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On Gradations of Decomposable Kac-Moody Lie Algebras by Kac-Moody Root Systems

Hechmi Ben Messaoud and Marwa Layouni

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Abstract. We are interested in the gradations of symmetrizable Kac-Moody Lie algebras $\mathfrak{g}$ by root systems Σ of Kac-Moody type. We first show that we can reduce to the case where the grading root system Σ is indecomposable. If the graded Kac-Moody Lie algebra $\mathfrak{g}$ is decomposable, then any indecomposable component of $\mathfrak{g}$ is either fictive (and contributes little to the gradation) or effective (and essentially Σ-graded). Based on work by G. Rousseau and the first-named author, we extend most of the results on finite gradations to the gradations of $\mathfrak{g}$ admitting adapted root bases. Namely, it is shown that, for such a gradation, there exists a regular standard Kac-Moody-subalgebra $\mathfrak{g}(I_{re})$ of $\mathfrak{g}$ containing the grading Kac-Moody Lie subalgebra $\mathfrak{m}$ and which is finitely really Σ-graded. This enables us to investigate the structure of the Weyl group and the Tits cone of the grading Kac-Moody Lie subalgebra $\mathfrak{m}$ in comparison with those of the graded Kac-Moody Lie algebra $\mathfrak{g}$ and to prove a conjugacy theorem on adapted pairs of root bases. We end the paper by providing a unified construction for the finite imaginary gradations of $\mathfrak{g}$.

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Key Words: Kac-Moody Lie algebra, gradation by a Kac-Moody root system, C-admissible pair.

1. Introduction

The concept of a complex Lie algebra $\mathfrak{g}$ graded by a finite irreducible reduced root system Σ was introduced by S. Berman and R. Moody [13], in 1992, where they classified, up to central isogeny, perfect Lie algebras graded by finite simply-laced root systems. Since then, many authors have been interested in the subject to extend the classification to the non simply-laced case (see [1], [11], [12], ...). In [20], E. Neher classified, in a unified way, perfect Lie algebras graded by 3-graded root systems (excluding root systems of type E_8, F_4 and G_2) in terms of Tits-Kantor-Koecher algebras in connection to Jordan pairs. This notion of root gradations was extended, in a topological setting, by K. H. Neeb [19] to locally convex Lie algebras graded by (infinite) locally finite root systems. This notion was also extended to the super setting by G. Benkart and A. Elduque ([8], [9] and [10]) where they studied Lie superalgebras graded by finite root supersystems. In [21], J. Nervi gave a complete classification of all simple finite-dimensional Lie algebras graded by a finite root system, she extended the classification in [22] to affine Kac-Moody Lie algebras graded by affine root systems. In her two articles she used the notion of C-admissible pairs in simple Lie algebras as introduced by H. Rubenthaler in the classification of

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dual pairs and that of prehomogeneous spaces of parabolic type ([29], [30]). In the Kac-Moody setting, G. Rousseau and the first author ([7]) have studied finite gradations of indecomposable Kac-Moody Lie algebras by Kac-Moody root systems. Following them, a Kac-Moody Lie algebra $\mathfrak{g}$ is said to be graded by a root system Σ of Kac-Moody type (or Σ-graded) if:

(Σ-G1) : The algebra $\mathfrak{g}$ contains a Kac-Moody subalgebra $\mathfrak{m}$ whose root system, relative to a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$, is Σ.

(Σ-G2) : The Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ is $\mathrm{ad}_{\mathfrak{g}}$-diagonalizable and

$$\mathfrak{g} = \bigoplus_{\bar{\alpha}\in\Sigma\cup\{0\}} V_{\bar{\alpha}},$$

with $V_{\bar{\alpha}} = \{x \in \mathfrak{g}; [a, x] = \bar{\alpha}(a)x,\ \forall a \in \mathfrak{a}\}$.

(Σ-FG) : If moreover the weight spaces $V_{\bar{\alpha}}$, $\bar{\alpha} \in \Sigma\cup\{0\}$, are all of finite dimension, we say that $\mathfrak{g}$ is finitely Σ-graded (or that the gradation is finite).

Thus, the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ is contained in a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and we denote by $\Delta := \Delta(\mathfrak{g}, \mathfrak{h})$ the corresponding root system. A root basis Π of Δ is said to be adapted to a fixed root basis $\Pi_{\mathfrak{a}}$ of Σ if, for any positive root $\alpha \in \Delta^+$ relative to Π, its restriction $\rho_{\mathfrak{a}}(\alpha)$ to $\mathfrak{a}$ is either zero or a positive root of Σ^+ relative to $\Pi_{\mathfrak{a}}$, in which case we say also that $(\Pi_{\mathfrak{a}}, \Pi)$ is an adapted pair for the gradation. If moreover, any simple root of Π, with nonzero restriction to $\mathfrak{a}$, is restricted in a simple root of $\Pi_{\mathfrak{a}}$, the gradation is said to be real (or $\mathfrak{g}$ is really Σ-graded). According to [7], finite gradations of an indecomposable Kac-Moody Lie algebra $\mathfrak{g}$ always possess adapted bases. Unlike the finite and the affine cases, Kac-Moody Lie algebras of indefinite type (and non hyperbolic) may possess imaginary gradations for which some simple roots of the graded algebra $\mathfrak{g}$ may be restricted in imaginary roots of the grading algebra $\mathfrak{m}$. As it was shown in [7], an indecomposable Kac-Moody Lie algebra having a finite gradation by a Kac-Moody root system Σ possesses a really Σ-graded standard Kac-Moody subalgebra which may be decomposable. Our focus here is on general gradations of decomposable Kac-Moody Lie algebras by Kac-Moody root systems. We characterize gradations having adapted root bases and add some structural results on (real or imaginary) gradations. The paper is organized as follows:

In the second section, we recall the basic definitions and facts on the structure of Kac-Moody Lie algebras and fix the notations.

In the third section, we present some results on the gradations of general Kac-Moody Lie algebras $\mathfrak{g}$ by Kac-Moody root systems Σ with weight spaces of arbitrary dimension. We prove that one can reduce to the case where Σ is indecomposable [Proposition 3.7] and then, if $\mathfrak{g}$ is decomposable, any indecomposable component of $\mathfrak{g}$ is either fictive (and contributes at most to the gradation only by imaginary roots) or effective (and essentially Σ-graded) [Proposition 3.11]. We characterize in particular the Kac-Moody Lie algebras graded by root systems of finite or affine type.

The fourth section is devoted to gradations admitting adapted pairs of root bases. Most of the results stated in [7] for finite gradations extend to the aforementioned setting with adapted pairs:

Theorem 1.1. (Propositions 4.3, 4.8, 4.11, 4.14) *Suppose that the Kac-Moody Lie algebra* $\mathfrak{g}$ *is graded by an indecomposable root system* Σ. *Let* $\Pi_{\mathfrak{a}} = \{\gamma_s, s \in \bar{I}\}$ *be a root basis of* Σ *and suppose that the root basis* $\Pi = \{\alpha_i, i \in I\}$ *of* Δ *is adapted to* $\Pi_{\mathfrak{a}}$. *Define:* $J := \{j \in I; \rho_{\mathfrak{a}}(\alpha_j) = 0\}$, $I' := I \setminus J$, $I'_{re} := \{k \in I'; \rho_{\mathfrak{a}}(\alpha_k) \in \Pi_a\}$, $I'_{im} := I' \setminus I'_{re}$. *For* $k \in I'$, *let* I_k *be the connected component of* $J \cup \{k\}$ *containing* k *and* $J_k := I_k \setminus \{k\}$. *Let* $I_{re} := \bigcup_{k \in I'_{re}} I_k$ *and* $J_{re} := \bigcup_{k \in I'_{re}} J_k$, *then we have:*

(1) $\rho_{\mathfrak{a}}^{-1}(\{0\}) \cap \Delta = \Delta_J$ *and the gradation is finite if and only if* J *is of finite type.*

(2) (I_{re}, J_{re}) *is a* C*-admissible pair and the standard regular subalgebra* $\mathfrak{g}(I_{re})$ *can be realized so that it contains the grading subalgebra* $\mathfrak{m}$ *and is then finitely* Σ*-graded.*

(3) *If* $I'_{im} \neq \emptyset$, *then* $\rho_{\mathfrak{a}}(\alpha_i) \in \Sigma^+_{im}$, $\forall i \in I'_{im}$. *In which case the gradation is called* imaginary.

Thus, we get the following characterization for the finite gradations of $\mathfrak{g}$:

Theorem 1.2. (Theorem 4.19) *Suppose the Kac-Moody algebra* $\mathfrak{g}$ *graded by an indecomposable root system* Σ. *The following assertions are equivalent:*

(1) *The Kac-Moody algebra* $\mathfrak{g}$ *is finitely* Σ*-graded,*

(2) V_0 *is finite dimensional,*

(3) $\rho_{\mathfrak{a}}(\Delta_{im}) \subset \Sigma_{im}$,

(4) *Any root basis* $\Pi_{\mathfrak{a}}$ *of* Σ *has an adapted root basis* $\Pi = \{\alpha_i, i \in I\}$ *of* Δ *such that* $V_0 = \mathfrak{h} + \mathfrak{g}_J$ *for a finite type subset* J *of* I.

Motivated by a desire to better understand the structure of the Weyl group $W_{\mathfrak{a}}$ and the Tits cone $X_{\mathfrak{a}}$ of the grading subalgebra $\mathfrak{m}$ in comparison with those of $\mathfrak{g}$, we prove [Theorem 4.30] that if the gradation possesses adapted pairs, then the Weyl group $W_{\mathfrak{a}}$ can be imbedded in the Weyl group W of $\mathfrak{g}$. This allows us to describe the stabilizer, in W, of the grading Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ as the semi-direct product $W_{\mathfrak{a}} \ltimes W_J$ of its fixator W_J and the Weyl group $W_{\mathfrak{a}}$ viewed as a subgroup of W. As a consequence, the Tits cone $X_{\mathfrak{a}}$ of the grading subalgebra $\mathfrak{m}$ can be viewed as a subcone of the Tits cone X of $\mathfrak{g}$. We end the section by a conjuguacy theorem on adapted pairs:

Theorem 1.3. (Theorem 4.38) *Suppose that the Kac-Moody Lie algebra* $\mathfrak{g}$ *is graded by an indecomposable root system* Σ *and the root basis* Π *of* Δ *is adapted to the root basis* $\Pi_{\mathfrak{a}}$ *of* Σ. *Suppose that* $\Delta_J := \rho_{\mathfrak{a}}^{-1}(\{0\})$ *does not contain any infinite fictive factor of the gradation. Let* $Stab_W(\mathfrak{a}) = W_{\mathfrak{a}} \ltimes W_J$ *be the stabilizer of* $\mathfrak{a}$ *in the Weyl group* W *of* Δ. *Then the group* $Stab_W(\mathfrak{a})$ *acts on the adapted pairs* $(\Pi'_{\mathfrak{a}}, \Pi')$, *where* $\Pi'_{\mathfrak{a}}$ *is a root basis of* Σ *and* Π' *is a root basis of* Δ *adapted to* $\Pi'_{\mathfrak{a}}$, *and any adapted pair* $(\Pi'_{\mathfrak{a}}, \Pi')$ *is* $Stab_W(\mathfrak{a})$*-conjugate to* $(\Pi_{\mathfrak{a}}, \Pi)$ *or* $(-\Pi_{\mathfrak{a}}, -\Pi)$. *Moreover, the adapted pairs* $(\Pi_{\mathfrak{a}}, \Pi)$ *and* $(-\Pi_{\mathfrak{a}}, -\Pi)$ *are* $Stab_W(\mathfrak{a})$*-conjugate if and only if* Σ *is of finite type.*

It was shown in [7] that finite real gradations are entirely classified in terms of C-admissible pairs and maximal gradations. We return, in the fifth section, to finite gradations to provide a unified construction of finite imaginary gradations and show that they can be realized as follows:

Theorem 1.4. (Theorem 5.1) *Suppose that there exists a proper subset* $I_{re} \subset I$ *such that the standard regular Kac-Moody subalgebra* $\mathfrak{g}(I_{re})$ *of* $\mathfrak{g}$ *is finitely really graded by an indecomposable Kac-Moody root system* Σ. *Let* $\mathfrak{m}$ *be the grading subalgebra of* $\mathfrak{g}(I_{re})$ *and* $\mathfrak{a}$ *be a Cartan subalgebra of* $\mathfrak{m}$ *contained in* $\mathfrak{h}(I_{re})$. *Let* $\Pi_{\mathfrak{a}} = \{\gamma_s, s \in \bar{I}\}$ *be a root basis of* $\Sigma = \Delta(\mathfrak{m}, \mathfrak{a})$ *and suppose that the standard root basis* $\Pi_{I_{re}}$ *of* $\Delta(I_{re})$ *is adapted to* $\Pi_{\mathfrak{a}}$. *Define:* $J := \{j \in I; \rho_{\mathfrak{a}}(\alpha_j) = 0\}$, $J_{re} := J \cap I_{re}$, $I' := I \setminus J$, $I'_{re} := I_{re} \cap I'$, $I'_{im} := I' \setminus I'_{re}$.

Suppose $I'_{im} \neq \emptyset$ *and add the following conditions:*

(FIG1) *The subset* J *is either empty or of finite type.*

(FIG2) *For any* $k \in I'_{im}$, $\bar{\alpha}_k := \rho_{\mathfrak{a}}(\alpha_k) \in \Sigma^+_{im}$.

(FIG3) *If* $k \neq l \in I'_{im}$ *are* J*-linked, then* $\bar{\alpha}_k + \bar{\alpha}_l \in \Sigma^+_{im}$.

Then the Kac-Moody Lie algebra $\mathfrak{g}$ *is finitely* Σ*-graded and the gradation is imaginary. The standard root basis* Π *of* Δ *is adapted to the root basis* $\Pi_{\mathfrak{a}}$ *and* Δ_J *is the annihilator of* $\mathfrak{a}$ *in* Δ.

We end the section by an application to the finite imaginary gradations by hyperbolic root systems.

2. Preliminaries

We recall the basic definitions and facts on the structure of Kac-Moody Lie algebras; for more details, we refer to the book of Kac [15].

2.1. Generalized Cartan matrices

1. Let $A = (a_{ij})_{i,j\in I}$ be an integral matrix of finite order. The matrix A is called a *generalized Cartan matrix* (GCM for short) if it satisfies:

(a) $a_{ii} = 2, \quad i \in I$,

(b) $a_{ij} \leq 0, \quad i \neq j$,

(c) $a_{ij} = 0 \Longleftrightarrow a_{ji} = 0$, $i \neq j \in I$.

Such a generalized Cartan matrix A is associated with a graph $S(A)$, called the Dynkin diagram of A, whose vertices are indexed by I and linked to each other according to the convention adopted in [15, 4.7].

2. A GCM $A = (a_{ij})_{i,j\in I}$ is called *indecomposable* if there is no permutation σ on I such that $A^\sigma := (a_{\sigma(i),\sigma(j)})$ is a direct sum of its non-trivial diagonal blocks. Thus A is indecomposable if and only if its Dynkin diagram $S(A)$ is a connected graph. Any indecomposable GCM A is of one of three mutually exclusive types: finite, affine and indefinite ([15, Th. 4.3]). This trichotomy result is valid for a larger class of indecomposable real matrices $M = (m_{ij})_{i,j\in I}$ of finite order (which includes the aforementioned GCMs) satisfying the conditions (m1), (m2) and (m3) of [15, Chap 4]. If M is such a matrix and D is a diagonal positive definite matrix of same order as M then M, MD and DM are of the same type.

3. A GCM A is called *symmetrizable* if there exists a real positive definite diagonal matrix D such that $B = D^{-1}A$ is symmetric. The entries of D can be chosen to be rational. If moreover the GCM A is indecomposable, then D is unique up to a positive scalar multiple. By a result of Carter ([14, 15.14]) the type of an indecomposable symmetrizable GCM A is given by the signature of the corresponding symmetric matrix B, which is of the same type as A.

4. An indecomposable GCM A is called *hyperbolic* (resp. strictly hyperbolic) if it is of indefinite type and the deletion of any vertex and the edges connected to it, in the corresponding Dynkin diagram, yields a disjoint union of Dynkin diagrams of finite or affine (resp. finite) type.

5. An indecomposable symmetrizable GCM is called *Lorentzian* if it is invertible and the corresponding symmetric matrix has signature $(+\ldots+-)$. According to [18], symmetrizable hyperbolic GCMs are Lorentzian.

2.2. Kac-Moody Lie algebras

Let $A=(a_{ij})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix.
Let $(\mathfrak{h}_{\mathbb{R}}, \Pi=(\alpha_i)_{i\in I}, \Pi\check{}=(\check{\alpha_i})_{i\in I})$ be a realization of A over the real field $\mathbb{R}$. Let $\mathfrak{h}:=\mathfrak{h}_{\mathbb{R}}\otimes\mathbb{C}$; then $(\mathfrak{h},\Pi,\Pi\check{})$ is a realization of A over the complex field $\mathbb{C}$. We recall that $\mathfrak{h}$ is a complex vector space, Π (resp. $\Pi\check{}$) is free in $\mathfrak{h}^*$ (resp. $\mathfrak{h}$) such that $\dim(\mathfrak{h})-|I|=corank(A)$ and $\langle\alpha_j,\check{\alpha_i}\rangle=a_{ij},\ i,j\in I$.

Let $\mathfrak{g}=\mathfrak{g}(A)$ be the symmetrizable Kac-Moody algebra over $\mathbb{C}$ associated with the GCM A (see [15]) it is generated by $\{\mathfrak{h}, e_i, f_i,\ i\in I\}$ with the following relations:

$$\begin{array}{lll}[\mathfrak{h},\mathfrak{h}]=0, & [e_i,f_j]=\delta_{i,j}\check{\alpha_i} & (i,j\in I);\\ [h,e_i]=\langle\alpha_i,h\rangle e_i, & [h,f_i]=-\langle\alpha_i,h\rangle f_i & (h\in\mathfrak{h});\\ (\mathrm{ad}e_i)^{1-a_{i,j}}(e_j)=0, & (\mathrm{ad}f_i)^{1-a_{i,j}}(f_j)=0 & (i\neq j).\end{array} \tag{1}$$

When the generalized Cartan matrix A is indecomposable, the type of the Kac-Moody algebra $\mathfrak{g}=\mathfrak{g}(A)$ is that of A.

The subalgebra $\mathfrak{h}$ is commutative, it acts diagonally on $\mathfrak{g}$ and is maximal for this property; it is called the standard Cartan subalgebra of $\mathfrak{g}$. The root system $\Delta=\Delta(\mathfrak{g},\mathfrak{h})$ of $\mathfrak{g}$ is the set of non-null weights of the adjoint representation of $\mathfrak{g}$. Thus $\mathfrak{g}=\mathfrak{h}\oplus(\oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha)$, where $\mathfrak{g}_\alpha$is the finite dimensional root space of $\mathfrak{g}$ corresponding to the root α.

Denote by $Q=\oplus_{i\in I}\mathbb{Z}\alpha_i$ the root lattice; $Q^+:=\sum_{i\in I}\mathbb{Z}^+\alpha_i$ and $Q^-=-Q^+$. Then Π is a root basis of Δ and $\Delta=\Delta^+\cup\Delta^-$, where $\Delta^\pm:=\Delta\cap Q^\pm$ is the set of positive (or negative) roots relative to the basis Π. The subspace $\mathfrak{c}:=\underset{i\in I}{\cap}\ker(\alpha_i)$ of $\mathfrak{h}$ is the center of $\mathfrak{g}$, it lies in $\mathfrak{h}':=\mathfrak{h}\cap\mathfrak{g}'=\underset{i\in I}{\oplus}\mathbb{C}\check{\alpha_i}$. The root basis Π induces a basis of the dual space $(\mathfrak{h}/\mathfrak{c})^*$ and let $(p_i)_{i\in I}$ its pre-dual basis in $\mathfrak{h}/\mathfrak{c}$ (that is $\langle\alpha_j,p_i\rangle=\delta_{i,j},\ i,j\in I$). The $p_i,\ i\in I$, are called the fundamental co-weights of Δ.

The Weyl group W of $\mathfrak{g}$ is the subgroup of $GL(\mathfrak{h})$ generated by the fundamental reflections $r_i,\ i\in I$, such that $r_i(h)=h-\langle\alpha_i,h\rangle\,\check{\alpha_i}$ for $h\in\mathfrak{h}$. The Weyl group W acts on $\mathfrak{h}^*$ and Δ; we denote by $\Delta_{re}=W(\Pi)$ the set of real roots of Δ and $\Delta_{im}=\Delta\backslash\Delta_{re}$ the set of imaginary roots. When the generalized Cartan matrix A is indecomposable, any root basis of Δ is W-conjugate to Π or $-\Pi$ (see[15]).

In [24], D. H. Peterson and V.G. Kac construct a group G, which is the connected and simply connected complex algebraic group associated to $\mathfrak{g}$ when $\mathfrak{g}$ is of finite type, depending only on the derived Lie algebra $\mathfrak{g}'$ and acting on $\mathfrak{g}$ via the adjoint representation $\mathrm{Ad}:\ G\to\mathrm{Aut}(\mathfrak{g})$. It is generated by the one-parameter subgroups $U_\alpha=\exp(\mathfrak{g}_\alpha),\ \alpha\in\Delta_{re}$, and $\mathrm{Ad}(U_\alpha)=\exp(\mathrm{ad}\mathfrak{g}_\alpha))$. In the definitions of J. Tits [31], G is the group of complex points of $\mathfrak{G}_D$ where D is the datum associated to A and the $\mathbb{Z}$-dual Λ of $\bigoplus_{i\in I}\mathbb{Z}\alpha_i^\vee$.

2.3. Standard Kac-Moody subalgebras and subgroups

A *Borel subalgebra* of $\mathfrak{g}$ is a maximal completely solvable subalgebra. A *parabolic subalgebra* of $\mathfrak{g}$ is a (proper) subalgebra containing a Borel subalgebra. The *standard positive (or negative) Borel subalgebra* is $\mathfrak{b}^{\pm} := \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta^{\pm}}\mathfrak{g}_{\alpha})$. A parabolic subalgebra $\mathfrak{p}^{+}$ (resp. $\mathfrak{p}^{-}$) containing $\mathfrak{b}^{+}$ (resp. $\mathfrak{b}^{-}$) is called *positive (resp. negative) standard parabolic subalgebra* of $\mathfrak{g}$; then there exists a subset J of I (called the type of $\mathfrak{p}^{\pm}$) such that $\mathfrak{p}^{\pm} = \mathfrak{p}^{\pm}_J := \oplus_{\alpha\in\Delta_J}\mathfrak{g}_{\alpha}) + \mathfrak{b}^{\pm}$, where $\Delta_J = \Delta \cap (\oplus_{j\in J}\mathbb{Z}\alpha_j)$ (cf. [17]). The normalizer $B^{\pm}$ of $\mathfrak{b}^{\pm}$ in G is the standard positive (or negative) Borel subgroup of G. A subgroup $P^{\pm}$ of G containing $B^{\pm}$ is called a standard positive (or negative) parabolic subgroup, it is the normalizer of a standard positive (or negative) parabolic subalgebra $\mathfrak{p}^{\pm} = \mathfrak{p}^{\pm}_J$, in which case the subgroup $P^{\pm} = P^{\pm}_J = B^{\pm}W_J B^{\pm}$ is its own normalizer in G, where W_J is the subgroup of W generated by fundamental reflections r_j, $j \in J$ (cf. [17] or [24]).

The Cartan subalgebras of $\mathfrak{g}$ are G-conjugate. If $\mathfrak{g}$ is indecomposable and not of finite type, there are exactly two conjugate classes (under the adjoint action of G) of Borel subalgebras: $G.\mathfrak{b}^{+}$ and $G.\mathfrak{b}^{-}$. A Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ which is G-conjugate to $\mathfrak{b}^{+}$ (resp. $\mathfrak{b}^{-}$) is called positive (resp. negative). It follows that any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is G-conjugate to a standard positive (or negative) parabolic subalgebra, in which case, we say that $\mathfrak{p}$ is positive (or negative). The same description hold for the corresponding Borel or parabolic subgroups.

Definition 2.1. Let J be a non-empty subset of I. Consider the generalized Cartan matrix $A_J = (a_{i,j})_{i,j\in J}$.

(1) The subset J is called *connected*, if the Dynkin subdiagram, with vertices indexed by J, is connected: that is, if $|J| \geq 2$ and $k \neq l \in J$, there exists a sequence $j_1, j_2, ..., j_m$ of J such that $j_1 = k$ and $j_m = l$ and $a_{j_i,j_{i+1}} < 0$ for all $0 < i < m$ (i.e, the corresponding vertices j_i and j_{i+1} are linked in the Dynkin diagram associated to A). This is equivalent to the fact that the corresponding generalized Cartan principal submatrix A_J is indecomposable; in such a case, J is called of finite type if A_J is.

(2) The subset J of I is also called of finite type if its connected components are all of finite type.

Proposition 2.2. ([7]) *Let J be a non-empty subset of I. Let $\Pi_J = \{\alpha_j,\ j \in J\}$ and $\Pi_J^{\vee} = \{\alpha_j^{\vee},\ j \in J\}$. Let $\mathfrak{h}'_J$ be the subspace of $\mathfrak{h}$ generated by $\Pi_J^{\vee}$, and $\mathfrak{h}^J = \Pi_J^{\perp} = \{h \in \mathfrak{h}, \langle\alpha_j, h\rangle = 0,\ \forall j \in J\}$. Let $\mathfrak{h}''_J$ be a supplementary subspace of $\mathfrak{h}'_J + \mathfrak{h}^J$ in $\mathfrak{h}$ and let $\mathfrak{h}_J = \mathfrak{h}'_J \oplus \mathfrak{h}''_J$. Then, we have:*

(1) *$(\mathfrak{h}_J,\ \Pi_J,\ \Pi_J^{\vee})$ is a realization of the generalized Cartan matrix A_J. Hence $\mathfrak{h}''_J = \{0\}$, $\mathfrak{h}_J = \mathfrak{h}'_J$ when A_J is regular (e.g., when J is of finite type).*

(2) *The subalgebra $\mathfrak{g}_J$ of $\mathfrak{g}$, generated by $\mathfrak{h}_J$ and the e_j, f_j, $j \in J$, is the Kac-Moody Lie algebra $\mathfrak{g}(J) := \mathfrak{g}(A_J)$ associated to the realization $(\mathfrak{h}_J,\ \Pi_J,\ \Pi_J^{\vee})$ of A_J.*

(3) *The corresponding root system $\Delta(J) = \Delta(\mathfrak{g}(J), \mathfrak{h}_J)$ can be identified with $\Delta_J := \Delta \cap (\oplus_{j\in J}\mathbb{Z}\alpha_j)$.*

(4) *The corresponding Weyl group $W(J)$ can be identified with the subgroup W_J of W generated by fundamental reflections r_j, $j \in J$.*

N.B. The subalgebra $\mathfrak{g}_J$ is called the *standard regular subalgebra* of $\mathfrak{g}$ associated to J. The derived algebra $\mathfrak{g}'_J$ of $\mathfrak{g}_J$ is generated by the e_j, f_j for $j \in J$; it does not depend of the choice of $\mathfrak{h}''_J$. The Cartan subalgebra $\mathfrak{h}_J$ of $\mathfrak{g}_J$ is sometimes denoted by $\mathfrak{h}(J)$ in the text.

In the same way, the subgroup G_J of G generated by $U_{\pm\alpha_j}$, $j \in J$, is equal to the Kac-Moody group associated to the generalized Cartan matrix A_J: it is clearly a quotient; the well known equality is proven explicitly in [27, 5.15.2], it may be deduced from [31, Th. 1], see also [25, 8.4.2].

2.4. The invariant bilinear form

We suppose from now on that the generalized Cartan matrix A is symmetrizable. Then there exists a non-degenerate $\mathrm{ad}(\mathfrak{g})$-invariant symmetric bilinear form $(.,.)$ on $\mathfrak{g}$, which is entirely determined by its restriction to $\mathfrak{h}$ (see [15]) such that

$$(\check{\alpha_i}, h) = \frac{(\check{\alpha_i}, \check{\alpha_i})}{2} \langle \alpha_i, h \rangle, \text{ with } (\check{\alpha_i}, \check{\alpha_i}) > 0 \quad i \in I, h \in \mathfrak{h},$$

There exists a supplementary subspace $\mathfrak{h}''$ of $\mathfrak{h}'$ in $\mathfrak{h}$ which is totally isotropic relative to the invariant bilinear form $(.,.)$. The non-degenerate invariant bilinear form $(.,.)$ induces an isomorphism $\nu : \mathfrak{h} \longrightarrow \mathfrak{h}^*$ and a non-degenerate symmetric bilinear form on $\mathfrak{h}^*$, noted also $(.,.)$, such that

$$\langle \nu(h), h_1 \rangle = (h, h_1), \quad \forall h, h_1 \in \mathfrak{h}$$

$$(\alpha, \beta) = \left(\nu^{-1}(\alpha), \nu^{-1}(\beta)\right), \ \forall \alpha, \beta \in \mathfrak{h}^*.$$

Note that any invariant symmetric bilinear form b on $\mathfrak{g}$ satisfies the following relation

$$b(\check{\alpha_i}, h) = \frac{b(\check{\alpha_i}, \check{\alpha_i})}{2} \langle \alpha_i, h \rangle, \ \forall i \in I, \ \forall h \in \mathfrak{h}. \tag{2}$$

In particular, if moreover $b(\check{\alpha_i}, \check{\alpha_i}) > 0$, $\forall i \in I$, then b is non-degenerate (see [15, 2.1]). It follows that, if $\mathfrak{g}$ is indecomposable, the restriction of b to $\mathfrak{g}'$ is proportional to that of $(.\,,\,.)$ and there exists an automorphism τ of $\mathfrak{g}$ fixing $\mathfrak{g}'$ such that b is proportional to $(\tau.\,,\,\tau.)$. If moreover A is non-singular, then the invariant bilinear form b is proportional to $(.\,,\,.)$.

2.5. The Tits cone

Let $C := \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha_i, h \rangle \geq 0, \forall i \in I\}$ be the fundamental chamber (relative to the root basis Π) and let $X := \underset{w \in W}{\cup} w(C)$ be the corresponding positive Tits cone.

We have the following description of the Tits cone (see [15, Chap. 3 and 5]):

(1) $X = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha, h \rangle < 0$ only for a finite number of $\alpha \in \Delta^+\}$.

(2) $X = \mathfrak{h}_{\mathbb{R}}$ if and only if the generalized Cartan matrix A is of finite type.

(3) If A is indecomposable of affine type, then $X = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \delta, h \rangle > 0\} \cup \mathbb{R}\nu^{-1}(\delta)$, where δ is the lowest imaginary positive root of Δ^+.

(4) If A is indecomposable of indefinite type, then the closure of the Tits cone, for the metric topology on $\mathfrak{h}_{\mathbb{R}}$, is $\overline{X} = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha, h \rangle \geq 0, \ \forall \alpha \in \Delta^+_{im}\}$.

(5) If $h \in X$, then h lies in the interior $\mathring{X}$ of X if and only if the fixator W_h of h, in the Weyl group W, is finite. Thus, $\mathring{X}$ is the union of finite type facets of X.

(6) If A is hyperbolic, then $\overline{X} \cup (-\overline{X}) = \{h \in \mathfrak{h}_{\mathbb{R}}; (h,h) \leq 0\}$ and the set of imaginary roots is $\Delta_{im} = \{\alpha \in Q \setminus \{0\}; (\alpha,\alpha) \leq 0\}$, where $Q = \mathbb{Z}\Pi$ is the root lattice.

Remark 2.3. Combining (3) and (4) one obtains, if A is not of finite type, that the closure of the Tits cone is $\overline{X} = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha, h\rangle \geq 0,\ \forall \alpha \in \Delta_{im}^{+}\}$ and $(-X) \cap X = \mathfrak{c}_{\mathbb{R}} := \mathfrak{c} \cap \mathfrak{h}_{\mathbb{R}}$.

3. Gradations of Kac-Moody Lie algebras by Kac-Moody root systems

Let $A = (a_{ij})_{i,j\in I}$ be a symmetrizable GCM. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the associated Kac-Moody Lie algebra, $\mathfrak{h}$ the standard Cartan subalgebra of $\mathfrak{g}$ and $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the corresponding root system.

Definition 3.1. Let Σ be a root system of Kac-Moody type. The Kac-Moody Lie algebra $\mathfrak{g}$ is said to be Σ*-graded* if:

(1) The Lie algebra $\mathfrak{g}$ contains a Kac-Moody subalgebra $\mathfrak{m}$ whose root system, relative to a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$, is Σ.

(2) The Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ is $\mathrm{ad}_{\mathfrak{g}}$-diagonalizable and $\mathfrak{g} = \oplus_{\bar{\alpha}\in\Sigma\cup\{0\}} V_{\bar{\alpha}}$, with

$$V_{\bar{\alpha}} = \{x \in \mathfrak{g}; [a, x] = \bar{\alpha}(a)x,\ \forall a \in \mathfrak{a}\}\,.$$

If moreover the subspaces $V_{\bar{\alpha}}$, $\bar{\alpha} \in \Sigma \cup \{0\}$, are all of finite dimension, we say that $\mathfrak{g}$ is *finitely* Σ*-graded*. The Lie subalgebra $\mathfrak{m}$ is called the *grading subalgebra* and $\mathfrak{g}$ the *graded Lie algebra*.

Example 3.2. Some examples of gradations:

(1) The Kac-Moody Lie algebra $\mathfrak{g}$ is naturally finitely graded by its own root system Δ.

(2) Other examples of finite gradations of $\mathfrak{g}$ are provided by the almost split real forms of $\mathfrak{g}$ with reduced restricted root systems of Kac-Moody type (see [7] and [5]).

(3) Let Σ_1 and Σ_2 be isomorphic Kac-Moody root systems and let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Kac-Moody Lie algebras graded by Σ_1 and Σ_2 respectively. For $i = 1, 2$, let $\mathfrak{m}_i$ be the grading subalgebra of $\mathfrak{g}_i$, $\mathfrak{a}_i$ be a Cartan subalgebra such that $\Sigma_i = \Delta(\mathfrak{m}_i, \mathfrak{a}_i)$. The grading subalgebras $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are isomorphic. Let $\phi\colon \mathfrak{m}_1 \to \mathfrak{m}_2$ be an isomorphism such that $\phi(\mathfrak{a}_1) = \mathfrak{a}_2$. Then, $\mathfrak{m} := \{(X_1, \phi(X_1)),\ X_1 \in \mathfrak{m}_1\}$ is a Kac-Moody subalgebra of $\mathfrak{g}_1 \times \mathfrak{g}_2$, isomorphic to $\mathfrak{m}_i$, $i = 1, 2$, and $\mathfrak{a} := \{(a_1, \phi(a_1)),\ a_1 \in \mathfrak{a}_1\}$ is a Cartan subalgebra of $\mathfrak{m}$. Let $\Sigma := \Delta(\mathfrak{m}, \mathfrak{a})$ the corresponding root system. Then $\mathfrak{g}_1 \times \mathfrak{g}_2$ is Σ-graded and, for $\bar{\alpha} \in \Sigma$, we have (with obvious notations):

$$V_{\bar{\alpha}} = V_{1,\bar{\alpha}} \times V_{2,\bar{\alpha}}.$$

In particular, the Kac-Moody algebra $\mathfrak{g}^2 := \mathfrak{g} \times \mathfrak{g}$ is finitely Δ-graded with grading subalgebra $\mathfrak{m} := \{(X, X),\ X \in \mathfrak{g}\}$ isomorphic to $\mathfrak{g}$. ■

Throughout this article we will assume that the symmetrizable Kac-Moody algebra $\mathfrak{g} := \mathfrak{g}(A)$ (which may be decomposable) is Σ-graded, with grading subalgebra $\mathfrak{m}$, and $\Sigma = \Delta(\mathfrak{m}, \mathfrak{a})$ is the grading root system relative to a Cartan subalgebra $\mathfrak{a}$ contained in the standard Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

We denote, as usual, by $\mathfrak{m}' := [\mathfrak{m}, \mathfrak{m}]$ the derived subalgebra of $\mathfrak{m}$, $\mathfrak{a}' := \mathfrak{a} \cap \mathfrak{m}'$ and $\mathfrak{a}''$ a supplementary subspace of $\mathfrak{a}'$ in $\mathfrak{a}$. Let $\rho_{\mathfrak{a}} : \mathfrak{h}^* \to \mathfrak{a}^*$ be the restriction map such that for $\alpha \in \Delta := \Delta(\mathfrak{g}, \mathfrak{h})$, $\rho_{\mathfrak{a}}(\alpha) := \alpha|\mathfrak{a}$ is its restriction to $\mathfrak{a}$. As $\mathfrak{g}$ is Σ-graded we get:

$$\rho_{\mathfrak{a}}(\Delta \cup \{0\}) = \Sigma \cup \{0\}, \quad \rho_{\mathfrak{a}}(\Delta_{im}) \subset \Sigma_{im} \cup \{0\}, \quad \rho_{\mathfrak{a}}^{-1}(\Sigma_{re}) \cap \Delta \subset \Delta_{re}.$$

The center of the grading algebra $\mathfrak{m}$ is $\mathfrak{c}_{\mathfrak{a}} = \mathfrak{c} \cap \mathfrak{a}$.
For $\gamma \in \Sigma$, put $\Gamma := \rho_{\mathfrak{a}}^{-1}(\{\gamma\}) \cap \Delta$ so that $V_\gamma = \underset{\alpha \in \Gamma}{\oplus} \mathfrak{g}_\alpha$.

Lemma 3.3. (1) *Let $\gamma \in \Sigma_{re}$ be a real root of the grading root system Σ and let $\alpha, \beta \in \Gamma := \rho_{\mathfrak{a}}^{-1}(\{\gamma\}) \cap \Delta$. Then $\alpha + \beta \notin \Delta$.*

(2) *The graded Kac-Moody Lie algebra $\mathfrak{g}$ is an integrable $\mathfrak{m}$-module via the adjoint action of the grading subalgebra $\mathfrak{m}$.*

Proof. Since $\gamma \in \Sigma_{re}$, assertion (1) follows from the fact that $2\gamma \notin \Sigma$. For assertion (2), let $\gamma \in \Sigma_{re}$ be a real root and let E_γ be a generator of $\mathfrak{m}_\gamma$. Write E_γ as a finite sum $E_\gamma = \sum_{\alpha \in \Gamma} e_\alpha$, with $e_\alpha \in \mathfrak{g}_\alpha$. Note that $\Gamma \subset \Delta_{re}$ and, by (1), the e_α, $\alpha \in \Gamma$, are commuting. It follows that $ad_{\mathfrak{g}}(E_\gamma)$ is locally nilpotent (as a sum of commuting locally nilpotent operators). ■

Lemma 3.4. *Let Δ_1, Δ_2, ..., Δ_k be the indecomposable components of the root system Δ and $\mathfrak{g}_1$, $\mathfrak{g}_2$, ..., $\mathfrak{g}_k$ the corresponding indecomposable Kac-Moody subalgebras. In particular $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus ... \oplus \mathfrak{g}_k$. For $i = 1, 2, ..., k$, let ψ_i be the projection of $\mathfrak{g}$ onto $\mathfrak{g}_i$ along $\oplus_{j \neq i} \mathfrak{g}_j$. For $\gamma \in \Sigma \cup \{0\}$, let $V_\gamma = \oplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ be the corresponding weight subspace with respect to $ad_{\mathfrak{g}}(\mathfrak{a})$. For $i = 1, 2, ..., k$, let $V_{\gamma,i} := V_\gamma \cap \mathfrak{g}_i$ and $\Gamma_i := \Gamma \cap \Delta_i$. Then we have:*

(1) *For $\gamma \in \Sigma \cup \{0\}$, $V_\gamma = \oplus_{i=1}^k V_{\gamma,i}$ with $V_{\gamma,i} = \psi_i(V_\gamma) = \underset{\alpha \in \Gamma_i}{\oplus} \mathfrak{g}_\alpha$.*

(2) *For $\gamma \in \Sigma$ and $\beta_i \in \Gamma_i := \Delta_i \cap \Gamma$, we have $\langle \gamma, a \rangle = \langle \beta_i, \psi_i(a) \rangle$, $\forall a \in \mathfrak{a}$.*

Proof. This follows from the fact that the $\mathfrak{g}_i$ are ideals of $\mathfrak{g}$, and so they are invariant under the action of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{a})$, and that the projections ψ_i are Lie algebra endomorphisms of $\mathfrak{g}$. ■

Proposition 3.5. *Suppose that the symmetrizable Kac-Moody algebra $\mathfrak{g}$ is Σ-graded. Let Δ_0 be the union of indecomposable components Δ_i of Δ such that $\rho_{\mathfrak{a}}(\Delta_i) = \{0\}$ and suppose that Δ_0 is nonempty. Let $\Delta^1 = \Delta \setminus \Delta_0$ and $\mathfrak{g}_0$ (resp. $\mathfrak{g}^1$) be the Kac-Moody subalgebra associated to Δ_0 (resp. Δ^1). Then there exists a Kac-Moody subalgebra $\mathfrak{m}^1$ of $\mathfrak{g}^1$, isomorphic to $\mathfrak{m}$, with $\mathfrak{m}^{1'} = \mathfrak{m}'$, which graduates $\mathfrak{g}$ and $\mathfrak{g}^1$, and commutes with $\mathfrak{g}_0$.*

Proof. Let ψ_0 be the projection along $\mathfrak{g}^1$ onto $\mathfrak{g}_0$. Note that ψ_0 is an endomorphism of the Lie algebra $\mathfrak{g}$ and for $\alpha_0 \in \Delta_0$ and $a \in \mathfrak{a}$, we have $\langle \alpha_0, a \rangle = \langle \alpha_0, \psi_0(a) \rangle$. Hence the condition $\rho_{\mathfrak{a}}(\Delta_0) = \{0\}$ implies that $\psi_0(\mathfrak{a})$ is contained in the center $\mathfrak{c}_0$ of $\mathfrak{g}_0$. It follows that $\psi_0([\mathfrak{a}, \mathfrak{m}]) \subset [\mathfrak{c}_0, \mathfrak{g}_0] = \{0\} = \psi_0(\mathfrak{m}')$, so $\psi_0(\mathfrak{m}) = \psi_0(\mathfrak{a}) \subset \mathfrak{c}_0$ and $\mathfrak{m}$ centralizes $\mathfrak{g}_0$. Since $\mathfrak{c}_0 \cap \mathfrak{m} \subset \mathfrak{c}_{\mathfrak{a}} \subset \mathfrak{m}' \subset \ker(\psi_0)$, we get $\mathfrak{g}_0 \cap \mathfrak{m} = \psi_0(\mathfrak{g}_0 \cap \mathfrak{m}) = \{0\}$. Hence, the projection $Id - \psi_0$ when restricted to $\mathfrak{m}$ is injective and the Kac-Moody subalgebra $\mathfrak{m}^1 := (Id - \psi_0)(\mathfrak{m})$ satisfies the desired properties. ■

Remark 3.6. By the Proposition 3.5, we are reduced to studying the gradations of $\mathfrak{g}$ for which the restriction map $\rho_{\mathfrak{a}}$ is nonzero on any indecomposable component of Δ. This can be achieved by requiring that the grading subalgebra $\mathfrak{m}$ does not centralize any indecomposable component of $\mathfrak{g}$.

Proposition 3.7. *With the notations introduced above, let Δ_1 be an indecomposable component of Δ such that $\rho_{\mathfrak{a}}(\Delta_1) \neq \{0\}$. Then there exists an indecomposable component Σ_1 of Σ such that $\rho_{\mathfrak{a}}(\Delta_1) \subset \Sigma_1 \cup \{0\}$. In particular, if $\mathfrak{g}$ is indecomposable then so is the grading subalgebra $\mathfrak{m}$.*

Proof. Let $\mathfrak{g}_1$ be the indecomposable Kac-Moody subalgebra associated to Δ_1 and let Σ_1 be an indecomposable component of Σ such that $\rho_{\mathfrak{a}}(\Delta_1) \cap \Sigma_1 \neq \emptyset$.

Define
$$\Delta_{1,1} := \rho_{\mathfrak{a}}^{-1}(\Sigma_1) \cap \Delta_1, \quad \Delta_{1,2} := \rho_{\mathfrak{a}}^{-1}(\Sigma \setminus \Sigma_1) \cap \Delta_1,$$
$$\mathfrak{g}_{1,1} := V_0 \cap \mathfrak{g}_1 \oplus \bigoplus_{\alpha \in \Delta_{1,1}} \mathfrak{g}_\alpha, \quad V_{1,2} := \bigoplus_{\alpha \in \Delta_{1,2}} \mathfrak{g}_\alpha.$$

We will see that $V_{1,2} = \{0\}$ (so $\Delta_{1,2} = \emptyset$ and $\rho_{\mathfrak{a}}(\Delta_1) \subset \Sigma_1 \cup \{0\}$). Note that $\mathfrak{g}_{1,1}$ is a nonzero subalgebra of $\mathfrak{g}_1$ and $V_{1,2}$ is a supplementary subspace of $\mathfrak{g}_{1,1}$ in $\mathfrak{g}_1$ satisfying
$$[\mathfrak{g}_\alpha, V_{1,2}] = \{0\}, \ \forall \alpha \in \Delta_{1,1}. \tag{3}$$

Let $\alpha_1 \in \Delta_{1,1}$ and $v_1 \in \mathfrak{g}_{\alpha_1} \setminus \{0\} \subset \mathfrak{g}_{1,1}$. Let $\mathfrak{I}_1$ be the ideal of $\mathfrak{g}_1$ generated by v_1. As $\mathfrak{g}_1$ is indecomposable, $\mathfrak{I}_1$ contains the derived subalgebra $\mathfrak{g}_1'$ of $\mathfrak{g}_1$ (see [15]). By the Poincaré-Birkhoff-Witt theorem and the relation (3) above, we get: $\mathfrak{I}_1 = \mathcal{U}(\mathfrak{g}_1).v_1 = \mathcal{U}(\mathfrak{g}_{1,1}).v_1 \subset \mathfrak{g}_{1,1}$, where $\mathcal{U}(\mathfrak{g}_1)$ denotes the enveloping algebra of $\mathfrak{g}_1$. Hence $\mathfrak{g}_{1,1}$ contains $\mathfrak{g}_1'$, $V_{1,2} = \{0\}$ and $\rho_{\mathfrak{a}}(\Delta_1) \subset \Sigma_1 \cup \{0\}$. If $\mathfrak{g}$ is indecomposable, then $\Delta_1 = \Delta$, $\Sigma_1 = \Sigma$ and so $\mathfrak{m}$ is also indecomposable. ■

Corollary 3.8. *Suppose that the restriction map $\rho_{\mathfrak{a}}$ is nonzero on any indecomposable component of Δ. Let $(\Sigma_i)_{1\le i\le q}$ be the indecomposable components of the grading root system Σ. Then the graded algebra $\mathfrak{g}$ is a direct sum of q Kac-Moody subalgebras $\tilde{\mathfrak{g}}_1$, $\tilde{\mathfrak{g}}_2$, ..., $\tilde{\mathfrak{g}}_q$ such that each $\tilde{\mathfrak{g}}_i$ is Σ_i-graded, $i = 1, 2, ...q$. Moreover, $\mathfrak{g}$ is finitely Σ-graded if and only if $\tilde{\mathfrak{g}}_i$ is finitely Σ_i-graded for $i = 1, 2, ...q$.*

Proof. For $i \in \{1, 2, ..., q\}$ let $\tilde{\Delta}_i$ be the union of all the indecomposable components of Δ with image by $\rho_{\mathfrak{a}}$ in $\Sigma_i \cup \{0\}$ and let $\tilde{\mathfrak{g}}_i$ be the corresponding Kac-Moody subalgebra. By the Proposition 3.7, one can see that
$$\Delta = \coprod_{1\le i\le q} \tilde{\Delta}_i$$
is a disjoint union and hence, for $i \neq j$, $(\tilde{\Delta}_i + \tilde{\Delta}_j) \cap \Delta = \emptyset$. Indeed, for $i \neq j$, $\rho_{\mathfrak{a}}(\tilde{\Delta}_i \cap \tilde{\Delta}_j) \subset (\Sigma_i \cup \{0\}) \cap (\Sigma_j \cup \{0\}) = \{0\}$; by the hypothesis of the corollary, $\tilde{\Delta}_i$ and $\tilde{\Delta}_j$ can not have a common indecomposable component and hence are disjoint. It follows that $\mathfrak{g}$ is the direct sum of the q Kac-Moody Lie subalgebras $\tilde{\mathfrak{g}}_i$, $i = 1, 2, ...q$, and since $\rho_{\mathfrak{a}}(\Delta \cup \{0\}) = \Sigma \cup \{0\}$, we have $\rho_{\mathfrak{a}}(\tilde{\Delta}_i \cup \{0\}) = \Sigma_i \cup \{0\}$ for any $i = 1, 2, ...q$, and thus the Kac-Moody subalgebra $\tilde{\mathfrak{g}}_i$ is Σ_i-graded. The necessary and sufficient condition for the gradation to be finite is obvious. ■

Remark 3.9. By Corollary 3.8, we are reduced to studying gradations of $\mathfrak{g}$ by indecomposable root systems.

Definition 3.10. Suppose that $\mathfrak{g}$ is graded by an indecomposable root system Σ. Let Δ_1 be an indecomposable component of Δ and let $\mathfrak{g}_1$ be the corresponding Kac-Moody subalgebra.

(1) Δ_1 (or $\mathfrak{g}_1$) is called a *trivial factor* of the gradation if $\rho_{\mathfrak{a}}(\Delta_1) = \{0\}$.

(2) Δ_1 (or $\mathfrak{g}_1$) is called a *fictive factor* of the gradation if $\rho_{\mathfrak{a}}(\Delta_1) \subset \Sigma_{im} \cup \{0\}$.

(3) Δ_1 (or $\mathfrak{g}_1$) is called an *effective factor* of the gradation if $\rho_{\mathfrak{a}}(\Delta_1 \cup \{0\}) = \Sigma \cup \{0\}$.

N.B.: (i) The case when $\mathfrak{g}_1$ is a trivial factor of the gradation was fulfilled in Proposition 3.5.

(ii) If $\mathfrak{g}_1$ is an effective factor, then $\mathfrak{g}_1 = \oplus_{\gamma\in\Sigma\cup\{0\}} V_{1,\gamma}$, with $V_{1,\gamma} := V_\gamma \cap \mathfrak{g}_1$ and $V_{1,\gamma} \neq \{0\}$, $\forall\gamma \in \Sigma$.

Proposition 3.11. *Suppose that $\mathfrak{g}$ is graded by an indecomposable root system Σ. Let Δ_1 be a non trivial factor for the gradation. Let $\Delta^1 = \Delta \setminus \Delta_1$ and $\mathfrak{g}_1$ (resp. $\mathfrak{g}^1$) be the Kac-Moody subalgebra associated with Δ_1 (resp. Δ^1). Let ψ_1 be the projection on $\mathfrak{g}_1$ with kernel $\mathfrak{g}^1$. Then $\mathfrak{g}_1$ is either a fictive or an effective factor of the gradation. More precisely, we have:*

(1) *The subalgebra $\mathfrak{g}_1$ is a fictive factor of the gradation if and only if $\psi_1(\mathfrak{m}') = \{0\}$, in which case Σ is of affine type and there exists $d_1 \in \mathfrak{h}_1 := \mathfrak{h} \cap \mathfrak{g}_1$ such that $\rho_{\mathfrak{a}}(\beta_1) = \langle\beta_1, d_1\rangle\delta$, $\forall\beta_1 \in \Delta_1$, where δ is a generator of Σ_{im}.*

(2) *The subalgebra $\mathfrak{g}_1$ is an effective factor of the gradation if and only if* $\ker(\psi_1)\cap\mathfrak{m}$ *is contained in the center $\mathfrak{c}_{\mathfrak{a}}$ of $\mathfrak{m}$.*

Proof. As $\mathfrak{m}$ is indecomposable, any ideal of $\mathfrak{m}$ is either contained in the center or contains $\mathfrak{m}'$ (cf. [15]). The dichotomy stems from the nature of the ideal $\ker(\psi_1)\cap\mathfrak{m}$ of $\mathfrak{m}$:

(1) Suppose that $\psi_1(\mathfrak{m}') = \{0\}$. For $\beta_1 \in \Delta_1$ and $a' \in \mathfrak{a}' := \mathfrak{a} \cap \mathfrak{m}'$, we have $\langle\beta_1, a'\rangle = \langle\beta_1, \psi_1(a')\rangle = 0$. As $\rho_{\mathfrak{a}}(\Delta_1) \neq \{0\}$, Σ has an imaginary root whose restriction to $\mathfrak{a}'$ is zero. It follows that Σ is of affine type and $\rho_{\mathfrak{a}}(\Delta_1) \subset \mathbb{Z}\delta$, where δ is a generator of Σ_{im}. Hence, Δ_1 is a fictive factor; more precisely let $d \in \mathfrak{a}$ be such that $\langle\delta, d\rangle = 1$ and let $d_1 = \psi_1(d)$; then $\rho_{\mathfrak{a}}(\beta_1) = \langle\rho_{\mathfrak{a}}(\beta_1), d\rangle\delta = \langle\beta_1, d_1\rangle\delta$, $\forall\beta_1 \in \Delta_1$. Conversely, suppose that Δ_1 is a fictive factor. It follows from the hypothesis that, for any real root $\gamma \in \Sigma_{re}$, $\mathfrak{m}_\gamma \subset \ker(\psi_1)$ (see Lemma 3.4) and therefore $\mathfrak{m}' \subset \ker(\psi_1)$.

(2) If Δ_1 is not a fictive factor, then by 1), $\psi_1(\mathfrak{m}') \neq \{0\}$ and the ideal $\ker(\psi_1)\cap\mathfrak{m}$ is contained in the center $\mathfrak{c}_{\mathfrak{a}}$ of $\mathfrak{m}$. Let $\mathfrak{c}_1$ be the center of $\mathfrak{g}_1$, then one can see in this case that $\psi_1^{-1}(\mathfrak{c}_1)\cap\mathfrak{m} = \mathfrak{c}_{\mathfrak{a}}$. Therefore, ψ_1 induces a monomorphism from $\mathfrak{m}/\mathfrak{c}_{\mathfrak{a}}$ into $\mathfrak{g}_1/\mathfrak{c}_1$ and we get $\rho_{\mathfrak{a}}(\Delta_1 \cup \{0\}) = \Sigma \cup \{0\}$. Thus Δ_1 is an effective factor of the gradation. The converse is straightforward. ■

Remark 3.12. (1) As $\mathfrak{g}$ is Σ-graded, it has at least one effective factor.

(2) The following example shows that gradations by affine root systems may include fictive factors. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a direct sum of two indecomposable Kac-Moody Lie algebras, with $\mathfrak{g}_0$ of affine type. Let $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ be a Cartan subalgebra of $\mathfrak{g}$. Write $\mathfrak{h}_0 = \mathbb{C}d_0 \oplus \mathfrak{h}_0'$, where $\langle\delta_0, d_0\rangle = 1$, and δ_0 is the shortest imaginary root (relative to a root basis) of $\Delta_0 := \Delta(\mathfrak{g}_0, \mathfrak{h}_0)$. Let $d_1 \in \mathfrak{h}_1$ such that $\langle\beta_1, d_1\rangle \in \mathbb{Z}$, for all $\beta_1 \in \Delta_1 := \Delta(\mathfrak{g}_1, \mathfrak{h}_1)$. Let $d = d_0 + d_1$ and $\mathfrak{m} = \mathbb{C}d \oplus \mathfrak{g}_0'$. Then, $\mathfrak{m}$ is a Kac-Moody subalgebra of $\mathfrak{g}$, isomorphic to $\mathfrak{g}_0$, and the corresponding root system graduates $\mathfrak{g}$.

The derived subalgebra $\mathfrak{m}'$ is contained in the kernel $\mathfrak{g}_0$ of the projection ψ_1 onto $\mathfrak{g}_1$ and so $\mathfrak{g}_1$ is a fictive factor for this gradation. If d_1 is not central in $\mathfrak{g}_1$, we get a nontrivial fictive factor (i.e., the grading subalgebra $\mathfrak{m}$ does not centralize $\mathfrak{g}_1$).

Corollary 3.13. *Under the assumptions of the proposition 3.11, suppose that* $\mathfrak{m}$ *is not of affine type. Then any indecomposable component* Δ_1 *of* Δ*, satisfying* $\rho_{\mathfrak{a}}(\Delta_1) \neq \{0\}$*, is an effective factor of the gradation. In particular, if* $\mathfrak{m} = \mathfrak{m}'$ *is perfect, then any non trivial factor* $\mathfrak{g}_1$ *of* $\mathfrak{g}$ *is* Σ*-graded.*

Proof. This follows from the proof of the Proposition 3.11. ∎

Proposition 3.14. *Suppose that* $\mathfrak{g}$ *is graded by an indecomposable root system* Σ*. Let* Δ^0 *(resp.* Δ^1*) be the union of fictive (resp. effective) factors of the gradation. Let* $\mathfrak{g}^0$ *(resp.* $\mathfrak{g}^1$*) be the Kac-Moody subalgebra associated to* Δ^0 *(resp.* Δ^1*). Let* ψ^1 *be the projection along* $\mathfrak{g}^0$ *onto* $\mathfrak{g}^1$*. Then the subalgebra* $\mathfrak{m}^1 := \psi^1(\mathfrak{m})$ *of* $\mathfrak{g}^1$ *is isomorphic to* $\mathfrak{m}$ *and the Kac-Moody subalgebra* $\mathfrak{g}^1$ *is* Σ*-graded with grading subalgebra* $\mathfrak{m}^1$*.*

N.B. This leads to studying gradations of $\mathfrak{g}$ with no fictive factor. Such gradations can be realized by requiring that the derived subalgebra $\mathfrak{m}'$, of the grading subalgebra $\mathfrak{m}$, does not centralize any indecomposable component of $\mathfrak{g}$.

Proof. The projection ψ^0 (resp. ψ^1) is the sum of the projections (given in Proposition 3.11) on the fictive (resp. effective) factors of the gradation . It follows that $\psi^0(\mathfrak{m}') = \{0\}$ and $\ker(\psi^1) \cap \mathfrak{m} \subset \mathfrak{c}_{\mathfrak{a}} \subset \ker(\psi^0)$ (see Propositions 3.5 and 3.11). Hence $\ker(\psi^1) \cap \mathfrak{m} \subset \ker(\psi^0) \cap \ker(\psi^1) = \{0\}$ and ψ^1 is injective on $\mathfrak{m}$. Thus, the Kac-Moody subalgebra $\mathfrak{m}^1 := \psi^1(\mathfrak{m})$ of $\mathfrak{g}^1$ is isomorphic to $\mathfrak{m}$. Since $\mathfrak{m}$ acts on $\mathfrak{g}^1$ as $\mathfrak{m}^1$, we get the required statement of the Proposition. ∎

Proposition 3.15. *Suppose that* $\mathfrak{g}$ *is graded by an indecomposable root system* Σ *of finite type and that* $\rho_{\mathfrak{a}}$ *is nonzero on any indecomposable component of* Δ*. Then any indecomposable component of* $\mathfrak{g}$ *is an effective factor of finite or affine type.*

Proof. Let Δ_1 be an indecomposable component of Δ on which $\rho_{\mathfrak{a}}$ is nonzero. Since the associated grading algebra $\mathfrak{m}$ is (perfect) of finite type, Δ_1 is an effective factor and the corresponding Kac-Moody subalgebra is Σ-graded (cf. Corollary 3.13). Suppose that Δ_1 is of indefinite type, then $\Delta_1 \subset Vect(\Delta_{1,im})$ (see [7], Lemma 3.5). As Σ is of finite type, $\rho_{\mathfrak{a}}(\Delta_{1,im}) = \{0\}$ and so $\rho_{\mathfrak{a}}(\Delta_1) = \{0\}$, contradicting the hypothesis of the proposition. Hence Δ_1 is of finite or affine type. ∎

N.B. (1) If $\mathfrak{g}$ is graded by an indecomposable root system Σ of finite type, then the (perfect) grading simple Lie subalgebra $\mathfrak{m}$ lies in the derived subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$. In particular, the perfect Lie algebras $\mathfrak{g}'$ and $\mathfrak{g}'/\mathfrak{c}$ are Σ-graded and the result is compatible with those of Berman and Moody ([13]) and Benkart and Zelmanov ([12]) on the classification of perfect Lie algebras graded by finite root systems.

(2) The gradations of Kac-Moody Lie algebras (of finite or affine type) by finite root systems were studied by Nervi in [21] and [23].

Proposition 3.16. *Suppose that* $\mathfrak{g}$ *is graded by an indecomposable root system* Σ *of affine type. Then any effective factor* $\mathfrak{g}_1$ *of the gradation is of affine type.*

N.B. Note that such a gradation may include fictive factors as described in Proposition 3.11. We will see below that any effective factor of the gradation is finitely Σ-graded (cf. Corollary 4.19).

Proof. As Σ is of affine type, any effective factor of the gradation can not be of finite type. Suppose that the gradation has an effective factor $\mathfrak{g}_1$ of indefinite type and let Δ_1 be the corresponding root system. The same argument used above shows that $\rho_{\mathfrak{a}}(\Delta_1) \subset Vect(\rho_{\mathfrak{a}}(\Delta_{1,im})) \subset \mathbb{C}\delta$, where δ is a generator of Σ_{im}. Thus, Δ_1 is a fictive factor, contradiction. Hence $\mathfrak{g}_1$ is of affine type. ■

4. Graded Kac-Moody Lie algebras admitting adapted root bases

Definition 4.1. Suppose that $\mathfrak{g}$ is graded by an indecomposable root system Σ. Let $\Pi_{\mathfrak{a}}$ be a root basis of Σ and let Σ^+ be the set of positive root relative to $\Pi_{\mathfrak{a}}$. A root basis Π of Δ (with positive roots Δ^+) is said to be *adapted to* $\Pi_{\mathfrak{a}}$ if $\rho_{\mathfrak{a}}(\Delta^+) \subset \Sigma^+ \cup \{0\}$. In such a case, we say also that $(\Pi_{\mathfrak{a}}, \Pi)$ is an *adapted pair* for the gradation.

N.B. As it was shown in [7], Theorem 3.10, adapted pairs (or adapted root bases) always exist for finite gradations of $\mathfrak{g}$.

Remark 4.2. Consider the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$ viewed as a real vector space and denote by $\psi_{re} : \mathfrak{h} \to \mathfrak{h}$ be the projection along $i\mathfrak{h}_{\mathbb{R}}$ onto $\mathfrak{h}_{\mathbb{R}}$. We recall that $\mathfrak{h}_{\mathbb{R}}$ is the standard Cartan subalgebra of the split real form of $\mathfrak{g}$ associated with the real realization $(\mathfrak{h}_{\mathbb{R}}, \Pi = (\alpha_i)_{i\in I}, \Pi\check{} = (\check{\alpha_i})_{i\in I})$ of the GCM A (see [26] or [3]).

(1) Note that $\mathfrak{h}_{\mathbb{R}}$ is invariant under the Weyl group W and hence ψ_{re} commutes to the action of W (i.e., $\psi_{re} \circ w = w \circ \psi_{re}$, $\forall w \in W$).

(2) Let $\mathfrak{c}_{\mathbb{R}} := \mathfrak{c} \cap \mathfrak{h}_{\mathbb{R}}$. If $\mathfrak{t}$ is a subset of $\mathfrak{h}$ such that $\langle \alpha, t\rangle \in \mathbb{R}$, $\forall (\alpha, t) \in \Delta \times \mathfrak{t}$, then $\mathfrak{t}$ lies in $\mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{c}_{\mathbb{R}}$ and we denote by $\tilde{\mathfrak{t}} := \psi_{re}(\mathfrak{t})$ its image in $\mathfrak{h}_{\mathbb{R}}$. Note that, for $(\alpha, t) \in \Delta \times \mathfrak{t}$ and $\tilde{t} := \psi_{re}(t)$, we have $\langle \alpha, t\rangle = \langle \alpha, \tilde{t}\rangle \in \mathbb{R}$.

(3) In particular, if $\mathfrak{g}$ is graded by a root system Σ and $\mathfrak{a}_{\mathbb{R}}$ is a real form of $\mathfrak{a}$ defining a split Cartan subalgebra of a split real form $\mathfrak{m}_{\mathbb{R}}$ of $\mathfrak{m}$, then $\langle \alpha, a\rangle = \langle \rho_{\mathfrak{a}}(\alpha), a\rangle \in \mathbb{R}$, $\forall (\alpha, a) \in \Delta \times \mathfrak{a}_{\mathbb{R}}$, and hence $\tilde{\mathfrak{a}}_{\mathbb{R}}$ is isomorphic to $\mathfrak{a}_{\mathbb{R}}/(\mathfrak{a}_{\mathbb{R}} \cap i\mathfrak{c}_{\mathbb{R}})$. In the sequel, we fix a split Cartan subalgebra $\mathfrak{a}_{\mathbb{R}}$ as above and, for any root basis $\Pi_{\mathfrak{a}}$ of Σ, we denote by $C_{\mathfrak{a}}$ (resp. $X_{\mathfrak{a}}$) the corresponding fundamental chamber (resp. Tits cone).

Proposition 4.3. *Suppose that $\mathfrak{g}$ is graded by a root system Σ. Let $\Pi_{\mathfrak{a}} = \{\gamma_s, s \in \bar{I}\}$ (resp. $\Pi = \{\alpha_i, i \in I\}$) be a root basis of Σ (resp. Δ) and let $C_{\mathfrak{a}}$ (resp. C) be the fundamental chamber associated to $\Pi_{\mathfrak{a}}$ (resp. Π). Then the root basis Π of Δ is adapted to $\Pi_{\mathfrak{a}}$ if and only if $\tilde{C}_{\mathfrak{a}} \subset C$; in which case, there exists a subset J of I such that $\rho_{\mathfrak{a}}^{-1}(\{0\}) \cap \Delta = \Delta_J$ and $\tilde{C}_{\mathfrak{a}}$ is contained in the closure of the facet of type J of C.*

Proof. The condition $\tilde{C}_{\mathfrak{a}} \subset C$ is obviously necessary (see Remark 4.2). For the converse, let $\bar{p} \in C_{\mathfrak{a}}$ such that $\langle \gamma_i, \bar{p}\rangle = 1$, $\forall i \in \bar{I}$ and let $\tilde{p} = \psi_{re}(\bar{p})$. By assumption $\tilde{p} \in C$ and let $J := \{j \in I; \langle \alpha_j, \tilde{p}\rangle = 0\}$. Hence, $\tilde{p}$ lies in the facet of type J of C and for $\alpha \in \Delta$, we have $\langle \alpha, \tilde{p}\rangle = \langle \alpha, \bar{p}\rangle = 0$ if and only if $\alpha \in \Delta_J := \Delta \cap (\bigoplus_{j\in J} \mathbb{Z}\alpha_j)$.

Note that for $\gamma \in \Sigma$, $\langle \gamma, \bar{p} \rangle = ht_{\mathfrak{a}}(\gamma)$ is the height of γ relative to $\Pi_{\mathfrak{a}}$ and $\gamma \in \Sigma^+$ if and only if $\langle \gamma, \bar{p} \rangle > 0$. It follows that Π is adapted to $\Pi_{\mathfrak{a}}$ and for $\alpha \in \Delta$, $\rho_{\mathfrak{a}}(\alpha) = 0$ if and only if $\langle \alpha, \bar{p} \rangle = 0$ or equivalently $\alpha \in \Delta_J$. It is clear that $\tilde{C}_{\mathfrak{a}}$ is contained in $C \cap \Delta_J^{\perp}$: the closure of the facet of type J. ∎

Lemma 4.4. *Suppose that $\mathfrak{g}$ is indecomposable of finite type and graded by an indecomposable root system Σ, of type A_1, and that the standard basis $\Pi = \{\alpha_i, i \in I\}$ is adapted to a root basis $\{\gamma\}$ of $\Sigma = \{-\gamma, \gamma\}$. For $i \in I$, denote by H_i the unique element of $\mathfrak{h}$ satisfying $\langle \alpha_j, H_i \rangle = 2\delta_{i,j}$, $\forall j \in I$. Then there exists a unique $k \in I$ such that $\rho_{\mathfrak{a}}(\alpha_k) = \gamma$. Moreover, H_k is the semi-simple element of an $\mathfrak{sl}_2$-triple $\{E_k, H_k, F_k\}$ generating the grading algebra $\mathfrak{m}$.*

Proof. Consider the highest positive root $\mu = \Sigma_{i \in I} n_i \alpha_i$ of Δ^+ relative to Π. As $\mathfrak{g}$ is Σ-graded and Π is adapted to the root basis $\{\gamma\}(= \Sigma^+)$, $\rho_{\mathfrak{a}}(\mu)$ is the unique positive root of Σ. Hence there exists a unique $k \in I$, with $n_k = 1$, such that $\rho_{\mathfrak{a}}(\alpha_k) = \rho_{\mathfrak{a}}(\mu) = \gamma$ and $\rho_{\mathfrak{a}}(\alpha_j) = 0$ for $j \neq k$. It follows that $H_k = \check{\gamma}$ is the semi-simple element of the $\mathfrak{sl}_2$-triple $\{E_k, H_k, F_k\}$ that generates the grading algebra $\mathfrak{m}$. ∎

This justifies the following definition:

Definition 4.5. Suppose that the Kac-Moody Lie algebra $\mathfrak{g}$ is indecomposable. Let $\Pi = \{\alpha_i, i \in I\}$ be the standard root basis of Δ and J be a subset of I such that $J \neq I$.

(1) Suppose that $\mathfrak{g}$ is of finite type; then (I, J) is called an *irreducible C-admissible pair* if $I \backslash J = \{i\}$ is reduced to a singleton and the unique element H_i of $\mathfrak{h}$, satisfying $\langle \alpha_j, H_i \rangle = 2\delta_{i,j}$, $\forall j \in I$, is the semi-simple element of an $\mathfrak{sl}_2$-triple $\{E_i, H_i, F_i\}$ whose root system Σ (of type A_1) graduates the Lie algebra $\mathfrak{g}$.

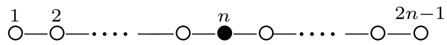
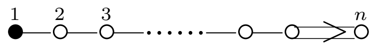
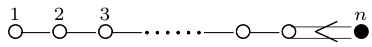
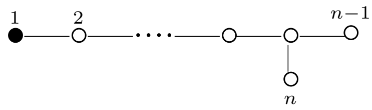
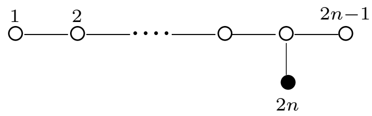
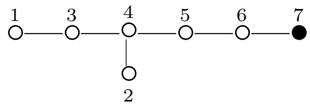
(2) Suppose that $\mathfrak{g}$ is of any type (not necessarily finite). For $k \in I \setminus J$, we denote by I_k the connected component of $J \cup \{k\}$ containing k and let $J_k := I_k \backslash \{k\}$. The pair (I, J) is called *C-admissible* if for all $k \in I \setminus J$, I_k is of finite type and (I_k, J_k) is an irreducible C-admissible pair for the standard simple Lie subalgebra $\mathfrak{g}(I_k)$.

Remark 4.6. (1) This definition of C-admissible pairs is equivalent to that introduced by H. Rubenthaler ([28], [29]) and then by J. Nervi ([21], [22]) in terms of (irreducible and commutative) regular prehomogeneous spaces of parabolic type (see [7], Definition 2.1). Table 1 below reproduces the classification, by Rubenthaler, of irreducible C-admissible pairs (I, J); the black vertex corresponds to that of $I \backslash J$. Note that if $\mathfrak{g}$ is a simple Lie algebra having an irreducible C-admissible pair (I, J), then the corresponding root system Δ is 3-graded as defined by Neher ([20]): that is $\Delta = \Delta_{-1} \cup \Delta_0 \cup \Delta_{+1}$ is a disjoint union deduced from the A_1-gradation of $\mathfrak{g}$ associated with the C-admissible pair (I, J).

(2) As it was shown in [7], this definition of C-admissible pairs (I, J) can be extended to decomposable Kac-Moody algebras: for $k \in I \setminus J$, I_k is connected and so is contained in a connected component of I. Hence, the pair (I, J) is C-admissible if and only if its trace on each connected component of I is C-admissible.

(3) If (I, J) is a C-admissible pair for $\mathfrak{g}$, any connected component of J is contained in one of the I_k, $k \in I \setminus J$, and therefore J is of finite type.

Table 1 (List of irreducible C-admissible pairs)

$A_{2n-1},\ n\geq 1$	
$B_n,\ n\geq 3$	
$C_n,\ n\geq 2$	
$D_{n,1},\ n\geq 4$	
$D_{2n,2},\ n\geq 2$	
E_7	

Lemma 4.7. *Let $J \subset I$ be a subset of I of finite type. Suppose that for any $k \in I \setminus J$, the connected component I_k, of $J \cup \{k\}$ containing k, is of finite type. For $k \in I \setminus J$, let H_k be the unique element of $\mathfrak{h}(I_k)$ satisfying $\langle \alpha_j, H_k \rangle = 2\delta_{j,k}$, $\forall j \in I_k$. Then $(H_k)_{k \in I \setminus J}$ is free in $\mathfrak{h}$.*

Proof. Let $\varphi^J := \frac{1}{|W_J|} \sum_{w \in W_J} w$ be the projection of $\mathfrak{h}$ along $\mathfrak{h}_J$ onto $\mathfrak{h}^J$ (see [4, 6.2]). For $k \in I \setminus J$, note $(p_{k,i})_{i \in I_k}$ the fundamental co-weights of the simple Lie algebra $\mathfrak{g}(I_k)$. One can write $p_{k,k} = \sum_{i \in I_k} n_{k,i} \check{\alpha_i}$, with $n_{k,i} > 0$, $\forall i \in I_k$. Since $H_k = 2p_{k,k} \in \mathfrak{h}^J$, we get $p_{k,k} = n_{k,k} \varphi^J(\check{\alpha_k})$. The Lemma follows from the fact that the restriction of φ^J to $\bigoplus_{k \in I \setminus J} \mathbb{C}\check{\alpha_k}$ is injective. ∎

Proposition 4.8. *Suppose that the Kac-Moody Lie algebra $\mathfrak{g}$ is graded by an indecomposable root system Σ. Let $\Pi_{\mathfrak{a}} = \{\gamma_s, s \in \bar{I}\}$ be a root basis of Σ and suppose that the root basis $\Pi = \{\alpha_i, i \in I\}$ of Δ is adapted to $\Pi_{\mathfrak{a}}$. Define $J := \{j \in I; \rho_{\mathfrak{a}}(\alpha_j) = 0\}$, $I' := I \setminus J$, $I'_{re} := \{k \in I'; \rho_{\mathfrak{a}}(\alpha_k) \in \Pi_a\}$, $I'_{im} := I' \setminus I'_{re}$.*
For $k \in I'$, we denote by I_k the connected component of $J \cup \{k\}$ containing k and $J_k := I_k \setminus \{k\}$. Let $I_{re} := \bigcup_{k \in I'_{re}} I_k$, $J_{re} := \bigcup_{k \in I'_{re}} J_k$ and $J^{\circ} := J \setminus J_{re}$.
For $s \in \bar{I}$, let $\Gamma_s := \{k \in I'; \rho_{\mathfrak{a}}(\alpha_k) = \gamma_s\}$. Then we have:
(1) *For any $k \in I'_{re}$, I_k is of finite type.*
(2) *For any $s \in \bar{I}$, $\Gamma_s \neq \emptyset$ and if $\alpha \in \Delta^+$ is a positive root such that $\rho_{\mathfrak{a}}(\alpha) = \gamma_s$, then there exists $k \in \Gamma_s$ such that supp$(\alpha) \subset I_k$ with coefficient 1 on α_k.*
(3) *If Γ_s has two distinct elements $l \neq k$, then I_k and I_l are not connected in I.*

(4) *For any simple root* $\gamma_s \in \Pi_{\mathfrak{a}}$, $V_{\gamma_s} = \oplus_{k\in\Gamma_s} V_{\gamma_s} \cap \mathfrak{g}(I_k)$ *is finite dimensional. In particular, the derived grading subalgebra* $\mathfrak{m}'$ *is contained in* $\mathfrak{g}'(I_{re})$.

(5) *For any* $k\in\Gamma_s$, (I_k, J_k) *is an irreducible* C*-admissible pair and* $\check{\gamma_s} = \sum_{k\in\Gamma_s} H_k$, *where* H_k *is the unique element of* $\mathfrak{h}(I_k)$ *satisfying* $\langle \alpha_i, H_k\rangle = 2\delta_{i,k}$, $\forall i \in I_k$. *In particular, the pair* (I_{re}, J_{re}) *is* C*-admissible (in the decomposable sense).*

(6) *The semi-simple Lie algebra* $\oplus_{k\in\Gamma_s} \mathfrak{g}(I_k)$ *is* A_1*-graded with grading algebra* $\mathfrak{m}_s$ *generated by the* $\mathfrak{sl}_2$*-triple* $(\bar{E}_s, \check{\gamma_s}, \bar{F}_s)$ *of* $\mathfrak{m}$ *associated to the simple root* γ_s *of* Σ.

(7) *Let* $k \in I'$ *such that* $\rho_{\mathfrak{a}}(\alpha_k)$ *is a (positive) real root of* Σ, *then* $\rho_{\mathfrak{a}}(\alpha_k) \in \Pi_{\mathfrak{a}}$ *(i.e.,* $k \in I'_{re}$*). In particular, if* $I'_{im} \neq \emptyset$, *then* $\rho_{\mathfrak{a}}(\alpha_i) \in \Sigma^+_{im}$, $\forall i \in I'_{im}$.

(8) *If* $J^\circ \neq \emptyset$, *then it is not connected to* I_{re}.

Proof. This is an extension to our setting of Proposition 3.12 and Proposition 3.14 in [7] which are essentially based on the existence of adapted bases:

(1) Suppose that there exists $k \in I'_{re}$ such that I_k is not of finite type. There exists an imaginary root $\beta = \sum_{i\in I_k} n_i\alpha_i$ with support I_k. Hence $\rho_{\mathfrak{a}}(\beta) = n_k\rho_{\mathfrak{a}}(\alpha_k)$ is an imaginary root, contradicting the fact that $k \in I'_{re}$.

(2) For $\gamma \in \Sigma^+$, let $ht_{\mathfrak{a}}(\gamma)$ be the height of γ relative to the root basis $\Pi_{\mathfrak{a}}$. Let $s \in \bar{I}$ and let $\alpha = \sum_{i\in I} n_i\alpha_i \in \Delta^+$ be such that $\rho_{\mathfrak{a}}(\alpha) = \gamma_s$. Then

$$ht_{\mathfrak{a}}(\rho_{\mathfrak{a}}(\alpha)) = \textstyle\sum_{i\in I'} n_i ht_{\mathfrak{a}}(\rho_{\mathfrak{a}}(\alpha_i)) = 1.$$

Hence there exists a unique $k \in I' \cap$ *supp*(α) such that $\rho_{\mathfrak{a}}(\alpha) = n_k\rho_{\mathfrak{a}}(\alpha_k) = \gamma_s$. Thus, $n_k = 1$, $\mathrm{supp}(\alpha) \subset I_k$ and $k \in \Gamma_s$ (since $\rho_{\mathfrak{a}}(\alpha) = \rho_{\mathfrak{a}}(\alpha_k) = \gamma_s$).

(3) Let $k \neq l \in \Gamma_s$ and suppose that $I_k \cup I_l$ is connected. Let $\beta \in \Delta^+$ be a positive root with support $I_k \cup I_l$; then $\rho_{\mathfrak{a}}(\beta)$ is a multiple of γ_s, contradicting the fact that $\gamma_s \in \Pi_{\mathfrak{a}}$ is a real root.

(4) In view of assertion 2, one can check that

$$V_{\gamma_s} = \oplus_{\alpha\in\rho_{\mathfrak{a}}^{-1}(\{\gamma_s\})} \mathfrak{g}_\alpha = \oplus_{k\in\Gamma_s} V_{\gamma_s} \cap \mathfrak{g}(I_k)$$

and thus V_{γ_s} is finite dimensional.

(5) Let $s \in \bar{I}$ and $k \in \Gamma_s$. Let $(\bar{E}_s, \check{\gamma_s}, \bar{F}_s)$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{m}$ associated to the simple root γ_s. Using (4) one can write $\bar{E}_s = \sum_{k\in\Gamma_s} E_k$ and $\bar{F}_s = \sum_{l\in\Gamma_s} F_l$, where $E_k \in V_{\gamma_s} \cap \mathfrak{g}(I_k)$ and $F_l \in V_{-\gamma_s} \cap \mathfrak{g}(I_l)$. By assertion (3) we get

$$\check{\gamma_s} = [\bar{E}_s, \bar{F}_s] = \textstyle\sum_{l\in\Gamma_s} \tilde{H}_l, \ \text{ where } \ \tilde{H}_l := [E_l, F_l] \in \mathfrak{h}_{I_l} \ \text{ for all } l \in \Gamma_s.$$

If $k \in \Gamma_s$, then $2 = \langle \gamma_s, \check{\gamma_s}\rangle = \langle \alpha_k, \check{\gamma_s}\rangle = \sum_{l\in\Gamma_s}\langle \alpha_k, \tilde{H}_l\rangle = \langle \alpha_k, \tilde{H}_k\rangle$ and for $j \in J_k$, $0 = \langle \alpha_j, \check{\gamma_s}\rangle = \sum_{l\in\Gamma_s}\langle \alpha_j, \tilde{H}_l\rangle = \langle \alpha_j, \tilde{H}_k\rangle$. Hence $(E_k, \tilde{H}_k = H_k, F_k)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}(I_k)$ and $\rho_{\mathfrak{a}}(\Delta(I_k) \cup \{0\}) = \{\pm\gamma_s\} \cup \{0\}$. Thus (I_k, J_k) is an irreducible C-admissible pair.

(6) follows from the fact that $\mathfrak{g}(I_k)$ is A_1-graded for all $k \in \Gamma_s$.

(7) Let $k \in I'$ such that $\rho_{\mathfrak{a}}(\alpha_k) \in \Sigma^+_{re}$. Suppose that $k \notin I'_{re}$. Then for all $s \in \bar{I}$, $k \notin \Gamma_s$, so $\langle \rho_{\mathfrak{a}}(\alpha_k), \check{\gamma_s}\rangle = \sum_{l\in\Gamma_s}\langle \alpha_k, H_l\rangle \leq 0$. By [15], Theorem 5.4, $\rho_{\mathfrak{a}}(\alpha_k) \in \Sigma^+_{im}$, contradicting the fact that $\rho_{\mathfrak{a}}(\alpha_k)$ is a real root of Σ. Hence $k \in I'_{re}$. The assertion (8) is obvious. ■

Definition 4.9. ([7]) (1) If $I_{re} = I$ (so $J_{re} = J$ and $I'_{im} = \emptyset$), the gradation of $\mathfrak{g}$ by the indecomposable root system Σ is called *real* (we say also that $\mathfrak{g}$ is really Σ-graded).

We will see in Proposition 4.14 below that such a gradation is necessarily finite.

(2) If moreover $J_{re} = J = \emptyset$, the real gradation of $\mathfrak{g}$ by Σ is called *maximal* (we say also that $\mathfrak{g}$ is maximally Σ-graded).

(3) If $I'_{im} \neq \emptyset$, the gradation is called *imaginary.*

Proposition 4.10. *Suppose that $\mathfrak{g}$ is graded by an indecomposable root system Σ. Let $\Pi_{\mathfrak{a}} = \{\gamma_s, s \in \bar{I}\}$ be a root basis of Σ and suppose that the root basis $\Pi = \{\alpha_i, i \in I\}$ of Δ is adapted to $\Pi_{\mathfrak{a}}$. For any effective factor $\mathfrak{g}_i$ of the gradation, the projection ψ_i (onto $\mathfrak{g}_i$ along the sum of the other factors) when restricted to $\mathfrak{m}$ is still injective and $\mathfrak{g}_i$ is Σ-graded with grading subalgebra $\mathfrak{m}_i := \psi_i(\mathfrak{m})$. Moreover, the root basis Π_i of $\mathfrak{g}_i$ induced by Π is adapted to the root basis $\Pi_{\mathfrak{a}}$ of Σ (viewed as the root system corresponding to $\mathfrak{m}_i$).*

Proof. For $s \in \bar{I}$, write $\psi_i(\check{\gamma_s}) = \sum_{k \in \Gamma_i(s)} H_k =: H_i^s$ (cf. Proposition 4.8, assertion 5). As $\mathfrak{g}_i$ is an effective factor of the gradation, $H_i^s \neq 0$ for all $s \in \bar{I}$. By Lemma 4.7, $(H_i^s)_{s \in \bar{I}}$ is free in $\mathfrak{h}$. Hence, the restriction of ψ_i to $\mathfrak{a}'$ is injective. Since $\mathfrak{m}$ is indecomposable, $\ker(\psi_i) \cap \mathfrak{m} \subset \mathfrak{c}_{\mathfrak{a}} \subset \mathfrak{a}'$ and thus ψ_i is injective on $\mathfrak{m}$. The same argument used above shows that $\mathfrak{g}_i$ is Σ-graded, with grading subalgebra $\mathfrak{m}_i := \psi_i(\mathfrak{m})$, and the root basis Π_i is clearly adapted to $\Pi_{\mathfrak{a}}$. ∎

Proposition 4.11. *Suppose that $\mathfrak{g}$ is graded by an indecomposable root system Σ. Let $\Pi_{\mathfrak{a}} = \{\gamma_s, s \in \bar{I}\}$ be a root basis of Σ. Suppose that the root basis $\Pi = \{\alpha_i, i \in I\}$ of Δ is adapted to $\Pi_{\mathfrak{a}}$ and let $J := \{j \in I; \rho_{\mathfrak{a}}(\alpha_j) = 0\}$. Then $\mathfrak{g}$ is finitely Σ-graded if and only if J is of finite type.*

Proof. The necessary condition is obvious since $V_0 = \mathfrak{h} + \mathfrak{g}_J$ should be finite dimensional. Conversely, suppose J of finite type (i.e, V_0 finite dimensional) and consider for $\gamma = \sum_{s \in \bar{I}} k_s \gamma_s \in \Sigma$ the corresponding weight space $V_\gamma = \sum_{\alpha \in \Gamma} \mathfrak{g}_\alpha$. We will see that Γ is finite. Using the Chevalley-Cartan involution ω, we may assume $\gamma \in \Sigma^+$ (and so $\Gamma \subset \Delta^+$). Let

$$\alpha = \sum_{j \in J} m_j(\alpha)\alpha_j + \sum_{r \in I'_{re}} n_r(\alpha)\alpha_r + \sum_{i \in I'_{im}} n_i(\alpha)\alpha_i \in \Gamma.$$

For $i \in I'_{im}$, write $\rho_{\mathfrak{a}}(\alpha_i) = \sum_{s \in \bar{I}} k_{s,i}\gamma_s$, then

$$\begin{aligned}\gamma = \rho_{\mathfrak{a}}(\alpha) &= \sum_{r \in I'_{re}} n_r(\alpha)\rho_{\mathfrak{a}}(\alpha_r) + \sum_{i \in I'_{im}} n_i(\alpha)\rho_{\mathfrak{a}}(\alpha_i) \\ &= \sum_{s \in \bar{I}} \Big(\sum_{r \in \Gamma_s} n_r(\alpha)\Big)\gamma_s + \sum_{i \in I'_{im}} n_i(\alpha) \sum_{s \in \bar{I}} k_{s,i}\gamma_s \\ &= \sum_{s \in \bar{I}} \Big(\sum_{r \in \Gamma_s} n_r(\alpha) + \sum_{i \in I'_{im}} n_i(\alpha)k_{s,i}\Big)\gamma_s.\end{aligned}$$

In particular we have, for $s \in \bar{I}$,

$$\sum_{r \in \Gamma_s} n_r(\alpha) + \sum_{i \in I'_{im}} n_i(\alpha)k_{s,i} = k_s$$

and $\{n_k(\alpha);\ k \in I', \alpha \in \Gamma\}$ is bounded.

It remains to see that $\{m_j(\alpha); j \in J, \alpha \in \Gamma\}$ is also bounded. Let $\check{\rho_J} = \frac{1}{2}\sum_{\alpha\in\Delta_J^+}\check{\alpha}$ be the half sum of positive coroots of Δ_J.

Note that $\langle \alpha_j, \check{\rho_J}\rangle = 1$ for all $j \in J$ and put $ht_J(\alpha) = \langle \alpha, \check{\rho_J}\rangle = \sum_{j\in J} m_j(\alpha)$: the height of α relatively to J. Since the subgroup W_J of the Weyl group W is finite and Γ is stable under W_J, we may assume that α is of minimal height relatively to J in its orbit $W_J\alpha$; in which case $\langle \alpha, \check{\alpha_j}\rangle \leq 0$, for all $j \in J$. Therefore, the condition $\langle \alpha, \check{\rho_J}\rangle \leq 0$ implies

$$\sum_{j\in J} m_j(\alpha) \leq -\sum_{i\in I'} n_i(\alpha)\,\langle \alpha_i, \check{\rho}_J\rangle$$

and $\{m_j(\alpha); j \in J, \alpha \in \Gamma\}$ is bounded. Hence, Γ is finite and V_γ is finite dimensional. ■

Lemma 4.12. ([7], Lemma 3.20) *Suppose that $\mathfrak{g}$ is Σ-graded (resp. finitely Σ-graded). Let $\mathfrak{l}$ be a Kac-Moody subalgebra of $\mathfrak{g}$ containing $\mathfrak{m}$. Then $\mathfrak{l}$ is Σ-graded (resp. finitely Σ-graded).*

Proof. This follows from [7], Lemma 3.20, which is obviously valid also for $\mathfrak{g}$ decomposable. ■

Proposition 4.13. *Suppose that $\mathfrak{g}$ is graded by an indecomposable root system Σ. Let $\Pi_\mathfrak{a} = \{\gamma_s, s \in \bar{I}\}$ be a root basis of Σ and suppose that the root basis $\Pi = \{\alpha_i, i \in I\}$ of Δ is adapted to $\Pi_\mathfrak{a}$. Then the restriction $(.\,,.)_\mathfrak{a}$ to $\mathfrak{m}$ of the invariant symmetric bilinear form $(.\,,.)$ of $\mathfrak{g}$ is nondegenerate. In particular, the grading subalgebra $\mathfrak{m}$ is symmetrizable and $\mathfrak{a}' := \mathfrak{a}\cap\mathfrak{m}' = \mathfrak{a}\cap\mathfrak{h}'$.*

Proof. Recall from Proposition 4.8 that, for $s \in \bar{I}$, $\check{\gamma_s} = \sum_{k\in\Gamma_s} H_k$ and thus $(\check{\gamma_s}, \check{\gamma_s})_\mathfrak{a} = \sum_{k\in\Gamma_s}(H_k, H_k) > 0$, $\forall s \in \bar{I}$. It follows that $(.\,,.)_\mathfrak{a}$ is nondegenerate (see the equation (2) in the subsection 2.4 above) and hence the grading subalgebra $\mathfrak{m}$ is symmetrizable. It is clear that $\mathfrak{a}' \subset \mathfrak{h}'$. Conversely, let $a \in \mathfrak{a}\cap\mathfrak{h}'$, then a lies in the orthogonal subspace $\mathfrak{c}^\perp$ of $\mathfrak{c}$ in $\mathfrak{h}$ relative to $(.\,,.)$. As the center $\mathfrak{c}_\mathfrak{a}$ of $\mathfrak{m}$ is contained in $\mathfrak{c}$, $a \in \mathfrak{c}_\mathfrak{a}^\perp \cap \mathfrak{a}$. Since $(.\,,.)_\mathfrak{a}$ is nondegenerate, $\mathfrak{c}_\mathfrak{a}^\perp \cap \mathfrak{a} = \mathfrak{a}'$ and thus $\mathfrak{a}\cap\mathfrak{h}' \subset \mathfrak{a}'$. ■

Proposition 4.14. *Suppose that $\mathfrak{g}$ is graded by an indecomposable root system Σ. Let $\Pi_\mathfrak{a} = \{\gamma_s, s \in \bar{I}\}$ be a root basis of Σ and suppose that the root basis $\Pi = \{\alpha_i, i \in I\}$ of Δ is adapted to $\Pi_\mathfrak{a}$.*

(1) *There exists a realization of the generalized Cartan submatrix $A(I_{re})$ for which the corresponding standard Kac-Moody subalgebra $\mathfrak{g}(I_{re})$ contains the grading subalgebra $\mathfrak{m}$. In such a case, the grading subalgebra $\mathfrak{m}$ is precisely contained in the C-admissible subalgebra $\mathfrak{g}(I_{re})^{J_{re}}$ of $\mathfrak{g}(I_{re})$ associated to the C-admissible pair (I_{re}, J_{re}).*[1]

(2) *The admissible subalgebra $\mathfrak{g}(I_{re})^{J_{re}}$ is , as well as the standard subalgebra $\mathfrak{g}(I_{re})$, finitely Σ-graded with no fictive factors.*

Proof. By Proposition 4.8, assertion 4, the derived subalgebra $\mathfrak{m}'$ is contained in $\mathfrak{g}'(I_{re})$. As it was shown in [7], Proposition 3.19, one can choose a realization

[1] For the definition of the C-admissible subalgebra, see the paragraph 4.2.1 below.

$(\mathfrak{h}(I_{re}), \Pi(I_{re}), \Pi^{\check{}}(I_{re}))$ of the generalized Cartan submatrix $A(I_{re})$ so that $\mathfrak{a}$ is contained in $\mathfrak{h}(I_{re})$ and hence $\mathfrak{m}$ is contained in $\mathfrak{g}(I_{re})$. As the pair (I_{re}, J_{re}) is C-admissible (cf. Proposition 4.8), J_{re} is of finite type and $\mathfrak{g}(I_{re})$ is finitely Σ-graded by Proposition 4.11. The indecomposable components of $\mathfrak{g}(I_{re})$ correspond to those of the C-admissible pair (I_{re}, J_{re}) and so are all effective factors for the gradation. The fact that $\mathfrak{m}$ is contained in the admissible subalgebra $\mathfrak{g}(I_{re})^{J_{re}}$ follows from [7], Proposition 3.19 which can be extended to the (decomposable) finitely Σ-graded Kac-Moody subalgebra $\mathfrak{g}(I_{re})$ by Proposition 4.10. ∎

4.1. Finite gradations revisited

In this section, we refine some results in [7] to characterize finite gradations of the Kac-Moody Lie algebra $\mathfrak{g}$ by indecomposable Kac-Moody root systems. We recall that the Kac-Moody Lie algebra $\mathfrak{g}$ may be here decomposable. Note that, for infinite dimensional (indecomposable) Kac-Moody Lie algebras, the existence of adapted bases for finite gradations as stated in [7], Theorem 3.10 and the advanced preliminary results, is based essentially on the fact $\rho_{\mathfrak{a}}(\Delta_{im}) \subset \Sigma_{im}$. We use these results here to extend them to our context.

Proposition 4.15. ([7], Lemma 3.4) *Suppose that $\mathfrak{g}$ is an infinite dimensional Kac-Moody Lie algebra graded by an indecomposable root system Σ and that the restriction map $\rho_{\mathfrak{a}}$ does not vanish on Δ_{im} (i.e., $\rho_{\mathfrak{a}}(\Delta_{im}) \subset \Sigma_{im}$). Let $\Pi_{\mathfrak{a}}$ be a root basis of Σ, then there exists a root basis Π of Δ such that $\rho_{\mathfrak{a}}(\Delta_{im}^{+}) \subset \Sigma_{im}^{+}$.*

N.B. As it was mentioned in the beginning of §3 in [7], Lemma 3.4 is available for infinite dimensional decomposable Kac-Moody Lie algebras.

Lemma 4.16. ([7], Lemma 3.9) *Suppose that $\mathfrak{g}$ is an infinite dimensional Kac-Moody Lie algebra. Let $p \in \overline{X}$ be such that $\langle \alpha, p\rangle \in \mathbb{Z}$, $\forall \alpha \in \Delta$, and $\langle \beta, p\rangle > 0$, $\forall \beta \in \Delta_{im}^{+}$. Then $p \in \mathring{X}$, where $\overline{X}$ (resp. $\mathring{X}$) is the closure (resp. interior) of the Tits cone X.*

Proof. Note that if $\mathfrak{g}$ is indecomposable, this result corresponds to Lemma 3.9 of [7]. If $\mathfrak{g}$ is decomposable, then each of the cones X, $\overline{X}$ or $\mathring{X}$ is the product of the analogous cones corresponding to the indecomposable components of $\mathfrak{g}$. If $\mathfrak{g}$ is indecomposable of finite type, then $\Delta_{im} = \emptyset$, $X = \overline{X} = \mathring{X}$ and the result is trivial in this case. Thus the Lemma follows from [7], Lemma 3.9, applied to the infinite-dimensional indecomposable components of $\mathfrak{g}$. ∎

Proposition 4.17. *Suppose that $\mathfrak{g}$ is graded by an indecomposable root system Σ and the restriction map $\rho_{\mathfrak{a}}$ does not vanish on Δ_{im} (i.e, $\rho_{\mathfrak{a}}(\Delta_{im}) \subset \Sigma_{im}$). Then any root basis Π_a of Σ has an adapted root basis Π of Δ and there exists a finite type subset J of I such that $V_0 = \mathfrak{h} + \mathfrak{g}_J$ is a reductive subalgebra of $\mathfrak{g}$. In particular, $\mathfrak{g}$ is finitely Σ-graded.*

Proof. Fix a root basis $\Pi_a = \{\gamma_i,\ i \in \bar{I}\}$ of Σ. By Proposition 4.15 there exists a root basis Π of Δ indexed by I such that $\rho_{\mathfrak{a}}(\Delta_{+}^{im}) \subset \Sigma_{+}^{im}$. Let $p \in \mathfrak{a}$ be such that $\langle \gamma_i, p\rangle = 1$, $\forall i \in \bar{I}$ and let $\tilde{p} := \psi_{re}(p) \in \mathfrak{h}_{\mathbb{R}}$ (cf. Remark 4.2). By assumption we have $\langle \alpha, \tilde{p}\rangle = \langle \alpha, p\rangle > 0$, $\forall \alpha \in \Delta_{im}^{+}$. Thus, $\tilde{p}$ satisfies the conditions of the Lemma 4.16 and hence lies in the interior $\mathring{X}$ of the Tits cone X of $\mathfrak{g}$. As $\mathring{X}$ is the union of finite type facets of X, we may assume that $\tilde{p}$ lies in the facet of finite type J of the fundamental chamber associated to the root basis Π. Therefore the root basis Π of Δ is adapted to Π_a and $V_0 = \mathfrak{h} + \mathfrak{g}_J$ is a reductive subalgebra of $\mathfrak{g}$. By Proposition 4.11, $\mathfrak{g}$ is finitely Σ-graded. ■

Proposition 4.18. *Suppose that the Kac-Moody algebra $\mathfrak{g}$ is graded by an indecomposable root system Σ. Then the subspace V_0 is finite dimensional if and only if $\mathfrak{g}$ is finite dimensional or $\rho_\mathfrak{a}(\Delta_{im}) \subset \Sigma_{im}$.*

Proof. The result is trivial if $\Delta_{im} = \emptyset$. We may assume that $\Delta_{im} \neq \emptyset$. If $\alpha \in \Delta_{im}$ is an imaginary root satisfying $\rho_\mathfrak{a}(\alpha) = 0$, then, for any positive integer n, $\mathfrak{g}_{n\alpha} \subset V_0$ and V_0 is thus infinite dimensional. Hence we get the necessary condition. Conversely, suppose that the restriction map $\rho_\mathfrak{a}$ does not vanish on Δ_{im}. By the Proposition 4.17, $\mathfrak{g}$ is finitely Σ-graded and thus V_0 is finite dimensional. ■

Theorem 4.19. *Suppose the Kac-Moody algebra $\mathfrak{g}$ graded by an indecomposable root system Σ. The following assertions are equivalent:*

(1) *The Kac-Moody algebra $\mathfrak{g}$ is finitely Σ-graded,*
(2) *V_0 is finite dimensional,*
(3) *Any root basis $\Pi_\mathfrak{a}$ of Σ has an adapted root basis $\Pi = \{\alpha_i, i \in I\}$ of Δ such that $V_0 = \mathfrak{h} + \mathfrak{g}_J$ for a finite type subset J of I.*

Proof. This follows from Propositions 4.18, 4.17 and 4.11. ■

Corollary 4.20. *Suppose that $\mathfrak{g}$ is graded by an indecomposable root system Σ of affine type. Then any effective factor $\mathfrak{g}_1$ of the gradation is finitely Σ-graded. We rediscover, in particular, the result of Nervi [22] on the finiteness of the gradations of affine Kac-Moody Lie algebras by affine root systems.*

Proof. Let Δ^1 be the union of effective factors of the gradation. Let $\mathfrak{g}^1$ be the Kac-Moody subalgebra associated to Δ^1. We know from Proposition 3.14 that $\mathfrak{g}^1$ is Σ-graded. Let $\mathfrak{g}_1$ be an effective factor of the gradation and let Δ_1 be the corresponding root system. We know from Proposition 3.16 that $\mathfrak{g}_1$ is of affine type. Let δ_1 be a generator of $\Delta_{1,im}$. As $\rho_\mathfrak{a}(\Delta_1) \cup \{0\} = \Sigma \cup \{0\}$, we have $\rho_\mathfrak{a}(\delta_1) \neq 0$ (otherwise, Σ will be finite) and so $\rho_\mathfrak{a}(\Delta_{1,im}) \subset \Sigma_{im}$. It follows, for $\mathfrak{g}^1$, that $\rho_\mathfrak{a}(\Delta^1_{im}) \subset \Sigma_{im}$. By Theorem 4.19, the subalgebra $\mathfrak{g}^1$ is finitely Σ-graded or equivalently any root basis $\Pi_\mathfrak{a}$ of Σ has an adapted root basis Π^1 of Δ^1. By Proposition 4.10, the effective factor $\mathfrak{g}_1$ is finitely Σ-graded. ■

4.2. The Weyl group of the grading algebra $\mathfrak{m}$ as a subgroup of that of $\mathfrak{g}$

In this section, we suppose that the symmetrizable Kac-Moody algebra $\mathfrak{g}$ is graded by an indecomposable root system Σ and that the root basis Π of Δ is adapted to a root basis $\Pi_\mathfrak{a}$ of Σ. We will see that the Weyl group of the grading subalgebra $\mathfrak{m}$ can be viewed as a subgroup of the Weyl group W of $\mathfrak{g}$ and give a description of the real roots of Δ which are restricted into real roots of Σ. This can be reduced to studying essentially gradations associated with C-admissible pairs and maximal gradations. The notations are those introduced in Definition 4.5 and Proposition 4.8.

4.2.1. The case of gradations associated with C-admissible pairs

Let (I, J) be a C-admissible pair of $\mathfrak{g}$. When the Kac-Moody Lie algebra $\mathfrak{g}$ is indecomposable, the first author and G. Rousseau constructed, in [7], a Kac-Moody subalgebra $\mathfrak{g}^J$ of $\mathfrak{g}$, called the C-admissible subalgebra associated with (I, J), whose root system Δ^J graduates finitely $\mathfrak{g}$. More precisely, we have the following theorem:

Theorem 4.21. ([7], 2.6, 2.11 and 2.14) *Suppose that the Kac-Moody Lie algebra $\mathfrak{g}$ is indecomposable. Let (I, J) be a C-admissible pair of $\mathfrak{g}$ and let $I' := I \setminus J$. For $k \in I'$ let (I_k, J_k) be the irreducible C-admissible pair such that the $\mathfrak{sl}_2$-triple $\{E_k, H_k, F_k\}$ graduates the simple Lie algebra $\mathfrak{g}(I_k)$.*

(1) *Let $\mathfrak{h}^J = \Pi_J^{\perp} = \{h \in \mathfrak{h}, \langle \alpha_j, h\rangle = 0,\ \forall j \in J\}$. For $k \in I'$, denote by $\alpha'_k = \alpha_k|\mathfrak{h}^J$ the restriction of α_k to the subspace $\mathfrak{h}^J$ of $\mathfrak{h}$, and define $\Pi^J = \{\alpha'_k;\ k \in I'\}$, $\Pi^{J\vee} = \{H_k;\ k \in I'\}$. For $k, l \in I'$, let $a'_{k,l} = \langle \alpha_l, H_k\rangle$ and $A^J = (a'_{k,l})_{k,l \in I'}$. Then A^J is an indecomposable and symmetrizable generalized Cartan matrix, $(\mathfrak{h}^J, \Pi^J, \Pi^{J\vee})$ is a realization of A^J and $corank(A^J) = corank(A)$.*

(2) *Let $\mathfrak{g}^J$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}^J$ and E_k, F_k, $k \in I'$. Then $\mathfrak{g}^J$ is the Kac-Moody Lie algebra associated to the realization $(\mathfrak{h}^J, \Pi^J, \Pi^{J\vee})$ of the generalized Cartan matrix A^J.*

(3) *Let $\Delta^J := \Delta(\mathfrak{g}^J, \mathfrak{h}^J)$ be the indecomposable root system of $\mathfrak{g}^J$, then the Kac-Moody Lie algebra $\mathfrak{g}$ is finitely Δ^J-graded, with grading subalgebra $\mathfrak{g}^J$.*

Remark 4.22. The notion of C-admissible subalgebra can be extended to decomposable Kac-Moody Lie algebras: if $\mathfrak{g}_i$, $i = 1, 2, ..., q$, are the indecomosable components of $\mathfrak{g}$ and (I, J) is a C-admissible pair of $\mathfrak{g}$ (that is, for $i = 1, 2, ..., q$, (I, J) induces a C-admissible pair (I^i, J^i) of the indecomposable Kac-Moody subalgebra $\mathfrak{g}_i$, see Remark 4.6) then the corresponding C-admissible subalgebra $\mathfrak{g}^J = \oplus_{1 \le i \le q} \mathfrak{g}_i^{J^i}$ (sum of Lie algebras) and the $\Delta_i^{J^i}$, $i = 1, 2, ..., q$, are the indecomposable components of Δ^J.

Proposition 4.23. *Let (I, J) be a C-admissible pair for the Kac-Moody Lie algebra $\mathfrak{g}$ (which may be decomposable). Let I'_1 be a non empty subset of $I' := I \setminus J$ and let $I^1 := \cup_{k \in I'_1} I_k$, $J^1 := I^1 \cap J = \cup_{k \in I'_1} J_k$ and $J^0 := J \setminus J^1$. The notations are those introduced in Proposition* 2.2 *and Theorem* 4.21.

(1) *There exists a realization $(\mathfrak{h}(I^1), \Pi(I^1), \Pi(I^1)\check{})$ of the generalized Cartan submatrix A_{I^1} of A such that $\mathfrak{h}(I^1) \subset \mathfrak{h}^{J^0}$ and $\mathfrak{h}(I^1)^{J^1} \subset \mathfrak{h}^J$.*

(2) *(I^1, J^1) is a C-admissible pair for the standard Kac-Moody Lie subalgebra $\mathfrak{g}(I^1)$ corresponding to the above realization $(\mathfrak{h}(I^1), \Pi(I^1), \Pi(I^1)\check{})$ of A_{I^1}.*

(3) *The C-admissible subalgebra $\mathfrak{g}(I^1)^{J^1}$ of $\mathfrak{g}(I^1)$, associated to the C-admissible pair (I^1, J^1), is the standard Kac-Moody subalgebra of $\mathfrak{g}^J$ corresponding to the subset I'_1 of I'.*

(4) *The Weyl group $W(I^1)^{J^1}$ of $\mathfrak{g}(I^1)^{J^1}$ can be identified with a subgroup of the Weyl group W^J of $\mathfrak{g}^J$.*

Proof. (1) Note that the subset J^0 of J is of finite type and not connected to I^1. Thus, we obtain:

$$\mathfrak{h} = \mathfrak{h}(J^0) \oplus \mathfrak{h}^{J^0}; \quad \mathfrak{h}'(I^1) \subset \mathfrak{h}^{J^0} \quad \text{and} \quad \mathfrak{h}(J^0) \subset \mathfrak{h}^{I^1}. \tag{4}$$

As $\mathfrak{h}'(I^1) + \mathfrak{h}^{I^1}$ contains $\mathfrak{h}(J^0)$, one can choose, in view of (4), a supplementary subspace $\mathfrak{h}(I^1)''$ of $\mathfrak{h}'(I^1) + \mathfrak{h}^{I^1}$ in $\mathfrak{h}$ such that $\mathfrak{h}(I^1)'' \subset \mathfrak{h}^{J^0}$. Therefore $\mathfrak{h}(I^1) := \mathfrak{h}'(I^1) \oplus \mathfrak{h}(I^1)''$ is contained in $\mathfrak{h}^{J^0}$. Since $J = J^0 \cup J^1$, we get $\mathfrak{h}(I^1)^{J^1} \subset \mathfrak{h}^{J^0} \cap \mathfrak{h}^{J^1} = \mathfrak{h}^J$ as desired.

(2) For any $k \in I_1'$, $I_k \subset I^1$ and (I_k, J_k) is by assumption C-admissible. Hence (I^1, J^1) is a C-admissible pair.

(3) Note that the generalized Cartan matrix $(A_{I^1})^{J^1} := (\langle \alpha_l, H_k \rangle)_{k,l \in I_1'}$ coincides with the principal submatrix $(A^J)_{I_1'}$ of A^J with $(\mathfrak{h}(I^1)^{J^1}, \Pi(I^1)^{J^1}, \Pi(I^1)^{J^1 \vee})$ as a realization. Hence the C-admissible subalgebra $\mathfrak{g}(I^1)^{J^1}$ which is generated by $\mathfrak{h}(I^1)^{J^1} (= \mathfrak{h}(I^1) \cap \mathfrak{h}^J)$ and E_k, F_k, $k \in I_1'$, is the standard Kac-Moody subalgebra of $\mathfrak{g}^J$ corresponding to the subset I_1' of I'.

(4) Actually, the Weyl group $W(I^1)^{J^1}$ of $\mathfrak{g}(I^1)^{J^1}$ coincides with the Weyl group $(W^J)(I_1')$ of the standard Kac-Moody subalgebra $\mathfrak{g}^J(I_1')$ of $\mathfrak{g}^J$ and can be identified with a subgroup of the Weyl group W^J of $\mathfrak{g}^J$ (cf. Proposition 2.2). ∎

Lemma 4.24. ([7], Lemma 2.7) *For $k \in I' := I \setminus J$, let w_k^J be the longest element of the Weyl group $W(I_k)$ generated by the fundamental reflections r_i, $i \in I_k$. Then w_k^J stabilizes $\mathfrak{h}^J$ and induces, by restriction, the fundamental reflection R_k of $\mathfrak{h}^J$ associated to the coroot H_k.*

Remark 4.25. Since the pair (I_k, J_k) is C-admissible, the longest element w_k^J of the Weyl subgroup $W(I_k)$ of W acts on $\mathfrak{h}(I_k)$ as $-\tau_k$, where τ_k is an involutive diagram automorphism of $\Delta(I_k)$ fixing k. In particular, τ_k permutes the elements of J_k and w_k^J is of order 2 as an element of $W(I_k)$ or W. Since I_k is not connected to $J \setminus J_k$, w_k^J (viewed as an element of W) fixes the simple roots α_j, $j \in J \setminus J_k$.

Proposition 4.26. *Let (I, J) be a C-admissible pair and let $W^J := W(\Delta^J)$ the Weyl group of the grading root system Δ^J generated with the fundamental reflexions R_k, $k \in I' := I \setminus J$. For $k \in I' := I \setminus J$, let w_k^J be the longest element of the Weyl group $W(I_k)$. Then we have:*

1. *For $k \neq l \in I'$, the element $R_k R_l$ of W^J is of finite order if and only if $I_k \cup I_l$ is of finite type.*
2. *For $k \neq l \in I'$, the element $w_k^J w_l^J$ of W has the same order as the element $R_k R_l$ of W^J.*

Proof. (1) The result is trivial when $I_k \cup I_l$ is not connected (i.e, I_k is not linked to I_l). We may assume that $I_k \cup I_l$ is connected. Let $I_{kl} := I_k \cup I_l$ and $J_{kl} := J_k \cup J_l$. By Proposition 4.23, the pair (I_{kl}, J_{kl}) is C-admissible and the subgroup $\langle R_k, R_l \rangle$ of W^J generated by R_k and R_l can be identified to the Weyl group $W(I_{kl})^{J_{kl}}$ of the C-admissible subalgebra $\mathfrak{g}(I_{kl})^{J_{kl}}$ associated to (I_{kl}, J_{kl}). Suppose that $R_k R_l$ is of finite order, then $W(I_{kl})^{J_{kl}}$ is finite (or equivalently, the C-admissible subalgebra $\mathfrak{g}(I_{kl})^{J_{kl}}$ is of finite type). By Theorem 4.21, 3) the corresponding gradation of $\mathfrak{g}(I_{kl})$ is finite, hence $\mathfrak{g}(I_{kl})$ is finite dimensional and $I_{kl} := I_k \cup I_l$ is of finite type. The sufficient condition follows from Lemma 4.24.

(2) Since $w_k^J, w_l^J \in W(I_{kl})$, we may assume, as in the previous assertion, that $I_{kl} = I$ and $J_{kl} = J$. Note that, by Lemma 4.24, the restriction of $w_k^J w_l^J$ to $\mathfrak{h}^J$ has the same order as $R_k R_l$. In view of the assertion (1), $w_k^J w_l^J$ is of infinite order if and only if $R_k R_l$ is. Suppose that $R_k R_l$ is of finite order m_{kl} (so is the restriction of $w_k^J w_l^J$ to $\mathfrak{h}^J$). Since $\mathfrak{h}(J)$ is stable by $w_k^J w_l^J$ and $\mathfrak{h} = \mathfrak{h}_J \oplus \mathfrak{h}^J$, it remains to see that $(w_k^J w_l^J)^{m_{kl}}$ is the identity on $\mathfrak{h}_J$. By the first assertion, $I = I_{kl}$ is of finite type and the C-admissible subalgebra $\mathfrak{g}^J$ is a simple Lie algebra of rank 2. The Table 2 below classifies, up to isomorphisms, the C-admissible subalgebras of rank 2 in simple Lie algebras, it is extracted from the list given by Rubenthaler in [29] (or Nervi [21]).

Note that the result is trivial if $J = \emptyset$ is empty (since in such a case the gradation is trivial and the admissible subalgebra coincides with the whole graded Lie algebra). We may assume that $J \neq \emptyset$. Recall from Remark 4.25 the description of the action of w_k^J on $\mathfrak{h}_J$, then a case-by-case study, based on the data in Table 2, confirms the requested result. Note that the case requiring more calculation corresponds to the following C-admissible pair (I, J) for $\mathfrak{g}$ of type A_{3p-1}, $p \geq 2$, where $k = p$, $l = 2p$ and so $J := I \setminus \{p, 2p\}$ has three connected components. The corresponding C-admissible subalgebra $\mathfrak{g}^J$ is of type A_2 and so $R_p R_{2p}$ is of order 3:

Let $w = w_p^J w_{2p}^J$, then for $j \in J = \{1, 2, ..., 3p-1\} \setminus \{p, 2p\}$, we have:

$$w_p^J(\alpha_j) = -\alpha_{2p-j},\ 1 \leq j \neq p \leq 2p-1;\quad w_p^J(\alpha_j) = \alpha_j,\ 2p+1 \leq j \leq 3p-1;$$

$$w_{2p}^J(\alpha_j) = \alpha_j,\ 1 \leq j \leq p-1;\quad w_{2p}^J(\alpha_j) = -\alpha_{4p-j},\ p+1 \leq j \neq 2p \leq 3p-1.$$

Hence $w(\alpha_j) = -\alpha_{2p-j}$, $1 \leq j < p$; $w(\alpha_j) = -\alpha_{4p-j}$, $p < j < 2p$; $w(\alpha_j) = \alpha_{j-2p}$, $2p < j < 3p$. Then one can verify easily that $w^3(\alpha_j) = \alpha_j$, for all $j \in J$ and so $w = w_p^J w_{2p}^J$ and $R_p R_{2p}$ are both of order 3. ■

Theorem 4.27. *Let (I, J) be a C-admissible pair of $\mathfrak{g}$. For $k \in I' := I \setminus J$, let w_k^J be the longest element of the Weyl group $W(I_k)$ generated by the fundamental reflections r_i, $i \in I_k$. Then the Weyl group W^J of the C-admissible subalgebra $\mathfrak{g}^J$ can be identified to the subgroup of W generated by $\{w_k^J,\ k \in I'\}$.*

Proof. Let $(m_{k,l})_{k,l \in I'}$ be the Coxeter matrix of the Weyl group W^J (which is a Coxeter group on $S^J := \{R_k,\ k \in I'\}$). As the relations $R_k^2 = 1$, $(R_k R_l)^{m_{k,l}} = 1$ when $m_{k,l}$ is finite, define a presentation of the group W^J, there exists, by Proposition 4.26, a unique group homomorphism $\iota : W^J \to W$ such that $\iota(R_k) = w_k^J$, $\forall k \in I'$. By Lemma 4.24, ι is an embedding. ■

Proposition 4.28. *Let (I, J) be a C-admissible pair of $\mathfrak{g}$. Let $\mathfrak{h}^J = \mathfrak{h}_J^\perp$ be the standard Cartan subalgebra of the C-admissible subalgebra $\mathfrak{g}^J$ and $W^J = W(\Delta^J)$ the corresponding Weyl group (viewed as a subgroup of W). Let W_J be the subgroup of W generated by the fundamental reflections r_j, $j \in J$. Then we have:*

(1) *The subgroup W_J is the fixator of $\mathfrak{h}^J$ in W.*

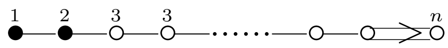
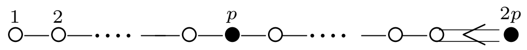
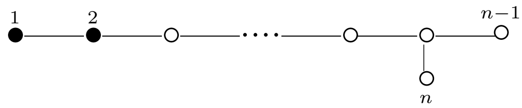
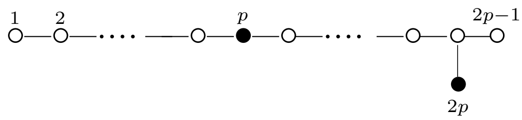
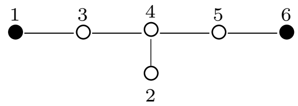
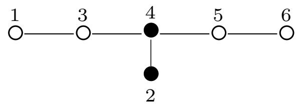
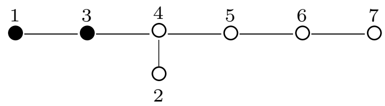
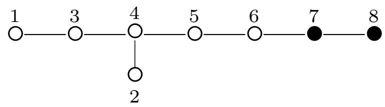
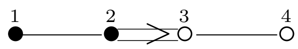
(2) *The subgroup W^J of W normalizes W_J and $W^J W_J = W^J \ltimes W_J$.*

N.B. We will see in Corollary 4.32 below that $W^J \ltimes W_J$ is exactly the stabilizer of $\mathfrak{h}^J$ in W.

Proof. (1) The fact that W_J fixes $\mathfrak{h}^J$ is obvious. Conversely, let $w \in W$ be an element fixing $\mathfrak{h}^J$, then w fixes pointwise the open fundamental chamber $Int(C^J)$ of $\mathfrak{g}^J$, relative to the root basis Π^J, which is the facet of type J of the closed fundamental chamber C of $\mathfrak{g}$. By [15, 3.12], $w \in W_J$.

(2) Let $k \in I' := I \setminus J$ and $j \in J_k$, then $R_k r_j R_k = w_k^J r_j w_k^J = r_{\tau_k(j)} \in W_J$ (cf. Remark 4.25). It is clear that w_k^J commutes with r_j for all $j \in J \setminus J_k$; hence W^J normalizes W_J. Since W_J is the fixator of $\mathfrak{h}^J$ in W, we get $W^J \cap W_J = \{1_W\}$ and then one can define the semi-direct product subgroup $W^J \ltimes W_J$. ■

Table 2 (List of 2 rank C-admissible subalgebras in simple Lie algebras)

$\mathfrak{g}$	C-admissible pair (I, J)	$\mathfrak{g}^J$
$A_n,\ _{n=3p-1\geq 2}$		A_2
$B_n,\ _{n\geq 3}$		B_2
$C_{2p},\ _{p\geq 1}$		C_2
$D_n,\ _{n\geq 4}$		B_2
$D_{2p},\ _{p\geq 4,\ p\ pair}$		B_2
E_6		A_2
E_6		G_2
E_7		G_2
E_8		G_2
F_4		G_2

4.2.2. The case of maximal gradations

We suppose here that the Kac-Moody Lie algebra $\mathfrak{g}$ is maximally Σ-graded. Recall that in this case the root basis $\Pi = \{\alpha_i, i \in I\}$ of Δ is adapted to the root basis $\Pi_{\mathfrak{a}} = \{\gamma_s, s \in \bar{I}\}$ of Σ, with $J = \emptyset$ and $\rho_{\mathfrak{a}}(\Delta) = \Sigma$. Let $\bar{W}$ denote the Weyl group of the grading root system Σ of $\mathfrak{m}$ generated by the fundamental reflexions $\bar{r}_s, s \in \bar{I}$.

Proposition 4.29. *For $s \in \bar{I}$, let $w_s := \Pi_{i\in\Gamma_s} r_i \in W$. Then we have:*
(a) $\bar{r}_s = w_s|\mathfrak{a}$, $s \in \bar{I}$.
(b) *For $s,t \in \bar{I}$, $\bar{r}_s\bar{r}_t$ has the same order as the element $w_s w_t$ of W.*
(c) *The Weyl group $\bar{W}$ of the grading subalgebra $\mathfrak{m}$ can be identified with the subgroup of W generated by $\{w_s,\ s \in \bar{I}\}$.*

Proof. (a) Let $s \in \bar{I}$. Recall that in this case $\gamma_s^\vee = \sum_{k\in\Gamma_s} \alpha_k^\vee$ is the sum of mutually orthogonal simple coroots, so w_s is the product of mutually commuting reflections and is as $\bar{r}_s$ of order 2. Hence $w_s(\gamma_s^\vee) = -\gamma_s^\vee = \bar{r}_s(\gamma_s^\vee)$. Note that $\ker\gamma_s = \bigcap\limits_{k\in\Gamma_s} \ker(\alpha_k) \cap \mathfrak{a}$, so w_s, like $\bar{r}_s$, fixes $\ker\gamma_s$ pointwise, proving (a).

(b) Note that w_s is like $\bar{r}_s$ of order 2. Let $s \neq t \in \bar{I}$. If $\bar{r}_s\bar{r}_t$ is of infinite order, then $w_s w_t$ is too by (a). Suppose now that $\bar{r}_s\bar{r}_t$ is of finite order $m_{s,t}$. By (a), $(w_s w_t)^{m_{s,t}}$ fixes $\mathfrak{a}$ pointwise; in particular $(w_s w_t)^{m_{s,t}}$ fixes the open fundamental chamber $\mathring{C}_\mathfrak{a}$, relative to the root basis $\Pi_\mathfrak{a}$ of Σ, as well as its image $\psi_{re}(\mathring{C}_\mathfrak{a})$ in $\mathfrak{h}_\mathbb{R}$ (cf. Remark 4.2 for the notations). As $\rho_\mathfrak{a}(\Delta^+) = \Sigma^+$, $\psi_{re}(\mathring{C}_\mathfrak{a})$ is contained in the open fundamental chamber $\mathring{C}$ relative to Π. Hence $(w_s w_t)^{m_{s,t}} = 1_W$ (cf. [15, 3.12]) and $(w_s w_t)$ is of the same order as $\bar{r}_s\bar{r}_t$.

(c) As the relations $\bar{r}_s{}^2 = 1$, $(\bar{r}_s\bar{r}_t)^{m_{s,t}} = 1$ when $m_{s,t}$ is finite, define a presentation for the Weyl group $\bar{W}$ of Σ, there exists following (b) a unique group homomorphism $\iota\colon \bar{W} \to W$ such that $\iota(\bar{r}_s) = w_s$, $s \in \bar{I}$, which is injective by (a). ∎

4.2.3. The general case

We return to the general situation where $\mathfrak{g}$ is graded by an indecomposable root system Σ, with grading subalgebra $\mathfrak{m}$, and the root basis Π of Δ is adapted to the root basis $\Pi_\mathfrak{a}$ of Σ. The notations are those introduced in Proposition 4.8. Recall that in such a case we have

$$\mathfrak{m} \subset \mathfrak{g}(I_{re})^{J_{re}} \subset \mathfrak{g}(I_{re}) \subset \mathfrak{g}$$

and, in view of Proposition 4.14 and Lemma 4.12, all the intermediate Kac-Moody subalgebras are Σ-graded, with $\mathfrak{g}(I_{re})^{J_{re}}$ maximally Σ-graded. Combining Proposition 4.29 and Theorem 4.27, we get:

Theorem 4.30. *Under the assumptions introduced above, denote by $W_\mathfrak{a}$ the Weyl group of the grading subalgebra $\mathfrak{m}$. Then the following embeddings hold:*

$$W_\mathfrak{a} \hookrightarrow W(I_{re})^{J_{re}} \hookrightarrow W(I_{re}) \hookrightarrow W.$$

In particular $W_\mathfrak{a}$ can be identified to a subgroup of $W(I_{re})$ and W.

Proposition 4.31. *Under the assumptions introduced above in Section 4.2.3, let W_J be the subgroup of W generated by the fundamental reflections r_j, $j \in J$. Then we have:*
(1) *$W_J = W_{J_{re}} W_{J^\circ} = W_{J^\circ} W_{J_{re}}$ is isomorphic to $W_{J_{re}} \times W_{J^\circ}$.*
(2) *The subgroup W_J is the fixator in W of the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$.*
(3) *The Weyl group $W_\mathfrak{a}$ of Σ, viewed as a subgroup of W, normalizes W_J and $W_\mathfrak{a} \cap W_J = \{1_W\}$.*

The stabilizer of $\mathfrak{a}$ in W is the subgroup $W_\mathfrak{a} \ltimes W_J$.

Proof. (1) This follows from the fact that J_{re} is not connected to J°.

(2) It is clear that W_J fixes $\mathfrak{a}$ pointwise since $\mathfrak{a} \subset \mathfrak{h}^J$. Conversely, let $w \in W$ be an element fixing $\mathfrak{a}$. Let p be an element of the open fundamental chamber $\mathring{C}_\mathfrak{a}$ of $\mathfrak{m}$ relative to the root basis $\Pi_\mathfrak{a}$ of Σ and let $\tilde{p} = \psi_{re}(p)$ its image in $\mathfrak{h}_\mathbb{R}$. By assumption, $\tilde{p}$ is fixed by w and lies in the facet of type J of the closed fundamental chamber C of $\mathfrak{h}_\mathbb{R}$ relative to Π. Thus, $w \in W_J$ by [15, 3.12].

(3) Since I_{re} is not connected to J°, $W_\mathfrak{a}$, viewed as a subgroup of W, commutes to W_{J°. It suffices to see, in view of assertion 1, that $W_\mathfrak{a}$ normalizes $W_{J_{re}}$. By Theorem 4.30, $W_\mathfrak{a}$ can be viewed as a subgroup of $W(I_{re})^{J_{re}}$ which normalizes $W_{J_{re}}$ by Proposition 4.28. Since W_J is the fixator of $\mathfrak{a}$ in W, we have $W_\mathfrak{a} \cap W_J = \{1_W\}$ and one can define the semi-direct product subgroup $W_\mathfrak{a} \ltimes W_J$.

(4) It is clear that the subgroup $W_\mathfrak{a} \ltimes W_J$ stabilizes $\mathfrak{a}$. Conversely, let $w \in W$ be an element stabilizing $\mathfrak{a}$, then w induces an automorphism $\bar{w}$ of the root system Σ and we have, for $\alpha \in \Delta$, $\rho_\mathfrak{a}(w(\alpha)) = \bar{w}(\rho_\mathfrak{a}(\alpha))$. In particular, $\rho_\mathfrak{a}(w(\alpha)) = 0$ if and only if $\rho_\mathfrak{a}(\alpha) = 0$ and thus we get $w(\Delta_J) = \Delta_J$ whenever $J \neq \emptyset$. As the grading subalgebra $\mathfrak{m}$ is indecomoposable, we may assume, by considering a left multiple of w by $W_\mathfrak{a}$, that $\bar{w}(\Pi_\mathfrak{a}) = \epsilon\Pi_\mathfrak{a}$, with $\epsilon = \pm 1$ (and $\epsilon = 1$ if $\mathfrak{m}$ is of finite type).

Suppose that $\mathfrak{m}$ is not of finite type and $\epsilon = -1$ (that is $\bar{w}(\Pi_\mathfrak{a}) = -\Pi_\mathfrak{a}$). As the root basis Π is adapted to $\Pi_\mathfrak{a}$, $\rho_\mathfrak{a}\big(\Delta^\pm \setminus \Delta_J\big) = \Sigma^\pm$ and we have by assumption, $\rho_\mathfrak{a}(w\big(\Delta^+ \setminus \Delta_J\big)) = \bar{w}(\Sigma^+) = \Sigma^-$. It follows that $w\big(\Delta^+ \setminus \Delta_J\big)$ is an infinite subset of Δ^-, contradicting the fact that the length $\ell(w)$ of w is finite in the Coxeter group W. [Recall that $\ell(w)$ is the cardinal of $\Delta^+_{re} \cap (-w\Delta^+_{re}) = \Delta^+ \cap (-w\Delta^+)$, cf. [16, 4.5] or [31, 5.6]].

Hence $\bar{w}(\Pi_\mathfrak{a}) = \Pi_\mathfrak{a}$ and $w\big(\Delta^\pm \setminus \Delta_J\big) \subset \Delta^\pm \setminus \Delta_J$. As $w(\Delta_J) = \Delta_J$, we get $w\big(\Delta^\pm \setminus \Delta_J\big) = \Delta^\pm \setminus \Delta_J$ and $w\big(\Delta^\pm \cup \Delta_J\big) = \Delta^\pm \cup \Delta_J$. In particular, w normalizes the standard positive parabolic subalgebra $\mathfrak{p}^+_J$ (and the corresponding positive parabolic subgroup P^+_J) of type J and then w lies in W_J (see, e.g., [17], 1.8 and 1.17). It follows that the initial element w lies in $W_\mathfrak{a} \ltimes W_J$. ∎

Corollary 4.32. *Let (I, J) be a C-admissible pair of $\mathfrak{g}$. Let $\mathfrak{h}^J = \mathfrak{h}^\perp_J$ be the standard Cartan subalgebra of the C-admissible subalgebra $\mathfrak{g}^J$ and $W^J = W(\Delta^J)$ the corresponding Weyl group (viewed as a subgroup of W). Let W_J be the subgroup of W generated by the fundamental reflections r_j, $j \in J$. Then the stabilizer of $\mathfrak{h}^J$ in W is the subgroup $W^J \ltimes W_J$.*

Proof. Since (I, J) is a C-admissible pair, $\mathfrak{g}$ is (finitely) Δ^J-graded, W^J is the Weyl group of the grading subalgebra $\mathfrak{g}^J$ and the root basis Π of Δ is adapted to the root basis Π^J of Δ^J. The result follows from Proposition 4.31. ∎

Proposition 4.33. *Under the assumptions made in Section 4.2.3, let $\alpha \in \Delta_{re} \setminus \Delta_J$ and let $\rho_\mathfrak{a}(\alpha)$ its restriction to $\mathfrak{a}$. Then $\rho_\mathfrak{a}(\alpha)$ is a real root of Σ if and only if there exist $k \in I'_{re}$ and $w \in W_\mathfrak{a}$ such that $w(\alpha) \in \Delta_{I_k} \setminus \Delta_{J_k}$.*

Proof. Note that, by Theorem 4.30 and Proposition 4.31, any element $w \in W_\mathfrak{a}$ can be viewed as an element of W stabilizing $\mathfrak{a}$ and thus

$$\rho_\mathfrak{a}(w(\alpha)) = w(\rho_\mathfrak{a}(\alpha)), \ \forall \alpha \in \Delta. \tag{5}$$

Let $\alpha \in \Delta_{re} \setminus \Delta_J$ such that $\rho_{\mathfrak{a}}(\alpha) \in \Sigma_{re}$, then there exists $w \in W_{\mathfrak{a}}$ such that $w(\rho_{\mathfrak{a}}\alpha)) = \gamma_s$ is a simple root of $\Pi_{\mathfrak{a}}$. Thus, $\rho_{\mathfrak{a}}(w(\alpha)) = w(\rho_{\mathfrak{a}}(\alpha)) = \gamma_s$. As the root bases Π and $\Pi_{\mathfrak{a}}$ are assumed to be adapted, $w(\alpha) \in \Delta^+ \setminus \Delta_J$ and, by Proposition 4.8, assertion 2, there exists $k \in \Gamma_s$ such that $w(\alpha) \in \Delta^+_{I_k} \setminus \Delta_{J_k}$. In view of the relation (5), the reciprocal is obvious since , for $k \in I'_{re}$, $\rho_{\mathfrak{a}}(\Delta_{I_k} \setminus \Delta_{J_k}) = \{\gamma_s, -\gamma_s\} \subset \Sigma_{re}$. ■

Proposition 4.34. *Suppose that the Kac-Moody Lie algebra $\mathfrak{g}$ is graded by an indecomposable root system Σ with grading subalgebra $\mathfrak{m}$. Let Π (resp. $\Pi_{\mathfrak{a}}$) be a root basis of Δ (resp. Σ) and X (resp. $X_{\mathfrak{a}}$) the corresponding Tits cone. Let $Aut(\mathfrak{g}, \mathfrak{g}')$ be the subgroup of the automorphisms of $\mathfrak{g}$ fixing $\mathfrak{g}'$ pointwise.*

1. *Suppose that the root basis Π of Δ is adapted to $\Pi_{\mathfrak{a}}$, then, by acting the group $Aut(\mathfrak{g}, \mathfrak{g}')$, one can modify the grading subalgebra $\mathfrak{m}$ so that $\mathfrak{a}_{\mathbb{R}} \subset \mathfrak{h}_{\mathbb{R}}$. In such a case we have $\tilde{X}_{\mathfrak{a}} = X_{\mathfrak{a}} \subset X$.*
2. *Suppose $\tilde{X}_{\mathfrak{a}} \subset X$, then there exists a W-conjugate of Π which is adapted to $\Pi_{\mathfrak{a}}$.*

Proof. (1) Let $\mathfrak{g}_{\mathbb{R}}$ be the split real form of $\mathfrak{g}$, generated by $\mathfrak{h}_{\mathbb{R}}$ and the Chevalley generators e_i, f_i, $i \in I$, and let σ' be the corresponding normal semi-involution. Let σ'_a be a normal semi-involution of $\mathfrak{m}$ stabilizing $\mathfrak{a}$ such that $\mathfrak{a}_{\mathbb{R}} = \mathfrak{a}^{\sigma'_a}$ is the corresponding split Cartan subalgebra. Note that $\mathfrak{a}'_{\mathbb{R}} = \sum_{s \in \bar{I}} \mathbb{R}\check{\gamma_s} \subset \mathfrak{h}'_{\mathbb{R}}$ and, by Proposition 4.13, $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{h}'$; hence $\mathfrak{a}'_{\mathbb{R}} = \mathfrak{a}_{\mathbb{R}} \cap \mathfrak{h}'$. Let $\mathfrak{a}''$ be a σ'_a-stable supplementary subspace of $\mathfrak{a}'$ in $\mathfrak{a}$ (that is $\mathfrak{a}''_{\mathbb{R}} := (\mathfrak{a}'')^{\sigma'_a}$ is a supplementary subspace of $\mathfrak{a}'_{\mathbb{R}}$ in $\mathfrak{a}_{\mathbb{R}}$ and $\mathfrak{a}'' = \mathfrak{a}''_{\mathbb{R}} \oplus i\mathfrak{a}''_{\mathbb{R}}$). Note that $\mathfrak{a}'' \cap \mathfrak{h}' = \{0\}$ and one can choose a supplementary subspace $\mathfrak{t}''$ of $\mathfrak{h}'$ in $\mathfrak{h}$ containing $\mathfrak{a}''$. As $\mathfrak{g}$ is Σ-graded, the real form of $\mathfrak{t}''$:

$$\mathfrak{t}''_{\mathbb{R}} := \{t \in \mathfrak{t}'';\ \ \alpha(t) \in \mathbb{R}, \forall \alpha \in \Delta\}$$

contains $\mathfrak{a}''_{\mathbb{R}}$ and hence $\mathfrak{t}_{\mathbb{R}} := \mathfrak{h}'_{\mathbb{R}} \oplus \mathfrak{t}''_{\mathbb{R}}$ is a real form of $\mathfrak{h}$ containing $\mathfrak{h}'_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}}$. Thus, $\mathfrak{t}_{\mathbb{R}}$ is the standard split Cartan subalgebra of the split real form $\mathfrak{g}^{\tau'}$ associated to the normal semi-involution τ' of $\mathfrak{g}$ fixing the Chevalley generators e_i, f_i, $i \in I$, and stabilizing $\mathfrak{h}$ so that $\mathfrak{t}_{\mathbb{R}} = \mathfrak{h}^{\tau'}$. As the two normal semi-involutions σ' and τ' coincide on $\mathfrak{g}'$, they are $Aut(\mathfrak{g}, \mathfrak{g}')$-conjugate by [26, 2.4]. (Note that the group $Aut(\mathfrak{g}, \mathfrak{g}')$ is denoted by $\mathrm{Tr}(\mathfrak{g}, \mathfrak{g}', \mathfrak{c})$ (or Tr for short) in [26, 2.4] or [3, 4.1]). So there exists $T \in Aut(\mathfrak{g}, \mathfrak{g}')$ such that $T(\mathfrak{t}_{\mathbb{R}}) = \mathfrak{h}_{\mathbb{R}}$ and, replacing $\mathfrak{m}$ by $T(\mathfrak{m})$, we may assume $\mathfrak{a}_{\mathbb{R}} \subset \mathfrak{h}_{\mathbb{R}}$. Let $C_{\mathfrak{a}}$ (resp. C) be the fundamental chamber with respect to $\Pi_{\mathfrak{a}}$ (resp. Π) in $\mathfrak{a}_{\mathbb{R}}$ (resp. $\mathfrak{h}_{\mathbb{R}}$). As $\mathfrak{a}_{\mathbb{R}}$ is contained in $\mathfrak{h}_{\mathbb{R}}$ and the root bases Π and $\Pi_{\mathfrak{a}}$ are assumed to be adapted, we get by Proposition 4.3, $C_{\mathfrak{a}} = \tilde{C}_{\mathfrak{a}} \subset C$. Hence $X_{\mathfrak{a}} = \cup_{w \in W_{\mathfrak{a}}} wC_{\mathfrak{a}}$ is contained in X by Theorem 4.30.

(2) Let p be an element of the open fundamental chamber $\overset{\circ}{C}_{\mathfrak{a}}$ with respect to $\Pi_{\mathfrak{a}}$; by assumption its image $\tilde{p}$ in $\mathfrak{h}_{\mathbb{R}}$ is contained in a chamber wC of X, with $w \in W$, then one can see easily that the root basis $w(\Pi)$ associated to wC is adapted to the root basis $\Pi_{\mathfrak{a}}$. ■

Corollary 4.35. *Suppose that the Kac-Moody Lie algebra $\mathfrak{g}$ is indecomposable of affine type and graded by an indecomposable finite root system Σ. Then there is no adapted root basis for the gradation.*

Proof. Let $\Pi_{\mathfrak{a}}$ be a root basis of Σ and suppose that there exists a root basis Π of Δ which is adapted to $\Pi_{\mathfrak{a}}$. By Proposition 4.34, we may assume that the corresponding Tits cones satisfy $X_{\mathfrak{a}} \subset X$. As Σ is assumed of finite type, $X_{\mathfrak{a}} = \mathfrak{a}_{\mathbb{R}}$ is a real vector subspace contained in X. It follows from the description of the Tits cone X in the affine case that $\mathfrak{a}_{\mathbb{R}}$ is central in $\mathfrak{g}$, contradiction. ■

Remark 4.36. (1) The proof of Corollary 4.35 can be also easily checked by considering the explicit description of the root system Δ in the affine case.

(2) If the Kac-Moody Lie algebra $\mathfrak{g}$ is graded by an indecomposable finite root system Σ, with adapted root bases Π and $\Pi_{\mathfrak{a}}$, then it follows from Proposition 3.15, Proposition 4.10 and Corollary 4.35 that any effective factor of the gradation is of finite type.

Proposition 4.37. *Suppose that the Kac-Moody Lie algebra $\mathfrak{g}$ is graded by an indecomposable root system Σ and that the root basis Π of Δ is adapted to the root basis $\Pi_{\mathfrak{a}}$ of Σ. Let W_J be the fixator of $\mathfrak{a}$ in the Weyl group W of Δ. Then, for $w \in W$, we have:*

1. $w\Pi$ *is adapted to* $\Pi_{\mathfrak{a}}$ *if and only if* $w \in W_J$.
2. *If* $w(-\Pi)$ *is adapted to* $\Pi_{\mathfrak{a}}$, *then any indecomposable component of* Δ *is either of finite type or contained in* Δ_J. *In particular, any effective factor of the gradation is of finite type as well as* Σ.

Proof. (1) The sufficient condition is obvious. For the necessary condition, let $C_{\mathfrak{a}}$ (resp. C) be the fundamental chamber with respect to $\Pi_{\mathfrak{a}}$ (resp. Π). As the two root bases Π and $w\Pi$ are assumed to be adapted to $\Pi_{\mathfrak{a}}$, we may assume $\tilde{C}_{\mathfrak{a}} := \psi_{re}(C_{\mathfrak{a}}) = C_{\mathfrak{a}}$, and thus $C_{\mathfrak{a}} \subset C \cap wC$ by Proposition 4.3. Suppose that $w \neq 1_W$ and let $\alpha \in \Delta^+$ such that $w^{-1}(\alpha) \in \Delta^-$, then $C \cap wC \subset \ker(\alpha)$. As the open fundamental chamber $\mathring{C}_{\mathfrak{a}}$ of $C_{\mathfrak{a}}$ is contained in the facet of type J of C, such a root α lies in Δ_J^+. Let $w^{-1} = r_{i_1} r_{i_2} ... r_{i_k}$ be a reduced expression of w^{-1}, then $w^{-1}(\alpha_{i_k}) < 0$ and hence $i_k \in J$. It follows that, for $w' := r_{i_k} w$, $w'(\Pi)$ is also adapted to $\Pi_{\mathfrak{a}}$, with $\ell(w') < \ell(w)$. Using induction on $\ell(w)$, we are done.

(2) Note that if Δ is indecomposable and infinite (and $\Delta \neq \Delta_J$) then, by considering an imaginary root with support I, one can see easily that $\Delta \setminus \Delta_J$ is also infinite (see, e.g., [26]). Recall that $w(\Delta^-) \cap \Delta^+$ is finite of cardinal $\ell(w)$. Suppose now that $w(-\Pi)$ is as Π adapted to $\Pi_{\mathfrak{a}}$, then we get $\rho_{\mathfrak{a}}^{-1}(\Sigma^+) = \Delta^+ \setminus \Delta_J = w(\Delta^-) \setminus \Delta_J$. It follows that $w^{-1}(\Delta^+ \setminus \Delta_J) \subset \Delta^-$ and hence $\Delta^+ \setminus \Delta_J$ is finite of cardinal less than $\ell(w)$. Let Δ_1 be an indecomoposable component of Δ and suppose that Δ_1 is not contained in Δ_J, that is $\rho_{\mathfrak{a}}(\Delta_1) \neq \{0\}$ and hence $\Delta_1 \setminus \Delta_J \neq \emptyset$. As we have mentioned above, the fact that $\Delta_1 \setminus \Delta_J$ is finite implies that Δ_1 is also finite. Thus, Δ_1 is of finite type. If moreover Δ_1 is an effective factor, then it is Σ-graded and so Σ is as Δ_1 of finite type. ■

Theorem 4.38. *Suppose that the Kac-Moody Lie algebra $\mathfrak{g}$ is graded by an indecomposable root system Σ and the root basis Π of Δ is adapted to the root basis $\Pi_{\mathfrak{a}}$ of Σ. Suppose that $\Delta_J := \rho_{\mathfrak{a}}^{-1}(\{0\}) \cap \Delta$ contains no infinite fictive factors of the gradation. Let $Stab_W(\mathfrak{a}) = W_{\mathfrak{a}} \ltimes W_J$ be the stabilizer of $\mathfrak{a}$ in the Weyl group W of Δ. Then the group $Stab_W(\mathfrak{a})$ acts on the adapted pairs $(\Pi'_{\mathfrak{a}}, \Pi')$, where $\Pi'_{\mathfrak{a}}$*

is a root basis of Σ *and* Π' *is a root basis of* Δ *adapted to* $\Pi'_{\mathfrak{a}}$. *Any adapted pair* $(\Pi'_{\mathfrak{a}}, \Pi')$ *is* $Stab_W(\mathfrak{a})$*-conjugate to* $(\Pi_{\mathfrak{a}}, \Pi)$ *or* $(-\Pi_{\mathfrak{a}}, -\Pi)$. *Moreover, the adapted pairs* $(\Pi_{\mathfrak{a}}, \Pi)$ *and* $(-\Pi_{\mathfrak{a}}, -\Pi)$ *are* $Stab_W(\mathfrak{a})$*-conjugate if and only if* Σ *is of finite type.*

Proof. Recall from Theorem 4.30 that the Weyl group $W_{\mathfrak{a}}$ of the grading root system Σ can be viewed as a subgroup of W. For $w = w_a w_j \in W_{\mathfrak{a}} \ltimes W_J$ and $(\Pi'_{\mathfrak{a}}, \Pi')$ an adapted pair, put $w(\Pi'_{\mathfrak{a}}, \Pi') := (w_a \Pi'_{\mathfrak{a}}, w\Pi')$. Thus, we get an action of the group $Stab_W(\mathfrak{a})$ on the set of adapted pairs.

Note that if $(\Pi'_{\mathfrak{a}}, \Pi')$ is an adapted pair, then so is $(-\Pi'_{\mathfrak{a}}, -\Pi')$. As Σ is indecomposable, the root basis $\Pi'_{\mathfrak{a}}$ is $W_{\mathfrak{a}}$-conjugate to $\epsilon\Pi_{\mathfrak{a}}$, with $\epsilon = \pm 1$ (and $\epsilon = 1$ if Σ is of finite type). Changing $(\Pi'_{\mathfrak{a}}, \Pi')$ in $(-\Pi'_{\mathfrak{a}}, -\Pi')$ if necessary and acting the subgroup $W_{\mathfrak{a}}$, we may assume $\Pi'_{\mathfrak{a}} = \Pi_{\mathfrak{a}}$. We claim that, in such a case, Π' is W-conjugate to Π and hence is W_J-conjugate to Π by Proposition 4.37.

If Σ is finite, then any effective factor of the gradation is of finite type (see Remark 4.36) and any fictive factor is contained in Δ_J. It follows from the assumption that any fictive factor is of finite type and Δ is finite. Thus, the claim is checked in this case.

Suppose now that Σ is infinite. It suffices to check the claim for each infinite indecomposable component Δ_1 of Δ. Let W_1 be the infinite Weyl subgroup corresponding to Δ_1 and Π_1 (resp. Π'_1) the root basis of Δ_1 induced by Π (resp. Π'). If Δ_1 is an effective factor, then it is Σ-graded and, with obvious notations, $(\Pi_{\mathfrak{a},1}, \Pi_1)$ and $(\Pi_{\mathfrak{a},1}, \Pi'_1)$ are the induced adapted bases. As Δ_1 is indecomposable and infinite, Π'_1 is W_1-conjugate to Π_1 (see the second assertion of Proposition 4.37).

If Δ_1 is an infinite fictive factor, then it is not contained in Δ_J by assumption. It follows from Proposition 3.11 that Σ is of affine type and there exists a non central element $d_1 \in \mathfrak{h}_1 := \mathfrak{h} \cap \mathfrak{g}_1$ such that $\rho_{\mathfrak{a}}(\beta_1) = \langle \beta_1, d_1 \rangle \delta$, $\forall \beta_1 \in \Delta_1$, where δ is a generator of Σ^+_{im}. The fact that $(\Pi_{\mathfrak{a}}, \Pi)$ is an adapted pair implies $\langle \beta_1, d_1 \rangle \geq 0$, $\forall \beta_1 \in \Delta_1^+$ and hence d_1 lies in the fundamental chamber C_1 associated to Π_1.

Suppose that the adapted basis $\Pi'_1 = -w_1 \Pi_1$, with $w_1 \in W_1$, then we get, in the same way, $\langle w_1(\beta_1), d_1 \rangle \leq 0$, $\forall \beta_1 \in \Delta_1^+$ and so d_1 lies in $w_1(-C_1)$, where $-C_1$ is the opposite fundamental chamber of C_1; contradiction (because, for the infinite indecomposable component Δ_1, $w_1(-C_1) \cap C_1$ is central in $\mathfrak{h}_{1,\mathbb{R}}$, see Remark 2.3). Hence Π'_1 is also W_1-conjugate to Π_1 in this case. The last assertion follows from Proposition 4.37. ■

5. Construction of finite imaginary gradations

In this section we provide a unified construction for finite imaginary gradations of the Kac-Moody Lie algebra $\mathfrak{g}$ as extension of finite real gradations on standard regular subalgebras $\mathfrak{g}(I_{re})$ of $\mathfrak{g}$. Recall that the Kac-Moody Lie algebra $\mathfrak{g}$ may be here decomposable. Suppose that there exists a proper subset $I_{re} \subset I$ such that the standard Kac-Moody subalgebra $\mathfrak{g}(I_{re})$ of $\mathfrak{g}$ is finitely really graded by an indecomposable Kac-Moody root system Σ. Let $\mathfrak{m}$ be the grading subalgebra of $\mathfrak{g}(I_{re})$ and $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{m}$ contained in $\mathfrak{h}(I_{re})$. Let $\Pi_{\mathfrak{a}} = \{\gamma_s, s \in \bar{I}\}$ be a root basis of $\Sigma = \Delta(\mathfrak{m}, \mathfrak{a})$ and suppose that the standard root basis $\Pi_{I_{re}}$ of $\Delta(I_{re})$ is adapted to $\Pi_{\mathfrak{a}}$. Let $\rho_{\mathfrak{a}} : \mathfrak{h}^* \to \mathfrak{a}^*$ be the restriction map.

Although we are not yet ensured that the Kac-Moody Lie algebra $\mathfrak{g}$ is Σ-graded, we will use the same notations introduced above for finite gradations:

$$J := \{j \in I;\, \rho_{\mathfrak{a}}(\alpha_j) = 0\}, \quad J_{re} := J \cap I_{re}, \quad J^{\circ} := J \setminus J_{re},$$
$$I' := I \setminus J, \quad I'_{re} := I_{re} \cap I', \quad I'_{im} := I' \setminus I'_{re}.$$

Note that $\rho_{\mathfrak{a}}$, when restricted to $\Pi(I'_{re})$, induces a quotient map $\bar{\rho} : I'_{re} \to \bar{I}$ such that, for $k' \in I'_{re}$, $\rho_{\mathfrak{a}}(\alpha_{k'}) = \gamma_{\bar{\rho}(k')}$ (cf. Proposition 4.8).

Theorem 5.1. *Under the assumptions introduced above suppose $I'_{im} \neq \emptyset$ and add the following conditions:*

(FIG1) *The subset $J := \{j \in I;\, \rho_{\mathfrak{a}}(\alpha_j) = 0\}$ is either empty or of finite type.*

(FIG2) *For any $k \in I'_{im}$, $\bar{\alpha}_k := \rho_{\mathfrak{a}}(\alpha_k) \in \Sigma^+_{im}$.*

(FIG3) *If $k \neq l \in I'_{im}$ are J-linked, then $\bar{\alpha}_k + \bar{\alpha}_l \in \Sigma^+_{im}$.*

Then the Kac-Moody Lie algebra $\mathfrak{g}$ is finitely Σ-graded and the gradation is imaginary. The standard root basis Π of Δ is adapted to the root basis $\Pi_{\mathfrak{a}}$ and Δ_J is the annihilator of $\mathfrak{a}$ in Δ.

N.B. (1) Note that for $k \in I'_{im}$, we have $\langle \bar{\alpha}_k, \check{\gamma_s} \rangle = \langle \alpha_k, \check{\gamma_s} \rangle \leq 0$, $\forall s \in \bar{I}$. It follows from the conditions (FIG2) and (FIG3) that if $k \neq l \in I'_{im}$ are J-linked, then $\mathbb{Z}^+\bar{\alpha}_k + \mathbb{Z}^+\bar{\alpha}_l$ is contained in $\Sigma^+_{im} \cup \{0\}$ (see [15]; Lemma 5.3).

(2) If $\mathfrak{g}$ is finitely imaginary Σ-graded with (I_{re}, J_{re}) the induced C-admissible pair (see Proposition 4.8) then the conditions (FIG1) , (FIG2) and (FIG3) are obviously satisfied and thus any finite imaginary gradation of $\mathfrak{g}$ can be realized in this way.

To prove the Theorem we need some preliminary results.

Definition 5.2. (1) Suppose that $|I'| \geq 2$, then two vertices $k' \neq l' \in I'$ are called *J-linked*, if there exists a sequence $j_0, j_1, j_2, ..., j_m, j_{m+1}$ of I such that $j_0 = k'$, $j_{m+1} = l'$, $j_s \in J$ for all $1 \leq s \leq m$, and $a_{i_j, i_{j+1}} < 0$ for all $0 \leq j \leq m$.

(2) A subset J' of I' is called *J-connected* if $|J'| = 1$ or, for any two arbitrary vertices $k' \neq l' \in J'$ there exists a sequence $i'_1, i'_2, ..., i'_m$ of J' such that $i'_1 = k'$, $i'_m = l'$ and i'_j, i'_{j+1} are J-linked for $1 \leq j < m$.

Note that J' is J-connected, if and only if it is contained in a connected component of $J' \cup J$.

Lemma 5.3. (1) *Let $l \in I \setminus I_{re}$. Then l is linked to I_{re} if and only if there exists $s \in \bar{I}$ such that $\langle \alpha_l, \check{\gamma_s} \rangle < 0$. In particular $J^{\circ} := J \setminus J_{re}$ is not linked to I_{re}.*

(2) *Let $i' \in I'_{re}$ and $k' \in I'$. If i' and k' are J-linked, then they are J_{re}-linked.*

Proof. As the Kac-Moody subalgebra $\mathfrak{g}(I_{re})$ is finitely really Σ-graded, the pair (I_{re}, J_{re}) is C-admissible and thus, for $k \in I'_{re}$, l is linked to I_k if and only if $\langle \alpha_l, H_k \rangle < 0$ (cf. [7], Lemma 2.5). Since, for $s \in \bar{I}$, $\check{\gamma_s} = \sum_{k \in \Gamma_s} H_k$, we get the result of the assertion 1. The assertion 2 follows from the fact that J° is not connected to I_{re}. ■

Lemma 5.4. *Let $k' \neq l' \in I' := I \setminus J$ be two J-linked vertices.*

1. *If $k', l' \in I'_{re}$, then $\bar{\rho}(k')$ and $\bar{\rho}(l')$ are linked in $\bar{I}$ with respect to the generalized Cartan matrix $\bar{A} := (\langle \gamma_t, \check{\gamma_s} \rangle)_{s,t \in \bar{I}}$.*
2. *If $k' \in I'_{re}$ and $l' \in I'_{im}$, then $\bar{\rho}(k')$ is linked to supp$(\bar{\alpha}_{l'})$ in $\bar{I}$.*
3. *If $k', l' \in I'_{im}$, then supp$(\bar{\alpha}_{k'})$ and supp$(\bar{\alpha}_{l'})$ are linked in $\bar{I}$.*

Proof. (1) If $k', l' \in I'_{re}$, put $s = \bar{\rho}(k')$ and $t = \bar{\rho}(l')$. By Lemma 5.3, k' and l' are J_{re}-linked and $\langle \gamma_t, \check{\gamma_s} \rangle = \langle \alpha_{l'}, \check{\gamma_s} \rangle < 0$ (cf. [7], Lemma 2.5).

(2) If $k' \in I'_{re}$ and $l' \in I'_{im}$, put $s = \bar{\rho}(k')$, then we have by Lemma 5.3, $\langle \bar{\alpha}_{l'}, \check{\gamma_s} \rangle = \langle \alpha_{l'}, \check{\gamma_s} \rangle < 0$. It follows that $\gamma := \bar{\alpha}_{l'} + \gamma_s \in \Sigma^+$ and $supp(\gamma) = \{s\} \cup supp(\bar{\alpha}_{l'})$ is a connected subset of $\bar{I}$.

(3) If $k', l' \in I'_{im}$, then, by the conditions (FIG2) and (FIG3), $\gamma := \bar{\alpha}_{k'} + \bar{\alpha}_{l'} \in \Sigma^+_{im}$ and $supp(\gamma) = supp(\bar{\alpha}_{k'}) \cup supp(\bar{\alpha}_{l'})$ is connected in $\bar{I}$. ■

Corollary 5.5. *Let $\bar{\rho} : I'_{re} \to \bar{I}$ be the quotient map induced by $\rho_{\mathfrak{a}}$. Let J' be a J-connected subset of I'. Let $J'_{re} := J' \cap I'_{re}$ and $J'_{im} := J' \cap I'_{im}$. Then $\bar{\rho}(J'_{re}) \cup \Big(\bigcup_{i' \in J'_{im}} supp(\bar{\alpha}_{i'}) \Big)$ is a connected subset of $\bar{I}$. In particular, if $\alpha \in \Delta^+ \setminus \Delta_J$ and $\bar{\alpha} := \rho_{\mathfrak{a}}(\alpha)$, then $supp(\bar{\alpha})$ is connected in $\bar{I}$.*

Definition 5.6. [17] Let $\mathfrak{p}$ be a Lie algebra. Let (φ, V) be a $\mathfrak{p}$-module. Then $\mathfrak{p}$ is *φ-finite* on V, if any vector $u \in V$ is contained in a finite dimensional $\mathfrak{p}$-submodule U of V.

Lemma 5.7. *Let $\mathfrak{g}$ be a Kac-Moody algebra and let $\tilde{\mathfrak{m}}$ be a Kac-Moody subalgebra of $\mathfrak{g}$. Suppose $\mathfrak{g}$ is an integrable $\tilde{\mathfrak{m}}$-module via the adjoint action of the subalgebra $\tilde{\mathfrak{m}}$ on $\mathfrak{g}$. Then any integrable $\mathfrak{g}$-module (φ, V) is an integrable $\tilde{\mathfrak{m}}$-module.*

Proof. Let $\tilde{\mathfrak{a}}$ be a Cartan subalgebra of $\tilde{\mathfrak{m}}$ and $\tilde{\Sigma} := \Delta(\tilde{\mathfrak{m}}, \tilde{\mathfrak{a}})$ the corresponding root system. Let $\gamma \in \tilde{\Sigma}_{re}$ and let $\tilde{\mathfrak{s}}_\gamma$ be the subalgebra of $\tilde{\mathfrak{m}}$ generated by an $\mathfrak{sl}_2$-triple $(\tilde{e}_\gamma, \tilde{h}_\gamma, \tilde{f}_\gamma)$ associated to γ. Let (φ, V) be an integrable $\mathfrak{g}$-module. As the Kac-Moody algebra $\mathfrak{g}$ is an integrable $\tilde{\mathfrak{m}}$-module, the subalgebra $\tilde{\mathfrak{s}}_\gamma$ is $ad_{\mathfrak{g}}$-finite. By a result of Kac and Wang ([17], 1.23) $\tilde{\mathfrak{s}}_\gamma$ is φ-finite. Furthermore, by the theory of finite dimensional representations of simple Lie algebras, $\varphi(\tilde{e}_\gamma)$ (resp. $\varphi(\tilde{f}_\gamma)$) is locally nilpotent on V. Hence (φ, V) is an integrable $\tilde{\mathfrak{m}}$-module. ■

Proposition 5.8. *Under the assumptions of Theorem 5.1, the Kac-Moody algebra $\mathfrak{g}$ is an integrable $\mathfrak{m}$-module with finite multiplicities.*

Proof. Since $\mathfrak{g}(I_{re})$ is Σ-graded, it is an integrable $\mathfrak{m}$-module by Lemma 3.3. The Kac-Moody Lie algebra $\mathfrak{g}$ is clearly an integrable $\mathfrak{g}(I_{re})$-module via the adjoint action of $\mathfrak{g}(I_{re})$. By Lemma 5.7, $\mathfrak{g}$ is an integrable $\mathfrak{m}$-module. One can deduce easily from the assumptions that $\rho_{\mathfrak{a}}(Q_+) \subset \bar{Q}_+ := \oplus_{s \in \bar{I}} \mathbb{Z}^+ \gamma_s$ and that any positive root $\alpha \in \Delta_+$ satisfying $\rho_{\mathfrak{a}}(\alpha) = 0$ lies in Δ_J. Since J is assumed to be of finite type, the same argument used in the proof of the Proposition 4.11 shows that the weight spaces of the integrable $\mathfrak{m}$-module $\mathfrak{g}$ are finite dimensional. ■

Proof of Theorem 5.1. Let $\bar{\Delta}^+ := \rho_{\mathfrak{a}}(\Delta^+ \setminus \Delta_J) \subset \bar{Q}_+ := \oplus_{s \in \bar{I}} \mathbb{Z}^+ \gamma_s$. We will see that $\bar{\Delta}^+$, as Σ^+, satisfies the three conditions (i), (ii) and (iii) below (see [15, Ex. 5.4]). This will prove that $\bar{\Delta}^+ = \Sigma^+$, hence $\mathfrak{g}$ is Σ-graded (finitely by Prop. 5.8).

(i) $\gamma_s \in \bar{\Delta}^+ \subset \bar{Q}_+$, $2\gamma_s \notin \bar{\Delta}^+$, $\forall s \in \bar{I}$.

(ii) if $\gamma \in \bar{\Delta}^+$, $\gamma \neq \gamma_s$, then the set $\{\gamma + k\gamma_s ; k \in \mathbb{Z}\} \cap \bar{\Delta}^+$ is a string of the form $\{\gamma - p\gamma_s, ..., \gamma + q\gamma_s\}$, where $p, q \in \mathbb{Z}^+$ and $p - q = \langle \gamma, \check{\gamma_s} \rangle$;

(iii) if $\gamma \in \bar{\Delta}^+$, then supp$(\gamma)$ is connected.

As the Kac-Moody subalgebra $\mathfrak{g}(I_{re})$ is really Σ-graded with the mentioned adapted bases, property (i) is satisfied.

By Proposition 5.8, the Kac-Moody Lie algebra $\mathfrak{g}$ is an integrable $\mathfrak{m}$-module with finite multiplicities. Hence, property (ii) follows from [15, 3.6].

By Corollary 5.5, property (iii) is satisfied.

As we have mentioned in the proof of Proposition 5.8, $\rho_{\mathfrak{a}}(Q_+) \subset \bar{Q}_+$, hence the standard root basis Π is adapted to $\Pi_{\mathfrak{a}}$ and Δ_J is the annihilator of $\mathfrak{a}$ in Δ. ■

Corollary 5.9. *In addition to the assumptions introduced at the beginning of this section, we further assume that the grading subalgebra* $\mathfrak{m}$ *of* $\mathfrak{g}(I_{re})$ *is hyperbolic. Suppose that the condition* (FIG1) *is satisfied and that we have for any* $k \in I'_{im}$, $\bar{\alpha}_k := \rho_{\mathfrak{a}}(\alpha_k) \in \bar{Q} := \bigoplus_{s \in \bar{I}} \mathbb{Z}\gamma_s$. *Then the Kac-Moody Lie algebra* $\mathfrak{g}$ *is finitely imaginary* Σ*-graded.*

We will need the following Lemma on hyperbolic GCMs:

Lemma 5.10. ([32], 3.5 and 3.6) *Let* A *be an* $n \times n$*-indecomposable GCM. The following assertions are equivalent:*

(i) *A is of hyperbolic type.*

(ii) *For any column vector* $u \in \mathbb{R}^n$, $Au \geq 0$ *implies* $u \leq 0$.

(iii) *A is invertible and its inverse* A^{-1} *has non positive entries* ($i.e.$, $A^{-1} \leq 0$).

N.B. Note that for the hyperbolic GCM A, one can check that any zero entry of the inverse matrix A^{-1} is on the main diagonal and the deletion of the corresponding vertex, in the Dynkin diagram associated with A, yields an affine type subdiagram. So A^{-1} has non-positive entries, with at most one zero entry per column (and per row).

Proof of Corollary 5.9. Recall that for $k \in I'_{im}$, we have:

$$\langle \bar{\alpha}_k, \check{\gamma_s} \rangle = \langle \alpha_k, \check{\gamma_s} \rangle \leq 0, \quad \forall s \in \bar{I}. \tag{6}$$

As Σ is assumed of hyperbolic type and, for $k \in I'_{im}$, $\bar{\alpha}_k := \rho_{\mathfrak{a}}(\alpha_k)$ is assumed to be in the root lattice $\bar{Q}$, we get $\bar{\alpha}_k \in \bar{Q}_+$ by Lemma 5.10. In view of the condition (6), we get by [15, Prop.5.10 c]:

$$\sum_{k \in I'_{im}} \mathbb{Z}^+ \bar{\alpha}_k \subset \Sigma^+_{im} \cup \{0\}.$$

Thus, the conditions (FIG2) and (FIG3) of the Theorem 5.1 are satisfied. ■

5.1. Finite imaginary gradations by hyperbolic regular subsystems

Suppose that the symmetrisable GCM $A = (a_{i,j})_{i,j \in I}$ is indecomposable of indefinite type and non hyperbolic; then the associated Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ possesses a proper standard regular Kac-Moody subalgebra $\mathfrak{m}$ of hyperbolic type: that is there exists a proper subset I_{re} of I such that $\mathfrak{m} := \mathfrak{g}(A_{I_{re}})$ is hyperbolic (just consider a minimal standard regular Kac-Moody subalgebra of indefinite type). Denote by $\Sigma := \Delta(I_{re})$ the hyperbolic regular subsystem of Δ associated to $\mathfrak{m}$.

Corollary 5.11. *Suppose that* $\mathfrak{g}$ *and* $\mathfrak{m}$ *are as above and that the following additional conditions are satisfied:*

(HG1) $J^\circ := \{j \in I \setminus I_{re}; a_{i,j} = 0, \ \forall i \in I_{re}\}$ *is either empty or of finite type.*

(HG2) *For any* $l \in I \setminus \{I_{re} \cup J^\circ\}$ *and any* $k \in I_{re}$ *linked to* l, $\det(A_{I_{re}})$ *divides* $a_{k,l}$.

Then, the Kac-Moody Lie algebra $\mathfrak{g}$ *is finitely imaginary* Σ*-graded with grading subalgebra* $\mathfrak{m} := \mathfrak{g}(I_{re})$.

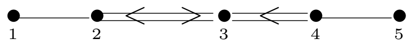
Proof. Note that the hyperbolic standard subalgebra $\mathfrak{m} = \mathfrak{g}(I_{re})$ is graded by its own root system of type $\Sigma := \Delta(I_{re})$ and so J_{re} is empty here. As $\mathfrak{a} := \mathfrak{h}(I_{re})$ is the standard Cartan subalgebra of $\mathfrak{m}$, it is immediately clear that the subset $J := \{j \in I;\ \rho_{\mathfrak{a}}(\alpha_j) = 0\} = J^\circ$ is either empty or of finite type by (HG1), hence the condition (FIG1) of Theorem 5.1 is satisfied. In view of Corollary 5.9, it suffices to see that, for $i \in I'_{im} := I \setminus \{I_{re} \cup J\}$, $\rho_{\mathfrak{a}}(\alpha_i) \in \bar{Q} := \mathbb{Z}\Delta(I_{re})$.

Let $(\bar{\omega}_i)_{i \in I_{re}}$ be the fundamental weights of $\mathfrak{g}(I_{re}) = \mathfrak{m}$ corresponding to the root basis $\Pi_{\mathfrak{a}} := \{\alpha_i;\ i \in I_{re}\}$ of $\Sigma := \Delta(I_{re})$. As the hyperbolic principal submatrix $A_{I_{re}}$ is invertible, the corresponding fundamental weights are given by its inverse matrix $A^{-1}_{I_{re}}$ (that is $\bar{\omega}_j = \sum_{i \in I_{re}} (A^{-1}_{I_{re}})_{i,j} \alpha_i$), $j \in I_{re}$). Hence, for $l \in I'_{im}$, we get $\rho_{\mathfrak{a}}(\alpha_l) = \sum_{k \in I_{re}} a_{k,l} \bar{\omega}_k$ which lies in $\bar{Q} = \sum_{k \in I_{re}} \mathbb{Z}\alpha_k$ by (HG2). The result follows from Corollary 5.9. ■

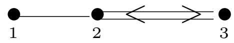
Remark 5.12. Note that there is a subclass of hyperbolic GCMs for which it is easy to compute the determinant, namely those which are obtained as Lorentzian extension of affine Kac-Moody Lie algebras (see [6, 5.1.1]): if the GCM is of type $H^m X_n^{(k)}$, then its determinant is $-m \det(\bar{X}_n)$, where $\bar{X}_n$ is the underlying Cartan matrix of $X_n^{(k)}$ obtained by deletion of the vertex 0 in the corresponding Dynkin diagram and m is the positive integer $\langle \alpha_{-1}, \check{\alpha_0} \rangle \times \langle \alpha_0, \check{\alpha_{-1}} \rangle$. Thus, we get for such indefinite Kac-Moody Lie algebras many examples of finite imaginary gradations by hyperbolic regular root subsystems.

5.2. Example

Consider the 5-rank Lorentzian Kac-Moody Lie algebra $\mathfrak{g}$ corresponding to the following Dynkin diagram:



Let $I = \{1, 2, 3, 4, 5\}$, $I_{re} = \{1, 2, 3\}$ and $J_{re} = \emptyset$. Let $\mathfrak{m} := \mathfrak{g}(I_{re})$ be the standard Kac-Moody subalgebra associated to the hyperbolic Dynkin subdiagram $HA_1^{(1)}$:



Note that $J^\circ = \{j \in I \setminus I_{re};\ \rho_{\mathfrak{a}}(\alpha_j) = 0\} = \{5\}$ here and (HG1) is satisfied. The hyperbolic GCM $A_{I_{re}}$ is of type $HA_1^{(1)}$: a Lorentzian extension of the affine matrix $A_1^{(1)}$ with determinant $(-1) \times \det(A_1) = -2$ (see Remark 5.12 above).

In particular $\det(A_{I_{re}})$ divides $a_{3,4} = -2$ and (HG2) is satisfied. By Corollary 5.11, the 5-rank Lorentzian Kac-Moody Lie algebra $\mathfrak{g}$ defined above is finitely imaginary $HA_1^{(1)}$-graded with grading subalgebra $\mathfrak{m} := \mathfrak{g}(I_{re})$.

Note that in this case,

$$A_{I_{re}}^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & -2 & -2 \\ -1 & -2 & -\frac{3}{2} \end{pmatrix}$$

and $a_{3,4}$ even is really necessary to get a gradation. ■

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Hechmi Ben Messaoud, Marwa Layouni, Dept. of Mathematics, Faculty of Sciences, University of Monastir, Tunisia; hechmi.benmessaoud@fsm.rnu.tn, marwa.layouni@fsm.rnu.tn.

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