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A Cohomological Approach to the Existence Problem of Compact Clifford-Klein Forms for Indecomposable Pseudo-Riemannian Symmetric Spaces

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Abstract. We give examples of indecomposable but reducible pseudo-Riemannian symmetric spaces of arbitrary signature which admit no compact Clifford-Klein forms by a cohomological approach. We also show some series of even dimensional indecomposable but reducible pseudo-Riemannian symmetric spaces of arbitrary signature admit compact Clifford-Klein forms. We give another proof for the classification of indecomposable but reducible symmetric spaces with signature $(2,2)$ which admit compact Clifford-Klein forms which is proved in a forthcoming paper [*Four dimensional compact Clifford–Klein forms of pseudo-Riemannian symmetric spaces with signature* $(2,2)$] by a different approach.

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1. Introduction

Let G be a Lie group, $H \subset G$ its closed subgroup and $\Gamma \subset G$ its discrete subgroup. Assume that Γ acts on a homogeneous space G/H properly discontinuously and (fixed point) freely. Then the quotient space $\Gamma\backslash G/H$ has a natural manifold structure and is called a *Clifford-Klein form* of G/H. If G/H admits a G-invariant Riemannian metric, that is, H is compact, any discrete subgroup $\Gamma \subset G$ acts on the space G/H properly discontinuously. On the other hand, if G/H is non-Riemannian, that is, H is not compact, this is not always true. In the late 1980s, a systematic study of Clifford–Klein forms for non-Riemannian homogeneous spaces was initiated by T. Kobayashi ([10]). The classification problem of homogeneous spaces which admit compact Clifford-Klein forms is one of the important open problems raised by T. Kobayashi.

Problem 1.1. [10] Classify homogeneous spaces which admit compact Clifford-Klein forms.

This problem is attacked by various approaches (see [1, 2, 11, 12, 15, 16, 18, 19]), but still widely open even for irreducible symmetric spaces. See the excellent survey papers [13] [14] by the founder of the field. So far, except for Riemannian cases, five series and seven sporadic types of irreducible symmetric spaces have been shown to

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admit compact Clifford-Klein forms ([13, 16]). It seems to be considered that this problem for irreducible symmetric spaces are most difficult.

However, this problem for irreducible symmetric spaces is also difficult. For indecomposable but reducible Lorentzian symmetric spaces, Kath-Olbrich ([8]) found a necessary and sufficient condition for the existence of compact Clifford-Klein forms. We study homogeneous spaces of the form $G_{D,D'}/H$ in this paper. The class of the spaces of the form $G_{D,D'}/H$ contains all indecomposable but reducible 1-connected Lorentzian symmetric spaces whose transvection group (see Definition 1.3 in [17]) is solvable. Moreover, it contains most of indecomposable but reducible 1-connected pseudo-Riemannian symmetric spaces with signature $(2,2)$ (see Fact 3.12 for the details, see also [8, 17]).

Definition 1.2. Let D, D' be two $n \times n$ invertible symmetric matrices and put:

$$W := \begin{pmatrix} & D' \\ D & \end{pmatrix} \in \mathfrak{sp}(n, \mathbb{R}).$$

Let $\mathfrak{h}_n := \langle X_1, ..., X_n, Y_1, ..., Y_n, Z\rangle_{\mathbb{R}}$ be the Heisenberg Lie algebra ($[X_i, Y_j] = \delta_{ij} Z$). We define a Lie algebra $\mathfrak{g}_{D,D'}$ by:

$$\mathfrak{g}_{D,D'} := \mathbb{R}W \ltimes_\rho \mathfrak{h}_n,$$

where the action ρ of $\mathbb{R}W$ on $\mathfrak{h}_n$ is given by:

$$\begin{aligned} \rho(W)X_i &= \sum_{j=1}^{n} s_{ij} Y_j \quad (D = (s_{ij})_{i,j}), \\ \rho(W)Y_i &= \sum_{j=1}^{n} s'_{ij} X_j \quad (D' = (s'_{ij})_{i,j}), \\ \rho(W)Z &= 0. \end{aligned}$$

We define a subalgebra $\mathfrak{h} := \langle Y_1, \ldots, Y_n\rangle_{\mathbb{R}}$. Denote by $G_{D,D'}$ the 1-connected Lie group with Lie algebra $\mathfrak{g}_{D,D'}$ and $H \subset G_{D,D'}$ the analytic subgroup having the Lie algebra $\mathfrak{h}$. ■

Remark that the Lie algebra $\mathfrak{g}_{D,D'}$ is not nilpotent but solvable. In Subsection 2.2, we see that the homogeneous space $G_{D,D'}/H$ admits a pseudo-Riemannian structure.

In this paper, we give a necessary condition for the existence of compact Clifford-Klein forms of the space $G_{D,D'}/H$.

Theorem 1.3. *For $a_1, ..., a_n, b_1, ..., b_n \in \mathbb{R}^\times$, we put $D := \mathrm{diag}(a_1, ..., a_n) \in GL(n, \mathbb{R})$ and $D' := \mathrm{diag}(b_1, \cdots, b_n) \in GL(n, \mathbb{R})$. The space $G_{D,D'}/H$ does not admit a compact Clifford-Klein forms unless there exist $\varepsilon_1, \cdots, \varepsilon_n \in \{\pm 1\}$ satisfying:*

$$\sum_{i=1}^{n} \varepsilon_i \sqrt{a_i b_i} = 0, \quad \textit{where } \sqrt{a_i b_i} \in \mathbb{R} \cup \sqrt{-1}\mathbb{R}.$$

We prove Theorem 1.3 by applying a cohomological obstruction to the existence of compact Clifford-Klein forms (Morita [18, Theorem 1.2(2)]). In the setting of Theorem 1.3, we can verify this obstruction by direct calculation (see Lemma 3.8).

Remark that this approach of Morita ([18]) does not give a necessary and sufficient condition [17], and we shall see in Example 1.5 the limitation of this approach.

Using this theorem, we can construct indecomposable but reducible symmetric spaces of any signature which admit no compact Clifford-Klein forms. Actually, the next corollary easily follows from Theorem 1.3.

Corollary 1.4. *Let p and q be positive integers. We set the matrices D and D' as $D := \mathrm{diag}(1, t, t^2, \cdots, t^{2p-4}, s, s^2, \cdots, s^{2q-4}) \in GL(p+q-2, \mathbb{R})$ for some $t, s \geq 2$ and $D' := I_{p-1,q-1} := \mathrm{diag}(\underbrace{1, 1, \cdots, 1}_{p-1}, \underbrace{-1, -1, \cdots, -1}_{q-1}) \in GL(p+q-2, \mathbb{R})$. The space $G_{D,D'}/H$ has indecomposable but reducible pseudo-Riemannian space structure with signature (p, q) and admits no compact Clifford-Klein forms.*

Example 1.5. Any indecomposable but reducible Lorentz symmetric space with signature $(4, 1)$ is written as $G_{D,D'}/H$, where $D = \mathrm{diag}(a_1, a_2, a_3)$ and $D' = I_3$ for some $a_1, a_2, a_3 \in \mathbb{R}^\times$. The parameter set for isomorphism classes for these spaces is given by the sphere $\{(a_1, a_2, a_3) \in (\mathbb{R}^\times)^3 \mid a_1^2 + a_2^2 + a_3^2 = 1\}$ (see [17, Proposition 4.9]). The necessary condition of Theorem 1.3 is

$$\begin{aligned}&(\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3})(\sqrt{a_1} - \sqrt{a_2} + \sqrt{a_3})(\sqrt{a_1} + \sqrt{a_2} - \sqrt{a_3})(\sqrt{a_1} - \sqrt{a_2} - \sqrt{a_3})\\ &= a_1^2 + a_2^2 + a_3^2 - 2a_1a_2 - 2a_2a_3 - 2a_3a_1 = 0,\end{aligned}$$

which is described by the two circles in the following figure. By Fact 1.6, the symmetric spaces which correspond to a dense subset of these circles admit compact Clifford-Klein forms.

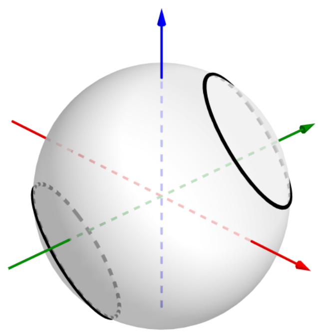

Figure 1: The parameter space of indecomposable but reducible Lorentz symmetric spaces with signature (4,1)

Fact 1.6. [8, Corollary 5.10] *Put $\mathcal{M}^c_{3,0}$ and $\mathcal{M}^0_{3,0}$ as follows. Then $\mathcal{M}^c_{3,0}$ is a dense subset of $\mathcal{M}^0_{3,0}$.*

$$\begin{aligned}\mathcal{M}^0_{3,0} &:= \{(a_1, a_2, a_3) \in (\mathbb{R}^\times)^3 \mid a_1^2 + a_2^2 + a_3^2 = 1,\ 2a_1a_2 + 2a_2a_3 + 2a_3a_1 = 1\}\\ &= \{(a_1, a_2, a_3) \in S^2 \mid a_1 + a_2 + a_3 = \pm\sqrt{2},\ a_1a_2a_3 \neq 0\},\\ \mathcal{M}^c_{3,0} &:= \left\{(a_1, a_2, a_3) \in M^0_{3,0} \;\middle|\; \begin{array}{l} G_{\mathrm{diag}(a_1,a_2,a_3),I_3}/H \quad \textit{admits} \\ \textit{compact Clifford-Klein forms} \end{array}\right\}.\end{aligned}$$

In [17, Theorem 1.5], we show that only two 1-connected indecomposable but reducible pseudo-Riemannian symmetric spaces G/H with signature $(2,2)$ admit compact Clifford-Klein forms, where G is the transvection group of the space G/H and it is solvable. We also give a part of this theorem another proof by a cohomological approach. The classification of the 1-connected pseudo-Riemannian symmetric spaces with signature $(2,2)$ is given by Kath-Olbrich (see Fact 3.12). We show the spaces correspond to the case (II) (a)(b)(c) in Fact 3.12 does not admit compact Clifford-Klein forms.

Moreover, we also construct some series of even dimensional indecomposable but reducible pseudo-Riemannian symmetric spaces which admit compact Clifford-Klein forms.

Theorem 1.7. *Let p and q be positive integers such that $p+q$ is even. Put $D := I_{p-1,q-1}$ and $D' := -I_{p-1,q-1}$. Then $G_{D,D'}/H$ admits compact Clifford-Klein forms.*

The paper is organized as follows: In Section 2, we review a cohomological obstruction to the existence of compact Clifford-Klein forms and give the space $G_{D,D'}/H$ a pseudo-Riemannian symmetric space structure. In Section 3, by using the cohomological obstruction, we give a proof of Theorem 1.3 and another proof for the classification of indecomposable but reducible symmetric spaces with signature $(2,2)$ which admit compact Clifford-Klein forms [17]. Finally, in Section 4, we show Theorem 1.7.

2. Preliminaries

In this section, we review a cohomological obstruction to the existence of compact Clifford-Klein forms and give a pseudo-Riemannian symmetric structure to the space $G_{D,D'}/H$.

2.1. Cohomological obstruction to the existence of compact Clifford-Klein forms

In this subsection, we review a cohomological obstruction to the existence of compact Clifford-Klein forms. This approach was initiated by Kobayashi-Ono([15]) and Kobayashi-Ono's cohomological obstruction has been used extensively by a number of mathematicians, and we use Morita ([18]) among others in this article. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{R}$. We use the following notation:

$$C^n(\mathfrak{g}) := \bigwedge^n \mathfrak{g}^*, \quad d_n : C^n(\mathfrak{g}) \to C^{n+1}(\mathfrak{g}) \quad \omega \mapsto d_n(\omega),$$

$$d_n(\omega)(X_1, \cdots, X_n) := \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_n),$$

$$C^n(\mathfrak{g}, \mathfrak{h}) := \Big(\bigwedge^n (\mathfrak{g}/\mathfrak{h})^* \Big)^{\mathfrak{h}},$$

where $\mathfrak{g}^*$ is the dual space of $\mathbb{R}$-vector space $\mathfrak{g}$. We also use the same notation d_n as its restriction to the subspace $C^n(\mathfrak{g}, \mathfrak{h})$. We denote the cohomology of $C^n(\mathfrak{g})$ and $C^n(\mathfrak{g}, \mathfrak{h})$ by $H^n(\mathfrak{g}) := H^n(C^n(\mathfrak{g}, \mathbb{R}), d_n)$ and $H^n(\mathfrak{g}, \mathfrak{h}) := H^n(C^n(\mathfrak{g}, \mathfrak{h}), d_n)$.

The next fact is essential for constructing a cohomological obstruction.

Fact 2.1. (Kobayashi-Ono [15]) *Let G be a Lie group and H its closed subgroup with finitely many connected components. Put $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{h} := \mathrm{Lie}(H)$. For a discontinuous group $\Gamma(\subset G)$ for G/H, if the Clifford-Klein form $\Gamma\backslash G/H$ is compact, the following map is injective.*

$$\eta : H^N(\mathfrak{g}, \mathfrak{h}) \to H^N(\Gamma\backslash G/H), \quad \textit{where } N := \dim(G/H).$$

Kobayashi-Ono [15] discovered a geometric cohomological obstruction by using characteristic classes by using Fact 2.1, and Morita ([18]) extended their obstruction algebraically based on Fact 2.1.

Fact 2.2. [18] *Let G be a Lie group, H its closed subgroup with finitely many connected components and K_H a maximal compact subgroup of H. Put $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$, and $\mathfrak{k}_H := \mathrm{Lie}(K_H)$. If the homogeneous space G/H admits compact Clifford-Klein forms, the following homomorphism induced by the natural projection $\pi : G/K_H \to G/H$ is injective.*

$$\pi^* : H^N(\mathfrak{g}, \mathfrak{h}) \to H^N(\mathfrak{g}, \mathfrak{k}_H), \quad \textit{where } N := \dim(G/H).$$

2.2. Pseudo-Riemannian structure of homogeneous spaces $G_{D,D'}/H$

In this subsection, we review a pseudo-Riemmanian symmetric space structure of $G_{D,D'}/H$ from [17]. It is enough to define an involution σ on $\mathfrak{g}_{D,D'}$ satisfying $\mathfrak{h} = \mathfrak{g}_{D,D'}^{\sigma}$ and an (indefinite) inner product on $\mathfrak{g}_{D,D'}^{-\sigma}$.

Proposition 2.3. [17, Proposition and Definition 4.6] *For the Lie algebra $\mathfrak{g}_{D,D'}$, we set a subspace $\mathfrak{q} := \langle W, X_1, \cdots, X_n, Z\rangle_{\mathbb{R}} \subset \mathfrak{g}_{D,D'}$. Remark that $\mathfrak{g}_{D,D'} = \mathfrak{h} \oplus \mathfrak{q}$. We put $\sigma := \mathrm{id}_{\mathfrak{h}} \oplus (-\mathrm{id}_{\mathfrak{q}}) : \mathfrak{g}_{D,D'} \to \mathfrak{g}_{D,D'}$ and set an inner product on $\mathfrak{q}$ by the following Gram matrix with respect to the basis $(W, X_1, \cdots, X_n, Z)$:*

$$g := \begin{pmatrix} 0 & & -1 \\ & D'^{-1} & \\ -1 & & 0 \end{pmatrix}.$$

The triple $(g_{D,D'}, \sigma, g)$ is an indecomposable but reducible symmetric triple with signature $(p+1, q+1)$. Then the homogeneous space $G_{D,D'}/H$ has an indecomposable but reducible pseudo-Riemannian symmetric space structure with signature $(p+1, q+1)$.

3. Cohomological obstruction to the pair $(\mathfrak{g}_{D,D'}, \mathfrak{h})$

In this section, we give a cohomological obstruction to the pair $(\mathfrak{g}_{D,D'}, \mathfrak{h})$ using the notation as in Subsection 2.2. We describe it for general case in Subsection 3.1, for the case that D and D' are diagonal matrices in Subsection 3.2, and for the case that D and D' are 2-dimensional symmetric matrices in Subsection 3.3.

3.1. The general case

In this subsection, we give a cohomological obstruction to the pair $(\mathfrak{g}_{D,D'}, \mathfrak{h})$ (Proposition 3.5).

For using a cohomological approach (Fact 2.2), we have to check if $H^{n+2}(\mathfrak{g}_{D,D'}, \mathfrak{h}; \mathbb{R})$ is non-trivial. We use the following:

Fact 3.1. [18] *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ its subalgebra.*

(1) *The $\mathfrak{h}$-action on $\mathfrak{g}/\mathfrak{h}$ is unimodular $\iff$ $C^{n+2}(\mathfrak{g}, \mathfrak{h}) \simeq \mathbb{R}$*

(2) *The $N(\mathfrak{h})$-action on $\mathfrak{g}/\mathfrak{h}$ is unimodular $\iff$ $H^{n+2}(\mathfrak{g}, \mathfrak{h}) \simeq \mathbb{R}$,*

where $N(\mathfrak{h})$ is the normalizer of $\mathfrak{h}$.

The next lemma follows from an easy calculation.

Lemma 3.2. *The action of $\mathfrak{h}$ on $\mathfrak{g}_{D,D'}/\mathfrak{h}$ and the action of $\mathfrak{h}_n = N(\mathfrak{h})$ on $\mathfrak{g}_{D,D'}/\mathfrak{h}$ are unimodular.*

We apply Fact 2.2 to the spaces $G_{D,D'}/H$. Since there are no non-trivial compact subgroups for 1-connected solvable Lie groups ([5, Theorem 2.3]), the following proposition follows from Fact 2.2.

Since there are no non-trivial compact subgroups in 1-connected solvable Lie groups ([6, Theorem 2.3]), to prove that $G_{D,D'}/H$ does not have a compact Clifford-Klein forms, it suffices to verify that the homomorphism

$$\pi^* : H^{n+2}(\mathfrak{g}_{D,D'}, \mathfrak{h}; \mathbb{R}) \to H^{n+2}(\mathfrak{g}_{D,D'}; \mathbb{R})$$

is not injective (Fact 2.2).

In the rest of this subsection, we simplify this condition (Proposition 3.5). We put $T := W^* \wedge X_1^* \wedge \cdots \wedge X_n^* \wedge Z^* \in C^{n+2}(\mathfrak{g}_{D,D'}, \mathfrak{h}; \mathbb{R})$. Note that $H^{n+2}(\mathfrak{g}_{D,D'}, \mathfrak{h}; \mathbb{R})$ is generated by $[T]$. Since $\pi^*(T) = T$ for the natural map

$$\pi^* : C^{n+2}(\mathfrak{g}_{D,D'}, \mathfrak{h}; \mathbb{R}) \to C^{n+2}(\mathfrak{g}_{D,D'}; \mathbb{R}),$$

the following equivalence holds:

$$\pi^* \text{ is injective} \iff T \notin \operatorname{Im}(d_{n+1}),$$

where d_{n+1} denotes the differential map $C^{n+1}(\mathfrak{g}_{D,D'}; \mathbb{R}) \to C^{n+2}(\mathfrak{g}_{D,D'}; \mathbb{R})$.

To simplify the condition $T \notin \operatorname{Im}(d_{n+1})$, we restrict d_{n+1} to some subspace of $C^{n+1}(\mathfrak{g}_{D,D'}; \mathbb{R})$.

Definition 3.3. For $\alpha, \beta \in \mathbb{N}$ satisfying $\alpha + \beta = n$, we define a subspace $V_{\alpha,\beta} \subset C^{n+1}(\mathfrak{g}_{D,D'}; \mathbb{R})$ as follows.

$$V_{\alpha,\beta} := \left\langle X_{i_1}^* \wedge \cdots \wedge X_{i_\alpha}^* \wedge Y_{j_1}^* \wedge \cdots \wedge Y_{j_\beta}^* \wedge Z^* \in C^{n+1}(\mathfrak{g}_{D,D'}; \mathbb{R}) \right\rangle_{\mathbb{R}}$$

where $1 \le i_1 < \cdots < i_\alpha \le n$ and $1 \le j_1 < \cdots < j_\beta \le n$.

We also define a linear map $\delta_{\alpha,\beta} : V_{\alpha,\beta} \to V_{\alpha-1,\beta+1} \oplus V_{\alpha+1,\beta-1}$ as follows:

$$\begin{aligned} &\delta(X_{i_1}^* \wedge \cdots \wedge X_{i_\alpha}^* \wedge Y_{j_1}^* \wedge \cdots \wedge Y_{j_\beta}^* \wedge Z^*) \\ &:= \sum_{l=1}^{\alpha} X_{i_1}^* \wedge \cdots \wedge \Big(\sum_k s_{ki_l} Y_k^* \Big) \wedge \cdots \wedge X_{i_\alpha}^* \wedge Y_{j_1}^* \wedge \cdots \wedge Y_{j_\beta}^* \wedge Z^* \\ &\quad + \sum_{l=1}^{\beta} X_{i_1}^* \wedge \cdots \wedge X_{i_\alpha}^* \wedge Y_{j_1}^* \wedge \cdots \wedge \Big(\sum_k s'_{kj_l} X_k^* \Big) \wedge \cdots \wedge Y_{j_\beta}^* \wedge Z^*, \end{aligned}$$

where $D = (s_{ij})$, $D' = (s'_{ij})$.

Put $V := \bigoplus_{\alpha+\beta=n} V_{\alpha,\beta} \subset C^{n+1}(\mathfrak{g}_{D,D'};\mathbb{R})$ and we use the notation $\delta := \bigoplus_{\alpha+\beta=n} \delta_{\alpha,\beta}$.

Remark 3.4. By an easy calculation, the following diagram is commutative.

$$\begin{array}{ccc} V_{\alpha,\beta} & \xrightarrow{\ d_{n+1}|_{V_{\alpha,\beta}}\ } & C^{n+2}(\mathfrak{g}_{D,D'}) \\ & \searrow^{\delta_{\alpha,\beta}} & \uparrow{\scriptstyle \iota|_{V_{\alpha+1,\beta-1}\oplus V_{\alpha-1,\beta+1}}} \\ & & V_{\alpha+1,\beta-1} \oplus V_{\alpha-1,\beta+1} \end{array}$$

Here, we set $\iota : V \to C^{n+2}(\mathfrak{g}_{D,D'}),\ X \mapsto -W^* \wedge X$. Note that $T \in \mathrm{Im}(\iota)$.

We are ready to rephrase the injectivity of π^* as follows.

Proposition 3.5. *The following equivalences hold.*

$$\pi^* \textit{is injective} \iff T \notin \mathrm{Im}(d_{n+1}|_V) \iff \iota^{-1}(T) \notin \mathrm{Im}(\delta).$$

Proof. Since the linear map ι is injective, it is enough to show that $\mathrm{Im}(d_{n+1}) = \mathrm{Im}(d_{n+1}|_V)$. For $U := \{W,\ X_1, \cdots,\ X_n,\ Y_1,\ \cdots,\ Y_n,\ Z\}$, put

$$W_1 := \langle \bigwedge_{s\in S} s^* \in C^{n+1}(\mathfrak{g}_{D,D'}) \mid W \in S \subset U,\ \#S = n+1\rangle_{\mathbb{R}},$$
$$W_2 := \langle \bigwedge_{s\in S} s^* \in C^{n+1}(\mathfrak{g}_{D,D'}) \mid Z \notin S \subset U,\ \#S = n+1\rangle_{\mathbb{R}},$$

then we have $C^{n+1}(\mathfrak{g}_{D,D'}) = (W_1 + W_2) \oplus V$. By an easy calculation, we obtain $d_{n+1}(W_1) = d_{n+1}(W_2) = \{0\}$ and so $\mathrm{Im}(d_{n+1}) = \mathrm{Im}(d_{n+1}|_V)$ holds. ■

3.2. Diagonal type

We devote this subsection to the proof of Theorem 1.3. We use the following:

Notations 3.6. For $n \in \mathbb{N}$ and $a_1, \cdots, a_n, b_1, \cdots, b_n \in \mathbb{R}^\times$, we write as follows:

$$D := \mathrm{diag}(a_i)_{i=1,\cdots,n},\ D' := \mathrm{diag}(b_i)_{i=1,\cdots,n},$$
$$T := W^* \wedge X_1^* \wedge \cdots \wedge X_n^* \wedge Z^* \in C^{n+2}(\mathfrak{g},\mathfrak{h}) \subset C^{n+2}(\mathfrak{g}),$$
$$[n] := \{1, \cdots, n\},$$
$$\mathcal{I}_k := \{I \subset [n] \mid n - \#I \equiv k \mod 2\},\ k \in \{0,1\},$$
$$\mathcal{E} := \{\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \in \{\pm 1\}^n \mid \varepsilon_1 = 1\}.$$

For a subset $I \subset [n], \varepsilon \in \mathcal{E}$, we set:

$$T_I := \Big(\bigwedge_{i\in I} X_i^*\Big) \wedge \Big(\bigwedge_{j\in I^C} Y_j^*\Big) \wedge Z^* \in V_{p,q} \quad (p = \#I,\ q = n-p),$$
$$V_k := \langle T_I \rangle_{\mathbb{R}} I \in \mathcal{I}_k \subset V, \quad \varepsilon_I := \prod_{i\in I} \varepsilon_i, \quad \varepsilon_\emptyset := 1,$$

where $\bigwedge_{i\in I} X_i := X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_s}$ for $I = \{i_1 < i_2 < \cdots, i_s\}$ and $I^C := [n] \setminus I$. We use the symbol $\cdot_{\mathbb{C}}$ for the complexification.

We present some fundamental properties.

Property 3.7. *The following conditions hold.*

(1) $T = T_{[n]} \in V_0$.

(2) $\{T_I\}_{I \in \mathcal{I}_k}$ *is a basis of* V_k $(k \in \{0,1\})$.

(3) $\delta(T_I) = \sum_{i \in I} b_i T_{I \setminus \{i\}} + \sum_{j \in I^C} a_j T_{I \cup \{j\}}$ $(I \subset [n])$.

(4) $\frac{1}{2^{n-1}} \sum_{I \in \mathcal{I}_k} \varepsilon_I \cdot \varepsilon'_I = \delta_{\varepsilon\varepsilon'}$ $(k \in \{0,1\},\ \varepsilon, \varepsilon' \in \mathcal{E})$.

Here $\delta_{\varepsilon\varepsilon'}$ *is Kronecker's delta.*

Proof. The properties (1), (2) and (3) directly follow from the definitions of T_I and δ (see Definition 3.3). Next we show the property (4). Since the case $n = 1$ is clear, we assume $n \geq 2$. We put $I' := \{i \in [n] \mid \varepsilon_i \neq \varepsilon'_i\}$, then we have:

$$\varepsilon_I \cdot \varepsilon'_I = \begin{cases} 1 & (\#(I \cap I') \equiv 0 \mod 2) \\ -1 & (\#(I \cap I') \equiv 1 \mod 2) \end{cases}.$$

Since the case $\varepsilon = \varepsilon'$ is easy, we omit the proof. We consider the case $\varepsilon \neq \varepsilon'$ and put $\ell := \#I'$. Note that $0 < \ell < n$. We have:

$$\#\{I \in \mathcal{I}_k \mid \#(I \cap I') \equiv 0 \mod 2\} = \left(\binom{\ell}{1} + \binom{\ell}{3} + \cdots + \binom{\ell}{m}\right) 2^{n-\ell-1} = 2^{n-2},$$

where m is the maximum odd number less than or equal to ℓ. On the other hand, we have $\#\{I \in \mathcal{I}_k \mid \#(I \cap I') \equiv 1 \mod 2\} = 2^{n-2}$ in the same way. ■

Next we prove Theorem 1.3, which easily follows from Lemma 3.8.

Lemma 3.8. *For diagonal matrices* $D := \mathrm{diag}(a_i)$ *and* $D' := \mathrm{diag}(b_i)$, *let* $\pi : G_{D,D'} \to G_{D,D'}/H$ *be the canonical surjection and we denote by* π^* *the induced homomorphism* $H^{n+2}(\mathfrak{g}_{D,D'}, \mathfrak{h}; \mathbb{R}) \to H^{n+2}(\mathfrak{g}_{D,D'}; \mathbb{R})$. *Then the following conditions are equivalent:*

(a) $T \notin \mathrm{Im}(d_{n+1}|_V)$,

(b) $T \notin \mathrm{Im}(d_{n+1}|_{V_1})$,

(c) $(\delta|_{V_1})_{\mathbb{C}}$ *is not invertible,*

(d) $\exists \varepsilon \in \{\pm 1\}^n$ *s.t.* $\sum_{i=1}^n \varepsilon_i \sqrt{a_i b_i} = 0$,

where $\sqrt{a_i b_i} \in \mathbb{R} \cup \sqrt{-1}\mathbb{R}$.

The proof of Lemma 3.8 uses the following lemma.

Lemma 3.9. *There exist basis* $\{T'_{1,\varepsilon}\}_{\varepsilon \in \mathcal{E}}$ *of* $(V_1)_{\mathbb{C}}$ *and* $\{T'_{0,\varepsilon}\}_{\varepsilon \in \mathcal{E}}$ *of* $(V_0)_{\mathbb{C}}$ *satisfying the following two conditions:*

(1) $\iota^{-1}(T) = \sum_{\varepsilon \in \mathcal{E}} \ell_\varepsilon T'_{0,\varepsilon}$, *where* $\ell_\varepsilon := \dfrac{1}{2^{n-1} \prod_{i \in [n]} \sqrt{a_i}} \varepsilon_{[n]} \in \mathbb{C}^\times$,

(2) $\delta_{\mathbb{C}}(T'_{0,\varepsilon}) = k_\varepsilon T'_{0,\varepsilon}$ *for each* $\varepsilon \in \mathcal{E}$, *where* $k_\varepsilon := \sum_{i \in [n]} \varepsilon_i \sqrt{a_i} \sqrt{b_i}$.

Proof. For $I \subset [n]$, $k = 0, 1$ and $\varepsilon \in \mathcal{E}$, we set T'_I and $T'_{k,\varepsilon}$ as follows:

$$T'_I := \Big(\bigwedge_{i\in I}\sqrt{a_i}X_i^*\Big) \wedge \Big(\bigwedge_{j\in I^C}\sqrt{b_j}Y_j^*\Big) \wedge Z^* \in \mathbb{C}^\times T_I,$$
$$T'_{k,\varepsilon} := \sum_{I\in\mathcal{I}_k} \varepsilon_I T'_I \in (V_k)_{\mathbb{C}}.$$

By Property 3.7 (2), $\{T'_I\}_{I\in\mathcal{I}_k}$ is a basis of $(V_k)_{\mathbb{C}}$. Therefore, $\{T'_{k,\varepsilon}\}_{\varepsilon\in\mathcal{E}}$ is also a basis of $(V_k)_{\mathbb{C}}$ by Property 3.7 (4). By the definition of $T'_{k,\varepsilon}$ and Property 3.7 (4), we have:

$$T'_I = \frac{1}{2^{n-1}}\sum_{\varepsilon\in\mathcal{E}} \varepsilon_I T'_{k,\varepsilon}.$$

By putting $I = [n]$, we obtain:

$$T = T_{[n]} = \frac{1}{\prod_{i\in[n]}\sqrt{a_i}} T'_{[n]} = \frac{1}{2^{n-1}\prod_{i\in[n]}\sqrt{a_i}} \sum_{\varepsilon\in\mathcal{E}} \varepsilon_{[n]} T'_{0,\varepsilon}.$$

Hence we have shown statement (1). Next we show statement (2).

To prove the representation matrix with respect to the basis is diagonal, it is enough to show the following:

Claim. $\delta_{\mathbb{C}}(T'_{1,\varepsilon}) = k_\varepsilon T'_{0,\varepsilon}$ for $\varepsilon \in \mathcal{E}$.

By Property 3.7 (3), for $I \subset [n]$, we get:

$$\begin{aligned}
\delta_{\mathbb{C}}(T'_I) &= \delta_{\mathbb{C}}\Big(\prod_{i\in I}\sqrt{a_i}\prod_{j\in I^C}\sqrt{b_j}T_I\Big) = \Big(\prod_{i\in I}\sqrt{a_i}\prod_{j\in I^C}\sqrt{b_j}\Big)\Big(\sum_{i\in I} b_i T_{I\setminus\{i\}} + \sum_{j\in I^C} a_j T_{I\cup\{j\}}\Big)\\
&= \sum_{i\in I}\sqrt{a_i}\sqrt{b_i}\Big(\prod_{k\in I\setminus\{i\}}\sqrt{a_k}\Big)\Big(\prod_{k\in I^C\cup\{i\}}\sqrt{b_k}\Big)T_{I\setminus\{i\}}\\
&\quad + \sum_{j\in I^C}\sqrt{a_j}\sqrt{b_j}\Big(\prod_{k\in I\cup\{j\}}\sqrt{a_k}\Big)\Big(\prod_{k\in I^C\setminus\{j\}}\sqrt{b_k}\Big)T_{I\cup\{j\}}\\
&= \sum_{i\in I}\sqrt{a_i}\sqrt{b_i}T'_{I\setminus\{i\}} + \sum_{j\in I^C}\sqrt{a_j}\sqrt{b_j}T'_{I\cup\{j\}}.
\end{aligned}$$

Using the above result, we have:

$$\begin{aligned}
\delta_{\mathbb{C}}(T'_{1,\varepsilon}) &= \sum_{I_1\in\mathcal{I}_1} \varepsilon_{I_1}\delta_{\mathbb{C}}(T'_{I_1})\\
&= \sum_{I_1\in\mathcal{I}_1} \varepsilon_{I_1}\Big(\sum_{i\in I_1}\sqrt{a_i}\sqrt{b_i}T'_{I_1\setminus\{i\}} + \sum_{j\in I_1^C}\sqrt{a_j}\sqrt{b_j}T'_{I_1\cup\{j\}}\Big)\\
&= \sum_{I_0\in\mathcal{I}_0}\Big(\sum_{i\in I_0}\varepsilon_{I_0\setminus\{i\}}\sqrt{a_i}\sqrt{b_i} + \sum_{j\in I_0^C}\varepsilon_{I_0\cup\{j\}}\sqrt{a_j}\sqrt{b_j}\Big)T'_{I_0}\\
&= \sum_{I_0\in\mathcal{I}_0}\Big(\sum_{i\in I_0}\varepsilon_i\sqrt{a_i}\sqrt{b_i} + \sum_{j\in I_0^C}\varepsilon_j\sqrt{a_j}\sqrt{b_j}\Big)\varepsilon_{I_0}T'_{I_0}\\
&= \sum_{I_0\in\mathcal{I}_0}\Big(\sum_{i\in[n]}\varepsilon_i\sqrt{a_i}\sqrt{b_i}\Big)\varepsilon_{I_0}T'_{I_0} = \Big(\sum_{i\in[n]}\varepsilon_i\sqrt{a_i}\sqrt{b_i}\Big)T'_{0,\varepsilon} = k_\varepsilon T'_{0,\varepsilon}.
\end{aligned}$$

■

Proof of Lemma 3.8. First, the equivalence (c)$\Leftrightarrow$(d) follows from Lemma 3.9 (2). Next, we show the equivalence (a)$\Leftrightarrow$(b). Recall that $d_{n+1} = \iota\delta$ and $V = V_1 \oplus V_0$, so it is enough to show the following three claims:

$$(1)\ \ \delta(V_0) \subset V_1, \qquad (2)\ \ \delta(V_1) \subset V_0, \qquad (3)\ \ T \in \iota(V_0).$$

Here, (1) and (2) come from the definition of δ and (3) from the definition of T (see Notation 3.6). Finally, we show the equivalence (b)$\Leftrightarrow$(c). Put $\{T_{1,\varepsilon}\}$ and $\{T_{0,\varepsilon}\}$ the basis of $(V_1)_{\mathbb{C}}$ and $(V_0)_{\mathbb{C}}$ and $P := \mathrm{diag}(k_\varepsilon) \in M(2^{n-1}, \mathbb{C})$ the representation matrix of $(\delta|_{V_1})_{\mathbb{C}}$ as in Lemma 3.9 (1). We have:

$$T \notin \mathrm{Im}(d_{n+1}|_{V_1}) \iff T \notin \mathrm{Im}(d_{n+1}|_{V_1})_{\mathbb{C}} \iff \mathrm{rank}\, P \neq \mathrm{rank}(P|\mathbf{T}),$$

where $(P|\mathbf{T})$ is the enlarged coefficient matrix and $\mathbf{T}$ denotes vector representation of $\iota^{-1}(T)$ with respect to the basis $\{T'_{0,\varepsilon}\}$. Since all components of $\mathbf{T}$ are non-zero by Lemma 3.9 (1) and P is diagonal, the above rank condition is equivalent to the condition $\det P = 0$. ■

As an application of Theorem 1.3, we give an alternative proof of the following result, originally proved by Kath and Olbrich in [8].

Fact 3.10. [8, Corollary 7.12] *Take any $p, q \in \mathbb{N}$ and $\lambda \in (\mathbb{R}_{>0})^p, \mu \in (\mathbb{R}_{>0})^q$. Put $n = p+q$ and $G' := (\mathbb{Z}/2\mathbb{Z} \times K) \ltimes G_{D,D'}$ for $D = \mathrm{diag}(\lambda, \mu)$, $D' = \mathrm{diag}(\lambda, -\mu)$ and $H' = (\mathbb{Z}_2 \times K) \ltimes H$. Here,*

$$K := \{\phi \in O(\mathbb{R}^{2n}) \mid \phi \textit{ preserves } \langle e_1, \cdots, e_n\rangle_{\mathbb{R}} \textit{ and } \langle e_{n+1}, \cdots, e_{2n}\rangle_{\mathbb{R}},\ W\phi = \phi W\},$$

and K naturally acts on $\langle X_1, \cdots, X_n, Y_1, \cdots, Y_n\rangle_{\mathbb{R}} \subset H_n$ and $1 \in \mathbb{Z}/2\mathbb{Z}$ acts on $\langle X_1, \cdots, X_n, Y_1, \cdots, Y_n\rangle_{\mathbb{R}} \subset H_n$ as $-\mathrm{id}_{\langle X_1 \cdots, X_n\rangle_{\mathbb{R}}} \oplus \mathrm{id}_{\langle Y_1, \cdots, Y_n\rangle_{\mathbb{R}}}$.
If G'/H' admits compact Clifford-Klein forms, there are choices of signs such that:

$$\sum_{i=1}^{p} \pm\lambda_i = 0 \quad \textit{and} \quad \sum_{j=1}^{q} \pm\mu_j = 0.$$

Proof. The maximal compact subgroup of H' is $\mathbb{Z}_2 \ltimes K$. Then by Theorem 2.2, if the natural map $\pi'^* : H^{n+2}(\mathfrak{g}', \mathfrak{h}'; \mathbb{R}) \to H^{n+2}(\mathfrak{g}', \mathfrak{k}; \mathbb{R})$ is not injective, G'/H' does not admit compact Clifford-Klein form, where $\mathfrak{g}' := \mathrm{Lie}(G')$, $\mathfrak{h}' := \mathrm{Lie}(H)$ and $\mathfrak{k} := \mathrm{Lie}(K)$. It is equivalent to the injectivity of

$$\pi^* : H^{n+2}(\mathfrak{g}_{D,D'}, \mathfrak{h}; \mathbb{R}) \to H^{n+2}(\mathfrak{g}_{D,D'}; \mathbb{R})$$

and the Fact follows from Theorem 1.3. ■

3.3. On the space with signature (2,2)

In this subsection, we give an easier proof to the following theorem ([17, Theorem 1.2]) by a cohomological approach.

Theorem 3.11. [17, Theorem 1.2] *Let G/H be a reducible and indecomposable 1-connected pseudo-Riemannian symmetric space with signature $(2,2)$ and assume that its transvection group G is solvable. Then G/H admits compact Clifford-Klein forms if and only if it is isometric to one of the two symmetric spaces $G_{\pm I_{1,1}, I_{1,1}}/H$ (see Definition 1.2).*

First of all, we give a list of the spaces which satisfy the above conditions.

Fact 3.12. ([7, Theorem 7.1], see also [17, Fact 1.4]) *Let $(G/H, g)$ be a 1-connected four-dimensional reducible and indecomposable pseudo-Riemannian solvable symmetric spaces with signature $(2,2)$, and assume that its transvection group G is solvable. Then the pseudo-Riemannian symmetric space $(G/H, g)$ is isometric to one of the following:*

(I) *Nilpotent symmetric spaces $(G_{nil}/H, \sigma, g_{\pm})$ (see [17, Definition 4.15]),*

(II) *Solvable symmetric spaces $(G_{D,D'}/H, \sigma, g)$ (see Definition 1.2), where:*

(a) $(D, D') = (\pm\mathrm{diag}(1,\nu), \mathrm{diag}(1,-\nu))$ $(\nu > 0)$,
$(D, D') = (\pm\mathrm{diag}(1,-\nu), \mathrm{diag}(1,-\nu))$ $(\nu > 0,\ \nu \neq 1)$,

(b) $(D, D') = (Q_\nu, Q_{-\nu})$ $(\nu > 0)$, *where* $Q_\nu := \begin{pmatrix} \nu & 1 \\ 1 & -\nu \end{pmatrix} \in M(2,\mathbb{R})$,

(c) $(D, D') = \left(\begin{pmatrix} \pm 1 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & \pm 1 \end{pmatrix} \right), \left(\begin{pmatrix} \pm 1 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & \mp 1 \end{pmatrix} \right)$,

(d) $(D, D') = (\pm I_{1,1}, I_{1,1})$.

In [17], only the spaces which correspond to the case (II)(d) admit compact Clifford-Klein forms. It is easy to check that G_{nil}/H (the case in list (I)) does not admit compact Clifford-Klein forms (see [17, Proposition 6.1]). In this subsection, we prove the following lemma.

Lemma 3.13. *For matrices D, D' in Fact 3.12(II), the space $G_{D,D'}/H$ admits no compact Clifford-Klein forms except for the case II(d).*

This lemma follows from the next:

Proposition 3.14. *Let D and D' be invertible symmetric matrices of dimension 2. Then the following equivalence holds:*

$$\pi^* \text{ is injective} \iff \exists\, k \in \mathbb{R}^\times \text{ such that } D' = kD^{-1}.$$

Proof. By Proposition 3.5, it is enough to show that $\iota^{-1}(T) \notin \mathrm{Im}(\delta)$ holds if and only if $k \in \mathbb{R}^\times$ s.t. $D' = kD^{-1}$. Set $D = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, $D' = \begin{pmatrix} a' & b' \\ b' & d' \end{pmatrix}$. Then the matrix representation of δ is as follows, where we choose the basis

$$(X_1^* \wedge Y_1^* \wedge Z^*,\ X_1^* \wedge Y_2^* \wedge Z^*,\ X_2^* \wedge Y_1^* \wedge Z^*,\ X_2^* \wedge Y_2^* \wedge Z^*) \text{ of } V_{1,1}$$

and the basis $(X_1^* \wedge X_2^* \wedge Z^*,\ Y_1^* \wedge Y_2^* \wedge Z^*)$ of $V_{2,0} \oplus V_{0,2}$,

where
$$\delta = \begin{pmatrix} b' & d' & -a' & -b' \\ -b & a & -d & b \end{pmatrix}.$$

Since $\iota^{-1}(T) = X_1^* \wedge X_2^* \wedge Z^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in V_{2,0}$, the condition $\iota^{-1}(T) \notin \mathrm{Im}(\delta)$ is equivalent to $(b', d', -a', -b') = \ell(-b, a, -d, b)$ for some $\ell \in \mathbb{R}$.

Therefore, we have $D' = \ell \begin{pmatrix} d & -b \\ -b & a \end{pmatrix} = \ell(\det D) D^{-1}$. Put $k := \ell(\det D)$, then we get $D' = kD^{-1}$. ■

4. On the existence of compact Clifford-Klein form for even dimensional indecomposable spaces

We devote this section to the proof of Theorem 1.7.

We use the following proposition which gives a necessary and sufficient condition for the existence of compact Clifford-Klein forms for homogeneous spaces of the form $G_{D,D'}/H$.

Proposition 4.1. [17, Proposition 5.26] *For symmetric and invertible matrices $D, D' \in M(n, \mathbb{R})$, the following conditions are equivalent:*

(a) *The symmetric space $G_{D,D'}/H$ admits compact Clifford-Klein forms.*

(b) *There exists $C \in M(n, \mathbb{R})$ satisfying the following conditions:*

  (i) *The matrix $A_t + B_t C$ is invertible for all $t \in \mathbb{R}$, where $A_t, B_t \in M(n, \mathbb{R})$ is determined by $\begin{pmatrix} A_t & B_t \\ * & * \end{pmatrix} := \exp t \begin{pmatrix} & D' \\ D & \end{pmatrix}$.*

  (ii) *The subgroup L_C has an $\mathcal{I}_\ell$-invariant lattice for some $\ell \in G_{D,D'} - H_n$, where $\mathcal{I}_\ell$ is the inner automorphism with respect to $\ell \in G_{D,D'}$.*

Proof of Theorem 1.7. Let n be a positive integer and put $D = -I_{p,q}$ and $D' = I_{p,q}$, where p and q are integers satisfying $p + q = 2n$. It is enough to find $C \in M(2n, \mathbb{R})$ and $\ell \in G_{D,D'} - H_n$ satisfying the conditions of Proposition 4.2.

If p is even, put
$$C := J_{2n} \in M(2n, \mathbb{R}),$$
otherwise, put
$$C := \begin{pmatrix} J_{p-1} & & \\ & J' & \\ & & J_{q-1} \end{pmatrix} \in M(2n, \mathbb{R}).$$

Here, $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $J' := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $J_{2m} := \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix}$.

The condition (b)(i) follows from:

$$\det(A_t + B_t C) = \det(\cos t I_{2n} - \sin t I_{p,q} C) = 1.$$

Finally we check the condition (b)(ii). Put $\ell := (2\pi, 0) \in G_{D,D'} - H_n$.

Since $\mathcal{I}_\ell = \mathrm{id}$, $L_C \simeq \begin{cases} H_n & (p \equiv 0 \mod 2) \\ \mathbb{R} \times H_{n-1} & (p \equiv 1 \mod 2) \end{cases}$ has an $\mathcal{I}_\ell$-invariant lattice.

Therefore $G_{D,D'}/H$ admits compact Clifford-Klein forms. ■

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