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On Lie Superalgebras with a Filiform Module as an Odd Part

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Abstract. The aim of this work is on one hand to characterise in any even dimension, via double extensions, a very special family of quadratic Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that $\mathfrak{g}_{\bar{1}}$ is a filiform $\mathfrak{g}_{\bar{0}}$-module (filiform type). On the other hand, we show that the study of quadratic Lie superalgebras of filiform type can be reduced to those that are solvable. Moreover, we obtain an inductive description of solvable quadratic Lie superalgebras of filiform type via both double extensions and odd double extensions of quadratic ones.

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1. Introduction

Quadratic Lie algebras have been studied in a deep way since they arised in connection with many problems derived from Geometry, Physics and other disciplines. The structure of quadratic Lie algebras plays an important role in conformal field theory and Sugawara construction exists precisely for quadratic Lie algebras [10]. The finite-dimensional quadratic Lie algebras over a field of characteristic zero are well known (see [4, 6, 9, 13, 19]). More recently, this class of algebras, also referred to as metric Lie algebras, is studied in [11, 19]. Concretely, in [17] Medina and Revoy introduced the idea of double extension and it allowed them to give a certain description of quadratic Lie algebras. More precisely, it is proved that every quadratic Lie algebra may be constructed as a direct sum of irreducible ones, and the latter by a sequence of double extensions. A construction that generalizes the semidirect product, namely the T^*-extension, was given in [6] to describe all solvable quadratic Lie algebras (indeed, this argument was made for more general classes of algebras). A different approach has been recently given by Kath and Olbrich in [16]. The first attempt to generalize this work to the context of the quadratic Lie superalgebras was done in [3]. In this case we note that it is more difficult to apply double extension to classify the quadratic Lie superalgebras than to describe inductively quadratic

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Lie algebras. Among the reasons we have that a 1-dimensional subspace of a Lie superalgebra $\mathfrak{g}$ is not necessarily a Lie subsuperalgebra of $\mathfrak{g}$ or there is no analog to the Lie Theorem for solvable Lie superalgebras. Despite the difficulties, in this work the authors shown that every irreducible quadratic Lie superalgebra in which the centre intersects the even part in a non-trivial way is a double extension. However, such a condition is rather restrictive. In [8] it is presented the structure of the Lie superalgebras with reductive even part and the action of the even part on the odd part completely reducible. After that, in [5], Benayadi obtained an inductive description of quadratic Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, and the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible. Moreover, he also gave an inductive description of the solvable quadratic Lie superalgebras such that the odd part is a completely reducible module on the even part. Recently, in [2] was generalized the notion of double extension of quadratic Lie superalgebras in order to present an inductive description of solvable quadratic Lie superalgebras. In particular, it is proved that all solvable quadratic Lie superalgebras are obtained by a sequence of generalized double extensions by 1-dimensional Lie superalgebras.

Throughout this paper we show that the study of quadratic Lie superalgebras of filiform type can be reduced to those that are solvable. Furthermore, we obtain an inductive description of solvable quadratic Lie superalgebras of filiform type via both double extensions and odd double extensions of quadratic ones.

The paper is organized as follows. After reviewing the background and introducing notation concerning the algebraic structures, in Section 3 we study the general case of quadratic Lie superalgebras of filiform type proving the convenience of studying the solvable case. After that, in Section 4 we obtain a family of solvable Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ in every even dimension m of $\mathfrak{g}_{\bar{1}}$ and $dim(\mathfrak{g}_{\bar{0}}) = 2$ via double extensions of quadratic ones. Let us note that this family is of filiform type (Theorem 4.6). Along sections 5 and 6 we provide with an inductive description of solvable quadratic Lie superalgebras of filiform type via double extensions and odd double extensions, respectively.

All Lie (super)algebras in this paper will be assumed finite-dimensional over the field of complex numbers $\mathbb{C}$. It should be noted that most of the results remain valid over an algebraically closed commutative field $\mathbb{F}$ of characteristic zero. Nevertheless, most of the results proved are also true over an arbitrary field of characteristic different from $2, 3$.

2. Basic concepts and preliminaries

This section is devoted to a review of the needed concepts in the sequel. For an integer i we denote by $\bar{i}$ its correspondent equivalence class in $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. For two integers i, j we use the well-defined notation $(-1)^{\bar{i}\bar{j}}$ for $(-1)^{ij} \in \{-1, 1\}$.

For a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ over the field $\mathbb{F}$, as usual, we write $V_{\bar{0}}$ for its *even part* and $V_{\bar{1}}$ for *odd part*. An element X of V is called *homogeneous* if either $X \in V_{\bar{0}}$ or $X \in V_{\bar{1}}$. In this work, all elements will be supposed to be homogeneous unless indicated otherwise.

A linear map $\phi\colon V \to W$ between two $\mathbb{Z}_2$-graded vector spaces is called *even* if $\phi(V_{\bar{0}}) \subset W_{\bar{0}}$ and $\phi(V_{\bar{1}}) \subset W_{\bar{1}}$. It is called *odd* if $\phi(V_{\bar{0}}) \subset W_{\bar{1}}$ and $\phi(V_{\bar{1}}) \subset W_{\bar{0}}$. Clearly, $Hom(V, W) = Hom(V, W)_{\bar{0}} \oplus Hom(V, W)_{\bar{1}}$, where the first summand comprises all the even linear maps, and the second all the odd. Tensor products

$V \otimes W$ are $\mathbb{Z}_2$-graded vector spaces, where its even part is $(V \otimes W)_{\bar{0}} = (V_{\bar{0}} \otimes W_{\bar{0}}) \oplus (V_{\bar{1}} \otimes W_{\bar{1}})$ and odd part $(V \otimes W)_{\bar{1}} = (V_{\bar{0}} \otimes W_{\bar{1}}) \oplus (V_{\bar{1}} \otimes W_{\bar{0}})$.

Definition 2.1. A *Lie superalgebra* is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, with an even bilinear operation $[\cdot,\cdot]$, which satisfies the conditions:

(i) $[X,Y] = -(-1)^{\overline{ij}}[Y,X]$,

(ii) $(-1)^{\overline{ik}}[X,[Y,Z]] + (-1)^{\overline{ij}}[Y,[Z,X]] + (-1)^{\overline{jk}}[Z,[X,Y]] = 0$,
(*super Jacobi identity*)

for any $X \in \mathfrak{g}_{\bar{i}}$, $Y \in \mathfrak{g}_{\bar{j}}$, $Z \in \mathfrak{g}_{\bar{k}}$, with $\bar{i}, \bar{j}, \bar{k} \in \mathbb{Z}_2$. ■

The general background on Lie superalgebras can be found in [18]. From the previous definition $\mathfrak{g}_{\bar{0}}$ is a Lie algebra, and $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module. The Lie superalgebra structure also contains the symmetric pairing $S^2\mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$, which is a $\mathfrak{g}_{\bar{0}}$-morphism and satisfies the super Jacobi identity applied to three elements of $\mathfrak{g}_{\bar{1}}$.

Definition 2.2. Let $\mathfrak{g}$ be a Lie superalgebra and $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ a bilinear form.

(i) B is *supersymmetric* if $B(X,Y) = (-1)^{\overline{ij}}B(Y,X)$, for any $X \in \mathfrak{g}_{\bar{i}}$, $Y \in \mathfrak{g}_{\bar{j}}$, with $\bar{i}, \bar{j} \in \mathbb{Z}_2$.

(ii) B is *invariant* if $B([X,Y],Z) = B(X,[Y,Z])$, for all $X,Y,Z \in \mathfrak{g}$.

(iii) B is *even* if $B(X,Y) = 0$, for any $X \in \mathfrak{g}_{\bar{0}}$, $Y \in \mathfrak{g}_{\bar{1}}$.

(iv) B is *non-degenerate* if $X \in \mathfrak{g}$ satisfies $B(X,Y) = 0$ for all $Y \in \mathfrak{g}$, then $X = 0$. Otherwise, B is called *degenerate*. ■

Definition 2.3. Let $\mathfrak{g}$ be a Lie superalgebra and B a bilinear form on $\mathfrak{g}$. A homogeneous superderivation D of $\mathfrak{g}$ of degree d is called *skew-supersymmetric* if

$$B(D(X),Y) = -(-1)^{dx}B(X,D(Y)), \quad X \in \mathfrak{g}_x,\ Y \in \mathfrak{g}.$$

We denote by $Der_a(\mathfrak{g})$ the subspace of $Der(\mathfrak{g})$ spanned by all the homogeneous skew-supersymmetric superderivations of $\mathfrak{g}$. ■

Definition 2.4. A Lie superalgebra $\mathfrak{g}$ is called *quadratic* if there exists a bilinear form $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ such that B is supersymmetric, invariant, even and non-degenerate. In this case, B is called an *invariant scalar product* on $\mathfrak{g}$. A quadratic Lie algebra is denoted by $(\mathfrak{g}, B)$ (simple by $\mathfrak{g}$ if no confusion is possible). ■

The semisimple Lie algebras endowed with the Killing form [14] and the classical Lie superalgebras [15, 18] are examples of quadratic Lie superalgebras. Some solvable Lie superalgebras are also quadratic Lie superalgebras [3, 17].

Definition 2.5. Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra.

(i) A graded ideal I of $\mathfrak{g}$ is *non-degenerate* (resp. *degenerate*) if the restriction of B to $I \times I$ is a non-degenerate (resp. degenerate) bilinear form.

(ii) A quadratic Lie superalgebra $(\mathfrak{g}, B)$ is *irreducible* if its unique non-degenerate graded ideals are $\{0\}, \mathfrak{g}$.

(iii) A graded ideal I of $\mathfrak{g}$ is called *irreducible* if I is non-degenerate and its unique non-degenerate graded ideals are $\{0\}, I$.

(iv) A graded ideal I of $\mathfrak{g}$ is said *isotropic* if $B(I, I) = \{0\}$. ■

Definition 2.6. Given a Lie superalgebra $\mathfrak{g}$, a $\mathbb{Z}_2$-graded vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a $\mathfrak{g}$-*module* if A is equipped with an even bilinear map $\mathfrak{g} \times A \to A$ (denoted by $(X, a) \mapsto Xa$, for $X \in \mathfrak{g}$ and $a \in A$) satisfying

$$[X, Y]a = X(Ya) - (-1)^{\bar{i}\bar{j}} Y(Xa),$$

for any $a \in A, X \in \mathfrak{g}_{\bar{i}}, Y \in \mathfrak{g}_{\bar{j}}$, with $\bar{i}, \bar{j} \in \mathbb{Z}_2$. ■

The following concept of filiform module was firstly introduced in [12] for defining filiform Lie superalgebras.

Definition 2.7. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra with $\dim(\mathfrak{g}_{\bar{1}}) = m > 0$. We said that $\mathfrak{g}_{\bar{1}}$ has the structure of *filiform* $\mathfrak{g}_{\bar{0}}$-*module* if the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ defines a *flag*, that is, a decreasing subsequence of vector subspaces in its underlying vector space $\mathfrak{g}_{\bar{1}}$,

$$\mathfrak{g}_{\bar{1}} = V_m \supset \cdots \supset V_1 \supset V_0,$$

with $\dim(V_i) = i$ and such that $[\mathfrak{g}_{\bar{0}}, V_{i+1}] = V_i$, for $i \in \{m-1, \ldots, 0\}$. To abbreviate, in the sequel we will refer to $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ as a Lie superalgebra of *filiform type.* ■

We recall now the *descending central sequence* of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ which is defined in the same way as for Lie algebras, that is,

$$\mathcal{C}^0(\mathfrak{g}) := \mathfrak{g}, \ \ \mathcal{C}^{k+1}(\mathfrak{g}) := [\mathcal{C}^k(\mathfrak{g}), \mathfrak{g}]$$

for $k \geq 0$. In case $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ for some $k \in \mathbb{N}$, then the Lie superalgebra is called *nilpotent.* Nevertheless, there are also defined two others descending sequences called $\mathcal{C}^k(\mathfrak{g}_{\bar{0}})$ and $\mathcal{C}^k(\mathfrak{g}_{\bar{1}})$ which will be also important in our study. They are defined as $\mathcal{C}^0(\mathfrak{g}_{\bar{i}}) := \mathfrak{g}_{\bar{i}}, \mathcal{C}^{k+1}(\mathfrak{g}_{\bar{i}}) := [\mathfrak{g}_{\bar{0}}, \mathcal{C}^k(\mathfrak{g}_{\bar{i}})]$, for $k \geq 0, \bar{i} \in \mathbb{Z}_2$.

Definition 2.8. If $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a solvable Lie superalgebra, then $\mathfrak{g}$ has *super-nilindex* or *s-nilindex* (p, q) if it satisfies

$$(\mathcal{C}^{p-1}(\mathfrak{g}_{\bar{0}})) \neq 0, \qquad (\mathcal{C}^{q-1}(\mathfrak{g}_{\bar{1}})) \neq 0, \qquad \mathcal{C}^p(\mathfrak{g}_{\bar{0}}) = \mathcal{C}^q(\mathfrak{g}_{\bar{1}}) = 0.$$ ■

Then we have an equivalent definition of filiform module structure.

Definition 2.9. If $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a solvable Lie superalgebra with *super-nilindex* $(-, \dim(\mathfrak{g}_{\bar{1}}))$, then $\mathfrak{g}_{\bar{1}}$ has filiform $\mathfrak{g}_{\bar{0}}$-module structure. ■

Remark 2.10. In particular whenever we have a nilpotent action over $\mathfrak{g}_{\bar{1}}$ the sequence $\mathcal{C}^0(\mathfrak{g}_{\bar{1}}) := \mathfrak{g}_{\bar{1}}, \mathcal{C}^{k+1}(\mathfrak{g}_{\bar{1}}) := [\mathfrak{g}_{\bar{0}}, \mathcal{C}^k(\mathfrak{g}_{\bar{1}})]$, $k \geq 0$, defines a *filtration* over $\mathfrak{g}_{\bar{1}}$, i.e a decreasing sequence of submodules $\mathcal{C}^0(\mathfrak{g}_{\bar{1}}) \supset \mathcal{C}^1(\mathfrak{g}_{\bar{1}}) \supset \mathcal{C}^2(\mathfrak{g}_{\bar{1}}) \ldots$

Lemma 2.11. *A necessary and sufficient condition to get a filiform* $\mathfrak{g}_{\bar{0}}$*-module structure over the odd part of a Lie superalgebra* $\mathfrak{g}$ *is* $\dim(\mathcal{C}^{i-1}(\mathfrak{g}_{\bar{1}})/\mathcal{C}^{i}(\mathfrak{g}_{\bar{1}})) = 1$ *for* $1 \leq i \leq \dim(\mathfrak{g}_{\bar{1}})$.

Note that henceforth we will note the center of $\mathfrak{g}$ by $\mathfrak{z}(\mathfrak{g})$.

Lemma 2.12. *Let* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ *be a quadratic Lie superalgebra with* $dim(\mathfrak{g}_{\bar{1}}) = m > 0$ *such that* $\mathfrak{g}_{\bar{1}}$ *has the structure of filiform* $\mathfrak{g}_{\bar{0}}$*-module concerning to the flag* $\mathfrak{g}_{\bar{1}} = V_m \supset \cdots \supset V_1 \supset V_0$. *Then* $\mathfrak{z}(\mathfrak{g})$ *intersects* $\mathfrak{g}_{\bar{1}}$, *more precisely,* $V_1 \subseteq \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}}$.

Proof. We have $[\mathfrak{g}_{\bar{0}}, V_1] = V_0 = \{0\}$. Because B is invariant we get immediately $B([\mathfrak{g}_{\bar{1}}, V_1], \mathfrak{g}_{\bar{0}}) = B(\mathfrak{g}_{\bar{1}}, [V_1, \mathfrak{g}_{\bar{0}}]) = \{0\}$. Since B is non-degenerate and $[\mathfrak{g}_{\bar{1}}, V_1] \subseteq \mathfrak{g}_{\bar{0}}$ then $[\mathfrak{g}_{\bar{1}}, V_1] = \{0\}$, so $V_1 \subseteq \mathfrak{z}(\mathfrak{g})$. ■

Lemma 2.13. *Let* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ *be a quadratic Lie superalgebra with* $dim(\mathfrak{g}_{\bar{1}}) = m > 0$ *such as* $\mathfrak{g}_{\bar{1}}$ *has the structure of filiform* $\mathfrak{g}_{\bar{0}}$*-module concerning to the flag* $\mathfrak{g}_{\bar{1}} = V_m \supset \cdots \supset V_1 \supset V_0$. *Then* V_1 *is ortogonal to* V_i *with respect to* B, *for all* $1 \leq i \leq m-1$.

Proof. As B is invariant, for $1 \leq i \leq m-1$ we have

$$B(V_i, V_1) = B([\mathfrak{g}_{\bar{0}}, V_{i+1}], V_1) = -B(V_{i+1}, [\mathfrak{g}_{\bar{0}}, V_1]) = \{0\},$$

as required. ■

Now, we will recall the concept of the double extension of Lie superalgebras which is both very useful to construct new quadratic Lie superalgebras and to give inductive descriptions of some important families of quadratic Lie superalgebras.

Theorem 2.14. [3, Theorem 1] *Let* $(\mathfrak{g}_1, B_1)$ *be a quadratic Lie superalgebra,* $\mathfrak{g}_2$ *a Lie superalgebra and* $\psi \colon \mathfrak{g}_2 \longrightarrow Der_a(\mathfrak{g}_1) \subset Der(\mathfrak{g}_1)$ *a morphism of Lie superalgebras. Let* φ *be the map from* $\mathfrak{g}_1 \times \mathfrak{g}_1$ *to* $\mathfrak{g}_2^*$, *defined by*

$$\varphi(X, Y)(Z) = (-1)^{(x+y)z} B_1(\psi(Z)(X), Y),$$

for any $X \in (\mathfrak{g}_1)_x$, $Y \in (\mathfrak{g}_1)_y$, $Z \in (\mathfrak{g}_2)_z$. *Let* π *be the coadjoint representation of* $\mathfrak{g}_2$. *Then the vector space* $\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2^*$ *with the product*

$$\begin{aligned}[X_2+X_1+f, Y_2+Y_1+g] = [X_2, Y_2]_{\mathfrak{g}_2} + [X_1, Y_1]_{\mathfrak{g}_1} + \psi(X_2)(Y_1) - (-1)^{xy}\psi(Y_2)(X_1) \\ + \pi(X_2)(g) - (-1)^{xy}\pi(Y_2)(f) + \varphi(X_1, Y_1),\end{aligned}$$

where $X_2 + X_1 + f$ *(resp.* $Y_2 + Y_1 + g$*) is homogeneous of degree* x *(resp.* y*) in* $\mathfrak{g}$, *is a Lie superalgebra.*
Moreover, if γ *is a* $\mathfrak{g}_2$*-invariant supersymmetric bilinear form, then the bilinear form* T, *defined on* $\mathfrak{g}$ *by*

$$T(X_2 + X_1 + f, Y_2 + Y_1 + g) = B_1(X_1, Y_1) + \gamma(X_2, Y_2) + f(Y_2) + (-1)^{xy} g(X_2)$$

where $X_2 + X_1 + f$ *and* $Y_2 + Y_1 + g$ *are homogeneous of respective degrees* x, y, *is an invariant scalar product on* $\mathfrak{g}$.
The Lie superalgebra $\mathfrak{g}$ *is called the* double extension *of* $(\mathfrak{g}_1, B_1)$ *by* $\mathfrak{g}_2$ *by means of* ψ.

3. Quadratic Lie superalgebras of filiform type: General case

Along this section we show that the study of quadratic Lie superalgebras of filiform type can be reduced to those that are solvable. Then, let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, [\,,\,], B)$ be a quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{1}}$ has the structure of filiform $\mathfrak{g}_{\bar{0}}$-module and $\mathfrak{g}_{\bar{1}} = V_m \supset \cdots \supset V_1 \supset V_0$ be the flag of the vector space $\mathfrak{g}_{\bar{1}}$ with the properties $[\mathfrak{g}_{\bar{0}}, V_i] = V_{i-1}$, for any $i \in \{1, \ldots, m\}$. Let us denote by $\rho_{\mathfrak{g}}$ the representation of Lie algebra $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ defined by:

$$\rho_{\mathfrak{g}}(X)(Y) := [X, Y], \quad \text{for any } X \in \mathfrak{g}_{\bar{0}} \text{ and } Y \in \mathfrak{g}_{\bar{1}}.$$

The kernel of this representation is $\mathrm{Ker}(\rho_{\mathfrak{g}}) := \{X \in \mathfrak{g}_{\bar{0}} : [X, Y] = 0, \forall Y \in \mathfrak{g}_{\bar{1}}\}$. Let us denote by $\mathcal{H}(\mathfrak{g})$ the set of the elements D of $(Der_a(\mathfrak{g}))_{\bar{0}}$ such that we have $D(\mathfrak{g}_{\bar{1}}) = \{0\}$. It is easy to verify that:

- $\mathcal{H}(\mathfrak{g})$ is an ideal of the Lie algebra $(Der_a(\mathfrak{g}))_{\bar{0}}$;
- Take D an even derivation of $\mathfrak{g}$. If $D(\mathfrak{g}_{\bar{1}}) = \{0\}$, then $D(\mathfrak{g}_{\bar{0}}) \subset \mathrm{Ker}(\rho_{\mathfrak{g}})$.

Definition 3.1. Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, [\,,\,], B)$ be a quadratic Lie superalgebra such that the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is filiform, $\mathfrak{h}$ a Lie algebra and $\psi\colon \mathfrak{h} \longrightarrow \mathcal{H}(\mathfrak{g})$ a morphism of Lie algebras such that: $\psi(x)(\mathfrak{g}_{\bar{1}}) = \{0\}$, whenever $x \in \mathfrak{h}$. Then the double extension $\mathfrak{L}$ of $(\mathfrak{g}, B)$ by $\mathfrak{h}$ by means of ψ is a quadratic Lie superalgebra such that the $\mathfrak{L}_{\bar{0}}$-module $\mathfrak{L}_{\bar{1}}$ is filiform, where $\mathfrak{L}_{\bar{0}} = \mathfrak{h} \oplus \mathfrak{g}_{\bar{0}}$ and $\mathfrak{L}_{\bar{1}} = \mathfrak{g}_{\bar{1}}$. The double extension $\mathfrak{L}$ is called an *elementary double extension of* $(\mathfrak{g}, B)$ *by the Lie algebra* $\mathfrak{h}$ *by means of* ψ. ■

We remark that for all $i \in \{1, ..., m\}$ we have $[\mathfrak{g}_{\bar{1}}, V_i]^{\perp} \cap \mathfrak{g}_{\bar{0}} = \{X \in \mathfrak{g}_{\bar{0}} : [X, V_i] = \{0\}\}$. In particular, $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]^{\perp} \cap \mathfrak{g}_{\bar{0}} = \mathrm{Ker}(\rho_{\mathfrak{g}})$.

Proposition 3.2. *Let* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, [\,,\,], B)$ *be a quadratic Lie superalgebra such that the* $\mathfrak{g}_{\bar{0}}$*-module* $\mathfrak{g}_{\bar{1}}$ *is filiform. Let* $\mathfrak{s}$ *be a Levi component of* $\mathfrak{g}_{\bar{0}}$*. Then we have the following assertions.*

(i) $\mathfrak{s}$ *is contained in Ker*$(\rho_{\mathfrak{g}})$.

(ii) *The greatest semisimple ideal* $\mathfrak{S}$ *of* $\mathfrak{g}_{\bar{0}}$ *is a non-degenerate graded ideal of* $\mathfrak{g}$*. Consequently,* $\mathfrak{g} = \mathfrak{S} \oplus \mathfrak{S}^{\perp}$ *and* $\mathfrak{S}^{\perp}$ *is a quadratic Lie superalgebra which contains no non-zero semisimple ideal in its even part and the odd part of* $\mathfrak{S}^{\perp}$ *is a filiform module on its even part.*

(iii) *If the graded ideal Ker*$(\rho_{\mathfrak{g}})$ *is non-degenerate, then* $(Ker(\rho_{\mathfrak{g}}))^{\perp}$ *is a nilpotent graded ideal of* $\mathfrak{g}$ *such that* $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ *is contained in* $((Ker(\rho_{\mathfrak{g}}))^{\perp})_{\bar{0}}$, $((Ker(\rho_{\mathfrak{g}}))^{\perp})_{\bar{1}}$ *is a filiform module on the even part* $((Ker(\rho_{\mathfrak{g}}))^{\perp})_{\bar{0}}$*, and the action of the ideal* $((Ker(\rho_{\mathfrak{g}}))^{\perp})_{\bar{0}}$ *on* $((Ker(\rho_{\mathfrak{g}}))^{\perp})_{\bar{1}}$ *is faithful. Then, in this case* $\mathfrak{g}$ *is a direct orthogonal sum of a quadratic Lie algebra and a nilpotent quadratic Lie superalgebra with odd part is a filiform module on the even part and the action of the even part on the odd part is faithful.*

Proof. (i) The quotient Lie algebra $\mathfrak{g}_{\bar{0}}/\mathrm{Ker}(\rho_{\mathfrak{g}})$ is isomorphic to nilpotent Lie algebra $\rho_{\mathfrak{g}}(\mathfrak{g}_{\bar{0}})$, consequently $\mathfrak{s}$ is contained in $\mathrm{Ker}(\rho_{\mathfrak{g}})$.

(ii) $\mathfrak{S}$ is contained in $\mathfrak{s}$, so $[\mathfrak{S}, \mathfrak{g}_{\bar{1}}] = \{0\}$. Consequently $\mathfrak{S}$ is an ideal of $\mathfrak{g}$. Since $\mathfrak{S} \cap \mathfrak{S}^{\perp}$ is an abelian ideal of $\mathfrak{S}$, then $\mathfrak{S} \cap \mathfrak{S}^{\perp} = \{0\}$. Therefore, $\mathfrak{S}$ is a non-degenerate graded ideal of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{S} \oplus \mathfrak{S}^{\perp}$. It is clear that all semisimple ideal in the even part of the graded ideal $\mathfrak{S}^{\perp}$ is also a semisimple ideal of $\mathfrak{g}_{\bar{0}}$, which proves that the even part of $\mathfrak{S}^{\perp}$ contains no non-zero semisimple ideals. The fact that $[\mathfrak{S}, \mathfrak{g}_{\bar{1}}] = \{0\}$, the odd part $(\mathfrak{S}^{\perp})_{\bar{1}} = \mathfrak{g}_{\bar{1}}$ of $\mathfrak{S}^{\perp}$ is a filiform module on its even part.

(iii) The invariance of B implies that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is contained in $((\mathrm{Ker}(\rho_{\mathfrak{g}}))^{\perp})_{\bar{0}}$. The rest of the assertion is clear. ∎

Remark 3.3. (i) The assertion of Proposition 3.2-ii reduces the study of quadratic Lie superalgebras where the odd part is a filiform module on the even part to those without a semisimple ideal in its even part.

(ii) In the assertion of Proposition 3.2-iii, the ideal $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ of $\mathfrak{g}_{\bar{0}}$ is nilpotent. In the following proposition, we will prove that without hypothesis of this assertion, $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is also a nilpotent ideal of $\mathfrak{g}_{\bar{0}}$.

Proposition 3.4. *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra such that the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is filiform. Then we have:*

(i) *V_1 is in the center of $\mathfrak{g}$.*

(ii) *For any $k \in \{1, \dots, m\}$, $\underbrace{[\mathfrak{g}_{\bar{0}}, \dots [\mathfrak{g}_{\bar{0}}}_{(k-1)m \text{ times}}, [\mathfrak{g}_{\bar{1}}, V_k]] \dots] = \{0\}$.*

(iii) *For any $k \in \{1, \dots, m\}$, $\mathcal{C}^{(k-1)m}([\mathfrak{g}_{\bar{1}}, V_k]) = \{0\}$.*

Proof. (i) The proof of Lemma 2.12 shows in particular that V_1 is contained in the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$.

(ii) V_1 is in the center of $\mathfrak{g}$, so $[\mathfrak{g}_{\bar{1}}, V_1] = \{0\}$. We have $[\mathfrak{g}_{\bar{0}}, [\mathfrak{g}_{\bar{1}}, V_2]] \subseteq [[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}], V_2]$ furthermore. Let $j \in \{1, \dots, m-1\}$. We assume that

$$\underbrace{[\mathfrak{g}_{\bar{0}}, \dots [\mathfrak{g}_{\bar{0}}}_{j \text{ times}}, [\mathfrak{g}_{\bar{1}}, V_2]] \dots] \subset [\underbrace{[\mathfrak{g}_{\bar{0}}, \dots [\mathfrak{g}_{\bar{0}}}_{j \text{ times}}, \mathfrak{g}_{\bar{1}}] \dots], V_2].$$

Since V_1 is a subset of $\mathfrak{z}(\mathfrak{g})$, then $\underbrace{[\mathfrak{g}_{\bar{0}}, \dots [\mathfrak{g}_{\bar{0}}}_{j+1 \text{ times}}, [\mathfrak{g}_{\bar{1}}, V_2]] \dots] \subset [\underbrace{[\mathfrak{g}_{\bar{0}}, \dots [\mathfrak{g}_{\bar{0}}}_{j+1 \text{ times}}, \mathfrak{g}_{\bar{1}}] \dots], V_2]$.

Therefore, we can conclude that $\underbrace{[\mathfrak{g}_{\bar{0}}, \dots [\mathfrak{g}_{\bar{0}}}_{m \text{ times}}, [\mathfrak{g}_{\bar{1}}, V_2]] \dots] = \{0\}$.

Now, let $k \in \{1, \dots, m-1\}$. Suppose that $\underbrace{[\mathfrak{g}_{\bar{0}}, \dots [\mathfrak{g}_{\bar{0}}}_{(k-1)m \text{ times}}, [\mathfrak{g}_{\bar{1}}, V_k]] \dots] = \{0\}$ and let us show that $\underbrace{[\mathfrak{g}_{\bar{0}}, \dots [\mathfrak{g}_{\bar{0}}}_{km \text{ times}}, [\mathfrak{g}_{\bar{1}}, V_{k+1}]] \dots] = \{0\}$. Since $\underbrace{[\mathfrak{g}_{\bar{0}}, \dots [\mathfrak{g}_{\bar{0}}}_{m \text{ times}}, \mathfrak{g}_{\bar{1}}] \dots] = \{0\}$, then $\underbrace{[\mathfrak{g}_{\bar{0}}, \dots [\mathfrak{g}_{\bar{0}}}_{km \text{ times}}, [\mathfrak{g}_{\bar{1}}, V_{k+1}]] \dots]$ is a subset of $[\mathfrak{g}_{\bar{1}}, V_k]$. It follows that

$$\underbrace{[\mathfrak{g}_{\bar{0}}, \dots [\mathfrak{g}_{\bar{0}}}_{km \text{ times}}, [\mathfrak{g}_{\bar{1}}, V_{k+1}]] \dots] \subset \underbrace{[\mathfrak{g}_{\bar{0}}, \dots [\mathfrak{g}_{\bar{0}}}_{(k-1)m \text{ times}}, [\mathfrak{g}_{\bar{1}}, V_k]] \dots] = \{0\}.$$

We conclude that for any $k \in \{1, \dots, m\}$, $\underbrace{[\mathfrak{g}_{\bar{0}}, \dots [\mathfrak{g}_{\bar{0}}}_{(k-1)m \text{ times}}, [\mathfrak{g}_{\bar{1}}, V_k]] \dots] = \{0\}$.

(iii) It is a direct consequence of previous item. ∎

Proposition 3.5. *If $\mathfrak{g}$ is not solvable and it contains no non-zero semisimple ideal in its even part, then $\mathfrak{g}$ is an elementary double extension of a quadratic Lie superalgebra, where its odd part is a filiform module on its even part, by a simple Lie algebra.*

Proof. Let $\mathfrak{s}$ be a Levi component of $\mathfrak{g}_{\bar{0}}$ and $\mathcal{R}(\mathfrak{g}_{\bar{0}})$ the solvable radical of $\mathfrak{g}_{\bar{0}}$. It is well known that $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_p$, where $(\mathfrak{s}_i)_{1 \leq i \leq p}$ are the simple ideals of $\mathfrak{s}$. Since $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is a nilpotent ideal of $\mathfrak{g}_{\bar{0}}$ and $[\mathfrak{s}_1, \mathfrak{g}_{\bar{1}}] = \{0\}$, then $\mathfrak{I} := \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_p \oplus \mathcal{R}(\mathfrak{g}_{\bar{0}}) \oplus \mathfrak{g}_{\bar{1}}$ is a maximal graded ideal of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{I} \oplus \mathfrak{s}_1$. Consequently, $\mathfrak{I}$ is a degenerate ideal of $\mathfrak{g}$ because $\mathfrak{g}_{\bar{0}}$ contains no non-zero semisimple ideals. It follows that $\mathfrak{I}^{\perp}$ is a minimal and a completely isotropic graded ideal of $\mathfrak{g}$. We conclude, by Theorem 2 of [3], that $\mathfrak{g}$ is a double extension of the quadratic Lie superalgebra $\mathfrak{h} := \mathfrak{I}/\mathfrak{I}^{\perp}$ by the simple Lie algebra $\mathfrak{s}_1$. Now, the fact $B([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}], \mathfrak{I}^{\perp}) = \{0\}$ implies, by invariance of B, that $B(\mathfrak{g}_{\bar{1}}, [\mathfrak{g}_{\bar{1}}, \mathfrak{I}^{\perp}]) = \{0\}$. It follows that $[\mathfrak{g}_{\bar{1}}, \mathfrak{I}^{\perp}] = \{0\}$. Therefore the $\mathfrak{h}_{\bar{0}}$-module $\mathfrak{h}_{\bar{1}}$ is filiform because the $\mathfrak{I}^{\perp}$ is contained in $\mathfrak{g}_{\bar{0}}$, $[\mathfrak{s}_1, \mathfrak{g}_{\bar{1}}] = \{0\}$, and the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$ is filiform. Since $[\mathfrak{s}_1, \mathfrak{g}_{\bar{0}}] \subseteq \mathrm{Ker}(\rho_{\mathfrak{g}})$ and $[\mathfrak{s}_1, \mathfrak{g}_{\bar{1}}] = \{0\}$, then $\mathfrak{g}$ is an elementary double extension of the quadratic Lie superalgebra $\mathfrak{h} := \mathfrak{I}/\mathfrak{I}^{\perp}$ by the simple Lie algebra $\mathfrak{s}_1$. ∎

Remark 3.6. The previous proposition reduces the study of quadratic Lie superalgebras where the odd part is a filiform module on the even part to those that are solvable.

4. A family of solvable Lie superalgebras of filiform type via double extensions of quadratic ones

Following the spirit of Example 2.1 of the paper [1] we obtain a family of solvable Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ in every even dimension m of $\mathfrak{g}_{\bar{1}}$ and $dim(\mathfrak{g}_{\bar{0}}) = 2$. Moreover, this family verifies that $\mathfrak{g}_{\bar{1}}$ has the structure of filiform $\mathfrak{g}_{\bar{0}}$-module.

We recall that an action is *completely reducible* if every invariant subspace has an invariant complement. A Lie algebra L is *reductive* if it is the direct sum of a semisimple Lie algebra and an abelian Lie algebra (or equivalently, its adjoint representation is completely reducible).
In [1], there are some examples of quadratic Lie superalgebras with a reductive even part and the action of the even part on the odd part is not completely reducible. Here we also present an example such that the action of the even part on the odd part is not completely reducible.

Example 4.1. Let $(\mathfrak{M}, B)$ be the 4-dimensional abelian quadratic Lie superalgebra such that $\mathfrak{M}_{\bar{0}} = \{0\}$ and $\{e_1, e_2, e_3, e_4\}$ is a basis of $\mathfrak{M}_{\bar{1}}$. Let $B \colon \mathfrak{M} \times \mathfrak{M} \longrightarrow \mathbb{F}$ be the bilinear form on $\mathfrak{M}$ defined by the only non-zero values

$$B(e_1, e_2) = B(e_2, e_3) = B(e_4, e_1) = 1,$$

$$B(e_2, e_1) = B(e_3, e_2) = B(e_1, e_4) = -1.$$

Notice that it can be easily seen that B is an invariant scalar product on $\mathfrak{M}$ and therefore $(\mathfrak{M}, B)$ is a quadratic Lie superalgebra with $\mathfrak{M}_{\bar{0}}$ a reductive Lie algebra and the action of $\mathfrak{M}_{\bar{0}}$ on $\mathfrak{M}_{\bar{1}}$ completely reducible. Next, we consider the linear map $D\colon \mathfrak{M} \longrightarrow \mathfrak{M}$ defined by $D(e_i) = e_{i+1}$, $1 \leq i \leq 3$ and $D(e_4) = 0$.

It can be easily checked that D is an even skew-supersymmetric superderivation of $(\mathfrak{M}, B)$. Applying Theorem 2.14 we can consider the double extension of the quadratic Lie superalgebra $(\mathfrak{M}, B)$ by the 1-dimensional Lie algebra $(\mathbb{F}e)_{\bar{0}}$ by means of $\psi\colon \mathbb{F}e \longrightarrow \mathrm{Der}_a(\mathfrak{M}, B)$ defined by $\psi(e) = D$. A straightforward calculation leads to the map $\varphi : \mathfrak{M} \times \mathfrak{M} \longrightarrow \mathbb{F}e^*$ with the only non-zero values that follow:

$$\varphi(e_1, e_1) = \varphi(e_2, e_2) = -e^* \quad \text{and} \quad \varphi(e_1, e_3) = \varphi(e_3, e_1) = e^*.$$

Consequently, we obtain the Lie superalgebra $\mathfrak{g} = \mathbb{F}e \oplus \mathfrak{M} \oplus \mathbb{F}e^*$, the double extension of $\mathfrak{M}$ by the 1-dimensional Lie algebra $(\mathbb{F}e)_{\bar{0}}$ by means of ψ. The non-zero bracket products of this Lie superalgebra expressed in the basis $\{e, e^*, e_1, e_2, e_3, e_4\}$ are :

$$\begin{aligned}
&[e_1, e_1] = -e^*, \\
&[e_1, e_3] = [e_3, e_1] = e^*, \\
&[e_2, e_2] = -e^*, \\
&[e, e_i] = -[e_i, e] = e_{i+1}, \quad 1 \leq i \leq 3,
\end{aligned}$$

where $\{e, e^*\}$ is a basis of $\mathfrak{g}_{\bar{0}} = \mathbb{F}e \oplus \mathbb{F}e^*$ and $\{e_1, e_2, e_3, e_4\}$ a basis of $\mathfrak{g}_{\bar{1}} = \mathfrak{M}$. Moreover, $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and $\mathfrak{g}_{\bar{1}}$ is not a completely reducible $\mathfrak{g}_{\bar{0}}$-module. In fact, if we consider in particular the flag defined over the underlying vector space of $\mathfrak{g}_{\bar{1}}$:

$$V_4 \supset V_3 \supset V_2 \supset V_1$$

where $V_4 = span\{e_1, e_2, e_3, e_4\}$, $V_3 = span\{e_2, e_3, e_4\}$, $V_2 = span\{e_3, e_4\}$ and $V_1 = span\{e_4\}$, it can be easily seen that $[\mathfrak{g}_{\bar{0}}, V_{i+1}] = V_i$ for $i \in \{3, 2, 1, 0\}$, and then $\mathfrak{g}_{\bar{1}}$ has structure of a filiform $\mathfrak{g}_{\bar{0}}$-module.

Remark 4.2. The 6-dimensional Lie superalgebra $\mathfrak{g}$ described in the previous example actually is a solvable filiform Lie superalgebra (see [12]).

Hereafter, we are going to extend the study for any even dimension.

Proposition 4.3. *Let $(\mathfrak{M}, B)$ be any m-dimensional abelian quadratic Lie superalgebra such that $\mathfrak{M}_{\bar{0}} = \{0\}$ and m even. Note by $\{e_1, e_2, \ldots, e_m\}$ the basis of $\mathfrak{M}$. Let $\mathfrak{g}$ be a double extension of the quadratic Lie superalgebra $(\mathfrak{M}, B)$ by the 1-dimensional Lie algebra $(\mathbb{F}e)_{\bar{0}}$ by means of $\psi\colon \mathbb{F}e \longrightarrow Der_a(\mathfrak{M}, B)$ with $\psi(e) = D$ and D an even skew-supersymmetric superderivation of $(\mathfrak{M}, B)$.*
Under these conditions, the simplest even skew-supersymmetric superderivation D of $(\mathfrak{M}, B)$ that provides the double extension $\mathfrak{g} = \mathbb{F}e \oplus \mathfrak{M} \oplus \mathbb{F}e^* = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a filiform structure (that is, $\mathfrak{g}_{\bar{1}}$ having a filiform $\mathfrak{g}_{\bar{0}}$-module structure) is defined by

$$D(e_i) = e_{i+1}, \; 1 \leq i \leq m-1 \;\; \textit{and} \;\; D(e_m) = 0,$$

up to reordering the basis vectors. Moreover, the invariant scalar product B satisfies the following equations:

$$B(e_{i+1}, e_j) = -B(e_i, e_{j+1}), \; 1 \leq i, j \leq m$$

assuming that $e_{m+1} = 0$.

Proof. Applying Theorem 2.14 in our case we have the double extension $\mathfrak{g} = \mathbb{F}e \oplus \mathfrak{M} \oplus \mathbb{F}e^*$ with $\mathfrak{g}_{\bar{0}} = \mathbb{F}e \oplus \mathbb{F}e^*$ and $\mathfrak{g}_{\bar{1}} = \mathfrak{M}$. Thus, the products $[e, e_i]$ which define the action of L_0 on L_1 are exactly

$$[e, e_i] = -[e_i, e] = \psi(e)(e_i) = D(e_i), \text{ for } i \in \{1, \dots, m\}$$

on account of the simplest action defining a filiform module is the one given by the action of the even part of $L^{n,m}$ over its odd part, being $L^{n,m}$ the simplest filiform Lie superalgebra or so-called the model one $L^{n,m}$ (for more details it can be consulted [7]). Recall briefly that $L^{n,m}$ is defined by the only non-zero bracket products that follow

$$L^{n,m} : \begin{cases} [X_0, X_i] = X_{i+1}, & 1 \le i \le n-1 \\ [X_0, Y_j] = Y_{j+1}, & 1 \le j \le m-1 \end{cases}$$

and the latter products $[X_0, Y_j] = Y_{j+1}$ provide the odd part with filiform module structure. Therefore we impose $[e, e_i] = D(e_i) = e_{i+1}$ and finally, a straightforward calculation leads to the condition for the invariant scalar product B. ∎

Remark 4.4. Under the conditions of the above proposition in addition to the fact that B is skew-symmetric, it can be seen that

$$B(e_i, e_j) = 0 \text{ if either } i + j \in \{2 + m, 3 + m, \dots, (m-1) + m\} \text{ or } i + j \text{ is even.}$$

For simplicity we denote $b_{ij} := B(e_i, e_j)$, then $b_{ij} = -b_{ji}$. Thus, for instance, if $m = 6$ the only remaining b_{ij}, with $i \le j$, are those involved in the following equations:

$$\begin{aligned} &b_{12} \\ &b_{14} = -b_{23} \\ &b_{16} = -b_{25} = b_{34} \end{aligned}$$

Since B is non-degenerate the last equation is always different from zero. Indeed, if the last equation is equal to one and the others vanish we will have the simplest B among all the possibles. In general, it can be writen as

$$B(e_i, e_{m-(i-1)}) = (-1)^{i-1} \text{ if } i \in \{1, \dots, m\}.$$

Also, note that B in Example 4.1 is not this simplest one.

Proposition 4.5. *Let $(\mathfrak{M}, B)$ be any m-dimensional abelian quadratic Lie superalgebra such that $\mathfrak{M}_{\bar{0}} = \{0\}$, m even and denote $\{e_1, e_2, \dots, e_m\}$ the basis of $\mathfrak{M}$. Let $\mathfrak{g}$ be any double extension of the quadratic Lie superalgebra $(\mathfrak{M}, B)$ by the 1-dimensional Lie algebra $(\mathbb{F}e)_{\bar{0}}$ by means of $\psi : \mathbb{F}e \longrightarrow Der_a(\mathfrak{M}, B)$ with $\psi(e) = D$ and D an even skew-supersymmetric superderivation of $(\mathfrak{M}, B)$.*
Under these conditions, the simplest double extension $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with $\mathfrak{g}_{\bar{1}}$ a filiform $\mathfrak{g}_{\bar{0}}$-module structure is (up to isomorphism) defined by the only non-zero bracket products

$$\begin{aligned} [e, e_i] &= -[e_i, e] = e_{i+1}, && 1 \le i \le m-1, \\ [e_i, e_j] &= [e_j, e_i] = C_{ij} e^*, && 1 \le i \le j \le m. \end{aligned}$$

where the structure constants verify

$$C_{ij} = C_{ji} = B(e_{i+1}, e_j), \ 1 \le i, j \le m$$

with $\{e, e^*\}$ is a basis of $\mathfrak{g}_{\bar{0}} = \mathbb{F}e \oplus \mathbb{F}e^*$ and $\{e_1, e_2, \dots, e_m\}$ a basis of $\mathfrak{g}_{\bar{1}} = \mathfrak{M}$.

Proof. The result follows from the Proposition 4.3, the definitions of φ (see [3]) and of the bracket products of the Lie superalgebra double extension. ■

Theorem 4.6. *Let L be under the conditions of Proposition 4.5. Then L is isomorphic to the Lie superalgebra with basis $\{e, e^*, e_1, e_2, \dots, e_m\}$ and multiplication table*

$$\begin{aligned} [e, e_i] &= -[e_i, e] = e_{i+1}, & 1 \le i \le m-1, \\ [e_j, e_{m-j}] &= (-1)^j e^*, & 1 \le j \le m-1. \end{aligned}$$

where the omitted products are equal to zero.

Proof. Remark 4.4 allows us to contruct the simplest family verifying the conditions of the Proposition 4.5. Then, considering the values for B given in Remark 4.4 we have a skew-symmetric bilinear map B with the only non-zero values

$$B(e_i, e_{m-(i-1)}) = (-1)^{i-1} \text{ if } i \in \{1, \dots, m\}.$$

But since $[e_i, e_j] = B(e_{i+1}, e_j)$, the only non-vanishing bracket products of this Lie superalgebra double extension, expressed in the basis $\{e, e^*, e_1, e_2, \dots, e_m\}$, with m even, are

$$\begin{aligned} [e, e_i] &= -[e_i, e] = e_{i+1}, & 1 \le i \le m-1, \\ [e_j, e_{m-j}] &= [e_{m-j}, e_j] = (-1)^j e^*, & 1 \le j \le \frac{m}{2}. \end{aligned}$$ ■

5. Solvable quadratic Lie superalgebras of filiform type via double extensions of quadratic ones

In this section we study the inductive description of solvable quadratic Lie superalgebras which the odd part is a filiform module. Since a Lie superalgebra $\mathfrak{g}$ is solvable if and only its even part $\mathfrak{g}_{\bar{0}}$ is solvable Lie algebra, we have then in our study quadratic Lie superalgebras with even part any solvable Lie algebra and the odd part a filiform module. Section 4 shows that this class is not empty.

Recall that, given a Lie (super)algebra $\mathfrak{g}$, its *derived sequence* is defined as

$$\mathcal{D}^0(\mathfrak{g}) := \mathfrak{g}, \;\; \mathcal{D}^{k+1}(\mathfrak{g}) := [\mathcal{D}^k(\mathfrak{g}), \mathcal{D}^k(\mathfrak{g})], \;\; \text{for } k \ge 0.$$

A Lie (super)algebra $\mathfrak{g}$ is *solvable* if there exist a $k_0 \in \mathbb{N}$ such that its derived sequence satisfies $\mathcal{D}^{k_0} = 0$.

First we consider that the starting quadratic Lie superalgebra is nilpotent (a subclass of solvable).

Proposition 5.1. *Let $(\mathfrak{M}, B)$ be a finite-dimensional nilpotent quadratic Lie superalgebra with odd part $\mathfrak{M}_{\bar{1}}$ a filiform $\mathfrak{M}_{\bar{0}}$-module. Let $\mathfrak{g}$ be a double extension of the quadratic Lie superalgebra $(\mathfrak{M}, B)$ by the 1-dimensional Lie algebra $(\mathbb{F}e)_{\bar{0}}$ by means of $\psi\colon \mathbb{F}e \to Der_a(\mathfrak{M}, B)$, defined by $\psi(e) = D$, and D an even skew-supersymmetric superderivation of $(\mathfrak{M}, B)$.*
Then we have two possibilities:

(i) *If the matrix associated to the linear map D is nilpotent then the Lie superalgebra double extension $\mathfrak{g}$ is a nilpotent Lie superalgebra with $\mathfrak{g}_{\bar{1}}$ a filiform $\mathfrak{g}_{\bar{0}}$-module.*

(ii) *If the matrix associated to the linear map D is non-nilpotent then the Lie superalgebra double extension $\mathfrak{g}$ is a solvable non-nilpotent Lie superalgebra. Moreover, if $D|_{\mathfrak{M}_{\bar{1}}}$ is nilpotent then $\mathfrak{g}_{\bar{1}}$ remains as filiform $\mathfrak{g}_{\bar{0}}$-module.*

Proof. Since $\mathfrak{M}_{\bar{0}}$ is a nilpotent Lie algebra and $\mathfrak{M}_{\bar{1}}$ has structure of $\mathfrak{M}_{\bar{0}}$-module, in general we have that the action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ defines a *flag*, that is, a decreasing subsequence of vector subspaces in its underlying vector space V,

$$V = V_{\dim \mathfrak{g}_{\bar{1}}} \supset V_{\dim \mathfrak{g}_{\bar{1}}-1} \supset \cdots \supset V_1 \supset V_0$$

with dimensions $\dim \mathfrak{g}_{\bar{1}}, \dim \mathfrak{g}_{\bar{1}} - 1, \ldots, 0$, respectively, $\dim \mathfrak{g}_{\bar{1}} > 0$ and such that $[\mathfrak{g}_{\bar{0}}, V_{i+1}] \subseteq V_i$, for $i \in \{\dim \mathfrak{g}_{\bar{1}} - 1, \ldots, 0\}$. In our case we have an additional condition, i.e., $\mathfrak{g}_{\bar{1}}$ is a filiform $\mathfrak{g}_{\bar{0}}$-module, which means that instead of $[\mathfrak{g}_{\bar{0}}, V_{i+1}] \subset V_i$ we get $[\mathfrak{g}_{\bar{0}}, V_{i+1}] = V_i$ for $i \in \{\dim \mathfrak{g}_{\bar{1}} - 1, \ldots, 0\}$. Applying Theorem 2.14 in our case we have the double extension $\mathfrak{g} = \mathbb{F}e \oplus \mathfrak{M} \oplus \mathbb{F}e^*$ with $\mathfrak{g}_{\bar{0}} = \mathbb{F}e \oplus \mathfrak{M}_{\bar{0}} \oplus \mathbb{F}e^*$ and $\mathfrak{g}_{\bar{1}} = \mathfrak{M}_{\bar{1}}$. By construction the following conditions hold for $X, Y \in \mathfrak{M}$:

$$\begin{aligned} [e, X] &= D(X) \\ [X, Y] &= [X, Y]_{\mathfrak{M}} + \varphi(X, Y) \\ [e^*, X] &= [e, e^*] = 0 \end{aligned}$$

Note that since $\varphi(X, Y) \in \mathbb{K}e^*$ and e^* belongs to the center of $\mathfrak{g}$ then the bracket products $[X, Y]$ do not change the nilpotent (resp. solvable) condition of $[X, Y]_{\mathfrak{M}}$. Moreover, the fact that $[e, X] = D(X)$ implies that if D is nilpotent then, in particular, the action of $\mathbb{K}e$ over $\mathfrak{M}_{\bar{1}} = \mathfrak{g}_{\bar{1}}$ is also nilpotent and at most we have a flag as aforementioned with $[\mathfrak{g}_{\bar{0}}, V_{i+1}] = V_i$ (filiform $\mathfrak{g}_{\bar{0}}$-module). As we have already the structure over $\mathfrak{M}_{\bar{1}}$ of filiform $\mathfrak{M}_{\bar{0}}$-module, then this structure remains for the Lie superalgebra double extension $\mathfrak{g}$.

On the other hand, if D is non-nilpotent then the adjoint operator ad_e is non-nilpotent. As Engle's Theorem and its corollaries remain valid for Lie superalgebras, then the Lie superalgebra double extension is non-nilpotent. Remark that we have $\mathfrak{g}_{\bar{0}}^2 = [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}] \subset \mathfrak{M}_{\bar{0}} \oplus \mathbb{F}e^*$ which is clearly nilpotent and then we get that $\mathfrak{g}_{\bar{0}}$ and then $\mathfrak{g}$ is solvable. In particular, if $D|_{\mathfrak{M}_{\bar{1}}}$ is nilpotent then $\mathfrak{g}_{\bar{1}}$ remains as filiform $\mathfrak{g}_{\bar{0}}$-module as above. ∎

A straightforward calculation leads to the following corollary provided that we are under the conditions of Proposition 5.1(ii).

Corollary 5.2. *The nilradical (maximal nilpotent ideal) of the Lie superalgebra double extension $\mathfrak{g}$ is exactly $\mathfrak{M} \oplus \mathbb{F}e^*$.*

Remark 5.3. Recall that for Lie superalgebras the Lie's Theorem is not verified in general and neither its corollaries. Therefore, for a solvable Lie superalgebra L we get that $L^2 := [L, L]$ can not be nilpotent.

6. Solvable Quadratic Lie superalgebras of filiform type via odd double extensions of quadratic ones

The aim of this section is to study quadratic Lie superalgebras with a filiform module as an odd part via odd double extensions of quadratic ones.

Theorem 6.1. *Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra such that $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$. Let $X_0 \in \mathfrak{z}(\mathfrak{g})\backslash\{0\}$ such that $B(X_0, X_0) = 0$. We denote by $\mathbb{F}e$ the 1-dimension $\mathbb{Z}_2$-graded vector space such that $\mathbb{F}e = (\mathbb{F}e)_{\bar{1}}$ and $\mathbb{F}e^*$ its dual vector space.*
Then the $\mathbb{Z}_2$-graded vector space $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_{\bar{0}} \oplus \widetilde{\mathfrak{g}}_{\bar{1}} = \mathbb{F}e \oplus \mathfrak{g} \oplus \mathbb{F}e^*$ with the even skew-symmetric bilinear map $[,] : \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}} \longrightarrow \widetilde{\mathfrak{g}}$ defined by

$$[e, e] = X_0,\ [e^*, \widetilde{\mathfrak{g}}] = \{0\},\ [e, X] = -B(X, X_0)e^*,\ [X, Y] = [X, Y]_{\mathfrak{g}},$$

for all $X, Y \in \mathfrak{g}$, and the supersymmetric bilinear form $\widetilde{B}\colon \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}} \longrightarrow \mathbb{F}$ defined by

$$\widetilde{B}|_{\mathfrak{g}\times\mathfrak{g}} = B,\ \widetilde{B}(e^*, e) = 1,\ \widetilde{B}(\mathfrak{g}, e) = \widetilde{B}(\mathfrak{g}, e^*) = \{0\},\ \widetilde{B}(e, e) = \widetilde{B}(e^*, e^*) = 0$$

is a quadratic Lie superalgebra of filiform type with $dim(\widetilde{\mathfrak{g}}_{\bar{1}}) = 2$, such that $\widetilde{\mathfrak{g}}_{\bar{1}}$ is a filiform $\widetilde{\mathfrak{g}}_{\bar{0}}$-module with the flag given by:

$$\widetilde{\mathfrak{g}}_{\bar{1}} = \widetilde{V}_2 \supset \widetilde{V}_1 \supset \widetilde{V}_0,$$

where $\widetilde{V}_2 = \mathbb{F}e \oplus \mathbb{F}e^*$, $\widetilde{V}_1 = \mathbb{F}e^*$, and $\widetilde{V}_0 = \{0\}$. The quadratic Lie superalgebra $(\widetilde{\mathfrak{g}} = \mathbb{F}e \oplus \mathfrak{g} \oplus \mathbb{F}e^*, \widetilde{B})$ is called the elementary odd double extension *of $(\mathfrak{g}, B)$ by the 1-dimensional Lie superalgebra $(\mathbb{F}e)_{\bar{1}}$ (by means of X_0).*

Remark 6.2. If we consider a Lie algebra $(\mathfrak{h}, [,]_{\mathfrak{h}})$ such that $[\mathfrak{h}, \mathfrak{h}]_{\mathfrak{h}} \neq \mathfrak{h}$ or $\mathfrak{z}(\mathfrak{h}) \neq \{0\}$ then $\mathfrak{g} := T^*\mathfrak{h} = \mathfrak{h} \oplus \mathfrak{h}^*$ is a quadratic Lie algebra such that $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$, because

$$\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{h}) \oplus \{f \in \mathfrak{h}^* : f([\mathfrak{h}, \mathfrak{h}]_{\mathfrak{h}}) = \{0\}\}.$$

For $X_0 \in \mathfrak{z}(\mathfrak{g})\backslash\{0\}$ we can construct a Lie superalgebra of filiform type. In particular, we can start from a non-zero solvabe Lie algebra $(\mathfrak{h}, [,]_{\mathfrak{h}})$.

Example 6.3. Consider in Remark 6.2 the 2-dimensional Lie algebra $\mathfrak{h} = \mathrm{vect}\{r, s\}$ such that $[r, s]_{\mathfrak{h}} = s$. Then $\mathfrak{g} := T^*\mathfrak{h} = \mathfrak{h} \oplus \mathfrak{h}^*$ is a quadratic Lie algebra equipped with the skew-symmetric bilinear map $[,]_{\mathfrak{g}}\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by

$$[r, s]_{\mathfrak{g}} = s,\ [r, s^*]_{\mathfrak{g}} = -s^* \text{ and } [s, s^*]_{\mathfrak{g}} = r^*.$$

As $\mathfrak{z}(\mathfrak{g}) = \langle r^* \rangle \neq \{0\}$ we can construct a Lie superalgebra of filiform type as follows. The 6-dimensional $\mathbb{Z}_2$-graded vector space $\widetilde{\mathfrak{g}} = \mathbb{F}e \oplus \mathfrak{g} \oplus \mathbb{F}e^*$ with the even skew-symmetric bilinear map $[,] : \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}} \longrightarrow \widetilde{\mathfrak{g}}$ defined by:
$[e, e] = r^*$, $[e^*, \widetilde{\mathfrak{g}}] = \{0\}$, $[e, r] = -e^*$, $[e, s] = 0$, $[e, r^*] = 0$, $[e, s^*] = 0$, $[r, s] = s$, $[r, s^*] = -s^*$ and $[s, s^*] = r^*$ and the supersymmetric bilinear form $\widetilde{B}\colon \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}} \longrightarrow \mathbb{F}$ defined by: $\widetilde{B}|_{\mathfrak{g}\times\mathfrak{g}} = B$, $\widetilde{B}(e^*, e) = 1$, $\widetilde{B}(\mathfrak{g}, e) = \widetilde{B}(\mathfrak{g}, e^*) = \{0\}$, $\widetilde{B}(e, e) = \widetilde{B}(e^*, e^*) = 0$ is a quadratic Lie superalgebra with $dim(\widetilde{\mathfrak{g}}_{\bar{1}}) = 2$, such that $\widetilde{\mathfrak{g}}_{\bar{1}}$ is a filiform $\widetilde{\mathfrak{g}}_{\bar{0}}$-module with the flag

$$\widetilde{\mathfrak{g}}_{\bar{1}} = \widetilde{V}_2 \supset \widetilde{V}_1 \supset \widetilde{V}_0,$$

where $\widetilde{V}_2 = \mathbb{F}e \oplus \mathbb{F}e^*$, $\widetilde{V}_1 = \mathbb{F}e^*$, and $\widetilde{V}_0 = \{0\}$.

Remark 6.4. Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic Lie superalgebra satisfying $\dim(\mathfrak{g}_{\bar{1}}) = m > 0$, such that $\mathfrak{g}_{\bar{1}}$ is a filiform $\mathfrak{g}_{\bar{0}}$-module with a flag

$$\mathfrak{g}_{\bar{1}} = V_m \supset \cdots \supset V_1 \supset V_0 .$$

We set $V_m/V_{m-1} = \mathbb{F}e_m$ and $V_1 = \mathbb{F}e_1$. Let D be an odd skew-supersymmetric superderivation of $(\mathfrak{g}, B)$. Under these conditions, if $e_m \in D(\mathfrak{g}_{\bar{0}})$, then we have $e_1 \notin [D(\mathfrak{g}_{\bar{0}})]^{\perp}$. Indeed, the result derives from the fact that $B(e_1, e_m) \neq 0$ together with the definition of odd skew-supersymmetric superderivation. Note also that as B is non-degenerate, we have $B(e_1, e_m) \neq 0$ as a consequence of Lemma 2.13.

The following theorem, which is one of the important results of this paper, allows us both to understand the structure of solvable quadratic Lie superalgebras with odd parts that are filiform modules on the even parts and to understand how to construct new quadratic Lie superalgebras of this type.

Theorem 6.5. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic Lie superalgebra satisfying $\dim(\mathfrak{g}_{\bar{1}}) = m > 0$ such that $\mathfrak{g}_{\bar{1}}$ is a filiform $\mathfrak{g}_{\bar{0}}$-module with a flag*

$$\mathfrak{g}_{\bar{1}} = V_m \supset \cdots \supset V_1 \supset V_0 .$$

Set $V_1 = \mathbb{F}e_1$ and $V_m/V_{m-1} = \mathbb{F}e_m$. Let D be an odd skew-supersymmetric superderivation of $(\mathfrak{g}, B)$, X_0 a non-zero element of $\mathfrak{g}_{\bar{0}}$ such that $D(X_0) = 0$, $B(X_0, X_0) = 0$, $D^2 = \frac{1}{2}[X_0, .]_{\mathfrak{g}}$ and $e_m \in D(\mathfrak{g}_{\bar{0}})$.
Then the generalized double extension $(\widetilde{\mathfrak{g}} = \mathbb{F}e \oplus \mathfrak{g} \oplus \mathbb{F}e^*, \widetilde{B})$ of $(\mathfrak{g}, B)$ by the 1-dimensional Lie superalgebra $(\mathbb{F}e)_{\bar{1}}$ (by means of the odd skew-supersymmetric superderivation D and X_0) is a quadratic Lie superalgebra with $\dim(\widetilde{\mathfrak{g}}_{\bar{1}}) = m+2 > 0$, such that $\widetilde{\mathfrak{g}}_{\bar{1}}$ is a filiform $\widetilde{\mathfrak{g}}_{\bar{0}}$-module with the flag defined by:

$$\widetilde{\mathfrak{g}}_{\bar{1}} = \widetilde{V}_{m+2} \supset \cdots \supset \widetilde{V}_1 \supset \widetilde{V}_0,$$

where $\widetilde{V}_{m+2} = \mathbb{F}e \oplus V_m \oplus \mathbb{F}e^*$, $\widetilde{V}_i = V_{i-1} \oplus \mathbb{F}e^*$, for $1 \leq i \leq m+1$, and $\widetilde{V}_0 = \{0\}$.
The product of $\widetilde{\mathfrak{g}}$ is defined by $[e, e] = X_0$, $[e^*, \widetilde{\mathfrak{g}}] = \{0\}$, $[e, X] = D(X) - B(X, X_0)e^*$, $[X, Y] = [X, Y]_{\mathfrak{g}} - B(D(X), Y)e^*$, for all $X, Y \in \mathfrak{g}$, and the supersymmetric bilinear form $\widetilde{B} \colon \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}} \longrightarrow \mathbb{F}$ is defined by $\widetilde{B}|_{\mathfrak{g}\times\mathfrak{g}} = B$, $\widetilde{B}(e^*, e) = 1$, $\widetilde{B}(\mathfrak{g}, e) = \widetilde{B}(\mathfrak{g}, e^*) = \{0\}$, $\widetilde{B}(e, e) = \widetilde{B}(e^*, e^*) = 0$. The quadratic Lie superalgebra $(\widetilde{\mathfrak{g}} = \mathbb{F}e \oplus \mathfrak{g} \oplus \mathbb{F}e^*, \widetilde{B})$ is called an odd double extension of $(\mathfrak{g}, B)$ by the 1-dimensional Lie superalgebra $(\mathbb{F}e)_{\bar{1}}$ (by means of the odd skew-supersymmetric superderivation D and X_0).

Remark 6.6. Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic Lie superalgebra satisfying $\dim(\mathfrak{g}_{\bar{1}}) = m > 0$, such that $\mathfrak{g}_{\bar{1}}$ is a filiform $\mathfrak{g}_{\bar{0}}$-module with a flag

$$\mathfrak{g}_{\bar{1}} = V_m \supset \cdots \supset V_1 \supset V_0 .$$

We set $V_m/V_{m-1} = \mathbb{F}e_m$ and $V_1 = \mathbb{F}e_1$. Let D be an odd skew-supersymmetric superderivation of $(\mathfrak{g}, B)$. Under these conditions, if $e_m \in D(\mathfrak{g}_{\bar{0}})$, then we have $e_1 \notin [D(\mathfrak{g}_{\bar{0}})]^{\perp}$. Indeed, the result derives from the fact that $B(e_1, e_m) \neq 0$ together with the definition of odd skew-supersymmetric superderivation. Note also that as B is non-degenerate, we have $B(e_1, e_m) \neq 0$ as a consequence of Lemma 2.13.

Remark 6.7. One condition of the above theorem is $e_m \in D(\mathfrak{g}_{\bar 0})$. By Remark 6.6 we have $e_1 \notin [D(\mathfrak{g}_{\bar 0})]^{\perp}$. Now we show that $e_1 \notin [D(\mathfrak{g}_{\bar 0})]^{\perp}$ if and only if $D(e_1) \neq 0$.

More generally, suppose $Y = Y_{\bar 0} + Y_{\bar 1} \in [D(\mathfrak{g}_{\bar 0})]^{\perp}$, since D is skew-symmetrical and B is non-degenerate then

$$Y = Y_{\bar 0} + Y_{\bar 1} \in [D(\mathfrak{g}_{\bar 0})]^{\perp} \Leftrightarrow B(Y_{\bar 1}, D(X)) = 0, \text{ for all } X \in \mathfrak{g}_{\bar 0}$$

$$\Leftrightarrow B(D(Y_{\bar 1}), X) = 0, \text{ for all } X \in \mathfrak{g}_{\bar 0} \Leftrightarrow D(Y_{\bar 1}) = 0.$$

Proof of Theorem 6.5. By [1, Theorem 1.3] we know that $\widetilde{\mathfrak{g}} = \mathbb{F}e \oplus \mathfrak{g} \oplus \mathbb{F}e^*$ is a quadratic Lie superalgebra with $\dim(\widetilde{\mathfrak{g}}_{\bar 1}) = m+2 > 0$. It remains to prove that $\widetilde{\mathfrak{g}}_{\bar 1}$ is a filiform $\widetilde{\mathfrak{g}}_{\bar 0}$-module with the flag defined above. More precisely, we have to prove that $[\widetilde{\mathfrak{g}}_{\bar 0}, \widetilde{V}_i] = \widetilde{V}_{i-1}$, for any $i \in \{1, \dots, m+2\}$. As $V_0 = \{0\}$ and $e^* \in \mathfrak{z}(\widetilde{\mathfrak{g}})$, we have $[\mathfrak{g}_{\bar 0}, V_0 \oplus \mathbb{F}e^*] = 0$, so $[\widetilde{\mathfrak{g}}_{\bar 0}, \widetilde{V}_1] = \widetilde{V}_0$.

Now let us assume that $i \in \{2, \dots, m+1\}$. As $V_1 = \mathbb{F}e_1$, we also have that $e_1 \in V_1 \subset V_{i-1} \subset \widetilde{V}_i$. Since $e_m \in D(\mathfrak{g}_{\bar 0})$ then $e_1 \notin [D(\mathfrak{g}_{\bar 0})]^{\perp}$ by Remark 6.6, therefore there exists $t_0 \in \mathfrak{g}_{\bar 0}$ such that $B(D(t_0), e_1) \neq 0$. Since $[\mathfrak{g}_{\bar 0}, V_1]_{\mathfrak{g}} = V_0 = \{0\}$, applying the product on $\widetilde{\mathfrak{g}}$ to t_0 and e_1, we get

$$[t_0, e_1] = [t_0, e_1]_{\mathfrak{g}} - B(D(t_0), e_1)e^* = -B(D(t_0), e_1)e^*,$$

so $e^* \in [\mathfrak{g}_{\bar 0}, \widetilde{V}_i]$. Now let $v \in V_{i-2}$. We know that $v \in [\mathfrak{g}_{\bar 0}, V_{i-1}]_{\mathfrak{g}} = V_{i-2}$, so there exist $x_k \in \mathfrak{g}_{\bar 0}$ and $w_k \in V_{i-1}$, for $1 \leq k \leq p$, with some natural p, and applying the product on $\widetilde{\mathfrak{g}}$ we get

$$\begin{aligned} v &= \sum_{k=1}^{p} [x_k, w_k]_{\mathfrak{g}} = \sum_{k=1}^{p} \Big([x_k, w_k] + B(D(x_k), w_k)e^* \Big) \\ &= \sum_{k=1}^{p} [x_k, w_k] + \Big(\sum_{k=1}^{p} (B(D(x_k), w_k) \Big) e^*. \end{aligned}$$

As $e^* \in [\mathfrak{g}_{\bar 0}, \widetilde{V}_i]$ we have $v \in [\widetilde{\mathfrak{g}}_{\bar 0}, \widetilde{V}_i]$, that is $V_{i-2} \subseteq [\widetilde{\mathfrak{g}}_{\bar 0}, \widetilde{V}_i]$. As $\widetilde{V}_{i-1} = V_{i-2} \oplus \mathbb{F}e^*$, we conclude $\widetilde{V}_{i-1} \subseteq [\widetilde{\mathfrak{g}}_{\bar 0}, \widetilde{V}_i]$.

Let us show the other inclusion. For any $a \in \widetilde{V}_i$ we may write $a = b + \lambda e^*$, where $b \in V_{i-1}$ and $\lambda \in \mathbb{F}$. Let $X \in \mathfrak{g}_{\bar 0}$, as $e^* \in \mathfrak{z}(\widetilde{\mathfrak{g}})$,

$$[X, a] = [X, b] = [X, b]_{\mathfrak{g}} - B(D(X), b)e^*,$$

where $[X, b]_{\mathfrak{g}} \in V_{i-2}$, so $[\widetilde{\mathfrak{g}}_{\bar 0}, \widetilde{V}_i] \subseteq V_{i-2} \oplus \mathbb{F}e^* = \widetilde{V}_{i-1}$. In conclusion, $[\widetilde{\mathfrak{g}}_{\bar 0}, \widetilde{V}_i] = \widetilde{V}_{i-1}$.

Finally, let us consider $i = m+2$. Let us also set $V_i/V_{i-1} = \mathbb{F}e_i$ for $2 \leq i \leq m-1$. Therefore $V_i = span\{e_1, \dots e_i\}$. Since $\widetilde{V}_{m+2} = \mathbb{F}e \oplus V_m \oplus \mathbb{F}e^*$, $e^* \in \mathfrak{z}(\widetilde{\mathfrak{g}})$ and e^* always appears in the bracket product $[t_0, e_1]$, with $t_0 \in \widetilde{\mathfrak{g}}_{\bar 0}$, described before, then

$$V_{m-1} \oplus \mathbb{F}e^* \subset [\widetilde{\mathfrak{g}}_{\bar 0}, \mathbb{F}e \oplus V_m] = [\widetilde{\mathfrak{g}}_{\bar 0}, \widetilde{V}_{m+2}].$$

Now, on account of $e_m \in D(\mathfrak{g}_{\bar 0})$ there exists $t_1 \in \mathfrak{g}_{\bar 0}$ such that

$$[e, t_1] = D(t_1) - B(t_1, X_0)e^* = \sum_{k=1}^{m} c_k e_k - B(t_1, X_0)e^*, \quad \text{with } c_m \neq 0,$$

then $e_m \in [\widetilde{\mathfrak{g}}_{\bar 0}, \mathbb{F}e \oplus V_m]$.

As $\widetilde{V}_{m+1} = V_m \oplus \mathbb{F}e^*$, we show that $\widetilde{V}_{m+1} \subset [\widetilde{\mathfrak{g}}_{\bar{0}}, \widetilde{V}_{m+2}]$.

It remains to prove that $[\widetilde{\mathfrak{g}}_{\bar{0}}, \widetilde{V}_{m+2}] \subset \widetilde{V}_{m+1}$. For any $a \in \widetilde{V}_{m+2}$, we may write $a = \mu e + b + \lambda e^*$, where $b \in V_m$ and $\mu, \lambda \in \mathbb{F}$. Let $X \in \mathfrak{g}_{\bar{0}}$, as $e^* \in \mathfrak{z}(\widetilde{\mathfrak{g}})$,

$$[X, a] = [X, \mu e + b] = \mu(D(X) - B(X, X_0)e^*) + [X, b]_{\mathfrak{g}} - B(D(X), b)e^*.$$

As $[X, b]_{\mathfrak{g}} \in V_{m-1}$ and D is an odd skew-supersymmetric superderivation of $(\mathfrak{g}, B)$ then $[\widetilde{\mathfrak{g}}_{\bar{0}}, \widetilde{V}_{m+2}] \subseteq V_m \oplus \mathbb{F}e^* = \widetilde{V}_{m+1}$, which concludes the proof of the theorem. $\square$

Theorem 6.8. *Suppose that* $(\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_{\bar{0}} \oplus \widetilde{\mathfrak{g}}_{\bar{1}}, \widetilde{B})$ *is a quadratic Lie superalgebra of a filiform type. Then* $(\widetilde{\mathfrak{g}}, \widetilde{B})$ *is an odd double extension of a quadratic Lie superalgebra* $(\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}, B)$ *of a filiform type (*$\dim \mathfrak{h} = \dim \widetilde{\mathfrak{g}} - 2$*) by a* 1*-dimensional Lie superalgebra with null even part.*

Proof. Let us assume that $(\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_{\bar{0}} \oplus \widetilde{\mathfrak{g}}_{\bar{1}}, \widetilde{B})$ is a quadratic Lie superalgebra such as $\widetilde{\mathfrak{g}}_{\bar{1}}$ has the structure of filiform $\widetilde{\mathfrak{g}}_{\bar{0}}$-module with the flag

$$\widetilde{\mathfrak{g}}_{\bar{1}} = V_m \supset V_{m-1} \supset \cdots \supset V_1 \supset V_0,$$

and we denote $V_1 = \mathbb{F}e_1$. By Lemma 2.12, we know that $V_1 \subseteq \mathfrak{z}(\widetilde{\mathfrak{g}}) \cap \widetilde{\mathfrak{g}}_{\bar{1}}$. By Lemma 2.13, we have $\widetilde{B}(\widetilde{\mathfrak{g}}_{\bar{0}} \oplus V_{m-1}, e_1) = \{0\}$ and as $\widetilde{B}$ is nondegenerate, then there exists $e_m \in V_m/V_{m-1}$ such that $\widetilde{B}(e_1, e_m) \neq 0$. We may assume that $\widetilde{B}(e_1, e_m) = 1$ (we recall that as $\widetilde{B}|_{\widetilde{\mathfrak{g}}_{\bar{1}} \times \widetilde{\mathfrak{g}}_{\bar{1}}}$ is skew-symmetric then $\widetilde{B}(e_m, e_m) = 0$ and $\widetilde{B}(e_1, e_1) = 0$). Let us consider $\mathfrak{h} := (\mathbb{F}e_1 \oplus \mathbb{F}e_m)^{\perp}$, that is, $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ is the $\mathbb{Z}_2$-graded vector space orthogonal to $\mathbb{F}e_1 \oplus \mathbb{F}e_m$ with respect to $\widetilde{B}$. It comes that $B := \widetilde{B}\,|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate. Then $\widetilde{\mathfrak{g}}_{\bar{0}} \subseteq \mathfrak{h}$ and $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_{\bar{0}} \oplus \big(\mathbb{F}e_m \oplus \mathfrak{h}_{\bar{1}} \oplus \mathbb{F}e_1\big)$, because $\widetilde{\mathfrak{g}} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$.

First we show that $(\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_{\bar{0}} \oplus \widetilde{\mathfrak{g}}_{\bar{1}}, \widetilde{B})$ is an odd double extension of the quadratic Lie superalgebra $(\mathfrak{h} = \widetilde{\mathfrak{g}}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}, [,]_{\mathfrak{h}}, B)$ by a 1-dimensional Lie superalgebra with null even part. We remark that $(\mathbb{F}e_1)^{\perp} = \widetilde{\mathfrak{g}}_{\bar{0}} \oplus \big(\mathfrak{h}_{\bar{1}} \oplus \mathbb{F}e_1\big)$, $\mathfrak{h} = (\mathbb{F}e_1)^{\perp}/\mathbb{F}e_1$ and $(\mathbb{F}e_1)^{\perp} = \mathbb{F}e_1 \oplus (\mathbb{F}e_1 \oplus \mathbb{F}e_m)^{\perp}$.

If $X, Y \in \mathfrak{h}$ we have

$$[X, Y] = [X, Y]_{\mathfrak{h}} + \phi(X, Y)e_m + \varphi(X, Y)e_1,$$

where $[X, Y]_{\mathfrak{h}} \in \mathfrak{h}$ and $\phi(X, Y), \varphi(X, Y) \in \mathbb{F}$. Further, for all $X \in \mathfrak{h}$,

$$[e_m, X] = D(X) + \alpha(X)e_m + \beta(X)e_1,$$

where $D(X) \in \mathfrak{h}$ and $\alpha(X), \beta(X) \in \mathbb{F}$. More, $[e_m, e_m] = X_0 \in \widetilde{\mathfrak{g}}_{\bar{0}} = \mathfrak{h}_{\bar{0}}$.

Claim 1. Then $(\mathfrak{h}, [,]_{\mathfrak{h}}, B)$ is a quadratic Lie superalgebra.

Proof of Claim 1: The graded skew-symmetry on $\widetilde{\mathfrak{g}}$ implies

$$\begin{aligned} [X, Y]_{\mathfrak{h}} &= -(-1)^{\overline{xy}}[Y, X]_{\mathfrak{h}}, \\ \varphi(X, Y) &= -(-1)^{\overline{xy}}\varphi(Y, X), \\ \phi(X, Y) &= -(-1)^{\overline{xy}}\phi(Y, X), \end{aligned} \tag{1}$$

for any $X \in \mathfrak{h}_{\overline{x}}$ and $Y \in \mathfrak{h}_{\overline{y}}$.

Moreover, by the graded Jacobi identity on $\widetilde{\mathfrak{g}}$ it follows that

$$\begin{aligned}(-1)^{\overline{xz}}[X,[Y,Z]_{\mathfrak{h}}]_{\mathfrak{h}}+(-1)^{\overline{xy}}[Y,[Z,X]_{\mathfrak{h}}]_{\mathfrak{h}}+(-1)^{\overline{yz}}[Z,[X,Y]_{\mathfrak{h}}]_{\mathfrak{h}} &= 0,\\ (-1)^{\overline{xz}}\varphi\big(X,\varphi(Y,Z)\big)+(-1)^{\overline{xy}}\varphi\big(Y,\varphi(Z,X)\big)+(-1)^{\overline{yz}}\varphi\big(Z,\varphi(X,Y)\big) &= 0, \quad (2)\\ (-1)^{\overline{xz}}\phi\big(X,\phi(Y,Z)\big)+(-1)^{\overline{xy}}\phi\big(Y,\phi(Z,X)\big)+(-1)^{\overline{yz}}\phi\big(Z,\phi(X,Y)\big) &= 0,\end{aligned}$$

for $X\in\mathfrak{h}_{\overline{x}}, Y\in\mathfrak{h}_{\overline{y}}$, and $Z\in\mathfrak{h}_{\overline{z}}$. By (1) and (2) we conclude that $(\mathfrak{h},[,]_{\mathfrak{h}})$ is a Lie superalgebra. By invariance of $\widetilde{B}$ in $\widetilde{\mathfrak{g}}$ it comes straightforward that

$$B\big([X,Y]_{\mathfrak{h}},Z\big) = B\big(X,[Y,Z]_{\mathfrak{h}}\big), \qquad \forall X,Y,Z\in\mathfrak{h},$$

which means that B is invariant in $\mathfrak{h}$. Therefore it is immediate that B is an odd-invariant scalar product on $\mathfrak{h}$ which shows Claim 1.
As $\widetilde{B}$ is invariant and $e_1\in\mathfrak{z}(\widetilde{\mathfrak{g}})$, for all $X\in\mathfrak{h}$

$$-\alpha(X)=\widetilde{B}(D(X)+\alpha(X)e_m+\beta(X)e_1,e_1)=\widetilde{B}([e_m,X],e_1)=\widetilde{B}(e_m,[X,e_1])=0,$$

so for all $X\in\mathfrak{h}$ we have $\alpha(X)=0$. Similarly, for all $X,Y\in\mathfrak{h}$ we get $\phi(X,Y)=0$. Using the invariance of $\widetilde{B}$, for all $X\in\mathfrak{h}_{\overline{x}}$

$$\begin{aligned}\beta(X)&=\widetilde{B}(D(X)+\beta(X)e_1,e_m)=\widetilde{B}([e_m,X],e_m)\\&=-(-1)^{\overline{x}}\widetilde{B}(X,[e_m,e_m])=-(-1)^{\overline{x}}\widetilde{B}(X,X_0),\end{aligned}$$

Claim 2. D is an odd skew-supersymmetric superderivation of $(\mathfrak{h},B)$ such that

$$D(X_0)=0 \ \text{ and } \ D^2=\frac{1}{2}[X_0,.]_{\mathfrak{h}}.$$

Moreover, $(\widetilde{\mathfrak{g}}=\widetilde{\mathfrak{g}}_{\bar 0}\oplus\widetilde{\mathfrak{g}}_{\bar 1},\widetilde{B})$ is an odd double extension of the quadratic Lie superalgebra $(\mathfrak{h}=\widetilde{\mathfrak{g}}_{\bar 0}\oplus\mathfrak{h}_{\bar 1},[,]_{\mathfrak{h}},B)$ by the 1-dimensional Lie superalgebra $(\mathbb{F}e_m)_{\bar 1}$ (by means of D and X_0).

Proof of Claim 2: We start by proving that D is an odd skew-supersymmetric superderivation of $(\mathfrak{h},B)$. It is immediate that D is a homogeneous linear map of degree $\bar 1$. Using the graded Jacobi identity

$$(-1)^{\overline{y}}[e_m,[X,Y]]+(-1)^{\overline{x}}[X,[Y,e_m]]+(-1)^{\overline{xy}}[Y,[e_m,X]] = 0,$$

we arrive at

$$\begin{aligned}D\big([X,Y]_{\mathfrak{h}}\big) &= [D(X),Y]_{\mathfrak{h}}+(-1)^{\overline{x}}[X,D(Y)]_{\mathfrak{h}}, \quad (3)\\ \varphi\big([X,Y]_{\mathfrak{h}}\big) &= \varphi\big(D(X),Y\big)+(-1)^{\overline{x}}\varphi\big(X,D(Y)\big),\end{aligned}$$

for any $X\in\mathfrak{h}_{\overline{x}}$ and $Y\in\mathfrak{h}_{\overline{y}}$. From (3) we say that $D\in\big(Der(\mathfrak{h})\big)_{\bar 1}$. Using the invariance of $\widetilde{B}$,

from $$\widetilde{B}([e_m,X],Y)=-(-1)^{x}\widetilde{B}(X,[e_m,Y]),$$

we obtain $$B\big(D(X),Y\big)=-(-1)^{x}B\big(X,D(Y)\big),$$

for any $X\in\mathfrak{h}_{\overline{x}}$ and $Y\in\mathfrak{h}$, which means that $D\in\big(Der_a(\mathfrak{h},B)\big)_{\bar 1}$.

From $$\widetilde{B}([e_m, X], Y) = \widetilde{B}(e_m, [X, Y]),$$

we infer that $$\varphi(X, Y) = -B(D(X), Y),$$

for any $X, Y \in \mathfrak{h}$. From graded Jacobi identity $[e_m, [e_m, e_m]] = 0$ we have $D(X_0) = 0$.

Finally $$(-1)^{\overline{x}}[e_m, [e_m, X]] - [e_m, [X, e_m]] + (-1)^{\overline{x}}[X, [e_m, e_m]] = 0,$$

leads to $$D^2(X) = \frac{1}{2}[X_0, X]_{\mathfrak{h}},$$

for any $X \in \mathfrak{h}_{\overline{x}}$, which we extend by linearity to all $\mathfrak{h}$, completing the proof of Claim 2. We proved that $(\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_{\bar{0}} \oplus \widetilde{\mathfrak{g}}_{\bar{1}}, \widetilde{B})$ is an odd double extension of the quadratic Lie superalgebra $(\mathfrak{h} = \widetilde{\mathfrak{g}}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}, [,]_{\mathfrak{h}}, B)$ by a 1-dimensional Lie superalgebra with null even part.

Now we construct on $\mathfrak{h}_{\bar{1}}$ a structure of filiform $\mathfrak{h}_{\bar{0}}$-module. Recall that $\widetilde{\mathfrak{g}}_{\bar{1}}$ has the structure of filiform $\widetilde{\mathfrak{g}}_{\bar{0}}$-module with the flag

$$\widetilde{\mathfrak{g}}_{\bar{1}} = V_m \supset V_{m-1} \supset \cdots \supset V_1 \supset V_0,$$

$V_1 = \mathbb{F}e_1$ and $V_m/V_{m-1} = \mathbb{F}e_m$, here $[\widetilde{\mathfrak{g}}_{\bar{0}}, V_{i+1}] = V_i$, for $i \in \{m-1, \dots 0\}$. For $1 \le i \le m-1$, we may write $V_i = U_{i-1} \oplus \mathbb{F}e_1$, where U_j is a j-dimensional vector space, for $j \in \{0, \dots m-2\}$, such that $\mathfrak{h}_{\bar{1}} \supset U_{m-2} \supset \cdots \supset U_1 \supset U_0$, because $V_{m-1} \subseteq (\mathbb{F}e_1)^{\perp}$. As $e_m \notin (\mathbb{F}e_1)^{\perp}$, $e_m \in V_m$ and $V_{m-1} \subseteq V_m$ then $V_m = \mathbb{F}e_m \oplus V_{m-1} = \mathbb{F}e_m \oplus U_{m-2} \oplus \mathbb{F}e_1$. We prove that there exists a decreasing subsequence of vector subspaces $U_{m-2} \supset \cdots \supset U_1 \supset U_0$ in underlying vector space $\mathfrak{h}_{\bar{1}}$ such that

$$\begin{aligned}
V_1 &= U_0 \oplus \mathbb{F}e_1 \\
V_2 &= U_1 \oplus \mathbb{F}e_1 \\
&\vdots \\
V_{m-1} &= U_{m-2} \oplus \mathbb{F}e_1 \\
V_m &= \mathbb{F}e_m \oplus U_{m-2} \oplus \mathbb{F}e_1.
\end{aligned}$$

Let us now show that $[\mathfrak{h}_{\bar{0}}, U_i]_{\mathfrak{h}} = U_{i-1}$, for $i \in \{m-2, \dots 1\}$. On one hand, as $e_1 \in \mathfrak{z}(\widetilde{\mathfrak{g}})$ we have

$$U_{i-1} \oplus \mathbb{F}e_1 = V_i = [\widetilde{\mathfrak{g}}_{\bar{0}}, V_{i+1}] = [\widetilde{\mathfrak{g}}_{\bar{0}}, U_i \oplus \mathbb{F}e_1] = [\widetilde{\mathfrak{g}}_{\bar{0}}, U_i] \subseteq [\mathfrak{h}_{\bar{0}}, U_i]_{\mathfrak{h}} \oplus \mathbb{F}e_1,$$

thus $U_{i-1} \subseteq [\mathfrak{h}_{\bar{0}}, U_i]_{\mathfrak{h}}$. On the other hand, from $[X, Y] = [X, Y]_{\mathfrak{h}} - B(D(X), Y)e_1$, for all $X \in \widetilde{\mathfrak{g}}_{\bar{0}}, Y \in V_{i+1}$, we have that $e_1 \in [\widetilde{\mathfrak{g}}_{\bar{0}}, V_{i+1}]$ and as $e_1 \in \mathfrak{z}(\widetilde{\mathfrak{g}})$

$$[\mathfrak{h}_{\bar{0}}, U_i]_{\mathfrak{h}} \subseteq [\mathfrak{h}_{\bar{0}}, U_i] = [\mathfrak{g}_{\bar{0}}, V_{i+1}] = V_i = U_{i-1} \oplus \mathbb{F}e_1$$

hence $[\mathfrak{h}_{\bar{0}}, U_i]_{\mathfrak{h}} \subseteq U_{i-1}$. Therefore $[\mathfrak{h}_{\bar{0}}, U_i]_{\mathfrak{h}} = U_{i-1}$.

We conclude that $(\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_{\bar{0}} \oplus \widetilde{\mathfrak{g}}_{\bar{1}}, \widetilde{B})$ is an odd double extension of the quadratic Lie superalgebra $(\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}, [,]_{\mathfrak{h}}, B)$ of a filiform type $(\dim \mathfrak{h} = \dim \mathfrak{g} - 2)$ by a 1-dimensional Lie superalgebra with null even part. ∎

In particular, the converse of Theorem 6.1 is a consequence of Theorem 6.8.

Corollary 6.9. *Suppose that* $(\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_{\bar 0} \oplus \widetilde{\mathfrak{g}}_{\bar 1}, \widetilde{B})$ *is a quadratic Lie superalgebra of a filiform type such that* $\dim \widetilde{\mathfrak{g}}_{\bar 1} = 2$*. Then* $(\widetilde{\mathfrak{g}}, \widetilde{B})$ *is an elementary odd double extension of a quadratic Lie algebra* $(\mathfrak{g}, B)$ *with non trivial center* $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$ *(*$\dim \mathfrak{g} = \dim \widetilde{\mathfrak{g}} - 2$*) by a* 1*-dimensional Lie superalgebra with null even part.*

Finally, we will present an inductive description of quadratic Lie superalgebras of a filiform type.

Theorem 6.10. *Let* $(\mathfrak{g}, B)$ *be a quadratic Lie superalgebra of a filiform type. Then* $\mathfrak{g}$ *is obtained by a finite sequence of double extensions of a quadratic Lie superalgebra of a filiform type by the* 1*-dimensional Lie superalgebra* $(\mathbb{F}e)_{\bar 1}$*, and the last one is an elementary odd double extension of a quadratic Lie algebra by the* 1*-dimensional Lie superalgebra* $(\mathbb{F}e)_{\bar 1}$*.*

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