2]}\right)$,

(TM4) $x_{(0)} \otimes \delta_A(x_{(1)}) + x_1 \otimes Q(x_2)$
$= \psi(x_{(0)}) \otimes x_{(1)} - \tau_{23}\left(\psi(x_{(0)}) \otimes x_{(1)}\right) + \psi(x_{<1>}) \otimes x_{<2>} + \tau_{12}\left(x_{<1>} \otimes \psi(x_{<2>})\right)$,

(TBB3) $a_{(-1)} \otimes \delta_A(a_{(0)}) + a_{[1]} \otimes Q(a_{[2]})$
$= \phi(a_1) \otimes a_2 + \tau_{12}\left(a_1 \otimes \phi(a_2)\right) + \psi(a_{(-1)}) \otimes a_{(0)} - \tau_{23}\left(\psi(a_{(-1)}) \otimes a_{(0)}\right)$,

(TBB4) $\delta_H(x_{(0)}) \otimes x_{(1)} + P(x_{<1>}) \otimes x_{<2>}$
$= x_1 \otimes \psi(x_2) + \tau_{23}\left(\psi(x_1) \otimes x_2\right) + x_{(0)} \otimes \psi(x_{(1)}) - \tau_{12}\left(x_{(0)} \otimes \psi(x_{(1)})\right)$.

In this case, (A,H) is called a cycle cross coproduct system.

The proof of the above Theorem 3.21 can be found in [30].
There are two cases for (A, δ_A) to be a Lie coalgebra. The first case is when $\phi \neq 0, Q = 0$, we obtain the following type (c1) unified product for Lie coalgebras.

Corollary 3.22. *Let (A,δ_A) be a Lie coalgebra and V a vector space. An extending datum of A by V of type* (c1) *is $\Omega^c(A,V)=(\phi,\psi,P,\delta_V)$ with linear maps*

$$\phi: A\to V\otimes A,\quad \psi: V\to V\otimes A,\quad P: A\to V\otimes V,\quad \delta_V: V\to V\otimes V.$$

Denote by $A^{\phi,P}\#^{\psi}V$ the vector space $E=A\oplus V$ with the linear map $\delta_E: E\to E\otimes E$ given by

$$\delta_E(a)=\delta_A(a)+\phi(a)-\tau\phi(a)+P(a),\quad \delta_E(x)=\delta_V(x)+\psi(x)-\tau\psi(x). \tag{18}$$

Then $A^{\phi,P}\#^{\psi}V$ is a Lie coalgebra with the Lie cobracket given by (18) *if and only if the following compatibility conditions hold:*

(C1) $P(a)=-\tau P(a),\quad \delta_V(x)=-\tau\delta_V(x)$,

(C2) $\delta_V(a_{(-1)})\otimes a_{(0)}+P(a_1)\otimes a_2$
$= a_{(-1)}\otimes\phi(a_{(0)})-\tau_{12}\left(a_{(-1)}\otimes\phi(a_{(0)})\right)+a_{[1]}\otimes\psi\left(a_{[2]}\right)+\tau_{23}\left(\psi(a_{[1]})\otimes a_{[2]}\right)$,

(C3) $x_{(0)}\otimes\delta_A(x_{(1)})=\psi(x_{(0)})\otimes x_{(1)}-\tau_{23}\left(\psi(x_{(0)})\otimes x_{(1)}\right)$,

(C4) $a_{(-1)}\otimes\delta_A(a_{(0)})=\phi(a_1)\otimes a_2+\tau_{12}\left(a_1\otimes\phi(a_2)\right)$
$+\psi(a_{(-1)})\otimes a_{(0)}-\tau_{23}\left(\psi(a_{(-1)})\otimes a_{(0)}\right)$,

(C5) $\delta_V(x_{(0)})\otimes x_{(1)}=x_1\otimes\psi(x_2)+\tau_{23}\left(\psi(x_1)\otimes x_2\right)$
$+x_{(0)}\otimes\psi(x_{(1)})-\tau_{12}\left(x_{(0)}\otimes\psi(x_{(1)})\right)$.

(C6) $a_{(-1)}\otimes P(a_{(0)})+\tau_{12}\tau_{23}\left(a_{(-1)}\otimes P(a_{(0)})\right)+\tau_{23}\tau_{12}\left(a_{(-1)}\otimes P(a_{(0)})\right)$
$=\delta(a_{[1]})\otimes a_{[2]}+\tau_{12}\tau_{23}\left(\delta(a_{[1]})\otimes a_{[2]}\right)+\tau_{23}\tau_{12}\left(\delta(a_{[1]})\otimes a_{[2]}\right)$,

(C7) $\delta(x_1)\otimes x_2+\tau_{12}\tau_{23}\left(\delta(x_1)\otimes x_2\right)+\tau_{23}\tau_{12}\left(\delta(x_1)\otimes x_2\right)$
$=x_{(0)}\otimes P(x_{(1)})+\tau_{12}\tau_{23}\left(x_{(0)}\otimes P(x_{(1)})\right)+\tau_{23}\tau_{12}\left(x_{(0)}\otimes P(x_{(1)})\right)$.

Note that in the above Corollary 3.22, (C2) is derived from (TM3), (C3) is derived from (TM4), (C4) is derived from (TBB3), (C5) is derived from (TBB4), (C6) is in fact (CC3) and (C7) is in fact (CC7).

Denote the set of all Lie coalgebra extending datum of A by V of type (c1) by $\mathcal{C}^{(1)}(A,V)$.

Lemma 3.23. *Let (A,δ_A) be a Lie coalgebra and E a vector space containing A as a subspace. Suppose that there is a Lie coalgebra structure (E,δ_E) on E such that $p: E\to A$ is a Lie coalgebra homomorphism. Then there exists a Lie coalgebra extending system $\Omega^c(A,V)$ of (A,δ_A) by V such that $(E,\delta_E)\cong A^{\phi,P}\#^{\psi}V$.*

Proof. Let $p: E\to A$ and $\pi: E\to V$ be the projection maps and $V=\ker(p)$. Then the extending datum of (A,δ_A) by V is defined as follows:

$$\begin{aligned}
\phi: A\to V\otimes A,&\qquad \phi(x)=(\pi\otimes p)\delta_E(a),\\
\psi: V\to V\otimes A,&\qquad \phi(x)=(\pi\otimes p)\delta_E(x),\\
\delta_V: V\to V\otimes V,&\qquad \delta_V(x)=(\pi\otimes\pi)\delta_E(x),\\
P: A\to V\otimes V,&\qquad P(a)=(\pi\otimes\pi)\delta_E(a).
\end{aligned}$$

One check that $\varphi: A^{\phi,P}\#^{\psi}V\to E$ given by $\varphi(a,x)=a+x$ for all $a\in A, x\in V$ is a Lie coalgebra isomorphism. ■

Lemma 3.24. *Let $\Omega^{(1)}(A,V)=(\phi,\psi,P,\delta_V)$ and $\Omega'^{(1)}(A,V)=(\phi',\psi',P',\delta'_V)$ be two Lie coalgebra extending datums of (A,δ_A) by V. Then there exists a bijection between the set of Lie coalgebra homomorphisms $\varphi: A^{\phi,P}\#^{\psi}V\to A^{\phi',P'}\#^{\psi'}V$ whose restriction on A is the identity map and the set of pairs (r,s), where $r:V\to A$ and $s:V\to V$ are two linear maps satisfying*

$$P'(a)=s(a_{[1]})\otimes s(a_{[2]}), \tag{19}$$
$$\phi'(a)=s(a_{(-1)})\otimes a_{(0)}+s(a_{[1]})\otimes r(a_{[2]}), \tag{20}$$
$$\delta'_A(a)=\delta_A(a)+r(a_{(-1)})\otimes a_{(0)}-a_{(0)}\otimes r(a_{(-1)})+r(a_{[1]})\otimes r(a_{[2]}) \tag{21}$$
$$\delta'_V(s(x))=(s\otimes s)\delta_V(x), \tag{22}$$
$$\psi'(s(x))=s(x_1)\otimes r(x_2)+s(x_{(0)})\otimes x_{(1)}, \tag{23}$$
$$\delta'_A(r(x))=r(x_1)\otimes r(x_2)+r(x_{(0)})\otimes x_{(1)}-x_{(1)}\otimes r(x_{(0)}). \tag{24}$$

Under the above bijection the Lie coalgebra homomorphism $\varphi=\varphi_{r,s}: A^{\phi,P}\#^{\psi}V\to A^{\phi',P'}\#^{\psi'}V$ to (r,s) is given by $\varphi(a,x)=(a+r(x),s(x))$ for all $a\in A$ and $x\in V$. Moreover, $\varphi=\varphi_{r,s}$ is an isomorphism if and only if $s:V\to V$ is a linear isomorphism.

Proof. Let $\varphi: A^{\phi,P}\#^{\psi}V\to A^{\phi',P'}\#^{\psi'}V$ be a Lie coalgebra homomorphism whose restriction on A is the identity map. Then φ is determined by two linear maps $r:V\to A$ and $s:V\to V$ such that $\varphi(a,x)=(a+r(x),s(x))$ for all $a\in A$ and $x\in V$. We will prove that φ is a homomorphism of Lie coalgebras if and only if the above conditions hold. First we easily see that $\delta'_E\varphi(a)=(\varphi\otimes\varphi)\delta_E(a)$

$$\delta'_E\varphi(a)=\delta'_E(a)=\delta'_A(a)+\phi'(a)-\tau\phi'(a)+P'(a)\quad\text{for all } a\in A\text{, and}$$

$$\begin{aligned}&(\varphi\otimes\varphi)\delta_E(a)\\&=(\varphi\otimes\varphi)\left(\delta_A(a)+\phi(a)-\tau\phi(a)+P(a)\right)\\&=\delta_A(a)+r(a_{(-1)})\otimes a_{(0)}+s(a_{(-1)})\otimes a_{(0)}-a_{(0)}\otimes r(a_{(-1)})-a_{(0)}\otimes s(a_{(-1)})\\&\quad+r(a_{[1]})\otimes r(a_{[2]})+r(a_{[1]})\otimes s(a_{[2]})+s(a_{[1]})\otimes r(a_{[2]})+s(a_{[1]})\otimes s(a_{[2]}).\end{aligned}$$

Thus we obtain that $\delta'_E\varphi(a)=(\varphi\otimes\varphi)\delta_E(a)$ if and only if the conditions (19), (20) and (21) hold. Then we consider that $\delta'_E\varphi(x)=(\varphi\otimes\varphi)\delta_E(x)$ for all $x\in V$.

$$\begin{aligned}\delta'_E\varphi(x)&=\delta'_E(r(x),s(x))=\delta'_E(r(x))+\delta'_E(s(x))\\&=\delta'_A(r(x))+\delta'_V(s(x))+\psi'(s(x))-\tau\psi'(s(x))),\end{aligned}$$

and
$$\begin{aligned}&(\varphi\otimes\varphi)\delta_E(x)\\&=(\varphi\otimes\varphi)(\delta_V(x)+\psi(x)-\tau\psi(x))\\&=(\varphi\otimes\varphi)(x_1\otimes x_2+x_{(0)}\otimes x_{(1)}-x_{(1)}\otimes x_{(0)})\\&=r(x_1)\otimes r(x_2)+r(x_1)\otimes s(x_2)+s(x_1)\otimes r(x_2)+s(x_1)\otimes s(x_2)\\&\quad+r(x_{(0)})\otimes x_{(1)}+s(x_{(0)})\otimes x_{(1)}-x_{(1)}\otimes r(x_{(0)})-x_{(1)}\otimes s(x_{(0)}).\end{aligned}$$

Thus we obtain that $\delta'_E\varphi(x)=(\varphi\otimes\varphi)\delta_E(x)$ if and only if the conditions (22), (23) and (24) hold. By the definition of $\varphi=\varphi_{r,s}$, we obtain that $\varphi=\varphi_{r,s}$ is an isomorphism if and only if $s:V\to V$ is a linear isomorphism. ∎

In the case $\phi = 0, Q \neq 0$, then from (TBB3) we get that $a_{[1]} \otimes Q(a_{[2]}) = 0$, since $Q \neq 0$ we assume $P = 0$ for simplicity, thus we obtain the following type (c2) unified product for Lie coalgebras.

Corollary 3.25. *Let (A, δ_A) be a Lie coalgebra and V a vector space. An extending datum of (A, δ_A) by V of type* (c2) *is $\Omega^{(2)}(A, V) = (\psi, Q, \delta_V)$ with linear maps*

$$\psi : V \to V \otimes A, \qquad Q : V \to A \otimes A, \qquad \delta_V : V \to V \otimes V.$$

Denote by $A\#^{\psi,Q}V$ the vector space $E = A \oplus V$ with the linear map $\delta_E : E \to E \otimes E$ given by

$$\delta_E(a) = \delta_A(a), \quad \delta_E(x) = \delta_V(x) + \psi(x) - \tau\psi(x) + Q(x). \tag{25}$$

Then $A\#^{\psi,Q}V$ is a Lie coalgebra with the Lie cobracket given by (25) if and only if the following compatibility conditions hold:

(D1) $Q(x) = -\tau Q(x), \quad \delta_V(x) = -\tau\delta_V(x)$,

(D2) $\delta_V(x_1) \otimes x_2 + \tau_{12}\tau_{23}\left(\delta_V(x_1) \otimes x_2\right) + \tau_{23}\tau_{12}\left(\delta_V(x_1) \otimes x_2\right) = 0$

(D3) $x_{(0)} \otimes \delta_A(x_{(1)}) + x_1 \otimes \psi(x_2)$
$= \psi(x_{(0)}) \otimes x_{(1)} - \tau_{23}\big(\psi(x_{(0)}) \otimes x_{(1)}\big) + \psi(x_{<1>}) \otimes x_{<2>} + \tau_{12}\left(x_{<1>} \otimes \psi(x_{<2>})\right)$,

(D4) $\delta_V(x_{(0)}) \otimes x_{(1)} = x_1 \otimes \psi(x_2) + \tau_{23}\left(\psi(x_1) \otimes x_2\right)$

(D5) $Q(x_{(0)}) \otimes x_{(1)} + \tau_{12}\tau_{23}\big(Q(x_{(0)}) \otimes x_{(1)}\big) + \tau_{23}\tau_{12}\big(Q(x_{(0)}) \otimes x_{(1)}\big)$
$= x_{<1>} \otimes \delta_A(x_{<2>}) + \tau_{12}\tau_{23}\left(x_{<1>} \otimes \delta_A(x_{<2>})\right) + \tau_{23}\tau_{12}\left(x_{<1>} \otimes \delta_A(x_{<2>})\right)$.

Note that in the above Corollary 3.25, (D2) is derived from (CC7), (D3) is derived from (TM4), (D4) is derived from (TBB4) and (D5) is in fact (CC4). From (D1) and (D2) we obtain that (V, δ_V) is a coalgebra.

Denote the set of all Lie coalgebra extending datum of A by V of type (c2) by $\mathcal{C}^{(2)}(A, V)$.

Similar as Lie algebra case, one show that any Lie coalgebra structure on E containing A as a Lie sub-coalgebra is isomorphic to such a unified coproduct.

Lemma 3.26. *Let (A, δ_A) be a Lie coalgebra and E a vector space containing A as a subspace. Suppose that there is a Lie coalgebra structure (E, δ_E) on E such that (A, δ_A) is a Lie sub-coalgebra of E. Then there exists a Lie coalgebra extending system $\Omega^{(2)}(A, V)$ of (A, δ_A) by V such that $(E, \delta_E) \cong A\#^{\psi,Q}V$.*

Proof. Let $p : E \to A$ and $\pi : E \to V$ be the projection map and $V = ker(p)$. Then the extending datum of (A, δ_A) by V is defined as follows:

$$\begin{aligned} \psi &: V \to V \otimes A, & \phi(x) &= (\pi \otimes p)\delta_E(x), \\ \delta_V &: V \to V \otimes V, & \delta_V(x) &= (\pi \otimes \pi)\delta_E(x), \\ Q &: V \to A \otimes A, & Q(x) &= (p \otimes p)\delta_E(x). \end{aligned}$$

One easily checks that $\varphi : A\#^{\psi,Q}V \to E$, given by $\varphi(a, x) = a + x$ for all $a \in A$, $x \in V$, is a Lie coalgebra isomorphism. ■

Lemma 3.27. *Let $\Omega^{(2)}(A,V)=(\psi,Q,\delta_V)$ and $\Omega'^{(2)}(A,V)=(\psi',Q',\delta'_V)$ be two Lie coalgebra extending datums of (A,δ_A) by V. Then there exists a bijection between the set of Lie coalgebra homomorphisms $\varphi: A\#^{\psi,Q}V\to A\#^{\psi',Q'}V$ whose restriction on A is the identity map and the set of pairs (r,s), where $r:V\to A$ and $s:V\to V$ are two linear maps satisfying for all $x\in V$*

$$\psi'(s(x))=s(x_1)\otimes r(x_2)+s(x_{(0)})\otimes x_{(1)}, \tag{26}$$

$$\delta'_V(s(x))=(s\otimes s)\delta_V(x), \tag{27}$$

$$\delta'_A(r(x))+Q'(s(x))=r(x_1)\otimes r(x_2)+r(x_{(0)})\otimes x_{(1)}-x_{(1)}\otimes r(x_{(0)})+Q(x). \tag{28}$$

Under the above bijection the Lie coalgebra homomorphism

$$\varphi=\varphi_{r,s}: A\#^{\psi,Q}V\to A\#^{\psi',Q'}V$$

to (r,s) is given by $\varphi(a,x)=(a+r(x),s(x))$ for all $a\in A$ and $x\in V$. Moreover, $\varphi=\varphi_{r,s}$ is an isomorphism if and only if $s:V\to V$ is a linear isomorphism.

Proof. The proof is similar as the proof of Lemma 3.24. Let

$$\varphi: A\#^{\psi,Q}V\to A\#^{\psi',Q'}V$$

be a Lie coalgebra homomorphism whose restriction on A is the identity map. First we see that $\delta'_E\varphi(a)=(\varphi\otimes\varphi)\delta_E(a)$ for all $a\in A$. Then we consider that $\delta'_E\varphi(x)=(\varphi\otimes\varphi)\delta_E(x)$ for all $x\in V$:

$$\begin{aligned}\delta'_E\varphi(x)&=\delta'_E(r(x),s(x))=\delta'_E(r(x))+\delta'_E(s(x))\\&=\delta'_A(r(x))+\delta'_V(s(x))+\psi'(s(x))-\tau\psi'(s(x))+Q'(s(x)),\end{aligned}$$

and

$$\begin{aligned}&(\varphi\otimes\varphi)\delta_E(x)\\&=(\varphi\otimes\varphi)(\delta_V(x)+\psi(x)-\tau\psi(x)+Q(x))\\&=(\varphi\otimes\varphi)(x_1\otimes x_2+x_{(0)}\otimes x_{(1)}-x_{(1)}\otimes x_{(0)}+Q(x))\\&=r(x_1)\otimes r(x_2)+r(x_1)\otimes s(x_2)+s(x_1)\otimes r(x_2)+s(x_1)\otimes s(x_2)\\&\quad+r(x_{(0)})\otimes x_{(1)}+s(x_{(0)})\otimes x_{(1)}-x_{(1)}\otimes r(x_{(0)})-x_{(1)}\otimes s(x_{(0)})+Q(x).\end{aligned}$$

Thus we obtain that $\delta'_E\varphi(x)=(\varphi\otimes\varphi)\delta_E(x)$ if and only if the conditions (26), (27) and (28) hold. By the definition of $\varphi=\varphi_{r,s}$, we obtain that $\varphi=\varphi_{r,s}$ is an isomorphism if and only if $s:V\to V$ is a linear isomorphism. ■

Let (A,δ_A) be a Lie coalgebra and V a vector space. Two Lie coalgebra extending systems $\Omega^{(i)}(A,V)$ and $\Omega'^{(i)}(A,V)$ are called equivalent if $\varphi_{r,s}$ is an isomorphism. We denote it by $\Omega^{(i)}(A,V)\equiv\Omega'^{(i)}(A,V)$. From the above lemmas, we obtain the following result.

Theorem 3.28. *Let (A,δ_A) be a Lie coalgebra, E a vector space containing A as a subspace and V be an A-complement in E.*
Define $\mathcal{HC}(V,A):=\mathcal{C}^{(1)}(A,V)\sqcup\mathcal{C}^{(2)}(A,V)/\equiv$. Then the map

$$\Psi:\mathcal{HC}^2_A(V,A)\to CExtd(E,A), \tag{29}$$

$$\overline{\Omega^{(1)}(A,V)}\mapsto A^{\phi,P}\#^{\psi}V,\quad \overline{\Omega^{(2)}(A,V)}\mapsto A\#^{\psi,Q}V \tag{30}$$

is bijective, where $\overline{\Omega^{(i)}(A,V)}$ is the equivalence class of $\Omega^{(i)}(A,V)$ under $\equiv$.

The next theorem says that we can obtain an ordinary Lie bialgebra from two braided Lie bialgebras.

Theorem 3.29. [30] *Let (A, H) be a cocycle cross product system and a cycle cross coproduct system. Then the Lie algebra $A_{\alpha,\theta}\#_{\beta,\sigma}H$ and Lie coalgebra $A^{\phi,P}\#^{\psi,Q}H$ fit together to form an ordinary Lie bialgebra if and only if the following compatibility conditions hold:*

(TBB5) $\delta_A(x\triangleright a)+Q(x\triangleleft a)=x\triangleright\delta_A(a)+x_{(0)}\triangleright a\otimes x_{(1)}-x_{(1)}\otimes x_{(0)}\triangleright a$
$+[Q(x),a]+\sigma(x,a_{(-1)})\otimes a_{(0)}-a_{(0)}\otimes\sigma(x,a_{(-1)})\,;$

(TBB6) $\delta_H(x\triangleleft a)+P(x\triangleright a)=\delta_H(x)\triangleleft a+a_{(-1)}\otimes x\triangleleft a_{(0)}-x\triangleleft a_{(0)}\otimes a_{(-1)}$
$+[x,P(a)]+x_{(0)}\otimes\theta(x_{(1)},a)-\theta(x_{(1)},a)\otimes x_{(0)}\,;$

(TBB7) $\phi([a,b])+\psi\theta(a,b)=a_{(-1)}\otimes[a_{(0)},b]+b_{(-1)}\otimes[a,b_{(0)}]+a_{(-1)}\triangleleft b\otimes a_{(0)}$
$-b_{(-1)}\triangleleft a\otimes b_{(0)}+\theta(a,b_1)\otimes b_2+\theta(a_1,b)\otimes a_2+a_{[1]}\otimes a_{[2]}\triangleright b-b_{[1]}\otimes b_{[2]}\triangleright a\,;$

(TBB8) $\psi([x,y])+\phi\sigma[x,y]=[x,y_{(0)}]\otimes y_{(1)}+[x_{(0)},y]\otimes x_{(1)}+y_{(0)}\otimes x\triangleright y_{(1)}$
$-x_{(0)}\otimes y\triangleright x_{(1)}+x_1\otimes\sigma(x_2,y)+y_1\otimes\sigma(x,y_2)+x\triangleleft y_{<1>}\otimes y_{<2>}$
$-y\triangleleft x_{<1>}\otimes x_{<2>}\,;$

(TLB1) $\delta_H\theta(a,b)+P[a,b]=a_{(-1)}\otimes\theta(a_{(0)},b)+b_{(-1)}\otimes\theta(a,b_{(0)})$
$-\theta(a,b_{(0)})\otimes b_{(-1)}-\theta(a_{(0)},b)\otimes a_{(-1)}+P(a)\triangleleft b-P(b)\triangleleft a\,;$

(TLB2) $\delta_A\sigma(x,y)+Q[x,y]=\sigma(x_{(0)},y)\otimes x_{(1)}+\sigma(x,y_{(0)})\otimes y_{(1)}$
$-x_{(1)}\otimes\sigma(x_{(0)},y)-y_{(1)}\otimes\sigma(x,y_{(0)})+x\triangleright Q(y)-y\triangleright Q(x)\,;$

(TLB3) $\delta_A([a,b])+Q\theta(a,b)=[\delta_A(a),b]+[a,\delta_A(b)]-b_{(-1)}\triangleright a\otimes b_{(0)}$
$-a_{(0)}\otimes a_{(-1)}\triangleright b+a_{(-1)}\triangleright b\otimes a_{(0)}+b_{(0)}\otimes b_{(-1)}\triangleright a\,;$

(TLB4) $\delta_H([x,y])+P\sigma(x,y)=[\delta_H(x),y]+[x,\delta_H([y])]-x_{(0)}\otimes y\triangleleft x_{(1)}$
$-x\triangleleft y_{(1)}\otimes y_{(0)}+y_{(0)}\otimes x\triangleleft y_{(1)}+y\triangleleft x_{(1)}\otimes x_{(0)}\,;$

(TYD) $\phi(x\triangleright a)+\psi(x\triangleleft a)=[x,a_{(-1)}]\otimes a_{(0)}+a_{(-1)}\otimes x\triangleright a_{(-1)}+x_1\otimes x_2\triangleright a$
$+x_{(0)}\otimes[x_{(1)},a]+x_{(0)}\triangleleft a\otimes x_{(1)}+x\triangleleft a_1\otimes a_2+a_{[1]}\otimes\sigma(x,a_{[2]})\,.$
$+x_{<1>}\otimes\theta(x_{<2>},a)\,.$

This Lie bialgebra is denote by $A^{\phi,P}_{\alpha,\theta}\#^{\psi,Q}_{\beta,\sigma}H$. We call it the unified product for braided Lie bialgebra A and H.

The above Theorem 3.29 can also be found in [30]. Note that (TBB5), …, (TBB8) are extended from (BB5), …, (BB8); (TLB3) and (TLB4) are extended from (LBS); (TYD) from (YDB). Thus Theorem 3.29 is a generalization of Theorem 3.9. In this case $\theta=0, P=0$, then $(A,[\cdot,\cdot])$ is a Lie algebra and (A,δ_A) is a Lie coalgebra and by (TLB3) we obtain that $(A,[\cdot,\cdot],\delta_A)$ is a braided Lie bialgebra in ${}^H_H\mathcal{M}$. In this case $\sigma=0, Q=0$, then $(H,[\cdot,\cdot])$ is a Lie algebra and (H,δ_H) is a Lie coalgebra and by (TLB4) we obtain that $(H,[\cdot,\cdot],\delta_H)$ is really a braided Lie bialgebra in $\mathcal{M}_A^A$. That is why in Theorem 3.29 we call $A^{\phi,P}_{\alpha,\theta}\#^{\psi,Q}_{\beta,\sigma}H$ the unified product for braided Lie bialgebras.

Put $\theta=0, Q=0$, then from (TLB3) we get that A is a braided Lie bialgebra. By the above Theorem 3.29, we obtain:

Theorem 3.30. *Let A be a braided Lie bialgebra and V a vector space. An extending datum of A by V is $\Omega^b(A,V)=(\alpha,\beta,\sigma,[\cdot,\cdot],\phi,\psi,Q,\delta_V)$ consisting of eight linear maps*

$$\alpha: V\times A\to A,\quad \beta: V\times A\to V,\quad \sigma: V\times V\to A,\quad [\cdot,\cdot]: V\times V\to V,$$
$$\phi: A\to V\otimes A,\quad \psi: V\to V\otimes A,\quad P: A\to V\otimes V,\quad \delta_V: V\to V\otimes V.$$

Then the unified product $A^{\phi}_{\alpha}\#^{\psi,Q}_{\beta,\sigma}V$ with bracket

$$[(a,x),(b,y)]:=([a,b]+x\triangleright b-y\triangleright a+\sigma(x,y),[x,y]+x\triangleleft b-y\triangleleft a) \tag{31}$$

and cobracket

$$\delta_E(a)=\delta_A(a)+\phi(a)-\tau\phi(a)+P(a),\quad \delta_E(x)=\delta_V(x)+\psi(x)-\tau\psi(x) \tag{32}$$

form a Lie bialgebra if and only if $A_{\alpha}\#_{\beta,\sigma}V$ form a Lie algebra, $A^{\phi}\#^{\psi,Q}V$ form a Lie coalgebra and the following conditions are satisfied:

(E1) $\delta_A(x\triangleright a)=x\triangleright\delta_A(a)+x_{(0)}\triangleright a\otimes x_{(1)}-x_{(1)}\otimes x_{(0)}\triangleright a$
$+\sigma(x,a_{(-1)})\otimes a_{(0)}-a_{(0)}\otimes\sigma(x,a_{(-1)})$,

(E2) $\delta_H(x\triangleleft a)+P(x\triangleright a)=\delta_H(x)\triangleleft a+a_{(-1)}\otimes x\triangleleft a_{(0)}-x\triangleleft a_{(0)}\otimes a_{(-1)}+[x,P(a)]$;

(E3) $\phi([a,b])=a_{(-1)}\otimes[a_{(0)},b]+b_{(-1)}\otimes[a,b_{(0)}]+a_{(-1)}\triangleleft b\otimes a_{(0)}-b_{(-1)}\triangleleft a\otimes b_{(0)}$
$+a_{[1]}\otimes a_{[2]}\triangleright b-b_{[1]}\otimes b_{[2]}\triangleright a$;

(E4) $\psi([x,y])+\phi\sigma[x,y]=[x,y_{(0)}]\otimes y_{(1)}+[x_{(0)},y]\otimes x_{(1)}+y_{(0)}\otimes x\triangleright y_{(1)}-x_{(0)}\otimes y\triangleright x_{(1)}$
$+x_1\otimes\sigma(x_2,y)+y_1\otimes\sigma(x,y_2)+x\triangleleft y_{<1>}\otimes y_{<2>}-y\triangleleft x_{<1>}\otimes x_{<2>}$;

(E5) $P[a,b]=P(a)\triangleleft b-P(b)\triangleleft a$;

(E6) $\delta_A\sigma(x,y)=\sigma(x_{(0)},y)\otimes x_{(1)}+\sigma(x,y_{(0)})\otimes y_{(1)}-x_{(1)}\otimes\sigma(x_{(0)},y)-y_{(1)}\otimes\sigma(x,y_{(0)})$;

(E7) $\delta_A([a,b])=[\delta_A(a),b]+[a,\delta_A(b)]-b_{(-1)}\triangleright a\otimes b_{(0)}-a_{(0)}\otimes a_{(-1)}\triangleright b$
$+a_{(-1)}\triangleright b\otimes a_{(0)}+b_{(0)}\otimes b_{(-1)}\triangleright a$;

(E8) $\delta_H([x,y])+P\sigma(x,y)=[\delta_H(x),y]+[x,\delta_H([y])]-x_{(0)}\otimes y\triangleleft x_{(1)}-x\triangleleft y_{(1)}\otimes y_{(0)}$
$+y_{(0)}\otimes x\triangleleft y_{(1)}+y\triangleleft x_{(1)}\otimes x_{(0)}$;

(E9) $\phi(x\triangleright a)+\psi(x\triangleleft a)=[x,a_{(-1)}]\otimes a_{(0)}+a_{(-1)}\otimes x\triangleright a_{(0)}+x_1\otimes x_2\triangleright a+x_{(0)}\otimes[x_{(1)},a]$
$+x_{(0)}\triangleleft a\otimes x_{(1)}+x\triangleleft a_1\otimes a_2+a_{[1]}\otimes\sigma(x,a_{[2]})$.

We denote in the following the set of all braided Lie bialgebra extending datum of A by V by $\mathcal{BLB}(A,V)$. In consequence of Lemmas 3.15, 3.18, 3.23 and 3.26, we have

Theorem 3.31. *Let $(A,[\cdot,\cdot],\delta_A)$ be a non-trivial braided Lie bialgebra, E a vector space. Suppose that there is a Lie bialgebra structure $(E,[\cdot,\cdot]_E,\delta_E)$ on E containing A as a Lie subalgebra and the projection map $p: E\to A$ as a Lie coalgebra homomorphism. Then there exists a Lie bialgebra extending system $\Omega^b(A,V)$ of A by V such that $(E,[\cdot,\cdot]_E,\delta_E)\cong A^{\phi}_{\alpha}\#^{\psi,Q}_{\beta,\sigma}V$.*

From Theorem 3.20, Theorem 3.28 we obtain

Theorem 3.32. *Let $(A, [\cdot,\cdot], \delta_A)$ be a non-trivial braided Lie bialgebra, $(E, [\cdot,\cdot]_E, \delta_E)$ is Lie bialgebra containing A as a Lie subalgebra and the projection map $p : E \to A$ as a Lie coalgebra homomorphism. Define $\mathcal{HBLB}(V, A) := \mathcal{BLB}(A, V)/\equiv$. Then the map*

$$\Phi : \mathcal{HBLB}(V, A) \to BExtd(E, A), \quad \overline{\Omega^b(A, V)} \mapsto A^{\phi}_{\alpha}\#^{\psi,Q}_{\beta,\sigma} V \tag{33}$$

is bijective, where $\overline{\Omega^b(A, V)}$ is the equivalence class of $\Omega^b(A, V)$ under $\equiv$.

4. Applications

In this section, we will study the extending problem and non-abelian extension problem for Lie bialgebra. We will find some special cases when the braided Lie bialgebra $(A, [\cdot,\cdot], \delta_A)$ is deduced to an ordinary Lie bialgebra. It is proved that these problems can be solved by using the non-abelian cohomology theory based on our unified product for braided Lie bialgebras in last section.

4.1. Unified products and extending problem for Lie bialgebras

A special case is that we assume $\alpha = 0, \sigma = 0, Q = 0$, in the above Theorem 3.29. In this case $(A, [\cdot,\cdot], \delta_A)$ is reduced to a Lie bialgebra, and we obtain the following result.

Theorem 4.1. *Let $(A, [\cdot,\cdot], \delta_A)$ be a Lie bialgebra and V a vector space. An extending datum of A by V of type (I) is $\Omega^{(1)}(A, V) = (\beta, \phi, \psi, P, [\cdot,\cdot]_V, \delta_V)$ consisting of linear maps*

$$\beta : V \times A \to V, \quad \theta : A \times A \to V, \quad [\cdot,\cdot]_V : V \times V \to V,$$
$$\phi : A \to V \otimes A, \quad \psi : V \to V \otimes A, \quad P : A \to V \otimes V, \quad \delta_V : V \to V \otimes V.$$

Then the unified product $A^{\phi,P}\#^{\psi}_{\beta,\sigma} V$ with bracket

$$[(a, x), (b, y)] := ([a, b], [x, y] + x \triangleleft b - y \triangleleft a + \theta(a, b)) \tag{34}$$

and cobracket

$$\delta_E(a) = \delta_A(a) + \phi(a) - \tau\phi(a) + P(a), \quad \delta_E(x) = \delta_V(x) + \psi(x) - \tau\psi(x) \tag{35}$$

form a Lie bialgebra if and only if $A\#_{\beta,\sigma}V$ form a Lie algebra, $A^{\phi,P}\#^{\psi} V$ form a Lie coalgebra and the following conditions are satisfied:

(F1) $\delta_V(x \triangleleft a) = \delta_V(x) \triangleleft a + a_{(-1)} \otimes x \triangleleft a_{(0)} - x \triangleleft a_{(0)} \otimes a_{(-1)} + [x, P(a)]$ $+x_{(0)} \otimes \theta(x_{(1)}, a) - \theta(x_{(1)}, a) \otimes x_{(0)}$;

(F2) $\phi([a, b]) + \psi\theta(a, b) = a_{(-1)} \otimes [a_{(0)}, b] + b_{(-1)} \otimes [a, b_{(0)}] + a_{(-1)} \triangleleft b \otimes a_{(0)}$ $-b_{(-1)} \triangleleft a \otimes b_{(0)} + \theta(a, b_1) \otimes b_2 + \theta(a_1, b) \otimes a_2$;

(F3) $\psi([x, y]) = [x, y_{(0)}] \otimes y_{(1)} + [x_{(0)}, y] \otimes x_{(1)}$;

(F4) $\delta_V\theta(a, b) + P[a, b] = a_{(-1)} \otimes \theta(a_{(0)}, b) + b_{(-1)} \otimes \theta(a, b_{(0)}) - \theta(a, b_{(0)}) \otimes b_{(-1)}$ $-\theta(a_{(0)}, b) \otimes a_{(-1)} + P(a) \triangleleft b - P(b) \triangleleft a$,

(F5) $\delta_V([x, y]) = [\delta_V(x), y] + [x, \delta_V([y)] - x_{(0)} \otimes y \triangleleft x_{(1)} - x \triangleleft y_{(1)} \otimes y_{(0)}$ $+y_{(0)} \otimes x \triangleleft y_{(1)} + y \triangleleft x_{(1)} \otimes x_{(0)}$;

(F6) $\psi(x \triangleleft a) = [x, a_{(-1)}] \otimes a_{(0)} + x_{(0)} \otimes [x_{(1)}, a] + x_{(0)} \triangleleft a \otimes x_{(1)} + x \triangleleft a_1 \otimes a_2$.

Conversely, any Lie bialgebra structure on E with the canonical projection map $p: E \to A$ both a Lie algebra homomorphism and a Lie coalgebra homomorphism is of this form.

Note that in this case, although $(A, [\cdot,\cdot], \delta_A)$ is not a Lie sub-bialgebra of $A^{\phi,P}\#_{\beta,\sigma}^{\psi,Q} V$, it is indeed a Lie bialgebra and a subspace $A^{\phi,P}\#_{\beta,\sigma}^{\psi,Q} V$. Denote the set of all Lie bialgebra extending datum of type (I) by $\mathcal{LB}^{(1)}(A,V)$.

Another special case is that we assume $\theta = 0, P = 0, \phi = 0$ in the above Theorem 3.29. In this case A is also a Lie bialgebra, and we obtain the following result.

Theorem 4.2. *Let A be a Lie bialgebra and V a vector space. An extending datum of A by V of type (II) is $\Omega^{(2)}(A,V) = (\alpha, \beta, \sigma, \psi, Q, [\cdot,\cdot]_V, \delta_V)$ consisting of linear maps*

$$\alpha: V \times A \to A, \quad \beta: V \times A \to V, \quad \sigma: V \times V \to A, \quad [\cdot,\cdot]_V: V \times V \to V,$$
$$\psi: V \to V \otimes A, \quad Q: V \to A \otimes A, \quad \delta_V: V \to V \otimes V.$$

Then the unified product $A_\alpha\#_{\beta,\sigma}^{\psi,Q} V$ with bracket

$$[(a,x),(b,y)] := ([a,b] + x \triangleright b - y \triangleright a + \sigma(x,y), [x,y] + x \triangleleft b - y \triangleleft a) \tag{36}$$

and cobracket

$$\delta_E(a) = \delta_A(a), \quad \delta_E(x) = \delta_V(x) + \psi(x) - \tau\psi(x) + Q(x) \tag{37}$$

form a Lie bialgebra if and only if $A_\alpha\#_{\beta,\sigma}V$ form a Lie algebra, $A\#^{\psi,Q}V$ form a Lie coalgebra and the following conditions are satisfied:

(G1) $\delta_A(x \triangleright a) + Q(x \triangleleft a) = x \triangleright \delta_A(a) + x_{(0)} \triangleright a \otimes x_{(1)} - x_{(1)} \otimes x_{(0)} \triangleright a$
$+[Q(x), a] + \sigma(x, a_{(-1)}) \otimes a_{(0)} - a_{(0)} \otimes \sigma(x, a_{(-1)})$;

(G2) $\delta_V(x \triangleleft a) = \delta_V(x) \triangleleft a$;

(G3) $\psi([x,y]) = [x, y_{(0)}] \otimes y_{(1)} + [x_{(0)}, y] \otimes x_{(1)} + y_{(0)} \otimes x \triangleright y_{(1)} - x_{(0)} \otimes y \triangleright x_{(1)}$
$+x_1 \otimes \sigma(x_2, y) + y_1 \otimes \sigma(x, y_2) + x \triangleleft y_{<1>} \otimes y_{<2>} - y \triangleleft x_{<1>} \otimes x_{<2>}$;

(G4) $\delta\sigma(x,y) + Q[x,y] = \sigma(x_{(0)}, y) \otimes x_{(1)} + \sigma(x, y_{(0)}) \otimes y_{(1)} - x_{(1)} \otimes \sigma(x_{(0)}, y)$
$-y_{(1)} \otimes \sigma(x, y_{(0)}) + x \triangleright Q(y) - y \triangleright Q(x)$;

(G5) $\delta([x,y]) = [x, \delta(y)] + [\delta(x), y] - x_{(0)} \otimes y \triangleleft x_{(1)} - x \triangleleft y_{(1)} \otimes y_{(0)}$
$+y_{(0)} \otimes x \triangleleft y_{(1)} + y \triangleleft x_{(1)} \otimes x_{(0)}$;

(G6) $\psi(x \triangleleft a) = x_1 \otimes x_2 \triangleright a + x_{(0)} \otimes [x_{(1)}, a] + x_{(0)} \triangleleft a \otimes x_{(1)} + x \triangleleft a_1 \otimes a_2$.

Conversely, any Lie bialgebra structure on E with the canonical injection map $i: A \to E$ both a Lie algebra homomorphism and a Lie coalgebra homomorphism is of this form.

Denote the set of all Lie bialgebra extending datum of type (II) by $\mathcal{LB}^{(2)}(A,V)$.

Note that $A^{\phi,P}\#_{\beta,\sigma}^{\psi} V$ and $A_\alpha\#_{\beta,\sigma}^{\psi,Q} V$ are all Lie bialgebra structures on E. Conversely, any Lie bialgebra extending system E of A through V is isomorphic to such a unified products of the two types. Now from Theorem 3.20, Theorem 3.28 in last section and Theorem 4.1, Theorem 4.2 we obtain the main result of in this section, which solve the extending problem for Lie bialgebra.

Theorem 4.3. *Let $(A,[\cdot,\cdot],\delta_A)$ be a Lie bialgebra, E a vector space containing A as a subspace and V be a complement of A in E. Denote by*

$$\mathcal{HLB}(V,A) := \mathcal{LB}^{(1)}(A,V) \sqcup \mathcal{LB}^{(2)}(A,V)/\equiv .$$

Then the map

$$\Upsilon : \mathcal{HLB}(V,A) \to BExtd(E,A), \tag{38}$$

$$\overline{\Omega^{(1)}(A,V)} \mapsto A^{\phi,P}\#^{\psi}_{\beta,\sigma}V, \quad \overline{\Omega^{(2)}(A,V)} \mapsto A_{\alpha}\#^{\psi,Q}_{\beta,\sigma}V \tag{39}$$

is bijective, where $\overline{\Omega^{(i)}(A,V)}$ is the equivalence class of $\Omega^{(i)}(A,V)$ under $\equiv$.

A very special case is that we assume $\theta=0, P=0$, $\alpha=0$, $\phi=0$ in the above Theorem 3.29. In this case A is also a Lie bialgebra, and we obtain the following result.

Theorem 4.4. *Let A be a Lie bialgebra and V a vector space. An extending datum of A by V is $\Omega(A,V)=(\alpha,\beta,\sigma,[\cdot,\cdot],\phi,\psi,Q,\delta_V)$ consisting of eight linear maps*

$$\beta : V\times A\to V, \qquad \sigma : V\times V\to A, \qquad [\cdot,\cdot] : V\times V\to V,$$
$$\psi : V\to V\otimes A, \quad Q : V\to A\otimes A, \quad \delta_V : V\to V\otimes V.$$

Then the unified product $A^{\phi}_{\alpha}\#^{\psi,Q}_{\beta,\sigma}V$ with bracket

$$[(a,x),(b,y)] := ([a,b]+\sigma(x,y), [x,y]+x\triangleleft b-y\triangleleft a) \tag{40}$$

and cobracket

$$\delta_E(a)=\delta_A(a), \quad \delta_E(x)=\delta_V(x)+\psi(x)-\tau\psi(x)+Q(x) \tag{41}$$

form a Lie bialgebra if and only if $A\#_{\beta,\sigma}V$ form a Lie algebra, $A\#^{\psi,Q}V$ form a Lie coalgebra and the following conditions are satisfied:

(H1) $Q(x\triangleleft a)=[Q(x),a]$*;*

(H2) $\delta_V(x\triangleleft a)=\delta_V(x)\triangleleft a$*;*

(H3) $\psi([x,y])=[x,y_{(0)}]\otimes y_{(1)}+[x_{(0)},y]\otimes x_{(1)}+x_1\otimes\sigma(x_2,y)+y_1\otimes\sigma(x,y_2)$
$+x\triangleleft y_{<1>}\otimes y_{<2>}-y\triangleleft x_{<1>}\otimes x_{<2>}$*;*

(H4) $\delta\sigma(x,y)+Q[x,y]=\sigma(x_{(0)},y)\otimes x_{(1)}+\sigma(x,y_{(0)})\otimes y_{(1)}$
$-x_{(1)}\otimes\sigma(x_{(0)},y)-y_{(1)}\otimes\sigma(x,y_{(0)})$*;*

(H5) $\delta([x,y])=[x,\delta(y)]+[\delta(x),y]-x_{(0)}\otimes y\triangleleft x_{(1)}-x\triangleleft y_{(1)}\otimes y_{(0)}$
$+y_{(0)}\otimes x\triangleleft y_{(1)}+y\triangleleft x_{(1)}\otimes x_{(0)}$*;*

(H6) $\psi(x\triangleleft a)=x_{(0)}\otimes[x_{(1)},a]+x_{(0)}\triangleleft a\otimes x_{(1)}+x\triangleleft a_1\otimes a_2$*.*

Yet an other special case is that we assume $\theta=0, P=0, \phi=0, \beta=0, [\cdot,\cdot]_A=0$, in the above Theorem 3.29. In this case A is an abelian Lie bialgebra, and we obtain the following [8, Theorem 2.10].

Corollary 4.5. [8] *Let A be an abelian Lie bialgebra and V a Lie bialgebra. An extending datum of A by V is $\Omega(A,V)=(\alpha,\sigma,\psi,Q)$ consisting of linear maps*

$$\alpha : V\times A\to A, \quad \sigma : V\times V\to A, \quad [\cdot,\cdot] : V\times V\to V,$$
$$\psi : V\to V\otimes A, \quad Q : V\to A\otimes A, \quad \delta_V : V\to V\otimes V.$$

Then the unified product $A_\alpha \#_\sigma^{\psi,Q} V$ *with bracket*

$$[(a,x),(b,y)] := (x \triangleright b - y \triangleright a + \sigma(x,y), [x,y]) \tag{42}$$

and cobracket

$$\delta_E(a) = \delta_A(a), \quad \delta_E(x) = \delta_V(x) + \psi(x) - \tau\psi(x) + Q(x) \tag{43}$$

form a Lie bialgebra if and only if $A_\alpha \#_\sigma V$ *form a Lie algebra,* $A\#^{\psi,Q} V$ *form a Lie coalgebra and the following conditions are satisfied:*

$$\begin{aligned} \delta_A(x \triangleright a) = {} & x \triangleright \delta_A(a) + x_{(0)} \triangleright a \otimes x_{(1)} - x_{(1)} \otimes x_{(0)} \triangleright a \\ & + \sigma(x, a_{(-1)}) \otimes a_{(0)} - a_{(0)} \otimes \sigma(x, a_{(-1)}), \end{aligned} \tag{44}$$

$$\begin{aligned} \psi([x,y]) = {} & [x, y_{(0)}] \otimes y_{(1)} + [x_{(0)}, y] \otimes x_{(1)} + y_{(0)} \otimes x \triangleright y_{(1)} - x_{(0)} \otimes y \triangleright x_{(1)} \\ & + x_1 \otimes \sigma(x_2, y) + y_1 \otimes \sigma(x, y_2) + x \triangleleft y_{<1>} \otimes y_{<2>} - y \triangleleft x_{<1>} \otimes x_{<2>}, \end{aligned} \tag{45}$$

$$\begin{aligned} \delta_A\sigma(x,y) + Q[x,y] = {} & \sigma(x_{(0)}, y) \otimes x_{(1)} + \sigma(x, y_{(0)}) \otimes y_{(1)} - x_{(1)} \otimes \sigma(x_{(0)}, y) \\ & - y_{(1)} \otimes \sigma(x, y_{(0)}) + x \triangleright Q(y) - y \triangleright Q(x). \end{aligned} \tag{46}$$

4.2. Flag extending systems

In this section, we study the case when V is a 1-dimensional vector space. This will be called flag extending system. Since V is a 1-dimensional vector space, then the bracket and cobracket of V is given by $[x,y] = 0$ and $\delta_V(x) = 0$ for all $x, y \in V$.

Lemma 4.6. *Let* $(A, [\cdot,\cdot], \delta_A)$ *be a braided Lie bialgebra and* $V = k\{x\}$ *be a 1-dimensional vector space. A flag datum consists of*

$$\lambda : A \to k, \quad D : A \to A, \quad T : A \to A, \quad a_0 \in A$$

satisfying the following compatibility conditions:

$$\lambda([a,b]) = 0, \tag{47}$$

$$D([a,b]) = [D(a), b] + [a, D(b)] + \lambda(a)D(b) - \lambda(b)D(a), \tag{48}$$

$$T([a,b]) = [T(a), b] + [a, T(b)] + \lambda(b)T(a) - \lambda(a)T(b), \tag{49}$$

$$T(D(a)) = D(T(a)) + [a_0, a] + \lambda(a_1)a_2. \tag{50}$$

The corresponding the extending datum $\Omega(A,V)$ *is given by:*

$$x \triangleright a = D(a), \quad x \triangleleft a = \lambda(a)x, \quad \phi(a) = x \otimes T(a), \quad \psi(x) = x \otimes a_0, \tag{51}$$

$$\sigma(x,x) = 0, \quad [x,x] = 0, \quad P(a) = 0, \quad \delta_V(x) = 0. \tag{52}$$

The unified product associated to this flag extending system is given by

$$[(a,x),(b,y)] = \Big([a,b] + D(a)y - D(b)x, \lambda(a)y - \lambda(b))x\Big), \quad \text{and} \tag{53}$$

$$\delta_E(a) = \delta_A(a) + x \otimes T(a) - T(a) \otimes x, \quad \delta_E(x) = x \otimes a_0 - a_0 \otimes x. \tag{54}$$

Denote the set of all flag datums of braided Lie bialgebra by $\mathcal{FB}(A)$.

Definition 4.7. Two flag datums (λ, D, T, a_0) and $(\lambda', D', T, a_0') \in \mathcal{FB}(A)$ are called *equivalent* if $\lambda' = \lambda$, $a_0' = a_0$ and there exist some element $r_0 \in A$ such that

$$\lambda(a)r_0 = [r_0, a] - D(a) + sD'(a), \tag{55}$$

$$\delta'_A(r_0) = r_0 \otimes a_0 - a_0 \otimes r_0, \tag{56}$$

$$\delta'_A(a) = \delta_A(a) + r_0 \otimes T(a) - T(a) \otimes r_0. \tag{57}$$

By the above lemma, we have

Theorem 4.8. *Let $(A, [\cdot,\cdot], \delta_A)$ be a braided Lie bialgebra and V be a 1-dimensional vector space. Then there is a bijection between the set $\mathcal{BLB}(A, V)$ of all Lie bialgebra extending systems of A by V and $\mathcal{FB}(A)$.*

Next, we consider flag extending systems for Lie bialgebras.

Lemma 4.9. *Let $(A, [\cdot,\cdot], \delta_A)$ be a Lie bialgebra. A flag datum of type (I) consists of*

$$\lambda : A \to k, \quad T : A \to A, \quad a_0 \in A$$

satisfying the following compatibility conditions:

$$\lambda([a, b]) = 0, \tag{58}$$

$$T([a, b]) = [T(a), b] + [a, T(b)], \tag{59}$$

$$[a_0, a] + \lambda(a_1)a_2 = 0. \tag{60}$$

The corresponding extending datum $\Omega^{(1)}(A, V)$ of type (I) *is given by:*

$$x \triangleleft a = \lambda(a)x, \quad \phi(a) = x \otimes T(a), \quad \psi(x) = x \otimes a_0, \tag{61}$$

$$\theta(a, b) = 0, \quad [x, x] = 0, \quad P(a) = 0, \quad \delta_V(x) = 0. \tag{62}$$

The unified product $A\#^{(1)}V$ associated to the flag extending system is given by

$$[(a, x), (b, y)] = \Big([a, b], \lambda(b)x - \lambda(a))y\Big), \quad \text{and} \tag{63}$$

$$\delta_E(a) = \delta_A(a) + x \otimes T(a) - T(a) \otimes x, \quad \delta_E(x) = x \otimes a_0 - a_0 \otimes x. \tag{64}$$

Denote the set of all flag datums of type (I) by $\mathcal{F}^{(1)}(A)$.

Lemma 4.10. *Let $(A, [\cdot,\cdot], \delta_A)$ be a Lie bialgebra. A flag datum of type* (II) *consists of*

$$\lambda : A \to k, \quad D : A \to A, \quad a_0 \in A, \quad Q \in A \wedge A$$

satisfying the following compatibility conditions:

$$\lambda([a, b]) = 0, \tag{65}$$

$$D([a, b]) = [D(a), b] + [a, D(b)] + \lambda(a)D(b) - \lambda(b)D(a), \tag{66}$$

$$[a_0, a] + \lambda(a_1)a_2 = 0, \tag{67}$$

$$\delta_A(D(a)) + \lambda(a)Q$$
$$= [a, Q] + D(a_1) \otimes a_2 + a_1 \otimes D(a_2) + D(a) \otimes a_0 - a_0 \otimes D(a), \tag{68}$$

$$a_0 \otimes Q - \tau_{12}(a_0 \otimes Q) + Q \otimes a_0 = (\mathrm{id} \otimes \delta - \tau_2(\mathrm{id} \otimes \delta) - \delta \otimes \mathrm{id})\, Q. \tag{69}$$

The corresponding extending datum $\Omega^{(2)}(A,V)$ *of type* (II) *is given by:*

$$x \triangleleft a = \lambda(a)x, \quad x \triangleright a = D(a), \quad \omega(x,x) = 0, \tag{70}$$

$$\psi(x) = x \otimes a_0, \quad Q(x) = Q. \tag{71}$$

The unified product $A\#^{(2)}V$ *is given by the bracket*

$$[(a,x),(b,y)] = \Big([a,b] + D(a)y - D(b)x, \lambda(a)y - \lambda(b))x\Big) \tag{72}$$

and cobracket $$\delta_E(a) = \delta_A(a), \quad \delta_E(x) = x \otimes a_0 - a_0 \otimes x + Q, \tag{73}$$

Denote the set of all flag datums of type (II) by $\mathcal{F}^{(2)}(A)$.
By the above two lemmas, we have

Theorem 4.11. *Let* $(A, [\cdot,\cdot], \delta_A)$ *be a Lie bialgebra and* $V = k\{x\}$ *be a 1-dimensional vector space. Then there is a bijection between the set* $\mathcal{LB}(A,V)$ *of all Lie bialgebra extending systems of* A *by* V *and* $\mathcal{F}(A) = \mathcal{F}^{(1)}(A) \sqcup \mathcal{F}^{(2)}(A)$.

Definition 4.12. Two flag datums (λ, T, a_0) and $(\lambda', T', a_0') \in \mathcal{F}^{(1)}(A)$ are called *equivalent* if $\lambda' = \lambda$, $a_0' = a_0$ and there exist some element $r_0 = r(x) \in A$ such that

$$\lambda(a)r_0 = [r_0, a], \tag{74}$$

$$\delta_A'(r_0) = r_0 \otimes a_0 - a_0 \otimes r_0, \tag{75}$$

$$\delta_A'(a) = \delta_A(a) + r_0 \otimes T(a) - T(a) \otimes r_0. \tag{76}$$

Definition 4.13. Two flag datums (λ, D, a_0, Q) and $(\lambda', D', a_0', Q') \in \mathcal{F}^{(2)}(A)$ are called equivalent if $\lambda' = \lambda$, $a_0' = a_0$ and there exist some element $r_0 = r(x) \in A$ and $s \in k^*$ such that

$$\lambda(a)r_0 = [r_0, a] - D(a) + sD'(a), \tag{77}$$

$$\delta_A(r_0) + sQ' = r_0 \otimes a_0 - a_0 \otimes r_0 + Q. \tag{78}$$

From the above discussion, we obtain:

Theorem 4.14. *Let* $(A, [\cdot,\cdot], \delta_A)$ *be a Lie bialgebra of codimension one in a vector space* E*. Then we have* $BExtd(E,A) \cong \mathcal{HLB}(V,A) \cong \mathcal{F}(A)/\equiv$.

Finally, we give an example to compute the flag extending datums.

Example 4.15. Let $A = \mathfrak{sl}(2) = span\{H, X, Y\}$ be the three dimensional Lie algebra with the Lie bracket given by

$$[H, X]=2X, \quad [H,Y]=-2Y, \quad [X,Y]=H.$$

There is a standard Lie bialgebra structure on A with the Lie co-bracket given by

$$\delta(H) = 0, \quad \delta(X) = X \wedge H, \quad \delta(Y) = Y \wedge H.$$

Then by the equation (58) $\lambda([a,b]) = 0$ and the fact that $[\mathfrak{sl}(2), \mathfrak{sl}(2)] = \mathfrak{sl}(2)$, we have $\lambda = 0$. Since $T([a,b]) = [T(a),b] + [a,T(b)]$, define

$$T(H,X,Y) = (H,X,Y)T = (H,X,Y)\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

By direct computations, we have $T=\left(\begin{array}{ccc}0 & b_1 & c_1\\ -2c_1 & b_2 & 0\\ -2b_1 & 0 & -b_2\end{array}\right)$.

Since $\lambda=0$, thus by equation (60) we have $[a_0,a]=0$ for all $a\in A$.
But the center of $\mathfrak{sl}(2)$ is zero, so we have $a_0=0$. Therefore we obtain the flag datum of type (I) by $\lambda=0, a_0=0$ and T as above.
Now we compute the flag datum of type (II). By similar discussion, we obtain $\lambda=0, a_0=0$ and D as above. Assume $Q=k_1X\wedge H+k_2Y\wedge H+k_3X\wedge Y$, then by equation (69) we get $(\mathrm{id}\otimes\delta)Q-\tau_{12}(\mathrm{id}\otimes\delta)Q-(\delta\otimes\mathrm{id})Q=0$. By direct computations, we have

$$(\mathrm{id}\otimes\delta-\tau_{12}(\mathrm{id}\otimes\delta)-\delta\otimes\mathrm{id})\,Q=k_3X\wedge Y\wedge H=0.$$

Thus we obtain $k_3=0$ and $Q=k_1X\wedge H+k_2Y\wedge H$.
Next, we check the equation (68): $\delta_A(D(a))=[a,Q]+D\left(a_1\right)\otimes a_2+a_1\otimes D\left(a_2\right)$.
When $a=H$, we have

$$\begin{aligned}
LHS&=\delta_A(D(H))=\delta_A\left(-2c_1X-2b_1Y\right)=-2c_1X\wedge H-2b_1Y\wedge H,\\
RHS&=[H,k_1X\wedge H+k_2Y\wedge H]=2k_1X\wedge H-2k_2Y\wedge H.
\end{aligned}$$

Thus we have $c_1=k_1$, $k_2=-b_1$. When $a=X$, we have

$$\begin{aligned}
LHS&=\delta_A(D(X))=\delta_A\left(b_1H+b_2X\right)=b_2X\wedge H,\\
RHS&=[X,k_1X\wedge H+k_2Y\wedge H]+D(X)\wedge H+X\wedge D(H)\\
&=-2k_2X\wedge Y+b_2X\wedge H-2b_1X\wedge Y\\
&\Rightarrow\quad -2k_2-2b_1=0\Rightarrow k_2=-b_1.
\end{aligned}$$

When $a=Y$, we have

$$\begin{aligned}
LHS&=\delta_A(D(Y))=\delta_A\left(c_1H-b_2Y\right)=c_1\delta_A(H)-b_2\delta_A(Y)=-b_2Y\wedge H\\
RHS&=[Y,k_1H\wedge x+k_2H\wedge Y]+D(Y)\wedge H+Y\wedge D(H)\\
&=2k_1Y\wedge X-b_2Y\wedge H-2c_1Y\wedge H\\
&\Rightarrow\quad c_1=k_1=0.
\end{aligned}$$

Therefore we obtain the flag datum of type (II) by

$$\lambda=0,\quad a_0=0,\quad D=\left(\begin{array}{ccc}0 & b_1 & 0\\ 0 & b_2 & 0\\ -2b_1 & 0 & -b_2\end{array}\right),\quad Q=-b_1Y\wedge H.$$

If we are given another flag datum of type (II) by

$$\lambda=0,\quad a_0=0,\quad D'=\left(\begin{array}{ccc}0 & b_1' & 0\\ 0 & b_2' & 0\\ -2b_1' & 0 & -b_2'\end{array}\right),\quad Q'=-b_1'Y\wedge H.$$

Let $r_0=Y, s=1$. By equation (78), $\delta(Y)=Q-sQ'=-(b_1-b_1')Y\wedge H$, we have $b_1-b_1'=-1$. Then by direct computations we get $[r_0,a]=\left(\begin{array}{ccc}0 & -1 & 0\\ 0 & 0 & 0\\ 2 & 0 & 0\end{array}\right)(a)$.

Thus by equation (77), $[r_0, a] = (D - D')(a)$, we obtain

$$(D - D') = \begin{pmatrix} 0 & b_1 - b_1' & 0 \\ 0 & b_2 - b_2' & 0 \\ -2(b_1 - b_1') & 0 & -(b_2 - b_2') \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Therefore these two flag datum (λ, D, a_0, Q) and (λ', D', a_0', Q') are equivalent if and only if $\lambda = \lambda' = 0, a_0 = a_0' = 0$, $b_1 + 1 = b_1'$ and $b_2 = b_2'$.

5. Conclusions and problems

In this paper, the theory unified product for (braided) Lie bialgebras was developed. An important type of braided Lie bialgebras is constructed from quasitriangular Lie bialgebras, see [18]. A natural interesting problems arise: when the unified product is a quasitriangular Lie bialgebra? On the other hand, the theory Yetter-Drinfeld modules for Hopf algebras have been well studied in recent years (see [24] and reference therein), but we have not seen any process in the Lie bialgebra direction. So another problem arise: How to classify the braided Lie bialgebras over a fixed Lie bialgebra? Finally, I conjecture that the symmetrizable Kac-Moody algebras can be realized by the unified product for some cocycle σ, θ or cycle P, Q. Since all these problems fall outside of the scope of this paper, their solutions are left to future investigations.

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