

Local and 2-Local Derivations on Lie Matrix Rings over Commutative Involutive Rings

Sh. A. Ayupov, F. N. Arzikulov, S. M. Umrzaqov

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Abstract. We prove that every 2-local inner derivation on the Lie ring of skew-adjoint matrices over a commutative $*$ -ring is an inner derivation. We also prove that every 2-local spatial derivation on various Lie algebras of skew-adjoint operator-valued maps on a set is a spatial derivation. We also show that every local spatial derivation on the Lie algebras mentioned above is a derivation.

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Key Words: Inner Lie derivation, 2-local Lie derivation, Lie ring, Lie algebras, Lie ring of skew-adjoint matrices.

Introduction

In the present paper we consider 2-local and local derivations on Lie rings and Lie algebras. The study of 2-local derivations began in the paper [17] of Šemrl. In [17] Šemrl introduced the notion of 2-local derivations and described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H . Later a number of papers were devoted to 2-local maps on different types of rings, algebras, Banach algebras and Banach spaces.

The main goal of this paper is to describe 2-local derivations on the Lie ring of skew-adjoint matrices over a commutative involutive ring. If in an associative algebra \mathcal{A} we take the Lie multiplication $[a, b] = ab - ba$, then we obtain the Lie algebra $(\mathcal{A}, [,])$. In this case, every 2-local inner derivation of the algebra \mathcal{A} is a derivation if and only if every 2-local inner derivation of the Lie algebra $(\mathcal{A}, [,])$ is a derivation. In general, for any Lie algebra $(\mathcal{L}, [,])$, the Lie multiplication $[,]$ is generated by an associative multiplication. Therefore in number of papers the proofs of results on 2-local Lie derivations are based on associative multiplication. For example, in [6] of Chen, Lu and Wang 2-local Lie derivations of operator algebras over Banach spaces are described. In the paper [5] of Ayupov, Kudaybergenov and Rakhimov 2-local derivations of Lie algebras, which are not generated by associative algebras, are described. They proved that every 2-local derivation on a finite-dimensional semi-simple Lie algebra \mathcal{L} over an algebraically closed field of characteristic zero is a derivation. They also showed that each finite-dimensional nilpotent Lie algebra \mathcal{L} with $\dim \mathcal{L} \geq 2$ admits a 2-local derivation which is not a derivation. At the same time, in [13] Lai and Chen give a description of 2-local Lie derivations for the case of

finite dimensional simple Lie algebras. Liu characterized 2-local Lie derivations on a semi-finite factor von Neumann algebra in [15]. In [9] it is proved by the authors that every 2-local Lie derivation on factor von Neumann algebras, UHF algebras and the Jiang-Su algebra is a Lie derivation. Recently, in [3], the first and second authors of the present paper proved that every 2-local inner derivation on the Lie ring $K_n(\mathfrak{R})$ of skew-symmetric $n \times n$ matrices over a commutative associative ring \mathfrak{R} is a derivation.

In the present paper we study inner derivations and 2-local inner derivations on Lie rings of skew-adjoint matrices over a commutative involutive ring.

The paper is organized as follows. In Section 2 we prove that each 2-local inner derivation on the Lie ring $K_n(\mathfrak{R})$ of skew-adjoint $n \times n$ matrices over a commutative unital involutive ring \mathfrak{R} is a derivation. As a corollary we establish that every 2-local inner derivation on the Lie algebra $K_n(\mathcal{A})$ of skew-adjoint $n \times n$ matrices over a commutative unital involutive algebra \mathcal{A} is a derivation.

We also study 2-local spatial derivations on various Lie subalgebras of the algebra $M(\Omega, B_{sk}(H))$ of all maps from Ω to $B_{sk}(H)$ for an arbitrary set Ω and the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H in Section 3. We prove that every 2-local spatial derivation on these subalgebras of the algebra $M(\Omega, B_{sk}(H))$, under some conditions, is a derivation. The problems considered here are firstly mentioned in [2] (Problem 1).

Sections 4 and 5 are devoted to the description of local inner derivations on the algebras mentioned in the previous paragraph. For this propose we apply a similar technique to these Lie algebras of skew-adjoint operator-valued maps on a set and prove that every local spatial derivation on such algebras is a spatial derivation. It should be noted that a number of results concerning local derivations are obtained, partially, in [4], [7], [8], [9], [10], [12], [14], [16].

1. Preliminaries

Let \mathfrak{R} be an associative ring. Recall that an additive map $D: \mathfrak{R} \rightarrow \mathfrak{R}$ is called a *derivation*, if $D(xy) = D(x)y + xD(y)$ for any two elements $x, y \in \mathfrak{R}$.

A map $\Delta: \mathfrak{R} \rightarrow \mathfrak{R}$ (not additive in general) is called a *2-local derivation*, if for any two elements $x, y \in \mathfrak{R}$ there exists a derivation $D_{x,y}: \mathfrak{R} \rightarrow \mathfrak{R}$ such that $\Delta(x) = D_{x,y}(x)$, $\Delta(y) = D_{x,y}(y)$.

A derivation D on \mathfrak{R} is called an *inner derivation*, if there exists an element $a \in \mathfrak{R}$ such that $D(x) = ax - xa$, where $x \in \mathfrak{R}$.

A map $\Delta: \mathfrak{R} \rightarrow \mathfrak{R}$ (not additive in general) is called a *2-local inner derivation*, if for any two elements $x, y \in \mathfrak{R}$ there exists an element $a \in \mathfrak{R}$ such that $\Delta(x) = ax - xa$, $\Delta(y) = ay - ya$.

Let \mathfrak{R} be a Lie ring with a Lie multiplication $[\cdot, \cdot]$. Recall that an additive map $D: \mathfrak{R} \rightarrow \mathfrak{R}$ is called a *derivation*, if $D([x, y]) = [D(x), y] + [x, D(y)]$ for any two elements $x, y \in \mathfrak{R}$.

Given an element a in \mathfrak{R} , the map $R_a(x) = [a, x]$, $x \in \mathfrak{R}$ is a *Lie derivation*. Such a derivation is called an *inner derivation* of \mathfrak{R} . A map Δ is called a *2-local inner derivation*, if for each pair of elements $x, y \in \mathfrak{R}$ there is an inner derivation R_a of \mathfrak{R} such that $\Delta(x) = R_a(x)$, $\Delta(y) = R_a(y)$.

Let \mathcal{A} be a unital associative ring. The vector space \mathcal{A} with respect to the Lie multiplication $[a, b] = ab - ba$, where $a, b \in \mathcal{A}$ is a Lie ring. This Lie ring will be denoted by $(\mathcal{A}, [,])$. For $a \in \mathcal{A}$ the map

$$R_a(x) = [a, x], \quad x \in \mathcal{A}.$$

is a derivation on \mathcal{A} both as an associative ring and as a Lie ring. Every inner derivation of the Lie ring $(\mathcal{A}, [,])$ is an inner derivation of the associative ring \mathcal{A} . And also every inner derivation $R_a = ax - xa$, $x \in \mathcal{A}$ is an inner derivation of the Lie ring $(\mathcal{A}, [,])$. Thus it is clear that every 2-local inner derivation of the Lie ring $(\mathcal{A}, [,])$ is a 2-local inner derivation of the associative ring \mathcal{A} . And vice-versa, every 2-local inner derivation of the associative ring \mathcal{A} is a 2-local inner derivation of the Lie ring $(\mathcal{A}, [,])$.

Definition 1.1. A **-ring* (or *involutive ring*, or *ring with involution*) is an associative ring with an involution $x \rightarrow x^*$:

$$(x^*)^* = x, \quad (x + y)^* = x^* + y^*, \quad (xy)^* = y^*x^*.$$

When \mathcal{A} is also an algebra, over a field with involution $\lambda \rightarrow \lambda^*$ (the identity involution is allowed), we assume further that $(\lambda x)^* = \lambda^*x^*$ and call \mathcal{A} a **-algebra*. Elements such that $x^* = -x$ are called *skew-adjoint*. ■

Let, now, \mathcal{A} be a *-ring and \mathcal{A}_k be the set of all skew-adjoint elements of \mathcal{A} . Then $(\mathcal{A}_k, [,])$ is a Lie ring. We take $a \in \mathcal{A}_k$ and the inner derivation

$$R_a(x) = [a, x], x \in \mathcal{A}_k.$$

Then R_a may be considered as an inner derivation on \mathcal{A} . Therefore every inner derivation of the Lie ring $(\mathcal{A}_k, [,])$ is extended to an inner derivation of the *-ring \mathcal{A} . Concerning 2-local inner derivations, it is not possible, in general, to extend a 2-local inner derivations from the Lie ring $(\mathcal{A}_k, [,])$ to a 2-local inner derivation on the involutive ring \mathcal{A} . Therefore we have to give a straightforward proofs of the main results below.

2. 2-Local derivations on the Lie ring of skew-adjoint matrices over a commutative *-ring

Let \mathfrak{R} be a unital associative ring, $M_n(\mathfrak{R})$ be the matrix ring over \mathfrak{R} , $n > 1$. Let $\{e_{i,j}\}_{i,j=1}^n$ be the set of matrix units in $M_n(\mathfrak{R})$, i.e. $e_{i,j}$ is a matrix with components $a^{i,j} = \mathbf{1}$ and $a^{k,l} = \mathbf{0}$ if $(i, j) \neq (k, l)$, where $\mathbf{1}$ is the identity element, $\mathbf{0}$ is the zero element of \mathfrak{R} , and a matrix $a \in M_n(\mathfrak{R})$ is written as $a = \sum_{k,l=1}^n a^{k,l}e_{k,l}$, where $a^{k,l} \in \mathfrak{R}$ for $k, l = 1, 2, \dots, n$.

Let \mathfrak{R} be a commutative unital involutive ring, $M_n(\mathfrak{R})$ be the associative ring of $n \times n$ matrices over \mathfrak{R} , $n > 1$. In this case the vector space

$$K_n(\mathfrak{R}) = \{(a^{i,j})_{i,j=1}^n \in M_n(\mathfrak{R}) : (a^{i,j})^* = -a^{j,i}, i, j = 1, 2, \dots, n\}$$

is a Lie ring with respect to the Lie multiplication

$$[a, b] = ab - ba, a, b \in K_n(\mathfrak{R}).$$

Throughout this section, let $s_{i,j} = e_{i,j} - e_{j,i}$ for every pair of different indices i, j in $\{1, 2, \dots, n\}$. Further we suppose that the ring \mathfrak{R} contains an imaginary unit I , i.e. $I^2 = -e$, where e is an identity element of \mathfrak{R} .

Lemma 2.1. *Let Δ be a 2-local derivation on $K_n(\mathfrak{R})$ and fix arbitrary pairwise different indices i, j, p .*

(1) *Let R_a, R_b be the derivations on $K_n(\mathfrak{R})$, generated by elements $a, b \in H_n(\mathfrak{R})$ such that $\Delta(s_{i,j}) = R_a(s_{i,j}) = R_b(s_{i,j})$. Then the following equalities are valid*

$$a^{i,j} + a^{j,i} = b^{i,j} + b^{j,i}, \quad a^{i,i} - a^{j,j} = b^{i,i} - b^{j,j}.$$

(2) *Let R_a, R_b be the derivations on $K_n(\mathfrak{R})$, generated by elements $a, b \in K_n(\mathfrak{R})$ such that $\Delta(s_{i,j}) = R_a(s_{i,j}), \Delta(s_{i,p}) = R_b(s_{i,p})$. Then the following equality is valid:*

$$a^{i,j} + a^{j,i} = b^{i,j} + b^{j,i}.$$

Proof. (1) From $R_a(s_{i,j}) = R_b(s_{i,j})$ it follows that $as_{i,j} - s_{i,j}a = bs_{i,j} - s_{i,j}b$.

Hence,
$$e_{j,j}as_{i,j}e_{j,j} - e_{j,j}s_{i,j}ae_{j,j} = e_{j,j}bs_{i,j}e_{j,j} - e_{j,j}s_{i,j}be_{j,j}$$

and
$$e_{j,j}ae_{i,j}e_{j,j} + e_{j,j}e_{j,i}ae_{j,j} = e_{j,j}be_{i,j}e_{j,j} + e_{j,j}e_{j,i}be_{j,j},$$

and
$$a^{j,i}e_{j,j} + a^{i,j}e_{j,j} = b^{j,i}e_{j,j} + b^{i,j}e_{j,j}.$$

Also,
$$e_{i,i}as_{i,j}e_{j,j} - e_{i,i}s_{i,j}ae_{j,j} = e_{i,i}bs_{i,j}e_{j,j} - e_{i,i}s_{i,j}be_{j,j}$$

and
$$e_{i,i}ae_{i,j}e_{j,j} - e_{i,i}e_{i,j}ae_{j,j} = e_{i,i}be_{i,j}e_{j,j} - e_{i,i}e_{i,j}be_{j,j},$$

and
$$a^{i,i}e_{i,j} - a^{j,j}e_{i,j} = b^{i,i}e_{i,j} - b^{j,j}e_{i,j}.$$

This ends the proof of (1).

(2) There exists $x \in K_n(\mathfrak{R})$ such that $\Delta(s_{i,j}) = R_x(s_{i,j}), \Delta(s_{i,p}) = R_x(s_{i,p})$.

We have $as_{i,j} - s_{i,j}a = xs_{i,j} - s_{i,j}x$, and $bs_{i,p} - s_{i,p}b = xs_{i,p} - s_{i,p}x$.

By assertion (1) of this lemma we obtain $a^{i,j} + a^{j,i} = x^{i,j} + x^{j,i}$.

We also have
$$e_{p,p}(bs_{i,p} - s_{i,p}b)e_{j,j} = e_{p,p}(xs_{i,p} - s_{i,p}x)e_{j,j},$$

and
$$e_{p,i}be_{j,j} = e_{p,i}xe_{j,j}, b^{i,j}e_{p,j} = x^{i,j}e_{p,j},$$

i.e. $b^{i,j} = x^{i,j}$. Similarly we have $b^{j,i} = x^{j,i}$. Hence $a^{i,j} + a^{j,i} = b^{i,j} + b^{j,i}$. ■

Lemma 2.2. *Let Δ be a 2-local derivation on $K_n(\mathfrak{R})$ and, for arbitrary pairwise different indices i, j, p , let R_a, R_b be the inner derivations on $K_n(\mathfrak{R})$, generated by elements $a, b \in K_n(\mathfrak{R})$ respectively, such that*

$$\Delta(s_{i,p}) = R_a(s_{i,p}), \quad \Delta(s_{p,j}) = R_b(s_{p,j}).$$

Then the following equalities hold: $e_{i,i}ae_{j,j} = e_{i,i}be_{j,j}, e_{j,j}ae_{i,i} = e_{j,j}be_{i,i}$.

Proof. There exists $x \in K_n(\mathfrak{R})$ such that

$$\Delta(s_{i,p}) = R_x(s_{i,p}), \quad \Delta(s_{p,j}) = R_x(s_{p,j}).$$

Then we obtain $as_{i,p} - s_{i,p}a = xs_{i,p} - s_{i,p}x$, $bs_{p,j} - s_{p,j}b = xs_{p,j} - s_{p,j}x$, $e_{j,j}ae_{i,p} = e_{j,j}xe_{i,p}$, $e_{p,i}ae_{j,j} = e_{p,i}xe_{j,j}$, $e_{i,i}be_{j,p} = e_{i,i}xe_{j,p}$, $e_{p,j}be_{i,i} = e_{p,j}xe_{i,i}$.

Hence $e_{j,j}ae_{i,i} = e_{j,j}xe_{i,i}$, $e_{i,i}ae_{j,j} = e_{i,i}xe_{j,j}$, $e_{i,i}be_{j,j} = e_{i,i}xe_{j,j}$, $e_{j,j}be_{i,i} = e_{j,j}xe_{i,i}$, and $e_{j,j}ae_{i,i} = e_{j,j}be_{i,i}$, $e_{i,i}ae_{j,j} = e_{i,i}be_{j,j}$. ■

Let Δ be a 2-local derivation on $K_n(\mathfrak{R})$ and, for arbitrary pairwise different indices i, j, p , let R_a be the derivation on $K_n(\mathfrak{R})$, generated by an element $a \in K_n(\mathfrak{R})$, such that $\Delta(s_{i,p}) = R_a(s_{i,p})$ and $\Delta(s_{p,j}) = R_a(s_{p,j})$.

By Lemma 2.2 the following elements are well-defined

$$a_{i,j} = e_{i,i}ae_{j,j} = a^{i,j}e_{i,j}, \quad a^{i,j} \in \mathfrak{R}, \quad \text{and} \quad a_{j,i} = e_{j,j}ae_{i,i} = a^{j,i}e_{j,i}, \quad a^{j,i} \in \mathfrak{R},$$

$$a = \sum_{i,j=1, i \neq j}^n a_{i,j}.$$

Using this notation the following lemma is valid.

Lemma 2.3. *Let \mathfrak{R} be a commutative unital involutive ring. Let Δ be a 2-local inner derivation on $K_n(\mathfrak{R})$. For arbitrary but fixed different indices i, j let*

$$\Delta(s_{i,j}) = R_d(s_{i,j})$$

for an appropriate element $d \in K_n(\mathfrak{R})$. Then

$$\Delta(s_{i,j}) = as_{i,j} - s_{i,j}a + (d^{i,i} - d^{j,j})e_{i,j} + (d^{j,j} - d^{i,i})e_{j,i}.$$

Proof. By (2) of Lemma 2.1 we have

$$\begin{aligned} \Delta(s_{i,j}) &= ds_{i,j} - s_{i,j}d = (1 - e_{i,i} - e_{j,j})ds_{i,j} - s_{i,j}d(1 - e_{i,i} - e_{j,j}) \\ &\quad + (e_{i,i} + e_{j,j})ds_{i,j} - s_{i,j}d(e_{i,i} + e_{j,j}) \\ &= (1 - e_{i,i} - e_{j,j})de_{i,j} - e_{i,j}d(1 - e_{i,i} - e_{j,j}) - (1 - e_{i,i} - e_{j,j})de_{j,i} \\ &\quad + e_{j,i}d(1 - e_{i,i} - e_{j,j}) + (e_{i,i} + e_{j,j})ds_{i,j} - s_{i,j}d(e_{i,i} + e_{j,j}) \\ &= (1 - e_{i,i} - e_{j,j})ae_{i,j} - e_{i,j}a(1 - e_{i,i} - e_{j,j}) - (1 - e_{i,i} - e_{j,j})ae_{j,i} \\ &\quad + e_{j,i}a(1 - e_{i,i} - e_{j,j}) + (e_{i,i} + e_{j,j})ds_{i,j} - s_{i,j}d(e_{i,i} + e_{j,j}). \end{aligned}$$

At the same time we get

$$\begin{aligned} &(e_{i,i} + e_{j,j})ds_{i,j} - s_{i,j}d(e_{i,i} + e_{j,j}) \\ &= e_{i,i}de_{i,j} - e_{i,i}de_{j,i} + e_{j,j}de_{i,j} - e_{j,j}de_{j,i} - e_{i,j}de_{i,i} + e_{j,i}de_{i,i} - e_{i,j}de_{j,j} + e_{j,i}de_{j,j} \\ &= (d^{i,i} - d^{j,j})e_{i,j} + (-d^{i,j} - d^{j,i})e_{i,i} + (d^{j,i} + d^{i,j})e_{j,j} + (d^{i,i} - d^{j,j})e_{j,i} \\ &= (d^{i,i} - d^{j,j})e_{i,j} + (-a^{i,j} - a^{j,i})e_{i,i} + (a^{j,i} + a^{i,j})e_{j,j} + (d^{i,i} - d^{j,j})e_{j,i} \\ &= (d^{i,i} - d^{j,j})e_{i,j} - a_{i,j}e_{j,i} - e_{i,j}a_{j,i} + a_{j,i}e_{i,j} + e_{j,i}a_{i,j} + (d^{i,i} - d^{j,j})e_{j,i} \end{aligned}$$

by (1) of Lemma 2.1. Hence

$$\begin{aligned} \Delta(s_{i,j}) &= ae_{i,j} - e_{i,j}a - ae_{j,i} + e_{j,i}a + (d^{i,i} - d^{j,j})e_{i,j} + (d^{j,j} - d^{i,i})e_{j,i} \\ &= as_{i,j} - s_{i,j}a + (d^{i,i} - d^{j,j})e_{i,j} + (d^{j,j} - d^{i,i})e_{j,i}. \end{aligned}$$

This ends the proof. ■

Let \mathfrak{R} be a unital involutive ring and, let Δ be a 2-local inner derivation on $K_n(\mathfrak{R})$. Consider the element $x_o = \sum_{k=1}^{n-1} s_{k,k+1} \in K_n(\mathfrak{R})$. Fix different indices i_o, j_o .

Let $c \in K_n(\mathfrak{R})$ be an element such that

$$\Delta(s_{i_o, j_o}) = [c, s_{i_o, j_o}], \Delta(x_o) = [c, x_o].$$

Define $c = \sum_{i,j=1}^n c^{i,j} e_{i,j} \in K_n(\mathfrak{R})$ and also $\bar{a} = \sum_{i,j=1, i \neq j}^n a_{i,j} + \sum_{i=1}^n a_{i,i}$, where $a_{i,i} = c^{i,i} e_{i,i}$, $i = 1, 2, \dots, n$.

Lemma 2.4. *Let \mathfrak{R} be a unital involutive ring and, let Δ be a 2-local inner derivation on $K_n(\mathfrak{R})$. For arbitrary different indices k, l , let $b \in K_n(\mathfrak{R})$ be an element such that $\Delta(s_{k,l}) = [b, s_{k,l}]$ and $\Delta(x_o) = [b, x_o]$. Then $c^{k,k} - c^{l,l} = b^{k,k} - b^{l,l}$.*

Proof. We may assume that $k < l$. We have $\Delta(x_o) = [c, x_o] = [b, x_o]$. Hence

$$cx_o - x_o c = bx_o - x_o b, \quad e_{k,k}(cx_o - x_o c)e_{k+1, k+1} = e_{k,k}(bx_o - x_o b)e_{k+1, k+1}$$

and

$$c^{k,k} - c^{k+1, k+1} = b^{k,k} - b^{k+1, k+1}.$$

Then for the sequence $(k, k + 1), (k + 1, k + 2) \dots (l - 1, l)$ we obtain

$$\begin{aligned} c^{k,k} - c^{k+1, k+1} &= b^{k,k} - b^{k+1, k+1}, \quad c^{k+1, k+1} - c^{k+2, k+2} = b^{k+1, k+1} - b^{k+2, k+2}, \\ &\dots, \quad c^{l-1, l-1} - c^{l, l} = b^{l-1, l-1} - b^{l, l}. \end{aligned}$$

Hence we get

$$\begin{aligned} c^{k,k} - b^{k,k} &= c^{k+1, k+1} - b^{k+1, k+1}, \quad c^{k+1, k+1} - b^{k+1, k+1} = c^{k+2, k+2} - b^{k+2, k+2}, \\ &\dots, \quad c^{l-1, l-1} - b^{l-1, l-1} = c^{l, l} - b^{l, l}. \end{aligned}$$

Therefore $c^{k,k} - b^{k,k} = c^{l,l} - b^{l,l}$, i.e. $c^{k,k} - c^{l,l} = b^{k,k} - b^{l,l}$. The proof is complete. ■

Theorem 2.5. *Let \mathfrak{R} be a commutative unital involutive ring, and let $K_n(\mathfrak{R})$ be the Lie ring of skew-adjoint $n \times n$ matrices over \mathfrak{R} . Then any 2-local inner derivation on the matrix Lie ring $K_n(\mathfrak{R})$ is an inner derivation.*

Proof. We prove that each 2-local inner derivation Δ on $K_n(\mathfrak{R})$ satisfies the condition

$$\Delta(x) = R_{\bar{a}}(x) = \bar{a}x - x\bar{a}, \quad x \in K_n(\mathfrak{R}),$$

where the element \bar{a} is defined before Lemma 2.3. Let x be an arbitrary element in $K_n(\mathfrak{R})$ and, let $d(ij) \in K_n(\mathfrak{R})$ be an element such that

$$\Delta(s_{i,j}) = R_{d(ij)}(s_{i,j}), \Delta(x) = R_{d(ij)}(x)$$

and $i \neq j$. Then $\Delta(s_{i,j}) = D_{d(ij)}(s_{i,j}), \Delta(x) = D_{d(ij)}(x)$, i.e.,

$$\Delta(s_{i,j}) = d(ij)(s_{i,j}) - (s_{i,j})d(ij) \quad \text{and} \quad \Delta(x) = d(ij)x - xd(ij).$$

By Lemma 2.3 we have the following equalities

$$\begin{aligned} \Delta(s_{i,j}) &= d(ij)(s_{i,j}) - (s_{i,j})d(ij) \\ &= (e_{i,i} + e_{j,j})d(ij)(s_{i,j}) - (s_{i,j})d(ij)(e_{i,i} + e_{j,j}) \\ &\quad + (1 - (e_{i,i} + e_{j,j}))d(ij)(s_{i,j}) - (s_{i,j})d(ij)(1 - (e_{i,i} + e_{j,j})) \\ &= as_{i,j} - s_{i,j}a + (d(ij)^{i,i} - d(ij)^{j,j})e_{i,j} + (d(ij)^{j,j} - d(ij)^{i,i})e_{j,i}. \end{aligned}$$

for all different indices i, j .

From this equality we get

$$(e_{i,i} + e_{j,j})d(ij)(s_{i,j}) - (s_{i,j})d(ij)(e_{i,i} + e_{j,j}) \\ = (a_{i,j} + a_{j,i})s_{i,j} - s_{i,j}(a_{i,j} + a_{j,i}) + (d(ij)^{i,i} - d(ij)^{j,j})e_{i,j} + (d(ij)^{j,j} - d(ij)^{i,i})e_{j,i}.$$

Hence,

$$(1 - (e_{i,i} + e_{j,j}))d(ij)e_{i,i} = (1 - (e_{i,i} + e_{j,j}))ae_{i,i}, \\ e_{i,i}d(ij)(1 - (e_{i,i} + e_{j,j})) = e_{i,i}a(1 - (e_{i,i} + e_{j,j})), \\ e_{j,j}d(ij)(1 - (e_{i,i} + e_{j,j})) = e_{j,j}a(1 - (e_{i,i} + e_{j,j})), \\ (1 - (e_{i,i} + e_{j,j}))d(ij)e_{j,j} = (1 - (e_{i,i} + e_{j,j}))ae_{j,j}$$

for all different i and j . At the same time

$$e_{j,j}d(ij)e_{i,j} + e_{j,i}d(ij)e_{j,j} = (d(ij)^{j,i} + d(ij)^{i,j})e_{j,j} = (a^{j,i} + a^{i,j})e_{j,j} = a_{j,i}e_{i,j} + e_{j,i}a_{i,j}, \\ e_{i,i}d(ij)e_{j,i} + e_{i,j}d(ij)e_{i,i} = (d(ij)^{i,j} + d(ij)^{j,i})e_{i,i} = (a^{i,j} + a^{j,i})e_{i,i} = a_{i,j}e_{j,i} + e_{i,j}a_{j,i}$$

by Lemma 2.1, and

$$e_{j,j}d(ij)e_{i,j} + e_{j,i}d(ij)e_{j,j} = e_{j,j}ae_{i,j} + e_{j,i}ae_{j,j}, \\ e_{i,i}d(ij)e_{j,i} + e_{i,j}d(ij)e_{i,i} = e_{i,i}ae_{j,i} + e_{i,j}ae_{i,i}.$$

Therefore

$$(1 - e_{i,i})d(ij)e_{i,i} = (1 - e_{i,i})ae_{i,i}, \\ e_{j,j}d(ij)(1 - e_{j,j}) = e_{j,j}a(1 - e_{j,j}).$$

Similarly we have

$$e_{i,i}d(ij)(1 - e_{i,i}) = e_{i,i}a(1 - e_{i,i}), \\ (1 - e_{j,j})d(ij)e_{j,j} = (1 - e_{j,j})ae_{j,j}.$$

Let $v \in K_n(\mathfrak{R})$ be elements such that

$$\Delta(s_{i,j}) = vs_{i,j} - s_{i,j}v \quad \text{and} \quad \Delta(x_o) = vx_o - x_ov.$$

Let $v = \sum_{k,q=1}^n v^{k,q}e_{k,q}$. Then we get $v^{i,i} - v^{j,j} = c^{i,i} - c^{j,j}$ by Lemma 2.4. We have $v^{i,i} - v^{j,j} = d(ij)^{i,i} - d(ij)^{j,j}$ since $vs_{i,j} - s_{i,j}v = d(ij)s_{i,j} - s_{i,j}d(ij)$.

Hence $c^{i,i} - c^{j,j} = d(ij)^{i,i} - d(ij)^{j,j}$, $c^{j,j} - c^{i,i} = d(ij)^{j,j} - d(ij)^{i,i}$.

Therefore

$$e_{j,j}\Delta(x)e_{i,i} = e_{j,j}(d(ij)x - xd(ij))e_{i,i} \\ = e_{j,j}d(ij)(1 - e_{j,j})xe_{i,i} + e_{j,j}d(ij)e_{j,j}xe_{i,i} - e_{j,j}x(1 - e_{i,i})d(ij)e_{i,i} - e_{j,j}xe_{i,i}d(ij)e_{i,i} \\ = e_{j,j} \sum_{\xi,\eta=1, \xi \neq \eta}^n a^{\xi,\eta}e_{\xi,\eta}xe_{i,i} - e_{j,j}x \sum_{\xi,\eta=1, \xi \neq \eta}^n a^{\xi,\eta}e_{\xi,\eta}e_{i,i} + e_{j,j}d(ij)e_{j,j}xe_{i,i} - e_{j,j}xe_{i,i}d(ij)e_{i,i} \\ = e_{j,j} \sum_{\xi,\eta=1, \xi \neq \eta}^n a^{\xi,\eta}e_{\xi,\eta}xe_{i,i} - e_{j,j}x \sum_{\xi,\eta=1, \xi \neq \eta}^n a^{\xi,\eta}e_{\xi,\eta}e_{i,i} + c^{j,j}e_{j,j}xe_{i,i} - e_{j,j}xe_{i,i}c^{i,i}e_{i,i} \\ = e_{j,j} \sum_{\xi,\eta=1, \xi \neq \eta}^n a^{\xi,\eta}e_{\xi,\eta}xe_{i,i} - e_{j,j}x \sum_{\xi,\eta=1, \xi \neq \eta}^n a^{\xi,\eta}e_{\xi,\eta}e_{i,i} \\ + e_{j,j}(\sum_{\xi=1}^n a^{\xi,\xi}e_{\xi,\xi})xe_{i,i} - e_{j,j}x(\sum_{\xi=1}^n a^{\xi,\xi}e_{\xi,\xi})e_{i,i}$$

$$= e_{j,j} \sum_{\xi,\eta=1}^n a^{\xi,\eta} e_{\xi,\eta} x e_{i,i} - e_{j,j} x \sum_{\xi,\eta=1}^n a^{\xi,\eta} e_{\xi,\eta} e_{i,i} = e_{j,j} (\bar{a}x - x\bar{a}) e_{i,i}.$$

Let $d(ii)$, v , $w \in K_n(\mathfrak{R})$ be elements such that

$$\Delta(Ie_{i,i}) = d(ii)Ie_{i,i} - Ie_{i,i}d(ii) \quad \text{and} \quad \Delta(x) = d(ii)x - xd(ii),$$

$$\Delta(Ie_{i,i}) = vIe_{i,i} - Ie_{i,i}v, \quad \Delta(s_{i,j}) = vs_{i,j} - s_{i,j}v$$

and

$$\Delta(Ie_{i,i}) = wIe_{i,i} - Ie_{i,i}w, \quad \Delta(s_{i,j}) = w(s_{i,j}) - (s_{i,j})w.$$

Then

$$(1 - e_{i,i})ae_{i,i} = (1 - e_{i,i})ve_{i,i} = (1 - e_{i,i})d(ii)e_{i,i}$$

and

$$e_{i,i}a(1 - e_{i,i}) = e_{i,i}w(1 - e_{i,i}) = e_{i,i}d(ii)(1 - e_{i,i}).$$

By the properties of the Peirce components we have

$$(1 - e_{i,i})ae_{i,i} = \sum_{\xi,\eta=1, \xi \neq \eta}^n a_{\xi,\eta} e_{i,i} \quad \text{and} \quad e_{i,i}a(1 - e_{i,i}) = e_{i,i} \sum_{\xi,\eta=1, \xi \neq \eta}^n a_{\xi,\eta}.$$

Hence,

$$\begin{aligned} e_{i,i}\Delta(x)e_{i,i} &= e_{i,i}(d(ii)x - xd(ii))e_{i,i} \\ &= e_{i,i}d(ii)(1 - e_{i,i})xe_{i,i} + e_{i,i}d(ii)e_{i,i}xe_{i,i} - e_{i,i}x(1 - e_{i,i})d(ii)e_{i,i} - e_{i,i}xe_{i,i}d(ii)e_{i,i} \\ &= e_{i,i}a(1 - e_{i,i})xe_{i,i} + e_{i,i}d(ii)e_{i,i}xe_{i,i} - e_{i,i}x(1 - e_{i,i})ae_{i,i} - e_{i,i}xe_{i,i}d(ii)e_{i,i} \\ &= e_{i,i} \sum_{\xi,\eta=1, \xi \neq \eta}^n a^{\xi,\eta} e_{\xi,\eta} x e_{i,i} - e_{i,i} x \sum_{\xi,\eta=1, \xi \neq \eta}^n a^{\xi,\eta} e_{\xi,\eta} e_{i,i} + e_{i,i} d(ii) e_{i,i} x e_{i,i} - e_{i,i} x e_{i,i} d(ii) e_{i,i} \\ &= e_{i,i} \sum_{\xi,\eta=1, \xi \neq \eta}^n a^{\xi,\eta} e_{\xi,\eta} x e_{i,i} - e_{i,i} x \sum_{\xi,\eta=1, \xi \neq \eta}^n a^{\xi,\eta} e_{\xi,\eta} e_{i,i} + c^{i,i} e_{i,i} x e_{i,i} - e_{i,i} x c^{i,i} e_{i,i} \\ &= e_{i,i} \sum_{\xi,\eta=1, \xi \neq \eta}^n a^{\xi,\eta} e_{\xi,\eta} x e_{i,i} - e_{i,i} x \sum_{\xi,\eta=1, \xi \neq \eta}^n a^{\xi,\eta} e_{\xi,\eta} e_{i,i} \\ &\quad + e_{i,i} \left(\sum_{\xi=1}^n a^{\xi,\xi} e_{\xi,\xi} \right) x e_{i,i} - e_{i,i} x \left(\sum_{\xi=1}^n a^{\xi,\xi} e_{\xi,\xi} \right) e_{i,i} \\ &= e_{i,i} \sum_{\xi,\eta=1}^n a^{\xi,\eta} e_{\xi,\eta} x e_{i,i} - e_{i,i} x \sum_{\xi,\eta=1}^n a^{\xi,\eta} e_{\xi,\eta} e_{i,i} = e_{i,i} (\bar{a}x - x\bar{a}) e_{i,i}. \end{aligned}$$

From this it follows that $\Delta(x) = \bar{a}x - x\bar{a}$ for each $x \in K_n(\mathfrak{R})$.

Therefore Δ is an inner derivation on $K_n(\mathfrak{R})$. The proof is complete. ■

Let $(\mathcal{L}, [,])$ be a Lie algebra. Recall that a linear map $D: \mathcal{L} \rightarrow \mathcal{L}$ is called a derivation, if $D([x, y]) = [D(x), y] + [x, D(y)]$ for any two elements $x, y \in \mathcal{L}$. A derivation D on \mathcal{L} is called an inner derivation, if there exists an element $a \in \mathcal{L}$ such that

$$D(x) = [a, x], \quad x \in \mathcal{L}.$$

A map $\Delta: \mathcal{L} \rightarrow \mathcal{L}$ (not linear in general) is called a 2-local inner derivation, if for any pair of elements $x, y \in \mathcal{L}$ there exists an element $a \in \mathcal{L}$ such that $\Delta(x) = [a, x]$, $\Delta(y) = [a, y]$.

Now, a $*$ -algebra \mathcal{A} is a $*$ -ring, with involution $*$ that is an associative algebra over a commutative $*$ -ring \mathfrak{R} with involution $'$, such that $(rx)^* = r'x^*$ for any $r \in \mathfrak{R}$, $x \in \mathcal{A}$. By Theorem 2.5 we have the following statement.

Theorem 2.6. *Let \mathcal{A} be a commutative unital involutive algebra, and let $K_n(\mathcal{A})$, $n > 1$, be the Lie algebra of all skew-adjoint $n \times n$ matrices over \mathcal{A} . Then any 2-local inner derivation on the matrix Lie algebra $K_n(\mathcal{A})$ is an inner derivation.*

3. 2-local derivations on Lie algebras of skew-adjoint operator-valued maps

Throughout the rest of the paper, let H be the infinite-dimensional separable complex Hilbert space and, let $B(H)$ be the von Neumann algebra of all bounded linear operators on H . Let $\{e_i\}_{i=1}^\infty$ be a maximal family of orthogonal minimal projections in $B(H)$ and, let $\{e_{i,j}\}_{i,j=1}^\infty$ be the family of matrix units defined by $\{e_i\}_{i=1}^\infty$, i.e. $e_{i,i} = e_i$, $e_{i,i} = e_{i,j}e_{j,i}$ and $e_{j,j} = e_{j,i}e_{i,j}$ for each pair i, j of natural numbers. Let $B_{sk}(H)$ be the vector space of all skew-adjoint operators in $B(H)$, i.e.

$$B_{sk}(H) = \{a \in B(H) : a^* = -a\}.$$

Then with respect to the Lie multiplication $[a, b] = ab - ba$, $a, b \in B_{sk}(H)$, $B_{sk}(H)$ is a Lie algebra.

Let, throughout the rest of the paper, Ω be an arbitrary set, $F(\Omega, B_{sk}(H))$ be the Lie algebra of all maps from Ω to $B_{sk}(H)$ with the Lie multiplication

$$[a, b](t) = [a(t), b(t)], t \in \Omega, a, b \in F(\Omega, B_{sk}(H)).$$

Put $\hat{e}_{i,j} = \mathbf{1}e_{i,j}$,

where $\mathbf{1}$ is the identity element of the algebra $F(\Omega)$ of all complex-valued maps on Ω . Throughout the rest sections, put $s_{i,j} = \hat{e}_{i,j} - \hat{e}_{j,i}$ for every pair of different natural numbers i, j , and, let I be the imaginary unit. Let (λ_n) be a sequence of nonzero numbers from \mathbb{C} such that

$$\sum_{n=1}^\infty \lambda_n \lambda_n^* < \infty \quad \text{and} \quad x_o = \sum_{n=1}^\infty \lambda_n s_{n,n+1} \in F(\Omega, B_{sk}(H)).$$

Let \mathcal{L} be a Lie subalgebra of $F(\Omega, B_{sk}(H))$ containing the element x_o and the family $\{I\hat{e}_{i,i}\}_{i=1}^\infty \cup \{s_{i,j}\}_{i,j=1}^\infty$.

Definition 3.1. Let A be a Lie algebra and, let B be a Lie subalgebra of A . A derivation D on B is said to be *spatial*, if D is implemented by an element in A , i.e.

$$D(x) = [a, x], x \in B,$$

for some $a \in A$. A 2-local derivation Δ on B is called 2-local spatial derivation implemented by elements from A , if for every two elements $x, y \in B$ there exists an element $a \in A$ such that $\Delta(x) = [a, x]$, $\Delta(y) = [a, y]$. ■

Then the following lemmas hold.

Lemma 3.2. *Let Δ be a 2-local spatial derivation on \mathcal{L} implemented by elements from $F(\Omega, B_{sk}(H))$ and i, j, p be arbitrary pairwise different natural numbers,*

(1) *Let R_a, R_b be the spatial derivations on \mathcal{L} , implemented by $a, b \in F(\Omega, B_{sk}(H))$, respectively, such that $\Delta(s_{i,j}) = R_a(s_{i,j}) = R_b(s_{i,j})$.*

Then the following equalities are valid

$$a^{i,j} + a^{j,i} = b^{i,j} + b^{j,i}, \quad a^{i,i} - a^{j,j} = b^{i,i} - b^{j,j}.$$

(2) let R_a, R_b be the spatial derivations on \mathcal{L} , implemented by $a, b \in F(\Omega, B_{sk}(H))$, respectively, such that $\Delta(s_{i,j}) = R_a(s_{i,j})$ and $\Delta(s_{i,p}) = R_b(s_{i,p})$.

Then the following equality is valid $a^{i,j} + a^{j,i} = b^{i,j} + b^{j,i}$.

Proof. The proof of this lemma is similar to the proof of lemma 2.1. ■

Lemma 3.3. Let Δ be a 2-local spatial derivation on \mathcal{L} implemented by elements from $F(\Omega, B_{sk}(H))$, i, j, p be arbitrary pairwise different natural numbers and, let R_a, R_b be the derivations on \mathcal{L} , generated by elements $a, b \in F(\Omega, B_{sk}(H))$ such that

$$\Delta(s_{i,p}) = R_a(s_{i,p}), \quad \Delta(s_{p,j}) = R_b(s_{p,j}).$$

Then the following equalities hold: $\hat{e}_{i,i}a\hat{e}_{j,j} = \hat{e}_{i,i}b\hat{e}_{j,j}$, $\hat{e}_{j,j}a\hat{e}_{i,i} = \hat{e}_{j,j}b\hat{e}_{i,i}$.

Proof. The proof of this lemma is similar to the proof of lemma 2.2. ■

Let Δ be a 2-local spatial derivation on \mathcal{L} , implemented by elements from the Lie algebra $F(\Omega, B_{sk}(H))$, i, j, p be arbitrary pairwise different natural numbers, and, let R_a be the derivation on \mathcal{L} , generated by an element $a \in F(\Omega, B_{sk}(H))$ such that

$$\Delta(s_{i,p}) = R_a(s_{i,p}), \quad \Delta(s_{p,j}) = R_a(s_{p,j}).$$

By Lemma 3.3 the following elements are well-defined:

$$a_{i,j} = \hat{e}_{i,i}a\hat{e}_{j,j} = a^{i,j}\hat{e}_{i,j}, \quad a^{i,j} \in \mathfrak{R}, \quad a_{j,i} = \hat{e}_{j,j}a\hat{e}_{i,i} = a^{j,i}\hat{e}_{j,i}, \quad a^{j,i} \in \mathfrak{R},$$

$$a = \sum_{i,j=1, i \neq j}^{\infty} a_{i,j}.$$

Using this notation the following lemma is valid.

Lemma 3.4. Let Δ be a 2-local spatial derivation on \mathcal{L} implemented by elements from $F(\Omega, B_{sk}(H))$, and let i, j be arbitrary different natural numbers and

$$\Delta(s_{i,j}) = R_d(s_{i,j})$$

for some element $d \in K_n(\mathfrak{R})$. Then

$$\Delta(s_{i,j}) = as_{i,j} - s_{i,j}a + (d^{i,i} - d^{j,j})\hat{e}_{i,j} + (d^{j,j} - d^{i,i})\hat{e}_{j,i}.$$

Proof. The proof of this lemma is similar to the proof of Lemma 2.3. ■

Fix different natural numbers i_o, j_o . Let $c \in F(\Omega, B_{sk}(H))$ be an element such that

$$\Delta(s_{i_o j_o}) = [c, s_{i_o j_o}], \quad \Delta(x_o) = [c, x_o].$$

Put $c = \sum_{i,j=1}^{\infty} c^{i,j}\hat{e}_{i,j} \in F(\Omega, B_{sk}(H))$ and $\bar{a} = \sum_{i,j=1, i \neq j}^{\infty} a_{i,j} + \sum_{i=1}^{\infty} a_{i,i}$, where $a_{i,i} = c^{i,i}\hat{e}_{i,i}$ for any i . Then $\bar{a} \in F(\Omega, B_{sk}(H))$. In the above notations we have

Lemma 3.5. *Let k, l be arbitrary different natural numbers, and let d be an element in $F(\Omega, B_{sk}(H))$ such that $\Delta(s_{k,l}) = [d, s_{k,l}]$, $\Delta(x_o) = [d, x_o]$. Then $c^{k,k} - c^{l,l} = d^{k,k} - d^{l,l}$.*

Proof. We may assume that $k < l$. We have

$$\Delta(x_o) = cx_o - x_o c = dx_o - x_o d.$$

Hence,

$$\begin{aligned} e_{k,k}(cx_o - x_o c)e_{k+1,k+1} &= e_{k,k}(dx_o - x_o d)e_{k+1,k+1}, \\ e_{k,k}(c\lambda_k s_{k,k+1} - \lambda_k s_{k,k+1}c)e_{k+1,k+1} &= e_{k,k}(d\lambda_k s_{k,k+1} - \lambda_k s_{k,k+1}d)e_{k+1,k+1}, \\ \lambda_k e_{k,k}c\hat{e}_{k,k+1} - \lambda_k \hat{e}_{k,k+1}ce_{k+1,k+1} &= \lambda_k e_{k,k}d\hat{e}_{k,k+1} - \lambda_k \hat{e}_{k,k+1}de_{k+1,k+1} \end{aligned}$$

and

$$c^{k,k} - c^{k+1,k+1} = d^{k,k} - d^{k+1,k+1}.$$

Thus, this lemma is proved similarly to the proof of Lemma 2.4. ■

Now we prove the key theorem of this section.

Theorem 3.6. *Let Ω be an arbitrary set and let $F(\Omega, B_{sk}(H))$ be the Lie algebra of all maps from Ω to $B_{sk}(H)$. Let \mathcal{L} be a Lie subalgebra of $F(\Omega, B_{sk}(H))$ containing the element x_o and the family $\{I\hat{e}_{i,i}\}_{i=1}^\infty \cup \{s_{i,j}\}_{i,j=1, i \neq j}^\infty$. Then any 2-local spatial derivation on \mathcal{L} implemented by elements from $F(\Omega, B_{sk}(H))$ is a spatial derivation.*

Proof. We prove that each 2-local inner derivation Δ on \mathcal{L} satisfies the condition

$$\Delta(x) = R_{\bar{a}}(x) = \bar{a}x - x\bar{a}, \quad x \in \mathcal{L}$$

for the element $\bar{a} \in F(\Omega, B_{sk}(H))$ defined above. Let i, j be arbitrary distinct natural numbers. The proofs of the equalities

$$\hat{e}_{j,j}\Delta(x)\hat{e}_{i,i} = \hat{e}_{j,j}(\bar{a}x - x\bar{a})\hat{e}_{i,i}, \quad \hat{e}_{i,i}\Delta(x)\hat{e}_{i,i} = \hat{e}_{i,i}(\bar{a}x - x\bar{a})\hat{e}_{i,i}$$

are the same as the proofs of the appropriate equalities in Theorem 2.5. If, for arbitrary elements y, z in $F(\Omega, B_{sk}(H))$,

$$\hat{e}_{\xi,\xi}y\hat{e}_{\eta,\eta} = \hat{e}_{\xi,\xi}z\hat{e}_{\eta,\eta}, \quad \xi, \eta = 1, 2, 3, \dots,$$

then $y = z$. Hence, $\Delta(x) = \bar{a}x - x\bar{a} \in F(\Omega, B_{sk}(H))$.

But, $\Delta(x) \in \mathcal{L}$, since $x \in \mathcal{L}$. Therefore $\Delta(x) = \bar{a}x - x\bar{a} \in \mathcal{L}$ for any element $x \in \mathcal{L}$. So, Δ is a spatial derivation on \mathcal{L} . This ends the proof. ■

In particular, as a corollary of Theorem 3.6 we have the following theorem.

Theorem 3.7. *Each 2-local inner derivation on the Lie algebra $F(\Omega, B_{sk}(H))$ is an inner derivation.*

Let Ω be a hyperstonean compact, $C(\Omega)$ denotes the algebra of all \mathbb{C} -valued continuous maps on Ω . There exists a subalgebra \mathcal{N} in $F(\Omega, B(H))$, containing the family $\{\hat{e}_{i,j}\}_{i,j=1}^\infty$, which is a von Neumann algebra with the center isomorphic to $C(\Omega)$ (see [1, Page 12]). More precisely \mathcal{N} is a von Neumann algebra of type I.

The vector space $\mathcal{K} = \{a \in \mathcal{N} : a^* = -a\}$ of all skew-adjoint elements in \mathcal{N} is a Lie algebra with respect to the Lie multiplication

$$[a, b] = ab - ba, \quad a, b \in \mathcal{K}.$$

The Lie algebra \mathcal{K} is a Lie subalgebra of $F(\Omega, B_{sk}(H))$ containing the element x_o and the family $\{I\hat{e}_{i,i}\}_{i \in \mathbb{N}}^\infty \cup \{s_{i,j}\}_{i,j=1, i \neq j}^\infty$. So, by Theorem 2.1, we have the following theorem.

Theorem 3.8. *Every 2-local spatial derivation Δ on \mathcal{K} implemented by elements from $F(\Omega, B_{sk}(H))$ is a spatial derivation, i.e., there exists an element $a \in F(\Omega, B_{sk}(H))$ such that $\Delta(x) = R_a(x)$, $x \in \mathcal{K}$.*

Remark 3.9. Note that it is not necessary that for each pair of elements $x \in \mathcal{K}$ and $y \in F(\Omega, B_{sk}(H))$ the elements yx , xy , $R_x(y)$ belong to \mathcal{K} . But, by theorem 3.6, in theorem 3.8, for every $x \in \mathcal{K}$, the element $R_a(x)$ belongs to \mathcal{K} .

Let $\mathcal{K}(H)$ be the C^* -algebra of all compact operators on the Hilbert space H over \mathbb{C} , $\mathcal{K}_K(H)$ be the Lie algebra of all skew-adjoint compact operators on the Hilbert space H . Let Ω be a topological space. Then the vector space $C(\Omega, \mathcal{K}_K(H))$ of all continuous maps from Ω to $\mathcal{K}_K(H)$ is a Lie subalgebra of $F(\Omega, B_{sk}(H))$ containing the element x_o and the family $\{I\hat{e}_{i,i}\}_{i=1}^\infty \cup \{s_{i,j}\}_{i,j=1, i \neq j}^\infty$. Therefore, by Theorem 3.6 we have the following result.

Theorem 3.10. *Every 2-local spatial derivation on the Lie algebra $C(\Omega, \mathcal{K}_K(H))$ implemented by elements from $F(\Omega, B_{sk}(H))$ is a spatial derivation, i.e., there exists an element $a \in F(\Omega, B_{sk}(H))$ such that $\Delta(x) = R_a(x)$, $x \in \mathcal{K}$.*

Remark 3.11. Note that, as in Remark 3.9, it is not necessary that for each pair of elements $x \in C(\Omega, \mathcal{K}_K(H))$ and $y \in F(\Omega, B_{sk}(H))$ the elements yx , xy , $R_x(y)$ belong to $C(\Omega, \mathcal{K}_K(H))$. But, by theorem 3.6, in theorem 3.10, for every $x \in C(\Omega, \mathcal{K}_K(H))$, the element $R_a(x)$ belongs to $C(\Omega, \mathcal{K}_K(H))$.

4. Local derivations on the Lie algebra of skew-adjoint operators on a complex Hilbert space

Let $(\mathcal{L}, [,])$ be a Lie algebra. A derivation D on \mathcal{L} is called an *inner derivation*, if there exists an element $a \in \mathcal{L}$ such that

$$D(x) = [a, x], \quad x \in \mathcal{L}.$$

A linear map $\Delta: \mathcal{L} \rightarrow \mathcal{L}$ is called a *local inner derivation*, if for any element $x \in \mathcal{L}$ there exists an element $a \in \mathcal{L}$ such that $\Delta(x) = [a, x]$.

Let ∇ be a local inner derivation on $B_{sk}(H)$. Let i, j be pairwise distinct natural numbers and $e = e_{i,i} + e_{j,j}$. We take the following map

$$\nabla_{i,j}(x) = e\nabla(x)e, \quad x \in eB_{sk}(H)e.$$

Then $\nabla_{i,j}$ is a local derivation on the subalgebra $eB_{sk}(H)e$, and, by [4], it is an inner derivation. Hence, there exists an element $a \in eB_{sk}(H)e$ such that

$$\nabla_{i,j}(x) = ax - xa, \quad x \in eB_{sk}(H)e.$$

Put $a_{i,j} = e_{i,i}ae_{j,j}$, $i, j = 1, 2, 3, \dots$

The set $\{a_{i,j}\}_{i,j=1}^\infty$ is well defined. Indeed, let $e_1 = \sum_{k=1}^m e_{i_k,j_k}$ for some subset $\{e_{i_k,j_l}\}_{k,l=1,2,\dots,m}$ of elements in $\{e_{\xi,\eta}\}_{\xi,\eta=1}^\infty$, including the elements $e_{i,i}$, $e_{j,j}$. Then, if b is an element in $e_1B_{sk}(H)e_1$ such that

$$e_1\nabla(x)e_1 = bx - xb, \quad x \in e_1B_{sk}(H)e_1,$$

then $a = ebe$, i.e.,

$$a_{i,i} = e_{i,i}be_{i,i}, \quad a_{i,j} = e_{i,i}be_{j,j}, \quad a_{j,i} = e_{j,j}be_{i,i}, \quad a_{j,j} = e_{j,j}be_{j,j}.$$

Lemma 4.1. *Let ∇ be a local inner derivation on $B_{sk}(H)$. Then for every pair of different natural numbers i, j*

- (1) $\bar{a} = \sum_{k=1}^\infty [(a_{i,k} + a_{j,k}) + (a_{k,i} + a_{k,j})] \in B_{sk}(H)$,
 - (2) $\nabla(Ie_{i,i}) = R_{\bar{a}}(Ie_{i,i})$, $\nabla(s_{i,j}) = R_{\bar{a}}(s_{i,j})$, $\nabla(I\bar{e}_{i,j}) = R_{\bar{a}}(I\bar{e}_{i,j})$, $\nabla(Ie_{j,j}) = R_{\bar{a}}(Ie_{j,j})$,
- where I is the imaginary unit and $\bar{e}_{i,j} = e_{i,j} + e_{j,i}$.

Proof. Let $d_{i,i}$, $d_{i,j}$, $\bar{d}_{i,j}$, $d_{j,j}$ be elements in $B_{sk}(H)$ such that

$$\begin{aligned} \nabla(Ie_{i,i}) &= d_{i,i}Ie_{i,i} - Ie_{i,i}d_{i,i}, & \nabla(s_{i,j}) &= d_{i,j}s_{i,j} - s_{i,j}d_{i,j}, \\ \nabla(I\bar{e}_{i,j}) &= \bar{d}_{i,j}I\bar{e}_{i,j} - I\bar{e}_{i,j}\bar{d}_{i,j}, & \nabla(Ie_{j,j}) &= d_{j,j}Ie_{j,j} - Ie_{j,j}d_{j,j}. \end{aligned}$$

Then, since

$$\begin{aligned} e_{k,k}(d_{i,i}e_{i,i} - e_{i,i}d_{i,i})e_{l,l} &= a_{k,i}e_{i,i}e_{l,l} - e_{k,k}e_{i,i}a_{i,l}, \\ e_{k,k}(d_{i,j}s_{i,j} - s_{i,j}d_{i,j})e_{l,l} &= (a_{k,i} + a_{k,j})s_{i,j}e_{l,l} - e_{k,k}s_{i,j}(a_{i,l} + a_{j,l}), \\ e_{k,k}(\bar{d}_{i,j}I\bar{e}_{i,j} - I\bar{e}_{i,j}\bar{d}_{i,j})e_{l,l} &= (a_{k,i} + a_{k,j})\bar{e}_{i,j}e_{l,l} - e_{k,k}\bar{e}_{i,j}(a_{i,l} + a_{j,l}), \\ e_{k,k}(d_{j,j}e_{j,j} - e_{j,j}d_{j,j})e_{l,l} &= a_{k,j}e_{j,j}e_{l,l} - e_{k,k}e_{j,j}a_{j,l} \end{aligned}$$

for every pair k, l of natural numbers, we have

$$\begin{aligned} e_{k,k}d_{i,i}e_{i,i} &= a_{k,i}, & e_{i,i}d_{i,i}e_{l,l} &= a_{i,l}, \\ e_{k,k}d_{j,j}e_{j,j} &= a_{k,j}, & e_{j,j}d_{j,j}e_{l,l} &= a_{j,l}, \end{aligned}$$

i.e., $\bar{a} \in B_{sk}(H)$, and

$$\begin{aligned} e_{k,k}(d_{i,i}e_{i,i} - e_{i,i}d_{i,i})e_{l,l} &= e_{k,k}(\bar{a}e_{i,i} - e_{i,i}\bar{a})e_{l,l}, \\ e_{k,k}(d_{i,j}s_{i,j} - s_{i,j}d_{i,j})e_{l,l} &= e_{k,k}(\bar{a}s_{i,j} - s_{i,j}\bar{a})e_{l,l}, \\ e_{k,k}(\bar{d}_{i,j}I\bar{e}_{i,j} - I\bar{e}_{i,j}\bar{d}_{i,j})e_{l,l} &= e_{k,k}(\bar{a}I\bar{e}_{i,j} - I\bar{e}_{i,j}\bar{a})e_{l,l}, \\ e_{k,k}(d_{j,j}e_{j,j} - e_{j,j}d_{j,j})e_{l,l} &= e_{k,k}(\bar{a}e_{j,j} - e_{j,j}\bar{a})e_{l,l} \end{aligned}$$

for every pair k, l of natural numbers. Hence,

$$\begin{aligned} \nabla(Ie_{i,i}) &= d_{i,i}Ie_{i,i} - Ie_{i,i}d_{i,i} = \bar{a}Ie_{i,i} - Ie_{i,i}\bar{a}, \\ \nabla(s_{i,j}) &= d_{i,j}s_{i,j} - s_{i,j}d_{i,j} = \bar{a}s_{i,j} - s_{i,j}\bar{a}, \\ \nabla(I\bar{e}_{i,j}) &= \bar{d}_{i,j}I\bar{e}_{i,j} - I\bar{e}_{i,j}\bar{d}_{i,j} = \bar{a}I\bar{e}_{i,j} - I\bar{e}_{i,j}\bar{a}, \\ \nabla(Ie_{j,j}) &= d_{j,j}Ie_{j,j} - Ie_{j,j}d_{j,j} = \bar{a}Ie_{j,j} - Ie_{j,j}\bar{a}. \end{aligned}$$

Theorem 4.2. *Every ultraweakly continuous local inner derivation on $B_{sk}(H)$ is an inner derivation.*

Proof. Let $\nabla: B_{sk}(H) \rightarrow B_{sk}(H)$ be a local inner derivation. Then there exist elements $a_{i,i} \in B_{sk}(H)$, $i = 1, 2, 3, \dots$ such that

$$\nabla(Ie_{i,i}) = a_{i,i}Ie_{i,i} - Ie_{i,i}a_{i,i}$$

for the elements $e_{i,i}$, $i = 1, 2, 3, \dots$. Let a_1 be an element in $B_{sk}(H)$ such that

$$\nabla(I\bar{e}_{i,k}) = a_1(I\bar{e}_{i,k}) - (I\bar{e}_{i,k})a_1$$

for $i, k = 1, 2, 3, \dots$. Then, by additivity of ∇ , we have

$$a_1(I\bar{e}_{i,k}) - (I\bar{e}_{i,k})a_1 = a_{i,i}Ie_{i,i} - Ie_{i,i}a_{i,i} + a_{k,k}Ie_{k,k} - Ie_{k,k}a_{k,k}.$$

From this equality it follows that $a_1^{i,k} - a_1^{k,i} = -a_{i,i}^{i,k} + a_{k,k}^{i,k}$.

Similarly $a_1^{k,i} - a_1^{i,k} = a_{i,i}^{k,i} - a_{k,k}^{k,i}$.

Hence $a_{i,i}^{i,k} = a_{k,k}^{i,k}$, $a_{i,i}^{k,i} = a_{k,k}^{k,i}$. (1)

Let $x_o = \sum_{k=1}^{\infty} (e_{k,k+1} - e_{k+1,k}) \in B_{sk}(H)$. Then $x_o \in B_{sk}(H)$ and there exists an element a_2 in $B_{sk}(H)$ such that

$$\nabla(x_o) = a_2x_o - x_oa_2. \tag{2}$$

We construct an element d as follows: its diagonal components are $d^{i,i} = a_2^{i,i}$, $i = 1, 2, 3, \dots$ and its nondiagonal components are $d^{i,j} = a_{i,i}^{i,j}$ when $i, j = 1, 2, 3, \dots$, $i \neq j$. By equalities (1) the element d is constructed correctly. It is clear that

$$\nabla(Ie_{i,i}) = a_{i,i}Ie_{i,i} - Ie_{i,i}a_{i,i} = dIe_{i,i} - Ie_{i,i}d \tag{3}$$

for $i = 1, 2, 3, \dots$. Now we prove that

$$\nabla(s_{i,k}) = d(s_{i,k}) - (s_{i,k})d \quad \text{and} \quad \nabla(I\bar{e}_{i,k}) = d(I\bar{e}_{i,k}) - (I\bar{e}_{i,k})d.$$

For this propose it is sufficient to prove that

$$\begin{aligned} \nabla(s_{i,k}) &= a_{i,k}(s_{i,k}) - (s_{i,k})a_{i,k} = d(s_{i,k}) - (s_{i,k})d, \\ \nabla(I\bar{e}_{i,k}) &= a_{i,k}(I\bar{e}_{i,k}) - (I\bar{e}_{i,k})a_{i,k} = d(I\bar{e}_{i,k}) - (I\bar{e}_{i,k})d. \end{aligned} \tag{4}$$

To prove the equalities (4) it is sufficient to show

$$a_{i,k}^{k,j} = d^{k,j}, \quad j \in \{1, \dots, n\} \setminus \{k\} \tag{5}$$

$$a_{i,k}^{j,i} = d^{j,i}, \quad j \in \{1, \dots, n\} \setminus \{i\} \tag{6}$$

$$a_{i,k}^{i,i} - a_{i,k}^{k,k} = d^{i,i} - d^{k,k}. \tag{7}$$

By the additivity of ∇ and Lemma 4.1 we have

$$\begin{aligned} a_3(Ie_{k,k} + (e_{k,i} - e_{i,k})) - (Ie_{k,k} + (e_{k,i} - e_{i,k}))a_3 \\ = dIe_{k,k} - Ie_{k,k}d + a_{i,k}(e_{k,i} - e_{i,k}) - (e_{k,i} - e_{i,k})a_{i,k}, \end{aligned} \tag{8}$$

$$a_3(Ie_{k,k} + I\bar{e}_{i,k}) - (Ie_{k,k} + I\bar{e}_{i,k})a_3 = dIe_{k,k} - Ie_{k,k}d + a_{i,k}I\bar{e}_{i,k} - I\bar{e}_{i,k}a_{i,k}. \quad (9)$$

$$a_3(Ie_{i,i} + (s_{i,k})) - (Ie_{i,i} + (s_{i,k}))a_3 = dIe_{i,i} - Ie_{i,i}d + a_{i,k}(s_{i,k}) - (s_{i,k})a_{i,k}, \quad (10)$$

$$a_3(Ie_{i,i} + I\bar{e}_{i,k}) - (Ie_{i,i} + I\bar{e}_{i,k})a_3 = dIe_{i,i} - Ie_{i,i}d + a_{i,k}I\bar{e}_{i,k} - I\bar{e}_{i,k}a_{i,k}. \quad (11)$$

Multiplying by $e_{j,j}$, $j \neq i$, $j \neq k$, the last equalities from the left side we have

$$\begin{aligned} Ia_3^{j,k}e_{j,k} + a_3^{j,k}e_{j,i} - a_3^{j,i}e_{j,k} &= Id^{j,k}e_{j,k} + a_{i,k}^{j,k}e_{j,i} - a_{i,k}^{j,i}e_{j,k}, \\ Ia_3^{j,k}e_{j,k} + Ia_3^{j,i}e_{j,k} + Ia_3^{j,k}e_{j,i} &= Id^{j,k}e_{j,k} + Ia_{i,k}^{j,i}e_{j,k} + Ia_{i,k}^{j,k}e_{j,i}, \\ Ia_3^{j,i}e_{j,i} + a_3^{j,i}e_{j,k} - a_3^{j,k}e_{j,i} &= Id^{j,i}e_{j,i} + a_{i,k}^{j,i}e_{j,k} - a_{i,k}^{j,k}e_{j,i}, \\ Ia_3^{j,i}e_{j,i} + Ia_3^{j,k}e_{j,i} + Ia_3^{j,i}e_{j,k} &= Id^{j,i}e_{j,i} + Ia_{i,k}^{j,k}e_{j,i} + Ia_{i,k}^{j,i}e_{j,k}. \end{aligned}$$

Hence,

$$\begin{aligned} a_3^{j,k} &= a_{i,k}^{j,k}, & a_3^{j,k} - a_3^{j,i} &= d^{j,k} - a_{i,k}^{j,i}, \\ a_3^{j,k} &= a_{i,k}^{j,k}, & a_3^{j,k} + a_3^{j,i} &= d^{j,k} + a_{i,k}^{j,i}, \\ a_3^{j,i} &= a_{i,k}^{j,i}, & a_3^{j,i} - a_3^{j,k} &= d^{j,i} - a_{i,k}^{j,k}, \\ a_3^{j,i} &= a_{i,k}^{j,i}, & a_3^{j,i} + a_3^{j,k} &= d^{j,i} + a_{i,k}^{j,k}, \end{aligned}$$

$$\text{i.e., } a_3^{j,k} = a_{i,k}^{j,k}, \quad a_3^{j,k} = d^{j,k}, \quad a_3^{j,i} = a_{i,k}^{j,i}, \quad a_3^{j,i} = d^{j,i}.$$

So, $a_{i,k}^{j,k} = d^{j,k}$, $a_{i,k}^{j,i} = d^{j,i}$, $j \neq i$, $j \neq k$.

Similarly, multiplying by $e_{j,j}$, $j \neq i$, $j \neq k$ the equalities (8)–(11) from the right side we have

$$a_{i,k}^{k,j} = d^{k,j}, \quad a_{i,k}^{i,j} = d^{i,j}, \quad j \neq i, \quad j \neq k.$$

Hence equalities (5), (6) are valid.

Similarly, multiplying by $e_{i,i}$ from the both right and left sides the equalities (8) and (9) we have

$$\begin{aligned} e_{i,i}a_3e_{k,i} + e_{i,k}a_3e_{i,i} &= e_{i,i}a_{i,k}e_{k,i} + e_{i,k}a_{i,k}e_{i,i}, \\ e_{i,i}a_3e_{k,i} - Ie_{i,k}a_3e_{i,i} &= e_{i,i}a_{i,k}Ie_{k,i} - Ie_{i,k}a_{i,k}e_{i,i}. \end{aligned}$$

$$\text{Hence, } a_3^{i,k} + a_3^{k,i} = a_{i,k}^{i,k} + a_{i,k}^{k,i} \quad \text{and} \quad Ia_3^{i,k} - Ia_3^{k,i} = a_{i,k}^{i,k}I - Ia_{i,k}^{k,i}$$

$$\text{and} \quad a_3^{i,k} = a_{i,k}^{i,k}, \quad a_3^{k,i} = a_{i,k}^{k,i}. \quad (12)$$

Multiplying the equalities (8) by $e_{i,i}$ from the left side and by $e_{k,k}$ from the right side we get

$$\begin{aligned} e_{i,i}a_3Ie_{k,k} - e_{i,i}a_3e_{i,k} + e_{i,k}a_3e_{k,k} &= e_{i,i}dIe_{k,k} - e_{i,i}a_{i,k}e_{i,k} + e_{i,k}a_{i,k}e_{k,k}, \\ e_{i,i}a_3Ie_{k,k} + Ie_{i,i}a_3e_{i,k} - Ie_{i,k}a_3e_{k,k} &= e_{i,i}dIe_{k,k} + e_{i,i}a_{i,k}Ie_{i,k} - Ie_{i,k}a_{i,k}e_{k,k}, \end{aligned}$$

Hence,

$$\begin{aligned} a_3^{i,k}Ie_{i,k} - a_3^{i,i}e_{i,k} + a_3^{k,k}e_{i,k} &= d^{i,k}Ie_{i,k} - a_{i,k}^{i,i}e_{i,k} + a_{i,k}^{k,k}e_{i,k}, \\ a_3^{i,k}Ie_{i,k} + Ia_3^{i,i}e_{i,k} - Ia_3^{k,k}e_{i,k} &= d^{i,k}Ie_{i,k} + a_{i,k}^{i,i}Ie_{i,k} - Ia_{i,k}^{k,k}e_{i,k}. \end{aligned}$$

From this it follows that $a_3^{i,k} = d^{i,k}$ and $a_{i,k}^{i,k} = d^{i,k}$ by (12).

Similarly, multiplying the equalities (10) and (11) by $e_{k,k}$ from the left side and by $e_{i,i}$ from the right side we get

$$Ia_3^{k,i} - a_3^{k,k} + a_3^{i,i} = Id^{k,i} - a_{k,k}^{k,i} + a_{i,k}^{i,i},$$

$$Ia_3^{k,i} + Ia_3^{k,k} - Ia_3^{i,i} = Id^{k,i} + Ia_{i,k}^{k,k} - Ia_{i,k}^{i,i}.$$

Hence, $a_3^{k,i} = d^{k,i}$ and $a_{i,k}^{k,i} = d^{k,i}$ by (12). Thus we have proved (5), (6).

Now, let us prove (7). Let i be a natural number and, let

$$x = x_o - s_{i-1,i} - s_{i,i+1} - s_{i+1,i+2}.$$

Then, by the additivity of ∇ and equality (2), the following equality holds

$$\begin{aligned} a_2x_o - x_oa_2 &= a_{i-1,i}s_{i-1,i} - s_{i-1,i}a_{m,i} + a_{i,i+1}s_{i,i+1} - s_{i,i+1}a_{i,i+1} \\ &\quad + a_{i+1,i+2}s_{i+1,i+2} - s_{i+1,i+2}a_{i+1,i+2} + bx - xb. \end{aligned}$$

Multiplying this equality by $e_{i,i}$ on the left side and by $e_{i+1,i+1}$ on the right side we get the equalities

$$a_{i,i+1}^{i,i} - a_{i,i+1}^{i+1,i+1} = a_2^{i,i} - a_2^{i+1,i+1} = d^{i,i} - d^{i+1,i+1}.$$

Let i, k be natural numbers. We suppose that $i < k$. Then we similarly have

$$\begin{aligned} a_{i,i+1}^{i,i} - a_{i,i+1}^{i+1,i+1} &= a_2^{i,i} - a_2^{i+1,i+1} = d^{i,i} - d^{i+1,i+1}, \\ a_{i+1,i+2}^{i+1,i+1} - a_{i+1,i+2}^{i+2,i+2} &= a_2^{i+1,i+1} - a_2^{i+2,i+2} = d^{i+1,i+1} - d^{i+2,i+2}, \\ a_{i+2,i+3}^{i+2,i+2} - a_{i+2,i+3}^{i+3,i+3} &= a_2^{i+2,i+2} - a_2^{i+3,i+3} = d^{i+2,i+2} - d^{i+3,i+3}, \\ &\dots \\ a_{k-2,k-1}^{k-2,k-2} - a_{k-2,k-1}^{k-1,k-1} &= a_2^{k-2,k-2} - a_2^{k-1,k-1} = d^{k-2,k-2} - d^{k-1,k-1}, \\ a_{k-1,k}^{k-1,k-1} - a_{k-1,k}^{k,k} &= a_2^{k-1,k-1} - a_2^{k,k} = d^{k-1,k-1} - d^{k,k}. \end{aligned}$$

But, for any natural number m and a projection e in $B(H)_{sk}$ with $e = \sum_{t=1}^l e_{m_t, m_t}$, $ee_{m,m} = e_{m,m}$, where $\{e_{m_1, m_1}, e_{m_2, m_2}, \dots, e_{m_l, m_l}\} \subset \{e_{i,i}\}_{i=1}^\infty$, the maps

$$x \rightarrow e_{m,m} \nabla(x) e_{m,m}, x \in e_{m,m} B(H)_{sk} e_{m,m}$$

$$x \rightarrow e \nabla(x) e, x \in e B(H)_{sk} e$$

are local derivations and, hence, are also derivations by [4]. Hence, there exist elements $v \in e_{m,m} B(H)_{sk} e_{m,m}$, $w \in e B(H)_{sk} e$ such that

$$e_{m,m} \nabla(x) e_{m,m} = vx - xv, x \in e_{m,m} B(H)_{sk} e_{m,m},$$

$$e \nabla(x) e = wx - xw, x \in e B(H)_{sk} e.$$

Besides, $e_{m,m} w e_{m,m} = v$.

Therefore we obtain

$$a_{i,i+1}^{i,i} = a_{i,k}^{i,i}, \quad a_{i,i+1}^{i+1,i+1} = a_{i+1,i+2}^{i+1,i+1}, \quad a_{i+1,i+2}^{i+2,i+2} = a_{i+2,i+3}^{i+2,i+2}, \quad a_{i+2,i+3}^{i+3,i+3} = \dots$$

$$\dots = a_{k-2,k-1}^{k-2,k-2}, \quad a_{k-2,k-1}^{k-1,k-1} = a_{k-1,k}^{k-1,k-1}, \quad a_{k-1,k}^{k,k} = a_{i,k}^{k,k}$$

by the first part of the present section and Lemma 4.1. Hence, we have equality (7):

$$a_{i,k}^{i,i} - a_{i,k}^{k,k} = a_2^{i,i} - a_2^{k,k} = d^{i,i} - d^{k,k}.$$

So, the equalities (4) are valid.

Now, let x be an arbitrary element in $B_{sk}(H)$ and

$$e_{i,i}x e_{j,j} + e_{j,j}x e_{i,i} = x_1^{i,j} s_{i,j} + Ix_2^{i,j} \bar{e}_{i,j} \quad \text{and} \quad e_{i,i}x e_{i,i} = Ix^{i,i} e_{i,i},$$

where $x_1^{i,j}, x_2^{i,j}, x^{i,i}$ are real numbers for every pair of different natural numbers i, j .

Then, by the equalities (3), (4), we have

$$\begin{aligned} \nabla(x_1^{i,j} s_{i,j}) &= x_1^{i,j} \nabla(s_{i,j}) = x_1^{i,j} R_d(s_{i,j}) = R_d(x_1^{i,j} s_{i,j}), \\ \nabla(Ix_2^{i,j} \bar{e}_{i,j}) &= x_2^{i,j} \nabla(I\bar{e}_{i,j}) = x_2^{i,j} R_d(I\bar{e}_{i,j}) = R_d(Ix_2^{i,j} \bar{e}_{i,j}), \\ \nabla(Ix^{i,i} e_{i,i}) &= x^{i,i} \nabla(Ie_{i,i}) = x^{i,i} R_d(Ie_{i,i}) = R_d(Ix^{i,i} e_{i,i}), \end{aligned}$$

since ∇ is linear. Let (x_α) be the net of all elements of the form

$$\sum_{k=1, l=2, k < l}^m (x_1^{i_k, i_l} s_{i_k, i_l} + Ix_2^{i_k, i_l} \bar{e}_{i_k, i_l}) + \sum_{k=1}^m x^{i_k, i_k} Ie_{i_k, i_k},$$

where $\alpha = \{i_1, i_2, \dots, i_m\}$ is an arbitrary finite subset of natural numbers. Then the net (x_α) ultraweakly converges to x . We have

$$\begin{aligned} &\nabla\left(\sum_{k=1, l=2, k < l}^m (x_1^{i_k, i_l} s_{i_k, i_l} + Ix_2^{i_k, i_l} \bar{e}_{i_k, i_l}) + \sum_{k=1}^m x^{i_k, i_k} Ie_{i_k, i_k}\right) \\ &= \sum_{k=1, l=2, k < l}^m (x_1^{i_k, i_l} \nabla(s_{i_k, i_l}) + x_2^{i_k, i_l} \nabla(I\bar{e}_{i_k, i_l})) + \sum_{k=1}^m x^{i_k, i_k} \nabla(Ie_{i_k, i_k}) \\ &= \sum_{k=1, l=2, k < l}^m (x_1^{i_k, i_l} R_d(s_{i_k, i_l}) + x_2^{i_k, i_l} R_d(I\bar{e}_{i_k, i_l})) + \sum_{k=1}^m x^{i_k, i_k} R_d(Ie_{i_k, i_k}) \\ &= R_d\left(\sum_{k=1, l=2, k < l}^m (x_1^{i_k, i_l} s_{i_k, i_l} + Ix_2^{i_k, i_l} \bar{e}_{i_k, i_l}) + \sum_{k=1}^m x^{i_k, i_k} Ie_{i_k, i_k}\right), \end{aligned}$$

i.e., $\nabla(x_\alpha) = R_d(x_\alpha)$ for every α . The Lie multiplication is ultraweakly continuous, so the net $(R_d(x_\alpha))$ is ultraweakly converges to $R_d(x)$. Since ∇ is ultraweakly continuous we have $\nabla(x) = R_d(x)$.

This completes the proof. ■

5. Local derivations on Lie algebras of skew-adjoint operator-valued maps

In this section we describe local derivations on some class of subalgebras of the Lie algebra $F(\Omega, B_{sk}(H))$.

Given $t \in \Omega$, $\phi_t: F(\Omega, B(H)) \rightarrow B(H)$ will denote the $*$ -homomorphism defined by $\phi_t(x) = x(t)$, $x \in F(\Omega, B(H))$. The space $F(\Omega, B(H))$ is an $B(H)$ -bimodule with products $(ax)(t) = ax(t)$ and $(xa)(t) = x(t)a$, for every $a \in B(H)$, $x \in F(\Omega, B(H))$. The map $\phi_t: F(\Omega, B(H)) \rightarrow B(H)$ is an $B(H)$ -module homomorphism.

Given an arbitrary set Ω , the $*$ -homomorphism which maps each element a in $B(H)$ to the constant function $\Omega \rightarrow \{a\}$ will be denoted by \hat{a} . The map

$$\psi(a) = \hat{a}, \quad a \in B(H),$$

is a $B(H)$ -module homomorphism from $B(H)$ to $F(\Omega, B(H))$.

Let A be a Lie algebra and, let B be a Lie subalgebra of A . A local derivation ∇ on B is called local spatial derivation implemented by elements from A , if for every element $x \in B$ there exists a spatial derivation D on B implemented by an element in A such that $\nabla(x) = D(x)$.

Let $\mathcal{A} = B_{sk}(H)$ or $\mathcal{A} = \mathcal{K}_{sk}(H)$, and let Ω be a topological space. Let $F(\Omega, \mathcal{A})$ be the Lie algebra of all maps from Ω to \mathcal{A} , and let $C(\Omega, \mathcal{A})$ be the Lie algebra of all continuous maps from Ω to \mathcal{A} . Then we have the following theorem

Theorem 5.1. *Let \mathcal{L} be one of the Lie algebras $F(\Omega, \mathcal{A})$ or $C(\Omega, \mathcal{A})$ and let ∇ be a local spatial derivation on \mathcal{L} implemented by elements from $F(\Omega, B_{sk}(H))$. Suppose that, for every $t \in \Omega$, the map $\phi_t \nabla \psi$ is ultraweakly continuous on $B_{sk}(H)$. Then ∇ is a spatial derivation.*

Proof. Similar to the proof of [11, Theorem 2.4] we can prove that the map $\phi_t \nabla \psi: B_{sk}(H) \rightarrow B_{sk}(H)$ is a local inner derivation for every $t \in \Omega$. Hence, by Theorem 4.2, $\phi_t \nabla \psi$ is an inner derivation on $B_{sk}(H)$ since $\phi_t \nabla \psi$ is ultraweakly continuous.

Let a_t be an element in $B_{sk}(H)$ such that

$$\phi_t \nabla \psi(x) = a_t x - x a_t, \quad x \in B_{sk}(H).$$

Let $\hat{a}(t) = a_t$, $t \in \Omega$. Then

$$\nabla(\mathbf{x})(t) = (\hat{a}\mathbf{x} - \mathbf{x}\hat{a})(t), \quad \mathbf{x} \in \mathcal{L}, \quad t \in \Omega,$$

i.e., $\nabla(\mathbf{x}) = \hat{a}\mathbf{x} - \mathbf{x}\hat{a}$, $\mathbf{x} \in \mathcal{L}$ and $\hat{a} \in F(\Omega, B_{sk}(H))$.

Hence, ∇ is a spatial derivation since $\nabla(\mathbf{x}) \in \mathcal{L}$ for any $\mathbf{x} \in \mathcal{L}$. ■

Remark 5.2. Let \mathcal{L} be one of the Lie algebras $F(\Omega, \mathcal{A})$ or $C(\Omega, \mathcal{A})$. Note that, it is not necessary that for each pair of elements $x \in \mathcal{L}$ and $y \in F(\Omega, B_{sk}(H))$ the elements yx , xy , $R_x(y)$ belong to \mathcal{L} . But, in theorem 5.1, for every $x \in \mathcal{L}$, the element $R_a(x)$ belongs to \mathcal{L} .

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Sh. A. Ayupov, V. I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan; and: National University of Uzbekistan, Tashkent, Uzbekistan; shavkat.ayupov@mathinst.uz.

F. N. Arzikulov, V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan; and: Andizhan State University, Andizhan, Uzbekistan; arzikulovfn@rambler.ru.

S. M. Umrzaqov, Andizhan State University, Andizhan, Uzbekistan; sardor.umrzaqov1986@gmail.com.

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