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On Extensions of Nilpotent Leibniz and Diassociative Algebras

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Abstract. Given a pair of nilpotent Lie algebras A and B, an extension $0 \to A \to L \to B \to 0$ is not necessarily nilpotent. However, if L_1 and L_2 are extensions which correspond to lifts of homomorphism $\Phi\colon B \to \mathrm{Out}(A)$, it has been shown that L_1 is nilpotent if and only if L_2 is nilpotent. In the present paper, we prove analogues of this result for each algebra of Loday. As an important consequence, we thereby gain its associative analogue as a special case of diassociative algebras.

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1. Introduction

Let A and B be nilpotent Lie algebras. In [7], Bill Yankosky proved that the nilpotency of an extension $0 \to A \to L \to B \to 0$ depends on A, B, and a homomorphism $\Phi : B \to \mathrm{Out}(A)$. In particular, given a pair of extensions L_1 and L_2 corresponding to lifts of Φ, L_1 is nilpotent if and only if L_2 is nilpotent. This result was based on the work of James A. Schafer, who proved the group analogue in [6].

Beyond Lie algebras, the objective of the present paper is to prove Schafer's result for six other types of algebras. Loday introduced three of these (Zinbiel, diassociative, and dendriform algebras [1, 2]) and generated interest in another (Leibniz algebras). The remaining two types are associative and commutative algebras. We observe that Yankosky's work rests on the assumptions of nonabelian 2-cocycles, also known as factor systems, which have long been known in the context of Lie algebras. As discussed in [3], factor systems are a tool for working on the extension problem of algebraic structures. The work herein is an application of factor systems, which were developed for all seven types of algebras in [3].

Our work is greatly reduced by generalizations of algebra types. Indeed, dendriform algebras are a generalization of Zinbiel algebras and diassociative algebras are a generalization of associative algebras. Furthermore, associative algebras are a generalization of commutative algebras. We refer to [3] for a full discussion of these relations. Finally, the dendriform case of our result follows similarly to the diassociative case after two key changes. First, replace $\dashv$ and $\vdash$ by $<$ and $>$ respectively.

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The ideal $A\Diamond B = A \dashv B + A \vdash B$ is thereby redefined by dendriform multiplications. Second, replace Lemma 2.6 in the diassociative case by the analogous Lemma 2.7. Thus, for the sake of this paper, it suffices to prove the Leibniz and diassociative cases.

The paper is structured as follows. For preliminaries, we define the relevant algebras and discuss notions of nilpotency. We state known lemmas concerning certain product algebras and briefly review extensions. We then derive Leibniz and diassociative analogues of the results found in [7]. We state the associative analogue as a corollary of the diassociative case. The final section of the paper contains several examples which highlight important intricacies in the results.

2. Preliminaries

Let $\mathbb{F}$ be a field. Throughout, all algebras will be $\mathbb{F}$-vector spaces equipped with bilinear multiplications which satisfy certain identities. First recall that a *Leibniz algebra* L is a nonassociative algebra with multiplication satisfying the *Leibniz identity* $x(yz) = (xy)z + y(xz)$ for all $x, y, z \in L$.

Definition 2.1. A *Zinbiel algebra* Z is a nonassociative algebra with multiplication satisfying what we will call the *Zinbiel identity* $(xy)z = x(yz) + x(zy)$ for all $x, y, z \in Z$.

Definition 2.2. A *diassociative algebra* (or *associative dialgebra*) D is a vector space equipped with two associative bilinear products $\dashv$ and $\vdash$ which satisfy the following identities for all $x, y, z \in D$:

(D1) $x \dashv (y \dashv z) = x \dashv (y \vdash z)$,

(D2) $(x \vdash y) \dashv z = x \vdash (y \dashv z)$,

(D3) $(x \dashv y) \vdash z = (x \vdash y) \vdash z$.

Definition 2.3. A *dendriform algebra* E is a vector space equipped with two bilinear products $<$ and $>$ which satisfy the following identities for all $x, y, z \in E$:

(E1) $(x < y) < z = x < (y < z) + x < (y > z)$,

(E1) $(x > y) < z = x > (y < z)$,

(E1) $(x < y) > z + (x > y) > z = x > (y > z)$.

The *lower central series* is a well-known sequence of ideals which is defined recursively, for a Leibniz algebra L, by $L^0 = L$ and $L^{k+1} = LL^k$ for $k \geq 0$. A Leibniz algebra is called *nilpotent of class* u, denoted $\operatorname{nil} L = u$, if $L^u = 0$ and $L^{u-1} \neq 0$ for some $u \geq 0$. The following lemma holds via induction and repeated application of the Leibniz identity.

Lemma 2.4. *Let L be a Leibniz algebra. Then $L^n L \subseteq LL^n$ for all n.*

For diassociative algebras, the definition of nilpotency is more involved. We take the following notions from [5]. Let A and B be subsets of a diassociative algebra D and define an ideal $A\Diamond B = A \dashv B + A \vdash B$ of D. There are notions of left, right, and general nilpotency for D which are based on the $\Diamond$ operator.

We define three sequences of ideals recursively for $k \geq 0$:

(i) $D^{\{0\}} = D$, $D^{\{k+1\}} = D \Diamond D^{\{k\}}$,

(ii) $D^{<0>} = D$, $D^{<k+1>} = D^{<k>} \Diamond D$,

(iii) $D^0 = D$, $D^{k+1} = D^0 \Diamond D^k + D^1 \Diamond D^{k-1} + \cdots + D^k \Diamond D^0$.

Definition 2.5. A diassociative algebra D is called

(i) *left nilpotent* if $D^{\{u\}} = 0$

(ii) *right nilpotent* if $D^{<u>} = 0$

(iii) *nilpotent* if $D^u = 0$

for some $u \geq 0$. In particular, D is *nilpotent of class u* if $D^u = 0$ and $D^{u-1} \neq 0$.

The following lemma from [5] is crucial for the diassociative case in this paper.

Lemma 2.6. *Let D be a diassociative algebra. For all n, $D^{\{n\}} = D^{<n>} = D^n$.*

The same definitions may be stated for dendriform algebras with the simple substitutions of $<$ and $>$ for $\dashv$ and $\vdash$ respectively. Let A and B be subsets of a dendriform algebra E. The dendriform analogue of Lemma 2.6 is shown in [4], where the three sequences $E^{\{n\}}$, $E^{<n>}$, and E^n of ideals in E are similarly defined.

Lemma 2.7. *Let E be a dendriform algebra. For all n, $E^{\{n\}} = E^{<n>} = E^n$.*

We now review extensions. Fix a type of algebra $\mathscr{P}$ and let A and B be $\mathscr{P}$ algebras. An *extension* of A by B is a short exact sequence of the form $0 \to A \xrightarrow{\sigma} L \xrightarrow{\pi} B \to 0$ where L is a $\mathscr{P}$ algebra and σ and π are *homomorphisms*, i.e. linear maps which preserve the $\mathscr{P}$ structure. An *isomorphism* of $\mathscr{P}$ algebras is a bijective homomorphism. A *section* of the extension is a linear map $T : B \to L$ such that $\pi T = \mathrm{id}_B$.

Definition 2.8. An extension $0 \to A \to L \to B \to 0$ of A by B is called *nilpotent* if L is nilpotent as an algebra.

3. Leibniz case

Consider a pair of nilpotent Leibniz algebras A and B and let $0 \to A \xrightarrow{\sigma} L \xrightarrow{\pi} B \to 0$ be an extension of A by B with section $T : B \to L$. We first define two ways for B to act on A. Let $\varphi : B \to \mathrm{Der}(A)$ by $\varphi(i)m = \sigma^{-1}(T(i)\sigma(m))$ and $\varphi' : B \to \mathscr{L}(A)$ by $\varphi'(i)m = \sigma^{-1}(\sigma(m)T(i))$ for $i \in B$, $m \in A$. Next, let $q : \mathrm{Der}(A) \to \mathrm{Der}(A)/\,\mathrm{ad}^l(A)$ and $q' : \mathscr{L}(A) \to \mathscr{L}(A)/\,\mathrm{ad}^r(A)$ denote the natural projections and define a pair $(\Phi, \Phi') = (q\varphi, q'\varphi')$. We say that the pair (φ, φ') is a *lift* of (Φ, Φ'). Any two lifts (φ, φ') and (ψ, ψ') of (Φ, Φ') are thus related by $\varphi(i) = \psi(i) + \mathrm{ad}^l_{m_i}$ and $\varphi'(i) = \psi'(i) + \mathrm{ad}^r_{m'_i}$ for $i \in B$ and some elements $m_i, m'_i \in A$ which depend on i. Our first proposition yields a criterion for when L is nilpotent which is based on the following recursive construction. Define $A_0 = A$ and $A_{k+1} = \sigma^{-1}(\sigma(A_k)L + L\sigma(A_k))$ for $k \geq 0$.

Proposition 3.1. *Let B be a nilpotent Leibniz algebra of class s. Then we have $L^{k+s} \subseteq \sigma(A_k) \subseteq L^k$ for all $k \geq 0$. Hence L is nilpotent if and only if $A_k = 0$ for some k.*

Proof. Since $\pi : L \to B$ is a homomorphism, one computes $\pi(L^s) = B^s = 0$, which implies that $L^s \subseteq \ker \pi = \sigma(A) = \sigma(A_0)$. Also, $\sigma(A_0) = \sigma(A) \subseteq L = L^0$. We therefore have a base case $L^s \subseteq \sigma(A_0) \subseteq L^0$ for $k = 0$. Now suppose $L^{n+s} \subseteq \sigma(A_n) \subseteq L^n$ for some $n \geq 0$. Then

$$\begin{aligned} L^{n+1+s} &= LL^{n+s} \subseteq L\sigma(A_n) \text{ [by induction] } \subseteq \sigma(A_n)L + L\sigma(A_n) \\ &\subseteq L^n L + LL^n \text{ [by induction] } \stackrel{*}{=} LL^n = L^{n+1} \end{aligned}$$

where $\sigma(A_n)L + L\sigma(A_n) = \sigma(A_{n+1})$ and the equality $*$ follows by Lemma 2.4. Thus $L^{s+k} \subseteq \sigma(A_k) \subseteq L^k$ for all $k \geq 0$ via induction. For the second statement, we first note that if L is nilpotent, then $\sigma(A_k) \subseteq L^k = 0$ for some $k \geq 0$. This means $A_k = 0$ since σ is injective. Conversely, if $A_k = 0$ for some $k \geq 0$, then $\sigma(A_k) = 0$ and thus $L^{k+s} = 0$. Hence L is nilpotent. ■

Again, let (φ, φ') be a lift of (Φ, Φ').

Definition 3.2. An ideal N of A is (φ, φ')*-invariant* if $\varphi(i)n, \varphi'(i)n \in N$ for all $i \in B$ and $n \in N$.

Lemma 3.3. *Let (φ, φ') and (ψ, ψ') be lifts of (Φ, Φ'). Then N is (φ, φ')-invariant if and only if N is (ψ, ψ')-invariant.*

Proof. Let $i \in B$. Since we have two lifts of the same pair, they are related by $\psi(i) = \varphi(i) + \mathrm{ad}^l_{m_i}$ and $\psi'(i) = \varphi'(i) + \mathrm{ad}^r_{m'_i}$ for some $m_i, m'_i \in A$. In one direction, assume N is (φ, φ')-invariant. Then $\varphi(i)n, \varphi'(i)n \in N$ for all $n \in N$ by definition. Also, $m_i n, nm'_i \in N$ for all $n \in N$ since N is an ideal. Thus $\psi(i)n, \psi'(i)n \in N$ and so N is (ψ, ψ')-invariant. The other direction is similar. ■

Definition 3.4. An ideal N of A is B*-invariant* if N is (φ, φ')-invariant for some, and hence all, lifts of (Φ, Φ').

In particular, A itself is B-invariant since $\varphi(i), \varphi'(i) \in \mathscr{L}(A)$ for all $i \in B$. Consider a B-invariant ideal N of A and let (φ, φ') be a lift of (Φ, Φ'). We define $\Gamma(N, \varphi, \varphi')$ to be the B-invariant ideal of A generated by AN, NA, and $\{\varphi(i)n, \varphi'(i)n \mid i \in B, n \in N\}$. Then $\Gamma(N, \varphi, \varphi') \subseteq N$ and we reach the following lemma.

Lemma 3.5. *If (φ, φ') and (ψ, ψ') are lifts of (Φ, Φ'), then*

$$\Gamma(N, \varphi, \varphi') = \Gamma(N, \psi, \psi').$$

Proof. It again suffices to show one direction. First note that AN and NA are contained in both sides of the equality by definition. For $i \in B$ and $n \in N$, we know $\psi(i)n = \varphi(i)n + m_i n$ and $\psi'(i)n = \varphi'(i)n + nm'_i$ for some $m_i, m'_i \in A$. These expressions clearly fall in $\Gamma(N, \varphi, \varphi')$ and therefore $\Gamma(N, \psi, \psi') \subseteq \Gamma(N, \varphi, \varphi')$. ■

We now fix a lift (φ, φ') of (Φ, Φ') and denote $\Gamma N = \Gamma(N, \varphi, \varphi')$. Given B, A, $\Phi : B \to \mathrm{Der}(A)/\mathrm{ad}^l(A)$, $\Phi' : B \to \mathscr{L}(A)/\mathrm{ad}^r(A)$, and a B-invariant ideal N of A, define a descending sequence of B-invariant ideals $\Gamma^B_k N$ of N by $\Gamma^B_0 N = N$ and $\Gamma^B_{k+1} N = \Gamma(\Gamma^B_k N)$ for $k \geq 0$.

Theorem 3.6. *Consider the extension* $0 \to A \xrightarrow{\sigma} L \to B \to 0$ *and our pair of maps* (Φ, Φ'). *If* $A_0 = A$ *and* $A_{k+1} = \sigma^{-1}(\sigma(A_k)L + L\sigma(A_k))$, *then* $A_k = \Gamma_k^B A$ *for all* $k \geq 0$.

Proof. By our work in [3], there exists a unique factor system (φ, φ', f) belonging to the extension $0 \to A \xrightarrow{\sigma} L \to B \to 0$. By construction, φ and φ' are the maps of our lift (φ, φ'). Next, there exists another extension $0 \to A \xrightarrow{\iota} L_2 \to B \to 0$ of A by B to which (φ, φ', f) belongs. Here, L_2 is the vector space $A \oplus B$ equipped with multiplication $(m, i)(n, j) = (mn + \varphi(i)n + \varphi'(j)m + f(i, j), ij)$, where $f : B \times B \to A$ is a bilinear form. Also $\iota(m) = (m, 0)$. Since (φ, φ', f) is equivalent to itself, the extensions are equivalent, and thus there exists an isomorphism $\tau : L \to L_2$ such that $\tau\sigma = \iota$.
We will now prove the statement via induction, first noting that the base case $A_0 = A = \Gamma_0^B A$ holds trivially. Assume that $A_n = \Gamma_n^B A$ for some $n \geq 0$. By definition, it suffices to show the inclusion of generating elements for each side of the equality. Generating elements of A_{n+1} have the forms $\sigma^{-1}(\sigma(m)x)$ and $\sigma^{-1}(x\sigma(m))$ for $x \in L$ and $m \in A_k$. Denote $\tau(x) = (m_x, i_x) \in L_2$. We compute

$$\begin{aligned}\sigma^{-1}(\sigma(m)x) &= \sigma^{-1}\tau^{-1}(\tau\sigma(m)\tau(x)) = \iota^{-1}((m, 0)(m_x, i_x)) \\ &= \iota^{-1}(mm_x + \varphi'(i_x)m, 0) = mm_x + \varphi'(i_x)m\end{aligned}$$

and

$$\begin{aligned}\sigma^{-1}(x\sigma(m)) &= \sigma^{-1}\tau^{-1}(\tau(x)\tau\sigma(m)) = \iota^{-1}((m_x, i_x)(m, 0)) \\ &= \iota^{-1}(m_x m + \varphi(i_x)m, 0) = m_x m + \varphi(i_x)m.\end{aligned}$$

Since $A_n = \Gamma_n^B A$, one has $m_x m \in A(\Gamma_n^B A)$ and $mm_x \in (\Gamma_n^B A)A$, which are both included in $\Gamma_{n+1}^B A$ since $\Gamma_{n+1}^B A$ is the B-invariant ideal generated by $(\Gamma_n^B A)A$, $A(\Gamma_n^B A)$, and $\{\varphi(i)m, \varphi'(i)m \mid m \in \Gamma_n^B A, i \in B\}$. Thus $\varphi'(i_x)m, \varphi(i_x)m \in \Gamma_{n+1}^B A$ as well and so $A_{n+1} \subseteq \Gamma_{n+1}^B A$. Conversely, one computes

$$(\Gamma_n^B A)A = \sigma^{-1}(\sigma(\Gamma_n^B A)\sigma(A)) \subseteq \sigma^{-1}(\sigma(A_n)L) \subseteq A_{n+1}$$

and

$$A(\Gamma_n^B A) = \sigma^{-1}(\sigma(A)\sigma(\Gamma_n^B A)) \subseteq \sigma^{-1}(L\sigma(A_n)) \subseteq A_{n+1}.$$

Also, let $i \in B$ and $m \in \Gamma_n^B A = A_n$. Then $\varphi(i)m = \sigma^{-1}(T(i)\sigma(m)) \in A_{n+1}$ and $\varphi'(i)m = \sigma^{-1}(\sigma(m)T(i)) \in A_{n+1}$ since $T(i) \in L$. Therefore $\Gamma_{n+1}^B A \subseteq A_{n+1}$. ■

Given B, A, $\Phi : B \to \mathrm{Der}(A)/\mathrm{ad}^l(A)$, and $\Phi' : B \to \mathscr{L}(A)/\mathrm{ad}^r(A)$, we define a new notion of nilpotency for A.

Definition 3.7. A is *B-nilpotent of class u*, written $\mathrm{nil}_B A = u$, if $\Gamma_u^B A = 0$ and $\Gamma_{u-1}^B A \neq 0$ for some $u \geq 0$.

The following two corollaries hold similarly to the Lie case. For their proofs, simply replace Proposition 2.1 and Theorem 3.1 of [7] by the analogous Proposition 3.1 and Theorem 3.6 of the present paper. The subsequent theorem is the main result, which follows from these corollaries and the same logic as Yankosky's proof.

Corollary 3.8. *L is nilpotent if and only if B is nilpotent and* $\Gamma_u^B A = 0$ *for some* $u \geq 1$.

Corollary 3.9. $\max(\operatorname{nil}_B A, \operatorname{nil} B) \leq \operatorname{nil} L \leq \operatorname{nil}_B A + \operatorname{nil} B$.

Theorem 3.10. *Let (φ, φ') and (ψ, ψ') be lifts of (Φ, Φ') corresponding to extensions $0 \to A \to L_{(\varphi,\varphi')} \to B \to 0$ and $0 \to A \to L_{(\psi,\psi')} \to B \to 0$ respectively. Then $L_{(\varphi,\varphi')}$ is nilpotent if and only if $L_{(\psi,\psi')}$ is nilpotent.*

4. Diassociative case

Consider a pair of nilpotent diassociative algebras A and B and an extension $0 \to A \xrightarrow{\sigma} L \xrightarrow{\pi} B \to 0$ of A by B with section $T\colon B \to L$. Throughout, we let $*$ range over $\dashv$ and $\vdash$ for the sake of brevity. We consider four natural ways for B to act on A. Define $\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash : B \to \mathscr{L}(A)$ by $\varphi_*(i)m = \sigma^{-1}(T(i) * \sigma(m))$ and $\varphi'_*(i)m = \sigma^{-1}(\sigma(m) * T(i))$ for $i \in B$, $m \in A$. Let $q_* : \mathscr{L}(A) \to \mathscr{L}(A)/\operatorname{ad}^l_*(A)$ and $q'_* : \mathscr{L}(A) \to \mathscr{L}(A)/\operatorname{ad}^r_*(A)$ be the natural projections and define a tuple of maps $\Phi = (\Phi_\dashv, \Phi_\vdash, \Phi'_\dashv, \Phi'_\vdash)$ by $\Phi_* = q_*\varphi_*$ and $\Phi'_* = q'_*\varphi'_*$. The tuple $\varphi = (\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash)$ is called a *lift* of Φ. Two lifts $\varphi = (\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash)$ and $\psi = (\psi_\dashv, \psi_\vdash, \psi'_\dashv, \psi'_\vdash)$ of Φ are related by

$$\psi_*(i) = \varphi_*(i) + \operatorname{ad}^l_*(m_{*,i}) \quad \text{and} \quad \psi'_*(i) = \varphi'_*(i) + \operatorname{ad}^r_*(m'_{*,i})$$

for $i \in B$ and some $m_{*,i}, m'_{*,i} \in A$ which depend on i. Finally, let $A_0 = A$ and define $A_{k+1} = \sigma^{-1}(\sigma(A_k)\Diamond L + L\Diamond\sigma(A_k))$ for $k \geq 0$.

Proposition 4.1. *Let B be a nilpotent diassociative algebra of class s. Then $L^{k+s} \subseteq \sigma(A_k) \subseteq L^k$ for all $k \geq 0$. Hence L is nilpotent if and only if $A_k = 0$ for some k.*

Proof. As with the Leibniz case, the base case $k = 0$ follows by our definitions and the properties of extensions. Suppose $L^{n+s} \subseteq \sigma(A_n) \subseteq L^n$ for some $n \geq 0$. We recall that $L^n = L^{<n>} = L^{\{n\}}$ by Lemma 2.6 and thereby compute

$$\begin{aligned} L^{n+1+s} &= L^{<n+1+s>} = L^{n+s}\Diamond L \subseteq \sigma(A_n)\Diamond L \text{ [by induction]} \\ &\subseteq \sigma(A_n)\Diamond L + L\Diamond\sigma(A_n) \subseteq L^{<n>}\Diamond L + L\Diamond L^{\{n\}} \text{ [by induction]} \ = L^{n+1} \end{aligned}$$

where $\sigma(A_n)\Diamond L + L\Diamond\sigma(A_n) = \sigma(A_{n+1})$. Thus $L^{s+k} \subseteq \sigma(A_k) \subseteq L^k$ for $k \geq 0$ via induction. The second statement follows by the same logic as the Leibniz case. ∎

Once more, let $\varphi = (\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash)$ be a lift of Φ.

Definition 4.2. An ideal N of A is *φ-invariant* if $\varphi_*(i)n, \varphi'_*(i)n \in N$ for all $i \in B$, $n \in N$.

Lemma 4.3. *Let φ and ψ be lifts of Φ. Then N is φ-invariant if and only if N is ψ-invariant.*

Proof. Let $i \in B$. Since $\varphi = (\varphi_\dashv, \varphi_\vdash, \varphi'_\dashv, \varphi'_\vdash)$ and $\psi = (\psi_\dashv, \psi_\vdash, \psi'_\dashv, \psi'_\vdash)$ are lifts of the same tuple, they are related by $\psi_*(i) = \varphi_*(i) + \operatorname{ad}^l_*(m_{*,i})$ and $\psi'_*(i) = \varphi'_*(i) + \operatorname{ad}^r_*(m'_{*,i})$ for some $m_{*,i}, m'_{*,i} \in A$. In one direction, suppose N is φ-invariant. Then $\psi_*(i)n, \psi'_*(i)n \in N$ for all $n \in N$ since N is a φ-invariant ideal in A. Therefore N is ψ-invariant. The converse is similar. ∎

Definition 4.4. An ideal N of A is *B-invariant* if N is φ-invariant for some, and hence all, lifts of Φ.

In particular, A is B-invariant since $\varphi_*(i), \varphi'_*(i) \in \mathscr{L}(A)$ for all $i \in B$. Now let N be a B-invariant ideal in A and φ be a lift of Φ. We denote by $\Gamma(N, \varphi)$ the B-invariant ideal generated by $N \dashv A$, $N \vdash A$, $A \dashv N$, $A \vdash N$, and the set $\{\varphi_*(i)n, \varphi'_*(i)n \mid i \in B, n \in N\}$. We thus have $\Gamma(N, \varphi) \subseteq N$ as well as the following lemma.

Lemma 4.5. *If φ and ψ are lifts of Φ, then $\Gamma(N, \varphi) = \Gamma(N, \psi)$.*

Proof. It suffices to show that $\Gamma(N, \psi) \subseteq \Gamma(N, \varphi)$. We first note that $N \dashv A$, $N \vdash A$, $A \dashv N$, and $A \vdash N$ are contained in both sides by definition. Similarly to the Leibniz case, the expressions for $\psi_*(i)n$ and $\psi'_*(i)n$ are clearly contained in $\Gamma(N, \varphi)$ for all $i \in B$ and $n \in N$. The converse holds without loss of generality. ∎

Fix a lift φ of Φ and denote $\Gamma N := \Gamma(N, \varphi)$. Given B, A, Φ, and a B-invariant ideal N of A, define a descending sequence of B-invariant ideals $\Gamma_k^B N$ of N by $\Gamma_0^B N := N$ and $\Gamma_{k+1}^B N := \Gamma(\Gamma_k^B N)$ for $k \geq 0$.

Theorem 4.6. *Consider $0 \to A \xrightarrow{\sigma} L \to B \to 0$ and let Φ be defined as above. If $A_0 = A$ and $A_{k+1} = \sigma^{-1}(\sigma(A_k) \Diamond L + L \Diamond \sigma(A_k))$, then $A_k = \Gamma_k^B A$ for all $k \geq 0$.*

Proof. As in the Leibniz case, the work with factor systems in [3] yields an equivalent extension $0 \to A \xrightarrow{\iota} L_2 \to B \to 0$ such that L_2 is the vector space $A \oplus B$ equipped with multiplications $(m, i)*(n, j) = (m*n+\varphi_*(i)n+\varphi'_*(j)m+f_*(i, j), i*j)$ and $\iota(m) = (m, 0)$. Here φ_* and φ'_* are the same maps as in our lift φ while $f_\dashv$ and $f_\vdash$ are the bilinear forms in some factor system of diassociative algebras. Let $\tau : L \to L_2$ be the equivalence.

The base case of this result is trivial since $A_0 = A = \Gamma_0^B A$ by definition. Now assume $A_n = \Gamma_n^B A$ for some $n \geq 0$. By definition, it suffices to show the inclusion of generating elements for each side of the equality. Generating elements in A_{n+1} have the forms $\sigma^{-1}(\sigma(m) * x)$ and $\sigma^{-1}(x * \sigma(m))$ for $m \in A_n$ and $x \in L$. Denote $\tau(x) = (m_x, i_x) \in L_2$. We compute

$$\begin{aligned}\sigma^{-1}(\sigma(m) * x) &= \sigma^{-1}\tau^{-1}(\tau\sigma(m) * \tau(x)) = \iota^{-1}((m, 0) * (m_x, i_x)) \\ &= m * m_x + \varphi'_*(i_x)m\end{aligned}$$

and
$$\begin{aligned}\sigma^{-1}(x * \sigma(m)) &= \sigma^{-1}\tau^{-1}(\tau(x) * \tau\sigma(m)) = \iota^{-1}((m_x, i_x) * (m, 0)) \\ &= m_x * m + \varphi_*(i_x)m.\end{aligned}$$

Since $A_n = \Gamma_n^B A$, one has $m_x * m \in A * (\Gamma_n^B A)$ and $m * m_x \in (\Gamma_n^B A) * A$, which are included in $\Gamma_{n+1}^B A$ since $\Gamma_{n+1}^B A$ is the B-invariant ideal generated by $(\Gamma_n^B A) * A$, $A * (\Gamma_n^B A)$, and $\{\varphi_*(i)m, \varphi'_*(i)m \mid m \in \Gamma_n^B A, i \in B\}$.

Thus $\varphi'_*(i_x)m, \varphi_*(i_x)m \in \Gamma_{n+1}^B A$ as well. Therefore $A_{n+1} \subseteq \Gamma_{n+1}^B A$. Conversely, one computes

$$(\Gamma_n^B A) * A = \sigma^{-1}(\sigma(\Gamma_n^B A) * \sigma(A)) \subseteq \sigma^{-1}(\sigma(A_n) * L) \subseteq A_{n+1}$$

and
$$A * (\Gamma_n^B A) = \sigma^{-1}(\sigma(A) * \sigma(\Gamma_n^B A)) \subseteq \sigma^{-1}(L * \sigma(A_n)) \subseteq A_{n+1}.$$

Also, let $i \in B$ and $m \in \Gamma_n^B A = A_n$. Then $\varphi_*(i)m = \sigma^{-1}(T(i) * \sigma(m)) \in A_{n+1}$ and $\varphi'_*(i)m = \sigma^{-1}(\sigma(m) * T(i)) \in A_{n+1}$ since $T(i) \in L$. Therefore $\Gamma_{n+1}^B A \subseteq A_{n+1}$. ∎

Definition 4.7. Given B, A, and the tuple Φ, we say that A is *B-nilpotent of class u*, written $\operatorname{nil}_B A = u$, if $\Gamma_u^B A = 0$ but $\Gamma_{u-1}^B A \neq 0$.

The following two corollaries hold by the same logic as the Lie and Leibniz cases. The subsequent theorem follows from these corollaries and the same logic used for the previous types of algebras.

Corollary 4.8. *L is nilpotent if and only if B is nilpotent and $\Gamma_u^B A = 0$ for some $u \geq 1$.*

Corollary 4.9. $\max(\operatorname{nil}_B A, \operatorname{nil} B) \leq \operatorname{nil} L \leq \operatorname{nil}_B A + \operatorname{nil} B$.

Theorem 4.10. *Let φ and ψ be lifts of $(\Phi_{\dashv}, \Phi_{\vdash}, \Phi'_{\dashv}, \Phi'_{\vdash})$ corresponding to extensions $0 \to A \to L_\varphi \to B \to 0$ and $0 \to A \to L_\psi \to B \to 0$ respectively. Then L_φ is nilpotent if and only if L_ψ is nilpotent.*

We now state the associative case as a corollary since we have not been able to find it written down. Let A and B be associative algebras and consider a pair of maps (Φ, Φ') such that $\Phi : B \to \mathscr{L}(A)/\operatorname{ad}^l(A)$ and $\Phi' : B \to \mathscr{L}(A)/\operatorname{ad}^r(A)$. Let *lifts* (φ, φ') and (ψ, ψ') of (Φ, Φ') be defined as in the Leibniz case and consider their corresponding extensions

$$0 \to A \to L_{(\varphi,\varphi')} \to B \to 0 \quad \text{and} \quad 0 \to A \to L_{(\psi,\psi')} \to B \to 0,$$

respectively.

Corollary 4.11. *$L_{(\varphi,\varphi')}$ is nilpotent if and only if $L_{(\psi,\psi')}$ is nilpotent.*

5. Examples

The first two examples demonstrate that extensions corresponding to lifts of the same tuple need not have the same nilpotency class. We provide an example for the non-Lie Leibniz case as well as for the diassociative case.

Example 5.1. Let $A = \langle x, y, z \rangle$ and $B = \langle w \rangle$ be abelian Leibniz algebras and consider two extensions L_1 and L_2 of A by B. Let $L_1 = \langle x, y, z, w \rangle$ have nonzero multiplications given by $w^2 = x$, $wx = y$, and $wy = z$. We note that L_1 is not a Lie algebra since $w^2 \neq 0$. One computes $L_1^2 = \langle x, y, z \rangle$, $L_1^3 = \langle y, z \rangle$, $L_1^4 = \langle z \rangle$, and $L_1^5 = 0$, making L_1 nilpotent of class 5.

Now let $L_2 = \langle x, y, z, w \rangle$ have nonzero multiplications given by $wx = y$ and $wy = z$. Then $L_2^2 = \langle y, z \rangle$, $L_2^3 = \langle z \rangle$, and $L_2^4 = 0$, making L_2 nilpotent of class 4. The key point here is that L_1 and L_2 are lifts of the same tuple yet have different nilpotency classes. Indeed, A is abelian, and hence $\operatorname{ad}^l(M)$ and $\operatorname{ad}^r(M)$ are zero, making $(\Phi, \Phi') = (\varphi, \varphi')$ for any lift of (Φ, Φ'). In this case, $\Phi(w)x = \varphi(w)x = y$ and $\Phi(w)y = \varphi(w)y = z$ for both. Also $\Phi'(w) = 0$.

We would also like to compute A_k and $\Gamma_k^B A$. Note that, since $A^2 = 0$, one needs only consider the actions of φ and φ' on A when computing $\Gamma_k^B A$.

As predicted, $A_k = \Gamma_k^B A$ for all k. One has

$$A_0 = A = \Gamma_0^B A, \quad A_1 = \langle y, z\rangle = \Gamma_1^B A, \quad A_2 = \langle z\rangle = \Gamma_2^B A,$$

$$A_3 = 0 = \Gamma_3^B A, \text{ and } A_k = 0 = \Gamma_k^B A \text{ otherwise.}$$

Example 5.2. Now for a diassociative example. Let $A = \langle x, y\rangle$ and $B = \langle u, v\rangle$ be abelian algebras and L_φ be an extension of A by B having nonzero multiplications $u \dashv u = x$, $u \vdash u = x+y$, $v \dashv v = y$, $v \vdash v = x+y$, and $v \vdash u = x+y = u \vdash v$. This diassociative algebra is a special case of the isomorphism type $Dias_4^1$ in Theorem 4.2 of [5]. One computes $L_\varphi^2 = \langle x, y\rangle$ and $L_\varphi = 0$. Hence L_φ is nilpotent of class 3. We also note that the action of B on A is entirely zero; i.e. $\varphi_\dashv = \varphi_\vdash = \varphi'_\dashv = \varphi'_\vdash = 0$. Moreover, A is again abelian, and hence all lifts of the natural (Φ, Φ') pair are equal. To finish the point, the abelian extension L_{ab} of A by B corresponds to the same zero-lift of this pair, but has nilpotency class 2.

We conclude with an example in which A is nonabelian and hence the lifts are allowed to vary by adjoint operators. In this example, however, our nilpotency classes turn out to be the same. We note that the algebras in this case are both associative and Leibniz.

Example 5.3. Let $A = \langle x, y, z\rangle$ and $B = \langle w\rangle$ be associative algebras with only nonzero multiplications $x^2 = y^2 = z$. Consider two extensions $L_{(\varphi,\varphi')}$ and $L_{(\psi,\psi')}$ of A by B. Let $L_{(\varphi,\varphi')}$ have nonzero multiplications given by $x^2 = y^2 = xw = z$, $wx = -z$ and let $L_{(\psi,\psi')}$ have nonzero multiplications given by $x^2 = y^2 = z$.

Both of these algebras are clearly nilpotent of class 3 since both have center $\langle z\rangle$ equal to their derived subalgebras. One computes $\varphi(w)x = -z$, $\varphi'(w)x = z$, and $\varphi(w)y = \varphi'(w)y = \varphi(w)z = \varphi'(w)z = 0$. Also $\psi(w) = \psi'(w) = 0$. Thus $\varphi(w) = \psi(w) - \mathrm{ad}^l(x)$ and $\varphi(w) = \psi'(w) + \mathrm{ad}^r(x)$, and so our lifts vary by adjoint operators.

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