

The Earliest Diamond of Finite Type in Nottingham Algebras

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Abstract. We prove several structural results on *Nottingham algebras*, a class of infinite-dimensional, modular, graded Lie algebras, which includes the graded Lie algebra associated to the Nottingham group with respect to its lower central series. Homogeneous components of a Nottingham algebra have dimension one or two, and in the latter case they are called *diamonds*. The first diamond occurs in degree 1, and the second occurs in degree q , a power of the characteristic. Each diamond past the second is assigned a *type*, which either belongs to the underlying field or is ∞ .

Nottingham algebras with a variety of diamond patterns are known. In particular, some have diamonds of both finite and infinite type. We prove that each of those known examples is uniquely determined by a certain finite-dimensional quotient. Finally, we determine how many diamonds of type ∞ may precede the earliest diamond of finite type in an arbitrary Nottingham algebra.

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1. Introduction

A *thin* Lie algebra is a graded Lie algebra $L = \bigoplus_{i=1}^{\infty} L_i$ with $\dim L_1 = 2$ and satisfying the following *covering property*: for each i , each nonzero $z \in L_i$ one has $[zL_1] = L_{i+1}$. (Note that we write Lie products without a comma.) This implies at once that homogeneous components of a thin Lie algebra are at most two-dimensional. Those components of dimension two are called *diamonds*, hence L_1 is a diamond, and if there are no other diamonds then L is a *graded Lie algebra of maximal class* [12, 14]. It is convenient, however, to explicitly exclude graded Lie algebras of maximal class from the definition of thin Lie algebras. Thus, a thin Lie algebra must have at least one further diamond besides L_1 (which we may call the *first* diamond of L), and we let L_k be the earliest such diamond (the *second* diamond). For the sake of simplicity in this introduction we assume all thin Lie algebras to have infinite dimension.

The most basic invariant of a thin Lie algebra is the degree k of the second diamond. It is known from [13] and [2] that if the characteristic p is different from 2 then k can only be one of 3, 5, q , or $2q-1$, where q is a power of p in case $p > 0$.

In this paper we focus on thin Lie algebras with second diamond L_q . One remarkable example of such thin Lie algebras arises as the graded Lie algebra associated to the lower central series of the *Nottingham group* over the prime field \mathbb{F}_p , for p

odd [8]. That algebra has its second diamond in degree p , but admits a natural generalization with a power q of p in place of p . For this reason thin Lie algebras with second diamond L_q have been called *Nottingham algebras* in the literature. However, because of exceptional behaviour in small characteristics here we reserve the name *Nottingham algebras* to thin Lie algebras of characteristic $p > 3$, having second diamond L_q with $q > 5$.

A wide variety of Nottingham algebras are known. Several arise from certain cyclic gradings of various simple Lie algebras of Cartan type. In particular, the thin Lie algebra associated with the Nottingham group arises from a cyclic grading of the *Witt algebra*. Further Nottingham algebras, and in fact uncountably many ones, are closely related to graded Lie algebras of maximal class. We refer the reader to Theorem 2.5 and the discussion which follows it for a comprehensive survey.

In an arbitrary Nottingham algebra each diamond past the first can be attached a *type*, which is an element of the underlying field, or ∞ . The second diamond L_q has invariably type -1 , and we assign no type to the first diamond L_1 . The type of a diamond L_m describes the adjoint action of L_1 on L_m , in such a way that knowledge of all degrees in which diamonds occur in L , and their types, determines L up to isomorphism. It is necessary to include *fake diamonds* in such a description. Those are in fact one-dimensional components, as we explain in Section 2, and correspond to types 0 and 1. We refer to diamonds which are not fake, and thus are two-dimensional, as *genuine diamonds*. Furthermore, the difference in degree of any two consecutive diamonds equals $q - 1$, see Section 3.

Various patterns of diamond types occur in Nottingham algebras. One possible pattern has all diamonds of infinite type, with the necessary exception of the first two. Such algebras were constructed in [17], they have second diamond L_q , of type -1 , and diamonds of infinite type in each higher degree congruent to 1 modulo $q - 1$. A complementary uniqueness result is proved in [7], where is showed that there exists a unique Nottingham algebra with second diamond in degree q and all other diamonds having infinite type.

Another possible pattern sees all diamond types follow an arithmetic progression in the underlying field. A special case of that arises for the algebra associated with the Nottingham group, where the sequence of types is constant. More generally, there are Nottingham algebras having bunches of diamonds of infinite type, separated by single occurrences of diamonds of finite type, where the finite types again follow an arithmetic progression. We give a more detailed description in Theorem 2.5, which is an existence result for Nottingham algebras with those diamond patterns and summarizes conclusions of several papers [8, 1, 3, 4]. Existence of the Nottingham algebras with all diamonds, but the second one, of infinite type also follows from [3] as a limit case.

Uniqueness results for the algebras of Theorem 2.5 where all diamonds have finite types were established in [11], in the sense that each of them is uniquely determined by an appropriate finite-dimensional quotient. This was done by providing finite presentations for those algebras, or some central (or second central) extensions of them. One of our goals is extending this to a uniqueness proof for the remaining algebras of Theorem 2.5, namely those with diamonds of both finite and infinite types.

Theorem 1.1. *There is a unique infinite-dimensional Nottingham algebra L with second diamond in degree q , with diamonds of infinite type in all degrees $k(q-1)+1$ for $1 < k \leq p^s$, where $s \geq 1$, and a diamond of finite type $\lambda \neq 0$ in degree $(p^s+1)(q-1)+1$. Furthermore, there is a central extension of L which is finitely presented.*

In Section 6 we prove Theorem 6.1, which is a more precise version of Theorem 1.1. We briefly discuss the exceptional case $\lambda = 0$ in Remark 6.3.

We give a further contribution to the classification project for Nottingham algebras with the following result, which determines the possible degree of the earliest diamond of finite type of a Nottingham algebra after a first bunch of diamonds of infinite type.

Theorem 1.2. *Let L be an infinite-dimensional Nottingham algebra with second diamond in degree q . Suppose L_{2q-1} is a diamond of infinite type, and suppose that the next diamond of finite type, that is, the earliest diamond of finite type past L_q , occurs in degree $a(q-1)+1$, and has type $\mu \in \mathbb{F}$. Then either $a-1$ equals a power of p , or a equals twice a power of p and $\mu = 1$.*

Note that according to the fact that the difference in degree of any two consecutive diamonds equals $q-1$, the degree of the earliest diamond of finite type past L_q must have the form $a(q-1)+1$ for some integer a . Theorem 1.2 will follow at once from a more precise and technical analogue where L is allowed to be finite-dimensional, Theorem 3.2. Of the two alternate conclusions on a stated in Theorem 1.2, the former occurs for the Nottingham algebras of Theorem 1.1, and the latter occurs for some of the Nottingham algebras studied by David Young in his PhD thesis [17]. We discuss some of those briefly at the end of Section 2. We now explain how our results complete a piece of a classification of Nottingham algebras. Thus, consider an infinite-dimensional Nottingham algebra L , with second diamond L_q , and make the additional assumption $p > 5$ (this restriction being inherited from [11]). Suppose that L has at least one further diamond of finite type past L_q , let L_m be the earliest, and assume that L_m is genuine.

Then either $m = 2q-1$, meaning there are no diamonds of infinite type before L_m , or $m = (p^s+1)(q-1)+1$ according to Theorem 1.2, for some $s \geq 1$. In the former case L is isomorphic to one of the algebras described in [8] and [11, Theorems 2.3 and 2.4]. In particular, L has diamonds of finite type in each degree congruent to 1 modulo $q-1$, with the diamond types following an arithmetic progression (including the fake ones). In the latter case L is isomorphic to one of the algebras described in Theorem 1.1. In particular, according to its full description recalled in Theorem 2.5, the algebra L has diamonds in all degrees of the form $t(q-1)+1$. Those diamonds have infinite type except for $t \equiv 1 \pmod{p^s}$, where the diamonds have finite types following an arithmetic progression. Allowing the parameter s to be zero we obtain the following uniform description of the structure of the Nottingham algebras under consideration.

Theorem 1.3. *Let L be an infinite-dimensional Nottingham algebra in characteristic $p > 5$, with second diamond in degree q . Suppose L has at least one genuine diamond of finite type past L_q , let L_m be the earliest, and assume that L_m has type $\lambda \neq 0, 1$. Then $m = (p^s+1)(q-1)+1$ for some $s \geq 0$.*

Furthermore, L has diamonds in all degrees congruent to 1 modulo $q - 1$. The diamond $L_{t(q-1)+1}$ has finite type (possibly fake) if $t \equiv 1 \pmod{p^s}$, and infinite type otherwise. The diamonds of finite type (which include fake diamonds if $\lambda \in \mathbb{F}_p$) follow an arithmetic progression. Furthermore, L itself (if $\lambda \notin \mathbb{F}_p$ and $s = 0$) or a central (if $\lambda \in \mathbb{F}_p$, $\lambda \neq -2$ and $s = 0$ or $\lambda \in \mathbb{F}$ and $s \geq 1$) or second central (if $\lambda = -2$) extension of L is finitely presented.

2. Nottingham algebras

Recall from the Introduction that a *thin* Lie algebra is a graded Lie algebra $L = \bigoplus_{i=1}^{\infty} L_i$, with $\dim L_1 = 2$ and satisfying the *covering property*: for each i , each nonzero $z \in L_i$ satisfies $[zL_1] = L_{i+1}$. In particular, if L has infinite dimension then its centre is trivial. If L has finite dimension then its centre coincides with its highest nonzero component. For convenience in Section 1 we summarized our results for L of infinite dimension, but our Theorem 3.2, which is a stronger and more precise version of Theorem 1.2, provides sharp information on certain finite-dimensional thin Lie algebras L .

As we mentioned in the Introduction, our definition of a Nottingham algebra includes restrictions on p and q , which we will partly justify below.

Definition 2.1. In this paper a *Nottingham algebra* is a thin Lie algebra, over a field of characteristic $p > 3$, with second diamond L_q , where $q > 5$ is a power of p .

In this paper we use the left-normed convention for iterated Lie products, hence $[abc]$ stands for $[[ab]c]$. We also use the shorthand $[ab^i] = [ab \cdots b]$, where b occurs i times.

Let L be a Nottingham algebra with second diamond L_q . We set up some standard notation. There is a nonzero element y of L_1 , unique up to a scalar multiple, such that $[L_2y] = 0$. Extending to a basis x, y of L_1 , this means $[yxy] = 0$. This choice of y implies $[Lyy] = 0$, which means $(\text{ad } y)^2 = 0$, see [16] for a more general result. It is not hard to deduce from this relation that no two consecutive components of L can both be diamonds, see [16] for a proof.

The element y centralizes each homogeneous component from L_2 up to L_{q-2} . That is an nontrivial assertion proved in [9], and relies on the theory of graded Lie algebras of maximal class established in [12, 14]. Consequently, L_i is spanned by $[yx^{i-1}]$ for $2 \leq i < q$. In particular, $v_1 = [yx^{q-2}]$ spans the component L_{q-1} and, in turn, $[v_1x]$ and $[v_1y]$ span the second diamond L_q . (The meaning of the subscript in v_1 will be revealed in Section 3.) It is now easy to see that one may redefine x in such a way that

$$[v_1xx] = 0 = [v_1yy] \quad \text{and} \quad [v_1yx] = -2[v_1xy],$$

see [3, Section 3] for a cleaner excerpt of the original argument in [8]. In the rest of this paper we refer to such x and y as *standard generators* of L . Each of them is only determined up to a scalar multiple, but a different choice will not affect our definitions below, in particular the definition of a diamond's type. Because $[yx^q] = [v_1xx] = 0$, we have $(\text{ad } x)^q = 0$. Indeed, since $(\text{ad } x)^q$ is a derivation of L , its kernel is a subalgebra, but then that must equal L as both generators x and y belong to it.

We recall the definition of *type* of a diamond as introduced in [11]. (Note that diamond types are defined differently for thin Lie algebras with second diamond L_{2q-1} , see [10].) We do not assign a type to the first diamond L_1 . Let then L_m be a diamond past L_1 , that is, a two-dimensional homogeneous component of L with $m > 1$, and assume $L_{m+1} \neq 0$ to avoid trivial cases. Because no two consecutive homogeneous components can be diamonds, L_{m-1} is one-dimensional, and so is L_{m+1} . If w spans L_{m-1} , then L_m is spanned by $[wx]$ and $[wy]$, and L_{m+1} is spanned by $[wxx]$, $[wxy]$, $[wyx]$ and $[wyy]$. The following definition encodes particular relations among these four elements.

Definition 2.2. Let L be a Nottingham algebra, with second diamond L_q and standard generators x and y . Let L_m be a diamond of L , with $m > 1$, and assume $L_{m+1} \neq 0$. Let w be a nonzero element in L_{m-1} .

(a) We say L_m is a diamond of *finite type* μ , where $\mu \in \mathbb{F}$, if

$$[wxx] = 0 = [wyy] \quad \text{and} \quad \mu[wyx] = (1 - \mu)[wxy].$$

(b) We say L_m is a diamond of *infinite type* if

$$[wxx] = 0 = [wyy] \quad \text{and} \quad [wyx] = -[wxy].$$

In particular, this definition applies to the second diamond L_q , which therefore has invariably type $\mu = -1$. It is shown in [7] that every diamond of L satisfies Definition 2.2 for some $\mu \in \mathbb{F} \cup \{\infty\}$.

The values $\mu = 0$ and $\mu = 1$ cannot actually occur in Definition 2.2. If $\mu = 0$ then the relations $[wxx] = 0 = [wxy]$ would imply that the element $[wx]$ is central. Similarly, if $\mu = 1$ then the element $[wy]$ would be central. However, because of the covering property and because $L_{m+1} \neq 0$, no nonzero element of L_m can be central. Hence $[wx] = 0$ if $\mu = 0$, and $[wy] = 0$ if $\mu = 1$, contradicting the two-dimensionality of L_m .

Thus, strictly speaking, diamonds of type 0 or 1 cannot occur, at least if we insist that a diamond should have dimension two, as in Definition 2.2. Nevertheless, it is convenient for a uniform description of the diamonds patterns in Nottingham algebras to allow ourselves to call *diamonds of type 0 or 1* certain one-dimensional homogeneous components L_m , as long as they satisfy the relations of Definition 2.2 with $\mu = 0$ or 1. This leads us to the following definition.

Definition 2.3. Let L be a Nottingham algebra, with second diamond L_q and standard generators x and y . Let L_{m-1} be a one-dimensional component, spanned by w , with $m > 1$, and assume $L_{m+1} \neq 0$.

(a) We say L_m is a diamond of *of type 1* if

$$[wxx] = 0 \quad \text{and} \quad [wy] = 0.$$

(b) We say L_m is a diamond of *of type 0* if

$$[wx] = 0 \quad \text{and} \quad [wyy] = 0.$$

We refer to diamonds of type 0 or 1 as *fake diamonds* to distinguish them from the *genuine diamonds* of dimension two. The necessity of including fake diamonds in a treatment of Nottingham algebras arises from the fact that in various notable instances (as in Theorem 2.5 below) diamonds occur at regular intervals, with types

following an arithmetic progression. When such arithmetic progression of types passes through 0 or 1, fake diamonds arise.

However, this carries an inherent ambiguity: whenever L_m satisfies the definition of a diamond of type 1 (which amounts to $[L_{m-1}y] = 0$ and $[L_mx] = 0$), the next homogeneous component L_{m+1} satisfies the definition of a diamond of type 0 (because $[L_my] = L_{m+1}$ due to the covering property, and then $[L_{m+1}y] = 0$ due to $[Lyy] = 0$). Thus, if w spans L_{m-1} then $[wx]$ spans L_m and $[wxy]$ spans L_{m+1} , and we have the relations

$$[wy] = 0, \quad [wxx] = 0, \quad [wxyy] = 0. \quad (1)$$

The first and second relations are those in part (a) of Definition 2.3, and the second and third relations are those in part (b) if we use $w' = [wx]$ instead of w in it. For various reasons it is inconvenient to simultaneously regard two consecutive components as fake diamonds, and so we adopt the following convention.

Convention 2.4. Whenever we have a diamond L_m of type 1, necessarily followed by a diamond L_{m+1} of type 0, we allow ourselves to call (fake) diamond precisely one of L_m and L_{m+1} , of the appropriate type, and not the other.

In several cases there is a natural choice between calling L_m a diamond of type 1, or L_{m+1} a diamond of type 0, which makes diamonds (including the fake ones) occur at regular distances, with a difference of $q - 1$ in degrees.

We illustrate that through the following existence result, which will be clarified and expanded in commentaries to follow.

Theorem 2.5. *There exist infinite-dimensional Nottingham algebras L with second diamond L_q , where (possibly fake) diamonds occur in each degree congruent to 1 modulo $q - 1$, and have types described by any of the following patterns:*

- (a) *all diamonds of type -1 [8];*
- (b) *all diamonds of finite types following a non-constant arithmetic progression [1, 3];*
- (c) *all diamonds of infinite type except for those in degrees $\equiv q \pmod{p^s(q-1)}$ for some $s > 0$, which have type -1 [3];*
- (d) *all diamonds of infinite type except for those in degrees $\equiv q \pmod{p^s(q-1)}$ for some $s > 0$, which have finite types following a non-constant arithmetic progression [4].*

Nottingham algebras as in case (a) of Theorem 2.5, thus with all diamonds having type -1 , were explicitly constructed in [8], using a certain cyclic grading of Zassenhaus algebras. The special case where $q = p$ is the graded Lie algebra associated with the lower central series of the Nottingham group, thus justifying their name. They were also shown in [8] to be uniquely determined by some finite-dimensional quotient. Here and in certain other cases such ‘uniqueness’ was proved by exhibiting a finite presentation for some central extension of L . (In most cases L is not itself finitely presented.)

Concerning case (b) of Theorem 2.5, Nottingham algebras including fake diamonds were first observed in [11]). More precisely, finite presentations for central extensions (and second-central in one case) of Nottingham algebras were given (with one exception on which we will expand below), where the diamonds occur in all degrees congruent to 1 modulo $q - 1$, and their types follow a non-constant arithmetic progression. If that arithmetic progression passes through 0, that is, if it runs through the prime field, then those diamonds include fake diamonds, of both types 0 and 1. Such Nottingham algebras were explicitly constructed, thus proving their existence, in [1] in case all types belong to the prime field, and in [3] otherwise. Those constructions use certain finite-dimensional simple modular Lie algebras of Cartan type, and certain gradings of them over a finite cyclic group.

Nottingham algebras where the third diamond has infinite type include those of cases (c) and (d). Again, their constructions in [3] and [4] used certain finite-dimensional simple modular Lie algebras of Cartan type, but special tools involving generalized exponentials of derivations had to be developed for producing the required gradings, in [15, 6, 5]. Proving uniqueness of those Nottingham algebras is one of the goals of the present paper, in Theorem 6.1, which implies Theorem 1.1.

In all cases of Theorem 2.5, each homogeneous component which is not a diamond or immediately precedes a diamond is centralized by y . Together with this information, the locations and types of all diamonds, as specified in each case of Theorem 2.5, give a complete description of those Nottingham algebras. Note that each of the Nottingham algebras of Theorem 2.5 has diamonds in each degree congruent to 1 modulo $q - 1$. In particular, the distance between consecutive diamonds in those particular Nottingham algebras is invariably $q - 1$, provided that we assign an appropriate type 0 or 1 to each fake diamond (making use of Convention 2.4). Several such features remain true for arbitrary Nottingham algebras, as we discuss in the next section.

All Nottingham algebras of Theorem 2.5 display a periodic structure which, however, is not a universal characteristic of Nottingham algebras. In fact, in his PhD thesis [17] David Young gave two procedures which allow one to produce two Nottingham algebras $\mathcal{T}_{q,1}(M)$ and $\mathcal{T}_{q,2}(M)$, both with second diamond L_q , starting from any given graded Lie algebra M of maximal class with at most two distinct two-step centralizers (see [12, 14] for the latter). The diamond patterns of $\mathcal{T}_{q,1}(M)$ and $\mathcal{T}_{q,2}(M)$ reflect the pattern of two-step centralizers of M , in two different ways. Because, over any given field of characteristic p , there are uncountably many such algebras M , and most of them are not periodic, corresponding assertions carry over to the class of Nottingham algebras, over a given field of characteristic p and for a fixed power q of p . Both algebras $\mathcal{T}_{q,1}(M)$ and $\mathcal{T}_{q,2}(M)$ have second diamond L_q (of type -1 by definition), and all remaining diamonds are fake or have infinite type. We focus here on $\mathcal{T}_{q,2}(M)$, which is the one relevant to this paper.

Diamonds of infinite type in $L = \mathcal{T}_{q,2}(M)$ occur in sequences, of lengths dictated in a certain way by the structure of M , separated by single occurrences of fake diamonds. The diamonds of infinite type in those sequences occur at distances of $q - 1$ in degree. However, if L_m is a diamond ending any such sequence, then L_{m+q-1} is a (fake) diamond of type 1, and then L_{m+2q-1} begins the next sequence of diamonds of infinite type. Thus, the degree difference between L_{m+q-1} and L_{m+2q-1} equals q , rather than $q - 1$ as in the examples from Theorem 2.5

which we discussed earlier. However, if we make use of the ambiguity which is intrinsic in the definition of fake diamonds, and view L_{m+q} as a diamond of type 0, then that has distance $q - 1$ from the next diamond L_{m+2q-1} . In conclusion, the existence of $\mathcal{T}_{q,2}(M)$ does not contradict a general claim (which is a theorem in [7], see Section 3 below) that the distance between two consecutive diamonds of a Nottingham algebra may always be interpreted to equal $q - 1$, provided that in the presence of fake diamonds we allow ourselves to choose which component we call fake diamond, according to Convention 2.4. We stress that, differently from the algebras considered in Theorem 2.5, a fake diamond in $\mathcal{T}_{q,2}(M)$ needs to be interpreted in two different ways (with the corresponding shift by one in degree), according to which distance we intend to measure (whether from the previous or to the next diamond). This double interpretation of the same fake diamond is required in $\mathcal{T}_{q,1}(M)$ as well, and also in other Nottingham algebras studied in [17].

3. The degree of the second diamond of finite type

It is not at all obvious that a two-dimensional component L_m of an arbitrary Nottingham algebra, with $m > q$, should satisfy the relations of Definition 2.2 for some μ , thus allowing type μ to be assigned to it. In fact, this is one of the main conclusions of [7]. More generally, it is shown there that whenever $[L_{m-2}y] = 0$ and $[L_{m-1}y] \neq 0$ for some $m > q$, either L_m is two-dimensional and can be assigned a type μ according to Definition 2.2, or L_m is a (fake) diamond of type 0 according to Definition 2.3. As we have observed right after that definition, the latter situation admits the alternate interpretation that L_{m-1} is a diamond of type 1.

Another main result of [7] is that any two consecutive diamonds in an arbitrary Nottingham algebra can always be assumed to have a difference of $q - 1$ in degrees, allowing appropriate interpretation in case fake diamonds are involved. We refer the reader to [7] for a deeper discussion of this and further results, and state here only the portions which we need in this paper.

Theorem 3.1. *Let L be a Nottingham algebra with second diamond L_q , and standard generators x and y . Let L_m be a (possibly fake) diamond of L , with $m \geq q$.*

- (a) *If L_m is a genuine diamond then L_{m+q-1} is a diamond.*
- (b) *If L_m is a diamond of type 1, then either L_{m+q-1} or L_{m+q} is a diamond.*
- (c) *If L_m is a diamond of type 0 and, in addition, L_{m-q+1} is a diamond of type different from 0, then L_{m+q-1} is a diamond.*

Furthermore, in each case y centralizes $L_{m+1}, \dots, L_{m+q-3}$, and also L_{m+q-2} if L_{m+q-1} is not a diamond in assertion (b).

Various clarifications are in order. The final assertion of Theorem 3.1, together with assertion (a) and (b), show that a Nottingham algebra y centralizes each homogeneous component which is not a diamond or immediately precedes a diamond. Thus, the locations and types of the diamonds (still making use of Convention 2.4) suffice to completely describe an arbitrary Nottingham algebra. Next, the two conclusions of assertion (b) are not disjoint, the common case being when L_{m+q-1} is a diamond of type 1, which means the same as L_{m+q} being a diamond of type 0. Also, the

hypothesis of assertion (b) can be alternately read as L_{m+1} being a diamond of type 0. Altogether, we see that the difference in degree between consecutive diamonds can always be read as $q - 1$, as long as we suitably interpret any fake diamonds involved. Finally, note that if we read the hypothesis of assertion (c) as L_{m-1} being a diamond of type 1, then assertion (b) would only tell us that either L_{m+q-2} or L_{m+q-1} is a diamond. In fact, inferring the stronger conclusion of assertion (c) requires information on the diamond which precedes L_m . All assertions of Theorem 3.1 were stated in [7] under a blanket hypothesis that Nottingham algebras have infinite dimension. However, as pointed out there, those assertions remain true for a finite-dimensional Nottingham algebra L as long as none of the homogeneous components they mention is the last nonzero homogeneous component of L or the preceding one.

After recalling such general features of Nottingham algebras established in [7], we focus on a major goal of this paper, Theorem 1.2, which determines the possible degrees t in which the (possibly fake) next diamond of finite type L_t may occur past L_q and a sequence of diamonds of infinite type in a Nottingham algebra L . We state here a more precise version of Theorem 1.2. While that more concise result assumes L to have infinite dimension, we relax that assumption in Theorem 3.2, thus turning the goal into proving finite-dimensionality of L unless the degree of that diamond of finite type has the particular form claimed in Theorem 1.2.

According to Theorem 3.1, all distances between consecutive diamonds up to L_t equal $q - 1$, and hence $t \equiv 1 \pmod{q - 1}$.

Theorem 3.2. *Let L be a Nottingham algebra with second diamond L_q and standard generators x and y . Suppose L_{2q-1} is a diamond of infinite type. Suppose that the earliest diamond of finite type past L_q occurs in degree $a(q - 1) + 1$, and has type $\mu \in \mathbb{F}$. Then the following assertions hold:*

- (a) a is even;
- (b) if $a \not\equiv 1 \pmod{p}$ then $L_{(a+1)(q-1)+3} = 0$, unless $a \equiv 0 \pmod{p}$, $\mu = 1$, and $[L_{(a+1)(q-1)}, y] = 0$;
- (c) if $a \equiv 1 \pmod{p}$ but $a - 1$ is not a power of p , then $L_{(a+p^s)(q-1)+2} = 0$, where p^s is the highest power of p which divides $a - 1$;
- (d) if $a \equiv 0 \pmod{p}$, $\mu = 1$, $[L_{(a+1)(q-1)}, y] = 0$, and $a/2$ is not a power of p , then $L_{(a+p^s)(q-1)+3} = 0$, where p^s is the highest power of p which divides a .

If the Nottingham algebra L of Theorem 3.2 has infinite dimension, it follows that the next diamond of finite type past L_q has degree $a(q - 1) + 1$ where either $a - 1$ is a power of p greater than 1 (coming from Assertion (c)), or a equals twice a power of p greater than 1 (coming from Assertion (d)). This is the content of Theorem 1.2. The former case occurs, in particular, for the Nottingham algebras of cases (c) and (d) of Theorem 2.5. The latter case occurs for the family of algebras $\mathcal{T}_{q,2}(M)$ found by David Young in [17], which we briefly introduced at the end of the previous section. In fact, if the graded Lie algebra M of maximal class, with at most two distinct two-step centralizers, from which $L = \mathcal{T}_{q,2}(M)$ is constructed, has *first constituent* of length $2p^s$ (as defined in [14]), then the earliest diamond of finite type past L_q occurs as fake of type 1 in degree $2p^s(q - 1) + 1$.

The proof of Theorem 3.2 is rather long and occupies the whole of Section 5. Before embarking in the proof proper, in the next section we set up some notation and perform some preliminary calculations in an arbitrary Nottingham algebra, which will also be useful in proving our second main result, in Section 6.

4. General calculations near the diamonds

Throughout the paper we make extensive use of the *generalized Jacobi identity*

$$[a[bc^n]] = \sum_{i=0}^n (-1)^i \binom{n}{i} [ac^i bc^{n-i}].$$

Two special instances which often occur are $[a[bc^q]] = [abc^q] - [ac^q b]$ (which amounts to $(\text{ad } c)^q$ being a derivation), and $[a[bc^{q-1}]] = \sum_{i=0}^{q-1} [ac^i bc^{q-1-i}]$, due to the equality $\binom{q-1}{i} \equiv (-1)^i \pmod{p}$. More generally, the binomial coefficients involved in the generalized Jacobi identity can be efficiently evaluated modulo p by means of Lucas' theorem: if q is a power of p and a, b, c, d are non-negative integers with $b, d < q$, then $\binom{aq+b}{cq+d} \equiv \binom{a}{c} \binom{b}{d} \pmod{p}$.

Now consider a Nottingham algebra L with second diamond in degree q . Besides $v_1 = [yx^{q-2}]$ as in the previous section, we set $v_2 = [v_1 xyx^{q-3}]$. Note that, according to Theorem 3.1, y centralizes L_{q+1}, \dots, L_{2q-3} , and L_{2q-1} is a diamond, possibly fake, spanned by $[v_2 x]$ and $[v_2 y]$. In the rest of the paper we will use v_k to denote a certain element spanning the homogeneous component which immediately precedes the $(k + 1)$ st diamond. Although in some previous papers a notation with the subscript increased by one was used, our choice appears slightly more convenient because v_k will have degree $k(q - 1)$.

Suppose L_m is a diamond of arbitrary type $\mu \in \mathbb{F} \cup \{\infty\}$, and assume v_k spans L_{m-1} . Assume L_{m+q-1} is a diamond (which is a consequence of Theorem 3.1 only when L_m is genuine). Then according to Theorem 3.1 the element y centralizes $L_{m+1}, \dots, L_{m+q-3}$. Define the element v_{k+1} in degree $m + q - 2$ as

$$v_{k+1} = \begin{cases} [v_k xyx^{q-3}] & \text{if } \mu \neq 0, \\ [v_k yx^{q-2}] & \text{otherwise.} \end{cases}$$

We start with describing the adjoint action of v_1 on the elements close to this diamond. We use the convention $\infty^{-1} = 0$.

Lemma 4.1. *Suppose $[v_k yx] = (\mu^{-1} - 1)[v_k xy]$ with $\mu \in \mathbb{F}^* \cup \{\infty\}$. Then we have*

$$\begin{aligned} [v_k v_1] &= (\mu^{-1} + 1)v_{k+1}, & [v_k x v_1] &= [v_{k+1} x], & [v_k y v_1] &= (1 - \mu^{-1})[v_{k+1} y], \\ [v_k x y v_1] &= -(2[v_{k+1} yx] + [v_{k+1} xy]), & [v_k x y x v_1] &= -(3[v_{k+1} yx^2] + 2[v_{k+1} xyx]). \end{aligned}$$

Proof. All the claimed equations are easy to prove using the generalized Jacobi identity, so we prove only a couple of them. The first equation is

$$[v_k v_1] = [v_k [yx^{q-2}]] = [v_k yx^{q-2}] + 2[v_k xyx^{q-3}] = (\mu^{-1} + 1)v_{k+1}.$$

The third equation is

$$[v_k y v_1] = [v_k y [yx^{q-2}]] = -[v_k yx^{q-2} y] = (1 - \mu^{-1})[v_{k+1} y]. \quad \blacksquare$$

Remark 4.2. In the case $\mu = 0$ excluded from Lemma 4.1, similar calculations show $[v_k v_1] = v_{k+1}$, $[v_k y v_1] = -[v_{k+1} y]$, $[v_k y x v_1] = -(2[v_{k+1} y x] + [v_{k+1} x y])$ and $[v_k y x^2 v_1] = -(3[v_{k+1} y x^2] + 2[v_{k+1} x y x])$.

As an immediate consequence of Lemma 4.1 we obtain the adjoint action of $[v_1 x]$ on elements close to diamonds.

Corollary 4.3. *Under the hypotheses of Lemma 4.1 we have*

$$[v_k [v_1 x]] = \mu^{-1} [v_{k+1} x], \quad [v_k x [v_1 x]] = 0,$$

$$[v_k y [v_1 x]] = (\mu^{-1} - 1)([v_{k+1} y x] + [v_{k+1} x y]), \quad [v_k x y [v_1 x]] = [v_{k+1} y x^2] + [v_{k+1} x y x].$$

Remark 4.4. In the excluded case $\mu = 0$ we find $[v_k [v_1 x]] = [v_{k+1} x]$, $[v_k y [v_1 x]] = [v_{k+1} y x] + [v_{k+1} x y]$ and $[v_k y x [v_1 x]] = [v_{k+1} y x^2] + [v_{k+1} x y x]$.

Now assume the diamond L_{m+q-1} has infinite type, and set $v_{k+2} = [v_{k+1} x y x^{q-3}]$. According to Theorem 3.1, the element y centralizes $L_{m+q}, \dots, L_{m+2q-4}$, and L_{m+2q-2} is a diamond. We show the adjoint action of v_2 on elements close to the diamond L_m .

Lemma 4.5. *Suppose $[v_k y x] = (\mu^{-1} - 1)[v_k x y]$, with $\mu \in \mathbb{F}^* \cup \{\infty\}$, and $[v_{k+1} y x] = -[v_{k+1} x y]$. Then we have*

$$[v_k v_2] = \mu^{-1} v_{k+2}, \quad [v_k x v_2] = 0 = [v_k y v_2],$$

$$[v_k x y v_2] = [v_{k+2} y x] + [v_{k+2} x y], \quad [v_k x y x v_2] = 2([v_{k+2} y x^2] + [v_{k+2} x y x]).$$

Furthermore, we have

$$[v_k x y x^{q-4} v_1] = [v_{k+1} x y x^{q-4}], \quad [v_k x y x^{q-4} [v_1 x]] = 0,$$

$$[v_k x y x^{q-4} v_2] = -3([v_{k+2} y x^{q-3}] + [v_{k+2} x y x^{q-4}]).$$

Proof. Using Corollary 4.3, the first equation in the first set is

$$[v_k v_2] = [v_k [v_1 x y x^{q-3}]] = [v_k [v_1 x y] x^{q-3}] + 3[v_k x [v_1 x y] x^{q-4}]$$

$$= [v_k [v_1 x] y x^{q-3}] - [v_k y [v_1 x] x^{q-3}] - 3[v_k x y [v_1 x] x^{q-4}] = \mu^{-1} v_{k+2}.$$

The first equation in the second set is

$$[v_k x y x^{q-4} v_1] = [v_k x y x^{q-4} [y x^{q-2}]] = 2[v_{k+1} y x^{q-3}] + 3[v_{k+1} x y x^{q-4}] = [v_{k+1} x y x^{q-4}].$$

Similar calculations yield the remaining equations. ■

Remark 4.6. In the case $\mu = 0$, which is excluded from Lemma 4.5, we find $[v_k v_2] = v_{k+2}$, $[v_k y v_2] = 0$, $[v_k y x v_2] = [v_{k+2} y x] + [v_{k+2} x y]$ and $[v_k y x^2 v_2] = 2([v_{k+2} y x^2] + [v_{k+2} x y x])$.

Now assume the diamond L_m has infinite type, and that the diamond L_{m+q-1} has finite type μ . Set $v_{k+2} = [v_{k+1} x y x^{q-3}]$, unless $\mu = 0$, in which case set $v_{k+2} = [v_{k+1} y x^{q-2}]$. According to Theorem 3.1, the element y centralizes $L_{m+q}, \dots, L_{m+2q-4}$. Assume L_{m+2q-2} is a diamond (which is a consequence of Theorem 3.1 only if $\mu \neq 1$). In case $\mu = 1$ we do assume here that L_{m+2q-2} is a diamond. We describe the adjoint action of v_2 on the elements close to the diamond L_m .

Lemma 4.7. *Suppose $[v_kxy] = -[v_kyx]$ and $[v_{k+1}yx] = (\mu^{-1} - 1)[v_{k+1}xy]$ for some $\mu \in \mathbb{F}^*$. Then we have*

$$\begin{aligned} [v_kv_2] &= -2\mu^{-1}v_{k+2}, & [v_kxv_2] &= -\mu^{-1}[v_{k+2}x], & [v_kyv_2] &= -\mu^{-1}[v_{k+2}y], \\ [v_kxyv_2] &= [v_{k+2}xy] + (2\mu^{-1} + 1)[v_{k+2}yx], \\ [v_kxyxv_2] &= 2[v_{k+2}xyx] + (3\mu^{-1} + 2)[v_{k+2}yx^2]. \end{aligned}$$

Proof. The first equation is

$$\begin{aligned} [v_kv_2] &= [v_k[v_1xy]x^{q-3}] + 3[v_kx[v_1xy]x^{q-4}] = -[v_ky[v_1x]x^{q-3}] - 3[v_kxy[v_1x]x^{q-4}] \\ &= -2([v_{k+1}yx^{q-2}] + v_{k+2}) = -2\mu^{-1}v_{k+2}. \end{aligned}$$

The remaining equations can be proved similarly. ■

Remark 4.8. In the excluded case $\mu = 0$ we find $[v_kv_2] = -2v_{k+2}$, $[v_kxv_2] = -[v_{k+2}x]$, $[v_kyv_2] = -[v_{k+2}y]$, $[v_kxyv_2] = 2[v_{k+2}yx]$ and $[v_kxyxv_2] = 3[v_{k+2}yx^2]$.

5. Proof of Theorem 3.2

Define recursively $v_{k+1} = [v_kxyx^{q-3}]$ for $1 \leq k \leq a - 1$. Further, define the element v_{a+1} in degree $(a + 1)(q - 1)$ as

$$v_{a+1} = \begin{cases} [v_axyx^{q-3}] & \text{if } \mu \neq 0, \\ [v_ayx^{q-2}] & \text{otherwise.} \end{cases}$$

(However, v_{a+1} will be redefined in the proof of Assertion (d).) Note that v_k has degree $k(q - 1)$. Theorem 3.1 can be inductively used to show that $[L_i, y] = 0$ except when i is congruent to 0 or 1 modulo $q - 1$.

5.1. Proving that a is even

According to the first equation proved in Lemma 4.1 the recursive definition of the elements v_k , for $3 \leq k \leq a$, can be replaced with the more compact formula $v_k = [v_2v_1^{k-2}]$. In particular, this allow us, together with the equations in Lemmas 4.1, 4.5, and 4.7, to compute the adjoint action of any v_k on homogeneous elements close to diamonds. A first instance of this type of calculation occurs in the proof of the following result.

Proposition 5.1. *Under the hypotheses of Theorem 3.2, if a is odd then the relation $[v_axy] + [v_ayx] = 0$ holds.*

Proof. We expand both sides of the equation

$$[v_{a-1}[v_1xy]] = -[v_1xyv_{a-1}]$$

using Corollary 4.3 and Lemmas 4.1 and 4.5. The left-hand side is

$$[v_{a-1}[v_1xy]] = [v_{a-1}[v_1x]y] - [v_{a-1}y[v_1x]] = -[v_{a-1}y[v_1x]] = [v_ayx] + [v_axy].$$

Expanding the right-hand side we have

$$\begin{aligned} [v_1xyv_{a-1}] &= [v_1xy[v_2v_1^{a-3}]] = \sum_{i=0}^{a-3} (-1)^i \binom{a-3}{i} [v_1xyv_1^i v_2v_1^{a-3-i}] \\ &= \sum_{i=0}^{a-3} (-1)^i \binom{a-3}{i} [v_{1+i}xyv_2v_1^{a-3-i}] = (-1)^{a-3} [v_{a-2}xyv_2] \\ &= (-1)^{a-3} ([v_a yx] + [v_a xy]). \end{aligned}$$

Having assumed a odd we conclude that $[v_a yx] + [v_a xy] = 0$. ■

The relation $[v_a xy] + [v_a yx] = 0$ proved in Proposition 5.1 contradicts the assumption that $L_{a(q-1)+1}$ has finite type. Hence a must be even, and Assertion (a) of Theorem 3.2 is proved.

5.2. The case where a is not congruent to 1 modulo p .

By hypothesis our algebra L has a diamond of finite type $\mu \in \mathbb{F}$ in degree $a(q-1)+1$, for some even integer $a > 2$, and diamonds of infinite type in all lower degrees congruent to 1 modulo $q - 1$, with the exception of q . We also know, from Theorem 3.1, that y centralizes all homogeneous component from $L_{a(q-1)+2}$ up to $L_{a(q-1)+q-2}$. In this subsection we gather some information on L up to the next diamond after $L_{a(q-1)+1}$ and a little past that. In particular, this will provide a proof of Assertion (b) of Theorem 3.2.

In this subsection we assume $L_{(a+1)(q-1)+3} \neq 0$. Suppose first that $L_{a(q-1)+1}$ is a diamond of type $\mu \neq 1$. Again according to Theorem 3.1, the homogeneous component $L_{(a+1)(q-1)+1}$ is a diamond. We now prove that it has infinite type, which amounts to the vanishing of $[v_{a+1}yx] + [v_{a+1}xy]$. Because of the covering property, it suffices to show that this element belongs to the centre. Being necessarily centralized by y because of $(\text{ad } y)^2 = 0$, we only need to show $[v_{a+1}yx^2] = -[v_{a+1}xyx]$.

Since $[v_2yx] + [v_2xy] = 0$ we have $[v_{a-1}x[v_2yx]] + [v_{a-1}x[v_2xy]] = 0$.

Assume first $\mu \neq 0$. We expand both terms on the left-hand side by means of Lemma 4.7. The first term is

$$\begin{aligned} [v_{a-1}x[v_2yx]] &= [v_{a-1}x[v_2y]x] = [v_{a-1}xv_2yx] - [v_{a-1}xyv_2x] \\ &= -(\mu^{-1} + 1)[v_{a+1}xyx] - (2\mu^{-1} + 1)[v_{a+1}yx^2], \end{aligned} \tag{2}$$

and the second term is

$$\begin{aligned} [v_{a-1}x[v_2xy]] &= [v_{a-1}x[v_2x]y] - [v_{a-1}xy[v_2x]] \\ &= [v_{a-1}xv_2xy] - [v_{a-1}xyv_2x] + [v_{a-1}xyxv_2] \\ &= [v_{a+1}xyx] + (\mu^{-1} + 1)[v_{a+1}yx^2]. \end{aligned}$$

Putting both terms together shows $[v_{a+1}yx^2] = -[v_{a+1}xyx]$. If $\mu = 0$ we still consider the equation $[v_{a-1}x[v_2yx]] + [v_{a-1}x[v_2xy]] = 0$. According to Remark 4.8, the first term on the left-hand side is

$$[v_{a-1}x[v_2yx]] = [v_{a-1}xv_2yx] - [v_{a-1}xyv_2x] = -[v_{a+1}xyx] - 2[v_{a+1}yx^2]. \tag{3}$$

The second term is

$$[v_{a-1}x[v_2xy]] = [v_{a-1}xv_2xy] - [v_{a-1}xyv_2x] + [v_{a-1}xyxv_2] = [v_{a+1}yx^2].$$

We deduce that $[v_{a+1}yx^2] = -[v_{a+1}xyx]$ also in this case. Thus, we have shown that if $\mu \neq 1$ then the diamond $L_{(a+1)(q-1)+1}$ has infinite type.

Now we consider the case $\mu = 1$. The equation $[v_{a-1}x[v_2yx]] = -[v_{a-1}x[v_2xy]]$ yields

$$[v_{a+1}x^2y] = -[v_{a+1}xyx] - [v_{a+1}yx^2].$$

According to Theorem 3.1 either $L_{(a+1)(q-1)+1}$ or $L_{(a+1)(q-1)+2}$ is a diamond. If $[v_{a+1}y] \neq 0$, then we are in the former case, and so we have $[v_{a+1}xx] = 0$. Then the above equation tells us that $[v_{a+1}xy] + [v_{a+1}yx]$ is centralized by x . Hence the diamond $L_{(a+1)(q-1)+1}$ has, again, infinite type.

Assume now $[v_{a+1}y] = 0$, so that $[v_{a+1}x^2y] = -[v_{a+1}xyx]$. Note that this element spans $L_{(a+1)(q-1)+3}$ according to Theorem 3.1. We will show $a \equiv 0 \pmod{p}$. Consider the equation $[v_2xy[v_{a-1}x]] = [v_{a-1}x[v_2yx]]$. We expand both sides by means of Lemmas 4.5 and 4.7. The right-hand side yields

$$[v_{a-1}x[v_2yx]] = [v_{a-1}xv_2yx] - [v_{a-1}xyv_2x] = -2[v_{a+1}xyx].$$

The left-hand side yields $[v_2xy[v_{a-1}x]] = [v_2xyv_{a-1}x] - [v_2xyxv_{a-1}]$. The first term of the difference is

$$\begin{aligned} [v_2xyv_{a-1}x] &= [v_2xy[v_2v_1^{a-3}x]] = (a-3)[v_{a-2}xyv_2v_1x] - [v_{a-1}xyv_2x] \\ &= -(a-2)[v_{a+1}xyx]. \end{aligned}$$

The second term of the difference is

$$\begin{aligned} [v_2xyxv_{a-1}] &= [v_2xyx[v_2v_1^{a-3}]] = (a-3)[v_{a-2}xyxv_2v_1] - [v_{a-1}xyxv_2] \\ &= 2(a-3)[v_{a+1}xyx] - 2[v_{a+1}xyx], \end{aligned}$$

where we have used the identity $[v_axyxv_1] = -2[v_{a+1}xyx] - [v_{a+1}x^2y] = -[v_{a+1}xyx]$. Therefore, we have $[v_2xy[v_{a-1}x]] = (a-2)[v_{a+1}xyx]$ and so $a[v_{a+1}xyx] = 0$, whence $a \equiv 0 \pmod{p}$ as claimed. Summarizing, in case $\mu = 1$ we have found that either $L_{(a+1)(q-1)+1}$ is a diamond of infinite type (same as in the case $\mu \neq 1$), or $L_{(a+1)(q-1)+2}$ is a diamond, in which case $a \equiv 0 \pmod{p}$.

We now proceed to prove Assertion (b) of Theorem 3.2.

Proposition 5.2. *Under the hypotheses of Theorem 3.2, if $a \not\equiv 1 \pmod{p}$ then $L_{(a+1)(q-1)+3} = 0$, unless $a \equiv 0 \pmod{p}$, $\mu = 1$, and $[L_{(a+1)(q-1)}, y] = 0$.*

Proof. We keep with the previous assumption that $L_{(a+1)(q-1)+3} \neq 0$. We have already proved that if $[L_{(a+1)(q-1)}, y] = 0$ then $a \equiv 0 \pmod{p}$ and $\mu = 1$, and so we may assume $L_{(a+1)(q-1)+1}$ to be a diamond of infinite type. Hence $[v_{a+1}yx] = -[v_{a+1}xy]$, and $[v_{a+1}xyx] \neq 0$.

We consider the equation

$$[v_2xy[v_{a-1}x]] = -[v_{a-1}x[v_2xy]] = [v_{a-1}x[v_2yx]]$$

and expand the first and the last Lie products by means of Lemmas 4.5 and 4.7. We need to distinguish two cases according as to whether μ equals zero or not.

According to Equations (2) and (3) we have $[v_{a-1}x[v_2yx]] = \mu^{-1}[v_{a+1}xyx]$ if $\mu \neq 0$, and $[v_{a-1}x[v_2yx]] = [v_{a+1}xyx]$ if $\mu = 0$. As to the other term we have

$$[v_2xy[v_{a-1}x]] = [v_2xyv_{a-1}x] - [v_2xyxv_{a-1}].$$

If $\mu \neq 0$, the former term in this difference is

$$\begin{aligned} [v_2xyv_{a-1}x] &= [v_2xy[v_2v_1^{a-3}]x] = (a - 3)[v_{a-2}xyv_2v_1x] - [v_{a-1}xyv_2x] \\ &= (a - 3)([v_axyv_1x] + [v_axyv_1x]) - (2\mu^{-1} + 1)[v_{a+1}yx^2] - [v_{a+1}xyx] \\ &= (a - 3)\mu^{-1}[v_{a+1}xyx] + 2\mu^{-1}[v_{a+1}xyx] \\ &= (a - 1)\mu^{-1}[v_{a+1}xyx]. \end{aligned}$$

The latter term is

$$\begin{aligned} [v_2xyxv_{a-1}] &= [v_2xyx[v_2v_1^{a-3}]] = (a - 3)[v_{a-2}xyxv_2v_1] - [v_{a-1}xyxv_2] \\ &= 2(a - 3)([v_axy^2v_1] + [v_axyxv_1]) - (3\mu^{-1} + 2)[v_{a+1}yx^2] - 2[v_{a+1}xyx] \\ &= 2(a - 3)\mu^{-1}[v_{a+1}xyx] + 3\mu^{-1}[v_{a+1}xyx] \\ &= (2a - 3)\mu^{-1}[v_{a+1}xyx]. \end{aligned}$$

Hence $[v_2xy[v_{a-1}x]] = (-a + 2)\mu^{-1}[v_{a+1}xyx]$, and we find $(a - 1)\mu^{-1}[v_{a+1}xyx] = 0$, which implies $a \equiv 1 \pmod{p}$ as desired. If $\mu = 0$ we have

$$\begin{aligned} [v_2xyv_{a-1}x] &= (a - 3)[v_{a-2}xyv_2v_1x] - [v_{a-1}xyv_2x] \\ &= (a - 3)[v_axyv_1x] - 2[v_{a+1}yx^2] = (a - 1)[v_{a+1}xyx], \end{aligned}$$

and

$$\begin{aligned} [v_2xyxv_{a-1}] &= (a - 3)[v_{a-2}xyxv_2v_1] - [v_{a-1}xyxv_2] \\ &= 2(a - 3)[v_axy^2v_1] - 3[v_{a+1}yx^2] \\ &= 2(a - 3)[v_{a+1}xyx] + 3[v_{a+1}xyx] = (2a - 3)[v_{a+1}xyx], \end{aligned}$$

Putting terms together we find $(a - 1)[v_{a+1}xyx] = 0$, whence $a \equiv 1 \pmod{p}$ in this case as well. ■

Thus, we have proved Assertion (b) of Theorem 3.2. Note, in particular, under our assumption $L_{(a+1)(q-1)+3} \neq 0$, our argument has also shown that if $a \equiv 1 \pmod{p}$ then $L_{(a+1)(q-1)+1}$ is a diamond of infinite type. This will serve as basis of induction in one argument of the next subsection.

5.3. If $a \equiv 1 \pmod{p}$ then $a - 1$ is a power of p .

Let p^s be the highest power of p dividing $a - 1$. Thus, $a = mp^s + 1$ with m not a multiple of p . In this subsection we prove that L has finite dimension unless $m = 1$. Thus, assume $m > 1$. We start with showing that the diamond $L_{a(q-1)+1}$ of finite type μ is followed by a number of diamonds of infinite type, an inductive argument which we will partly reuse later in Section 6. Then we will use that to show finite-dimensionality of L in Proposition 5.4, which amounts to the more precise Assertion (c) of Theorem 3.2.

Define recursively the elements

$$v_{a+k} = [v_{a+k-1}xyx^{q-3}] \quad \text{for } 1 < k \leq p^s,$$

which extends the corresponding definition for $k = 1$. We prove by induction on k that $[v_{a+k}x]$ and $[v_{a+k}y]$ span a diamond of type infinity, for $1 \leq k \leq p^s$, unless the next homogeneous component equals zero. According to Theorem 3.1 we only need to show

$$[v_{a+k}yx] + [v_{a+k}xy] = 0 \quad \text{for } 1 \leq k \leq p^s.$$

The case $k = 1$ was proved in the previous subsection, thus let $k > 1$ and assume the conclusion to hold up to $k - 1$. Because $a > p^s + 1$ we have $[v_{k+1}yx] = -[v_{k+1}xy]$ for $k \leq p^s$. Expanding each side of $[v_{a-1}[v_{k+1}yx]] = -[v_{a-1}[v_{k+1}xy]]$ we obtain

$$[v_{a-1}v_{k+1}yx] + [v_{a-1}v_{k+1}xy] = 2[v_{a-1}xv_{k+1}y] + 2[v_{a-1}yv_{k+1}x]. \tag{4}$$

We distinguish two cases according to the value of μ . Suppose first $\mu \neq 0$. Because $[v_{a-1}v_{k+1}] = -\mu^{-1}(k + 1)v_{a+k}$, $[v_{a-1}xv_{k+1}] = -\mu^{-1}[v_{a+k}x]$, and $[v_{a-1}yv_{k+1}] = -\mu^{-1}[v_{a+k}y]$, substituting into Equation (4) we get

$$(k - 1)([v_{a+k}yx] + [v_{a+k}xy]) = 0.$$

We find the same conclusion when $\mu = 0$, because in that case a direct calculation shows $[v_{a-1}v_{k+1}] = -(k + 1)v_{a+k}$, $[v_{a-1}xv_{k+1}] = -[v_{a+k}x]$, and $[v_{a-1}yv_{k+1}] = -[v_{a+k}y]$. This proves the desired conclusion $[v_{a+k}yx] + [v_{a+k}xy] = 0$ as long as $k \not\equiv 1 \pmod p$.

If $k \equiv 1 \pmod p$, write $k = hp^t + 1$ where p^t is the highest power of p dividing $k - 1$, whence $h \not\equiv 0 \pmod p$. Because $h \leq p^{s-t} - 1$ we have $k + p^t \leq p^s + 1$. Since $a > p^s + 1$ we have $[v_{k+p^t}yx] = -[v_{k+p^t}xy]$. The equation $[v_{a-p^t}[v_{k+p^t}yx]] = -[v_{a-p^t}[v_{k+p^t}xy]]$ yields

$$[v_{a-p^t}v_{k+p^t}yx] + [v_{a-p^t}v_{k+p^t}xy] = 2[v_{a-p^t}xv_{k+p^t}y] + 2[v_{a-p^t}yv_{k+p^t}x]. \tag{5}$$

Once again we need to distinguish two cases according to the value of μ . Assume first $\mu \neq 0$. We have

$$\begin{aligned} [v_{a-p^t}v_{k+p^t}] &= [v_{a-p^t}[v_2v_1^{k+p^t-2}]] \\ &= \binom{k+p^t-2}{p^t-1} [v_{a-1}v_2v_1^{k-1}] - \binom{k+p^t-2}{p^t} [v_a v_2 v_1^{k-2}] = -\mu^{-1}(2+h)v_{a+k}, \end{aligned}$$

because $\binom{k+p^t-2}{p^t-1} = \binom{hp^t+p^t-1}{p^t-1} \equiv 1 \pmod p$ and $\binom{k+p^t-2}{p^t} = \binom{hp^t+p^t-1}{p^t} \equiv h \pmod p$.

Similarly, we have

$$[v_{a-p^t}xv_{k+p^t}] = \binom{k+p^t-2}{p^t-1} [v_{a-1}xv_2v_1^{k-1}] = -\mu^{-1}[v_{a+k}x],$$

and
$$[v_{a-p^t}yv_{k+p^t}] = \binom{k+p^t-2}{p^t-1} [v_{a-1}yv_2v_1^{k-1}] = -\mu^{-1}[v_{a+k}y].$$

Substituting into Equation (5) we get $h([v_{a+k}yx] + [v_{a+k}xy]) = 0$.

We find the same conclusion when $\mu = 0$ because $[v_{a-p^t}v_{k+p^t}] = -(2+h)v_{a+k}$, $[v_{a-p^t}xv_{k+p^t}] = -[v_{a+k}x]$ and $[v_{a-p^t}yv_{k+p^t}] = -[v_{a+k}y]$.

Thus, we have proved $[v_{a+k}yx] + [v_{a+k}xy] = 0$ for $1 \leq k \leq p^s$. In particular, $[v_{a+p^s}yx] + [v_{a+p^s}xy] = 0$ and $[v_{a+p^s}yx]$ spans $L_{(a+p^s)(q-1)+2}$. In the next result, which amounts to Assertion (c) of Theorem 3.2, we show that $[v_{a+p^s}yx]$ must actually vanish, under our present assumption $m > 1$.

Remark 5.3. In case $m = 1$, that is, when $a - 1$ is a power of p , one may still show that $[v_{a+k}x]$ and $[v_{a+k}y]$ span a diamond of infinite type for $1 \leq k < p^s$, but the above argument would fail because induction breaks down whenever $k + p^t = p^s + 1$. However, in those cases one may use a different calculation, as we will do with a more general scope in the proof of Theorem 6.1.

Proposition 5.4. *Under the hypotheses of Theorem 3.2, if $a \equiv 1 \pmod{p}$ but $a - 1$ is not a power of p , then $L_{(a+p^s)(q-1)+2} = 0$, where p^s is the highest power of p which divides $a - 1$.*

Proof. We have $a = mp^s + 1$ with a even, whence m is odd. Written $m = 2h + 1$, assume $h \neq 0$. We will prove that $[v_{a+p^s}yx] = 0$. Set $b = (h + 1)p^s$ and consider the element $u = [v_bxyx^{(q-3)/2}]$ in degree $((a + p^s)(q - 1) + 2)/2$. We expand

$$\begin{aligned} 0 = [uu] &= [u[v_bxyx^{(q-3)/2}]] = \sum_{i=0}^{(q-3)/2} (-1)^i \binom{(q-3)/2}{i} [ux^i[v_bxy]x^{(q-3)/2-i}] \\ &= \sum_{i=0}^{(q-3)/2} (-1)^i \binom{(q-3)/2}{i} [ux^i[v_bx]yx^{(q-3)/2-i}] \\ &\quad - \sum_{i=0}^{(q-3)/2} (-1)^i \binom{(q-3)/2}{i} [ux^iy[v_bx]x^{(q-3)/2-i}]. \end{aligned}$$

All terms of the former sum in the final difference vanish except, possibly, for $i = (q - 5)/2$ and $i = (q - 3)/2$. All terms of the latter sum vanish except, possibly, for $i = (q - 3)/2$. Consequently, we find

$$\begin{aligned} 0 = \pm[uu] &= \frac{q-3}{2} [v_bxyx^{q-4}v_bxyx] - \frac{q-3}{2} [v_{b+1}v_byx] - [v_{b+1}v_bxy] + \\ &\quad + [v_{b+1}xv_by] + [v_{b+1}yv_bx] + [v_{b+1}xyv_bx]. \end{aligned}$$

We now expand the Lie products in the above equation by means of Lemmas 4.5 and 4.7. We start with

$$[v_{b+1}v_b] = [v_{b+1}[v_2v_1^{b-2}]] = \sum_{i=0}^{b-2} (-1)^i \binom{b-2}{i} [v_{b+1}v_1^i v_2 v_1^{b-2-i}].$$

The Lie product in the sum vanishes with the possible exceptions of when $b + 1 + i = a - 1$, whence $i = b - p^s - 1 = hp^s - 1$, or $b + 1 + i = a$, whence $i = b - p^s = hp^s$. In the former case the binomial coefficient vanishes modulo p because $\binom{b-2}{b-p^s-1} = \binom{hp^s+p^s-2}{(h-1)p^s+p^s-1}$, and in the latter case the binomial coefficient is congruent to 1 modulo p because $\binom{b-2}{b-p^s} = \binom{hp^s+p^s-2}{hp^s}$. Hence $[v_{b+1}v_b] = (-1)^{hp^s} \mu^{-1} v_{a+p^s}$ if $\mu \neq 0$, and $[v_{b+1}v_b] = (-1)^{hp^s} v_{a+p^s}$ otherwise.

Next, we expand $[v_{b+1}xv_b]$ and obtain

$$[v_{b+1}xv_b] = [v_{b+1}x[v_2v_1^{b-2}]] = \sum_{i=0}^{b-2} (-1)^i \binom{b-2}{i} [v_{b+1}xv_1^i v_2 v_1^{b-2-i}].$$

The Lie product in the sum vanishes, except possibly when $i = b - p^s - 1$. However, in that case the binomial coefficient vanishes, hence $[v_{b+1}xv_b] = 0$, irrespectively of the value of $\mu \in \mathbb{F}$. Similarly, we find $[v_{b+1}yv_b] = 0$ and $[v_{b+1}xyv_b] = 0$.

Finally, we expand

$$[v_bxyx^{q-4}v_b] = [v_bxyx^{q-4}[v_2v_1^{b-2}]] = (-1)^{hp^s} [v_{a-1}xyx^{q-4}v_2v_1^{p^s-2}].$$

We obtain $(-1)^{hp^s} 2\mu^{-1}[v_{a+p^s-1}xyx^{q-4}]$ if $\mu \neq 0$, and $(-1)^{hp^s} 2[v_{a+p^s-1}xyx^{q-4}]$ otherwise. Here we have used the equation $[v_{a-1}xyx^{q-4}v_2] = 2\mu^{-1}[v_{a+1}xyx^{q-4}]$ if $\mu \neq 0$, and the equation $[v_{a-1}xyx^{q-4}v_2] = 2[v_{a+1}xyx^{q-4}]$ otherwise. Both can be verified by direct calculation.

Substituting all the Lie products found into the equation $0 = [uu]$ as expanded above, we conclude $[v_{a+p^s}yx] = 0$ as desired. ■

5.4. If $a \equiv 0 \pmod{p}$ then a equals twice a power of p

In this final subsection we prove Assertion (d) of Theorem 3.2. Thus, suppose the next diamond of finite type past L_q is $L_{a(q-1)+1}$ with $a \equiv 0 \pmod{p}$. According to Assertion (b) of Theorem 3.2, the algebra L is then finite-dimensional unless $\mu = 1$ and $[L_{(a+1)(q-1)}, y] = 0$. Here we show that, assuming those conditions as well, one still concludes that L has finite dimension unless a equals twice a power of p . The proof follows a similar pattern as that of Assertion (c), and begins with showing that the diamond $L_{a(q-1)+1}$, here of type $\mu = 1$, is followed by a certain number of diamonds of infinite type. However, the equation $[L_{(a+1)(q-1)}, y] = 0$ forces $L_{(a+1)(q-1)+2}$ to be the first such diamond, rather than $L_{(a+1)(q-1)+1}$ as before. This shift in degree will require adapting notation and calculations used previously.

Thus, assume $a \equiv 0 \pmod{p}$, $\mu = 1$, and $[L_{(a+1)(q-1)}, y] = 0$. Taking into account that a is even, write $a = 2mp^s$ with m not a multiple of p . Hence p^s is the highest power of p dividing a . We will prove $L_{(a+p^s)(q-1)+3} = 0$ when $m > 1$.

We redefine the element v_{a+1} as $v_{a+1} = [v_a xyx^{q-2}]$.

Note that here v_{a+1} spans $L_{(a+1)(q-1)+1}$, rather than $L_{(a+1)(q-1)}$ as previously. According to Theorem 3.1 the component $L_{(a+1)(q-1)+2}$ is a diamond, and in Subsection 5.2 we have actually proved that it has infinite type, hence $[v_{a+1}yx] = -[v_{a+1}xy]$. Define recursively the elements

$$v_{a+k} = [v_{a+k-1}xyx^{q-3}] \quad \text{for } 1 < k \leq p^s.$$

We will prove by induction on k that $[v_{a+k}x]$ and $[v_{a+k}y]$ span a diamond of type infinity, for $1 \leq k \leq p^s$, unless the next homogeneous component equals zero. According to Theorem 3.1 we only need to show

$$[v_{a+k}yx] + [v_{a+k}xy] = 0 \quad \text{for } 1 \leq k \leq p^s. \tag{6}$$

Before that we need to review our results in Section 3 on the adjoint action of v_1 and v_2 on the elements close to the diamond $L_{a(q-1)+1}$, taking into account the updated definition of the element v_{a+1} . As an immediate consequence of Lemmas 4.1 and 4.7, we have $[v_a xv_1] = v_{a+1}$, $[v_a xyv_1] = -[v_{a+1}y]$, $[v_{a-1} xv_2] = -v_{a+1}$, $[v_{a-1} xyv_2] = [v_{a+1}y]$, and $[v_{a-1} xyxv_2] = 2[v_{a+1}yx] = -2[v_{a+1}xy]$.

Lemma 5.5. *Under the above hypotheses we have*

$$\begin{aligned} [v_a xyxv_1] &= [v_{a+1}xy], & [v_a xv_2] &= v_{a+2}, & [v_a xyv_2] &= 0, \\ [v_a xyxv_2] &= [v_{a+2}yx] + [v_{a+2}xy]. \end{aligned}$$

Proof. The second equation is

$$[v_a xv_2] = [v_a x[v_1 xyx^{q-3}]] = [v_a x[v_1 xy]x^{q-3}] = [v_a x[v_1 x]yx^{q-3}] = [v_{a+1}xyx^{q-3}] = v_{a+2},$$

where we have used $[v_a xy[v_1 x]] = [v_a xy[yx^{q-1}]] = [v_{a+1}yx] + [v_{a+1}xy] = 0$ and $[v_a x[v_1 x]] = [v_{a+1}x]$. The remaining equations are similar. ■

We now begin an inductive proof of Equation (6). The case $k = 1$ was known from previous subsections, thus let $k > 1$ and assume the conclusion to hold up to $k - 1$. Note that $[v_k yx] + [v_k xy] = 0$ for every k in the range under consideration. Expanding the equation

$$[v_a x[v_k yx]] + [v_a x[v_k xy]] = 0$$

we obtain $[v_a xv_k yx] - 2[v_a xyv_k x] + [v_a xv_k xy] + [v_a xyxv_k] = 0$. Since $[v_a xv_k] = [v_a x[v_2 v_1^{k-2}]] = [v_a xv_2 v_1^{k-2}] = v_{a+k}$, $[v_a xyv_k] = 0$ and

$$[v_a xyxv_k] = (-1)^k [v_a xyxv_1^{k-2} v_2] = (-1)^k ([v_{a+k} yx] + [v_{a+k} xy])$$

we conclude $(1 + (-1)^k) ([v_{a+k} yx] + [v_{a+k} xy]) = 0$,

which gives the desired conclusion as long as k is even.

The case where k is odd, as we assume from now on, is harder and requires longer arguments. We start with expanding

$$[v_{a-1} x[v_{k+1} yx]] + [v_{a-1} x[v_{k+1} xy]] = 0,$$

thus obtaining $[v_{a-1} xv_{k+1} yx] - 2[v_{a-1} xyv_{k+1} x] + [v_{a-1} xv_{k+1} xy] + [v_{a-1} xyxv_{k+1}] = 0$. We have

$$\begin{aligned} [v_{a-1} xv_{k+1}] &= [v_{a-1} x[v_2 v_1^{k-1}]] = [v_{a-1} xv_2 v_1^{k-1}] - (k-1)[v_{a-1} xv_1 v_2 v_1^{k-2}] \\ &= -v_{a+k} - (k-1)v_{a+k} = -kv_{a+k}, \end{aligned}$$

and $[v_{a-1} xyv_{k+1}] = [v_{a-1} xyv_2 v_1^{k-1}] = [v_{a+k} y]$. Furthermore, taking into account that k is odd, we have

$$\begin{aligned} [v_{a-1} xyxv_{k+1}] &= [v_{a-1} xyxv_2 v_1^{k-1}] + [v_{a-1} xyxv_1^{k-1} v_2] \\ &= 2(2[v_{a+k} yx] + [v_{a+k} xy]) - 2([v_{a+k} yx] + [v_{a+k} xy]) \\ &= 2[v_{a+k} yx]. \end{aligned}$$

Hence we find $k([v_{a+k} yx] + [v_{a+k} xy]) = 0$, which yields the desired conclusion as long as k is not a multiple of p .

If k is a multiple of p (and is odd as throughout), write $k = hp^t$ with h not a multiple of p . Note that $[v_{k+p^t} yx] + [v_{k+p^t} xy] = 0$. In particular, when $k = p^s$ we have $[v_{2p^s} yx] + [v_{2p^s} xy] = 0$, since we are assuming $a > 2p^s$. Expanding $[v_{a-p^t} x[v_{k+p^t} yx]] + [v_{a-p^t} x[v_{k+p^t} xy]] = 0$ we find

$$[v_{a-p^t} xv_{k+p^t} yx] - 2[v_{a-p^t} xyv_{k+p^t} x] + [v_{a-p^t} xv_{k+p^t} xy] + [v_{a-p^t} xyxv_{k+p^t}] = 0.$$

Consider now the individual terms. We have

$$\begin{aligned} [v_{a-p^t} xv_{k+p^t}] &= [v_{a-p^t} x[v_2 v_1^{k+p^t-2}]] \\ &= \sum_{i=0}^{k+p^t-2} (-1)^i \binom{k+p^t-2}{i} [v_{a-p^t} xv_1^i v_2 v_1^{k+p^t-2-i}] \\ &= -h[v_{a-p^t} xv_1^{p^t} v_2 v_1^{k-2}] = -hv_{a+k}, \end{aligned}$$

where we have used $\binom{k+p^t-2}{p^t-1} = \binom{hp^t+p^t-2}{p^t-1} \equiv 0 \pmod{p}$ and $\binom{k+p^t-2}{p^t} \equiv h \pmod{p}$.

Similarly, we have

$$[v_{a-p^t}xyv_{k+p^t}] = -[v_{a-p^t}xyv_1^{p^t-2}v_2v_1^k] = [v_{a+k}y],$$

because $\binom{k+p^t-2}{p^t-2} \equiv 1 \pmod{p}$. Finally, considering that $k + p^t - 2$ is even, we have

$$[v_{a-p^t}xyxv_{k+p^t}] = -[v_{a-p^t}xyxv_1^{p^t-2}v_2v_1^k] + [v_{a-p^t}xyxv_1^{k+p^t-2}v_2] = 2[v_{a+k}yx].$$

We conclude $h([v_{a+k}yx] + [v_{a+k}xy]) = 0$.

Thus, we have proved $[v_{a+k}yx] + [v_{a+k}xy] = 0$ for $2 \leq k \leq p^s$. We are now ready to prove Assertion (d) of Theorem 3.2, which we state again as follows.

Proposition 5.6. *Under the hypotheses of Theorem 3.2, if $a \equiv 0 \pmod{p}$, $\mu = 1$, $[L_{(a+1)(q-1)}, y] = 0$, and $a/2$ is not a power of p , then $L_{(a+p^s)(q-1)+3} = 0$, where p^s is the highest power of p which divides a .*

Proof. Because $[v_{a+p^s}yx] + [v_{a+p^s}xy] = 0$, the desired conclusion $L_{(a+p^s)(q-1)+3} = 0$ now amounts to $[v_{a+p^s}yx] = 0$. That will follow if we show the stronger equation $[v_{a+p^s}y] = 0$. We expand

$$0 = [v_{p^s}x[v_a y]] = [v_{p^s}xv_a y] - [v_{p^s}xyv_a].$$

The former Lie product in this difference is

$$[v_{p^s}xv_a y] = [v_{p^s}x[v_2v_1^{a-2}]y] = -[v_{p^s}xv_1^{a-p^s}v_2v_1^{p^s-2}y] = -[v_{a+p^s}y],$$

where we have used $\binom{a-2}{a-1-p^s} \equiv 0 \pmod{p}$ and $\binom{a-2}{a-p^s} \equiv 1 \pmod{p}$. The latter Lie product in the difference is

$$[v_{p^s}xyv_a] = -(2m - 1)[v_{p^s}xyv_1^{a-2-p^s}v_2v_1^{p^s}] = (2m - 1)[v_{a+p^s}y],$$

where we have used $\binom{a-2}{a-2-p^s} \equiv 2m - 1 \pmod{p}$. We obtain $2m[v_{a+p^s}y] = 0$, which yields the desired conclusion. ■

6. Nottingham algebras with diamonds of finite and infinite type

Let L be an infinite-dimensional Nottingham algebra with second diamond L_q and standard generators x and y . Suppose that L has diamonds of infinite type in all degrees $k(q - 1) + 1$ for $1 < k \leq p^s$, where $s \geq 1$, and a diamond of finite type $\lambda \in \mathbb{F}$, with $\lambda \neq 0$, in degree $(p^s + 1)(q - 1) + 1$.

In this section we prove that L is uniquely determined by these prescriptions. It has diamonds in all degrees of the form $t(q - 1) + 1$. If $t \not\equiv 1 \pmod{p^s}$ the corresponding diamond is of infinite type. If $t \equiv 1 \pmod{p^s}$, say $t = rp^s + 1$, the corresponding diamond has finite type $r\lambda + r - 1$. The diamonds of finite type (including the fake ones if $\lambda \in \mathbb{F}_p$) follow an arithmetic progression. We prove this uniqueness result by showing that if a Lie algebra N is defined by a finite presentation encoding part of the above prescriptions (that is, up to specifying the type of the second diamond of finite type), then the quotient L of N modulo its centre is a Nottingham algebra and has the structure stated above.

Theorem 6.1. *Let $q > 5$ be a power of a prime $p > 3$, let \mathbb{F} a field of characteristic p . Fix $\lambda \in \mathbb{F}^*$, and a positive integer s . Let $N = \bigoplus_{i=1}^{\infty} N_i$ be the Lie algebra over \mathbb{F} on two generators x and y subject to the following relations, and graded assigning degree 1 to x and y , where v_k is defined recursively by $v_1 = [yx^{q-2}]$ and $v_k = [v_{k-1}xyx^{q-3}]$ for $k > 1$:*

$$\begin{aligned} [yx^i y] &= 0 && \text{for } 0 < i < q - 2, \\ [v_1 x x] &= 0 = [v_1 y y], \quad [v_1 y x] = -2[v_1 x y], \\ [v_1 y x^i y] &= 0 && \text{for } 0 < i < q - 2, \\ [v_k y x] + [v_k x y] &= 0 && \text{for } 2 \leq k \leq p^s \text{ with } k \text{ even,} \\ \lambda[v_{p^s+1} y x] &= (1 - \lambda)[v_{p^s+1} x y] \end{aligned}$$

Then $N/Z(N)$ is a Nottingham algebra and has the diamond structure described above in the text.

Note in passing that the presentation in Theorem 6.1 does not include relations $[v_k y x] + [v_k x y] = 0$ for k odd in the range $2 < k \leq p^s$, because those are consequences of the remaining relations, as we will see in its proof.

Naturally, we will prove Theorem 6.1 by induction, deducing homogeneous relations in each degree j from those already proved in lower degree. Hence, in essence, we will deduce relations in degree j from already established properties of N/N^j . In doing so we can make use of certain arguments in [7], as long as they do not assume $N/N^{j+1}Z(N)$ or even larger quotients of $N/Z(N)$ to be thin, which is something we actually need to prove here. Thus, we extract from [7] a result, adapted to our present setting, which follows from those arguments. It is an adapted version of [7, Theorem 10].

Theorem 6.2. *Let $N = \bigoplus_{i=1}^{\infty} N_i$ be a graded Lie algebra, generated by two elements x and y of N_1 .*

Suppose its quotient $M = N/N^{m+2}Z(N)$ is a Nottingham algebra with second diamond M_q and standard generators the images of x and y , where $m \geq 2q - 1$.

Suppose M_m is a (possibly fake) diamond of M , of type μ . If μ equals -1 or 0 , assume in addition that M_{m-q+1} is a diamond with a type λ , and in case $\mu = 0$ assume $\lambda \neq 0$.

Then $[N_i y] \subseteq Z(N)$ for $m < i \leq m + q - 3$.

Proof of Theorem 6.1. Set $L = N/Z(N)$. We will prove, inductively, that L is thin, and has the claimed structure. More precisely, we will show that for all $k \geq 1$ we have

$$[L_{(k-1)(q-1)+1+i} y] = 0 \quad \text{for } 0 < i < q - 2, \tag{7}$$

$$[v_k x x] = 0 = [v_k y y], \tag{8}$$

$$[v_k y x] + [v_k x y] = 0 \quad \text{if } k \not\equiv 1 \pmod{p^s}, \tag{9}$$

$$\mu_r [v_{rp^s+1} y x] = (1 - \mu_r) [v_{rp^s+1} x y] \quad \text{where } \mu_r = r(\lambda + 1) - 1, \tag{10}$$

$$[v_{rp^s+1} y] = 0 \quad \text{if } \mu_r = 1, \tag{11}$$

$$[v_{rp^s+1} x] = 0 \quad \text{if } \mu_r = 0, \tag{12}$$

where v_k denotes any nonzero element of $L_{k(q-1)}$. We will naturally prove them by induction on the degree of those equations. There will be a main induction on r , then an induction on $0 < k \leq p^s$ to prove Equations (8) and (9) on the diamond types, with Equations (10), (11) and (12) concerning the finite types, and for each k an induction on i to prove Equation (7).

The presentation of N tells us explicitly that the quotient N/N^{2q} is thin (note $N_{2q} = [v_2N_1N_1]$), and thus a Nottingham algebra, with second diamond in degree q . We start our proof with showing that $L/L^{(p^s+1)(q-1)+3}$ is thin. (It is actually the case that $N/N^{(p^s+1)(q-1)+3}$ itself is thin, but we do not need that here.) Note $L_{(p^s+1)(q-1)+3} = [v_{p^s+1}L_1L_1L_1]$. Thus, we show by induction that, in L , we have

$$\begin{aligned} [v_{k-1}yx^i y] &= 0 && \text{for } 0 < i < q - 2, \\ [v_k xx] &= 0 = [v_k yy], \end{aligned}$$

for $2 \leq k \leq p^s + 1$. For $k = 2$ the first set of relations is included in the presentation of N . Because

$$0 = [v_1xyx^{q-4}[xyy]] = [v_2yy]$$

and
$$0 = [v_1[v_1xx]] = [v_1[yx^q]] = [v_1yx^q] - [v_1x^qy] = [v_1yx^q] = [v_2xx],$$

we have $[v_2yy] = 0$ and $[v_2xx] = 0$. Hence N/N^{2q+1} is thin, and has a diamond in degree $2q - 1$, of infinite type as imposed by the presentation of N . Now let $2 < k \leq p^s + 1$ and assume the conclusions hold for all smaller values of k . According to Theorem 6.2 with $m = (k - 1)(q - 1) + 1$, in L we have

$$[v_{k-1}yx^i y] = 0 \quad \text{for } 0 < i < q - 2.$$

As before, because

$$0 = [v_{k-1}xyx^{q-4}[xyy]] = [v_kyy]$$

and

$$0 = [v_{k-1}[v_1xx]] = [v_{k-1}[yx^q]] = [v_{k-1}yx^q] - [v_{k-1}x^qy] = [v_{k-1}yx^q] = [v_kxx]$$

we find $[v_kyy] = 0$, and $[v_kxx] = 0$. Thus, $L/L^{k(q-1)+3}$ is thin, and has a diamond in degree $k(q - 1) + 1$. That diamond has infinite type if $k \leq p^s$, as imposed by the presentation of N for even k , and because of Proposition 5.1 for odd k , and type λ if $k = p^s + 1$. This completes our induction, and hence $L/L^{(p^s+1)(q-1)+3}$ is thin, with the diamond structure announced.

In the rest of the proof we will proceed inductively, by successive spans of $p^s(q-1)$ in degree, to prove L is thin and has the diamond structure announced. In particular, letting $a_r = rp^s + 1$ for $r \geq 1$, we will proceed by induction on r , assuming $\mu_r[v_{a_r}yx] = (1 - \mu_r)[v_{a_r}xy]$ where $\mu_r = r(\lambda + 1) - 1$, hence $[v_{a_r}x]$ and $[v_{a_r}y]$ span a diamond of type μ_r , preceded by a string of $p^s - 1$ diamonds of infinite type, in the appropriate degrees. Note that what we have proved so far constitutes a virtual case $r = 0$. To simplify notation we will denote a_r by a (with a warning that its meaning is more general than how it was used in Section 5, the proof of Theorem 3.2).

In our description of L at the beginning of this proof we have called v_k any nonzero element of $L_{k(q-1)}$, hence defined only up to a scalar multiple. Now we refine this

by making a definite choice of scalar. Thus, define the element v_{a+1} in degree $(a + 1)(q - 1)$ as

$$v_{a+1} = \begin{cases} [v_a xyx^{q-3}] & \text{if } \mu_r \neq 0, \\ [v_a yx^{q-2}] & \text{otherwise,} \end{cases}$$

and, recursively, the elements

$$v_{a+k} = [v_{a+k-1} xyx^{q-3}] \quad \text{for } 1 < k \leq p^s.$$

We now prove that every homogeneous component of degree $(a + k)(q - 1) + 1$ in L is a diamond of infinite type, for $1 \leq k < p^s$, and the homogeneous component of degree $(a + p^s)(q - 1) + 1$ is a diamond of (finite) type $\mu_{r+1} = r\lambda + r + \lambda$. Therefore, we prove

$$\begin{aligned} [L_{(a+k-1)(q-1)+1+iy}] &= 0 && \text{for } 1 \leq k \leq p^s \text{ and } 0 < i < q - 2, \\ [v_{a+k}xx] &= 0 = [v_{a+k}yy] && \text{for } 1 \leq k \leq p^s, \\ [v_{a+k}yx] + [v_{a+k}xy] &= 0 && \text{for } 1 \leq k \leq p^s - 1, \\ \mu_{r+1}[v_{a+p^s}yx] &= (1 - \mu_{r+1})[v_{a+p^s}xy], \end{aligned}$$

in L . In our overall induction on r we will prove the induction base at the same time as the induction step, relying on a virtual case $r = 0$ which essentially amounts to what we have proved so far.

We now proceed by nested inductions on k and i . Thus, let $1 \leq k \leq p^s$ and assume the conclusions hold for all smaller values of k . (This means none when $k = 1$, where, however, the required information is available from the previous part of the proof.) According to Theorem 6.2 with $m = (a + k - 1)(q - 1) + 1$, in L we have

$$[L_{(a+k-1)(q-1)+1+iy}] = 0 \quad \text{for } 0 < i < q - 2.$$

Letting u a homogeneous element with $[ux] = v_{a+k}$, because $[uy] = 0$ we obtain

$$0 = [u[xyy]] = [v_{a+k}yy].$$

Unless $k = 1$ and $\mu_r = 1$, we obtain

$$0 = [v_{a+k-1}[yx^q]] = [v_{a+k-1}yx^q] - [v_{a+k-1}x^qy] = [v_{a+k}xx].$$

We will deal with the excluded case below, where $[v_{a+1}xx]$ will turn out to only be central in N .

Now we complete the case $k = 1$ of our induction by proving $[v_{a+1}yx] + [v_{a+1}xy] = 0$ in L . If $\mu_r \neq 1$ we may proceed as in Subsection 5.2. Thus, the equation $[v_{a-1}x[v_2yx]] + [v_{a-1}x[v_2xy]] = 0$ yields $[v_{a+1}yx^2] + [v_{a+1}xyx] = 0$. Consequently, the element $[v_{a+1}yx] + [v_{a+1}xy]$ is central in N and the conclusion follows. If $\mu_r = 1$, then the equation $[v_{a-1}x[v_2yx]] + [v_{a-1}x[v_2xy]] = 0$ yields

$$[v_{a+1}xxy] + [v_{a+1}yx^2] + [v_{a+1}xyx] = 0.$$

Furthermore, we obtain $0 = [v_ax[yx^q]] = [v_{a+1}x^3]$. Consequently, once we prove $[v_{a+1}yx] + [v_{a+1}xy] = 0$ it will also follow that the element $[v_{a+1}x^2]$ is central in N , as we announced above. Let v_{a-p^s} be a nontrivial element in degree $(a - p^s)(q - 1)$,

then v_{a-p^s} is just above a diamond of (finite) type $\mu_{r-1} = \mu_r - (\lambda + 1) = -\lambda$. We expand both terms of the equation

$$\lambda[v_{a-p^s}[v_{p^s+1}yx]] = (1 - \lambda)[v_{a-p^s}[v_{p^s+1}xy]], \tag{13}$$

by means of Lemmas 4.5 and 4.7. The Lie bracket on the left-hand side is

$$\begin{aligned} & [v_{a-p^s}[v_{p^s+1}yx]] \\ &= [v_{a-p^s}v_{p^s+1}yx] - [v_{a-p^s}yv_{p^s+1}x] - [v_{a-p^s}xv_{p^s+1}y] + [v_{a-p^s}xyv_{p^s+1}] \\ &= \lambda^{-1}[v_{a+1}yx] + [v_{a+1}xy] \end{aligned}$$

because $[v_{a-p^s}v_{p^s+1}] = -2v_{a+1}$, $[v_{a-p^s}yv_{p^s+1}] = -(\lambda^{-1} + 1)[v_{a+1}y]$, $[v_{a-p^s}xv_{p^s+1}] = -[v_{a+1}x]$ and $[v_{a-p^s}xyv_{p^s+1}] = [v_{a+1}yx]$. The Lie product at the right-hand side is

$$\begin{aligned} [v_{a-p^s}[v_{p^s+1}xy]] &= [v_{a-p^s}v_{p^s+1}xy] - [v_{a-p^s}xv_{p^s+1}y] \\ &\quad - [v_{a-p^s}yv_{p^s+1}x] + [v_{a-p^s}y xv_{p^s+1}] - [v_{a+1}xy], \end{aligned}$$

where we have used $[v_{a-p^s}yx] = -(\lambda^{-1} + 1)[v_{a-p^s}xy]$ due to this diamond's type. Substituting in Equation (13) we get $[v_{a+1}yx] = -[v_{a+1}xy]$, as desired.

We continue our induction on k and prove $[v_{a+k}yx] + [v_{a+k}xy] = 0$ in L for $1 < k \leq p^s - 1$. Because $[v_{k+1}yx] + [v_{k+1}xy] = 0$, expanding the equation $[v_{a-1}[v_{k+1}yx]] + [v_{a-1}[v_{k+1}xy]] = 0$ as in Subsection 5.3 yields $[v_{a+k}yx] + [v_{a+k}xy] = 0$ when $k \not\equiv 1 \pmod{p}$. The argument used in Subsection 5.3 to cover the case $k \equiv 1 \pmod{p}$ works in the present setting as well, unless $k + p^t = p^s + 1$ for some t with $1 \leq t \leq s - 1$. If $k = p^s - p^t + 1$ then we use the equation

$$[v_{a-1}x[v_{k+1}yx]] + [v_{a-1}x[v_{k+1}xy]] = 0. \tag{14}$$

The former term in the sum is $[v_{a-1}x[v_{k+1}yx]] = [v_{a-1}xv_{k+1}yx] - [v_{a-1}xyv_{k+1}x]$. The latter term is $[v_{a-1}x[v_{k+1}xy]] = [v_{a-1}xv_{k+1}xy] - [v_{a-1}xyv_{k+1}x] + [v_{a-1}xyxv_{k+1}]$. Assume first $\mu_r \neq 0$, and recall $[v_{a-1}xv_{k+1}] = -\mu_r^{-1}[v_{a+k}x]$ from Subsection 5.3. Taking into account that k is odd we find

$$\begin{aligned} [v_{a-1}xyv_{k+1}] &= [v_{a-1}xy[v_2v_1^{k-1}]] = [v_{a-1}xyv_2v_1^{k-1}] + [v_{a-1}xyv_1^{k-1}v_2] \\ &= -2\mu_r^{-1}[v_{a+k-1}xyv_1] - (2\mu_r^{-1} - 1)[v_{a+k-2}xyv_2] \\ &= (2\mu_r^{-1} + 1)[v_{a+k}yx] + [v_{a+k}xy], \end{aligned}$$

and we deduce $[v_{a-1}x[v_{k+1}yx]] = -(2\mu_r^{-1} + 1)[v_{a+k}yx^2] - (\mu_r^{-1} + 1)[v_{a+k}xyx]$. Since

$$\begin{aligned} [v_{a-1}xyxv_{k+1}] &= [v_{a-1}xyx[v_2v_1^{k-1}]] = [v_{a-1}xyxv_2v_1^{k-1}] + [v_{a-1}xyxv_1^{k-1}v_2] \\ &= -3\mu_r^{-1}[v_{a+k-1}xyxv_1] - (3\mu_r^{-1} - 1)[v_{a+k-2}xyxv_2] \\ &= (3\mu_r^{-1} + 2)[v_{a+k}yx^2] + 2[v_{a+k}xyx], \end{aligned}$$

we have $[v_{a-1}x[v_{k+1}xy]] = (\mu_r^{-1} + 1)[v_{a+k}yx^2] + [v_{a+k}xyx]$. We conclude

$$\mu_r^{-1}([v_{a+k}yx^2] + [v_{a+k}xyx]) = 0,$$

and so the element $[v_{a+k}yx] + [v_{a+k}xy]$ is central in N . In the excluded case $\mu_r = 0$, Equation (14) yields $[v_{a+k}yx^2] + [v_{a+k}xyx] = 0$, because $[v_{a-1}xv_{k+1}] = -[v_{a+k}x]$, $[v_{a-1}xyxv_{k+1}] = 2[v_{a+k}yx]$, and $[v_{a-1}xyxv_{k+1}] = 3[v_{a+k}yx^2]$.

Therefore we have proved that each homogeneous component of L of the degree $(a + k)(q - 1) + 1$ is a diamond of infinite type, for $1 \leq k \leq p^s - 1$. To complete the proof we show

$$\mu_{r+1}[v_{a+p^s}yx] = (1 - \mu_{r+1})[v_{a+p^s}xy],$$

in N . To this purpose we expand both sides of the equation

$$\lambda[v_{a-1}[v_{p^s+1}yx]] = (1 - \lambda)[v_{a-1}[v_{p^s+1}xy]], \tag{15}$$

and assume first $\mu_r \neq 0$. When $k = p^s$ the calculations we did in the previous paragraph for k odd yield $[v_{a-1}xv_{p^s+1}] = -\mu_r^{-1}[v_{a+p^s}x]$ and

$$[v_{a-1}xyv_{p^s+1}] = (2\mu_r^{-1} + 1)[v_{a+p^s}yx] + [v_{a+p^s}xy].$$

Recalling the equations $[v_{a-1}v_{p^s+1}] = -\mu_r^{-1}v_{a+p^s}$ and $[v_{a-1}yv_{p^s+1}] = -\mu_r^{-1}[v_{a+p^s}y]$ from Subsection 5.3, we find

$$\lambda\mu_r[v_{a-1}[v_{p^s+1}yx]] = \lambda(2 + \mu_r)[v_{a+p^s}yx] + \lambda(1 + \mu_r)[v_{a+p^s}xy],$$

and

$$(1 - \lambda)\mu_r[v_{a-1}[v_{p^s+1}xy]] = (\lambda - 1)(1 + \mu_r)[v_{a+p^s}yx] + (\lambda - 1)\mu_r[v_{a+p^s}xy].$$

Substituting in Equation (15) and multiplying both sides by μ_r we obtain

$$(\mu_r + \lambda + 1)[v_{a+p^s}yx] = -(\mu_r + \lambda)[v_{a+p^s}xy],$$

as desired. Now we expand Equation (15) assuming $\mu_r = 0$, whence $\mu_{r+1} = \lambda + 1$.

The above calculations for k odd (and $\mu_r = 0$) give $[v_{a-1}xv_{p^s+1}] = -[v_{a+p^s}x]$ and $[v_{a-1}xyv_{p^s+1}] = 2[v_{a+p^s}yx]$ when $k = p^s$. Recalling the equations $[v_{a-1}v_{p^s+1}] = -v_{a+p^s}$ and $[v_{a-1}yv_{p^s+1}] = -[v_{a+p^s}y]$ from Subsection 5.3, we find

$$\lambda[v_{a-1}[v_{p^s+1}yx]] = 2\lambda[v_{a+p^s}yx] + \lambda[v_{a+p^s}xy]$$

and

$$(1 - \lambda)[v_{a-1}[v_{p^s+1}xy]] = (\lambda - 1)[v_{a+p^s}yx].$$

We obtain $(\lambda + 1)[v_{a+p^s}yx] = -\lambda[v_{a+p^s}xy]$, thus concluding the proof. ■

Remark 6.3. The conclusion of Theorem 6.1 extends to the case $\lambda = 0$ excluded once the additional relation $[v_{2p^s+2}xx] = 0$ is included, at the expense of some additional calculations, which we now outline. The proof of Theorem 6.1 only fails to show that if v_a is just above a diamond of type $\mu_r = 1$ then v_{a+1} lies just above a diamond of infinite type. As in the general case, one has

$$[v_{a+1}xxy] + [v_{a+1}yx^2] + [v_{a+1}xyx] = 0 = [v_{a+1}x^3].$$

However, expanding Equation (13) is inconclusive when $\lambda = 0$. To get around this, at the first occurrence of a diamond of type $\mu_r = 1$, which occurs for $a = 2p^s + 1$, the additional relation $[v_{2p^s+2}xx] = 0$ allows one to conclude. For $a > 2p^s + 1$ one uses the additional relation to expand $[v_{a-2p^s-1}y[v_{2p^s+2}xx]] = 0$.

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