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Hardy Inequalities for Fractional (k, a)-Generalized Harmonic Oscillators

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Abstract. We will define a-deformed Laguerre operators $L_{a,\alpha}$ and a-deformed Laguerre holomorphic semigroups on $L^2\left((0,\infty), d\mu_{a,\alpha}\right)$. Then we give a spherical harmonic expansion, which reduces to the Bochner-type identity when taking the boundary value $z = \pi i/2$, of the (k, a)-generalized Laguerre semigroup introduced by Ben Saïd, Kobayashi and Ørsted. We prove a Hardy inequality for fractional powers of the a-deformed Dunkl harmonic oscillator $\triangle_{k,a} := |x|^{2-a}\triangle_k - |x|^a$ using this expansion. When $a = 2$, the fractional Hardy inequality reduces to that of Dunkl-Hermite operators given by Ciaurri, Roncal and Thangavelu. The operators $L_{a,\alpha}$ also give a tangible characterization of the radial part of the (k, a)-generalized Laguerre semigroup on each k-spherical component $\mathcal{H}_k^m\left(\mathbb{R}^N\right)$ for

$$\lambda_{k,a,m} := \frac{2m + 2\langle k\rangle + N - 2}{a} \geq -\frac{1}{2}$$

defined via a decomposition of the unitary representation.

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1. Introduction

Dunkl theory is a far-reaching generalization of Euclidean Fourier analysis associated with root system with a rich structure parallel to ordinary Fourier analysis, where finite reflection groups play the role of orthogonal groups in Euclidean Fourier analysis. The Lebesgue measure was replaced by a weighted measure $dm_k(x) = h_k(x)dx$ invariant under the reflection group and parameterized by a multiplicity function k, and the ordinary partial derivatives were replaced by a kind of differential-difference operators using the finite reflection groups and the multiplicity functions. Such differential-difference operators, called Dunkl operators, gave an explicit expression of the radial part of the Laplacian on a flat Riemann symmetric space. This theory has drawn considerable attention and there have been a lot of works on Dunkl's analysis in the last twenty years.

More recently, Ben Saïd, Kobayashi and Ørsted [4] gave a further far-reaching generalization of Dunkl theory by introducing a parameter $a > 0$ arisen from the "interpolation" of the two $sl(2, \mathbb{R})$ actions on the Weil representation of the metaplectic group $Mp(N, \mathbb{R})$ and the minimal unitary representation of the conformal group $O(N + 1, 2)$. They deformed an sl_2 triple studied in [2] via the parameter

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a such that the a-deformed Dunkl harmonic oscillator $\triangle_{k,a} := |x|^{2-a}\triangle_k - |x|^a$ is symmetric on the Hilbert space $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$, where $\vartheta_{k,a}(x) = |x|^{a-2}h_k(x)$. In the case of $k \equiv 0$, such a-deformed harmonic oscillator is also a deformation of the operator $|x|\triangle - |x|$ studied by Kobayashi and Mano in [12, 13]. Motivated by the definition of the classical Fourier transform on $L^2\left(\mathbb{R}^N\right)$ given by Howe [11] via classical harmonic oscillators, they then proved the existence of a (k,a)-generalized holomorphic semigroup $\mathcal{I}_{k,a}(z)$, $\Re z \geq 0$ with infinitesimal generator $\frac{1}{a}\triangle_{k,a}$ acting on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$. The (k,a)-generalized Laguerre semigroup $\mathcal{I}_{k,a}(z)$ generalizes the Hermite semigroup studied by Howe [11] ($k \equiv 0$ and $a = 2$), the Laguerre semigroup studied by Kobayashi and Mano [12, 13] ($k \equiv 0$ and $a = 1$), and the Dunkl Hermite semigroup studied by Rösler [19] ($k \geq 0$, $a = 2$ and $z = 2t$, $t > 0$). When taking the boundary value $z = \frac{\pi i}{2}$, the semigroup $\mathcal{I}_{k,a}(z)$ reduces to the so-called (k,a)-generalized Fourier transform $F_{k,a}$, i.e.,

$$F_{k,a} = c\mathcal{I}_{k,a}\left(\frac{\pi i}{2}\right), \tag{1}$$

where $c = e^{i\pi\left(\frac{2\langle k\rangle+N+a-2}{2a}\right)}$. The generalized Fourier transform includes the Fourier transform ($k \equiv 0$ and $a = 2$), the Kobayashi-Mano Hankel transform ($k \equiv 0$ and $a = 1$), and the Dunkl transform [8] ($k \geq 0$ and $a = 2$).

We will then define a one-dimensional a-deformed Laguerre holomorphic semigroup $I_{a,\alpha;z} := e^{-\frac{z}{a}L_{a,\alpha}}$ with the infinitesimal generator $-\frac{1}{a}L_{a,\alpha}$, where $L_{a,\alpha}$ is the a-deformed Laguerre operator, and show that it reduces to a-deformed Hankel transform $H_{a,\alpha}$ when taking the boundary value $z = \frac{\pi i}{2}$. The operators $L_{a,\alpha}$ also give an explicit expression of the radial part $\Omega_{k,a}^{(m)}(\gamma_z)$ of the (k,a)-generalized Laguerre semigroup on each k-spherical component $\mathcal{H}_k^m\left(\mathbb{R}^N\right)$ defined via decomposition of unitary representation in [4, Section 4.1], i.e.,

$$\Omega_{k,a}^{(m)}(\gamma_z)\,f(s) = s^m I_{a,\lambda_{k,a,m};z}\left((\cdot)^{-m} f\right)(s),\ \Re z \geq 0,\ s > 0,$$

where $\lambda_{k,a,m} := \dfrac{2m + 2\langle k\rangle + N - 2}{a}$, for $f \in L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$ and $\lambda_{k,a,m} \geq -\frac{1}{2}$ as will be shown in Section 5. We define $\lambda_a := \dfrac{2\langle k\rangle + N - 2}{a}$.

Theorem 1.1. *For any function $f \in L^2\left((0,\infty), d\mu_{a,\alpha}\right)$, $\alpha \geq -1/2$, we have*

$$e^{(\alpha+1)\pi i/2} I_{a,\alpha;\frac{\pi i}{2}}(f) = H_{a,\alpha}(f),$$

where the a-deformed Hankel transform is defined as

$$H_{a,\alpha}(f)(r) = \frac{1}{a^\alpha\Gamma(\alpha+1)}\int_0^\infty f(s) j_\alpha\left(\frac{2}{a} r^{\frac{a}{2}} s^{\frac{a}{2}}\right) s^{a\alpha+a-1}\,ds$$

and $j_\alpha(t) = 2^\alpha\Gamma(\alpha+1)t^{-\alpha}J_\alpha(t)$ is the normalized Bessel function.

We will then give a spherical harmonic expansion of the (k,a)-generalized Laguerre semigroup.

Theorem 1.2. (Spherical harmonic expansion of the (k,a)-generalized Laguerre semigroup) *For* $f \in L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$, $4\langle k\rangle + 2N + a - 4 \geq 0$, *and* $x \in \mathbb{R}^N$, $x = rx'$, *with* $r \in \mathbb{R}^+$, $x' \in \mathbb{S}^{N-1}$, *we have*

$$\mathcal{I}_{k,a}(z)\, f(x) = \sum_{m,j} Y_{m,j}(x') r^m I_{a,\lambda_{k,a,m};z}\left((\cdot)^{-m} f_{m,j}\right)(r), \tag{2}$$

where $\Re z \geq 0$. *Specially, the* (k,a)*-generalized Laguerre semigroup reduces to the one dimensional* a*-deformed Laguerre holomorphic semigroup for radial functions, that is, for* $f = f_0(|\cdot|)$, $f_0 \in L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$ *and* $r = |x|$, *we have*

$$\mathcal{I}_{k,a}(z)\, f(x) = \left(\mathcal{I}_{k,a}(z)\, f\right)_0(r), \quad \left(\mathcal{I}_{k,a}(z)\, f\right)_0(r) = I_{a,\lambda_a;z}(f_0)(r).$$

Remark 1.3. (i) This theorem, together with (1) and Theorem 1.1, imply the Bochner-type identity in [4, Theorem 5.21], which was used in [10] for Schwartz functions to prove Pitt's inequalities for the generalized Fourier transform. That is, taking the boundary value $z = \frac{\pi i}{2}$, the expansion reduces to

$$F_{k,a} f(x) = \sum_{m,j} e^{-i\pi m/a} Y_{m,j}(x') r^m H_{a,\lambda_{k,a,m}}\left((\cdot)^{-m} f_{m,j}\right)(r). \tag{3}$$

This theorem also generalizes the result in [22] that Hermite semigroups reduce to Laguerre semigroups of type $\frac{N}{2}-1$ (the case of $a=2$ and $k=0$) for radial functions on $\mathbb{R}^N$.

(ii) When $a = 2$ and $z = 2t$, $t > 0$, the expansion reduces to the formula given in Theorem 4.5 in [6], but our proof is different from that in [6] even in this case because we used the new tools introduced by Ben Saïd, Kobayashi and Ørsted [4] in the development of (k,a)-generalized Fourier analysis. ■

We will be interested in Hardy inequalities of the form

$$\int_X \frac{|f(x)|^2}{\left(1+|x|^2\right)^\sigma} d\eta(x) \leq B_\sigma \langle L^\sigma f, f\rangle \tag{4}$$

(or the Hardy inequality with homogeneous potential) for given $0 < \sigma < 1$, where L^σ is the fractional powers of a non-negative self-adjoint operator L and B_σ is a constant. It is a generalization of the classical Hardy inequality on $\mathbb{R}^N$

$$\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|f(x)|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla f(x)|^2\, dx, \ N \geq 3.$$

In [6], Ciaurri, Roncal and Thangavelu worked with *conformally invariant* fractional powers of Dunkl-Hermite operators $\mathbf{H}_k = -\triangle_k + |x|^2$, where $\triangle_k$ is a generalization of the classical Laplacian on Euclidean space called Dunkl Laplacian, and proved the fractional Hardy inequalities for these operators of form (4) using ground state representation. The conformal invariant fractional powers was borrowed from the context of sublaplacians on Heisenberg groups (see [17]). They also deduced the Hardy inequalities for pure fractional powers of Dunkl-Hermite operators $\mathbf{H}_k^\sigma$ (see [6, Corollary 1.5]) as a consequence of the conformally invariant fractional Hardy inequalities.

We will prove a Hardy inequality of type (4) for fractional powers of the a-deformed Dunkl-Hermite operator $\triangle_{k,a} = |x|^{2-a}\triangle_k - |x|^a$ using the spherical harmonic expansion of the (k,a)-generalized Laguerre semigroup (2).

Theorem 1.4. *Let us define the constant* $B^{\delta}_{\alpha,\sigma} := \delta^{\sigma}\dfrac{\Gamma\left(\frac{\alpha+2+\sigma}{2}\right)}{\Gamma\left(\frac{\alpha+2-\sigma}{2}\right)}$.
If $0<\sigma<1$, $\delta>0$, $4\langle k\rangle+2N+a-4\geq 0$ *we have for all* $f\in C_0^\infty(\mathbb{R}^N)$

$$\left(\frac{a}{2}\right)^{\sigma} B^{\delta}_{\lambda_a,\sigma}\int_{\mathbb{R}^N}\frac{|f(x)|^2}{\left(\delta+\frac{2}{a}|x|^a\right)^{\sigma}}\vartheta_{k,a}(x)dx \leq \left\langle \left(-\triangle_{k,a}\right)_{\sigma} f, f\right\rangle_{L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)dx\right)}.$$

When $a=2$, this inequality reduces to the fractional Hardy inequality in [6], which was proved using Dunkl-Hermite expansions. The definition of the modified fractional operator $\left(-\triangle_{k,a}\right)_{\sigma}$ will be given analogously as in [6] in Section 4. We can also deduce the Hardy inequalities for pure fractional powers of the operator $\left(-\triangle_{k,a}\right)^{\sigma}$ analogous to Corollary 1.5 in [6] from this Hardy inequality. An uncertainty principle for fractional powers of $\triangle_{k,a}$ can also be deduced from this Hardy inequality as in [17].

There have also been several other studies of Hardy inequalities of form (4). For example, Gorbachev, Ivanov and Tikhonov [9] proved a sharp Pitt's inequality for Dunkl transform in $L^2\left(\mathbb{R}^N\right)$. Such Pitt's inequalities can imply a Hardy inequality of the form (4) for fractional powers of the Dunkl Laplacian $\triangle_k$. They also proved a sharp Pitt's inequality for the generalized Fourier transform $F_{k,a}$ in [10] using the Bochner-type identity (3), a particular case of the expansion (2) we will use. By the formula (5.6 b) in [4], the fractional powers of $-|x|^{2-a}\Delta_k$ can be naturally defined as follows,

$$F_{k,a}\left(\left(-|\cdot|^{2-a}\Delta_k\right)^{\beta} f\right)(\xi) = \left(|\xi|^a\right)^{\beta} F_{k,a}(f)(\xi).$$

And then from the inversion formula [4, Theorem 5.3] of the (k,a)-generalized Fourier transform, the Pitt's inequality in [10] implies also a Hardy inequality of the form (4) for $L=-|x|^{2-a}\Delta_k$ for $a=\frac{2}{n}$, $n\in\mathbb{N}_+$. When $a=2$, this Hardy inequality reduces to that for fractional powers of the Dunkl Laplacian in [9]. The two Pitt's inequalities imply the logarithmic uncertainty principle for the Dunkl transform and $F_{k,a}$, respectively.

This paper is organized as follows. In Section 2, we recall the tools and concepts we will use to prove the main theorems. We refer to [3, 4, 7] for the tools and concepts. In Section 3, we give the definitions of the a-deformed Laguerre convolution and the fractional a-deformed Laguerre operators, and then prove a Hardy inequality for the fractional a-deformed Laguerre operators, which reduces to the Hardy inequality for fractional Laguerre operators given in [6] when $a=2$. We will prove Theorem 1.1 in Section 3 as well. In Section 4 we give the proof of Theorem 1.2 using the tools introduced by Ben Saïd, Kobayashi and Ørsted [4] and then prove Theorem 1.4. In Section 5 we will give the tangible characterization of the radial part of the (k,a)-generalized Laguerre semigroup on each k-spherical component $\mathcal{H}^m_k\left(\mathbb{R}^N\right)$ for $\lambda_{k,a,m}\geq -1/2$.

2. Preliminaries

2.1. Dunkl operators and Dunkl transform

Given a root system R in the Euclidean space $\mathbb{R}^N$, denote by G the finite subgroup of $O(N)$ generated by the reflections σ_α associated to the root system. Define a *multiplicity function* $k\colon R\to\mathbb{C}$ such that k is G-invariant, that is, $k(\alpha)=k(\beta)$

if σ_α and σ_β are conjugate. We assume $k \geqslant 0$ in this paper. The *Dunkl operators* T_ξ, $\xi \in \mathbb{R}^N$, which were introduced in [7], are defined by the following deformations by difference operators of directional derivatives ∂_ξ:

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R^+} k(\alpha) \langle \alpha,\ \xi \rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha,\ x \rangle},$$

where R^+ is any fixed positive system of R. They commute pairwise and are skew-symmetric with respect to the G-invariant measure $dm_k(x) = h_k(x)dx$, where

$$h_k(x) = \prod_{\alpha \in R} |\langle \alpha,\ x \rangle|^{k(\alpha)}.$$

Denote by $\mathbf{N} = N + 2\langle k \rangle$ the homogeneous dimension of the root system, where $\langle k \rangle := \sum_{\alpha \in R^+} k(\alpha)$. Let $e_j,\ j = 1, 2, ..., N,$ be the canonical orthonormal basis in $\mathbb{R}^N$ and denote $T_j = T_{e_j}$. The *Dunkl Laplacian* is defined by $\triangle_k = \sum_{j=1}^N T_j^2$ and it can be expressed explicitly.

The Dunkl kernel $E(x, y)$ is the unique analytic solution to the differential-difference equation system

$$T_\xi f = \langle \xi, y \rangle f, \quad f(0) = 1,$$

for any fixed $y \in \mathbb{R}^N$. For $f \in L^1(m_k)$ the Dunkl transform is defined by

$$F(f)(\xi) = \frac{1}{c_k} \int_{\mathbb{R}^N} f(x) E(-i\xi,\ x) dm_k(x),\ c_k = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{2}} dm_k(x).$$

It is a generalization of and has similar properties with the classical Fourier transform.

2.2. An orthonormal basis in $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\, dx\right)$

An h-harmonic polynomial of degree m is a homogeneous polynomial p on $\mathbb{R}^N$ of degree m satisfying $\triangle_k p = 0$. Denote by $\mathcal{H}_k^m\left(\mathbb{R}^N\right)$ the space of h-harmonic polynomials of degree m. Spherical h-harmonics (or just h-harmonics) of degree m are then defined as the restrictions of $\mathcal{H}_k^m\left(\mathbb{R}^N\right)$ to the unit sphere $\mathbb{S}^{N-1}$. The spaces $\mathcal{H}_k^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}},\ m = 0, 1, 2, \ldots$ are finite dimensional and orthogonal to each other with respect to the measure $h_k(x')\, d\sigma(x')$. And there is the spherical harmonics decomposition

$$L^2\left(\mathbb{S}^{N-1}, h_k(x')\, d\sigma(x')\right) = \sum_{m \in \mathbb{N}}^{\oplus} \mathcal{H}_k^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}. \tag{5}$$

Consider the weight function $\vartheta_{k,a}(x) = |x|^{a-2} h_k(x)$. It reduces to $h_k(x)$ when $a = 2$ and for any $x' \in \mathbb{S}^{N-1}$ we then have $\vartheta_{k,a}(x') = h_k(x')$. For the polar coordinates $x = rx'$ $(r > 0,\ x' \in \mathbb{S}^{N-1})$ we have

$$\vartheta_{k,a}(x)\, dx = r^{2\langle k \rangle + N + a - 3} \vartheta_{k,a}(x')\, dr d\sigma(x').$$

From the spherical harmonic decomposition (5) of $L^2\left(\mathbb{S}^{N-1}, h_k(x')\, d\sigma(x')\right)$, there is a unitary isomorphism (see [4, (3.25)])

$$\sum_{m \in \mathbb{N}}^{\oplus} (\mathcal{H}_k^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}) \otimes L^2\left(\mathbb{R}_+, r^{\,2\langle k \rangle + N + a - 3} dr\right) \xrightarrow{\sim} L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\, dx\right).$$

Define the Laguerre polynomial as

$$L_l^{\mu}(t) := \sum_{j=0}^{l} \frac{(-1)^j \Gamma(\mu+l+1)}{(l-j)!\Gamma(\mu+j+1)} \frac{t^j}{j!}, \ \mathrm{Re}\mu > -1.$$

Proposition 2.1. ([4, Proposition 3.15])) *For fixed $m \in \mathbb{N}$, $a > 0$, and a multiplicity function k satisfying $\lambda_{k,a,m} > -1$. Set*

$$\psi_{l,m}^{(a)}(r) := \left(\frac{2^{\lambda_{k,a,m}+1}\Gamma(l+1)}{a^{\lambda_{k,a,m}}\Gamma(\lambda_{k,a,m}+l+1)} \right)^{1/2} r^m L_l^{\lambda_{k,a,m}} \left(\frac{2}{a} r^a \right) \exp\left(-\frac{1}{a} r^a \right). \quad (6)$$

Then $\left\{ \psi_{l,m}^{(a)}(r) : \ l \in \mathbb{N} \right\}$ forms an orthonormal basis in $L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$.

For each fixed $m \in \mathbb{N}$, we take an orthonormal basis of $\mathcal{H}_k^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}$ as

$$\left\{ Y_i^m : \ i = 1, 2, \cdots, d(m) \right\}, \quad (7)$$

where $d(m) = \dim\left(\mathcal{H}_k^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}\right)$. They are the eigenvectors of the generalized Laplace-Beltrami operator $\triangle_{k;0}$. Proposition 2.1 yields the orthonormal basis in $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ immediately.

Corollary 2.2. ([4, Corollary 3.17]) *Suppose that $a > 0$ and that k satisfy the inequality $2m + 2\langle k\rangle + N + a - 2 > 0$. Set*

$$\Phi_{l,m,j}^{(a)}(x) := Y_j^m\left(\frac{x}{|x|}\right)\psi_{l,m}^{(a)}(|x|). \quad (8)$$

Then

$$\left\{ \Phi_{l,m,j}^{(a)} \,\middle|\, l \in \mathbb{N}, \ m \in \mathbb{N}, \ j = 1, 2, \cdots, d(m) \right\} \quad (9)$$

forms an orthonormal basis of $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$.

2.3. The (k,a)-generalized Laguerre semigroup and Fourier transform

Define $W_{k,a}\left(\mathbb{R}^N\right) := \mathbb{C}\text{-}span\left\{ \Phi_l^{(a)}(p,\cdot) \,\middle|\, l \in \mathbb{N}, \ m \in \mathbb{N}, \ p \in \mathcal{H}_k^m\left(\mathbb{R}^N\right) \right\}$, where

$$\Phi_l^{(a)}(p,x) = p(x')r^m L_l^{\lambda_{k,a,m}}\left(\frac{2}{a}r^a\right)\exp\left(-\frac{1}{a}r^a\right)$$

for $x = rx'$ ($r > 0$, $x' \in \mathbb{S}^{N-1}$). It is a dense subset of the Hilbert space $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$. Define the a-deformed Dunkl-type harmonic oscillator with domain $W_{k,a}\left(\mathbb{R}^N\right)$ as follows (see [3, 4]),

$$\Delta_{k,a} = |x|^{2-a}\Delta_k - |x|^a, a > 0.$$

It is an essentially self-adjoint operator on $L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$ with only negative discrete spectrum. And so $\frac{1}{a}\triangle_{k,a}$ is the infinitesimal generator of the (k,a)-generalized Laguerre semigroup $\mathcal{I}_{k,a}(z) := \exp\left(\frac{z}{a}\triangle_{k,a}\right)$, $\Re z \geq 0$. The semigroup $\mathcal{I}_{k,a}(z)$ can also be defined by a unitary representation, i.e., $\mathcal{I}_{k,a}(z) := \Omega_{k,a}(\gamma_z)$, $\Re z \geq 0$ (see [4] for the detailed definition of $\Omega_{k,a}(\gamma_z)$). By Schwartz kernel theorem, $\mathcal{I}_{k,a}(z)$ has an integral representation by means of a distribution kernel $\Lambda_{k,a}(x,y;z)$.

We refer to [4] for the details on the distribution kernel. The boundary value $z=\frac{\pi i}{2}$ of $\mathcal{I}_{k,a}(z)$ gives the definition of the (k,a)-generalized Fourier transform $F_{k,a}$. The operator $F_{k,a}$ is a bijective linear operator such that the Plancherel formula holds for $f\in L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$.

3. A Hardy inequality for the fractional *a*-deformed Laguerre operator

The Laguerre translation $\mathcal{T}_r^\alpha$ was introduced by McCully [15] for $\alpha=0$ and was extended to $\alpha\geq -1/2$ (see [1] or [21, Chapter 6]). We define the *a-deformed Laguerre translation* as

$$\mathcal{T}_r^{a,\alpha}f(s):=\frac{\Gamma(\alpha+1)2^\alpha}{\sqrt{2\pi}}\int_0^\pi f\left(\left(r^a+s^a+2r^{\frac{a}{2}}s^{\frac{a}{2}}\cos\theta\right)^{1/a}\right) \cdot J_{\alpha-1/2}\left(\frac{2}{a}r^{\frac{a}{2}}s^{\frac{a}{2}}\sin\theta\right)\left(\frac{2}{a}r^{\frac{a}{2}}s^{\frac{a}{2}}\sin\theta\right)^{-(\alpha-1/2)}(\sin\theta)^{2\alpha}\,d\theta$$

for $r,s>0$ and $\alpha\geq -1/2$, where J_ν is the Bessel function of order ν. When $a=2$, it reduces to the Laguerre translation $\mathcal{T}_r^\alpha$ in [6]. The results in [6] are also valid for the critical case when $\alpha=-1/2$ since the definition of the Laguerre translation can be extended to this case. If f and g are functions defined on $(0,\infty)$, the *a-deformed Laguerre convolution* $f*_{a,\alpha}g$ is given by

$$f*_{a,\alpha}g(r)=\int_0^\infty \mathcal{T}_r^{a,\alpha}f(s)g(s)s^{a\alpha+a-1}\,ds. \tag{10}$$

By changing variables $\quad r=\left(\frac{a}{2}\right)^{1/a}r_1^{2/a},\quad s=\left(\frac{a}{2}\right)^{1/a}s_1^{2/a}$

and setting $\quad f_1=f\left(\left(\frac{a}{2}\right)^{1/a}(\cdot)^{2/a}\right),\quad g_1=g\left(\left(\frac{a}{2}\right)^{1/a}(\cdot)^{2/a}\right),$

we have

$$\begin{aligned}&\int_0^\infty \mathcal{T}_r^{a,\alpha}f(s)g(s)s^{a\alpha+a-1}\,ds=\left(\frac{a}{2}\right)^{\alpha+1}\int_0^\infty \mathcal{T}_{r_1}^{\alpha}f_1(s_1)g_1(s_1)s_1^{2\alpha+1}\,ds_1\\&=\left(\frac{a}{2}\right)^{\alpha+1}f_1*_\alpha g_1(r_1)=\left(\frac{a}{2}\right)^{\alpha+1}g_1*_\alpha f_1(r_1)=\left(\frac{a}{2}\right)^{\alpha+1}\int_0^\infty \mathcal{T}_{r_1}^{\alpha}g_1(s_1)f_1(s_1)s_1^{2\alpha+1}\,ds_1\\&=\int_0^\infty \mathcal{T}_r^{a,\alpha}g(s)f(s)s^{a\alpha+a-1}\,ds,\end{aligned}$$

where $f*_\alpha g$ is the Laguerre convolution defined in [21, Chapter 6]. Thus we obtain $f*_{a,\alpha}g(r)=g*_{a,\alpha}f(r)$. Let

$$\varphi_l^{a,\alpha}(r):=L_l^\alpha\left(\frac{2}{a}r^a\right)\exp\left(-\frac{1}{a}r^a\right),\quad l=0,1,\cdots.$$

Then substituting r as $\sqrt{\frac{2}{a}}r^{\frac{a}{2}}$ and s as $\sqrt{\frac{2}{a}}s^{\frac{a}{2}}$ in the formula (3.2) in [6], we get

$$\mathcal{T}_r^{a,\alpha}\varphi_n^{a,\alpha}(s)=\frac{n!}{(\alpha+1)_n}\varphi_n^{a,\alpha}(r)\varphi_n^{a,\alpha}(s),\qquad \alpha\geq -1/2. \tag{11}$$

The Laguerre operator $\quad L_\alpha=-\frac{d^2}{dr^2}+r^2-\frac{2\alpha+1}{r}\frac{d}{dr} \quad (12)$

studied in [6] is a symmetric operator on $L^2((0,\infty), d\mu_\alpha)$, where $\alpha \geq -1/2$ and $d\mu_\alpha(r) = r^{2\alpha+1}dr$. The functions

$$\widetilde{\varphi}_l^\alpha(r) = \left(\frac{2\Gamma(l+1)}{\Gamma(\alpha+l+1)}\right)^{1/2} L_l^\alpha\left(r^2\right)\exp\left(-\frac{1}{2}r^2\right),\ l = 0, 1, \cdots$$

are eigenfunctions of L_α with eigenvalues $2(2l+\alpha+1)$.

Substituting r by $u = \sqrt{\frac{2}{a}}r^{\frac{a}{2}}$ in (12),

$$\begin{aligned}-\frac{d^2}{du^2} + u^2 - \frac{2\alpha+1}{u}\frac{d}{du} &= -\frac{2}{a}\left(\frac{1}{r^{a-2}}\frac{d^2}{dr^2} + \left(1-\frac{a}{2}\right)\frac{1}{r^{a-1}}\frac{d}{dr}\right) + \frac{2}{a}r^a - \frac{2\alpha+1}{r^{a-1}}\frac{d}{dr}\\ &= \frac{2}{a}\left(-\frac{1}{r^{a-2}}\frac{d^2}{dr^2} + r^a - (a\alpha+1)\frac{1}{r^{a-1}}\frac{d}{dr}\right).\end{aligned}$$

The a-deformed Laguerre differential operator can then be defined as

$$L_{a,\alpha} = -\frac{1}{r^{a-2}}\frac{d^2}{dr^2} + r^a - (a\alpha+1)\frac{1}{r^{a-1}}\frac{d}{dr}. \tag{13}$$

It is symmetric on $L^2(0,\infty)$ with respect to the measure $d\mu_{a,\alpha}(r) = r^{a\alpha+a-1}dr$, $\alpha \geq -1/2$. When $a = 2$, the operator reduces to the Laguerre operator (12).

Define the Laguerre functions of type α as

$$\widetilde{\varphi}_l^{a,\alpha}(r) = \left(\frac{2^{\alpha+1}\Gamma(l+1)}{a^\alpha\Gamma(\alpha+l+1)}\right)^{1/2} L_l^\alpha\left(\frac{2}{a}r^a\right)\exp\left(-\frac{1}{a}r^a\right),\ l = 0, 1, \cdots,$$

where $\alpha \geq -1/2$. Then they form an orthonormal basis of $L^2((0,\infty), d\mu_{a,\alpha})$ (this is also the case of Proposition 2.1 when $\alpha = \lambda_{k,a,m}$) and are the eigenfunctions of the a-deformed Laguerre operator (13). Indeed,

$$L_{a,\alpha}\widetilde{\varphi}_l^{a,\alpha} = a(2l+\alpha+1)\widetilde{\varphi}_l^{a,\alpha},\quad l = 0, 1, \cdots.$$

It suffices to substitute r by $\sqrt{\frac{2}{a}}r^{\frac{a}{2}}$ in the conclusions of [6, Section 3] to get this.

The Laguerre expansion of $f \in L^2((0,\infty), d\mu_{a,\alpha})$, namely the expansion

$$f = \sum_{l=0}^{\infty}\left(\frac{2^{\alpha+1}\Gamma(l+1)}{a^\alpha\Gamma(\alpha+l+1)}\right)\langle f, \varphi_l^{a,\alpha}\rangle_{d\mu_{a,\alpha}}\varphi_l^{a,\alpha}$$

can be written in a compact form in terms of Laguerre convolution.

Lemma 3.1. *For a function $f \in L^2((0,\infty), d\mu_{a,\alpha})$, $\varphi_l^{a,\alpha}$ is an eigenfunction of f, i.e.*

$$f *_{a,\alpha} \varphi_l^{a,\alpha} = \frac{\Gamma(\alpha+1)\Gamma(l+1)}{\Gamma(\alpha+l+1)}\langle f, \varphi_l^{a,\alpha}\rangle_{d\mu_{a,\alpha}}\varphi_l^{a,\alpha}.$$

In particular,

$$\delta_{nj}\varphi_n^{a,\alpha} = \frac{2^{\alpha+1}}{a^\alpha\Gamma(\alpha+1)}\varphi_n^{a,\alpha} *_{a,\alpha} \varphi_j^{a,\alpha}. \tag{14}$$

Proof. Omitted. It is only a slight modification of the proof of Lemma 3.1 in [6]. ∎

Thus $f *_{a,\alpha} \varphi_l^{a,\alpha}$ are eigenfunctions of $L_{a,\alpha}$ with the eigenvalues $a(2l+\alpha+1)$ for $l=0,1,\cdots$ and we have the spectral decomposition of the a-deformed Laguerre operator

$$L_{a,\alpha} f = \frac{2^{\alpha+1}}{a^\alpha \Gamma(\alpha+1)} \sum_{l=0}^{\infty} a(2l+\alpha+1) f *_{a,\alpha} \varphi_l^{a,\alpha}.$$

It is then natural to define fractional powers of Laguerre operators as

$$L_{a,\alpha}^{\sigma} f = \frac{2^{\alpha+1}}{a^\alpha \Gamma(\alpha+1)} \sum_{l=0}^{\infty} \left(a(2l+\alpha+1)\right)^{\sigma} f *_{a,\alpha} \varphi_l^{a,\alpha}, \quad \alpha \geq -1/2.$$

But it suits better to work with the modified fractional operator $L_{a,\alpha;\sigma}$ with the spectrum $4^\sigma S_l^{a,\alpha;\sigma}$, i.e.

$$L_{a,\alpha,\sigma} f = \frac{2^{\alpha+1}}{a^\alpha \Gamma(\alpha+1)} \sum_{l=0}^{\infty} (2a)^{\sigma} S_l^{a,\alpha;\sigma} f *_{a,\alpha} \varphi_l^{a,\alpha}, \quad \alpha \geq -1/2,$$

where

$$S_l^{a,\alpha;\sigma} = \frac{\Gamma\left(\frac{a(2l+\alpha+1)}{2a} + \frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{a(2l+\alpha+1)}{2a} + \frac{1-\sigma}{2}\right)},$$

because such fractional powers of the operator correspond to the conformally invariant fractional powers of sublaplacian $\mathcal{L}$ on Heisenberg groups when we consider the conformally invariant fractional powers $\mathcal{L}_\sigma$ (see [17]) acting on the functions of the form $e^{it} f(|z|)$. In short, we write

$$L_{a,\alpha;\sigma} = (2a)^{\sigma} \frac{\Gamma\left(\frac{L_{a,\alpha}}{2a} + \frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{L_{a,\alpha}}{2a} + \frac{1-\sigma}{2}\right)}.$$

The motivation for this definition goes back to [5, (1.33)], for instance.
For $\delta > 0$ and $\alpha \geq -1/2$, define

$$\omega_{\alpha,\sigma}^{\delta,a}(r) := c_{\alpha,\sigma} \left(\delta + \tfrac{2}{a} r^a\right)^{-(\alpha+1+\sigma)/2} K_{(\alpha+1+\sigma)/2}\left(\frac{\delta + \frac{2}{a} r^a}{2}\right),$$

where K_ν is the Macdonald's function of order ν (see [14, Chapter 5, Section 5.7]), and $c_{\alpha,\sigma}$ is the constant

$$c_{\alpha,\sigma} := \frac{\sqrt{\pi}\, 2^{1-\sigma}}{\Gamma\left((\alpha+2+\sigma)/2\right)}.$$

In [6], the authors proved a Hardy inequality for the fractional Laguerre operator for the case of $a=2$ using ground state representation.

Theorem 3.2. ([6, Theorem 1.1]) *Let* $0 < \sigma < 1$, $\delta > 0$, *and* $2\alpha + 1 > 0$. *Then*

$$B_{\alpha,\sigma}^{\delta} \int_0^\infty \frac{|f(r)|^2}{(\delta + r^2)^\sigma} d\mu_\alpha(r) \leq \frac{4^\sigma}{\delta^\sigma} \left(B_{\alpha,\sigma}^{\delta}\right)^2 \int_0^\infty |f(r)|^2 \frac{\omega_{\alpha,\sigma}^{\delta}(r)}{\omega_{\alpha,-\sigma}^{\delta}(r)} d\mu_\alpha(r) \leq \left\langle L_{\alpha,\sigma} f, f \right\rangle_{d\mu_\alpha}$$

for all $f \in C_0^\infty(0,\infty)$.

Taking f as the Laguerre functions for the case of $a=2$, and then substituting r by $\sqrt{\frac{2}{a}}r^{\frac{a}{2}}$, we get for $\alpha \geq -1/2$

$$B^{\delta}_{\alpha,\sigma}\int_0^\infty \frac{|\widetilde{\varphi}_l^{a,\alpha}(r)|^2}{\left(\delta+\frac{2}{a}r^a\right)^\sigma}d\mu_{a,\alpha}(r) \leq \frac{4^\sigma}{\delta^\sigma}\left(B^{\delta}_{\alpha,\sigma}\right)^2\int_0^\infty |\widetilde{\varphi}_l^{a,\alpha}(r)|^2\frac{\omega^{\delta,a}_{\alpha,\sigma}(r)}{\omega^{\delta,a}_{\alpha,-\sigma}(r)}d\mu_{a,\alpha}(r)$$
$$\leq \left\langle \left(\frac{2}{a}L_{a,\alpha}\right)_\sigma \widetilde{\varphi}_l^{a,\alpha}, \widetilde{\varphi}_l^{a,\alpha}\right\rangle_{d\mu_{a,\alpha}}.$$

Here $\left(\frac{2}{a}L_{a,\alpha}\right)_\sigma = 4^\sigma\frac{\Gamma\left(\frac{\frac{2}{a}L_{a,\alpha}}{4}+\frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{\frac{2}{a}L_{a,\alpha}}{4}+\frac{1-\sigma}{2}\right)}$, which is equal to $\left(\frac{2}{a}\right)^\sigma L_{a,\alpha;\sigma}$.

Then using the expansion via Laguerre functions, we derive the Hardy inequality for the fractional a-deformed Laguerre operator.

Theorem 3.3. *Let $0<\sigma<1$, $\delta>0$, and $\alpha\geq -1/2$. Then, for all $f\in C_0^\infty(0,\infty)$,*

$$\left(\frac{a}{2}\right)^\sigma B^{\delta}_{\alpha,\sigma}\int_0^\infty \frac{|f(r)|^2}{\left(\delta+\frac{2}{a}r^a\right)^\sigma}d\mu_{a,\alpha}(r) \leq \left(\frac{2a}{\delta}\right)^\sigma\left(B^{\delta}_{\alpha,\sigma}\right)^2\int_0^\infty |f(r)|^2\frac{\omega^{\delta,a}_{\alpha,\sigma}(r)}{\omega^{\delta,a}_{\alpha,-\sigma}(r)}d\mu_{a,\alpha}(r)$$
$$\leq \langle L_{a,\alpha;\sigma}f,f\rangle_{d\mu_{a,\alpha}}.$$

The holomorphic semigroup related to the a-deformed Laguerre operator $L_{a,\alpha}$ is defined on $L^2((0,\infty),\ d\mu_{a,\alpha})$ by

$$I_{a,\alpha;z}f = e^{-\frac{z}{a}L_{a,\alpha}}f, \quad \Re z\geq 0. \tag{15}$$

From the spectral decomposition of $L_{a,\alpha}$ this is equal to

$$\frac{2^{\alpha+1}}{a^\alpha\Gamma(\alpha+1)}\sum_{l=0}^\infty e^{-z(2l+\alpha+1)}f *_{a,\alpha}\varphi_l^{a,\alpha}.$$

Proof of Theorem 1.1. Define

$$q_{a,\alpha;z}(r) := \frac{2^{\alpha+1}}{a^\alpha\Gamma(\alpha+1)}\sum_{l=0}^\infty e^{-z(2l+\alpha+1)}\varphi_l^{a,\alpha}(r) = \left(\frac{2}{a}\right)^\alpha q_{2,\alpha;z}\left(\sqrt{\frac{2}{a}}r^{\frac{a}{2}}\right).$$

Then we can write $e^{-\frac{z}{a}L_{a,\alpha}}f = f *_{a,\alpha} q_{a,\alpha;z}$.

We give the kernel of the holomorphic semigroup $I_{a,\alpha;z}$.

Lemma 3.4. *Let $\alpha\geq -1/2$, $\Re z\geq 0$ and $z\neq 0$, we have that*

$$\mathcal{T}_r^{a,\alpha}q_{a,\alpha;z}(s) = \frac{e^{-\frac{\coth z}{a}(r^a+s^a)}}{(r^{\frac{a}{2}}s^{\frac{a}{2}})^\alpha\sinh z}I_\alpha\left(\frac{\frac{2}{a}r^{\frac{a}{2}}s^{\frac{a}{2}}}{\sinh z}\right),$$

where I_α is the modified Bessel function of the first kind and order α, see [14, Chapter 5, Section 5.7].

Proof. For the case when $a=2$, we take $w=e^{-z}$ in the equality (see [21, p. 83])

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\varphi_n^{\alpha}(r)\varphi_n^{\alpha}(s)w^{2n}\\&\qquad=(1-w^2)^{-1}(rsw)^{-\alpha}\exp\Big\{-\frac{1}{2}\Big(\frac{1+w^2}{1-w^2}\Big)(r^2+s^2)\Big\}I_\alpha\Big(\frac{2wrs}{1-w^2}\Big),|w|<1.\end{aligned}$$

Then we get the Lemma for $a=2$. And it reduces to Lemma 3.2 in [6] when $z=2t,\ t>0$ in this case. For the general case of $a>0$, changing variables

$$r=\left(\frac{a}{2}\right)^{1/a}r_1^{2/a},\quad s=\left(\frac{a}{2}\right)^{1/a}s_1^{2/a},$$

leads to

$$\begin{aligned}\mathcal{T}_r^{a,\alpha}q_{a,\alpha;z}(s)&=\left(\frac{2}{a}\right)^{\alpha}\mathcal{T}_{r_1}^{\alpha}q_{2,z;\alpha}(s_1)\\&=\left(\frac{2}{a}\right)^{\alpha}\frac{e^{-\frac{\coth\ z}{2}(r_1^2+s_1^2)}}{(r_1s_1)^{\alpha}\sinh\ z}I_\alpha\left(\frac{r_1s_1}{\sinh\ z}\right)=\frac{e^{-\frac{\coth\ z}{2}\frac{2}{a}(r^a+s^a)}}{(r^{\frac{a}{2}}s^{\frac{a}{2}})^{\alpha}\sinh\ z}I_\alpha\left(\frac{\frac{2}{a}r^{\frac{a}{2}}s^{\frac{a}{2}}}{\sinh\ z}\right).\end{aligned}$$

The proof of Lemma 3.4 is therefore complete. This lemma can also be deduced from the Hille-Hardy identity directly. ■

Let $z=i\frac{\pi}{2}$. Then from formula (5.7.4) in [14],

$$I_\alpha\left(\frac{\frac{2}{a}r^{\frac{a}{2}}s^{\frac{a}{2}}}{\sinh i\frac{\pi}{2}}\right)=e^{-\alpha\pi i/2}J_\alpha\left(\frac{2}{a}r^{\frac{a}{2}}s^{\frac{a}{2}}\right)=e^{-\alpha\pi i/2}\frac{\left(\frac{2}{a}r^{\frac{a}{2}}s^{\frac{a}{2}}\right)^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}j_\alpha\left(\frac{2}{a}r^{\frac{a}{2}}s^{\frac{a}{2}}\right).$$

So

$$\begin{aligned}I_{a,\alpha;i\frac{\pi}{2}}f(r)=f*_{a,\alpha}q_{a,\alpha;i\frac{\pi}{2}}(r)&=\int_0^\infty f(s)\mathcal{T}_r^{a,\alpha}q_{a,\alpha;i\frac{\pi}{2}}(s)s^{a\alpha+a-1}\,ds\\&=e^{-(\alpha+1)\pi i/2}\frac{1}{a^{\alpha}\Gamma(\alpha+1)}\int_0^\infty f(s)j_\alpha\left(\frac{2}{a}r^{\frac{a}{2}}s^{\frac{a}{2}}\right)s^{a\alpha+a-1}\,ds\\&=e^{-(\alpha+1)\pi i/2}H_{a,\alpha}(f)(r).\end{aligned}$$

The proof of Theorem 1.1 is therefore complete. ■

4. Proof of Theorem 1.4

Consider the orthonormal basis (7) of $\mathcal{H}_k^m\left(\mathbb{R}^N\right)|_{\mathbb{S}^{N-1}}$. Accordingly, we have the h-harmonic expansion for $f\in L^2\left(\mathbb{R}^N,\vartheta_{k,a}(x)\,dx\right)$,

$$f(rx')=\sum_{m=0}^{\infty}\sum_{i=1}^{d(m)}f_{m,i}(r)Y_i^m(x'),\tag{16}$$

where
$$f_{m,i}(r)=\int_{\mathbb{S}^{N-1}}f(rx')\,Y_i^m(x')\vartheta_{k,a}(x')d\sigma(x').$$

In [4], the authors proved that $\Phi_{l,m,j}^{(a)}$ (see (8)) are eigenfunctions for $-\Delta_{k,a}$ by interpreting $\Delta_{k,a}$ in the framework of the (infinite dimensional) representation of the Lie algebra $sl(2,\mathbb{R})$ (see (3.9 a) and (3.32 a) in [4]) on its dense domain $W_{k,a}\left(\mathbb{R}^N\right)$, i.e.

$$-\Delta_{k,a}\Phi_{l,m,j}^{(a)}(x)=a\left(2l+\lambda_{k,a,m}+1\right)\Phi_{l,m,j}^{(a)}(x).\tag{17}$$

Then we have the spectral decomposition of $\mathcal{I}_{k,a}(z)f$ via the basis (9) in terms of Laguerre polynomials for $f \in L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)\,dx\right)$,

$$\mathcal{I}_{k,a}(z)f(x) = \sum_{l,m,j} e^{-z\left(2l+\lambda_{k,a,m}+1\right)} \left\langle f, \Phi_{l,m,j}^{(a)} \right\rangle_{k,a} \Phi_{l,m,j}^{(a)}(x), \tag{18}$$

where $\langle f, g\rangle_{k,a} = \int_{\mathbb{R}^N} f(x)g(x)\vartheta_{k,a}(x)dx$.

Proof of Theorem 1.2. By Lemma 3.1, the a-deformed Laguerre holomorphic semigroup can also be written as

$$I_{a,\alpha;z}f = \sum_{l=0}^{\infty} e^{-z(2l+\alpha+1)} \langle f, \widetilde{\varphi}_l^{a,\alpha}\rangle_{d\mu_{a,\alpha}} \widetilde{\varphi}_l^{a,\alpha}.$$

We then apply the spherical harmonic expansion (16) to the spectral definition (18) of $T_t^{k,a}f(x)$. By (8) and by noticing that

$$\widetilde{\varphi}_l^{a,\lambda_{k,a,m}}(r) = r^{-m}\psi_{l,m}^{(a)}(r)$$

when $\lambda_{k,a,m} \geq -1/2$, we have

$$\begin{aligned}
\mathcal{I}_{k,a}(z)f(x) &= \sum_{l,m,j} e^{-z\left(2l+\lambda_{k,a,m}+1\right)} \left\langle f, \Phi_{l,m,j}^{(a)} \right\rangle_{k,a} \Phi_{l,m,j}^{(a)}(x) \\
&= \sum_{m=0}^{\infty}\sum_{j=0}^{d(m)}\sum_{l=0}^{\infty} \int_0^{\infty} f_{m,j}(r)\,\psi_{l,m}^{(a)}(r)\, r^{2\langle k\rangle+N+a-3}dr \\
&\qquad \cdot e^{-z\left(2l+\lambda_{k,a,m}+1\right)}\psi_{l,m}^{(a)}(r)\,Y_{m,j}(x') \\
&= \sum_{m=0}^{\infty}\sum_{j=0}^{d(m)}\sum_{l=0}^{\infty} \int_0^{\infty} f_{m,j}(r)\, r^{-m}\widetilde{\varphi}_l^{a,\lambda_{k,a,m}}(r)\, r^{a\lambda_{k,a,m}+a-1}dr \\
&\qquad \cdot e^{-z\left(2l+\lambda_{k,a,m}+1\right)}\widetilde{\varphi}_l^{a,\lambda_{k,a,m}}(r)\, r^m Y_{m,j}(x') \\
&= \sum_{m,j} Y_{m,j}(x') r^m I_{a,\lambda_{k,a,m};z}\left((\cdot)^{-m} f_{m,j}\right)(r).
\end{aligned}$$

For $f(x) = Y_{m,j}(x')\,\psi(r)$, $\psi(r) \in L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$, $x = rx'$, we have the following Hecke-Bochner identity for the (k,a)-generalized Laguerre semigroups,

$$\mathcal{I}_{k,a}(z)f(x) = Y_{m,j}(x') r^m I_{a,\lambda_{k,a,m};z}\left((\cdot)^{-m}\psi\right)(r).$$

Taking $m = 0$, we get the special case for radial functions. The proof of Theorem 1.2 is therefore complete. ∎

Define the a-deformed Dunkl-Hermite heat semigroup with infinitesimal generator $\Delta_{k,a}$ as $T_t^{k,a}f := \mathcal{I}_{k,a}(ta)f$, $t > 0$ and the a-deformed Laguerre heat semigroup as $T_{a,\alpha;t}f := I_{a,\alpha;ta}f$, $t > 0$. Then from Theorem 4.1,

$$T_t^{k,a}f(x) = \sum_{m,j} Y_{m,j}(x') r^m T_{a,\lambda_{k,a,m},t}\left((\cdot)^{-m} f_{m,j}\right)(r). \tag{19}$$

It reduces to the equation in Theorem 4.5 in [6] when $a = 2$.

Remark 4.1. The case of $a = 2$ of the above argument gives a new proof of Theorem 4.5 in [6]. In [6] the authors proved Theorem 4.5 by using the Dunkl-Hermite expansions and proving the identity for Dunkl-Hermite projections first.

But if we use the basis given in terms of Laguerre polynomials, which are also the eigenfunctions of Dunkl-Hermite operators, the theorem can be proven directly from the above. For radial functions it was shown in [22] in classical case that Hermite expansions reduce to Laguerre expansions. The Heisenberg uncertainty principle for Dunkl transforms was also proved using the two different expansions successively. It was first proved by Rösler using the Dunkl-Hermite expansions (see [18]), and was then proved in [4, Section 5.7] using the tools we refer to in this paper as well (see [20] also for a proof using the basis given by Dunkl [8] in terms of Laguerre polynomials). ■

Now we use the following Lemma (see [6]) to give the expansion of the fractional (k,a)-generalized harmonic ocillator into fractional a-deformed Laguerre operator (there is a constant missed in [6, Lemma 3.4]. Here we give the corrected Lemma).

Lemma 4.2. ([6, Lemma 3.4]) *Let* $0<\sigma<1$*, and* $\lambda\in\mathbb{R}$ *such that* $\lambda+\sigma>-1$*. Then,*

$$2^{\sigma}\left|\Gamma\left(-\sigma\right)\right|\frac{\Gamma\left(\frac{\lambda}{2}+\frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{\lambda}{2}+\frac{1-\sigma}{2}\right)}$$

$$=\int_0^{\infty}\left(\cosh\ t-1\right)\left(\sinh\ t\right)^{-\sigma-1}dt+\int_0^{\infty}\left(1-e^{-t\lambda}\right)\left(\sinh\ t\right)^{-\sigma-1}dt.$$

Define $E_{\sigma}:=\frac{a^{\sigma}}{|\Gamma(-\sigma)|}\int_0^{\infty}\left(\cosh\ t-1\right)\left(\sinh\ t\right)^{-\sigma-1}dt$. Then

$$\begin{aligned}L_{a,\alpha;\sigma}f\left(r\right)&=\frac{2^{\alpha+1}}{a^{\alpha}\Gamma(\alpha+1)}\sum_{l=0}^{\infty}(2a)^{\sigma}S_l^{a,\alpha;\sigma}f*_{a,\alpha}\varphi_l^{a,\alpha}(r)\\&=E_{\sigma}f\left(r\right)+\frac{a^{\sigma}}{\left|\Gamma\left(-\sigma\right)\right|}\int_0^{\infty}\left(f\left(r\right)-T_{a,\alpha;t/a}f\left(r\right)\right)\left(\sinh\ t\right)^{-\sigma-1}dt.\end{aligned}$$

Given $0<\sigma<1$, we define conformally invariant fractional (k,a)-generalized harmonic ocillator $\left(-\triangle_{k,a}\right)_{\sigma}$ to be the operator

$$\left(-\triangle_{k,a}\right)_{\sigma}=(2a)^{\sigma}\frac{\Gamma\left(\frac{-\triangle_{k,a}}{2a}+\frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{-\triangle_{k,a}}{2a}+\frac{1-\sigma}{2}\right)}.$$

So, in view of (17), $\left(-\triangle_{k,a}\right)_{\sigma}$ corresponds to the spectral multiplier

$$(2a)^{\sigma}\Gamma\left(\frac{2l+\lambda_{k,a,m}+1}{2}+\frac{1+\sigma}{2}\right)/\ \Gamma\left(\frac{2l+\lambda_{k,a,m}+1}{2}+\frac{1-\sigma}{2}\right)$$

and is equal to

$$E_{\sigma}f\left(x\right)+\frac{a^{\sigma}}{\left|\Gamma\left(-\sigma\right)\right|}\int_0^{\infty}\left(f\left(x\right)-T_{t/a}^{k,a}f\left(x\right)\right)\left(\sinh\ t\right)^{-\sigma-1}dt$$

from Lemma 4.2. For $a=2$, it should coincide with the fractional Dunkl-Hermite operator in [6] (there is a constant factor missed in the definition given in [6]).
By formula (19),

$$\left(-\triangle_{k,a}\right)_{\sigma}f\left(x\right)=E_{\sigma}f\left(x\right)+\frac{a^{\sigma}}{\left|\Gamma\left(-\sigma\right)\right|}\int_0^{\infty}\left(f\left(x\right)-T_{t/a}^{k,a}f\left(x\right)\right)\left(\sinh\ t\right)^{-\sigma-1}dt$$

$$= \sum_{m,j} Y_{m,j}(x')r^m \Big[E_\sigma r^{-m} f_{m,j}(r) + \frac{a^\sigma}{|\Gamma(-\sigma)|} \int_0^\infty \left(r^{-m} f_{m,j}(r) - T_{a,\lambda_{k,a,m},t/a} \left((\cdot)^{-m} f_{m,j} \right)(r) \right) (\sinh\ t)^{-\sigma-1}\, dt \Big]$$

$$= \sum_{m,j} Y_{m,j}(x')r^m \Big[E_\sigma g_{m,j}(r) + \frac{a^\sigma}{|\Gamma(-\sigma)|} \int_0^\infty \left(g_{m,j}(r) - T_{a,\lambda_{k,a,m},t/a} g_{m,j}(r) \right) (\sinh\ t)^{-\sigma-1}\, dt \Big]$$

$$= \sum_{m,j} Y_{m,j}(x')r^m L_{a,\lambda_{k,a,m};\sigma} g_{m,j}(r),$$

where $g_{m,j}(r) = r^{-m} f_{m,j}(r)$.

The following Lemma was found by Yafaev [23] for $v = m/2,\ m \in \mathbb{N}$, and was then proved in [10] for any $v > 0$.

Lemma 4.3. ([10, Lemma 2.3]) *If $v > 0$, then*

$$\frac{\Gamma(t+v)}{\Gamma(\tau+v)} < \frac{\Gamma(t)}{\Gamma(\tau)}, \qquad 0 < t < \tau.$$

By Theorem 3.3 we have

$$\left\langle (-\triangle_{k,a})_\sigma f, f \right\rangle_{L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)dx\right)} = \sum_{m=0}^\infty \sum_{j=1}^{d(m)} \left\langle L_{a,\lambda_{k,a,m};\sigma} g_{m,j}, g_{m,j} \right\rangle_{L^2\left((0,\infty), d\mu_{a,\lambda_{k,a,m}}(r)\right)}$$

$$\geq \sum_{m=0}^\infty \sum_{j=1}^{d(m)} \left(\frac{2a}{\delta}\right)^\sigma \left(B^\delta_{\lambda_{k,a,m},\sigma}\right)^2 \int_0^\infty |g_{m,j}(r)|^2 \frac{\omega^{\delta,a}_{\lambda_{k,a,m},\sigma}(r)}{\omega^{\delta,a}_{\lambda_{k,a,m},-\sigma}(r)} d\mu_{a,\lambda_{k,a,m}}(r)$$

$$= \sum_{m=0}^\infty \sum_{j=1}^{d(m)} \left(\frac{2a}{\delta}\right)^\sigma \left(B^\delta_{\lambda_{k,a,m},\sigma}\right)^2 \int_0^\infty |f_{m,j}(r)|^2 \frac{\omega^{\delta,a}_{\lambda_{k,a,m},\sigma}(r)}{\omega^{\delta,a}_{\lambda_{k,a,m},-\sigma}(r)} d\mu_{a,\lambda_a}(r).$$

Then by Lemma 4.3 and a similar argument as in the end of the proof in [6],

$$\left(\frac{2a}{\delta}\right)^\sigma \left(B^\delta_{\lambda_{k,a,m},\sigma}\right)^2 \frac{\omega^{\delta,a}_{\lambda_{k,a,m},\sigma}(r)}{\omega^{\delta,a}_{\lambda_{k,a,m},-\sigma}(r)}$$

$$= \left(\frac{a}{2}\right)^\sigma \delta^\sigma \frac{\Gamma\left(\frac{\lambda_{k,a,m}+2+\sigma}{2}\right)}{\Gamma\left(\frac{\lambda_{k,a,m}+2-\sigma}{2}\right)} \frac{K_{(\lambda_{k,a,m}+1+\sigma)/2}\left((\delta+\frac{2}{a}r^a)/2\right)}{K_{(\lambda_{k,a,m}+1-\sigma)/2}\left((\delta+\frac{2}{a}r^a)/2\right)} \left(\delta+\tfrac{2}{a}r^a\right)^{-\sigma}$$

$$\geq \left(\frac{a}{2}\right)^\sigma \delta^\sigma \frac{\Gamma\left(\frac{\lambda_a+2+\sigma}{2}\right)}{\Gamma\left(\frac{\lambda_a+2-\sigma}{2}\right)} \left(\delta+\tfrac{2}{a}r^a\right)^{-\sigma} = \left(\frac{a}{2}\right)^\sigma B^\delta_{\lambda_a,\sigma} \left(\delta+\tfrac{2}{a}r^a\right)^{-\sigma}.$$

Therefore,

$$\left\langle (-\triangle_{k,a})_\sigma f, f \right\rangle_{L^2\left(\mathbb{R}^N, \vartheta_{k,a}(x)dx\right)} \geq \sum_{m,j} \left(\frac{2a}{\delta}\right)^\sigma \left(B^\delta_{\lambda_{k,a,m},\sigma}\right)^2 \int_0^\infty |f_{m,j}(r)|^2 \frac{\omega^{\delta,a}_{\lambda_{k,a,m},\sigma}(r)}{\omega^{\delta,a}_{\lambda_{k,a,m},-\sigma}(r)} d\mu_{a,\lambda_a}(r)$$

$$\geq \left(\frac{a}{2}\right)^{\sigma} B^{\delta}_{\lambda_a,\sigma} \sum_{m,j} \int_0^\infty |f_{m,j}(r)|^2 \left(\delta + \tfrac{2}{a} r^a\right)^{-\sigma} d\mu_{a,\lambda_a}(r)$$

$$= \left(\frac{a}{2}\right)^{\sigma} B^{\delta}_{\lambda_a,\sigma} \int_{\mathbb{R}^N} \frac{|f(x)|^2}{\left(\delta + \frac{2}{a}|x|^a\right)^{\sigma}} \vartheta_{k,a}(x)\, dx.$$

The proof of Theorem 1.4 is complete. ■

5. Characterization of $\Omega_{k,a}(\gamma_z)$ on each k-spherical component $\mathcal{H}_k^m(\mathbb{R}^N)$

In Section 4.1 of [4] the authors gave the definition of the radial part $\Omega^{(m)}_{k,a}(\gamma_z)$, $\Re z \geq 0$ of the holomorphic semigroup $\Omega_{k,a}(\gamma_z) = \mathcal{I}_{k,a}(z)$ on each k-spherical component $\mathcal{H}_k^m(\mathbb{R}^N)$ via a decomposition of unitary representation $\Omega_{k,a}(\gamma_z)$ of the universal covering group $\widetilde{SL(2,\mathbb{R})}$ on $L^2\left(\mathbb{R}^N \vartheta_{k,a}(x)dx\right)$ (see [4, Section 4.1] for the detailed definition of $\Omega^{(m)}_{k,a}(\gamma_z)$). And they showed that the unitary operator $\Omega^{(m)}_{k,a}(\gamma_z)$ on $L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$ can be expressed as

$$\Omega^{(m)}_{k,a}(\gamma_z) f(r) = \int_0^\infty \Lambda^{(m)}_{k,a}(r,s;z) f(s) s^{2\langle k\rangle+N+a-3} ds, \tag{20}$$

where $\Lambda^{(m)}_{k,a}(r,s;z)$ has its closed formula (see [4, (4.11)])

$$\Lambda^{(m)}_{k,a}(r,s;z) = \frac{(rs)^{-\langle k\rangle - \frac{N}{2}+1}}{\sinh z} e^{-\frac{\coth z}{a}(r^a+s^a)} I_{\lambda_{k,a,m}}\left(\frac{\frac{2}{a} r^{\frac{a}{2}} s^{\frac{a}{2}}}{\sinh z}\right).$$

The integral on the right hand side of (20) converges for $f \in L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$ if $\Re z > 0$ and for all f in the dense subspace of $L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$ spanned by the functions $\left\{\psi^{(a)}_{l,m}(r) : l \in \mathbb{N}\right\}$ if $\Re z = 0$ (see (6) for the definition of $\psi^{(a)}_{l,m}(r)$). We give an explicit expression of $\Omega^{(m)}_{k,a}(\gamma_z)$ in this section via the a-deformed Laguerre operator $L_{a,\alpha}$ (see [2] for the case of $a = 2$ on such expression).

Theorem 5.1. *Assume $\lambda_{k,a,m} \geq -1/2$, $\Re z \geq 0$ and $s > 0$. Then $\Omega^{(m)}_{k,a}(\gamma_z)$ acting on $L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$ has the form*

$$\Omega^{(m)}_{k,a}(\gamma_z) f(s) = s^m I_{a,\lambda_{k,a,m};z}\left((\cdot)^{-m} f\right)(s), \ \ f \in L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right).$$

Thus

$$\left.\frac{d}{dz}\right|_{z=0} \Omega^{(m)}_{k,a}(\gamma_z) f(s) = -s^m \frac{1}{a} L_{a,\lambda_{k,a,m}}\left((\cdot)^{-m} f\right)(s).$$

Proof. We can take α as $\lambda_{k,a,m}$ in Lemma 3.4, then we get

$$\mathcal{T}_r^{a,\lambda_{k,a,m}} q_{a,\lambda_{k,a,m};z}(s) = (rs)^{-m} \Lambda^{(m)}_{k,a}(r,s;z). \tag{21}$$

For every f in the dense subspace of $L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$ spanned by the functions $\left\{\psi_{l,m}^{(a)}(r) : \; l \in \mathbb{N}\right\}$, we have

$$\begin{aligned}\Omega_{k,a}^{(m)}\left(\gamma_z\right) f(s) &= \int_0^\infty f(r)\Lambda_{k,a}^{(m)}\left(r,s;z\right) r^{\,2\langle k\rangle+N+a-3}dr \\ &= s^m \int_0^\infty r^{-m} f(r) \mathcal{T}_r^{a,\lambda_{k,a,m}} q_{a,\lambda_{k,a,m};z}(s) r^{\,2m+2\langle k\rangle+N+a-3}dr \\ &= s^m I_{a,\lambda_{k,a,m};z}\left(\left(\cdot\right)^{-m} f\right)(s)\,.\end{aligned}$$

Then from the boundedness of the operator $\Omega_{k,a}^{(m)}\left(\gamma_z\right)$ on $L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$, we get

$$\Omega_{k,a}^{(m)}\left(\gamma_z\right) f(s) = s^m I_{a,\lambda_{k,a,m};z}\left(\left(\cdot\right)^{-m} f\right)(s)$$

for all $f \in L^2\left(\mathbb{R}_+, r^{\,2\langle k\rangle+N+a-3}dr\right)$. ■

Remark 5.2. (i) From this theorem, the spherical harmonic expansion of the (k,a)-generalized Laguerre semigroup (2) can be derived directly by (4.3) in [4].

(ii) Taking $m = 0$, we get the formula of $\triangle_{k,a}$ on radial Schwartz functions $f = f_0\left(|\cdot|\right),\ f_0 \in \mathcal{S}\left(\mathbb{R}_+\right)$,

$$\triangle_{k,a} f(x) = -L_{a,\lambda_a}\left(f_0\right)(r)\,,\ r = |x|\,.$$

This is equivalent to the formula of Dunkl Laplacian $\triangle_k$ on radial functions in [16, Proposition 4.15].

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