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Minimal Parabolic Subgroups and Automorphism Groups of Schubert Varieties

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Abstract. Let G be a simple simply-laced algebraic group of adjoint type over the field $\mathbb{C}$ of complex numbers, B be a Borel subgroup of G containing a maximal torus T of G. In this article, we show that ω_α is a minuscule fundamental weight if and only if for any parabolic subgroup Q containing B properly, there is no Schubert variety $X_Q(w)$ in G/Q such that the minimal parabolic subgroup P_α of G is the connected component, containing the identity automorphism of the group of all algebraic automorphisms of $X_Q(w)$.

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1. Introduction

We recall that if X is a projective variety over $\mathbb{C}$, the connected component, containing the identity automorphism of the group of all algebraic automorphisms of X is an algebraic group (see [15, Theorem 3.7, p. 17]). On the other hand, M. Brion proved that every connected algebraic group H over $\mathbb{C}$ is the connected component, containing identity automorphism of the group of all algebraic automorphisms of some normal projective variety X (see [4, Theorem 1]). Let G be a simple algebraic group of adjoint type over $\mathbb{C}$. Let T be a maximal torus of G, and let R be the set of roots with respect to T. Let $R^+ \subset R$ be a set of positive roots. Let B^+ be the Borel subgroup of G containing T, corresponding to R^+. Let B be the Borel subgroup of G opposite to B^+ determined by T. Let $W = N_G(T)/T$ denote the Weyl group of G with respect to T. For $w \in W$, let $X(w) := \overline{BwB/B}$ denote the Schubert variety in G/B corresponding to w. In [8], Demazure studied the automorphism group of the homogeneous space G/P, where P is a parabolic subgroup of G. The connected component, containing the identity automorphism of the group of all algebraic automorphisms of G/P is G, provided (G, P) is not one of the following:

1. G is of type B_n and P is the maximal parabolic subgroup corresponding to the simple root α_n.
2. G is of type C_n and P is the maximal parabolic subgroup corresponding to the simple root α_1.
3. G is of type G_2 and P is the maximal parabolic subgroup corresponding to the simple root α_1.

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The Lie algebra of G may be identified with the Lie algebra of global vector fields $H^0(G/P, T_{G/P})$. Let $\mathrm{Aut}^0(X(w))$ denote the connected component, containing the identity automorphism of the group of all algebraic automorphisms of $X(w)$. Let α_0 denote the highest root of G with respect to T and B^+. For the left action of G on G/B, let P_w denote the stabilizer of $X(w)$ in G. In [13, p. 772, Theorem 4.2(2)], the first named author proved that if G is simply-laced and $X(w)$ is smooth, then we have $P_w = \mathrm{Aut}^0(X(w))$ if and only if $w^{-1}(\alpha_0) < 0$. Therefore, it is a natural question to ask whether given any parabolic subgroup P of G containing B properly, is there a Schubert variety $X(w)$ in G/B such that $P = \mathrm{Aut}^0(X(w))$? If $P = B$, there is no such Schubert variety in G/B. In [14], authors gave an affirmative answer to this question. Also, authors gave some partial results for Schubert varieties in partial flag varieties of type A_n. In this article, we study minimal parabolic subgroup P_i for which there exists a Schubert variety $X(w_i)$ in a partial flag variety such that $P_i = \mathrm{Aut}^0(X(w_i))$. We prove the following.

Theorem 1.1. (see Theorem 9.2) *Assume that G is simply-laced. A fundamental weight ω_α is minuscule if and only if for any parabolic subgroup Q containing B properly, there is no Schubert variety $X_Q(w)$ in G/Q such that $P_\alpha = \mathrm{Aut}^0(X_Q(w))$.*

The organization of the paper is as follows. In Section 2, we recall some preliminaries on algebraic groups and Lie algebras, a result from [6] and some results on cohomology of vector bundles on Schubert varieties (see [7, p. 271–272]). Further, we recall a lemma on indecomposable modules from [1] which will be used in computing the cohomology modules. We conclude this section by proving a generalization of the result [13, Theorem 4.2, p. 772]. In Section 3, we prove some results on minuscule fundamental weights and co-minuscule simple roots. In Section 4, we prove some results on non minuscule fundamental weights. In Section 5, we prove that for any non minuscule fundamental weight ω_i in type D, there exists a Schubert variety $X_{P_i}(w_i)$ in G/P_i such that P_i is the connected component, containing the identity automorphism of the group of all algebraic automorphisms of $X_{P_i}(w_i)$ (for precise notation see section 2). In Section 6, we prove that for any non minuscule fundamental weight ω_i in type E_6, there exists a Schubert variety $X_{P_4}(w_i)$ in G/P_4 such that P_i is the connected component, containing the identity automorphism of the group of all algebraic automorphisms of $X_{P_4}(w_i)$ (for precise notation see section 2). In Section 7, we prove that for any non minuscule fundamental weight ω_i in type E_7, there exists a Schubert variety $X_{P_3}(w_i)$ in G/P_3 such that P_i is the connected component, containing the identity automorphism of the group of all algebraic automorphisms of $X_{P_3}(w_i)$ (for precise notation see section 2). In Section 8, we prove that for any fundamental weight ω_i in type E_8, there exists a Schubert variety $X_{P_7}(w_i)$ in G/P_7 such that P_i is the connected component, containing the identity automorphism of the group of all algebraic automorphisms of $X_{P_7}(w_i)$ (for precise notation see Section 2). In Section 9, we prove Theorem 1.1.

2. Notation and preliminaries

In this section, we set up some notation and preliminaries. We refer to [5], [10], [11], [12] for preliminaries in algebraic groups and Lie algebras.

Let G, B, T, R, R^+, and W be as in the introduction. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of simple roots in R^+. Every $\beta \in R$ can be expressed uniquely in the form $\sum_{i=1}^{n} k_i\alpha_i$ with integral coefficients k_i all non-negative sign or non-positive. This allows us to define the *height* of a root (relative to S) by $\mathrm{ht}(\beta) = \sum_{i=1}^{n} k_i$. For $\beta = \sum_{i=1}^{n} k_i\alpha_i \in R$, we define the *support* of β to be the set $\{\alpha_i : k_i \neq 0\}$.

The *simple reflection* in W corresponding to α_i is denoted by s_i. Then (W, S) is a Coxeter group (see [11, Theorem 29.4, p. 180]). There is a natural length function ℓ defined on W. Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of T and $\mathfrak{b} \subset \mathfrak{g}$ be the Lie algebra of B. Let $X(T)$ denote the group of all characters of T. We have $X(T) \otimes \mathbb{R} = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$, the dual of the real form of $\mathfrak{h}$. The positive definite W-invariant form on $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$ induced by the Killing form of $\mathfrak{g}$ is denoted by $(\, , \,)$. We use the notation $\langle \, , \, \rangle$ to denote $\langle \mu, \alpha \rangle = \frac{2(\mu,\alpha)}{(\alpha,\alpha)}$, for every $\mu \in X(T) \otimes \mathbb{R}$ and $\alpha \in R$. For a subset J of S, we denote by W_J the subgroup of W generated by $\{s_\alpha : \alpha \in J\}$. Let $W^J := \{w \in W : w(\alpha) \in R^+ \textit{ for all } \alpha \in J\}$.

We denote by w_0 the longest element of W. Note that for $J \subseteq S$, and $w \in W$, there are unique elements $w_J \in W_J$ and w^J in W^J such that $w = w^J w_J$. We denote by $w_{0,J}$ the longest element of W_J. Note that $w_{0,J} = (w_0)_J$. Further, we denote by w_0^J the minimal representative in W^J of w_0. For $w \in W$, let $R^+(w) := \{\beta \in R^+ : w(\beta) < 0\}$. For a root α, let U_α be the root subgroup of G corresponding α.

For each $w \in W_J$, we choose a representative element $n_w \in N_G(T)$ and define $N_J := \{n_w : w \in W_J\}$. Let $P_J := BN_JB$. For a simple root α_i, we denote by P_i the minimal parabolic subgroup P_{α_i} of G. Let $\{\omega_i : 1 \leq i \leq n\}$ be the set of fundamental dominant weights corresponding to $\{\alpha_i : 1 \leq i \leq n\}$. For $1 \leq i \leq n$, let $h(\alpha_i) \in \mathfrak{h}$ be the fundamental co-weight corresponding to α_i. That is, $\alpha_i(h(\alpha_j)) = \delta_{ij}$, where δ_{ij} is Kronecker delta.

We recall the following definition and facts (see [2, p. 119–120]):

A fundamental weight ω is said to be *minuscule* if ω satisfies $\langle \omega, \beta \rangle \leq 1$ for all $\beta \in R^+$. The following is the complete list of minuscule weights in the simply-laced root systems.

Table 1: Minuscule weight in simply-laced root system

no.	Root System	Minuscule weight
1.	A_n	$\omega_1, \omega_2, ..., \omega_n$
2.	D_n	$\omega_1, \omega_{n-1}, \omega_n$
3.	E_6	ω_1, ω_6
4.	E_7	ω_7
5.	E_8	none

A simple root α is said to be *co-minuscule* if α occurs with coefficient 1 in the expression of the highest root α_0. A fundamental weight ω_α associated to a simple root α is said to be *co-minuscule* if α is *co-minuscule*.

Since G is simply-laced, the list of minuscule weights is also the list of co-minuscule weights (see Lemma 3.1).

We recall the following Proposition from [6, Proposition 7.1, p. 342–343].

Let $\alpha_0 = \sum\limits_{i=1}^{n} c_i\alpha_i$, and $\check{\alpha_0} = \sum\limits_{i=1}^{n} \check{c_i}\check{\alpha_i}$. We have

$$\check{\alpha_0} = \frac{2\alpha_0}{(\alpha_0, \alpha_0)} = \frac{2}{(\alpha_0, \alpha_0)} \sum_{i=1}^{n} c_i \frac{(\alpha_i, \alpha_i)}{2} \check{\alpha_i},$$

hence $\check{c_i} = \frac{(\alpha_i,\alpha_i)}{(\alpha_0,\alpha_0)} c_i$. If G is simply-laced, we have $c_i = \check{c_i}$. The dual Coxeter number of $\mathfrak{g}$ is

$$g = 1 + \sum_{i=1}^{n} \check{c_i}.$$

Proposition 2.1. *Let α be any long root. Then we have*

(1) *There is a unique element u_α in W of minimal length such that $u_\alpha^{-1}(\alpha_0) = \alpha$.*

(2) *If α is in S, then $\ell(u_\alpha) = g - 2$.*

Proof. See [6, Proposition 7.1, p. 342–343]. ∎

Corollary 2.2. *If G is simply-laced and α is in S, then we have $\ell(v_\alpha) = \mathrm{ht}(\alpha_0)$ where $v_\alpha = u_\alpha s_\alpha$.*

Proof. By Proposition 2.1, we have $\ell(u_\alpha) = g - 2$. Further, we note that $\ell(v_\alpha) = \ell(u_\alpha)+1 = g-1 = \sum_{i=1}^{n} \check{c_i}$. If G is simply-laced, we have $(\alpha_0, \alpha_0) = (\alpha_i, \alpha_i)$ for all $1 \leq i \leq n$. Therefore, $\ell(u_\alpha) = \sum_{i=1}^{n} c_i = \mathrm{ht}(\alpha_0)$, as $\check{c_i} = \frac{(\alpha_i,\alpha_i)}{(\alpha_0,\alpha_0)} c_i = c_i$. ∎

Now we discuss some preliminaries on the cohomology of vector bundles on Schubert varieties associated to the rational B-modules.

Let V be a rational B-module. Let $\phi : B \longrightarrow GL(V)$ be the corresponding homomorphism of algebraic groups. The total space of the vector bundle $\mathcal{L}(V)$ on G/B is defined by the set of equivalence classes $\mathcal{L}(V) = G \times_B V$ corresponding to the following equivalence relation on $G \times V$:

$$(g, v) \sim (gb, \phi(b^{-1}) \cdot v) \text{ for } g \in G, b \in B, v \in V.$$

We denote by the restriction of $\mathcal{L}(V)$ to $X(w)$ also by $\mathcal{L}(V)$. We denote the cohomology modules $H^i(X(w), \mathcal{L}(V))$ by $H^i(w, V)$ $(i \in \mathbb{Z}_{\geq 0})$. If $V = \mathbb{C}_\lambda$ is a one dimensional representation $\lambda : B \longrightarrow \mathbb{C}^\times$ of B, then we denote $H^i(w, V)$ by $H^i(w, \lambda)$.

Let L_α denote the Levi subgroup of P_α containing T. Note that L_α is the product of T and the homomorphic image G_α of $SL(2, \mathbb{C})$ via a homomorphism $\psi\colon SL(2, \mathbb{C}) \longrightarrow L_\alpha$ (see [Jan, II, 1.3]). We denote the intersection of L_α and B by B_α. We note that the morphism $L_\alpha/B_\alpha \hookrightarrow P_\alpha/B$ induced by the inclusion $L_\alpha \hookrightarrow P_\alpha$ is an isomorphism. Therefore, to compute the cohomology modules $H^i(P_\alpha/B, \mathcal{L}(V))$ $(0 \leq i \leq 1)$ for any B-module V, we treat V as a B_α-module and we compute $H^i(L_\alpha/B_\alpha, \mathcal{L}(V))$.

For $\lambda \in X(T)$ and $w \in W$, we define the dot action by the rule $w \cdot \lambda = w(\lambda + \rho) - \rho$, where ρ is the half sum of positive roots of G.

The following lemma is due to Demazure (see [7, p. 271-272]). We use this lemma to compute cohomology modules.

Lemma 2.3. *Let $w = \tau s_\alpha$, $\ell(w) = \ell(\tau) + 1$, and λ be a character of B. Then*

(1) *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^j(\tau, H^0(s_\alpha, \lambda))$ for all $j \geq 0$.*

(2) *If $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^{j+1}(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$.*

(3) *If $\langle \lambda, \alpha \rangle \leq -2$, then $H^{j+1}(w, \lambda) = H^j(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$.*

(4) *If $\langle \lambda, \alpha \rangle = -1$, then $H^j(w, \lambda)$ vanishes for every $j \geq 0$.*

Let $\pi\colon \hat{G} \longrightarrow G$ be the simply connected covering of G. Let $\hat{L}_\alpha$ (respectively, $\hat{B}_\alpha$) be the inverse image of L_α (respectively, of B_α) in $\hat{G}$. Note that $\hat{L}_\alpha/\hat{B}_\alpha$ is isomorphic to L_α/B_α. We make use of this isomorphism to use the same notation for the vector bundle on L_α/B_α associated to a $\hat{B}_\alpha$-module. Let V be an irreducible $\hat{L}_\alpha$-module and λ be a character of $\hat{B}_\alpha$.

Then, we have

Lemma 2.4.

(1) *If $\langle \lambda, \alpha \rangle \geq 0$, then the $\hat{L}_\alpha$-module $H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda)$ is isomorphic to the tensor product of V and $H^0(L_\alpha/B_\alpha, \mathbb{C}_\lambda)$.*
Further, we have $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$ for every $j \geq 1$.

(2) *If $\langle \lambda, \alpha \rangle \leq -2$, then we have $H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$. Further, the $\hat{L}_\alpha$-module $H^1(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda)$ is isomorphic to the tensor product of V and $H^0(L_\alpha/B_\alpha, \mathbb{C}_{s_\alpha \cdot \lambda})$.*

(3) *If $\langle \lambda, \alpha \rangle = -1$, then $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) = 0$ for every $j \geq 0$.*

Proof. By [12, I, Proposition 4.8, p. 53] and [12, I, Proposition 5.12, p. 77] for $j \geq 0$, we have the following isomorphism as $\hat{L}_\alpha$-modules:

$$H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}_\lambda) \simeq V \otimes H^j(L_\alpha/B_\alpha, \mathbb{C}_\lambda).$$

Now the proof of the lemma follows from Lemma 2.3 by taking $w = s_\alpha$ and the fact that $L_\alpha/B_\alpha \simeq P_\alpha/B$. ∎

We now state the following lemma on indecomposable $\hat{B}_\alpha$ (respectively, B_α) modules which will be used in computing the cohomology modules (see [1, Corollary 9.1, p. 130]).

Lemma 2.5.

(1) *Any finite dimensional indecomposable $\hat{B}_\alpha$-module V is isomorphic to $V' \otimes \mathbb{C}_\lambda$ for some irreducible representation V' of $\hat{L}_\alpha$, and some character λ of $\hat{B}_\alpha$.*

(2) *Any finite dimensional indecomposable B_α-module V is isomorphic to $V' \otimes \mathbb{C}_\lambda$ for some irreducible representation V' of $\hat{L}_\alpha$, and some character λ of $\hat{B}_\alpha$.*

Proof. Proof of part (1) follows from [1, Corollary 9.1, p. 130].
Proof of part (2) follows from the fact that every B_α-module can be viewed as a $\hat{B}_\alpha$-module via the natural homomorphism. ∎

We conclude this section by proving a generalization of [13, Theorem 4.2, p. 772] which describes the automorphism group of a smooth Schubert variety in the simply laced case. We use this result frequently in our article.

Assume that G is simply-laced. Let $P = P_I$ be the standard parabolic subgroup of G corresponding to a subset $I \subseteq S$. Let $w \in W^I$, and $X_P(w) := \overline{BwP/P}$ be the Schubert variety in G/P corresponding to w. For the left action of G on G/P, let P_w denote the stabilizer of $X_P(w)$ in G.

Theorem 2.6. *Then we have*

(1) *The homomorphism* $\varphi_w\colon P_w \longrightarrow \mathrm{Aut}^0(X_P(w))$ *induced by the action of* P_w *on* $X_P(w)$ *is surjective.*

(2) $\varphi_w : P_w \longrightarrow \mathrm{Aut}^0(X_P(w))$ *is an isomorphism if and only if* $w^{-1}(\alpha_0)$ *is a negative root.*

Proof. (1) Let $T_{X_P(w)}$ (respectively, $T_{G/P}$) be the tangent sheaf of $X_P(w)$ (respectively, of G/P). We denote the restriction of $T_{G/P}$ to $X_P(w)$ also by $T_{G/P}$. Note that $T_{G/P} = \mathcal{L}(\mathfrak{g}/\mathfrak{p})$, and $T_{X_P(w)}$ is a subsheaf of $T_{G/P}$.

Let $\pi : G/B \longrightarrow G/P$ be the natural map. Then the restriction of π to $X(w)$, i.e., $\pi\colon X(w) \longrightarrow X_P(w)$ is a birational surjective morphism. Further, by [5, Theorem 3.3.4(a), p. 96], $\pi^*\colon H^j(X_P(w), \mathcal{L}_P(V)) \longrightarrow H^j(X(w), \pi^*\mathcal{L}_P(V))$ $(j \geq 0)$ is an isomorphism for any P-module V. Thus we denote the cohomology modules $H^j(X_P(w), \mathcal{L}_P(V))(\simeq H^j(X(w), \pi^*\mathcal{L}_P(V)))$ by simply $H^j(w, V)$.

Now we consider the short exact sequence

$$0 \longrightarrow \mathfrak{p}/\mathfrak{b} \longrightarrow \mathfrak{g}/\mathfrak{b} \longrightarrow \mathfrak{g}/\mathfrak{p} \longrightarrow 0$$

of B-modules. Let $R^+(P)$ be the set of positive roots of the Levi factor of P containing T. Observe that $\mathfrak{p}/\mathfrak{b}$ has a filtration of B-submodules such that the successive quotients are of the form $\mathbb{C}_\beta$, where $\beta \in R^+(P)$. Further, by [13, Corollary 3.6, p. 771] we have $H^j(w, \beta) = 0$ $(j \geq 1)$ for any positive root β.

Hence, by using the long exact sequence associated to the above short exact sequence, we see that $H^0(w, \mathfrak{g}/\mathfrak{b}) \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p})$ is surjective.

On the other hand, by using [13, Lemma 3.5, p. 770], the restriction map

$$H^0(w_0, \mathfrak{g}/\mathfrak{b}) \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{b})$$

is surjective. Hence, the restriction map $H^0(w_0^I, \mathfrak{g}/\mathfrak{b}) \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{b})$ is surjective.

Further, we have the following commutative diagram:

$$\begin{array}{ccc} H^0(w_0^I, \mathfrak{g}/\mathfrak{b}) & \twoheadrightarrow & H^0(w_0^I, \mathfrak{g}/\mathfrak{p}) \\ \downarrow\!\!\downarrow & & \downarrow r \\ H^0(w, \mathfrak{g}/\mathfrak{b}) & \twoheadrightarrow & H^0(w, \mathfrak{g}/\mathfrak{p}) \end{array}$$

Hence, the natural restriction map $r : H^0(w_0^I, \mathfrak{g}/\mathfrak{p}) \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p})$ is surjective.

Since P_w is a parabolic subgroup of G containing B, the Lie algebra $\mathfrak{p}_w$ is a Lie subalgebra of $\mathfrak{g} = H^0(w_0^I, \mathfrak{g}/\mathfrak{p})$ containing $\mathfrak{b}$. Further, since $X_P(w)$ is a P_w-stable subvariety for the left action of P_w on G/P, we have the following commutative diagram of B-modules:

$$\begin{array}{ccc} \mathfrak{p}_w & \xrightarrow{d\varphi_w} & H^0(X_P(w), T_{X_P(w)}) \\ \downarrow & & \downarrow \\ H^0(w_0^I, \mathfrak{g}/\mathfrak{p}) & \xrightarrow{r} & H^0(w, \mathfrak{g}/\mathfrak{p}) \end{array}$$

Now, let $\mathfrak{q} = r^{-1}(H^0(X_P(w), T_{X_P(w)}))$. Note that since $\mathfrak{q}$ is a B-submodule of $\mathfrak{g}$ containing $\mathfrak{p}_w$, $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{p}_w$. We denote the restriction of r to $\mathfrak{q}$ also by r. We now show that $\mathfrak{p}_w = \mathfrak{q}$. Since $\mathfrak{g} = H^0(w_0^I, \mathfrak{g}/\mathfrak{p})$, every element $x \in \mathfrak{q} \subseteq \mathfrak{g}$ is a tangent vector field on G/P. Further, by the definition of $\mathfrak{q}$; the ideal sheaf of $X_P(w)$ is x stable for every $x \in \mathfrak{q}$. Therefore, $\mathfrak{q}$ is contained in the Lie algebra $\mathfrak{p}_w$ of the stabiliser P_w of $X_P(w)$ in G. Thus, we have $\mathfrak{p}_w = \mathfrak{q}$.

Clearly, $r\colon \mathfrak{p}_w = \mathfrak{q} \longrightarrow H^0(X_P(w), T_{X_P(w)})$ is a homomorphism of Lie algebras. Hence $d\varphi_w\colon \mathfrak{p}_w \longrightarrow H^0(X_P(w), T_{X_P(w)})$ is surjective. Let $A = \varphi_w(P_w) \subseteq \mathrm{Aut}^0(X_P(w))$. Consequently we have $d\varphi_w(\mathfrak{p}_w) = \mathrm{Lie}(A) \subseteq H^0(X_P(w), T_{X_P(w)})$. Since the mapping $d\varphi_w\colon \mathfrak{p}_w \longrightarrow H^0(X_P(w), T_{X_P(w)})$ is surjective, we have $\mathrm{Lie}(A) = H^0(X_P(w), T_{X_P(w)})$. Therefore $A = \mathrm{Aut}^0(X_P(w))$ and hence, $\varphi_w\colon P_w \longrightarrow \mathrm{Aut}^0(X_P(w))$ is surjective.

(2) Assume that $w^{-1}(\alpha_0)$ is a negative root. Then we claim that $H^0(w, \mathfrak{p}) = 0$.

Note that $\mathfrak{p}$ has a filtration of B-submodules such that the successive quotients are of the form $\mathbb{C}_0$ or $\mathbb{C}_\beta$ for some $\beta \in R^+(P) \cup R^-$. So, every weight of $H^0(w, \mathfrak{p})$ is a weight of $H^0(w, 0)$ or $H^0(w, \beta)$ for some $\beta \in R^+(P) \cup R^-$. Now, if $-\alpha_0$ is a weight of $H^0(w, \mathfrak{p})$, then $-\alpha_0$ is a weight of $H^0(w, \beta)$ for some $\beta \in R^+(P) \cup R^-$. Further, every weight μ of $H^0(w, \beta)$ satisfies $\mu \geq -\alpha_0$. Thus, we have $w(\beta) = -\alpha_0$. Since $w^{-1}(\alpha_0) < 0$, it follows that $\beta \notin R^-$. So, we have $\beta \in R^+(P)$.

Now we show that $\beta \in R^+ \setminus R^+(P)$. Since $w \in W^I$, we have $R^+(w) \cap R^+(w_{0,I}) = \emptyset$. Now, since $\beta \in R^+(w)$, we have $\beta \notin R^+(w_{0,I})$, i.e., $\beta \notin R^+(P)$, which is a contradiction to the fact that $\beta \in R^+(P)$. Hence, $-\alpha_0$ is not a weight of $H^0(w, \mathfrak{p})$.

Now since $H^0(w, \mathfrak{p})$ is a B-submodule of $H^0(w, \mathfrak{g}) = \mathfrak{g}$ such that $H^0(w, \mathfrak{p})_{-\alpha_0}$ is zero, and $\mathfrak{g}_{-\alpha_0}$ is the unique B-stable line in $\mathfrak{g}$, it follows that $H^0(w, \mathfrak{p}) = 0$. Therefore, the map $H^0(w, \mathfrak{g}) \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p})$ induced by the map $\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{p}$ is injective. Further, by [13, Lemma 3.4, p. 770], the map $H^0(w, \mathfrak{g}) \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p})$ is surjective. Thus, the map $H^0(w, \mathfrak{g}) \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p})$ is an isomorphism. Hence, $H^0(w, \mathfrak{g}/\mathfrak{p})$ is isomorphic to $\mathfrak{g}$ as B-module. Therefore, $H^0(w, \mathfrak{g}/\mathfrak{p})$ has a unique B-stable line, namely $\mathfrak{g}_{-\alpha_0}$. Note that since $w^{-1}(\alpha_0) < 0$, we have $w \neq id$.

Therefore, the action of P_w on $X_P(w)$ is non trivial. Hence, the homomorphism $d\varphi_w : \mathfrak{p}_w \longrightarrow H^0(X_P(w), T_{X_P(w)})$ of B-modules is non-zero. Therefore, the B-stable line $H^0(w, \mathfrak{g}/\mathfrak{p})_{-\alpha_0}$ is in the image $d\varphi_w(\mathfrak{p}_w) \subset H^0(X_P(w), T_{X_P(w)}) \subset H^0(w, \mathfrak{g}/\mathfrak{p})$. Hence, we have $\mathfrak{g}_{-\alpha_0} \cap \ker(d\varphi_w) = 0$. Thus, $d\varphi_w : \mathfrak{p}_w \longrightarrow H^0(X_P(w), T_{X_P(w)})$ is injective. Since the base field is $\mathbb{C}$, it follows that φ_w is separable. Hence, the kernel of $d\varphi_w$ is the Lie algebra of the kernel of φ_w. Therefore, $\varphi_w : P_w \longrightarrow \mathrm{Aut}^0(X_P(w))$ is injective. Now, φ_w is an isomorphism follows from (1).

Conversely, if $\varphi_w : P_w \longrightarrow \mathrm{Aut}^0(X_P(w))$ is an isomorphism, then, the induced homomorphism $d\varphi_w : \mathfrak{p}_w \longrightarrow H^0(X_P(w), T_{X_P(w)}) \subseteq H^0(w, \mathfrak{g}/\mathfrak{p})$ is injective. In particular, the $-\alpha_0$-weight space $H^0(w, \mathfrak{g}/\mathfrak{p})_{-\alpha_0}$ is non-zero.

We now show that $w^{-1}(\alpha_0) < 0$. For this, we first note that the B-module $\mathfrak{g}/\mathfrak{p}$ has a composition series of B-modules with each successive simple quotient is isomorphic to $\mathbb{C}_\alpha$, where $\alpha \in R^+ \setminus R^+(P)$. Applying [13, Corollary 3.6, p. 771] we see that the B-module $H^0(w, \mathfrak{g}/\mathfrak{p})$ has a filtration of B-submodules with each successive quotient is isomorphic to $H^0(w, \alpha)$ for some $\alpha \in R^+ \setminus R^+(P)$. Now, since $H^0(w, \mathfrak{g}/\mathfrak{p})_{-\alpha_0} \neq 0$, $H^0(w, \alpha)_{-\alpha_0} \neq 0$ for some $\alpha \in R^+ \setminus R^+(P)$. On the other hand, there is a $v \in W$ such that $v(\alpha_0) = \alpha$. Without loss of generality, we may assume that v is of minimum length among such elements. It follows from the Demazure character formula that if $H^0(wv,\alpha_0)_\mu \neq 0$, then μ is in the convex hull of the set $\{x(\alpha_0) : x \leq wv\}$ (see [5, Theorem 3.3.8, p. 97, equation (3)] and [12, Proposition 14.18(b), p. 379]). Note that the convention for the signature of the weights in the Demazure character formula in this paper is the same as the one in [12]. Further, using the above arguments, we see that $\mathbb{C}_\alpha$ is a B-submodule of $H^0(v,\alpha_0)$. Therefore, $H^0(w,\alpha)$ is a B-submodule of $H^0(w,H^0(v,\alpha_0)) = H^0(wv,\alpha_0)$. Hence, every weight μ of $H^0(w,\alpha)$ satisfies $\mu \geq w(\alpha)$. Clearly, $w(\alpha) \geq -\alpha_0$. Therefore, since $H^0(w,\alpha)_{-\alpha_0} \neq 0$, we have $w(\alpha) = -\alpha_0$. Thus, $w^{-1}(\alpha_0) < 0$. ■

3. Preliminaries on minuscule fundamental weights

Now onwards we assume that G is simply-laced. Note that $-w_0$ is an automorphism of the root system R. Let σ be the Dynkin diagram automorphism induced by $-w_0$ i.e., $\alpha_i \mapsto \alpha_{\sigma(i)} = -w_0(\alpha_i)$ for $1 \leq i \leq n$.

Lemma 3.1. *Let $1 \leq r \leq n$. Then α_r is co-minuscule root if and only if ω_r is minuscule.*

Proof. Let $\alpha_0 = \sum_{i=1}^{n} c_i\alpha_i$. Suppose that the coefficient of α_r in α_0 is 1. Then we have $\langle \omega_r, \alpha_0 \rangle = 1$, as G is simply laced. Since G is simply laced, we have $\langle \omega_r, \alpha_0 \rangle \geq \langle \omega_r, \beta \rangle$ for all $\beta \in R^+$. Hence ω_r is minuscule.

Conversely, we assume that ω_r is minuscule. Since G is simply laced, we have $\langle \omega_r, \alpha_0 \rangle = c_r$. Thus we have $c_r = 1$, as ω_r is minuscule. ■

Lemma 3.2. *ω_r is minuscule if and only if $w_{0,S\setminus\{\alpha_r\}}(\alpha_r) = \alpha_0$.*

Proof. Assume that ω_r is minuscule. Note that $-\alpha_r$ is $L_{S\setminus\{\alpha_r\}}$ dominant. Therefore, $w_{0,S\setminus\{\alpha_r\}}(-\alpha_r)$ is $L_{S\setminus\{\alpha_r\}}$ negative dominant. Further, the coefficient of α_r in $w_{0,S\setminus\{\alpha_r\}}(-\alpha_r)$ is -1. On the other hand, if $\langle w_{0,S\setminus\{\alpha_r\}}(-\alpha_r), \alpha_r \rangle \geq 1$ the coefficient of α_r in $s_r(w_{0,S\setminus\{\alpha_r\}}(-\alpha_r))$ is ≤ -2. Since $-\alpha_0 \leq s_r w_{0,S\setminus\{\alpha_r\}}(-\alpha_r)$, the coefficient of α_r in $-\alpha_0$ is ≤ -2. This is a contradiction to Lemma 3.1. Hence $w_{0,S\setminus\{\alpha_r\}}(-\alpha_r)$ is negative dominant. Thus we have $w_{0,S\setminus\{\alpha_r\}}(\alpha_r) = \alpha_0$.

Conversely, suppose that $w_{0,S\setminus\{\alpha_r\}}(\alpha_r) = \alpha_0$.

Let $\alpha_0 = \sum_{i=1}^{n} c_i\alpha_i$. Since G is simply-laced, we have $\langle \omega_r, \alpha_0 \rangle = c_r$. Further, since $w_{0,S\setminus\{\alpha_r\}}(\alpha_r) = \alpha_0$, we have $\langle \omega_r, \alpha_0 \rangle = \langle \omega_r, w_{0,S\setminus\{\alpha_r\}}(\alpha_r) \rangle$. As $\langle \cdot, \cdot \rangle$ is W-invariant, we have $\langle \omega_r, w_{0,S\setminus\{\alpha_r\}}(\alpha_r) \rangle = \langle w_{0,S\setminus\{\alpha_r\}}(\omega_r), \alpha_r \rangle$. Since $w_{0,S\setminus\{\alpha_r\}}(\omega_r) = \omega_r$, we have $\langle w_{0,S\setminus\{\alpha_r\}}(\omega_r), \alpha_r) \rangle = \langle \omega_r, \alpha_r \rangle = 1$. Thus we have $c_r = 1$. Therefore, by Lemma 3.1, ω_r is minuscule. ■

Lemma 3.3. *Assume that ω_r is minuscule. Let $v \in W^{S\setminus\{\alpha_r\}}$. Then we have $v = w_0^{S\setminus\{\alpha_r\}}$ if and only if $v(\alpha_0) < 0$.*

Proof. Assume that $v = w_0^{S\setminus\{\alpha_r\}}$.

Since $w_0 = w_0^{S\setminus\{\alpha_r\}} w_{0,S\setminus\{\alpha_r\}}$, we have $w_0^{S\setminus\{\alpha_r\}} = w_0 w_{0,S\setminus\{\alpha_r\}}$.

Therefore, by Lemma 3.2 we have $w_0^{S\setminus\{\alpha_r\}}(\alpha_0) = w_0(\alpha_r) = -\alpha_{\sigma(r)}$.

Conversely, suppose that $v(\alpha_0) < 0$.

By Lemma 3.2, we have $s_{\sigma(r)} w_0{}^{S\setminus\{\alpha_r\}}(\alpha_0) = \alpha_{\sigma(r)}$.

Therefore, we have $v \not\leq s_{\sigma(r)} w_0^{S\setminus\{\alpha_r\}}$. Otherwise $v(\alpha_0) \geq s_{\sigma(r)} w_0^{S\setminus\{\alpha_r\}}(\alpha_0) = \alpha_{\sigma(r)}$. Therefore, $v(\alpha_0)$ is a positive root, a contradiction. Since $v, s_{\sigma(r)} w_0^{S\setminus\{\alpha_r\}} \in W^{S\setminus\{\alpha_r\}}$, we have $s_{\sigma(r)} {w_0}^{S\setminus\{\alpha_r\}} < v$. Therefore, we have $v = w_0^{S\setminus\{\alpha_r\}}$. ■

4. Preliminaries on non minuscule fundamental weights

In this section, we prove a crucial lemma for a non minuscule weight of type D or E associated to the fundamental weight α_0. We recall the Dynkin diagram of D_n, E_6, E_7, E_8 (see[10, Theorem 11.4, p. 57–58]):

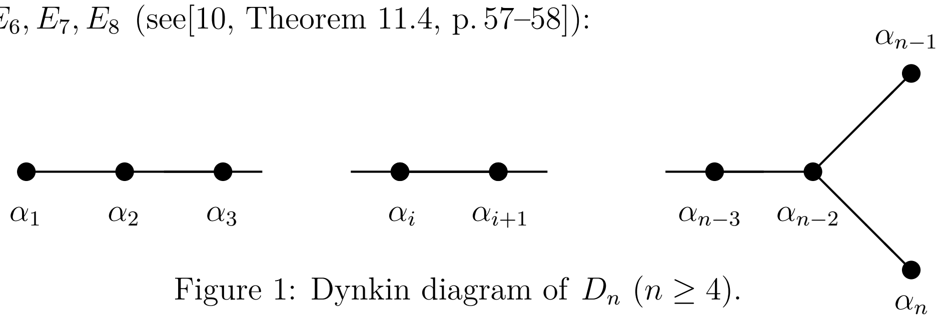


Figure 1: Dynkin diagram of D_n $(n \geq 4)$.

$$\alpha_0 = \alpha_1 + 2(\alpha_2 + \alpha_3 + \cdots + \alpha_{n-3} + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n = \omega_2.$$

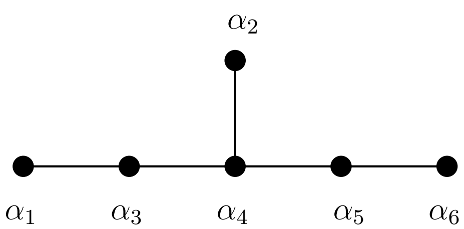


Figure 2: Dynkin diagram of E_6.

$$\alpha_0 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \omega_2.$$

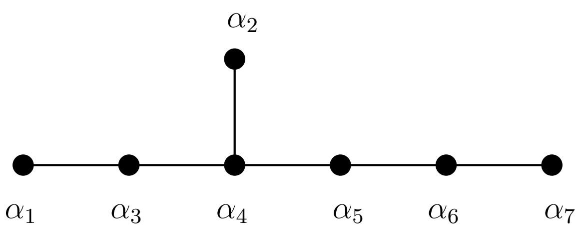


Figure 3: Dynkin diagram of E_7.

$$\alpha_0 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = \omega_1.$$

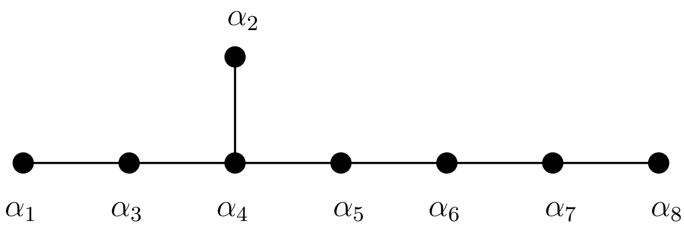


Figure 4: Dynkin diagram of E_8.

$$\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 = \omega_8.$$

In type D_n, E_6, E_7, or E_8, we note that α_0 is a non minuscule weight. We now prove

Lemma 4.1. *Assume that G is of type D or E. Then, we have $w_{0,S\setminus\{\alpha_r\}}(\alpha_r) = \alpha_0 - \alpha_r$, where α_r is the simple root such that $\alpha_0 = \omega_r$.*

Proof. Note that $-\alpha_r$ is $L_{S\setminus\{\alpha_r\}}$ dominant. Therefore, $w_{0,S\setminus\{\alpha_r\}}(-\alpha_r)$ is $L_{S\setminus\{\alpha_r\}}$ negative dominant. Next we claim that $\langle w_{0,S\setminus\{\alpha_r\}}(-\alpha_r), \alpha_r\rangle \geq 1$. On the contrary, if $\langle w_{0,S\setminus\{\alpha_r\}}(-\alpha_r), \alpha_r\rangle \leq 0$, then $w_{0,S\setminus\{\alpha_r\}}(-\alpha_r)$ is negative dominant for S. Since there is exactly one negative dominant root $-\alpha_0$ in the root system, we have $w_{0,S\setminus\{\alpha_r\}}(-\alpha_r) = -\alpha_0$. Therefore, by Lemma 3.2, $\alpha_0 = \omega_r$ is a minuscule, a contradiction. Since G is simply-laced, we have $\langle w_{0,S\setminus\{\alpha_r\}}(-\alpha_r), \alpha_r\rangle = 1$. Thus $\langle s_r w_{0,S\setminus\{\alpha_r\}}(-\alpha_r), \alpha_r\rangle = -1$. Next we prove the Lemma by studying case by case.

Case I: G is of type D_n.

Then we have $\alpha_0 = \omega_2$. Since $\langle \alpha_i, \alpha_2\rangle = 0$ for $i \neq 1, 2, 3$, and $w_{0,S\setminus\{\alpha_2\}}(-\alpha_2)$ is $L_{S\setminus\{\alpha_2\}}$ negative dominant, we have $\langle s_2 w_{0,S\setminus\{\alpha_2\}}(-\alpha_2), \alpha_i\rangle \leq 0$ for $i \neq 1, 3$. Further, $\langle s_2 w_{0,S\setminus\{\alpha_2\}}(-\alpha_2), \alpha_i\rangle = \langle w_{0,S\setminus\{\alpha_2\}}(-\alpha_2), \alpha_2+\alpha_i\rangle$ for $i = 1, 3$. By the above discussion we have $\langle w_{0,S\setminus\{\alpha_2\}}(-\alpha_2), \alpha_2\rangle = 1$. Moreover, since $w_{0,S\setminus\{\alpha_2\}}(\alpha_i) = -\alpha_i$ for $i = 1, 3$, we have $\langle s_2 w_{0,S\setminus\{\alpha_2\}}(-\alpha_2), \alpha_i\rangle = 0$ for $i = 1, 3$. Thus $s_2 w_{0,S\setminus\{\alpha_2\}}(-\alpha_2)$ is negative dominant for S. Therefore, we have $s_2 w_{0,S\setminus\{\alpha_2\}}(-\alpha_2) = -\alpha_0$. So, we have $w_{0,S\setminus\{\alpha_2\}}(\alpha_2) = \alpha_0 - \alpha_2$.

Case II: G is of type E_6.

Then we have $\alpha_0 = \omega_2$. Since $\langle \alpha_i, \alpha_2\rangle = 0$ for $i \neq 2, 4$, and $w_{0,S\setminus\{\alpha_2\}}(-\alpha_2)$ is $L_{S\setminus\{\alpha_2\}}$ negative dominant, we have $\langle s_2 w_{0,S\setminus\{\alpha_2\}}(-\alpha_2), \alpha_i\rangle \leq 0$ for $i \neq 2, 4$. We note that $\langle s_2 w_{0,S\setminus\{\alpha_2\}}(-\alpha_2), \alpha_4\rangle = \langle w_{0,S\setminus\{\alpha_2\}}(-\alpha_2), \alpha_2 + \alpha_4\rangle$. By the above discussion we have $\langle w_{0,S\setminus\{\alpha_2\}}(-\alpha_2), \alpha_2\rangle = 1$.

Since $w_{0,S\setminus\{\alpha_2\}}(\alpha_4) = -\alpha_4$, $\langle s_2 w_{0,S\setminus\{\alpha_2\}}(-\alpha_2), \alpha_4\rangle = 0$. Thus $s_2 w_{0,S\setminus\{\alpha_2\}}(-\alpha_2)$ is negative dominant for S. Therefore, $s_2 w_{0,S\setminus\{\alpha_2\}}(-\alpha_2) = -\alpha_0$. So, we have $w_{0,S\setminus\{\alpha_2\}}(\alpha_2) = \alpha_0 - \alpha_2$.

Case III: G is of type E_7.

Then we have $\alpha_0 = \omega_1$. Since $\langle \alpha_i, \alpha_1\rangle = 0$ for $i \neq 1, 3$, and $w_{0,S\setminus\{\alpha_1\}}(-\alpha_1)$ is $L_{S\setminus\{\alpha_1\}}$ negative dominant, we have $\langle s_{\alpha_1} w_{0,S\setminus\{\alpha_1\}}(-\alpha_1), \alpha_i\rangle \leq 0$ for $i \neq 1, 3$. By the above discussion we have $\langle w_{0,S\setminus\{\alpha_1\}}(-\alpha_1), \alpha_1\rangle = 1$.

Therefore, $\langle s_1 w_{0,S\setminus\{\alpha_1\}}(-\alpha_1), \alpha_1\rangle = -1$. On the other hand,

$$\langle s_1 w_{0,S\setminus\{\alpha_1\}}(-\alpha_1), \alpha_3\rangle = \langle w_{0,S\setminus\{\alpha_1\}}(-\alpha_1), \alpha_1 + \alpha_3\rangle.$$

Since $w_{0,S\setminus\{\alpha_1\}}(\alpha_3) = -\alpha_3$, we have $\langle s_1 w_{0,S\setminus\{\alpha_1\}}(-\alpha_1), \alpha_3\rangle = 0$. Thus $s_1 w_{0,S\setminus\{\alpha_1\}}(-\alpha_1)$ is negative dominant for S. Therefore, $s_1 w_{0,S\setminus\{\alpha_1\}}(-\alpha_1) = -\alpha_0$. So, we have $w_{0,S\setminus\{\alpha_1\}}(\alpha_1) = \alpha_0 - \alpha_1$.

Case IV: G is of type E_8.

Then we have $\alpha_0 = \omega_8$. Since $\langle \alpha_i, \alpha_8\rangle = 0$ for $i \neq 7, 8$, and $w_{0,S\setminus\{\alpha_8\}}(-\alpha_8)$ is $L_{S\setminus\{\alpha_8\}}$ negative dominant, we have $\langle s_8 w_{0,S\setminus\{\alpha_8\}}(-\alpha_8), \alpha_i\rangle \leq 0$ for $i \neq 7, 8$.

Further, $\langle s_8 w_{0,S\setminus\{\alpha_8\}}(-\alpha_8), \alpha_7\rangle = \langle w_{0,S\setminus\{\alpha_8\}}(-\alpha_8), \alpha_8+\alpha_7\rangle$. By the above discussion we have $\langle w_{0,S\setminus\{\alpha_8\}}(-\alpha_8), \alpha_8\rangle = 1$. Moreover, since $w_{0,S\setminus\{\alpha_8\}}(\alpha_7) = -\alpha_7$, we have $\langle s_8 w_{0,S\setminus\{\alpha_8\}}(-\alpha_8), \alpha_7\rangle = 0$. Thus $s_8 w_{0,S\setminus\{\alpha_8\}}(-\alpha_8)$ is negative dominant for S. Therefore, $s_8 w_{0,S\setminus\{\alpha_8\}}(-\alpha_8) = -\alpha_0$. So, we have $w_{0,S\setminus\{\alpha_8\}}(\alpha_8) = \alpha_0 - \alpha_8$. ■

5. G is of type D_n ($n \geq 4$)

In this section, we prove the following proposition:

Proposition 5.1. *Assume that ω_i is non minuscule. Then there exists a Schubert variety $X_{P_i}(w_i)$ in G/P_i such that $P_i = \mathrm{Aut}^0(X_{P_i}(w_i))$.*

We recall that there exists a unique element of minimal length v_i in W such that $v_i^{-1}(\alpha_0) = -\alpha_i$ for all $1 \leq i \leq n$ (see Corollary 2.2).

For all $1 \leq i \leq n-2$, let $u_i = s_i s_{i+1} \cdots s_{n-2} s_{n-1} s_n s_{n-2} s_{n-3} \cdots s_{i+1} s_i$. Note that u_i is self inverse i.e., $u_i^{-1} = u_i$.
Let $w_2 = u_2 s_1$, $w_3 = u_3 v_2$, and finally $w_i = u_i(s_{i-1}u_i)\cdots(s_3\cdots s_{i-1}u_i)v_{i-1}$ for all $4 \leq i \leq n-2$.

Lemma 5.2. *For $1 \leq i \leq n-2$, let u_i be as above. Then we have the following:*

(1) $u_i(\alpha_j) = \alpha_j$ *for all* $1 \leq j \leq i-2$.

(2) (i) $u_i(\alpha_{i-1}) = \alpha_{i-1} + 2\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ *for all* $2 \leq i \leq n-2$.

(ii) $u_i(\alpha_i) = -(\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n)$ *for all* $1 \leq i \leq n-2$.

In particular, we have $u_i(\alpha_{i-1} + \alpha_i) = \alpha_{i-1} + \alpha_i$ *for all* $2 \leq i \leq n-2$.

(3) (i) $u_i(\alpha_j) = \alpha_j$ *for all* $i+1 \leq j \leq n-2$.

(ii) $u_i(\alpha_{n-1}) = \alpha_n$, *and* $u_i(\alpha_n) = \alpha_{n-1}$ *for all* $1 \leq i \leq n-2$.

In particular, we have $u_i(\alpha_{n-1} + \alpha_n) = \alpha_{n-1} + \alpha_n$ *for all* $1 \leq i \leq n-2$.

Proof. (1) Since $1 \leq i \leq n-2$, and $1 \leq j \leq i-2$, we have $\langle \alpha_j, \alpha_k \rangle = 0$ for all $i \leq k \leq n$. Therefore, $u_i(\alpha_j) = s_i s_{i+1} \cdots s_{n-2} s_{n-1} s_n s_{n-2} s_{n-3} \cdots s_i(\alpha_j) = \alpha_j$.

(2) Follows from the usual calculation using the description of u_i.

(3) We note that $u_i = s_i \cdots s_j u_{j+1} s_j s_{j-1} \cdots s_i$ for all $i+1 \leq j \leq n-3$. Since $1 \leq i \leq n-2$, and $i+1 \leq j \leq n-2$, we have $s_j s_{j-1} \cdots s_i(\alpha_j) = \alpha_{j-1}$. By (1) we have $u_{j+1}(\alpha_{j-1}) = \alpha_{j-1}$. Further, since $s_j s_{j-1} \cdots s_i(\alpha_j) = \alpha_{j-1}$, we have $s_i \cdots s_j(\alpha_{j-1}) = \alpha_j$. Therefore, the proof of (3)(i) follows.

(3)(ii) follows from the usual calculation. ■

Lemma 5.3. *Let v_i be as above for $1 \leq i \leq n-3$. Then we have the following:*

(1) $v_i = (s_2 s_3 \cdots s_{n-2} s_{n-1} s_n s_{n-2} s_{n-3} \cdots s_{i+1})(s_1 \cdots s_i)$ *for all* $1 \leq i \leq n-3$.

(2) $v_i(\alpha_{i+1}) = \alpha_1 + \alpha_2 + \cdots + \alpha_i + \alpha_{i+1}$ *for all* $1 \leq i \leq n-3$.

(3) $w_{i+1}(\alpha_{i+1}) = \alpha_1 + \alpha_2 + \cdots + \alpha_i + \alpha_{i+1}$ *for all* $1 \leq i \leq n-3$.

In particular, $w_{i+1}(\alpha_{i+1})$ *is a non simple positive root for* $1 \leq i \leq n-3$.

Proof. (3) Assume $v_i' = (s_2 s_3 \cdots s_{n-2} s_{n-1} s_n s_{n-2} s_{n-3} \cdots s_{i+1})(s_1 \cdots s_i)$ for all $1 \leq i \leq n-3$. Then we note that $(s_2 s_3 \cdots s_{n-2} s_{n-1} s_n s_{n-2} s_{n-3} \cdots s_{i+1})(s_1 \cdots s_i)$ is a reduced expression v_i'. Further, we observe that $v_i'^{-1}(\alpha_0) = -\alpha_2$. Since $\ell(v'_2) = \ell(v_2)$, by Proposition 2.1 we have $v_2 = v_2'$.

(2) Follows from the usual calculation using the description of v_i as in (1).

(3) Note that $w_{i+1} = u_{i+1}(s_i u_{i+1}) \cdots (s_3 \cdots s_i u_{i+1}) v_i$. By (2) we have $v_i(\alpha_{i+1}) = \alpha_1 + \alpha_2 + \cdots + \alpha_i + \alpha_{i+1}$. By using Lemma 5.2 (1), (2) we have

$$u_{i+1}(\alpha_1 + \alpha_2 + \cdots + \alpha_i + \alpha_{i+1}) = \alpha_1 + \alpha_2 + \cdots + \alpha_i + \alpha_{i+1}.$$

Since $\alpha_1+\alpha_2+\cdots+\alpha_i+\alpha_{i+1}$ is orthogonal to α_k for all $3\le k\le i$, we have $s_3\cdots s_i(\alpha_1+\alpha_2+\cdots+\alpha_i+\alpha_{i+1})=\alpha_1+\alpha_2+\cdots+\alpha_i+\alpha_{i+1}$. Similarly, we have $s_l\cdots s_i(\alpha_1+\alpha_2+\cdots+\alpha_i+\alpha_{i+1})=\alpha_1+\alpha_2+\cdots+\alpha_i+\alpha_{i+1}$ for all $3\le l\le i$. Therefore, we have

$$w_i(\alpha_1+\alpha_2+\cdots+\alpha_i+\alpha_{i+1})=\alpha_1+\alpha_2+\cdots+\alpha_i+\alpha_{i+1}. \qquad \blacksquare$$

Recall that $v_i=(s_2\cdots s_{n-2}s_{n-1}s_ns_{n-2}\cdots s_{i+1})s_1\cdots s_i$ for all $1\le i\le n-3$.

Lemma 5.4. *Then v_i satisfies the following conditions:*

(1) $v_i^{-1}(\alpha_1)=\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n$ *for all* $1\le i\le n-3$.

(2) (i) $v_1^{-1}(\alpha_2)=-(\alpha_1+2\alpha_2+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n)$.
(ii) $v_i^{-1}(\alpha_2)=-(\alpha_1+\alpha_2+\cdots+2\alpha_i+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n)$ *for all* $2\le i\le n-3$.

(3) $v_i^{-1}(\alpha_j)=\alpha_{j-2}$ *for all* $1\le i\le n-3$, *and* $3\le j\le i$.

(4) $v_i^{-1}(\alpha_{i+1})=\alpha_{i-1}+\alpha_i+\alpha_{i+1}$ *for all* $2\le i\le n-3$.

(5) (i) $v_i^{-1}(\alpha_j)=\alpha_j$ *for all* $1\le i\le n-3$, *and* $i+2\le j\le n-2$.
(ii) $v_i^{-1}(\alpha_{n-1})=\alpha_n$, *and* $v_i^{-1}(\alpha_n)=\alpha_{n-1}$ *for all* $1\le i\le n-3$.
In particular, we have $v_i^{-1}(\alpha_{n-1}+\alpha_n)=\alpha_{n-1}+\alpha_n$ *for all* $1\le i\le n-3$.

Proof. (1) Follows from the usual calculation.

(2) Follows from the usual calculation.

(3) Since $3\le j\le i$, we have $s_{i+1}\cdots s_{n-2}s_ns_{n-1}s_{n-2}\cdots s_2(\alpha_j)=\alpha_{j-1}$. Further, since $3\le j\le i$, we have $s_i\cdots s_1(\alpha_{j-1})=\alpha_{j-2}$. So, we have $v_i^{-1}(\alpha_j)=\alpha_{j-2}$ for all $3\le j\le i$.

(4) For $2\le i\le n-3$, we have $s_{i+1}\cdots s_{n-2}s_ns_{n-1}s_{n-2}\cdots s_2(\alpha_{i+1})=\alpha_i+\alpha_{i+1}$. Further, we have $s_i\cdots s_1(\alpha_i+\alpha_{i+1})=\alpha_{j-2}$. So, we have $v_i^{-1}(\alpha_j)=\alpha_{i-1}+\alpha_i+\alpha_{i+1}$ for all $2\le i\le n-3$.

(5) For $i+2\le j\le n-2$, we have

$$s_{i+1}\cdots s_{n-2}s_ns_{n-1}s_{n-2}\cdots s_2(\alpha_j)=\alpha_j.$$

Further, we have $s_i\cdots s_1(\alpha_j)=\alpha_j$ for all $i+2\le j\le n-2$. So, we have $v_i^{-1}(\alpha_j)=\alpha_j$ for all $2+i\le j\le n-2$. We note that $s_{i+1}\cdots s_{n-2}s_ns_{n-1}s_{n-2}\cdots s_2(\alpha_{n-1})=\alpha_n$. Further, we have $s_i\cdots s_1(\alpha_n)=\alpha_n$. Thus we have $v_i^{-1}(\alpha_{n-1})=\alpha_n$.

Similarly, we note that $s_{i+1}\cdots s_{n-2}s_ns_{n-1}s_{n-2}\cdots s_2(\alpha_n)=\alpha_{n-1}$. Furthermore, we have $s_i\cdots s_1(\alpha_{n-1})=\alpha_{n-1}$. So, $v_i^{-1}(\alpha_n)=\alpha_{n-1}$. ■

Lemma 5.5. *For $2\le i\le n-2$, let w_i be as above. Then we have $w_i^{-1}(\alpha_j)$ is a positive root for $j\ne i$, and $w_i^{-1}(\alpha_i)$ is a negative root.*

Proof. Case I: $i=2$.

Note that $w_2=u_2s_1$. By Lemma 5.2(2), we have

$$u_2^{-1}(\alpha_1)=\alpha_1+2\alpha_2+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n.$$

Therefore, we have $w_2^{-1}(\alpha_1)=\alpha_1+2\alpha_2+\cdots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n$.

By Lemma 5.2(3), for $3\le j\le n$, $u_2^{-1}(\alpha_j)$ is a positive root whose support does not contain α_1. Hence, $w_2^{-1}(\alpha_j)$ is a positive root for all $3\le j\le n$.

On the other hand, by Lemma 5.2(2), $u_2^{-1}(\alpha_2)$ is negative of a non simple root. Therefore, $w_2^{-1}(\alpha_2)$ is a negative root.

Case II: $i = 3$.

Note that $w_3 = u_3 v_2$. By Lemma 5.2(2), we have

$$u_3{}^{-1}(\alpha_2) = \alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n.$$

By using Lemma 5.4(2),(3),(4) and (5), we have

$$v_2{}^{-1}(\alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n) = \alpha_1.$$

Therefore, we have $w_3^{-1}(\alpha_2) = \alpha_1$.

By Lemma 5.2, for $j = 1$ or $4 \leq j \leq n$, $u_3^{-1}(\alpha_j)$ is a positive root whose support does not contain α_2. Therefore, by Lemma 5.2, $w_3^{-1}(\alpha_j) = v_2^{-1}u_3^{-1}(\alpha_j)$ is a positive root for all $4 \leq j \leq n$.

On the other hand, by Lemma 5.2, $u_3^{-1}(\alpha_3)$ is a non simple negative root whose support does not contain α_2. Therefore, by Lemma 5.4, $w_3^{-1}(\alpha_3) = v_2^{-1}u_3^{-1}(\alpha_3)$ is a negative root.

Case III: $4 \leq i \leq n-2$.

Note that for $4 \leq i \leq n-2$, we have $w_i = u_i(s_{i-1}u_i)\cdots(s_3\cdots s_{i-1}u_i)v_{i-1}$.

Since $i \geq 4$, by using Lemma 5.2(1), we have

$$(s_3\cdots s_{i-1}u_i)^{-1}\cdots(s_{i-1}u_i)^{-1}u_i{}^{-1}(\alpha_1) = \alpha_1.$$

On the other hand, by Lemma 5.4(1), we have

$$v_{i-1}{}^{-1}(\alpha_1) = \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n.$$

Thus we have $w_i^{-1}(\alpha_1) = \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ for all $4 \leq i \leq n-2$.

Since $i \geq 4$, by using Lemma 5.2(1), we have

$$\begin{aligned}&(s_3\cdots s_{i-1}u_i)^{-1}\cdots(s_{i-1}u_i)^{-1}u_i{}^{-1}(\alpha_2)\\ &\qquad = \alpha_2 + \alpha_3 + \cdots + \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n.\end{aligned}$$

On the other hand, by Lemma 5.4, we have

$$v_{i-1}{}^{-1}(\alpha_2 + \alpha_3 + \cdots + \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n) = \alpha_{i-2}.$$

Thus we have $w_i^{-1}(\alpha_2) = \alpha_{i-2}$ for all $4 \leq i \leq n-2$.

For $3 \leq j \leq i-2$, we have $u_i^{-1}(\alpha_j) = \alpha_j$, $(s_k\cdots s_{i-1}u_i)^{-1}(\alpha_j) = \alpha_j$ for all $j+2 \leq k \leq i-1$. On the other hand, we note that $(s_{j+1}\cdots s_{i-1}u_i)^{-1}(\alpha_j) = u_i^{-1}(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{i-1})$. Therefore, by Lemma 5.2(1),(2), we have

$$\begin{aligned}&u_i^{-1}(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{i-1})\\ &= \alpha_j + \cdots + \alpha_{i-2} + \alpha_{i-1} + 2\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n.\end{aligned}$$

Now,

$$\begin{aligned}&(s_js_{j+1}\cdots s_{i-1}u_i)^{-1}(\alpha_j + \cdots + \alpha_{i-2} + \alpha_{i-1} + 2\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n)\\ &\qquad = u_i^{-1}(\alpha_{i-1} + 2\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n) = \alpha_{i-1}\end{aligned}$$

(see Lemma 5.2(2)). Hence $(s_{j-1}s_js_{j+1}\cdots s_{i-1}u_i)^{-1}(\alpha_{i-1}) = u_i^{-1}(\alpha_{i-2}) = \alpha_{i-2}$ (see Lemma 5.2(1)).

Similarly, by recursion we have

$$(s_3 \cdots s_j s_{j+1} \cdots s_{i-1} u_i)^{-1} \cdots (s_{i-1} u_i)^{-1} u_i^{-1}(\alpha_j) = \alpha_{i-j+2}.$$

By using Lemma 5.4(3), we have $v_{i-1}^{-1}(\alpha_{i-j+2}) = \alpha_{i-j}$. Therefore, we have $w_i^{-1}(\alpha_j) = \alpha_{i-j}$ for all $i \geq 4$, and $3 \leq j \leq i-2$. For $i+1 \leq j \leq n$, by Lemma 5.2(3), $(u_i(s_{i-1}u_i) \cdots (s_3 \cdots s_{i-1}u_i))^{-1}(\alpha_j)$, is a positive root whose support does not contain α_2. Thus by Lemma 5.4, $v_{i-1}^{-1}(u_i(s_{i-1}u_i) \cdots (s_3 \cdots s_{i-1}u_i))^{-1}(\alpha_j)$ is a positive root. Therefore, $w_i^{-1}(\alpha_j)$ is a positive root for all $i+1 \leq j \leq n$.

Now, we consider $i \geq 4$, and $j = i-1$. Then by Lemma 5.2(2), we have $u_i^{-1}(\alpha_{i-1}) = \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$. Again, by Lemma 5.2, we have $(s_{i-1}u_i)^{-1}(\alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n) = \alpha_{i-1}$. Now, $(s_{i-2}s_{i-1}u_i)^{-1}(\alpha_{i-1}) = u_i^{-1}(\alpha_{i-2}) = \alpha_{i-2}$. Similarly, by recursion we have $(s_3 \cdots s_j s_{j+1} \cdots s_{i-1}u_i)^{-1}(\alpha_j) = \alpha_3$. By using Lemma 5.4(3) we have $v_{i-1}^{-1}(\alpha_3) = \alpha_1$. Therefore, we have $w_i^{-1}(\alpha_{i-1}) = \alpha_1$.

By Lemma 5.2(2), we have $u_i^{-1}(\alpha_i) = -(\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n)$. Now let $\beta_i = \alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$. Therefore, by Lemma 5.2(2), and Lemma 5.2(3), we have $(s_{i-1}u_i)^{-1}(-\beta_i) = u_i^{-1}(-\beta_i - \alpha_{i-1}) = -(\alpha_{i-1} + \beta_i)$. Then by using Lemma 5.2(1),(2), and (3), we have

$$(s_{i-2}s_{i-1}u_i)^{-1}(-(\alpha_{i-1} + \beta_i)) = -u_i^{-1}(\alpha_{i-2} + \alpha_{i-1} + \beta_i) = -(\alpha_{i-2} + \alpha_{i-1} + \beta_i).$$

Thus by recursion we have

$$(s_4 \cdots s_{i-1}u_i)^{-1} \cdots (s_{i-1}u_i)^{-1}u_i^{-1}(\alpha_i) = -(\alpha_4 + \cdots + \alpha_{i-1} + \beta_i).$$

Therefore,

$$s_3 \cdots s_{i-1}u_i)^{-1}(-(\alpha_4 + \cdots + \alpha_{i-1} + \beta_i))$$
$$= -u_i^{-1}(\alpha_3 + \alpha_4 + \cdots + \alpha_{i-1} + \beta_i) = -(\alpha_3 + \alpha_4 + \cdots + \alpha_{i-1} + \beta_i).$$

Since the support of $\alpha_3 + \alpha_4 + \cdots + \alpha_{i-1} + \beta_i$ does not contain α_2, by Lemma 5.4, $v_{i-1}^{-1}(-(\alpha_3 + \alpha_4 + \cdots + \alpha_{i-1} + \beta_i)$ is a negative root. Thus $w_i^{-1}(\alpha_i)$ is a negative root. Therefore, combining Case I, Case II, and Case III proof of the lemma follows. ■

Proof of Proposition 5.1. We note that ω_i is not minuscule if and only if $2 \leq i \leq n-2$. Fix an integer $2 \leq i \leq n-2$. Then by Lemma 5.3(3), we conclude that $w_i(\alpha_i)$ is a non simple positive root. On the other hand, by Lemma 5.5, $w_i^{-1}(\alpha_i)$ is a negative root, and $w_i^{-1}(\alpha_j)$ is a positive root for $j \neq i$. Thus P_i is the stabilizer of $X_{P_i}(w_i)$ in G/P_i for the natural action of G. Since $s_2 \nleq u_i(s_{i-1}u_i) \cdots (s_3 \cdots s_{i-1}u_i)$, and $\alpha_0 = \omega_2$, we further have $(s_3 \cdots s_{i-1}u_i)^{-1} \cdots (s_{i-1}u_i)^{-1}u_i^{-1}(\alpha_0) = \alpha_0$. Therefore, $w_i^{-1}(\alpha_0) = v_{i-1}^{-1}(\alpha_0)$ is a negative root. Therefore, by using Theorem 2.6 the natural homomorphism $P_i \longrightarrow \mathrm{Aut}^0(X_{P_i}(w_i))$ is an isomorphism of algebraic groups. ■

6. G is of type E_6

In this section, we prove our result for E_6. Note that by Lemma 3.1, the non-minuscule weights are ω_i where $i = 2, 3, 4, 5$. In this section, we prove the following:

Proposition 6.1. *Assume that ω_i is non-minuscule. Then there exists a Schubert variety $X_{P_4}(w_i)$ in G/P_4 such that $P_i = \mathrm{Aut}^0(X_{P_4}(w_i))$.*

Recall that there exists a unique element $v_2 \in W$ of smallest length such that $v_2^{-1}(\alpha_0) = -\alpha_2$ (see Corollary 2.2).

Lemma 6.2. *Let $v_2 \in W$ be as above. Then we have the following:*

(1) $v_2 = s_2s_4s_5s_3s_6s_4s_1s_3s_5s_4s_2$. *In particular, we have* $v_2^{-1} = v_2$.

(2) $v_2(\alpha_1) = \alpha_5,\ v_2(\alpha_3) = \alpha_6,\ v_2(\alpha_4) = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$, *and* $v_2(\omega_2 - \alpha_2) = \omega_2 - \alpha_2$.

(3) (i) $w_{0,S\setminus\{\alpha_2\}}v_2(\alpha_4) = \omega_2 - (\alpha_2+\alpha_3+2\alpha_4+\alpha_5) = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6$.

(ii) $w_{0,S\setminus\{\alpha_2,\alpha_i\}}w_{0,S\setminus\{\alpha_2\}}v_2(\alpha_4)$ *is a non simple positive root for* $i = 3, 4, 5$.

Proof. (1) Let $v'_2 = s_2s_4s_5s_3s_6s_4s_1s_3s_5s_4s_2$. Note that ${v'_2}^{-1}(\alpha_0) = -\alpha_2$. Since $\ell(v'_2) = \ell(v_2)$, by Corollary 2.2, we have $v_2 = v'_2$.

(2) Follows from the usual calculation.

(3)(i) By (2), we have $v_2(\alpha_4) = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$. We observe that the Dynkin subdiagram of E_6 corresponding to $S \setminus \{\alpha_2\}$ is of type A_5. Since $w_{0,S\setminus\{\alpha_2\}}$ is the longest element of $W_{S\setminus\{\alpha_2\}}$, we have $w_{0,S\setminus\{\alpha_2\}}(\alpha_1) = -\alpha_6$, $w_{0,S\setminus\{\alpha_2\}}(\alpha_3) = -\alpha_5$, and $w_{0,S\setminus\{\alpha_2\}}(\alpha_4) = -\alpha_4$. Thus by using Lemma 4.1, we have $w_{0,S\setminus\{\alpha_2\}}v_2(\alpha_4) = \omega_2 - (\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)$.

(3)(ii) By (3)(i), we have $w_{0,S\setminus\{\alpha_2\}}v_2(\alpha_4) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$. Since the support of $w_{0,S\setminus\{\alpha_2\}}v_2(\alpha_4)$ contains α_i, $1 \le i \le 6$, the proof of (3)(ii) follows. ■

Lemma 6.3. *We have the following:*

(1) (i) $w_{0,S\setminus\{\alpha_2,\alpha_3\}}(\alpha_2) = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$.

(ii) $w_{0,S\setminus\{\alpha_2\}}w_{S\setminus\{\alpha_2,\alpha_3\}}(\alpha_2) = \omega_2 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$.

(iii) $v_2w_{0,S\setminus\{\alpha_2\}}w_{S\setminus\{\alpha_2,\alpha_3\}}(\alpha_2) = \omega_2-(2\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6) = \alpha_1+\alpha_3+\alpha_4$.

(2) (i) $w_{S\setminus\{\alpha_2,\alpha_4\}}(\alpha_2) = \alpha_2$.

(ii) $w_{0,S\setminus\{\alpha_2\}}w_{S\setminus\{\alpha_2,\alpha_4\}}(\alpha_2) = \omega_2 - \alpha_2$.

(iii) $v_2w_{0,S\setminus\{\alpha_2\}}w_{S\setminus\{\alpha_2,\alpha_4\}}(\alpha_2) = \omega_2 - \alpha_2$.

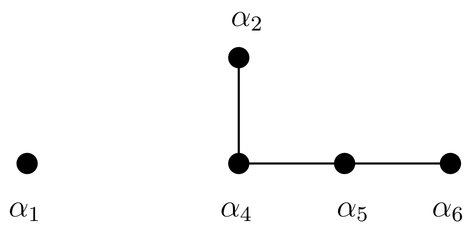
(3) (i) $w_{S\setminus\{\alpha_2,\alpha_5\}}(\alpha_2) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$.

(ii) $w_{0,S\setminus\{\alpha_2\}}w_{S\setminus\{\alpha_2,\alpha_5\}}(\alpha_2) = \omega_2 - (\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6)$.

(iii) $v_2w_{0,S\setminus\{\alpha_2\}}w_{S\setminus\{\alpha_2,\alpha_5\}}(\alpha_2) = \omega_2-(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5) = \alpha_4+\alpha_5+\alpha_6$.

(4) $w_{S\setminus\{\alpha_2,\alpha_i\}}(\alpha_i) = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ *for* $i = 3, 4, 5$.

Proof. (1)(i) Note that the Dynkin subdiagram of E_6, corresponding to the subset $S \setminus \{\alpha_3\}$ of S (see Figure 2):

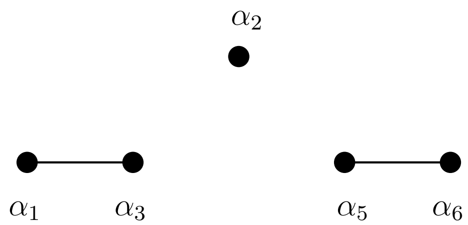


Let $I = S \setminus \{\alpha_3\}$. Now we observe that $w_{0,S\setminus\{\alpha_2,\alpha_3\}}(\alpha_2) = w_{0,I\setminus\{\alpha_2\}}(\alpha_2)$. We note that the connected component of the Dynkin subdiagram associated to I, containing α_2 is of type A_4. Since α_2 is minuscule in type A_4, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_2\}}(\alpha_2) = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$.

(ii) Note that the Dynkin subdiagram of E_6 corresponding to $S\setminus\{\alpha_2\}$ is of type A_5. Since $w_{0,S\setminus\{\alpha_2\}}$ is the longest element of $W_{S\setminus\{\alpha_2\}}$, we have $w_{0,S\setminus\{\alpha_2\}}(\alpha_1)=-\alpha_6$, $w_{0,S\setminus\{\alpha_2\}}(\alpha_3)=-\alpha_5$, and $w_{0,S\setminus\{\alpha_2\}}(\alpha_4)=-\alpha_4$. By using the above discussion together with Lemma 4.1 proof of (ii) follows.

(iii) By (ii), we have $w_{0,S\setminus\{\alpha_2\}}w_{0,S\setminus\{\alpha_2,\alpha_3\}}(\alpha_2)=\omega_2-(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$. Hence, by Lemma 6.2(1),(2), we get $v_2w_{0,S\setminus\{\alpha_2\}}w_{0,S\setminus\{\alpha_2,\alpha_3\}}=\omega_2-(2\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6)$. Further, simplifying we have $v_2w_{S\setminus\{\alpha_2\}}w_{0,S\setminus\{\alpha_2,\alpha_3\}}=\alpha_1+\alpha_3+\alpha_4$.

(2)(i) Next we consider the Dynkin subdiagram of E_6, corresponding to the subset $S\setminus\{\alpha_4\}$ of S:

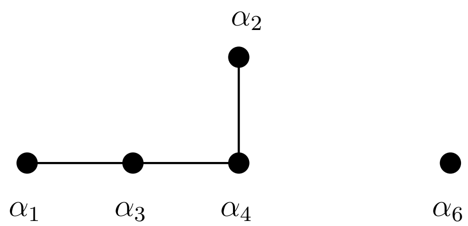


Let $I=S\setminus\{\alpha_4\}$. Then we observe that $w_{0,S\setminus\{\alpha_2,\alpha_4\}}(\alpha_2)=w_{0,I\setminus\{\alpha_2\}}(\alpha_2)$. Further, we note that the connected component of the Dynkin subdiagram associated to I, containing α_2 is of type A_1. Since α_2 is minuscule in type A_1, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_2\}}(\alpha_2)=\alpha_2$.

(ii) Proof is similar to the proof of (1)(ii).

(iii) By using Lemma 6.2(2) and (ii), proof of (iii) follows.

(3)(i) Next we consider the Dynkin subdiagram of E_6, corresponding to the subset $S\setminus\{\alpha_5\}$ of S :



Let $I=S\setminus\{\alpha_5\}$. Then we observe that $w_{0,S\setminus\{\alpha_2,\alpha_5\}}(\alpha_2)=w_{0,I\setminus\{\alpha_2\}}(\alpha_2)$. Further, we note that the connected component of the Dynkin subdiagram associated to I, containing α_2 is of type A_4. Since α_2 is minuscule in type A_4, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_2\}}(\alpha_2)=\alpha_1+\alpha_2+\alpha_3+\alpha_4$.

(ii) Proof is similar to the proof of (1)(ii).

(iii) By using Lemma 6.2(2) and (ii), proof of (iii) follows.

(4) For a fixed i in $\{3,4,5\}$, let $I=S\setminus\{\alpha_2\}$. Then we have $w_{0,S\setminus\{\alpha_2,\alpha_i\}}(\alpha_i)=w_{0,I\setminus\{\alpha_i\}}(\alpha_i)$. Then we observe that the Dynkin subdiagram associated to I is of type A_5. Since α_i is minuscule in type A_5, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_i\}}(\alpha_i)=\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6$. ∎

Lemma 6.4. *Let $w_i=w_{0,S\setminus\{\alpha_2,\alpha_i\}}w_{0,S\setminus\{\alpha_2\}}v_2$ for $i\neq 1,6$. Then we have $w_i^{-1}(\alpha_j)$ is a positive root for $j\neq i$, and $w_i^{-1}(\alpha_i)$ is negative root.*

Proof. Note that for $i=2$ we have $w_2=v_2$. Then by Lemma 6.2(1),(2), we are done. For $i\neq 1,2,6$, let $w_{0,S\setminus\{\alpha_2,\alpha_i\}}(\alpha_i)=\beta$. Then β is a positive root whose support does not contain α_2. Since $w_{0,S\setminus\{\alpha_2\}}$ is the longest element of $W_{S\setminus\{\alpha_2\}}$, $w_{0,S\setminus\{\alpha_2\}}(\beta)$ is a negative root whose support does not contain α_2.

Further, since the support of $w_{0,S\setminus\{\alpha_2\}}(\beta)$ does not contain α_2, by Lemma 6.2(2), we have $v_2 w_{0,S\setminus\{\alpha_2\}}(\beta)$ is a negative root. Hence, we have $w_i^{-1}(\alpha_i)$ is a negative root for $i \neq 1, 2, 6$.

On the other hand, for $i \neq 1, 2, 6$ and $j \neq 2, i$, $w_{0,S\setminus\{\alpha_2,\alpha_i\}}(\alpha_j) = -\alpha_k$, where α_k is a simple root different from α_2. Therefore, $w_{0,S\setminus\{\alpha_2\}}(-\alpha_k) = \alpha_l$ for some $l \neq 2$. Further, since α_l is different from α_2, by Lemma 6.2(2), $v_2(\alpha_l)$ is a positive root. Hence, $w_i^{-1}(\alpha_j)$ is a positive root when $i \neq 1, 2, 6$ and $j \neq 2, i$. Moreover, by Lemma 6.3, we have $w_i^{-1}(\alpha_2)$ is a positive root for $i \neq 2, 1, 6$. Hence, we have $w_i^{-1}(\alpha_j)$ is a positive root for $i \neq 1, 6$ and $j \neq i$. ∎

Proof of Proposition 6.1. We note that ω_i is not minuscule if and only if $i \neq 1, 6$. Assume that ω_i is not minuscule. Recall that $w_i = w_{0,S\setminus\{\alpha_2,\alpha_i\}} w_{0,S\setminus\{\alpha_2\}} v_2$. By Lemma 6.2(3), we conclude that $w_i(\alpha_4)$ is a non simple positive root. On the other hand, by Lemma 6.4, $w_i^{-1}(\alpha_i)$ is a negative root and $w_i^{-1}(\alpha_j)$ is a positive root for $j \neq i$. Thus P_i is the stabilizer of $X_{P_4}(w_i)$ in G. Since $\alpha_0 = \omega_2$, and $v_2^{-1}(\alpha_0) = -\alpha_2$, $w_i^{-1}(\alpha_0) = v_2^{-1}(\alpha_0)$ (as $w_{0,S\setminus\{\alpha_2\}} w_{0,S\setminus\{\alpha_2,\alpha_i\}}(\alpha_0) = \alpha_0$) is a negative root. Therefore, by using Theorem 2.6 the natural homomorphism $P_i \longrightarrow \mathrm{Aut}^0(X_{P_4}(w_i))$ is an isomorphism of algebraic groups. ∎

7. G is of type E_7

We note that by Lemma 3.1, ω_i is not minuscule of E_7 if and only if $i \neq 7$. In this section, our goal is to prove the following proposition:

Proposition 7.1. *Assume that ω_i is not minuscule. Then there exists a Schubert variety $X_{P_3}(w_i)$ in G/P_3 such that $P_i = \mathrm{Aut}^0(X_{P_3}(w_i))$.*

We recall that by Corollary 2.2, there exists a unique element v_4 in W of minimal length such that $v_4^{-1}(\alpha_0) = -\alpha_4$. We note that $w_{0,S\setminus\{\alpha_1\}}$ is the longest element of the Dynkin subdiagram of E_7 corresponding to the subset $S \setminus \{\alpha_1\}$ of S. Thus we have $w_{0,S\setminus\{\alpha_1\}}(\alpha_i) = -\alpha_i$ for $i \neq 1$.

Lemma 7.2. *Let v_4 be as above. Then we have the following:*

(1) $v_4 = s_1s_3s_4s_5s_2s_4s_3s_6s_5s_4s_1s_2s_3s_7s_6s_5s_4$.

(2) (i) $v_4(\alpha_3) = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$.

(ii) $w_{0,S\setminus\{\alpha_1\}} v_4(\alpha_3) = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$.

(iii) $w_{0,S\setminus\{\alpha_1,\alpha_i\}} w_{0,S\setminus\{\alpha_1\}} v_4(\alpha_3)$ *is a non simple positive root for* $i \neq 7$.

(3) $v_4^{-1}(\alpha_1) = -(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)$, $v_4^{-1}(\alpha_2) = \alpha_7$, $v_4^{-1}(\alpha_3) = \alpha_2 + \alpha_4 + \alpha_5$, $v_4^{-1}(\alpha_4) = \alpha_6$, $v_4^{-1}(\alpha_5) = \alpha_3 + \alpha_4 + \alpha_5$, $v_4^{-1}(\alpha_6) = \alpha_1$, $v_4^{-1}(\alpha_7) = \alpha_2 + \alpha_3 + \alpha_4$.

Proof. (1) Let $v'_4 = s_1s_3s_4s_5s_2s_4s_3s_6s_5s_4s_1s_2s_3s_7s_6s_5s_4$. Note that ${v'_4}^{-1}(\alpha_0) = -\alpha_4$. Since $\ell(v'_4) = \ell(v_4)$, by Corollary 2.2, we have $v_4 = v'_4$.

(2) (i) Follows from the usual calculation. (ii) Follows from (2)(i), and using Lemma 4.1. Since the support of $w_{0,S\setminus\{\alpha_1\}} v_4(\alpha_3)$ contains α_i $i \neq 7$, the proof of (2)(iii) follows.

(3) Follows from the usual calculation by using the description of v_4 as in (1). ∎

Lemma 7.3. *We have the following:*

(1) (i) $w_{0,S\setminus\{\alpha_1,\alpha_2\}}(\alpha_1) = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$.
(ii) $w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_2\}}(\alpha_1) = \omega_1 - (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)$.
(iii) ${v_4}^{-1}w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_2\}}(\alpha_1) = \alpha_5 + \alpha_6 + \alpha_7$.

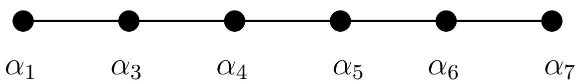
(2) (i) $w_{0,S\setminus\{\alpha_1,\alpha_3\}}(\alpha_1) = \alpha_1$.
(ii) $w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_3\}}(\alpha_1) = \omega_1 - \alpha_1$.
(iii) ${v_4}^{-1}w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_3\}}(\alpha_1) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$.

(3) (i) $w_{0,S\setminus\{\alpha_1,\alpha_4\}}(\alpha_1) = \alpha_1 + \alpha_3$.
(ii) $w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_4\}}(\alpha_1) = \omega_1 - (\alpha_1 + \alpha_3)$.
(iii) ${v_4}^{-1}w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_4\}}(\alpha_1) = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$.

(4) (i) $w_{0,S\setminus\{\alpha_1,\alpha_5\}}(\alpha_1) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$.
(ii) $w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_5\}}(\alpha_1) = \omega_1 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$.
(iii) ${v_4}^{-1}w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_5\}}(\alpha_1) = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$.

(5) (i) $w_{0,S\setminus\{\alpha_1,\alpha_6\}}(\alpha_1) = \alpha_1 + 2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5$.
(ii) $w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_6\}}(\alpha_1) = \omega_1 - (\alpha_1 + 2\alpha_3 + 2\alpha_4 + \alpha_2 + \alpha_5)$.
(iii) ${v_4}^{-1}w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_6\}}(\alpha_1) = \alpha_1 + \alpha_3$.

Proof. (1)(i) We consider the Dynkin subdiagram of E_7, corresponding to the subset $S \setminus \{\alpha_2\}$ of S (see Figure 3):



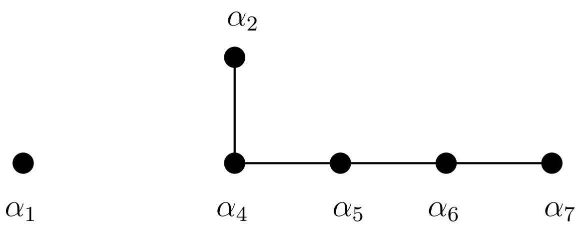
Let $I = S \setminus \{\alpha_2\}$. Now we observe that $w_{0,S\setminus\{\alpha_1,\alpha_2\}}(\alpha_1) = w_{0,I\setminus\{\alpha_1\}}(\alpha_1)$. We note that the connected component of the Dynkin subdiagram associated to I, containing α_1 is of type A_6. Since α_1 is minuscule in type A_6, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_1\}}(\alpha_1) = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$.

(ii) Note that the Dynkin subdiagram of E_7, corresponding to $S \setminus \{\alpha_1\}$ is of type D_6. Since $w_{0,S\setminus\{\alpha_1\}}$ is the longest element of $W_{S\setminus\{\alpha_1\}}$, we have $w_{0,S\setminus\{\alpha_1\}}(\alpha_i) = -\alpha_i$ for all $i \neq 1$. Therefore, from the above discussion together with Lemma 4.1 proof of (1)(ii) follows.

(iii) By (1)(ii) we have $w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_2\}}(\alpha_1) = \omega_1 - (\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7)$. Therefore, by using Lemma 7.2(3), we get

$$\begin{aligned}{v_4}^{-1}w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_2\}}(\alpha_1) &= -\alpha_4 + (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7) \\ &- (\alpha_2 + \alpha_4 + \alpha_5) - \alpha_6 - (\alpha_3 + \alpha_4 + \alpha_5) - \alpha_1 - (\alpha_2 + \alpha_3 + \alpha_4) = \alpha_5 + \alpha_6 + \alpha_7.\end{aligned}$$

(2)(i) Next we consider the Dynkin subdiagram of E_7, corresponding to the subset $S \setminus \{\alpha_3\}$ of S :



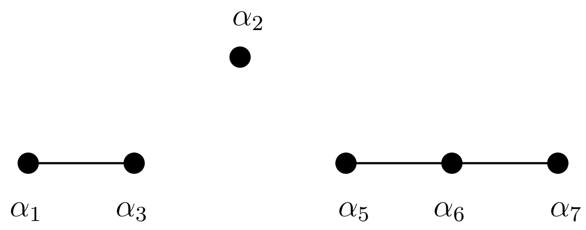
Let $I = S \setminus \{\alpha_3\}$. Then we observe that $w_{0,S\setminus\{\alpha_1,\alpha_3\}}(\alpha_1) = w_{0,I\setminus\{\alpha_1\}}(\alpha_1)$. Further, we note that the connected component of the Dynkin subdiagram associated to I, containing α_1 is of type A_1. Since α_1 is minuscule in type A_1, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_1\}}(\alpha_1) = \alpha_1$

(ii) Similar to the proof of (1)(ii).

(iii) By (2)(ii) we have $w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_2\}}(\alpha_1) = \omega_1 - \alpha_1$. Therefore, by using Lemma 7.2(3), we get

$$\begin{aligned} {v_4}^{-1}w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_2\}}(\alpha_1) &= -\alpha_4 + (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7) \\ &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7. \end{aligned}$$

(3)(i) Next we consider the Dynkin subdiagram of E_7, corresponding to the subset $S \setminus \{\alpha_4\}$ of S :



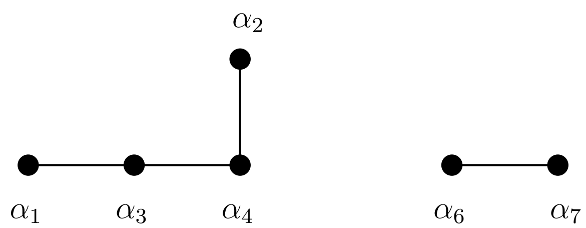
Let $I = S \setminus \{\alpha_4\}$. Then we observe that $w_{0,S\setminus\{\alpha_1,\alpha_4\}}(\alpha_1) = w_{0,I\setminus\{\alpha_1\}}(\alpha_1)$. Further, we note that the connected component of the Dynkin subdiagram associated to I, containing α_1 is of type A_2. Since α_1 is minuscule in type A_1, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_1\}}(\alpha_1) = \alpha_1 + \alpha_3$

(ii) Similar to the proof of (1)(ii).

(iii) By (3)(ii) we obtain $w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_3\}}(\alpha_1) = \omega_1 - (\alpha_1 + \alpha_3)$. Therefore, by using Lemma 7.2(3), we get

$$\begin{aligned} &v_4^{-1}w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_3\}}(\alpha_1) \\ &= -\alpha_4 + (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7) - (\alpha_2 + \alpha_4 + \alpha_5) \\ &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7. \end{aligned}$$

(4)(i) Next we consider the Dynkin subdiagram of E_7, corresponding to the subset $S \setminus \{\alpha_5\}$ of S:



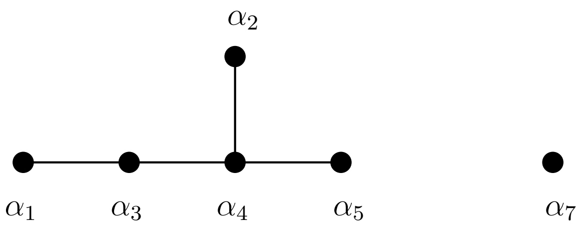
Let $I = S \setminus \{\alpha_5\}$. Then we observe that $w_{0,S\setminus\{\alpha_1,\alpha_5\}}(\alpha_1) = w_{0,I\setminus\{\alpha_1\}}(\alpha_1)$. Further, we note that the connected component of the Dynkin subdiagram associated to I, containing α_1 is of type A_4. Since α_1 is minuscule in type A_4, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_1\}}(\alpha_1) = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_2$.

(ii) Similar to the proof of (1)(i).

(iii) By (4)(ii) we have $w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_3\}}(\alpha_1) = \omega_1 - (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_2)$. Therefore, by using Lemma 7.2(3), we get

$$\begin{aligned}&v_4^{-1}w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_3\}}(\alpha_1)\\&=-\alpha_4+(\alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7)-(\alpha_2+\alpha_4+\alpha_5)-\alpha_6-\alpha_7\\&=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6.\end{aligned}$$

(5)(i) Next we consider the Dynkin subdiagram of E_7, corresponding to the subset $S\setminus\{\alpha_6\}$ of S :



Let $I=S\setminus\{\alpha_6\}$. Then we observe that $w_{0,S\setminus\{\alpha_1,\alpha_4\}}(\alpha_1)=w_{0,I\setminus\{\alpha_1\}}(\alpha_1)$. Further, we note that the connected component of the Dynkin subdiagram associated to I, containing α_1 is of type D_5. Since α_1 is minuscule in type D_5, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_1\}}(\alpha_1)=\alpha_1+2(\alpha_3+\alpha_4)+\alpha_2+\alpha_5$.

(ii) Similar to the proof of (1)(ii).

(iii) By (5)(ii) we have

$$w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_3\}}(\alpha_1)=\omega_1-(\alpha_1+2\alpha_3+2\alpha_4+\alpha_2+\alpha_5).$$

Therefore, by using Lemma 7.2(3), we get

$$\begin{aligned}&v_4^{-1}w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_3\}}(\alpha_1)\\&=-\alpha_4+(\alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7)-2(\alpha_2+\alpha_4+\alpha_5)\\&\qquad-2\alpha_6-\alpha_7-(\alpha_3+\alpha_4+\alpha_5)\\&=\alpha_1+\alpha_3.\end{aligned}$$

■

Lemma 7.4. *Let $w_i=w_{0,S\setminus\{\alpha_1,\alpha_i\}}w_{0,S\setminus\{\alpha_1\}}v_4$ for $i\neq 7$. Then $w_i^{-1}(\alpha_j)$ is a positive root for $j\neq i$, and $w_i^{-1}(\alpha_i)$ is a negative root.*

Proof. Note that for $i=1$, we have $w_1=v_4$. Then by Lemma 7.2(3), we are done. For $i\neq 1$, let $w_{0,S\setminus\{\alpha_1,\alpha_i\}}(\alpha_i)=\beta$. Then β is a positive root whose supports does not contain α_1. Since $w_{0,S\setminus\{\alpha_1\}}$ is the longest element of $W_{S\setminus\{\alpha_1\}}$ (Weyl group of type D_6), we have $w_{0,S\setminus\{\alpha_1\}}(\alpha_i)=-\alpha_i$ for all $i\neq 1$. Thus we have $w_{0,S\setminus\{\alpha_1\}}(\beta)=-\beta$. Further, since the support of β does not contain α_1, by Lemma 7.2(3) we have $v_4^{-1}(-\beta)$ is a negative root. Hence we have $w_i^{-1}(\alpha_i)$ is a negative root for $i\neq 1$.

On the other hand, for $i\neq 1,7$ and $j\neq 1,i$, $w_{0,S\setminus\{\alpha_1,\alpha_i\}}(\alpha_j)=-\alpha_k$, where α_k is a simple root different from α_1. Therefore, $w_{0,S\setminus\{\alpha_1\}}(-\alpha_k)=\alpha_k$. Further, since α_k is different from α_1, by Lemma 7.2(3), $v_4^{-1}(\alpha_k)$ is a positive root. Hence $w_i^{-1}(\alpha_j)$ is a positive root when $i\neq 1,7$ and $j\neq 1,i$. Moreover, by Lemma 7.3, we have $w_i^{-1}(\alpha_1)$ is a positive root for $i\neq 1,7$. Hence, we have $w_i^{-1}(\alpha_j)$ is a positive root for $i\neq 7$ and $j\neq i$. ■

Proof of Proposition 7.1. We note that ω_i is not minuscule if and only if $i\neq 7$. Assume that ω_i is not minuscule. Recall that $w_i=w_{0,S\setminus\{\alpha_1,\alpha_i\}}w_{0,S\setminus\{\alpha_1\}}v_4$. By Lemma 7.2(2) and (3), we conclude that $w_i(\alpha_3)$ is a non simple positive root. On the other hand, by Lemma 7.4, $w_i^{-1}(\alpha_i)$ is a negative root and $w_i^{-1}(\alpha_j)$ is a positive root for $j \neq i$. Thus P_i is the stabilizer of $X_{P_3}(w_i)$ in G. Since $\alpha_0 = \omega_1$, and $v_4^{-1}(\alpha_0)$ is a negative root, $w_i^{-1}(\alpha_0) = v_4^{-1}(\alpha_0)$ (as $w_{0,S\setminus\{\alpha_1\}}w_{0,S\setminus\{\alpha_1,\alpha_i\}}(\alpha_0) = \alpha_0$) is a negative root. Therefore, by using Theorem 2.6 the natural homomorphism $P_i \longrightarrow \mathrm{Aut}^0(X_{P_3}(w_i))$ is an isomorphism of algebraic groups. ■

8. G is of type E_8

By Lemma 3.1, we note that all simple roots are non-minuscule. In this section, our goal is to prove the following proposition

Proposition 8.1. *Assume that ω_i is non minuscule. Then there exists $X_{P_7}(w_i)$ in G/P_7 such that $P_i = \mathrm{Aut}^0(X_{P_7}(w_i))$.*

We recall that there exists a unique element v_6 in W of minimal length such that $v_6^{-1}(\alpha_0) = -\alpha_6$ (see Corollary 2.2). Further, we observe that $w_{0,S\setminus\{\alpha_8\}}$ is the longest element of the Dynkin subdiagram removing the node α_8, which is of type E_7. Thus $w_{0,S\setminus\{\alpha_8\}}(\alpha_i) = -\alpha_i$ for all $i \neq 8$. We use these observations very frequently.

Lemma 8.2. *Let v_6 be as above. Then we have the following:*

(1) $v_6 = s_8s_7s_6s_5s_4s_2s_3s_1s_4s_3s_5s_6s_4s_2s_5s_7s_4s_6s_5s_3s_4s_2s_8s_7s_1s_3s_4s_5s_6$.

(2) (i) $v_6(\alpha_7) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$.

(ii) $w_{0,S\setminus\{\alpha_8\}}v_6(\alpha_7) = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$.

(iii) $w_{0,S\setminus\{\alpha_8,\alpha_i\}}w_{0,S\setminus\{\alpha_8\}}v_6(\alpha_7)$ *is a non simple positive root for* $1 \leq i \leq 8$.

(3) $v_6^{-1}(\alpha_1) = \alpha_8$, $v_6^{-1}(\alpha_2) = \alpha_2$, $v_6^{-1}(\alpha_3) = \alpha_5 + \alpha_6 + \alpha_7$, $v_6^{-1}(\alpha_4) = \alpha_4$, $v_6^{-1}(\alpha_5) = \alpha_3$, $v_6^{-1}(\alpha_6) = \alpha_1$, $v_6^{-1}(\alpha_7) = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$ $v_6^{-1}(\alpha_8) = -(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + +6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7 + \alpha_8)$.

Proof. (1) Let

$$v'_6 = s_8s_7s_6s_5s_4s_2s_3s_1s_4s_3s_5s_6s_4s_2s_5s_7s_4s_6s_5s_3s_4s_2s_8s_7s_1s_3s_4s_5s_6 .$$

Note that ${v'_6}^{-1}(\alpha_0) = -\alpha_6$. Since $\ell(v'_6) = \ell(v_6)$, by Proposition 2.1 we have $v_6 = v'_6$.

(2) (i) follows from the usual calculation using description of v_6 as in (1). (ii) follows from (i), and using Lemma 4.1. Since support of $w_{0,S\setminus\{\alpha_1\}}v_4(\alpha_3)$ contains α_i $(1 \leq i \leq 8)$, (iii) follows.

(3) Follows from the usual calculation using the description of v_6 as in (i). ■

Lemma 8.3. *We have the following:*

(1) (i) $w_{0,S\setminus\{\alpha_1,\alpha_8\}}(\alpha_8) = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$.

(ii) $w_{0,S\setminus\{\alpha_8\}}w_{0,S\setminus\{\alpha_1,\alpha_8\}}(\alpha_8) = \omega_8 - (\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8)$.

(iii) ${v_6}^{-1}w_{0,S\setminus\{\alpha_8\}}w_{0,S\setminus\{\alpha_1,\alpha_8\}}(\alpha_8) = \alpha_7 + \alpha_8$.

(2) (i) $w_{0,S\setminus\{\alpha_2,\alpha_8\}}(\alpha_8) = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$.

(ii) $w_{0,S\setminus\{\alpha_8\}}w_{0,S\setminus\{\alpha_2,\alpha_8\}}(\alpha_8) = \omega_8 - (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)$.

(iii) ${v_6}^{-1}w_{0,S\setminus\{\alpha_8\}}w_{0,S\setminus\{\alpha_2,\alpha_8\}}(\alpha_8) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$.

(3) (i) $w_{0,S\setminus\{\alpha_3,\alpha_8\}}(\alpha_8) = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$.
(ii) $w_{0,S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_3,\alpha_8\}}(\alpha_8) = \omega_8 - (\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)$.
(iii) ${v_6}^{-1} w_{0,S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_3,\alpha_8\}}(\alpha_8) = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$.

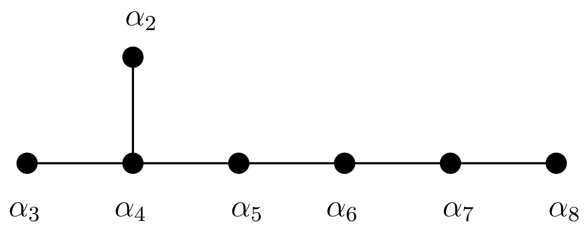
(4) (i) $w_{0,S\setminus\{\alpha_4,\alpha_8\}}(\alpha_8) = \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$.
(ii) $w_{0,S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_4,\alpha_8\}}(\alpha_8) = \omega_8 - (\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)$.
(iii) ${v_6}^{-1} w_{0,S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_4,\alpha_8\}}(\alpha_8) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$.

(5) (i) $w_{0,S\setminus\{\alpha_5,\alpha_8\}}(\alpha_8) = \alpha_6 + \alpha_7 + \alpha_8$.
(ii) $w_{0,S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_5,\alpha_8\}}(\alpha_8) = \omega_8 - (\alpha_6 + \alpha_7 + \alpha_8)$.
(iii) ${v_6}^{-1} w_{0,S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_5,\alpha_8\}}(\alpha_8) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$.

(6) (i) $w_{0,S\setminus\{\alpha_6,\alpha_8\}}(\alpha_8) = \alpha_7 + \alpha_8$.
(ii) $w_{0,S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_5,\alpha_8\}}(\alpha_8) = \omega_8 - (\alpha_7 + \alpha_8)$.
(iii) ${v_6}^{-1} w_{0,S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_5,\alpha_8\}}(\alpha_8) = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$.

(7) (i) $w_{0,S\setminus\{\alpha_7,\alpha_8\}}(\alpha_8) = \alpha_8$.
(ii) $w_{0,S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_5,\alpha_8\}}(\alpha_8) = \omega_8 - \alpha_8$.
(iii) ${v_6}^{-1} w_{0,S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_5,\alpha_8\}}(\alpha_8) = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$.

Proof. (1)(i) We consider the Dynkin subdiagram of E_8, corresponding to the subset $S \setminus \{\alpha_1\}$ of S (see Figure: 4):



Let $I = S \setminus \{\alpha_1\}$. Now we observe that $w_{0,S\setminus\{\alpha_1,\alpha_8\}}(\alpha_8) = w_{0,I\setminus\{\alpha_8\}}(\alpha_8)$. We note that the connected Dynkin subdiagram of E_8 associated to I containing α_8 is of type D_7. Since α_8 is minuscule in type D_7, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_8\}}(\alpha_8) = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8$.

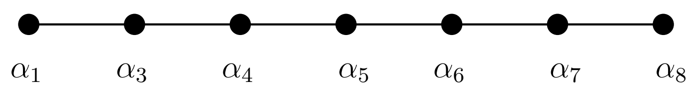
(ii) Follows from the observation that $w_{0,S\setminus\{\alpha_8\}}(\alpha_i) = -\alpha_i$ for all $i \neq 8$, and by using Lemma 4.1.

(iii) By (ii) we have

$$w_{0,S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_1,\alpha_8\}}(\alpha_8) = \omega_8 - (\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8).$$

Therefore, by using Lemma 8.2(3), we get ${v_6}^{-1} w_{S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_1,\alpha_8\}} = \alpha_7 + \alpha_8$.

(2)(i) Next we consider the Dynkin subdiagram of E_8, corresponding to the subset $S \setminus \{\alpha_2\}$ of S:



Let $I = S \setminus \{\alpha_2\}$. Now we observe that $w_{0,S\setminus\{\alpha_2,\alpha_8\}}(\alpha_8) = w_{0,I\setminus\{\alpha_8\}}(\alpha_8)$. We note that the connected component of the Dynkin subdiagram of E_8 associated to I containing α_8 is of type A_7. Since α_8 is minuscule in type A_7, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_8\}}(\alpha_8) = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$.

(ii) Follows from the observation that $w_{0,S\setminus\{\alpha_8\}}(\alpha_i) = -\alpha_i$ for all $i \neq 8$, and by using Lemma 4.1.

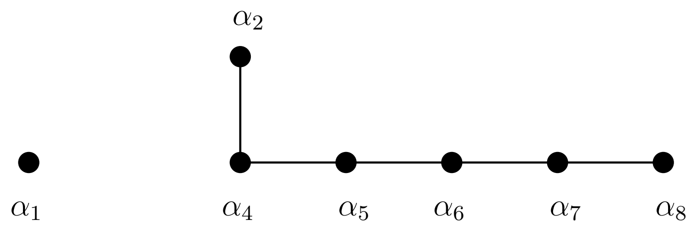
(iii) By (ii) we have

$$w_{0,S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_2,\alpha_8\}}(\alpha_8) = \omega_8 - (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8).$$

Therefore, by using Lemma 8.2(3), we get

$$v_6{}^{-1} w_{S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_2,\alpha_8\}} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7.$$

(3)(i) Next we consider the Dynkin subdiagram of E_8, corresponding to the subset $S \setminus \{\alpha_3\}$ of S :



Let $I = S \setminus \{\alpha_3\}$. Then we observe that $w_{0,S\setminus\{\alpha_3,\alpha_8\}}(\alpha_8) = w_{0,I\setminus\{\alpha_8\}}(\alpha_8)$. Further, we note that the connected component of the Dynkin subdiagram associated to I containing α_8 is of type A_6. Since α_8 is minuscule in type A_6, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_8\}}(\alpha_8) = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$.

(ii) Follows from the observation that $w_{0,S\setminus\{\alpha_8\}}(\alpha_i) = -\alpha_i$ for all $i \neq 8$, and by using Lemma 4.1.

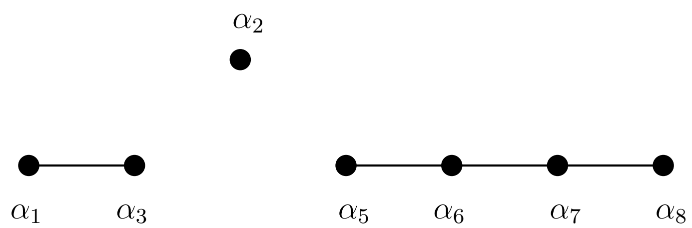
(iii) By (ii) we have

$$w_{0,S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_3,\alpha_8\}}(\alpha_8) = \omega_8 - (\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8).$$

Therefore, by using Lemma 8.2(3), we get

$$v_6{}^{-1} w_{S\setminus\{\alpha_8\}} w_{0,S\setminus\{\alpha_3,\alpha_8\}} = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8.$$

(4)(i) Next we consider the Dynkin subdiagram of E_8, corresponding to the subset $S \setminus \{\alpha_4\}$ of S :



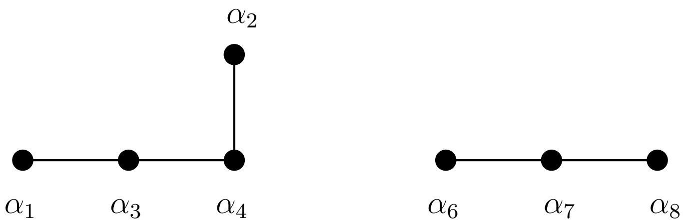
Let $I = S \setminus \{\alpha_4\}$. Then we observe that $w_{0,S\setminus\{\alpha_4,\alpha_8\}}(\alpha_8) = w_{0,I\setminus\{\alpha_8\}}(\alpha_8)$. Further, we note that the connected component of the Dynkin subdiagram associated to I containing α_8 is of type A_4. Since α_8 is minuscule in type A_4, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_8\}}(\alpha_8) = \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8$.

(ii) Follows from the observation that $w_{0,S\setminus\{\alpha_8\}}(\alpha_i) = -\alpha_i$ for all $i \neq 8$, and by using Lemma 4.1.

(iii) By 4(ii) we have $w_{0,S\setminus\{\alpha_8\}}w_{0,S\setminus\{\alpha_4,\alpha_8\}}(\alpha_8) = \omega_8 - (\alpha_5+\alpha_6+\alpha_7+\alpha_8)$. Therefore, by using Lemma 8.2(3), we get

$$v_6{}^{-1}w_{S\setminus\{\alpha_8\}}w_{0,S\setminus\{\alpha_4,\alpha_8\}} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8.$$

(5)(i) Next we consider the Dynkin subdiagram of E_8, corresponding to the subset $S \setminus \{\alpha_5\}$ of S :



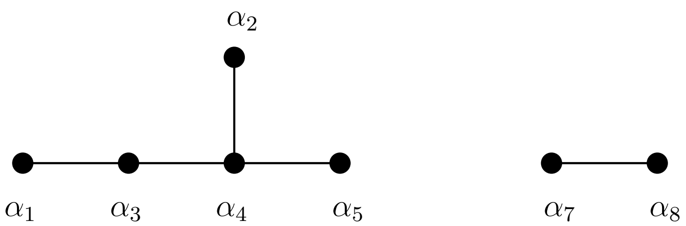
Let $I = S \setminus \{\alpha_5\}$. Then we observe that $w_{0,S\setminus\{\alpha_5,\alpha_8\}}(\alpha_8) = w_{0,I\setminus\{\alpha_8\}}(\alpha_8)$. Further, we note that the connected component of the Dynkin subdiagram associated to I containing α_8 is of type A_3. Since α_8 is minuscule in type A_3, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_8\}}(\alpha_8) = \alpha_6 + \alpha_7 + \alpha_8$.

(ii) Follows from the observation that $w_{0,S\setminus\{\alpha_8\}}(\alpha_i) = -\alpha_i$ for all $i \neq 8$, and by using Lemma 4.1.

(iii) By 5(ii) we have $w_{0,S\setminus\{\alpha_8\}}w_{0,S\setminus\{\alpha_5,\alpha_8\}}(\alpha_8) = \omega_8 - (\alpha_6 + \alpha_7 + \alpha_8)$. Therefore, by using Lemma 8.2(3), we get

$$v_6{}^{-1}w_{S\setminus\{\alpha_8\}}w_{0,S\setminus\{\alpha_5,\alpha_8\}} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8.$$

(6)(i) Next we consider the Dynkin subdiagram of E_8, corresponding to the subset $S \setminus \{\alpha_6\}$ of S:



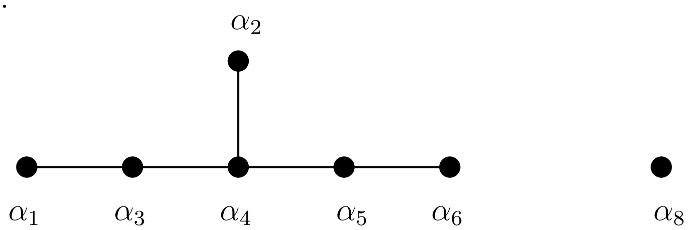
Let $I = S \setminus \{\alpha_6\}$. Then we observe that $w_{0,S\setminus\{\alpha_6,\alpha_8\}}(\alpha_8) = w_{0,I\setminus\{\alpha_8\}}(\alpha_8)$. Further, we note that the connected component of the Dynkin subdiagram associated to I containing α_8 is of type A_2. Since α_8 is minuscule in type A_2, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_8\}}(\alpha_8) = \alpha_7 + \alpha_8$.

(ii) Follows from the observation that $w_{0,S\setminus\{\alpha_8\}}(\alpha_i) = -\alpha_i$ for all $i \neq 8$, and by using Lemma 4.1.

(iii) By 6(ii) we have $w_{0,S\setminus\{\alpha_8\}}w_{0,S\setminus\{\alpha_6,\alpha_8\}}(\alpha_8) = \omega_8 - (\alpha_7+\alpha_8)$. Therefore, by using Lemma 8.2(3), we get

$$v_6{}^{-1}w_{S\setminus\{\alpha_8\}}w_{0,S\setminus\{\alpha_6,\alpha_8\}} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8.$$

(7)(i) Next we consider the Dynkin subdiagram of E_8, corresponding to the subset $S \setminus \{\alpha_7\}$ of S:



Let $I = S \setminus \{\alpha_7\}$. Then we observe that $w_{0,S\setminus\{\alpha_7,\alpha_8\}}(\alpha_8) = w_{0,I\setminus\{\alpha_8\}}(\alpha_8)$. Further, we note that the connected component of the Dynkin subdiagram associated to I containing α_8 is of type A_1. Since α_8 is minuscule in type A_1, by Lemma 3.2, we have $w_{0,I\setminus\{\alpha_8\}}(\alpha_8) = \alpha_8$.

(ii) Follows from the observation that $w_{0,S\setminus\{\alpha_8\}}(\alpha_i) = -\alpha_i$ for all $i \neq 8$, and by using Lemma 4.1.

(iii) By 7(ii) we have $w_{0,S\setminus\{\alpha_8\}}w_{0,S\setminus\{\alpha_7,\alpha_8\}}(\alpha_8) = \omega_8 - \alpha_8$. Therefore, by using Lemma 8.2(3), we get

$$v_6{}^{-1}w_{S\setminus\{\alpha_8\}}w_{0,S\setminus\{\alpha_7,\alpha_8\}} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8. \qquad \blacksquare$$

Lemma 8.4. *Let $w_i = w_{0,S\setminus\{\alpha_i,\alpha_8\}}w_{0,S\setminus\{\alpha_8\}}v_6$ for $1 \leq i \leq 8$. Then we have $w_i^{-1}(\alpha_j)$ is a positive root for $j \neq i$, and $w_i^{-1}(\alpha_i)$ is a negative root.*

Proof. Note that for $i = 8$ we have $w_8 = v_6$. Then by Lemma 8.2, we are done. For $i \neq 8$, let $w_{0,S\setminus\{\alpha_i,\alpha_8\}}(\alpha_i) = \beta$. Then β is positive root whose support does not contain α_8. Since $w_{0,S\setminus\{\alpha_8\}}$ is the longest element of $W_{S\setminus\{\alpha_8\}}$, $w_{0,S\setminus\{\alpha_8\}}(\alpha_i) = -\alpha_i$ for $i \neq 8$. Thus we have $w_{0,S\setminus\{\alpha_8\}}(\beta) = -\beta$. Further, since the support of β does not contain α_8, by Lemma 7.2(3) we have $v_6^{-1}(-\beta)$ is a negative root. Hence we have $w_i^{-1}(\alpha_i)$ is a negative root for $i \neq 8$.

On the other hand, for $i \neq 8$, and $j \neq 8, i$, $w_{0,S\setminus\{\alpha_i,\alpha_8\}}(\alpha_j) = -\alpha_k$, where α_k is a simple root different from α_8. Therefore, $w_{0,S\setminus\{\alpha_8\}}(-\alpha_k) = \alpha_k$. Further, since α_k is different from α_8, by Lemma 8.2(3), $v_6^{-1}(\alpha_k)$ is a positive root. Hence $w_i^{-1}(\alpha_j)$ is a positive root when $i \neq 8$, and $j \neq i, 8$. Moreover, by Lemma 8.3, we have $w_i^{-1}(\alpha_8)$ is a positive root for $i \neq 8$. Hence, we have $w_i^{-1}(\alpha_j)$ is a positive root for $i \neq 8$, and $j \neq i$. $\blacksquare$

Proof of Proposition 8.1. Recall that $w_i = w_{0,S\setminus\{\alpha_i,\alpha_8\}}w_{0,S\setminus\{\alpha_8\}}v_6$. Note that $w_i(\alpha_7) = w_{S\setminus\{\alpha_i,\alpha_8\}}w_{0,S\setminus\{\alpha_8\}}v_6(\alpha_7)$. By Lemma 8.2(2), we conclude that $w_i(\alpha_7)$ is a non simple positive root. On the other hand, by Lemma 8.4, $w_i^{-1}(\alpha_i)$ is a negative root and $w_i^{-1}(\alpha_j)$ is a positive root for $j \neq i$. Thus P_i is the stabilizer of $X_{P_7}(w_i)$ in G. Since $\alpha_0 = \omega_8$, and $v_6^{-1}(\alpha_0)$ is a negative root, $w_i^{-1}(\alpha_0) = v_6^{-1}(\alpha_0)$ (as $w_{0,S\setminus\{\alpha_8\}}w_{0,S\setminus\{\alpha_i,\alpha_8\}}(\alpha_0) = \alpha_0$) is a negative root. Therefore, by using Theorem 2.6 the natural homomorphism $P_i \longrightarrow \mathrm{Aut}^0(X_{P_7}(w_i))$ is an isomorphism of algebraic groups. $\blacksquare$

9. Main theorem

In this section, our goal is to prove the following. A fundamental weight ω_i is minuscule if and only if for any parabolic subgroup Q containing B properly, there is no Schubert variety $X_Q(w)$ in G/Q such that $P_i = \mathrm{Aut}^0(X_Q(w))$.

Theorem 9.1. *Assume that ω_r is minuscule. If there exists a Schubert variety $X_Q(w)$ in G/Q (for some parabolic subgroup Q of G containing B) such that $P_r = \mathrm{Aut}^0(X_Q(w))$, then we have $Q = B$.*

Proof. Assume that $P_r = \mathrm{Aut}^0(X_Q(w))$ where $X_Q(w)$ is a Schubert variety in G/Q, for some $Q \neq B$. Then there exists a non-empty subset J of S such that $Q = P_J$. Since $P_r = \mathrm{Aut}^0(X_Q(w))$, we have $w^{-1}(\alpha_j) > 0$ for all $j \neq r$, and we have

$w^{-1} \in W^{S\setminus\{\alpha_r\}}$. Since the action P_r on $X_Q(w)$ is faithful, by Theorem 2.6 we have $w^{-1}(\alpha_0) < 0$. Further, since $w^{-1}(\alpha_0) < 0$ and $w^{-1} \in W^{S\setminus\{\alpha_r\}}$, by Lemma 3.3 we have $w^{-1} = w_0^{S\setminus\{\alpha_r\}}$. Thus we have $w = (w_0^{S\setminus\{\alpha_r\}})^{-1}$.

Note that $(w_0^{S\setminus\{\alpha_r\}})^{-1} = w_{0,S\setminus\{\alpha_r\}}w_0 = w_0(w_0w_{0,S\setminus\{\alpha_r\}}w_0) = w_0w_{0,S\setminus\{\alpha_{\sigma(r)}\}}$.

Therefore, we have $w = w_0^{S\setminus\{\alpha_{\sigma(r)}\}}$. Hence, we have $\sigma(J) \subset S \setminus \{\alpha_r\}$. Let τ be the Dynkin diagram automorphism of $S \setminus \{\alpha_r\}$ induced by $-w_{0,S\setminus\{\alpha_r\}}$. Now since J is nonempty, there exists $\alpha_i \in J$ such that $\alpha_{\sigma(i)} \neq \alpha_r$. So there exists $\alpha_j \in S \setminus \{\alpha_r\}$ such that $\alpha_{\tau(j)} = \alpha_{\sigma(i)}$. That is we have $\alpha_{\sigma\tau(j)} = \alpha_i$. Now we have $w^{-1}s_jw = w_0w_{0,S\setminus\{\alpha_r\}}s_jw_{0,S\setminus\{\alpha_r\}}w_0 = s_{\sigma\tau(j)} = s_i$. This is a contradiction to the fact that P_r is the stabilizer of $X_Q(w)$ in G. Hence, we have $Q = B$. ∎

Theorem 9.2. *A fundamental weight ω_α is minuscule if and only if for every parabolic subgroup Q containing B properly, there is no Schubert variety $X_Q(w)$ in G/Q such that $P_\alpha = \mathrm{Aut}^0(X_Q(w))$.*

Proof. Follows from Proposition 5.1, Proposition 6.1, Proposition 7.1, Proposition 8.1, and Theorem 9.1. ∎

In the following remark we label the simple roots for B_2 as in [10, p. 58], namely $\langle\alpha_1, \alpha_2\rangle = -2$, and $\langle\alpha_2, \alpha_1\rangle = -1$.

Remark 9.3. In non-simply-laced type, Theorem 9.2 may not hold. For instance consider $G = SO(5, \mathbb{C})$ and $w = s_2s_1$. Here, ω_2 is the unique minuscule weight. Then, $X_{P_2}(w)$ is a Schubert variety in G/P_2 such that $P_2 = \mathrm{Aut}^0(X_{P_2}(w))$.

Proof. Note that the tangent bundle of G/P_2 is $\mathcal{L}(\mathfrak{g}/\mathfrak{p}_2)$. Further, $\mathfrak{g}/\mathfrak{p}_2$ has a filtration of B-submodules with successive quotients are of the form $\mathbb{C}_\beta$ for some $\beta \in \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$. Then we use Lemma 2.4, Lemma 2.5, and Lemma 2.3, to compute the following cohomology modules:

$$H^0(s_1, \mathfrak{g}/\mathfrak{p}_2) = \mathbb{C}_{\alpha_1} \oplus \mathbb{C}h(\alpha_1) \oplus \mathbb{C}_{-\alpha_1} \oplus \mathbb{C}_{\alpha_1+\alpha_2} \oplus \mathbb{C}_{\alpha_2} \oplus \mathbb{C}_{\alpha_1+2\alpha_2}$$

and

$$\begin{aligned} H^0(w, \mathfrak{g}/\mathfrak{p}_2) = H^0(s_2s_1, \mathfrak{g}/\mathfrak{p}_2) = H^0(s_2, H^0(s_1, \mathfrak{g}/\mathfrak{p}_2)) = \mathbb{C}_{\alpha_1} \oplus \mathbb{C}_{\alpha_1+\alpha_2} \oplus \mathbb{C}_{\alpha_1+2\alpha_2} \\ \oplus \mathbb{C}h(\alpha_1) \oplus \mathbb{C}_{-\alpha_1} \oplus \mathbb{C}_{-\alpha_1-\alpha_2} \oplus \mathbb{C}_{-\alpha_1-2\alpha_2} \oplus \mathbb{C}_{\alpha_2} \oplus \mathbb{C}h(\alpha_2) \oplus \mathbb{C}_{-\alpha_2} \simeq \mathfrak{g} \end{aligned}$$

as T-module. Note that $\mathbb{C}_{\alpha_1}$ is a B-submodule of $\mathfrak{g}/\mathfrak{p}_2$. Thus, $H^0(w, \alpha_1)$ is a B-submodule of $H^0(w, \mathfrak{g}/\mathfrak{p}_2)$.

But, $H^0(w, \alpha_1)_{-\alpha_0} \neq 0$, as $s_2s_1(\alpha_1) = -\alpha_0$. Hence, $H^0(w, \mathfrak{g}/\mathfrak{p}_2)_{-\alpha_0} \neq 0$. Therefore, the restriction map $\varphi : H^0(G/P_2, \mathcal{L}(\mathfrak{g}/\mathfrak{p}_2)) \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p}_2)$ is injective. Furthermore, since G acts on G/P_2, we have $\mathfrak{g} \subseteq H^0(G/P_2, \mathcal{L}(\mathfrak{g}/\mathfrak{p}_2))$. Thus by the above discussion, φ is an isomorphism and hence we have $H^0(G/P_2, \mathcal{L}(\mathfrak{g}/\mathfrak{p}_2)) = \mathfrak{g}$ and $H^0(w, \mathfrak{g}/\mathfrak{p}_2) = \mathfrak{g}$ as B-modules.

On the other hand, $T_{X_{P_2}(w)}$ is a subsheaf of the restriction of the tangent bundle T_{G/P_2} to $X_{P_2}(w)$. Hence, $H^0(X_{P_2}(w), T_{X_{P_2}(w)})$ is a B-submodule of $H^0(w, \mathfrak{g}/\mathfrak{p}_2)$. Further, since $H^0(w, \mathfrak{g}/\mathfrak{p}_2) = \mathfrak{g}$, $\mathrm{Lie}(\mathrm{Aut}^0(X_{P_2}(w)))$ is a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. Thus $\mathrm{Aut}^0(X_{P_2}(w))$ is a parabolic subgroup of G containing B.

Let $P' = \mathrm{Aut}^0(X_{P_2}(w)) \subseteq G$. Let α be a simple root such that $s_\alpha \in W(P')$, Weyl group of P'. Let $n_\alpha \in N_{P'}(T)$ be a representative of s_α. Then $n_\alpha \cdot wP_2/P_2$ is a T-fixed point in $X_{P_2}(w)$.

Further, for any dominant character χ of P_2, T acts on the fiber of the $\mathcal{L}(\chi)$ over the point $n_\alpha \cdot wP_2/P_2$ by the character $s_\alpha w(\chi)$. Therefore, $n_\alpha \cdot wP_2/P_2 = s_\alpha wP_2/P_2$. Since $s_\alpha wP_2/P_2 \in X_{P_2}(w)$, we have $s_\alpha w \in W^{\{\alpha_2\}}$ and $s_\alpha w < w$ or $s_\alpha wP_2 = wP_2$.

Thus, n_α is in the stabilizer of $X_{P_2}(w)$ in G. Since P' is generated by B and n_α, where $\alpha \in S$ such that $s_\alpha \in W(P')$, it follows that P' is the stabilizer of $X_{P_2}(w)$ in G. Hence, $\mathrm{Aut}^0(X_{P_2}(w))$ is equal to the stabilizer of $X_{P_2}(w)$ in G. Now, the remark follows from the fact that stabilizer of $X_{P_2}(w)$ in G is P_2.

We now describe $X_{P_2}(w)$ geometrically. Consider the Schubert variety $X(w)$ in G/B. Consider the open subsets $BwB/B \subseteq X(w)$ and $BwP_2/P_2 \subseteq X_{P_2}(w)$.

Let $U = \prod_{\alpha \in R^+(w^{-1})} U_{-\alpha}$. The morphisms $U \longrightarrow BwB/B$ sending $u \mapsto uwB/B$ and $U \longrightarrow BwP_2/P_2$ sending $u \mapsto uwP_2/P_2$ are isomorphisms (see [5, Section 2.1, (2) and (3), p. 62]). Therefore, the class groups of BwB/B and BwP_2/P_2 are trivial.

We further see that $X_{P_2}(w) \setminus BwP_2/P_2 = X_{P_2}(s_1)$ is an irreducible divisor. Also, $X(w) \setminus BwB/B$ is union of two irreducible divisors $X(s_1)$ and $X(s_2)$. By [9, Proposition 6.5(c), p. 133], $\mathrm{rank}(\mathrm{Cl}(X(w)) \leq 2$ and $\mathrm{rank}(\mathrm{Cl}(X_{P_2}(w))) \leq 1$, where $\mathrm{Cl}(X(w))$ (respectively, $\mathrm{Cl}(X_{P_2}(w))$) denotes the class group of $X(w)$ (respectively, of $X_{P_2}(w)$).

Hence, $\mathrm{rank}(\mathrm{Pic}(X(w)) \leq 2$ and $\mathrm{rank}(\mathrm{Pic}(X_{P_2}(w))) \leq 1$, where $\mathrm{Pic}(X(w))$ (respectively, $\mathrm{Pic}(X_{P_2}(w))$) denotes the Picard group of $X(w)$ (respectively, of $X_{P_2}(w)$). Further, the restriction of the line bundles $\mathcal{L}(\omega_1)$ and $\mathcal{L}(\omega_2)$ to $X(w)$ are linearly independent. Therefore, $\mathrm{rank}(\mathrm{Pic}(X(w))) = 2$ and $\mathrm{rank}(\mathrm{Pic}(X_{P_2}(w))) = 1$.

Now, consider the restriction of the morphism $\pi : G/B \longrightarrow G/P_2$ to $X(w)$, which also we denote it by π. So, $\pi : X(w) \longrightarrow X_{P_2}(w)$ is a surjective birational morphism such that $\pi_* \mathcal{O}_{X(w)} = \mathcal{O}_{X_{P_2}(w)}$ (see [5, Theorem 3.3.4(a), p. 96]). Therefore, by [3, Corollary 2.2, p. 45], π induces homomorphism $\pi_* : \mathrm{Aut}^0(X(w)) \longrightarrow \mathrm{Aut}^0(X_{P_2}(w))$ of algebraic groups. On the other hand, since $w^{-1}(\alpha_0) < 0$, by [13, Theorem 6.6, p. 781] $P_2 \subseteq \mathrm{Aut}^0(X(w))$. Therefore, $\pi_* : \mathrm{Aut}^0(X(w)) \longrightarrow \mathrm{Aut}^0(X_{P_2}(w)) = P_2$ is an isomorphism. ■

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