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Composition Series and Unitary Subquotients of Representations Induced from Essentially Speh and Cuspidal Representations

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Abstract. We consider representations of either symplectic or special odd-orthogonal groups over a non-archimedean local field. We obtain the composition series of a representation induced from essentially Speh and cuspidal representations under certain conditions. Using previous results of the author, we obtain irreducible unitary representations of the considered groups at the ends of complementary series.

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1. Introduction

The fundamental object of harmonic analysis on a locally compact group G is the set of equivalence classes of its irreducible unitary representations. It is called the unitary dual of G and denoted $\widehat{G}$. The notable example of the classification of $\widehat{G}$ for $G = \mathrm{GL}_n$ can be found in [23]. Besides some low-rank cases (e.g. [7],[11]), it is still an open question when G is a classical group.

The aim of this paper is to give an insight into the structure of $\widehat{G}$ for a symplectic or special odd-orthogonal group G over a non-archimedean local field F. Our approach follows the strategy proposed by Harish-Chandra: classify the equivalence classes of smooth irreducible representations of G and then identify unitarizable classes among them. The first step is also known as the problem of the non-unitary dual. Results in the scope of the first step are obtained in this paper and in author's previous work [4]. Using these results, we determine the first reducibility point of certain complementary series and thus identify unitary irreducible representations of G.

More precisely, using the Langlands classification (cf. Preliminaries), we define representation

$$\pi_S = L(\delta([\nu^a\rho, \nu^b\rho]), \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]))$$

of a general linear group, for a character ν of a general linear group such that $\nu(g) = |\det(g)|_F$, cuspidal representation $\rho \in \mathrm{Irr}(GL)$ and real numbers a, b such that $b - a \in \mathbb{Z}_{\geq 0}$. Note that π_S belongs to the class of essentially Speh representations.

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They play a role of the building blocks of the unitary dual of general linear groups. This motivates the study of the induced representation

$$\mathrm{Ind}_P^G(\pi_S \otimes \sigma_c) =: \pi_S \rtimes \sigma_c,$$

for a cuspidal representation $\sigma_c \in \mathrm{Irr}(G)$ and a maximal standard parabolic subgroup P of G. Since the composition series of the representation $\pi_S \rtimes \sigma_c$ such that $a > 0$ is determined in [4], it suffices to consider the case $a \leq 0$.

Recently, in [2], Atobe determined the socle and the reducibility criterion for representations induced from an essentially Speh representation on the general linear part and an Arthur type representation on the classical part.

We describe the method of obtaining unitary representations at the ends of complementary series. We denote by π_S^u a Speh representation such that $\pi_S^u \rtimes \sigma_c$ is an irreducible representation. From the Section 3 of [24], we see that representations $\nu^x \pi_S^u \rtimes \sigma_c$ are unitarizable for $0 \leq x < x_0$, where x_0 is the minimal positive number such that the induced representation $\nu^{x_0} \pi_S^u \rtimes \sigma_c$ is reducible. Moreover, all irreducible subquotients of $\nu^{x_0} \pi_S^u \rtimes \sigma_c$ are unitarizable.

The irreducible subquotients of the considered induced representations are determined in terms of the Langlands classification, see [3]. For the description of the associated tempered parameters, we use Mœglin-Tadić's classification of discrete series representations [17] and Tadić's classification of tempered representations [26]. In the course of determining candidates for the subquotients of $\pi_S \rtimes \sigma_c$, it became evident that we need to determine irreducible tempered subquotients of a representation $\pi \rtimes \sigma_c$. Here π is isomorphic to $L(\delta([\nu^a\rho, \nu^b\rho]), \delta([\nu^{x+1}\rho, \nu^{b+1}\rho]))$, for a real number x such that $0 \leq x \leq b$ and $-a \leq b$.

Among the used methods, the most prominent role is taken by the structural formula for the Jacquet modules of induced representations from [25]. It is an algebraization of the Geometric Lemma of Bernstein and Zelevinsky, which enables us to narrow down the set of possible subquotients of considered representations. Also, counting multiplicities of certain constituents of Jacquet modules enables us to prove the existence of subquotients of the considered induced representations. Additionally, we often use the embeddings of representations and the Frobenius reciprocity. This is motivated with Lemma 2.13 and the fact that the embeddings are used in the classifications of irreducible tempered representations.

Let us briefly describe the content of the paper. In the second section we introduce the notation and results which are used throughout the paper. The third section deals with the description of the irreducible tempered subquotients of the induced representation $\pi \rtimes \sigma_c$. In the last two sections we determine the composition series of the induced representation $\pi_S \rtimes \sigma_c$. Moreover, we determine unitary representations as subquotients of the end of the considered complementary series. For a reader's convenience, we state the theorem about unitary irreducible subquotients of $\pi_S \rtimes \sigma_c$.

Let us fix cuspidal representations $\rho \in \mathrm{Irr}(GL)$ and $\sigma_c \in \mathrm{Irr}(G)$ such that ρ is selfdual and $\alpha \geq 2$ is the unique non-negative real number such that $\nu^\alpha \rho \rtimes \sigma_c$ reduces. If an induced representation $\delta([\nu^{\alpha-1}\rho, \nu^x\rho]) \times \delta([\nu^\alpha\rho, \nu^y\rho]) \rtimes \sigma_c$ has a strongly positive discrete series subquotient, we denote it $\sigma_{(x,y)}$. We define the following representations:

$$\delta([\nu^{-(\alpha-2)}\rho,\nu^{\alpha-2}\rho])\rtimes\sigma_{(b,b+1)}\simeq\tau_+\oplus\tau_-,$$
$$\pi_0=L(\delta([\nu^{-\alpha}\rho,\nu^{-\alpha+2b+1}\rho]),\delta([\nu^{-\alpha+1}\rho,\nu^{-\alpha+2b+2}\rho]);\sigma_c),$$
$$\pi_1=L(\delta([\nu^{-(\alpha-1)}\rho,\nu^{-\alpha+2b+1}\rho]),\delta([\nu^{-(\alpha-2)}\rho,\nu^{-\alpha+2b+2}\rho]);\sigma_{(\alpha-1,\alpha)}).$$

Theorem 1.1. *The irreducible subquotients of the induced representation*

$$L(\delta([\nu^{\alpha-2b-2}\rho,\nu^{\alpha-1}\rho]),\delta([\nu^{\alpha-2b-1}\rho,\nu^{\alpha}\rho]))\rtimes\sigma_c, \tag{1}$$

for a real number b such that $\alpha-b\in\mathbb{Z}$ and $-\frac{1}{2}\leq b<\alpha-1$, are unitarizable. The composition series of the induced representation (1) is equal to:

- $\pi_0+L(\nu^{-1}\rho;\tau_+)+L(\delta([\nu^{-2}\rho,\nu\rho]);\sigma_c)$, *if* $\alpha-2=b=0$,
- $\pi_0+L(\delta([\nu^{-(\alpha-1)}\rho,\nu^{\alpha-3}\rho]);\tau_-)$, *if* $\alpha-2=b\geq\frac{1}{2}$,
- $\pi_0+\pi_1+L(\delta([\nu^{-\alpha}\rho,\nu^{\alpha-1}\rho]);\sigma_c)$, *if* $\alpha-2=2b\geq 1$,
- $\pi_0+\pi_1$, *otherwise.*

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2. Preliminaries

In the sequel, F will denote a local non-archimedean field of characteristic zero. We denote by GL_n the general linear group of rank n over F. Also, G_n denotes a rank n group from a fixed series of symplectic or split special odd-orthogonal group over F. The matrix realizations of these groups can be found in [25, Sections 3,6].

Let $\mathcal{C}(G)$ denote a category of admissible representations of finite length of group G, where G is isomorphic to GL_n, G_n or to a Levi factor of some parabolic subgroup of any of these two groups. Let us denote the Grothendieck group of $\mathcal{C}(G)$ with $R(G)$. For $\pi\in\mathcal{C}(G)$ and its irreducible subquotients $\pi_1,\dots,\pi_k$, in $R(G)$ we have $\pi=\pi_1+\ldots+\pi_k$. The sum on the right hand side of the equality is called the semisimplification of π. We denote $R(G)=\oplus_{n\geq 0}R(G_n)$, $R(GL)=\oplus_{n\geq 0}R(GL_n)$.

Also, the set of irreducible representations in $R(G)$ or $R(GL)$ is denoted by $\mathrm{Irr}(G)=\oplus_{n\geq 0}$, $\mathrm{Irr}(G_n)$, $\mathrm{Irr}(GL)=\oplus_{n\geq 0}\mathrm{Irr}(GL_n)$. The contragredient of a representation π in $R(GL)$ or $R(G)$ is denoted by $\widetilde{\pi}$. We say that π is selfdual if $\widetilde{\pi}\simeq\pi$. For $\pi,\pi_1,\pi_2\in\mathrm{Irr}(GL)$, in $R(GL)$ we have $\pi_1\times\pi_2=\pi_2\times\pi_1$. Moreover, for $\sigma\in R(G)$, in $R(G)$ we have $\pi_1\times\pi_2\rtimes\sigma=\pi_2\times\pi_1\rtimes\sigma$ and $\widetilde{\pi}\rtimes\sigma=\pi\rtimes\sigma$. The cuspidal support of $\pi\in\mathcal{C}(G)$ is denoted $[\pi]$. The multiset $[\pi]_{\mathrm{GL}}$ contains all elements of $[\pi]$ which are representations of general linear groups.

We fix a Borel subgroup B of upper triangular matrices in G, where G equals GL_n or G_n. Consider standard parabolic subgroups which contain B. Let $P=MN$ be a standard parabolic subgroup of G with its Levi decomposition. The main constructions with representations $\pi\in\mathcal{C}(G)$ and $\sigma\in\mathcal{C}(M)$ are the normalized parabolic induction $\mathrm{Ind}_P^G(\sigma)$ and the normalized Jacquet module $JM_N(\pi)$.

The mappings $\mathrm{Ind}_P^G : \mathcal{C}(M) \to \mathcal{C}(G)$ and $JM_N : \mathcal{C}(G) \to \mathcal{C}(M)$ are exact and transitive functors. Their interplay is described by the Frobenius reciprocity. For a representation $\pi_1 \otimes \ldots \otimes \pi_k \otimes \sigma \in \mathcal{C}(M)$, where the Levi factor M is isomorphic to $GL_{m_1} \times \ldots \times GL_{m_k} \times G_{n-|\beta|}$ for $\beta = (m_1, \ldots, m_k)$, the normalized parabolically induced representation $\mathrm{Ind}_P^{G_n}(\pi_1 \otimes \ldots \otimes \pi_k \otimes \sigma)$ of G_n is denoted $\pi_1 \times \ldots \times \pi_k \rtimes \sigma$.

For a k-tuple $\beta = (m_1, \ldots, m_k)$, which corresponds to the parabolic subgroup P_β of G_n, and a representation $\pi \in \mathcal{C}(G)$, the normalized Jacquet module of π is denoted $r_\beta(\pi)$. Also, we define

$$m^*(\pi) = \sum_{k=0}^{n} r_{(k)}(\pi) \text{ for } \pi \in \mathcal{C}(GL) \quad \text{and} \quad \mu^*(\pi) = \sum_{k=0}^{n} r_{(k)}(\pi) \text{ for } \pi \in \mathcal{C}(G).$$

For a cuspidal irreducible representation ρ of the group GL_n, we denote the set $\{\rho, \nu\rho, \ldots, \nu^m\rho\}$ for $m \in \mathbb{N}$ with $[\rho, \nu^m\rho]$ and call it a *segment*. The set of all segments is denoted $\mathcal{S}$. The set $D(GL)$ of all essentially square-integrable representations of general linear groups is described in terms of segments. Namely, the induced representation $\nu^m\rho \times \cdots \times \nu\rho \times \rho$ has the unique irreducible quotient denoted $\delta([\rho, \nu^m\rho])$. Then $D(GL) = \{\delta(\Delta) : \Delta \in \mathcal{S}\}$. For $\delta \in D(GL)$, there is a unique real number $e(\delta)$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. If $\delta \simeq \delta([\nu^x\rho, \nu^y\rho])$, then $e(\delta) = \frac{x+y}{2}$. To obtain a uniform description of results, we define $\delta([\nu^x\rho, \nu^y\rho]) \simeq 1$ for $y = x-1$, where 1 stands for one-dimensional representation of the trivial group. Moreover, $\delta \in D(GL)$ is square-integrable if and only if $e(\delta) = 0$.

Let us recall the Langlands classifications for groups GL_n and G_n. For $\delta_1, \ldots, \delta_l$ in $D(GL)$ such that $e(\delta_1) \leq \ldots \leq e(\delta_l)$, the induced representation $\delta_1 \times \cdots \times \delta_l$ has the unique irreducible subrepresentation, which we denote $L(\delta_1, \ldots, \delta_l)$. For every non-tempered representation $\pi \in \mathrm{Irr}(GL)$ there is the unique representation $L(\delta_1, \ldots, \delta_l)$, of the form as above, such that $\pi \simeq L(\delta_1, \ldots, \delta_l)$. For tempered $\tau \in \mathrm{Irr}(G)$ and $\delta_1, \ldots, \delta_l \in D(GL)$ such that $e(\delta_1) \leq \ldots \leq e(\delta_l) < 0$, the induced representation $\delta_1 \times \cdots \times \delta_l \rtimes \tau$ has the unique irreducible subrepresentation which we denote $L(\delta_1, \ldots, \delta_l; \tau)$. For every non-tempered representation $\sigma \in \mathrm{Irr}(G)$ there is the unique representation $L(\delta_1, \ldots, \delta_l; \tau)$, of the form as above, such that $\sigma \simeq L(\delta_1, \ldots, \delta_l; \tau)$.

Definition 2.1. For a positive integer m and a square-integrable representation $\delta \in D(GL)$, an irreducible representation

$$L(\nu^{-\frac{m-1}{2}}\delta, \, \nu^{\frac{m-1}{2}+1}\delta, \, \ldots, \, \nu^{\frac{m-1}{2}}\delta)$$

is called the *Speh representation* and denoted $u(\delta, m)$.

Definition 2.2. A *ladder representation* is a representation of the form

$$L(\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]), \ldots, \delta([\nu^{a_m}\rho, \nu^{b_m}\rho])) \tag{2}$$

for $m \in \mathbb{N}$ and representations $\delta([\nu^{a_i}\rho, \nu^{b_i}\rho]) \in D(GL)$ such that $a_1 < \ldots < a_m$ and $b_1 < \ldots < b_m$ for $i = 1, \ldots, m-1$. If $a_{i+1} = a_i + 1$ and $b_{i+1} = b_i + 1$ for $i = 1, \ldots, m-1$, then we call (2) the essentially Speh representation.

Throughout the paper, we fix cuspidal representations $\rho \in \mathrm{Irr}(GL)$ and $\sigma_c \in \mathrm{Irr}(G)$ and define for real numbers a, b with $b-a \in \mathbb{Z}_{\geq 0}$ the essentially Speh representation

$$\pi_S = L(\delta([\nu^a\rho, \nu^b\rho]), \delta([\nu^{a+1}\rho, \nu^{b+1}\rho])).$$

The following two lemmas are easy consequences of the representation theory of general linear groups, mainly developed by Zelevinsky in [27].

Lemma 2.3. *Let $L(\delta_1^{(i)}, \ldots, \delta_{n_i}^{(i)})$ for $i = 1, 2$ denote two ladder representations such that the cuspidal support of*

$$\pi_0 = L(\delta_1^{(1)}, \ldots, \delta_{n_1}^{(1)}) \times L(\delta_1^{(2)}, \ldots, \delta_{n_2}^{(2)})$$

is a segment Δ. If there exist $i_0 \in \{1, 2\}$ and $j_0 \in \{1, \ldots, n_{i_0} - 1\}$ such that $\delta_{j_0}^{(i_0)} \times \delta_{j_0+1}^{(i_0)}$ reduces, then π_0 does not contain essentially square-integrable subquotient.

Lemma 2.4. *Let $\pi \simeq L(\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]), \ldots, \delta([\nu^{a_n}\rho, \nu^{b_n}\rho]))$, $\pi_1 \simeq \delta([\nu^{a_m}\rho, \nu^{c_m}\rho])$ and $\pi_2 \simeq L(\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]), \ldots, \delta([\nu^{c_m+1}\rho, \nu^{b_m}\rho]), \ldots, \delta([\nu^{a_n}\rho, \nu^{b_n}\rho]))$ denote ladder representations for some $m \in \{1, \ldots, n\}$. Then π is a subquotient of $\pi_1 \times \pi_2$ of multiplicity one.*

Results of [9] give the semisimplification of $m^*(\pi)$ for a ladder representation π.

Theorem 2.5. *For a ladder representation $\pi = L(\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]), \ldots, \delta([\nu^{a_m}\rho, \nu^{b_m}\rho]))$, $m^*(\pi)$ is in $R(GL)$ equal to*

$$\sum_{Lad(\pi)} L(\delta([\nu^{c_1+1}\rho, \nu^{b_1}\rho]), \ldots, \delta([\nu^{c_m+1}\rho, \nu^{b_m}\rho])) \otimes L(\delta([\nu^{a_1}\rho, \nu^{c_1}\rho]), \ldots, \delta([\nu^{a_m}\rho, \nu^{c_m}\rho])), \tag{3}$$

where $Lad(\pi)$ denotes the set of all m-tuples $(c_1, \ldots, c_m)$ of real numbers such that $c_1 < \ldots < c_m$, $a_i - 1 \leq c_i \leq b_i$ and $c_i - a_i \in \mathbb{Z}$ for $i = 1, \ldots, m$.

The Geometric Lemma implies that the map m^* is multiplicative, that is, whenever $\pi_1, \pi_2 \in R(GL)$ we have $m^*(\pi_1 \times \pi_2) = m^*(\pi_1) \times m^*(\pi_2)$. We define a map

$$M^* \colon R(GL) \to R(GL) \otimes R(GL),\ M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*,$$

where m is a parabolic induction $\times \colon R(GL) \otimes R(GL) \to R(GL)$ and $s : \sum x_i \otimes y_i \mapsto \sum y_i \otimes x_i$. Note that M^* is a multiplicative map. The next theorem is the main result of [25].

Theorem 2.6. *For $\pi \in R(GL)$ and $\sigma \in R(G)$ we have*

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma),$$

where the right hand side of the equality is computed as $(\pi_1 \otimes \pi_2) \rtimes (\pi' \otimes \sigma') = (\pi_1 \times \pi') \otimes (\pi_2 \rtimes \sigma')$.

Let π denote the ladder representation $L(\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]), \ldots, \delta([\nu^{a_m}\rho, \nu^{b_m}\rho]))$. We define $\mathrm{Lad}(\pi)'$ as the subset consisting of pairs of m-tuples in $\mathrm{Lad}(\pi) \times \mathrm{Lad}(\pi)$ such that for every $\big((c_1, \ldots, c_m), (d_1, \ldots, d_m)\big) \in \mathrm{Lad}(\pi)'$ holds $c_i \leq d_i$ for all $i = 1, \ldots, m$. Since we know $m^*(\pi)$ from (3), for $\sigma \in \mathrm{Irr}(G)$ we have

$$\begin{aligned}\mu^*(\pi \rtimes \sigma) = \sum\nolimits_{\mathrm{Lad}(\pi)'} & L(\delta([\nu^{-c_m}\widetilde{\rho}, \nu^{-a_m}\widetilde{\rho}]), \ldots, \delta([\nu^{-c_1}\widetilde{\rho}, \nu^{-a_1}\widetilde{\rho}])) \\ & \times L(\delta([\nu^{d_1+1}\rho, \nu^{b_1}\rho]), \ldots, \delta([\nu^{d_m+1}\rho, \nu^{b_m}\rho])) \\ & \otimes L(\delta([\nu^{c_1+1}\rho, \nu^{d_1}\rho]), \ldots, \delta([\nu^{c_m+1}\rho, \nu^{d_m}\rho])) \rtimes \mu^*(\sigma).\end{aligned} \tag{4}$$

We shall now describe the Mœglin-Tadić classification of square-integrable representations in $\mathrm{Irr}(G)$, also called discrete series representations, which is obtained in [15, 17]. These results now hold unconditionally, due to [1], [16, Théorème 3.1.1] and [5, Theorem 7.8]. The classification attaches to a discrete series representation three invariants: a partial cuspidal support, Jordan block and ϵ-function, and claims it is uniquely determined by them.

Let σ denote a discrete series representation of group G_n. We say a cuspidal representation $\sigma_{cusp} \in \mathrm{Irr}(G)$ is a partial cuspidal support of σ if there exists $\pi \in \mathrm{Irr}(GL)$ such that $\sigma \hookrightarrow \pi \rtimes \sigma_{cusp}$. The second invariant of σ is the set $\mathrm{Jord}(\sigma)$ of pairs (a, ρ), for $a \in \mathbb{Z}_{>0}$ and selfdual cuspidal ρ from $\mathrm{Irr}(GL_{m_\rho})$, such that we have the following: a is even if and only if $L(s, \rho, r)$, defined by Shahidi in [20, 21], has a pole in $s = 0$ and an induced representation $\delta([\nu^{-\frac{a-1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \sigma$ is irreducible.

Remark 2.7. (*Basic Assumption*) Let $\rho \in \mathrm{Irr}(GL)$ and $\sigma_{cusp} \in \mathrm{Irr}(G)$ denote cuspidal representations such that ρ is selfdual. Results of [22] imply the existence of the unique $\alpha_{\rho,\sigma_{cusp}} \geq 0$ such that $\nu^{\alpha_{\rho,\sigma_{cusp}}}\rho \rtimes \sigma_{cusp}$ reduces. Moreover, the results of [16, Théorème 3.1.1] and [5, Theorem 7.8] imply $2\alpha_{\rho,\sigma_{cusp}} \in \mathbb{Z}$. The set $\mathrm{Jord}_\rho(\sigma_{cusp})$ is equal to $\{2(\alpha - \lceil\alpha\rceil + 1) + 1, \ldots, 2(\alpha - 1) + 1\}$, if $2\alpha + 1$ is even or $\{1, 2(\alpha - \lceil\alpha\rceil + 1) + 1, \ldots, 2(\alpha - 1) + 1\}$, if $2\alpha + 1$ is odd.

Let us describe the set Trip of all triplets $(\mathrm{Jord}, \sigma', \epsilon)$. Jord is a finite set of pairs (a, ρ) such that ρ is a selfdual cuspidal representation from $\mathrm{Irr}(GL)$ and $a \in \mathbb{Z}_{>0}$ is even if and only if $L(s, \rho, r)$ has a pole in $s = 0$. For a fixed cuspidal representation $\rho \in \mathrm{Irr}(GL)$, we write $\mathrm{Jord}_\rho = \{a : (a, \rho) \in \mathrm{Jord}\}$. For $a \in \mathrm{Jord}_\rho$, $a\_$ denotes the maximal element of Jord_ρ which is strictly less than a, if such exists. $\sigma' \in \mathrm{Irr}(G)$ is a cuspidal representation. ϵ is a function defined on a subset of $\mathrm{Jord} \cup (\mathrm{Jord} \times \mathrm{Jord})$ with codomain $\{\pm 1\}$, under certain conditions which can be found in [17]. We note that to define the ϵ-function on the elements of $\mathrm{Jord}(\sigma) \times \mathrm{Jord}(\sigma)$, it is enough to define the ϵ-function on the elements of the form $((a\_, \rho), (a, \rho))$. Let $(\mathrm{Jord}, \sigma', \epsilon)$ and $(a, \rho) \in \mathrm{Jord}$, such that $a\_$ is defined, and $\epsilon(a, \rho)\epsilon(a\_, \rho)^{-1} = 1$.

If we denote $\mathrm{Jord}' = \mathrm{Jord} \setminus \{(a, \rho), (a\_, \rho)\}$ and ϵ' a restriciton of ϵ-function on $\mathrm{Jord}' \cup (\mathrm{Jord}' \times \mathrm{Jord}')$, then it follows $(\mathrm{Jord}', \sigma', \epsilon') \in \mathrm{Trip}$. We say that the triple $(\mathrm{Jord}', \sigma', \epsilon')$ is *subordinated* to $(\mathrm{Jord}, \sigma', \epsilon)$. We say that $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}$ is an *admissible triple of alternated type* if $\epsilon(a, \rho)\epsilon(a\_, \rho)^{-1} = -1$, when $a\_$ is defined, and if there is an increasing bijection $\phi_\rho : \mathrm{Jord}_\rho \to \mathrm{Jord}'_\rho(\sigma')$, where

$$\mathrm{Jord}'_\rho(\sigma') = \begin{cases} \mathrm{Jord}_\rho(\sigma') \cup \{0\} & , \text{ if } a \text{ is even and } \epsilon(\min\mathrm{Jord}_\rho, \rho) = 1, \\ \mathrm{Jord}_\rho(\sigma') & , \text{ otherwise.} \end{cases}$$

The set of all admissible triples of alternated type is denoted Trip_{alt}. We say that $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}$ *dominates* a triple $(\mathrm{Jord}'', \sigma', \epsilon'') \in \mathrm{Trip}$ if there is a sequence of triples $(\mathrm{Jord}_i, \sigma', \epsilon_i)$ for $i = 1, \ldots, k$ such that $(\mathrm{Jord}, \sigma', \epsilon) = (\mathrm{Jord}_1, \sigma', \epsilon_1)$, $(\mathrm{Jord}_{i+1}, \sigma', \epsilon_{i+1})$ is subordinated to $(\mathrm{Jord}_i, \sigma', \epsilon_i)$ for $i = 1, \ldots, k-1$, $(\mathrm{Jord}'', \sigma', \epsilon'') = (\mathrm{Jord}_k, \sigma', \epsilon_k)$. We say that a triple $(\mathrm{Jord}, \sigma', \epsilon) \in \mathrm{Trip}$ is *admissible* if it dominates a triple of alternated type. The set of all admissible triples is denoted Trip_{adm}. Thus we have $\mathrm{Trip}_{alt} \subset \mathrm{Trip}_{adm} \subset \mathrm{Trip}$. Now we state the main theorem of classification.

Theorem 2.8. *There is a bijection between the set of discrete series representations in Irr(G) and the set of all triples* $(Jord, \sigma', \epsilon) \in Trip_{adm}$*, denoted* $\sigma = \sigma_{(Jord,\sigma',\epsilon)}$*, such that*

(i) $Jord(\sigma) = Jord$ *and* $\sigma_{cusp} \simeq \sigma'$.

(ii) *Let* $(Jord, \sigma', \epsilon) \in Trip_{alt}$*. Then we can give an explicit description of* σ*. For every* ρ *such that* $Jord_\rho \neq \emptyset$*, denote elements of* $Jord_\rho$ *in an increasing order* $a_1^\rho < a_2^\rho < \ldots < a_{k_\rho}^\rho$*. Now* σ *is the unique irreducible subrepresentation of*

$$\times_\rho \times_{i=1}^{k_\rho} \delta([\nu^{(\phi_\rho(a_i^\rho)+1)/2}\rho, \nu^{(a_i^\rho-1)/2}\rho]) \rtimes \sigma'.$$

(iii) *Let* $(Jord, \sigma', \epsilon) \in Trip_{adm}$ *and* $(2b+1, \rho) \in Jord$ *such that* $2b\_+1 := (2b+1)\_$ *is defined and* $\epsilon(2b+1,\rho)\epsilon(2b\_+1,\rho)^{-1} = 1$*. Denote*

$$Jord'' = Jord \setminus \{(2b+1,\rho),\ (2b\_+1,\rho)\}$$

and ϵ'' *a restriction of* ϵ *on* $Jord'' \cup (Jord'' \times Jord'')$*. Then it follows that* $(Jord'', \sigma', \epsilon'') \in Trip_{adm}$ *and*

$$\sigma \hookrightarrow \delta([\nu^{-b\_}\rho, \nu^b\rho]) \rtimes \sigma'',$$

where $\sigma'' = \sigma_{(Jord'',\sigma',\epsilon'')}$*. Additionally,* $\delta([\nu^{-b\_}\rho, \nu^{b\_}\rho]) \rtimes \sigma''$ *is a direct sum of two non-equivalent irreducible tempered representations* $\tau_\pm$ *and there is a unique* $\tau \in \{\tau_-, \tau_+\}$ *such that*

$$\sigma \hookrightarrow \delta([\nu^{b\_+1}\rho, \nu^b\rho]) \rtimes \tau.$$

Definition 2.9. We say that a representation $\sigma \in \mathrm{Irr}(G)$ is a *strongly positive representation* if for every embedding $\sigma \hookrightarrow \nu^{x_1}\rho_1 \times \cdots \times \nu^{x_k}\rho_k \rtimes \sigma_{cusp}$, for unitary $\rho_i \in \mathrm{Irr}(GL_{n_i})$ for $i = 1, \ldots, k$ and cuspidal representation $\sigma_{cusp} \in \mathrm{Irr}(G)$, we have $x_i > 0$ for every i.

Strongly positive representations in $\mathrm{Irr}(G)$ are the discrete series representations which are in a bijection with the alternated triples. We will see that the only interesting case is when the earlier fixed $\rho \in \mathrm{Irr}(GL)$ is selfdual. In that case, we denote by α the unique non-negative real number such that $\nu^\alpha\rho \rtimes \sigma_c$ reduces. Also, denote $r = \lceil \alpha \rceil$. We say that r-tuple of real numbers $(i_1, \ldots, i_r)$ is sp-acceptable if $i_1 < \ldots < i_r$, $i_j - \alpha \in \mathbb{Z}$ and $\alpha - r + j - 1 \leq i_j$ for $j = 1, \ldots, r$. For sp-acceptable r-tuple $(x_1, \ldots, x_r)$, we denote $\sigma = \sigma_{(x_1,\ldots,x_r)}$, if σ is the unique irreducible subrepresentation of

$$\delta([\nu^{\alpha-r+1}\rho, \nu^{x_1}\rho]) \times \ldots \times \delta([\nu^\alpha\rho, \nu^{x_r}\rho]) \rtimes \sigma_c.$$

For the simplicity, the non-cuspidal discrete series representation $\sigma_{(x_1,\ldots,x_r)}$ is denoted $\sigma_{(x_t,\ldots,x_r)}$, where t is the minimal element of the set $\{1, \ldots, r\}$ such that $\alpha - r + t \leq x_t$. We will now describe the classification of tempered representations in $\mathrm{Irr}(G)$ obtained in [26]. According to the results of [6], every tempered $\tau \in \mathrm{Irr}(G)$ is a subrepresentation of an induced representation of the form $\delta_1 \times \ldots \times \delta_k \rtimes \sigma$, where the square-integrable representations $\delta_i \in \mathrm{Irr}(GL)$ for $i = 1, \ldots, k$ and $\sigma \in \mathrm{Irr}(G)$ are unique up to a permutation. Assume that $\delta_i \rtimes \sigma \simeq \sigma_{\delta_i} \oplus \sigma_{-\delta_i}$ for every $i = 1, \ldots, k$. Let $j_1, \ldots, j_k \in \{\pm\}$. Then there is the unique irreducible subrepresentation τ of $\delta_1 \times \ldots \times \delta_k \rtimes \sigma$, which is a subrepresentation of

$$\delta_1 \times \ldots \times \delta_{i-1} \times \delta_{i+1} \times \ldots \times \delta_k \rtimes \sigma_{j_i\delta_i}, \text{ for every } i = 1, \ldots, k.$$

Such a representation τ is denoted $\sigma_{j_1\delta_1,\dots,j_k\delta_k}$. In this way, it suffices to parametrize σ_δ to obtain the classification of all tempered representations from $\mathrm{Irr}(G)$. It is the content of Theorem 1.2 from [26] which we now state.

Theorem 2.10. *Let σ denote a square-integrable representation from $Irr(G)$, ρ cuspidal selfdual representation from $Irr(GL)$ and $b \in \mathbb{Z}_{>0}$. Assume that the induced representation $\delta(\rho,b) \rtimes \sigma$ reduces, where $\delta(\rho,b) \simeq \delta([\nu^{-\frac{b-1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho])$.*

(i) *Assume $Jord_\rho(\sigma) \cap [1,b] \neq \emptyset$ and define $a = max(Jord_\rho(\sigma) \cap [1,b])$.*

Then there is a unique irreducible subrepresentation of $\delta(\rho,b) \rtimes \sigma$, denoted σ_δ, such that the following is equivalent:

(1) *σ_δ embeds into $\delta([\nu^{\frac{a-1}{2}+1}\rho, \nu^{\frac{b-1}{2}}\rho])^2 \times \delta([\nu^{-\frac{a-1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \sigma$.*

(2) *σ_δ embeds into $\delta([\nu^{\frac{a-1}{2}+1}\rho, \nu^{\frac{b-1}{2}}\rho])^2 \rtimes \lambda$ for some $\lambda \in Irr(G)$.*

(ii) *Assume $Jord_\rho(\sigma) \cap [1,b] = \emptyset$. Assume that b is even.*

Then there is a unique irreducible subrepresentation of $\delta(\rho,b) \rtimes \sigma$, denoted σ_δ, such that the following is equivalent:

(1) *σ_δ embeds into $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho])^2 \rtimes \sigma$.*

(2) *σ_δ embeds into $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{b-1}{2}}\rho])^2 \rtimes \lambda$ for some $\lambda \in Irr(G)$.*

Assume that b is odd and $Jord_\rho \neq \emptyset$. Define $a := min(Jord_\rho(\sigma))$.

Then there is a unique irreducible subrepresentation of $\delta(\rho,b) \rtimes \sigma$, denoted σ_δ, such that the following is equivalent:

(1) *σ_δ embeds into $\delta([\nu\rho, \nu^{\frac{b-1}{2}}\rho])^2 \times \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \times \rho \rtimes \sigma'$ for a square-integrable representation $\sigma' \in Irr(G)$.*

(2) *σ_δ embeds into $\delta([\nu\rho, \nu^{\frac{b-1}{2}}\rho])^2 \times \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \lambda$ for some $\lambda \in Irr(G)$.*

Moreover, $\sigma \hookrightarrow \delta([\nu\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \sigma'$ for the square-integrable representation σ' as in the last item (1).

In the sequel we denote $\tau_+ := \sigma_\delta$ i $\tau_- := \sigma_{-\delta}$.

Remark 2.11. The representation σ' from Theorem 2.10 can be described in terms of the classification of discrete series representations, as in [26, Theorem 8.2 (5)]. For σ, ρ as in the theorem, let $a \in \mathrm{Jord}_\rho(\sigma)$ and $a \geq 3$. Assume that there is $k \in \mathbb{Z}_{>0}$ such that $[a-2k, a-2] \cap \mathrm{Jord}_\rho(\sigma) = \emptyset$.

Then σ embeds into
$$\nu^{\frac{a-1}{2}}\rho \times \nu^{\frac{a-3}{2}}\rho \times \ldots \times \nu^{\frac{a-2(k-1)-1}{2}}\rho \rtimes \sigma' \tag{5}$$

for square-integrable $\sigma' \in \mathrm{Irr}(G)$. Let $\sigma' \in \mathrm{Irr}(G)$ be an arbitrary square-integrable representation such that σ embeds into (5). If $\sigma'' \in \mathrm{Irr}(G)$ such that

$$\sigma \hookrightarrow \nu^{\frac{a-1}{2}}\rho \times \nu^{\frac{a-3}{2}}\rho \times \ldots \times \nu^{\frac{a-2(k-1)-1}{2}}\rho \rtimes \sigma'',$$

then $\sigma' \simeq \sigma''$. Hence, σ'' is uniquely determined with σ.

We will now introduce some restrictions on the cuspidal representation ρ and real numbers a, b from the definition of π_S. Note that from [12, Lemma 4.2] we get that in $R(G)$ we have

$$L(\delta([\nu^x\rho,\nu^y\rho]),\delta([\nu^{x+1}\rho,\nu^{y+1}\rho]))\rtimes\sigma_c \\ = L(\delta([\nu^{-y-1}\widetilde{\rho},\nu^{-x-1}\widetilde{\rho}]),\delta([\nu^{-y}\widetilde{\rho},\nu^{-x}\widetilde{\rho}]))\rtimes\sigma_c. \quad (6)$$

The equality (6) enables us to pose additional conditions on real numbers a and b. Note that knowing the composition series of $\pi_S \rtimes \sigma_c$ in case $-a = |a| \leq b$ is equivalent to knowing it in the case $|-b-1| \leq -a-1$. Since the composition series of $\pi_S \rtimes \sigma_c$ is determined for $b+1 < 0$ in [4], we can assume $b+1 \geq 0$. In this way, assuming $-a \leq b$, we also solve the case $-a \geq b+2$. Therefore, determining the composition series of $\pi_S \rtimes \sigma_c$ in case $-a < b+2$ solves the problem in general. The next lemma can be easily shown using the Langlands classification and the classification of irreducible tempered representations.

Lemma 2.12. *Assume $\rho \not\simeq \widetilde{\rho}$ or $a \notin \frac{1}{2}\mathbb{Z}$. Then the induced representation $\pi_S \rtimes \sigma_c$ is irreducible.*

In the sequel we assume $\rho \simeq \widetilde{\rho}$ and $a \in \frac{1}{2}\mathbb{Z}$. From Lemma 2.12 and earlier discussion we conclude that it suffices to determine a composition series of a representation $\pi_S \rtimes \sigma_c$ for $-a \leq b$ or $-a = b+1$. For the sake of completeness, we state often used Lemma 5.5 from [8].

Lemma 2.13. *Let π denote an irreducible representation of G_n and M, L Levi factors of parabolic subgroups of G_n such that $M \subset L$. Let λ denote an irreducible representation of M such that $\pi \hookrightarrow Ind_M^{G_n}(\lambda)$. Then there exists an irreducible representation ρ of L such that*

(1) $\pi \hookrightarrow Ind_L^{G_n}(\rho)$,

(2) *ρ is a subquotient of $Ind_M^L(\lambda)$.*

3. Representation $\pi_x \rtimes \sigma_c$

Let us define the representation

$$\pi_x = L(\delta([\nu^a\rho,\nu^b\rho]),\delta([\nu^{x+1}\rho,\nu^{b+1}\rho]))$$

for real numbers x, a, b such that $0 \leq x \leq b$ and $-a \leq b$. In this section we classify irreducible tempered subquotients of the representation $\pi_x \rtimes \sigma_c$. The obtained results will be extensively used to determine the composition series of $\pi_S \rtimes \sigma_c$.

Since $a \leq 0$ and $b \geq 0$, we conclude that $[\pi_x \rtimes \sigma_c]$ contains ρ, or $\nu^{\frac{1}{2}}\rho$ with multiplicity two. In any case, it follows that $\pi_x \rtimes \sigma_c$ does not contain a strongly positive subquotient. Hence, if τ is an irreducible tempered subquotient of the representation $\pi_x \rtimes \sigma_c$, then τ embeds into an induced representation of the form $\delta([\nu^{-z}\rho,\nu^y\rho]) \rtimes \tau'$. Here z, y are real numbers such that $0 \leq z \leq y$ and $\tau' \in \mathrm{Irr}(G)$ is a tempered representation. We will first determine irreducible tempered subquotients of $\pi_x \rtimes \sigma_c$ which are not discrete series representations.

Firstly, we determine all constituents of $\mu^*(\pi_x \rtimes \sigma_c)$ of the form $\delta([\nu^{-y}\rho,\nu^y\rho]) \otimes \pi$, for a non-negative real number y and some $\pi \in \mathrm{Irr}(G)$.

In $R(G)$, the semi-simplification of $\mu^*(\pi_x \rtimes \sigma_c)$ equals

$$\sum_{\mathrm{Lad}(\pi_x)'} L(\delta([\nu^{-c_2}\rho, \nu^{-x-1}\rho]), \delta([\nu^{-c_1}\rho, \nu^{-a}\rho])) \times L(\delta([\nu^{d_1+1}\rho, \nu^{b}\rho]), \delta([\nu^{d_2+1}\rho, \nu^{b+1}\rho])) \quad (7)$$
$$\otimes L(\delta([\nu^{c_1+1}\rho, \nu^{d_1}\rho]), \delta([\nu^{c_2+1}\rho, \nu^{d_2}\rho])) \rtimes \sigma_c. \quad (8)$$

Since $\delta([\nu^{-y}\rho, \nu^{y}\rho])$ is a subquotient of (7), we have $y \in \{-a, b, b+1\}$. We analyze the candidates in each of these cases.

If $y = b+1$, then we have $d_1 + 1 \leq b$ and $d_2 + 1 \leq b+1$ for $\nu^{b+1}\rho$ to be contained in the cuspidal support of the representation (7). From Lemma 2.3 now it directly follows that the representation (7) does not have an essentially square-integrable subquotient. Thus we see that this case is not possible.

Let us now assume $[\nu^{d_1+1}\rho, \nu^{b}\rho] \neq \emptyset$ and $y = b$. Then we have $d_2 = b+1$. Since $-a \leq b$, Lemma 2.3 implies $(c_1, c_2) \in \{(b, x), (a-1, b), (a-1, x)\}$.

- If $(c_1, c_2) = (b, x)$, then $b = c_1 < c_2 = x$, which contradicts the assumption $x \leq b$.
- If $(c_1, c_2) = (a-1, b)$, then $d_1 = -x-1$. The constituents we search for are the subquotients of $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \otimes \nu^{b+1}\rho \times \delta([\nu^{x+1}\rho, \nu^{-a}\rho]) \rtimes \sigma_c$. Here we assume $x \leq -a$, since $a - 1 = c_1 \leq d_1 = -x-1$. If $x = -a$, then necessarily $b+1 = \alpha$. If $0 \leq x \leq -a-1$, then [14, Lemma 3.6] implies that the candidates are equal to $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \otimes \sigma_{(b+1)}$, for $x = \alpha - 1$ and $-a = b$, or $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \otimes \sigma_{(b,b+1)}$, for $-a = b = \alpha - 1$ and $x = \alpha - 2$.
- If $(c_1, c_2) = (a-1, x)$, then $d_1 + 1 = -b$. From the assumption $c_1 \leq d_1$ it follows $b \leq -a$. Hence, we have $b = -a$ and all candidates are the subquotients of $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{x+1}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$. Now it follows $x = \alpha - 1$ and it is easy to notice that these constituents are contained in the second item.

For the meaning of notation $\sigma_{(b)}$ and $\sigma_{(b,b+1)}$, we remind the reader of the notations below the Definition 2.9.

Let us now assume $[\nu^{-c_1}\rho, \nu^{-a}\rho] \neq \emptyset$ and $y = -a$. Then we have $d_1 = b$ and $d_2 = b+1$. Now Lemma 2.3 implies $c_2 = x$ and $c_1 = -a$, so that the constituents in this case are subquotients of

$$\delta([\nu^{a}\rho, \nu^{-a}\rho]) \otimes L(\delta([\nu^{-a+1}\rho, \nu^{b}\rho]), \delta([\nu^{x+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c.$$

Note that from $c_1 < c_2$ it follows $-a < x$. Since on the general linear part we have a ladder representation with positive exponents in the cuspidal support, the irreducible tempered subquotients of

$$L(\delta([\nu^{-a+1}\rho, \nu^{b}\rho]), \delta([\nu^{x+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c$$

are determined in [4, Theorem 3.1]. Thus it is a strongly positive representation and the assumption $x \leq b$ implies $-a+1 = \alpha - 1$ and $x + 1 = \alpha$. Thus we have proved the following proposition.

Proposition 3.1. *If the induced representation $\pi_x \rtimes \sigma_c$ contains an irreducible tempered subquotient which is not discrete series representation, then we have one of the following disjoint cases:*

(i) $x = -a$ *and* $b = \alpha - 1$, (ii) $x = \alpha - 1 \leq -a - 1$ *and* $-a = b$,
(iii) $x = -a + 1 = \alpha - 1$, (iv) $x = \alpha - 2$ *and* $-a = b = \alpha - 1$.

Let us define tempered representations which occur in the analysis in case $y = b$. Assume $1 \leq \alpha \leq b$. For a real number x_2 such that $\alpha - 1 \leq x_2 \leq b$, the results of [17, Section 13] imply

$$\delta([\nu^{-x_2}\rho, \nu^{x_2}\rho]) \rtimes \sigma_{(b+1)} \simeq \tau_{2,+}^{(x_2)} \oplus \tau_{2,-}^{(x_2)}. \tag{9}$$

From [18, Theorem 4.1 (ii)] it follows that the representation $\delta([\nu^{-(\alpha-1)}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$ has a common tempered subquotient with $\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma_{(b+1)}$, which is its unique irreducible subrepresentation. We parametrize the tempered representations $\tau_{2,\pm}^{(x_2)}$ following Theorem 2.10.

If $\alpha \geq 2$, then $\mathrm{Jord}_\rho(\sigma_{(b+1)}) \cap [1, 2x_2+1] \neq \emptyset$. Now the parametrization of irreducible subrepresentations of $\delta([\nu^{-x_2}\rho, \nu^{x_2}\rho]) \rtimes \sigma_{(b+1)}$ follows from Theorem 2.10 (i). More precisely, $\tau_{2,+}^{(x_2)}$ embeds into

$$\delta([\nu^{\alpha-1}\rho, \nu^{x_2}\rho]) \times \delta([\nu^{\alpha-1}\rho, \nu^{x_2}\rho]) \times \delta([\nu^{-(\alpha-2)}\rho, \nu^{\alpha-2}\rho]) \rtimes \sigma_{(b+1)}.$$

If $\alpha \in \{1, \frac{3}{2}\}$, then $\mathrm{Jord}_\rho(\sigma_{(b+1)}) \cap [1, 2x_2 + 1] = \emptyset$. For the parametrizations in this case we use Theorem 2.10 (ii).

- If $\alpha = \frac{3}{2}$, we get $\tau_{2,+}^{(x_2)} \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{x_2}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{x_2}\rho]) \rtimes \sigma_{(b+1)}$, which is a special case of analysis in the case $\alpha \geq 2$.
- If $\alpha = 1$, we get $\tau_{2,+}^{(x_2)} \hookrightarrow \delta([\nu\rho, \nu^{x_2}\rho]) \times \delta([\nu\rho, \nu^{x_2}\rho]) \times \delta([\nu\rho, \nu^{b+1}\rho]) \times \rho \rtimes \pi'$. In this embedding $\pi' \in \mathrm{Irr}(G)$ is a discrete series representation such that $\sigma_{(b+1)} \hookrightarrow \delta([\nu\rho, \nu^{b+1}\rho]) \rtimes \pi'$. From the embedding $\sigma_{(b+1)} \hookrightarrow \delta([\nu\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$ and the Remark 2.11, we conclude $\pi' \simeq \sigma_c$.

Note that for $\alpha \geq \frac{3}{2}$ and $x_2 = \alpha - 1$, a representation of the form $\nu^{\alpha-1}\rho \otimes \nu^{\alpha-1}\rho \otimes \pi''$ for $\pi'' \in \mathrm{Irr}(G)$ is a constituent of the Jacquet module of $\tau_{2,+}^{(\alpha-1)}$ with respect to the appropriate parabolic subgroup. Nevertheless, the same is not true for Jacquet modules of $\delta([\nu^{-(\alpha-1)}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$. This follows easily from the formula (7) and the assumption $b + 1 \geq \alpha$. Thus $\tau_{2,-}^{(\alpha-1)}$ is the unique irreducible subrepresentation of $\delta([\nu^{-(\alpha-1)}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$.

We prove the next proposition under the assumptions from Proposition 3.1 (i). Here the representation π_x is isomorphic to $L(\delta([\nu^{-x}\rho, \nu^{\alpha-1}\rho]), \delta([\nu^{x+1}\rho, \nu^{\alpha}\rho]))$.

Proposition 3.2. *Assume $b = \alpha - 1$ and $-a = x$. The induced representation $\pi_x \rtimes \sigma_c$ contains an irreducible tempered subquotient which is not a discrete series representation if and only if $x = \alpha - 1$. In that case it is unique, embeds into $\pi_x \rtimes \sigma_c$ and for $\alpha = 1$ is isomorphic to $\tau_{2,-}^{(0)}$, while for $\alpha \geq \frac{3}{2}$ is isomorphic to $\tau_{2,+}^{(\alpha-1)}$.*

Proof. If $\alpha = 1$, then since $0 \leq x \leq b = \alpha - 1 = 0$, we have $\pi_x \simeq L(\rho, \nu\rho)$, $\tau_{2,+}^{(0)} \oplus \tau_{2,-}^{(0)} \simeq \rho \rtimes \sigma_{(1)}$ and $\tau_{2,+}^{(0)}$ is parametrized by embedding into $\nu\rho \times \rho \rtimes \sigma_c$. From Lemma 2.13 it follows that $\tau_{2,-}^{(0)}$ embeds into $\delta([\rho, \nu\rho]) \rtimes \sigma_c$ or $L(\rho, \nu\rho) \rtimes \sigma_c \simeq \pi_x \rtimes \sigma_c$.

Because of the embeddings $\tau_{2,-}^{(0)} \hookrightarrow \delta([\rho,\nu\rho]) \rtimes \sigma_c \hookrightarrow \nu\rho \times \rho \rtimes \sigma_c$ and the definition of $\tau_{2,+}^{(0)}$, we see that the first option is not possible. Thus it follows that $\tau_{2,-}^{(0)}$ is a subrepresentation of $\pi_x \rtimes \sigma_c$.

If $\alpha \geq \frac{3}{2}$, then we have two possibilities: $x = \alpha - 1$ and $0 \leq x \leq \alpha - 2$. For $x = \alpha - 1$ we have $\pi_x \simeq L(\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]), \nu^\alpha\rho)$. Let us define the induced representation $\Pi = \delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \times \nu^\alpha\rho \rtimes \sigma_c$. Note that the embeddings

$$\tau_{2,+}^{(\alpha-1)} \oplus \tau_{2,-}^{(\alpha-1)} \simeq \delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma_{(\alpha)} \hookrightarrow \delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \times \nu^\alpha\rho \rtimes \sigma_c$$

and Lemma 2.13 imply that irreducible representations $\tau_{2,+}^{(\alpha-1)}$ and $\tau_{2,-}^{(\alpha-1)}$ embed into $\pi_x \rtimes \sigma_c$ or $\delta([\nu^{-(\alpha-1)}\rho, \nu^\alpha\rho]) \rtimes \sigma_c$. Since we have seen earlier that $\tau_{2,+}^{(\alpha-1)}$ is not a subquotient of $\delta([\nu^{-(\alpha-1)}\rho, \nu^\alpha\rho]) \rtimes \sigma_c$, we obtain that $\tau_{2,+}^{(\alpha-1)}$ is a subrepresentation of $\pi_x \rtimes \sigma_c$.

For $0 \leq x \leq \alpha-2$ we prove that the induced representation $\pi_x \rtimes \sigma_c$ does not contain a tempered subquotient $\tau_{2,\pm}^{(\alpha-1)}$. Namely, if it contains $\tau_{2,\pm}^{(\alpha-1)}$, then the Frobenius reciprocity implies that $\pi_x \rtimes \sigma_c$ contains $\nu^{\alpha-1}\rho \otimes \nu^{\alpha-1}\rho \otimes \delta([\nu^{-(\alpha-2)}\rho, \nu^{\alpha-2}\rho]) \otimes \sigma_{(b+1)}$ or $\delta([\nu^{-(\alpha-1)}\rho, \nu^\alpha\rho]) \otimes \sigma_c$ in the Jacquet module with respect to the appropriate parabolic subgroup. We respectively show that it is not possible.

Note that $\big((-(x+1), x), (\alpha-2, \alpha)\big)$ is the unique element of $\mathrm{Lad}(\pi_x)'$ for which the corresponding constituent in $\mu^*(\pi_x \rtimes \sigma_c)$ is of the form $\nu^{\alpha-1}\rho \otimes \pi'$ for some $\pi' \in R(G)$. Let us denote $\pi_1 = L(\delta([\nu^{-x}\rho, \nu^{\alpha-2}\rho]), \delta([\nu^{x+1}\rho, \nu^\alpha\rho]))$ and observe that $\pi' \simeq \pi_1 \rtimes \sigma_c$. Furthermore, $\mu^*(\pi_1 \rtimes \sigma_c)$ does not contain a constituent of the form $\nu^{\alpha-1}\rho \otimes \pi''$ for some $\pi'' \in R(G)$. Note that for every element $\big((c_1, c_2), (d_1, d_2)\big)$ of $\mathrm{Lad}(\pi_1)'$, the induced representation

$$L(\delta([\nu^{-c_2}\rho, \nu^{-(x+1)}\rho]), \delta([\nu^{-c_1}\rho, \nu^{x}\rho])) \times L(\delta([\nu^{d_1+1}\rho, \nu^{\alpha-2}\rho]), \delta([\nu^{d_2+1}\rho, \nu^\alpha\rho]))$$

is not isomorphic to $\nu^{\alpha-1}\rho$ since every element of $\{-(x+1), x, \alpha-2, \alpha\}$ is different from $\alpha - 1$.

On the other hand, $\big((c,d),(c,d)\big)$ for real numbers c, d such that $c < d$ are the only elements of $\mathrm{Lad}(\pi_x)'$ such that π_2 from $\pi_1 \otimes \pi_2 \leq \mu^*(\pi_x \rtimes \sigma_c)$ is isomorphic to σ_c. Note that Lemma 2.3 implies that if representation

$$L(\delta([\nu^{-d}\rho, \nu^{-(x+1)}\rho]), \delta([\nu^{-c}\rho, \nu^{x}\rho])) \times L(\delta([\nu^{c+1}\rho, \nu^{\alpha-1}\rho]), \delta([\nu^{d+1}\rho, \nu^\alpha\rho]))$$

contains $\delta([\nu^{-(\alpha-1)}\rho, \nu^\alpha\rho])$, then $c = \alpha - 1$ or $d = \alpha$. Since $\delta([\nu^{-(\alpha-1)}\rho, \nu^\alpha\rho])$ contains $\nu^\alpha\rho$ in a cuspidal support and we have $0 \leq x \leq \alpha - 2$, it follows that $c = \alpha - 1$. From $c < d \leq \alpha$ it follows $d = \alpha$, a contradiction.

Finally, let us denote with τ the tempered representation $\tau_{2,\pm}^{(\alpha-1)}$ for which we showed that it is a subrepresentation of $\pi_x \rtimes \sigma_c$. If it has the multiplicity in $\pi_x \rtimes \sigma_c$ greater than or equal to two, we have the following embeddings:

$$\tau \oplus \tau \hookrightarrow \pi_x \rtimes \sigma_c \hookrightarrow \delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \times \nu^\alpha\rho \rtimes \sigma_c.$$

Using Lemma 2.13 and [18, Proposition 3.1], we have two possibilities: $\tau \oplus \tau$ embeds into $\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma_{(\alpha)}$ or $\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes L(\nu^{-\alpha}\rho; \sigma_c)$. The first embedding is not possible since τ has the multiplicity one in that induced representation. The second embedding is not possible because of the following embeddings and the Casselman's criterion:

$$\tau \oplus \tau \hookrightarrow \delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes L(\nu^{-\alpha}\rho; \sigma_c) \hookrightarrow \delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \times \nu^{-\alpha}\rho \rtimes \sigma_c.$$

This finishes the proof of the proposition. ∎

Let us assume the conditions from Proposition 3.1 (ii) in the next proposition. It is proved in a similar manner as Proposition 3.2.

Proposition 3.3. *Assume $-a = b$ and $x = \alpha - 1 \leq -a - 1$. The induced representation $\pi_x \rtimes \sigma_c$ contains the unique irreducible tempered subquotient which is not discrete series representation. It embeds into $\pi_x \rtimes \sigma_c$ and for $\alpha = 1$ is isomorphic to $\tau_{2,-}^{(b)}$, while for $\alpha \geq \frac{3}{2}$ is isomorphic to $\tau_{2,+}^{(b)}$.*

Let us define tempered representations which occur in the analysis in case $y = -a$. Assume $0 \leq \alpha - 1 \leq b$. For a real number x_1 such that $\alpha - 2 \leq x_1 \leq b - 1$, the results of [17, Section 13] imply

$$\delta([\nu^{-x_1}\rho, \nu^{x_1}\rho]) \rtimes \sigma_{(b,b+1)} \simeq \tau_{1,+}^{(x_1)} \oplus \tau_{1,-}^{(x_1)}.$$

From Theorem 4.1 (ii) in [18] it follows that $\delta([\nu^{-(\alpha-2)}\rho, \nu^{b}\rho]) \rtimes \sigma_{(b+1)}$ has a common tempered subquotient with $\delta([\nu^{-(\alpha-2)}\rho, \nu^{\alpha-2}\rho]) \rtimes \sigma_{(b,b+1)}$, which is its unique irreducible subrepresentation. Let us parametrize representations $\tau_{1,\pm}^{(\alpha-2)}$.
If $\alpha \geq 3$, then $\mathrm{Jord}_\rho(\sigma_{(b,b+1)}) \cap [1, 2(\alpha-2)+1] \neq \emptyset$. According to Theorem 2.10 (i), the representation $\tau_{1,+}^{(\alpha-2)}$ embeds into the induced representation

$$\nu^{\alpha-2}\rho \times \nu^{\alpha-2}\rho \times \delta([\nu^{-(\alpha-3)}\rho, \nu^{\alpha-3}\rho]) \rtimes \sigma_{(b,b+1)}.$$

If $\alpha \in \{2, \frac{5}{2}\}$, then $\mathrm{Jord}_\rho(\sigma_{(b,b+1)}) \cap [1, 2(\alpha-2)+1] = \emptyset$. For the parametrizations in this case we use Theorem 2.10 (ii).

- If $\alpha = \frac{5}{2}$, it follows $\tau_{1,+}^{(\alpha-2)} \hookrightarrow \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{(b,b+1)}$, which is a special case of the parametrization in case $\alpha \geq 3$.
- If $\alpha = 2$, it follows $\tau_{1,+}^{(0)} \hookrightarrow \delta([\nu\rho, \nu^{b}\rho]) \times \rho \rtimes \pi'$, where $\pi' \in \mathrm{Irr}(G)$ is such that $\sigma_{(b,b+1)} \hookrightarrow \delta([\nu\rho, \nu^{b}\rho]) \rtimes \pi'$. The embedding $\sigma_{(b,b+1)} \hookrightarrow \delta([\nu\rho, \nu^{b}\rho]) \rtimes \sigma_{(b+1)}$ and Remark 2.11 imply $\pi' \simeq \sigma_{(b+1)}$.

Note that if $\alpha \geq \frac{5}{2}$, then a representation of the form $\nu^{\alpha-2}\rho \otimes \nu^{\alpha-2}\rho \otimes \pi''$, for some $\pi'' \in \mathrm{Irr}(G)$, is a constituent of the Jacquet module of $\tau_{1,+}^{(\alpha-2)}$ with respect to the appropriate parabolic subgroup. This is not a case with the Jacquet modules of $\delta([\nu^{-(\alpha-2)}\rho, \nu^{b}\rho]) \rtimes \sigma_{(b+1)}$, which easily follows from the formula (7) and the assumption $b \geq \alpha - 1$. Thus $\tau_{1,-}^{(\alpha-2)} \hookrightarrow \delta([\nu^{-(\alpha-2)}\rho, \nu^{b}\rho]) \rtimes \sigma_{(b+1)}$.
Let us assume the conditions from Proposition 3.1 (iii) in the next proposition. Proof of the following proposition follows the lines of the proof of Proposition 3.2.

Proposition 3.4. *Assume $x = -a + 1 = \alpha - 1$. The induced representation $\pi_x \rtimes \sigma_c$ contains the unique irreducible tempered subquotient which is not a discrete series representation. It embeds into $\pi_x \rtimes \sigma_c$ and for $\alpha = 2$ is isomorphic to $\tau_{1,+}^{(0)}$, while for $\alpha \geq \frac{5}{2}$ is isomorphic to $\tau_{1,-}^{(\alpha-2)}$.*

We have seen that, under the assumptions from Proposition 3.1 (iv), the irreducible induced representation $\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma_{(\alpha-1,\alpha)}$ is a candidate for a subquotient of representation $\pi_x \rtimes \sigma_c$. Here, the representation π_x is isomorphic to $L(\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]), \delta([\nu^{\alpha-1}\rho, \nu^{\alpha}\rho]))$. Note that a representation of the form $\nu^{\alpha-1}\rho \otimes \nu^{\alpha-1}\rho \otimes \nu^{\alpha-1}\rho \otimes \pi'$, for some $\pi' \in \mathrm{Irr}(G)$, is a constituent of the Jacquet module of $\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma_{(\alpha-1,\alpha)}$ with respect to the appropriate parabolic subgroup. This follows easily from the formula (3) and the fact that $\nu^{\alpha-1}\rho \otimes \sigma_{(\alpha)}$ is a constituent of $\mu^*(\sigma_{(\alpha-1,\alpha)})$. On the other hand, this is not the case for the Jacquet modules of $\pi_x \rtimes \sigma_c$. Namely, every constituent of the form $\nu^{\alpha-1}\rho \otimes \nu^{\alpha-1}\rho \otimes \pi''$, for some $\pi'' \in \mathrm{Irr}(G)$, of the Jacquet module of $\pi_x \rtimes \sigma_c$ with respect to the appropriate parabolic subgroup is a subquotient of

$$\nu^{\alpha-1}\rho \otimes \nu^{\alpha-1}\rho \otimes L(\delta([\nu^{-(\alpha-2)}\rho, \nu^{\alpha-2}\rho]), \delta([\nu^{\alpha-1}\rho, \nu^{\alpha}\rho])) \rtimes \sigma_c.$$

But from the formula (3) we see that if $\pi_1 \otimes \pi_2$ is a constituent of

$$\mu^*(L(\delta([\nu^{-(\alpha-2)}\rho, \nu^{\alpha-2}\rho]), \delta([\nu^{\alpha-1}\rho, \nu^{\alpha}\rho])) \rtimes \sigma_c)$$

such that $\nu^{\alpha-1}\rho \in [\pi_1]$, then $\nu^{\alpha}\rho \in [\pi_1]$. Hence, $\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma_{(\alpha-1,\alpha)}$ is not a subquotient of $\pi_x \rtimes \sigma_c$. In this way, we have finished the analysis of irreducible tempered subquotients of the induced representation $\pi_x \rtimes \sigma_c$.

Let us now analyze the discrete series subquotients of $\pi_x \rtimes \sigma_c$. We use the classification of discrete series representations to determine possible discrete series subquotients of $\pi_x \rtimes \sigma_c$. More precisely, a non-strongly positive discrete series representation σ embeds into an induced representation of the form

$$\delta([\nu^{-z}\rho, \nu^{y}\rho]) \rtimes \sigma',$$

where σ' is a discrete series representation and y, z are real numbers such that $0 \leq z < y$. Moreover, in $\mathrm{Jord}_\rho(\sigma)$ we have $(2y+1)\_ = 2z+1$. The Frobenius reciprocity implies that $\delta([\nu^{-z}\rho, \nu^{y}\rho]) \otimes \sigma'$ is a constituent of $\mu^*(\sigma)$. In this way, we determine candidates for discrete series subquotients of $\pi_x \rtimes \sigma_c$ according to the constituents of $\mu^*(\pi_x \rtimes \sigma_c)$ of described form.

In the same way as in the analysis of tempered subquotients of $\pi_x \rtimes \sigma_c$ which are not discrete series representations, we get $y \in \{-a, b\}$. Let us first analyze the case where $[\nu^{d_1+1}\rho, \nu^{b}\rho] \neq \emptyset$ and $y = b$. The following constituents of $\mu^*(\pi_x \rtimes \sigma_c)$ are of the required form:

1. $\delta([\nu^{-c_1}\rho, \nu^{b}\rho]) \otimes L(\delta([\nu^{c_1+1}\rho, \nu^{-a}\rho]), \delta([\nu^{x+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c$, for $d_1 = -a$, $c_2 = x$ and $c_1 \geq 0$,
2. $\delta([\nu^{-c_2}\rho, \nu^{b}\rho]) \otimes L(\delta([\nu^{a}\rho, \nu^{-x-1}\rho]), \delta([\nu^{c_2+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c$, for $d_1 = -x-1$, $c_1 = a-1$ and $c_2 \geq 0$,
3. $\delta([\nu^{d_1+1}\rho, \nu^{b}\rho]) \otimes L(\delta([\nu^{a}\rho, \nu^{d_1}\rho]), \delta([\nu^{x+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c$, for $c_1 = a-1$, $c_2 = x$ and $d_1 + 1 \leq 0$.

In the first case, from $c_1 + 1 \geq 1$, $x + 1 \geq 1$ and $c_1 < c_2 = x$, it follows that

$$L(\delta([\nu^{c_1+1}\rho, \nu^{-a}\rho]), \delta([\nu^{x+1}\rho, \nu^{b+1}\rho]))$$

is a ladder representation with exponents in the cuspidal support greater than or equal to one.

Theorem 3.1 from [4] describes irreducible tempered subquotients of the induced representation $L(\delta([\nu^{c_1+1}\rho,\nu^{-a}\rho]),\delta([\nu^{x+1}\rho,\nu^{b+1}\rho]))\rtimes\sigma_c$.

Let us consider the case $c_1=-a$. We necessarily have $x=\alpha-1$. Hence, the candidates for discrete series representations are subquotients of $\delta([\nu^a\rho,\nu^b\rho])\rtimes\sigma_{(b+1)}$. The classification of discrete series representations implies $(2b+1)\_=2(-a)+1$. Now from the form of the set $\mathrm{Jord}_\rho(\sigma_{(b+1)})$ we get $\alpha-1=x\le -a$. This is a contradiction with the assumptions $-a=c_1<c_2=x$.

On the other hand, if both square-integrable parameters are non-trivial representations, then $c_1+1=\alpha-1$ and $x+1=\alpha$. Hence, the candidates for discrete series representations are subquotients of $\delta([\nu^{-(\alpha-2)}\rho,\nu^b\rho])\rtimes\sigma_{(-a,b+1)}$. Note that we have $(2b+1)\_=2(\alpha-2)+1,\ 2(-a)+1\in\mathrm{Jord}_\rho(\sigma_{(-a,b+1)})$ and $\alpha-1\le -a\le b$. This is a contradiction with the classification of discrete series representations. Hence, in the first case we do not have candidates for discrete series subquotients of $\pi_x\rtimes\sigma_c$.

In the second case, the result [14, Lemma 3.6] determines discrete series subquotients of the induced representation $L(\delta([\nu^a\rho,\nu^{-x-1}\rho]),\delta([\nu^{c_2+1}\rho,\nu^{b+1}\rho]))\rtimes\sigma_c$. In the Grothendieck group $R(G)$, it is equal to

$$\delta([\nu^{x+1}\rho,\nu^{-a}\rho])\times\delta([\nu^{c_2+1}\rho,\nu^{b+1}\rho])\rtimes\sigma_c. \tag{10}$$

If $0\le x\le -a-1$, then the discrete series subquotient of (10) is isomorphic to $\sigma_{(-a,b+1)}$ or it embeds into $\delta([\nu^a\rho,\nu^b\rho])\rtimes\sigma_{(b+1)}$. Repeating the arguments from the first case, we see that it is not isomorphic to $\sigma_{(-a,b+1)}$. The second option assumes $x=\alpha-1$ and $c_2=-a$. If $x=-a$, then the discrete series subquotient of (10) embeds into $\delta([\nu^{-(\alpha-1)}\rho,\nu^b\rho])\rtimes\sigma_{(b+1)}$. Here we assume $c_2=\alpha-1$. Altogether, note that the conditions under which we have obtained these candidates are $-a\ge\alpha$ and $x=\alpha-1$, if $0\le x\le -a-1$, and $-a=x\le\alpha-1$, if $-a\le x$.

In the third case, we need to determine the discrete series subquotients of the induced representation $L(\delta([\nu^a\rho,\nu^{d_1}\rho]),\delta([\nu^{x+1}\rho,\nu^{b+1}\rho]))\rtimes\sigma_c$. In the Grothendieck group $R(G)$, it is equal to

$$\delta([\nu^{-d_1}\rho,\nu^{-a}\rho])\times\delta([\nu^{x+1}\rho,\nu^{b+1}\rho])\rtimes\sigma_c. \tag{11}$$

Again by the result [14, Lemma 3.6], the induced representation (11) contains a discrete series subquotient if $(d_1,x)\in\{(-\alpha,-a),(a-1,\alpha-1),(-\alpha+1,\alpha-1)\}$. If $(d_1,x)=(-\alpha+1,\alpha-1)$, the candidates are discrete series subrepresentations of $\delta([\nu^{-(\alpha-2)}\rho,\nu^b\rho])\rtimes\sigma_{(-a,b+1)}$ obtained by the classification. We have seen in the first case that there are no such candidates. Hence, we are left to analyze the possibilities with $\sigma_{(b+1)}$ on the classical part. We consider discrete series representations which embed into the following induced representations:

- $\delta([\nu^{-(\alpha-1)}\rho,\nu^b\rho])\rtimes\sigma_{(b+1)}$, if $(d_1,x)=(-\alpha,-a)$,
- $\delta([\nu^a\rho,\nu^b\rho])\rtimes\sigma_{(b+1)}$, if $(d_1,x)=(a-1,\alpha-1)$.

Note that $-a\ge\alpha-1$ is the common assumption under which we obtain these discrete series representations for the candidates.

Note that we have showed that $\delta([\nu^a\rho,\nu^b\rho])\otimes\sigma_{(b+1)}$ is a constituent of $\mu^*(\pi_x\rtimes\sigma_c)$ of multiplicity two, if $0\le x\le -a-1$ and $x=\alpha-1$. If $x=-a$, the representation $\delta([\nu^{-(\alpha-1)}\rho,\nu^b\rho])\otimes\sigma_{(b+1)}$ is a constituent of $\mu^*(\pi_x\rtimes\sigma_c)$ of multiplicity one.

Let us finally analyze the case where $[\nu^{-c_1}\rho, \nu^{-a}\rho] \neq \emptyset$ and $y = -a$. We obtain candidates from the constituent

$$\delta([\nu^{-c_1}\rho, \nu^{-a}\rho]) \otimes L(\delta([\nu^{c_1+1}\rho, \nu^{b}\rho]), \delta([\nu^{x+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c \tag{12}$$

of $\mu^*(\pi_x \rtimes \sigma_c)$ for $\big((c_1, x), (b, b+1)\big) \in \mathrm{Lad}(\pi_x)'$. Since we have assumed $c_1 \geq 0$, the only candidate for tempered subquotient of a representation on the second factor of (12) is a strongly positive representation. From Theorem 3.1 in [4] it follows $c_1 + 1 = \alpha - 1$ and $x + 1 = \alpha$. According to [18, Theorem 2.1], if $-a < b$, the induced representation $\delta([\nu^{-(\alpha-2)}\rho, \nu^{-a}\rho]) \rtimes \sigma_{(b,b+1)}$ contains exactly two discrete series subrepresentations. We conclude that at most one of them is a subquotient of $\pi_x \rtimes \sigma_c$. Namely, we have shown that all candidates are subquotients of (12) and $\sigma_{(b,b+1)}$ is the multiplicity-one subquotient of the representation on the classical part. If $-a \leq x$, then from $-a \leq x = \alpha - 1$ and $\alpha - 2 < -a$ it follows $-a = x = \alpha - 1$. If $0 \leq x \leq -a - 1$, then from $x = \alpha - 1$ it follows $\alpha \leq -a$. The next lemma will be useful throughout the paper.

Lemma 3.5. *Let σ denote a discrete series representation which embeds into the induced representation $\delta([\nu^{-y}\rho, \nu^{b}\rho]) \rtimes \sigma_{(b+1)}$ for real numbers y, b such that $\max\{0, \alpha - 1\} \leq y < b$ and $\epsilon_\sigma((2b+1, \rho), (2(b+1)+1, \rho)) = -1$.*
Then the representation of the form $\nu^{b+1}\rho \otimes \pi'$ for some $\pi' \in Irr(G)$ is not a constituent of $\mu^*(\sigma)$.

Proof. Firstly, note that Lemma 2.13 and the following embeddings

$$\sigma \hookrightarrow \delta([\nu^{-y}\rho, \nu^{b}\rho]) \rtimes \sigma_{(b+1)} \hookrightarrow \delta([\nu^{-y}\rho, \nu^{b}\rho]) \times \nu^{b+1}\rho \rtimes \sigma_{(b)}$$

imply that σ embeds into $\delta([\nu^{-y}\rho, \nu^{b+1}\rho]) \rtimes \sigma_{(b)}$ or $L(\delta([\nu^{-y}\rho, \nu^{b}\rho]), \nu^{b+1}\rho) \rtimes \sigma_{(b)}$. If it embeds into $\delta([\nu^{-y}\rho, \nu^{b+1}\rho]) \rtimes \sigma_{(b)}$, then from the result [13, Proposition 3.2] it follows $\sigma \simeq \sigma_{ds}$, for the discrete series representation σ_{ds} defined there. This is not possible since the values of ϵ_σ and $\epsilon_{\sigma_{ds}}$ on $((2b+1, \rho), (2(b+1)+1, \rho))$ differentiate. Hence, $\sigma \hookrightarrow L(\delta([\nu^{-y}\rho, \nu^{b}\rho]), \nu^{b+1}\rho) \rtimes \sigma_{(b)}$. The formula (3) implies that $\mu^*(L(\delta([\nu^{-y}\rho, \nu^{b}\rho]), \nu^{b+1}\rho) \rtimes \sigma_{(b)})$ does not contain a constituent of the form $\nu^{b+1}\rho \otimes \pi'$ for some $\pi' \in \mathrm{Irr}(G)$. This proves the lemma. ∎

Let us denote a discrete series subrepresentation of $\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \sigma_{(b+1)}$ such that $\epsilon_{\sigma_0}((2b+1, \rho), (2(b+1)+1, \rho)) = -1$ with σ_0. The other one is denoted σ_0'.

Proposition 3.6. *If the induced representation $\pi_x \rtimes \sigma_c$ has a discrete series subquotient, then $0 \leq x \leq -a < b$ and $x = \alpha - 1$. In that case, if $\alpha \geq 2$, discrete series representation embeds into $\delta([\nu^{-(\alpha-2)}\rho, \nu^{-a}\rho]) \rtimes \sigma_{(b,b+1)}$, while if $\alpha \in \{1, \frac{3}{2}\}$, it is isomorphic to σ_0.*

Proof. Analyzing the constituents of $\mu^*(\pi_x \rtimes \sigma_c)$, we have seen that candidates for discrete series subquotients of $\pi_x \rtimes \sigma_c$ are subrepresentations of

(i) $\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \sigma_{(b+1)}$ or $\delta([\nu^{-(\alpha-2)}\rho, \nu^{-a}\rho]) \rtimes \sigma_{(b,b+1)}$, if $x \leq -a$ and $x = \alpha - 1$,

(ii) $\delta([\nu^{-(\alpha-1)}\rho, \nu^{b}\rho]) \rtimes \sigma_{(b+1)}$, if $x = -a$ and $x \neq \alpha - 1$.

Note that the Jordan blocks of discrete series representations in (i) are equal. Namely, we have $\mathrm{Jord}_\rho = \{2(-a)+1, 2b+1, 2(b+1)+1\} \cup \mathrm{Jord}_\rho(\sigma_c) \setminus \{2(\alpha-1)+1\}$.

The discrete series subrepresentations of $\delta([\nu^a\rho,\nu^b\rho])\rtimes\sigma_{(b+1)}$ are characterized by the values of the ϵ-function:

$$\epsilon((2(\alpha-2)+1,\rho),(2(-a)+1,\rho))\cdot\epsilon((2b+1,\rho),(2(b+1)+1,\rho))=-1,$$

$\epsilon((2(-a)+1,\rho),(2b+1,\rho))=1$ and $\epsilon((x\_,\rho),(x,\rho))=-1$ on other $x\in\mathrm{Jord}_\rho$.

The discrete series representation σ_0' such that

$$\epsilon_{\sigma_0'}((2(\alpha-2)+1,\rho),(2(-a)+1,\rho))=-1,\ \epsilon_{\sigma_0'}((2b+1,\rho),(2(b+1)+1,\rho))=1,$$

embeds into $\delta([\nu^{-b}\rho,\nu^{b+1}\rho])\rtimes\sigma_c\hookrightarrow\nu^{b+1}\rho\times\delta([\nu^{-b}\rho,\nu^{b}\rho])\rtimes\sigma_c$. The Frobenius reciprocity now implies that $\mu^*(\sigma_0')$ has a constituent of the form $\nu^{b+1}\rho\otimes\pi'$ for some $\pi'\in\mathrm{Irr}(G)$. The formula (3) shows that the same is not the case for the Jacquet modules of $\pi_x\rtimes\sigma_c$. Hence, the only candidate is the discrete series representation σ_0 such that

$$\epsilon_{\sigma_0}((2(\alpha-2)+1,\rho),(2(-a)+1,\rho))=1,\ \epsilon_{\sigma_0}((2b+1,\rho),(2(b+1)+1,\rho))=-1.$$

Hence, we see that for $\alpha\geq 2$ discrete series representation σ_0 also embeds into $\delta([\nu^{-(\alpha-2)}\rho,\nu^{-a}\rho])\rtimes\sigma_{(b,b+1)}$.

Next we show that, in case $x=-a$ and $x\neq\alpha-1$, the discrete series subquotients of representation in (ii) are not subquotients of $\pi_x\rtimes\sigma_c$. Let us denote by σ a discrete series subquotient of (ii) such that $\epsilon_\sigma((2b+1,\rho),(2(b+1)+1,\rho))=-1$. For the other discrete series subquotient σ', in the analogous way as for σ_0', one can see that it is not a subquotient of $\pi_x\rtimes\sigma_c$. In the following cases we prove that σ is also not a subquotient of $\pi_x\rtimes\sigma_c$.

Assume that $\alpha=1$. According to the classification of discrete series representations, σ embeds into $\delta([\nu\rho,\nu^b\rho])\rtimes\tau^{(0)}_{2,\pm}$. The parametrization of $\tau^{(0)}_{2,+}$ implies $\tau^{(0)}_{2,+}\hookrightarrow\delta([\nu\rho,\nu^{b+1}\rho])\times\rho\rtimes\sigma_c$. Hence, we have the following embeddings

$$\begin{aligned}\delta([\nu\rho,\nu^b\rho])\rtimes\tau^{(0)}_{2,+}&\hookrightarrow\delta([\nu\rho,\nu^b\rho])\times\delta([\nu\rho,\nu^{b+1}\rho])\times\rho\rtimes\sigma_c\\&\simeq\delta([\nu\rho,\nu^{b+1}\rho])\times\delta([\nu\rho,\nu^b\rho])\times\rho\rtimes\sigma_c\hookrightarrow\nu^{b+1}\rho\rtimes\pi'',\end{aligned}$$

for some $\pi''\in\mathrm{Irr}(G)$. From Lemma 3.5 it follows $\sigma'\hookrightarrow\delta([\nu\rho,\nu^b\rho])\rtimes\tau^{(0)}_{2,+}$. Moreover, from the parametrization of $\tau^{(0)}_{2,+}$ and Lemma 2.13, it follows that $\tau^{(0)}_{2,-}$ embeds into $L(\rho,\delta([\nu\rho,\nu^{b+1}\rho]))\rtimes\sigma_c$. If σ is a subquotient of $\pi_x\rtimes\sigma_c$, then $\delta([\nu\rho,\nu^b\rho])\otimes\tau^{(0)}_{2,-}$ is a constituent of $\mu^*(\pi_x\rtimes\sigma_c)$. From the analysis of the constituents of $\mu^*(\pi_x\rtimes\sigma_c)$ of the form $\delta([\nu\rho,\nu^b\rho])\otimes\pi$ for some $\pi\in R(G)$, it follows that π is isomorphic to $L(\delta([\nu^{-x}\rho,\rho]),\delta([\nu^{x+1}\rho,\nu^{b+1}\rho]))\rtimes\sigma_c$ and $\tau^{(0)}_{2,-}\leq\pi$. Since $x\neq 0$, in $R(GL)$ we have

$$\begin{aligned}L(\delta([\nu^{-x}\rho,\rho]),\delta([\nu^{x+1}\rho,\nu^{b+1}\rho]))&=\delta([\rho,\nu^x\rho])\times\delta([\nu^{x+1}\rho,\nu^{b+1}\rho])\\&=\delta([\rho,\nu^{b+1}\rho])+L(\delta([\rho,\nu^x\rho]),\delta([\nu^{x+1}\rho,\nu^{b+1}\rho])).\end{aligned}$$

It is easy to see that we necessarily have $\tau^{(0)}_{2,-}\leq L(\delta([\rho,\nu^x\rho]),\delta([\nu^{x+1}\rho,\nu^{b+1}\rho]))\rtimes\sigma_c$. On the other hand, one can see that $L(\rho,\delta([\nu\rho,\nu^{b+1}\rho]))\otimes\sigma_c$ is not a constituent of $\mu^*(L(\delta([\rho,\nu^x\rho]),\delta([\nu^{x+1}\rho,\nu^{b+1}\rho]))\rtimes\sigma_c)$.

Now assume $\alpha\geq\frac{3}{2}$. We will show that σ embeds into a representation of the form $\nu^{\alpha-1}\rho\rtimes\pi'$ for some $\pi'\in\mathrm{Irr}(G)$. The Frobenius reciprocity then implies that σ is not a subquotient of $\pi_x\rtimes\sigma_c$. Precisely, the assumptions $x\neq\alpha-1$ and $\alpha-1<b$ imply that $\mu^*(\pi_x\rtimes\sigma_c)$ does not have a constituent of the form $\nu^{\alpha-1}\rho\otimes\pi'$. We have shown $\tau^{(\alpha-1)}_{2,-}\hookrightarrow\delta([\nu^{-(\alpha-1)}\rho,\nu^{b+1}\rho])\rtimes\sigma_c$.

Because of the isomorphism

$$\delta([\nu^{\alpha}\rho,\nu^{b}\rho])\times\delta([\nu^{-(\alpha-1)}\rho,\nu^{b+1}\rho])\rtimes\sigma_c\simeq\delta([\nu^{-(\alpha-1)}\rho,\nu^{b+1}\rho])\times\delta([\nu^{\alpha}\rho,\nu^{b}\rho])\rtimes\sigma_c$$

and Lemma 3.5, we have $\sigma\hookrightarrow\delta([\nu^{\alpha}\rho,\nu^{b}\rho])\rtimes\tau_{2,+}^{(\alpha-1)}$. The parametrization of $\tau_{2,+}^{(\alpha-1)}$ implies $\sigma\hookrightarrow\delta([\nu^{\alpha}\rho,\nu^{b}\rho])\times\nu^{\alpha-1}\rho\times\nu^{\alpha-1}\rho\rtimes\pi'$, for some $\pi'\in\mathrm{Irr}(G)$. If σ embeds into $\delta([\nu^{\alpha-1}\rho,\nu^{b}\rho])\times\nu^{\alpha-1}\rho\rtimes\pi'$, then the claim is true because of the isomorphism $\delta([\nu^{\alpha-1}\rho,\nu^{b}\rho])\times\nu^{\alpha-1}\rho\simeq\nu^{\alpha-1}\rho\times\delta([\nu^{\alpha-1}\rho,\nu^{b}\rho])$. Otherwise, the claim follows from the embedding $L(\nu^{\alpha-1}\rho,\delta([\nu^{\alpha}\rho,\nu^{b}\rho]))\hookrightarrow\nu^{\alpha-1}\rho\times\delta([\nu^{\alpha}\rho,\nu^{b}\rho])$. This finishes the proof of the proposition. ∎

Proposition 3.7. *The induced representation $\pi_x\rtimes\sigma_c$ has a discrete series subquotient if and only if $0\le x\le -a<b$ and $x=\alpha-1$. In that case, $\pi_x\rtimes\sigma_c$ has the unique discrete series subquotient; it is isomorphic to σ_0 and embeds into $\pi_x\rtimes\sigma_c$.*

Proof. If the induced representation $\pi_x\rtimes\sigma_c$ has a discrete series subquotient, then Proposition 3.6 implies $0\le x\le -a<b$ and $x=\alpha-1$. Conversely, assume that these conditions hold. Because of the following embeddings

$$\sigma_0\hookrightarrow\delta([\nu^{a}\rho,\nu^{b}\rho])\rtimes\sigma_{(b+1)}\hookrightarrow\delta([\nu^{a}\rho,\nu^{b}\rho])\times\delta([\nu^{\alpha}\rho,\nu^{b+1}\rho])\rtimes\sigma_c$$

and Lemma 2.13, the discrete series representation σ_0 embeds into the induced representations $\pi_x\rtimes\sigma_c$ or $\delta([\nu^{a}\rho,\nu^{b+1}\rho])\times\delta([\nu^{\alpha}\rho,\nu^{b}\rho])\rtimes\sigma_c$. If σ_0 embeds into

$$\delta([\nu^{a}\rho,\nu^{b+1}\rho])\times\delta([\nu^{\alpha}\rho,\nu^{b}\rho])\rtimes\sigma_c\hookrightarrow\nu^{b+1}\rho\times\delta([\nu^{a}\rho,\nu^{b}\rho])\times\delta([\nu^{\alpha}\rho,\nu^{b}\rho])\rtimes\sigma_c,$$

the Frobenius reciprocity implies that $\mu^*(\sigma_0)$ contains a constituent of the form $\nu^{b+1}\rho\otimes\pi'$, for some $\pi'\in\mathrm{Irr}(G)$. But according to Lemma 3.5, this is not possible, and thus σ_0 embeds into $\pi_x\rtimes\sigma_c$.

Now we are in a position to prove that the other discrete series representation of $\delta([\nu^{-(\alpha-2)}\rho,\nu^{-a}\rho])\rtimes\sigma_{(b,b+1)}$ is not a subquotient of $\pi_x\rtimes\sigma_c$. We have shown that $\delta([\nu^{-(\alpha-2)}\rho,\nu^{-a}\rho])\otimes\sigma_{(b,b+1)}$ is a constituent of $\mu^*(\pi_x\rtimes\sigma_c)$ of multiplicity one. From the exactness of Jacquet modules, it follows that it is obtained from $\mu^*(\sigma_0)$ since σ_0 is a subquotient of $\pi_x\rtimes\sigma_c$. ∎

Results of this section are summarized in the following theorem.

Theorem 3.8. *The induced representation*

$$\pi_x\rtimes\sigma_c=L(\delta([\nu^{a}\rho,\nu^{b}\rho]),\delta([\nu^{x+1}\rho,\nu^{b+1}\rho]))\rtimes\sigma_c$$

such that $0\le x\le b$ and $-a\le b$ has

(i) *an irreducible tempered subquotient which is not a discrete series representation if and only if $x=-a+1=\alpha-1$ and $x=\alpha-1\le -a=b$.*

(ii) *a discrete series representation if and only if $x=\alpha-1\le -a<b$.*

In that case, the irreducible tempered subquotient is unique and it is a subrepresentation of $\pi_x\rtimes\sigma_c$. We denote it by τ.

If $x=-a+1=\alpha-1$, τ is isomorphic to $\tau_{1,+}^{(0)}$ for $\alpha=2$ and to $\tau_{1,-}^{(\alpha-2)}$ for $\alpha\ge\frac{5}{2}$.

If $x=\alpha-1\le -a=b$, τ is isomorphic to $\tau_{2,-}^{(b)}$ for $\alpha=1$ and to $\tau_{2,+}^{(b)}$ for $\alpha\ge\frac{3}{2}$.

If $x=\alpha-1\le -a<b$, τ is isomorphic to σ_0.

Proof. Note that the proof of this theorem follows from Proposition 3.4 in the first case, from Propositions 3.2 and 3.3 in the second case and from Proposition 3.7 in the third case. ■

4. Tempered subquotients of $\pi_S \rtimes \sigma_c$

In this section we determine irreducible tempered subquotients of the induced representation $\pi_S \rtimes \sigma_c$. Recall, the representation π_S is isomorphic to

$$L(\delta([\nu^a\rho, \nu^b\rho]), \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]))$$

for real numbers a, b such that $a \leq 0$ and $-a \leq b$ or $b = -a-1$. The following lemma will be important in the rest of this section. It is a direct consequence of the classification of discrete series representations.

Lemma 4.1. *Let π denote a representation in $R(G)$ such that the multiset $[\pi]_{GL}$ is non-empty and its elements are of the form $\nu^x\rho$ for real number x and the cuspidal representation ρ fixed earlier. Let M denote the maximal exponent in $[\pi]_{GL}$.*

(i) If a discrete series representation σ is a subquotient of π, then $2M+1$ is an element of $Jord_\rho(\sigma)$.

(ii) *If M is of multiplicity at least two in $[\pi]_{GL}$, then π does not contain a discrete series subquotient.*

We will first resolve the case $b = -a-1$. Note that π_S is the Speh representation since it is isomorphic to the Langlands subrepresentation of

$$\nu^{-\frac{1}{2}}\delta([\nu^{a+\frac{1}{2}}\rho, \nu^{-a-\frac{1}{2}}\rho]) \times \nu^{\frac{1}{2}}\delta([\nu^{a+\frac{1}{2}}\rho, \nu^{-a-\frac{1}{2}}\rho]).$$

From the formula (3), it follows that an irreducible tempered subquotient τ of $\pi_S \rtimes \sigma_c$ embeds into

$$\delta([\nu^{-z}\rho, \nu^y\rho]) \rtimes \tau',$$

where τ' is an irreducible tempered representation and y, z are real numbers such that $0 \leq z \leq y$ and $y \in \{-a-1, -a\}$.

If $y = -a$, then we have $\tau \hookrightarrow \delta([\nu^{-z}\rho, \nu^{-a}\rho]) \rtimes \tau' \hookrightarrow \nu^{-a}\rho \times \delta([\nu^{-z}\rho, \nu^{-a-1}\rho]) \rtimes \tau'$. Hence, $\mu^*(\tau)$ has a constituent of the form $\nu^{-a}\rho \otimes \pi'$, for some $\pi'' \in R(G)$. This is not the case for $\mu^*(\pi_S \rtimes \sigma_c)$. In this way we see that the tempered subrepresentations which occur in that case are not subquotients of $\pi_S \rtimes \sigma_c$.

Let us analyze the case $y = -a-1$. In $\mu^*(\pi_S \rtimes \sigma_c)$, the representations of the form $\delta([\nu^{-y}\rho, \nu^{-a-1}\rho]) \otimes \tau'$ are subquotients of

$$\delta([\nu^{d_1+1}\rho, \nu^{-a-1}\rho]) \otimes L(\delta([\nu^a\rho, \nu^{d_1}\rho]), \delta([\nu^{a+1}\rho, \nu^{-a}\rho])) \rtimes \sigma_c \tag{13}$$

for $((a-1, a), (d_1, -a)) \in \mathrm{Lad}(\pi_S)'$ and

$$\delta([\nu^{-c_2}\rho, \nu^{-a-1}\rho]) \otimes L(\delta([\nu^a\rho, \nu^{-a-1}\rho]), \delta([\nu^{c_2+1}\rho, \nu^{-a}\rho])) \rtimes \sigma_c \tag{14}$$

for $((a-1, c_2), (-a-1, -a)) \in \mathrm{Lad}(\pi_S)'$. Let us denote with $\pi_1 \otimes \pi_2$ any of the representations (13) and (14). Since $-a$ is the maximal exponent in $[\pi_2]_{\mathrm{GL}}$ of multiplicity at least two, from Lemma 4.1 (ii) it follows that π_2 does not contain a discrete series subquotient.

In this way, we have reduced the set of candidates for the tempered subquotient of $\pi_S \rtimes \sigma_c$ to the set of subquotients of

$$\delta([\nu^{a+1}\rho, \nu^{-a-1}\rho]) \times L(\delta([\nu^{a}\rho, \nu^{-a-1}\rho]), \nu^{-a}\rho) \rtimes \sigma_c$$

obtained for $d_1 = a$ or $c_2 = -a-1$. Furthermore, note that the candidates for tempered subquotients of $L(\delta([\nu^{a}\rho, \nu^{-a-1}\rho]), \nu^{-a}\rho) \rtimes \sigma_c$ are not obtained from $\delta([\nu^{a+1}\rho, \nu^{-a-1}\rho]) \otimes \nu^{a}\rho \times \nu^{-a}\rho \rtimes \sigma_c$. Namely, the assumption $-a \geq 1$ implies that the induced representation $\nu^{a}\rho \times \nu^{-a}\rho \rtimes \sigma_c$ does not contain a tempered subquotient. Hence, the tempered subquotients necessarily embed into $\delta([\nu^{a}\rho, \nu^{-a}\rho]) \rtimes \tau'''$, for some tempered representation $\tau'' \in \mathrm{Irr}(G)$. Thus τ embeds into

$$\delta([\nu^{a+1}\rho, \nu^{-a-1}\rho]) \times \delta([\nu^{a}\rho, \nu^{-a}\rho]) \rtimes \tau'' \simeq \delta([\nu^{a}\rho, \nu^{-a}\rho]) \times \delta([\nu^{a+1}\rho, \nu^{-a-1}\rho]) \rtimes \tau''.$$

In an analogous way as in the beginning of the case, the embedings show that τ is not a subquotient of $\pi_S \rtimes \sigma_c$.

In the rest of this section we consider the case $-a \leq b$. Firstly, we determine tempered subquotients of $\pi_S \rtimes \sigma_c$ which are not discrete series representations. As earlier, denote with τ an irreducible tempered representation which is a candidate for a subquotient of $\pi_S \rtimes \sigma_c$. Note that $y \neq -a$, if $-a < b$, and $y \neq b+1$. Let us analyze the candidates for $y = -a-1$. All such candidates are subquotients of

$$\delta([\nu^{a+1}\rho, \nu^{-a-1}\rho]) \otimes L(\delta([\nu^{a}\rho, \nu^{b}\rho]), \delta([\nu^{-a}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c \tag{15}$$

obtained for $\big((a-1, -a-1), (b, b+1)\big) \in \mathrm{Lad}(\pi_S)'$. In terms of Theorem 3.8, for $x = -a-1$, the induced representation

$$L(\delta([\nu^{a}\rho, \nu^{b}\rho]), \delta([\nu^{-a}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c \tag{16}$$

contains an irreducible tempered subquotient if and only if $x = -a-1 = \alpha - 1$. If $-a = b$, then it is isomorphic to $\tau_{2,-}^{(b)}$ for $\alpha = 1$ and to $\tau_{2,+}^{(b)}$ for $\alpha \geq \frac{3}{2}$. If $-a < b$, then the representation (16) has the discrete series subquotient σ_0. Since the tempered subquotients from Theorem 3.8 are unique and the representation (15) contains all constituents of $\mu^*(\pi_S \rtimes \sigma_c)$ of the form $\delta([\nu^{a+1}\rho, \nu^{-a-1}\rho]) \otimes \pi'$, for some $\pi' \in \mathrm{Irr}(G)$, we conclude that there is at most one candidate for tempered subquotient of $\pi_S \rtimes \sigma_c$ of the desired form.

We will first deal with the case $-a < b$. Let us denote candidates for tempered subquotients of $\pi_S \rtimes \sigma_c$ with $\tau_{5,\pm}$, such that $\tau_{5,+} \oplus \tau_{5,-} \simeq \delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma_0$. From Theorem 2.10 we get the parametrization of $\tau_{5,+}$:

- $\tau_{5,+} \hookrightarrow \nu\rho \times \rho \rtimes \pi'$, for a discrete series representation π', if $\alpha = 1$,
- $\tau_{5,+} \hookrightarrow \nu^{\alpha-1}\rho \times \nu^{\alpha-1}\rho \times \delta([\nu^{-(\alpha-2)}\rho, \nu^{\alpha-2}\rho]) \rtimes \sigma_0$, if $\alpha \geq \frac{3}{2}$.

Proposition 4.2. *Assume $1 \leq \alpha = -a < b$. Then the induced representation $\pi_S \rtimes \sigma_c$ contains the unique irreducible tempered subquotient of $\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma_0$, which is a subrepresentation of $\pi_S \rtimes \sigma_c$ and it is isomorphic to $\tau_{5,-}$.*

Proof. From the embedding $\sigma_0 \hookrightarrow \delta([\nu^{-\alpha}\rho, \nu^{b}\rho]) \rtimes \sigma_{(b+1)}$, we conclude

$$\begin{aligned}\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma_0 &\hookrightarrow \delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \times \delta([\nu^{-\alpha}\rho, \nu^{b}\rho]) \rtimes \sigma_{(b+1)}\\ &\simeq \delta([\nu^{-\alpha}\rho, \nu^{b}\rho]) \times \delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma_{(b+1)}.\end{aligned}$$

Let us analyze the case $\alpha = 1$. From the embeddings above, we conclude that $\tau_{5,\pm}$ embed into $\delta([\nu^{-1}\rho, \nu^b\rho]) \rtimes \tau_{2,\pm}^{(0)}$. Note that these representations have $\rho \otimes \sigma_0$ in the Jacquet modules with respect to appropriate parabolic subgroup with the multiplicity one. Hence, each representation $\tau_{5,\pm}$ embeds into exactly one representation $\delta([\nu^{-1}\rho, \nu^b\rho]) \rtimes \tau_{2,\pm}^{(0)}$. From the parametrization of $\tau_{2,+}^{(0)}$ in Theorem 2.10(ii), it follows that $\tau_{2,+}^{(0)} \hookrightarrow \delta([\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$. Using Lemma 3.5 and the embedding $\tau_{5,\pm} \hookrightarrow \rho \rtimes \sigma_0$, we see that $\mu^*(\tau_{5,\pm})$ do not have a constituent of the form $\nu^{b+1}\rho \otimes \pi'$, for some $\pi' \in \mathrm{Irr}(G)$. Hence, the representation from the set $\{\tau_{5,+}, \tau_{5,-}\}$, which embeds into

$$\delta([\nu^{-1}\rho, \nu^b\rho]) \rtimes \tau_{2,+}^{(0)} \hookrightarrow \delta([\nu^{-1}\rho, \nu^b\rho]) \times \delta([\rho, \nu^{b+1}\rho]) \rtimes \sigma_c,$$

also embeds into $\pi_S \rtimes \sigma_c$. We will show that $\tau_{5,-}$ is the wanted representation. From the formula (3), it is easy to note that the constituent of the form $\nu\rho \otimes \pi'$, for some $\pi' \in \mathrm{Irr}(G)$, is not a constituent of $\mu^*(\pi_S \rtimes \sigma_c)$. Therefore, $\tau_{5,+}$ is not a subquotient of $\pi_S \rtimes \sigma_c$. Hence, $\tau_{5,-} \hookrightarrow \pi_S \rtimes \sigma_c$.

In the rest of the proof, we assume $\alpha \geq \frac{3}{2}$. The tempered representations $\tau_{2,\pm}^{(\alpha-1)}$ embed into $\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \times \delta([\nu^{\alpha}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$. According to Theorem 2.10, we have $\tau_{2,+}^{(\alpha-1)} \hookrightarrow \nu^{\alpha-1}\rho \times \nu^{\alpha-1}\rho \times \delta([\nu^{-(\alpha-2)}\rho, \nu^{\alpha-2}\rho]) \rtimes \sigma_{(b+1)}$. The common irreducible tempered subquotient of $\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma_{(b+1)}$ and $\delta([\nu^{-(\alpha-1)}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$, as in [18, Theorem 4.1 (ii)], is isomorphic to $\tau_{2,-}^{(\alpha-1)}$. We conclude that

$$\tau_{5,-} \hookrightarrow \delta([\nu^{-\alpha}\rho, \nu^b\rho]) \rtimes \tau_{2,-}^{(\alpha-1)} \hookrightarrow \delta([\nu^{-\alpha}\rho, \nu^b\rho]) \times \delta([\nu^{-(\alpha-1)}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c.$$

It remains to notice that $\tau_{5,-}$ does not embed into the induced representation $\delta([\nu^{-\alpha}\rho, \nu^{b+1}\rho]) \times \delta([\nu^{-(\alpha-1)}\rho, \nu^b\rho]) \rtimes \sigma_c$. On the contrary, $\mu^*(\tau_{5,-})$ would contain a constituent of the form $\nu^{b+1}\rho \otimes \pi'$, for some $\pi' \in \mathrm{Irr}(G)$. But then the same holds for $\mu^*(\sigma_0)$, since $\alpha - 1 < b + 1$. According to Lemma 3.5, this is not possible, and we conclude $\tau_{5,-} \hookrightarrow \pi_S \rtimes \sigma_c$. ■

Let us now assume $-a = b$. The candidates for irreducible tempered subquotients of $\pi_S \rtimes \sigma_c$ are subrepresentations of induced representations

- $\rho \rtimes \tau_{2,-}^{(1)}$, for $\alpha = 1$,
- $\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \tau_{2,+}^{(\alpha)}$, for $\alpha \geq \frac{3}{2}$.

Since the representations $\tau_{2,\pm}^{(\alpha)}$ embed into $\delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \rtimes \sigma_{(\alpha+1)}$, the parametrization of the tempered candidate for a subquotient of $\pi_S \rtimes \sigma_c$ is determined with its embedding into $\delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \rtimes \tau_{2,\pm}^{(\alpha-1)}$. We define the following representations:

- τ_4, which embeds into $\delta([\nu^{-1}\rho, \nu\rho]) \rtimes \tau_{2,+}^{(0)}$, for $\alpha = 1$,
- τ_5, which embeds into $\delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \rtimes \tau_{2,-}^{(\alpha-1)}$, for $\alpha \geq \frac{3}{2}$.

We prove the following proposition in a similar manner as Proposition 4.2.

Proposition 4.3. *Let $-a = b = \alpha \geq 1$. Then the induced representation $\pi_S \rtimes \sigma_c$ contains the unique irreducible tempered subquotient of $\rho \rtimes \tau_{2,-}^{(1)}$, for $\alpha = 1$, or $\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \tau_{2,+}^{(\alpha)}$, for $\alpha \geq \frac{3}{2}$. It is a subrepresentation of $\pi_S \rtimes \sigma_c$ and it is isomorphic to τ_4, for $\alpha = 1$, and to τ_5, for $\alpha \geq \frac{3}{2}$.*

We are left to analyze the subquotients of the form $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \otimes \pi$, for some $\pi \in \mathrm{Irr}(G)$. They are subquotients of the following constituents of $\mu^*(\pi_S \rtimes \sigma_c)$:

- $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \otimes L(\delta([\nu^{a}\rho, \nu^{-a-1}\rho]), \nu^{b+1}\rho) \rtimes \sigma_c$, for $\big((a-1, b), (-a-1, b+1)\big)$ in $\mathrm{Lad}(\pi_S)'$,
- $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{-(b-1)}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$, for $\big((a-1, a), (a-1, b+1)\big)$ in $\mathrm{Lad}(\pi_S)'$, if $-a = b$.

An easy argument shows that, in case $-a < b$, the representation

$$L(\delta([\nu^{a}\rho, \nu^{-a-1}\rho]), \nu^{b+1}\rho) \rtimes \sigma_c \tag{17}$$

does not contain a tempered subquotient. Let us determine its irreducible tempered subquotients in case $-a = b$. In the Grothendieck group $R(G)$, we have

$$\begin{aligned}
&L(\delta([\nu^{-b}\rho, \nu^{b-1}\rho]), \nu^{b+1}\rho) \rtimes \sigma_c \simeq \delta([\nu^{-b}\rho, \nu^{b-1}\rho]) \times \nu^{b+1}\rho \rtimes \sigma_c \\
&\simeq \nu^{b+1}\rho \times \delta([\nu^{-b}\rho, \nu^{b-1}\rho]) \rtimes \sigma_c = \nu^{b+1}\rho \times \delta([\nu^{-(b-1)}\rho, \nu^{b}\rho]) \rtimes \sigma_c \\
&= \delta([\nu^{-(b-1)}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c + L(\delta([\nu^{-(b-1)}\rho, \nu^{b}\rho]), \nu^{b+1}\rho) \rtimes \sigma_c.
\end{aligned}$$

In case $b \geq 1$, Theorem 3.8 and [18, Theorems 2.1, 4.1] determine tempered subquotients of (17).

Let us first resolve the cases where $b \in \{0, \frac{1}{2}\}$. The representation (17) contains a tempered subquotient in case $b = 0$ if and only if $\alpha = 1$, in which case it is isomorphic to $\sigma_{(1)}$. Hence, the tempered representations $\tau_{2,\pm}^{(0)}$ are candidates for subquotients of $\pi_S \rtimes \sigma_c$. According to Theorem 2.10 (ii), we have $\tau_{2,+}^{(0)} \hookrightarrow \nu\rho \times \rho \rtimes \sigma_c$. As earlier, an easy application of Lemma 2.13 gives us the next proposition.

Proposition 4.4. *Let $a = b = 0$. The induced representation $\pi_S \rtimes \sigma_c$ contains an irreducible tempered subquotient which is not a discrete series representation if and only if $\alpha = 1$. In that case, it is unique, embeds into $\pi_S \rtimes \sigma_c$ and it is isomorphic to $\tau_{2,-}^{(0)}$.*

The representation (17) contains a tempered subquotient in case $b = \frac{1}{2}$ if and only if α equals $\frac{1}{2}$ or $\frac{3}{2}$. If $\alpha = \frac{1}{2}$, then it is isomorphic to $\sigma_{(\frac{3}{2})}$, and if $\alpha = \frac{3}{2}$, then to $\sigma_{(\frac{1}{2}, \frac{3}{2})}$. The representation π_S is isomorphic to $L(\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]), \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]))$. Tempered representations $\tau_{2,\pm}^{(\frac{1}{2})}$ are candidates for subquotients of $\pi_S \rtimes \sigma_c$. According to Theorem 2.10 (ii), we have $\tau_{2,+}^{(\frac{1}{2})} \hookrightarrow \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_{(\frac{3}{2})}$.

Proposition 4.5. *Let $-a = b = \frac{1}{2}$. The induced representation $\pi_S \rtimes \sigma_c$ contains an irreducible tempered subquotient which is not a discrete series representation if and only if $\alpha = \frac{1}{2}$. In that case, it is unique, embeds into $\pi_S \rtimes \sigma_c$ and it is isomorphic to $\tau_{2,-}^{(\frac{1}{2})}$.*

Proof. Let us first show that, for $\alpha = \frac{1}{2}$, there is an irreducible tempered subquotient of $\pi_S \rtimes \sigma_c$. As before, Lemma 2.13 and the embedding

$$\tau_{2,\pm}^{(\frac{1}{2})} \hookrightarrow \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma_c$$

imply that each of them is a subrepresentation of $\pi_S \rtimes \sigma_c$ or $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_c$.

Note that in $R(G)$ we have $\nu^{\frac{1}{2}}\rho \rtimes \sigma_c = \sigma_{(\frac{1}{2})} + L(\nu^{-\frac{1}{2}}\rho; \sigma_c)$. Assume that any of these tempered representations embeds into $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_c$ and denote it τ. Then the Casselman's criterion implies that τ embeds into $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma_{(\frac{1}{2})}$. From the result [18, Theorem 4.1 (ii)] it follows that the induced representation $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma_{(\frac{1}{2})}$ has a unique irreducible tempered subquotient. As we will see, τ is isomorphic to $\tau_{2,+}^{(\frac{1}{2})}$. From the formula (3), it follows that $\nu^{\frac{1}{2}}\rho \otimes \nu^{\frac{1}{2}}\rho \otimes \sigma_{(\frac{3}{2})}$ is not a constituent of the Jacquet module of $\pi_S \rtimes \sigma_c$ with respect to the appropriate parabolic subgroup. Thus $\tau_{2,+}^{(\frac{1}{2})} \hookrightarrow \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_c$ and $\tau_{2,-}^{(\frac{1}{2})} \hookrightarrow \pi_S \rtimes \sigma_c$.
On the other hand, we can easily see that, for $\alpha = \frac{3}{2}$, the irreducible tempered representation $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{(\frac{1}{2},\frac{3}{2})}$ is not a subquotient of $\pi_S \rtimes \sigma_c$. It is again a consequence of the fact that a representation of the form $\nu^{\frac{1}{2}}\rho \otimes \nu^{\frac{1}{2}}\rho \otimes \pi'$, for $\pi' \in \mathrm{Irr}(G)$, is a constituent of the Jacquet module of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{(\frac{1}{2},\frac{3}{2})}$, while the same is not true for the Jacquet modules of $\pi_S \rtimes \sigma_c$. ■

Let us now consider the case where $b \geq 1$. In the analysis of the induced representation

$$\delta([\nu^{-(b-1)}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c,$$

we differentiate the cases when the set $[2(b-1)+1, 2(b+1)+1] \cap \mathrm{Jord}_\rho(\sigma_c)$ is empty or not.
In case $b-1 \geq \alpha$, from [18, Theorem 2.1], it follows that it has two non-equivalent discrete series representations σ_3, σ_4.
In case of non-empty intersection, we have $2(b-1)+1 \in \mathrm{Jord}_\rho(\sigma_c)$, since the elements of $\mathrm{Jord}_\rho(\sigma_c)$ are consecutive integers of the same parity. If $2(b+1)+1 \in \mathrm{Jord}_\rho(\sigma_c)$ it follows immediately from [18, Theorem 4.1 (i)] that the induced representation $\delta([\nu^{-(b-1)}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$ does not have any tempered subquotient. In this way, the result [18, Theorem 4.1 (ii)] solves the only remaining case. It shows that the induced representation $\delta([\nu^{-(b-1)}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$ has an irreducible tempered subquotient if and only if $b = \alpha$. As we have seen in the proof of Proposition 4.2, it is isomorphic to $\tau_{2,+}^{(0)}$, for $\alpha = 1$, and $\tau_{2,-}^{(\alpha-1)}$, for $\alpha \geq \frac{3}{2}$.
On the other hand, from Theorem 3.8, it follows that the induced representation $L(\delta([\nu^{-(b-1)}\rho, \nu^{b}\rho]), \nu^{b+1}\rho) \rtimes \sigma_c$ has an irreducible tempered subquotient if and only if $b = \alpha - 1$. Precisely, it is isomorphic to $\tau_{1,+}^{(0)}$, for $\alpha = 2$, and to $\tau_{1,-}^{(\alpha-2)}$, for $\alpha \geq \frac{5}{2}$.
All together, for $b \geq 1$, we have the following candidates:

- $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \rtimes \tau_{1,+}^{(0)}$, for $b = \alpha - 1 = 1$,
- $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \rtimes \tau_{1,-}^{(\alpha-2)}$, for $b = \alpha - 1 \geq \frac{3}{2}$,
- $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \rtimes \tau_{2,+}^{(0)}$, for $b = \alpha = 1$,
- $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \rtimes \tau_{2,-}^{(\alpha-1)}$, for $b = \alpha \geq \frac{3}{2}$,
- $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \rtimes \sigma_3$ and $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \rtimes \sigma_4$, for $b \geq \alpha + 1$.

In case $b \geq \alpha+1$, we define tempered representations $\tau_{i,+} \oplus \tau_{i,-} = \delta([\nu^{-b}\rho, \nu^{b}\rho]) \rtimes \sigma_i$, for $i = 3, 4$. Since $2(b-1)+1 \in \mathrm{Jord}_\rho(\sigma_i) \cap [1, 2b+1]$, we are in the setting of Theorem 2.10 (i). Hence, $\tau_{i,+}$ is the unique irreducible tempered subrepresentation of $\delta([\nu^{-b}\rho, \nu^{b}\rho]) \rtimes \sigma_i$ which embeds into $\nu^{b}\rho \times \nu^{b}\rho \times \delta([\nu^{-(b-1)}\rho, \nu^{b-1}\rho]) \rtimes \sigma_i$.

Proposition 4.6. *Let $-a = b \geq 1$. The induced representation $\pi_S \rtimes \sigma_c$ has a tempered subquotient which is not a discrete series representation if and only if $b \geq \alpha$. If $b = \alpha$, then the unique irreducible tempered subquotient of $\pi_S \rtimes \sigma_c$ which is not a discrete series representation is isomorphic to τ_4, for $\alpha = 1$, and to τ_5, for $\alpha \geq \frac{3}{2}$. If $b \geq \alpha + 1$, then the irreducible tempered subquotients of $\pi_S \rtimes \sigma_c$ are isomorphic to $\tau_{3,-}$ and $\tau_{4,-}$. In each case, tempered subquotients are subrepresentations of $\pi_S \rtimes \sigma_c$.*

Proof. We need to analyze if the candidates from the list before the statement of this proposition are subquotients of $\pi_S \rtimes \sigma_c$.

If $b = \alpha - 1$, then the candidate τ is subrepresentation of a representation of the form $\delta([\nu^{-(\alpha-2)}\rho, \nu^{\alpha-2}\rho]) \rtimes \tau'$, for some tempered $\tau' \in \mathrm{Irr}(G)$. Since $-a-1 = \alpha - 2$, the earlier analysis implies that, if τ is a subquotient of $\pi_S \rtimes \sigma_c$, then τ' is a subquotient of the induced representation (16). This is not possible since representation (16) has a tempered subquotient if and only if $-a = \alpha$. Hence, these candidates do not give subquotients of $\pi_S \rtimes \sigma_c$.

If $b = \alpha$, the candidates for subquotients of $\pi_S \rtimes \sigma_c$ are tempered subrepresentations of $\delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \rtimes \tau_{2,+}^{(0)}$, for $\alpha = 1$, and $\delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \rtimes \tau_{2,-}^{(\alpha-1)}$, for $\alpha \geq \frac{3}{2}$. We have seen in Proposition 4.3 that τ_4 and τ_5, defined earlier, are their only subrepresentations which are subquotients of $\pi_S \rtimes \sigma_c$.

If $b \geq \alpha + 1$, from $\sigma_i \hookrightarrow \delta([\nu^{-(b-1)}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$, we get

$$\tau_{i,\pm} \hookrightarrow \delta([\nu^{-b}\rho, \nu^{b}\rho]) \times \delta([\nu^{-(b-1)}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c,$$

for $i = 3, 4$. Then Lemma 2.13 implies that representation $\tau \in \{\tau_{3,\pm}, \tau_{4,\pm}\}$ embeds into $\pi_S \rtimes \sigma_c$ or $\delta([\nu^{-(b-1)}\rho, \nu^{b}\rho]) \times \delta([\nu^{-b}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$. The following intertwining operators and the parametrization of tempered representations $\tau_{i,\pm}$, for $i = 3, 4$, imply that $\tau_{3,-}$ and $\tau_{4,-}$ are subrepresentations of $\pi_S \rtimes \sigma_c$.

$$\begin{aligned}
&\delta([\nu^{-(b-1)}\rho, \nu^{b}\rho]) \times \delta([\nu^{-b}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c \\
&\hookrightarrow \delta([\nu^{-(b-1)}\rho, \nu^{b}\rho]) \times \delta([\nu^{-(b-1)}\rho, \nu^{b+1}\rho]) \times \nu^{-b}\rho \rtimes \sigma_c \\
&\simeq \delta([\nu^{-(b-1)}\rho, \nu^{b}\rho]) \times \delta([\nu^{-(b-1)}\rho, \nu^{b+1}\rho]) \times \nu^{b}\rho \rtimes \sigma_c \qquad (18)\\
&\simeq \nu^{b}\rho \times \delta([\nu^{-(b-1)}\rho, \nu^{b}\rho]) \times \delta([\nu^{-(b-1)}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c \hookrightarrow \\
&\hookrightarrow \nu^{b}\rho \times \nu^{b}\rho \times \delta([\nu^{-(b-1)}\rho, \nu^{b-1}\rho]) \times \delta([\nu^{-(b-1)}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c,
\end{aligned}$$

where we have the isomorphism (18) because of the assumption $b \geq \alpha + 1$. Since a representation of the form $\nu^{b}\rho \otimes \nu^{b}\rho \otimes \pi'$, for $\pi' \in \mathrm{Irr}(G)$, is not in the Jacquet module of $\pi_S \rtimes \sigma_c$ with respect to the appropriate parabolic subgroup, we conclude that $\tau_{i,+}$, for $i = 3, 4$, are not subrepresentations of $\pi_S \rtimes \sigma_c$. ■

Let us now consider discrete series subquotients of $\pi_S \rtimes \sigma_c$. Let $-a \in \{0, \frac{1}{2}\}$. Assume that the representation $\pi_S \rtimes \sigma_c$ has a discrete series subquotient σ. Counting the multiplicities of ρ or $\nu^{\frac{1}{2}}\rho$ in the cuspidal support of π_S, we conclude that σ embeds into $\delta([\nu^{-z}\rho, \nu^{y}\rho]) \rtimes \sigma_{sp}$. Here σ_{sp} denotes a strongly positive representation and y, z are real numbers such that $0 \leq z < y$ and $z = y\_$. In this way, a representation of the form $\nu^{y}\rho \otimes \pi'$, for a positive number y and $\pi' \in \mathrm{Irr}(G)$, is a constituent of $\mu^*(\pi_S \rtimes \sigma_c)$. According to the formula (4), we have $y = b$. Since $b + 1$ is the maximal exponent in $[\pi_S]$, we conclude $[\nu^{\alpha}\rho, \nu^{b+1}\rho] \subseteq [\sigma_{sp}]$. The Frobenius reciprocity implies that $\delta([\nu^{-z}\rho, \nu^{b}\rho]) \otimes \sigma_{sp}$ is a constituent of $\mu^*(\pi_S \rtimes \sigma_c)$.

Thus it is necessarily a subquotient of

- $\delta([\nu^{-(\alpha-1)}\rho, \nu^{-1}\rho]) \times \delta([\rho, \nu^{b}\rho]) \otimes \delta([\nu^{\alpha}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$, for $a = 0$,
- $\delta([\nu^{-(\alpha-1)}\rho, \nu^{-\frac{1}{2}}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \otimes \nu^{-\frac{1}{2}}\rho \times \delta([\nu^{\alpha}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$, for $-a = \frac{1}{2}$ and $\alpha \geq \frac{3}{2}$,
- $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{\frac{1}{2}}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$, for $-a = \frac{1}{2}$ and $\alpha = \frac{1}{2}$.

In each case we have $z = \alpha - 1$.

If $a = 0$ and $\alpha \geq 2$, then $\{2(\alpha-2)+1, 2(\alpha-1)+1, 2b+1, 2(b+1)+1\} \subseteq \mathrm{Jord}_{\rho}(\sigma)$. From the following values of the ϵ-function:

$$\epsilon_{\sigma}((2(\alpha-2)+1,\rho),(2(b+1)+1,\rho)) = -1,$$
$$\epsilon_{\sigma}((2(\alpha-1)+1,\rho),(2b+1,\rho)) = 1,$$
$$\epsilon_{\sigma}((2b+1,\rho),(2(b+1)+1,\rho)) = -1,$$

we get $\epsilon_{\sigma}((2(\alpha-2)+1,\rho),(2(\alpha-1)+1,\rho)) = 1$. Hence, $\nu^{\alpha-1}\rho \otimes \pi'$ is a constituent of $\mu^{*}(\pi_S \rtimes \sigma_c)$. This implies $\alpha - 1 = b$, contradicting the assumption $\alpha - 1 < b$. Hence, $\alpha = 1$.

If $-a = \frac{1}{2}$, the induced representation $\nu^{-\frac{1}{2}}\rho \times \delta([\nu^{\alpha}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$ has a strongly positive subquotient if and only if $\alpha = \frac{3}{2}$. But then 2 is an element of the set $\mathrm{Jord}_{\rho}(\sigma) \setminus \{2, 2b+1\}$, which is impossible. Hence, $\alpha = \frac{1}{2}$.

For $a = 0$ and $\alpha = 1$, the candidate for a discrete series subquotient embeds into $\delta([\rho, \nu^{b}\rho]) \rtimes \sigma_{(b+1)}$. From [18, Theorem 2.1], it follows that there are two such representations. We denote them σ_5 and σ_5'. Clearly,

$$\mathrm{Jord}_{\rho}(\sigma_5) = \mathrm{Jord}_{\rho}(\sigma_5') = \{1, 2b+1, 2(b+1)+1\}$$

and their partial cuspidal support is σ_c. The epsilon functions ϵ_{σ_5} and $\epsilon_{\sigma_5'}$ are defined as follows: $\epsilon_{\sigma_5}((1,\rho),(2b+1,\rho)) = \epsilon_{\sigma_5'}((1,\rho),(2b+1,\rho)) = 1$,

$$\epsilon_{\sigma_5}((2b+1,\rho),(2(b+1)+1,\rho)) = -1, \ \epsilon_{\sigma_5'}((2b+1,\rho),(2(b+1)+1,\rho)) = 1.$$

Proposition 4.7. *Let $a = 0 < b$ and $\alpha = 1$. The induced representation $\pi_S \rtimes \sigma_c$ has the unique irreducible tempered subquotient, which is its subrepresentation and is isomorphic to σ_5.*

Proof. Because of $\epsilon_{\sigma_5'}((2b+1,\rho),(2(b+1)+1,\rho)) = 1$, we have that σ_5' embeds into $\delta([\nu^{-b}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$. Hence, $\mu^{*}(\sigma_5')$ has a constituent of the form $\nu^{b+1}\rho \otimes \pi'$ for $\pi' \in \mathrm{Irr}(G)$. Since the same is not true for $\mu^{*}(\pi_S \rtimes \sigma_c)$, we conclude that σ_5' is not a subquotient of $\pi_S \rtimes \sigma_c$. Let us now show that σ_5 embeds into $\pi_S \rtimes \sigma_c$. From the embeddings

$$\begin{aligned}\sigma_5 \hookrightarrow \delta([\rho, \nu^{b}\rho]) \times \sigma_{(b+1)} &\hookrightarrow \delta([\rho, \nu^{b}\rho]) \times \delta([\nu\rho, \nu^{b+1}\rho]) \rtimes \sigma_c \\ &= \pi_S \rtimes \sigma_c + \delta([\rho, \nu^{b+1}\rho]) \times \delta([\nu\rho, \nu^{b}\rho]) \rtimes \sigma_c \qquad (19)\end{aligned}$$

and Lemma 2.13, we get that σ_5 embeds into one of representations from (19). From Lemma 3.5 we know that $\mu^{*}(\sigma_5)$ does not contain $\nu^{b+1}\rho \otimes \pi'$ for $\pi' \in \mathrm{Irr}(G)$. Thus it necessarily embeds into $\pi_S \rtimes \sigma_c$.

It remains to verify the uniqueness of the irreducible tempered subquotient of $\pi_S \rtimes \sigma_c$. Note that we have shown earlier in the section that under the assumptions of the proposition there is not an irreducible tempered subquotient which is not a discrete series representation. ∎

For $a = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$, the candidates which are discrete series representations embed into $\delta([\nu^{-\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma_{(b+1)}$. We denote them σ_6 and σ_6'. Clearly,

$$\mathrm{Jord}_\rho(\sigma_6) = \mathrm{Jord}_\rho(\sigma_6') = \{2, 2b+1, 2(b+1)+1\}$$

and their partial cuspidal support is σ_c. The epsilon functions are defined as follows:

$$\epsilon_{\sigma_6}(2,\rho) = -1,\ \epsilon_{\sigma_6}((2,\rho),(2b+1,\rho)) = 1,\ \epsilon_{\sigma_6}((2b+1,\rho),(2(b+1)+1,\rho)) = -1,$$
$$\epsilon_{\sigma_6'}(2,\rho) = 1,\ \epsilon_{\sigma_6'}((2,\rho),(2b+1,\rho)) = 1,\ \epsilon_{\sigma_6'}((2b+1,\rho),(2(b+1)+1,\rho)) = 1.$$

The proof of the next proposition follows the lines of the proof of Proposition 4.7.

Proposition 4.8. *Let $-a = \frac{1}{2} < b$ and $\alpha = \frac{1}{2}$. The induced representation $\pi_S \rtimes \sigma_c$ has the unique irreducible tempered subquotient, which is its subrepresentation and is isomorphic to σ_6.*

In the rest of this section, we assume $a + 1 \leq 0$. Again, we analyze constituents of $\mu^*(\pi_S \rtimes \sigma_c)$ of the form $\delta([\nu^{-z}\rho, \nu^y\rho]) \otimes \sigma$, for a discrete series representation $\sigma \in \mathrm{Irr}(G)$ and $0 \leq z < y$. Also, the only possibilities for y are $-a-1$ and b.
If $y = -a-1$, the wanted constituents are subquotients of

$$\delta([\nu^{-c_2}\rho, \nu^{-a-1}\rho]) \otimes L(\delta([\nu^a\rho, \nu^b\rho]), \delta([\nu^{c_2+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c,$$

obtained from $\big((a-1, c_2), (b, b+1)\big) \in \mathrm{Lad}(\pi_S)'$. In the notation of Theorem 3.8, we have $x = c_2$. Additionally, from Theorem 3.8 it follows that the representation $L(\delta([\nu^a\rho, \nu^b\rho]), \delta([\nu^{x+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c$ has a discrete series subquotient if and only if $x = \alpha - 1 \leq -a < b$. Hence, if $\alpha - 1 < -a-1$, the candidates which are discrete series representations embed into $\delta([\nu^{-(\alpha-1)}\rho, \nu^{-a-1}\rho]) \rtimes \sigma_0$. Again, from [18, Theorem 2.1], it follows that the candidates are two discrete series representations obtained by the classification. They are characterized by the expansion of ϵ-function of σ_0 such that $\epsilon((2(\alpha-1)+1,\rho),(2(-a-1)+1,\rho)) = 1$ and

$$\epsilon((2(\alpha-2)+1,\rho),(2(\alpha-1)+1,\rho)) \cdot \epsilon((2(-a-1)+1,\rho),(2(-a)+1,\rho)) = 1.$$

If $\epsilon((2(-a-1)+1,\rho),(2(-a)+1,\rho)) = 1$, then the discrete series representation embeds into $\delta([\nu^{a+1}\rho, \nu^{-a}\rho]) \rtimes \sigma'$, for discrete series representation $\sigma' \in \mathrm{Irr}(G)$.

Thus it has a constituent of the form $\nu^{-a}\rho \otimes \pi'$ in the Jacquet module with respect to the appropriate parabolic subgroup. We have already noted that this is not true for Jacquet modules of $\pi_S \rtimes \sigma_c$. In this way, for the only discrete series candidate we have

$$\epsilon((2(\alpha-2)+1,\rho),(2(\alpha-1)+1,\rho)) = \epsilon((2(-a-1)+1,\rho),(2(-a)+1,\rho)) = -1.$$

This discrete series representation embeds into $\delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$. As we will soon show, this property is true for every discrete series representation of $\pi_S \rtimes \sigma_c$.

The candidates in the case $y = b$ are obtained from the following constituents of $\mu^*(\pi_S \rtimes \sigma_c)$:

$$\delta([\nu^{d_1+1}\rho, \nu^b\rho]) \otimes L(\delta([\nu^a\rho, \nu^{d_1}\rho]), \delta([\nu^{a+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c \tag{20}$$

for $\big((a-1,a),(d_1,b+1)\big) \in \mathrm{Lad}(\pi_S)'$ and

$$\delta([\nu^{-c_2}\rho, \nu^b\rho]) \otimes L(\delta([\nu^a\rho, \nu^{-a-1}\rho]), \delta([\nu^{c_2+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c \tag{21}$$

for $\big((a-1,c_2),(-a-1,b+1)\big) \in \mathrm{Lad}(\pi_S)'$.

Lemma 4.9. *Let σ denote a discrete series subquotient of $\pi_S \rtimes \sigma_c$. Then*

$$\sigma \hookrightarrow \delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c.$$

Proof. Counting multiplicities of ρ or $\nu^{\frac{1}{2}}\rho$ in $[\pi_S]$, we conclude that σ embeds into an induced representation of the form $\delta([\nu^{-x_1}\rho, \nu^{y_1}\rho]) \times \delta([\nu^{-x_2}\rho, \nu^{y_2}\rho]) \rtimes \sigma_{sp}$. Here σ_{sp} is a strongly positive representation and x_i, y_i, $i = 1, 2$ are real numbers such that $0 \le x_i < y_i$ and $y_1 < y_2$. Also, we assume that this embedding is by the classification of discrete series. In particular, $(y_1)\_ = x_1$. Since $\nu^{y_1}\rho \otimes \pi'$ is a constituent of $\mu^*(\pi_S \rtimes \sigma_c)$, we get $y_1 = -a-1$ or $y_1 = b$. Since the case $y_1 = -a-1$ is proved above the lemma, we consider $y_1 = b$. Also, $y_2 = b+1$, because $b+1$ is the maximal exponent in $[\pi_S]$ of multiplicity one. Let us analyze the candidates obtained from the constituents (20) and (21) of $\mu^*(\pi_S \rtimes \sigma_c)$. If we denote $-z = d_1 + 1$, the representation (20) equals

$$\delta([\nu^{-z}\rho, \nu^b\rho]) \otimes L(\delta([\nu^a\rho, \nu^{-z-1}\rho]), \delta([\nu^{a+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c \tag{22}$$

for $0 \le z \le -a$. For $z = -a$, the representation (22) is isomorphic to the representation $\delta([\nu^a\rho, \nu^b\rho]) \otimes \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$. Hence, we get the desired induced representation.

Let σ' denote a discrete series representation such that $\sigma' \hookrightarrow \delta([\nu^{-x_2}\rho, \nu^{b+1}\rho]) \rtimes \sigma_{sp}$ and $\sigma \hookrightarrow \delta([\nu^{-x_1}\rho, \nu^b\rho]) \rtimes \sigma'$. Assume that σ' is a subquotient of

$$L(\delta([\nu^a\rho, \nu^{-z-1}\rho]), \delta([\nu^{a+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c \tag{23}$$

for $0 \le z \le -a-1$. If $z = -a-1$, the representation (22) is isomorphic to

$$\delta([\nu^{a+1}\rho, \nu^b\rho]) \otimes L(\nu^a\rho, \delta([\nu^{a+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c.$$

Let $\pi_1 \otimes \pi_2$ be a constituent of $\mu^*(L(\nu^a\rho, \delta([\nu^{a+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c)$ such that the representation $\delta([\nu^{-x_2}\rho, \nu^{b+1}\rho])$ is a subquotient of π_1. From Lemma 2.3, it follows that π_1 is not isomorphic to a representation of the form

$$L(\delta([\nu^{-c_2}\rho, \nu^{-a-1}\rho]), \nu^{-a}\rho) \times \pi_1',$$

for a real number c_2 and $\pi_1' \in \mathrm{Irr}(GL)$. Thus we have $-x_2 = a$ or $\nu^{-a}\rho \in [\pi_2]$, which together with Lemma 4.1 (i) implies $2(-a)+1 \in \mathrm{Jord}_\rho(\sigma_{sp})$. From the equality $(2b+1)\_ = 2(-a-1)+1$, we get a contradiction to $2(-a)+1 \in \mathrm{Jord}_\rho(\sigma)$.

If $0 \le z \le -a-2$, then Remark 2.11 and the fact that $\mu^*(\pi_S \rtimes \sigma_c)$ does not contain a constituent of the form $\nu^z\rho \otimes \pi'$ imply $-x_2 = -z+1$. Again, let

$\pi_1 \otimes \pi_2$ be a constituent of $\mu^*(L(\delta([\nu^a\rho, \nu^{-z-1}\rho]), \delta([\nu^{a+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c)$ such that $\delta([\nu^{-z+1}\rho, \nu^{b+1}\rho])$ is a subquotient of π_1. Since $-z-1 \geq a+1$, we conclude that $\nu^{-a-1}\rho$ appears in the cuspidal support of representation (23) with the multiplicity three. On the other hand, $[\pi_1]$ is a segment so there is $\nu^{-a-1}\rho$ in $[\pi_2]$. Therefore, the maximal exponent x in $[\pi_2]$ is greater than or equal to $-a-1$. In consequence of Lemma 4.1(i) we obtain $2x+1 \in \mathrm{Jord}_\rho(\sigma)$. This is in a contradiction with assumptions $(2b+1)\_ = 2z+1$ and $z \leq -a-2$. Hence, we also do not have candidates which are discrete series representations in this case.

Analyzing the candidates for discrete series representations obtained from (21), we see that for $c_2 = -a$ they embed into $\delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$.

If $c_2 \geq -a+1$, there is no constituent $\pi_1 \otimes \pi_2$ of

$$\mu^*(L(\delta([\nu^a\rho, \nu^{-a-1}\rho]), \delta([\nu^{c_2+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c)$$

such that $\delta([\nu^{-x_2}\rho, \nu^{b+1}\rho])$ is a subquotient of π_1.

If $0 \leq c_2 \leq -a-1$, then $\nu^{-a}\rho$ appears in the cuspidal support of the representation $L(\delta([\nu^a\rho, \nu^{-a-1}\rho]), \delta([\nu^{c_2+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c$ with the multiplicity two. Again, from Remark 2.11, it follows $-x_2 = -c_2+1 \geq a+2$ so $\nu^{-a}\rho \in [\sigma_{sp}]$ and $-a$ is the maximal exponent in $[\sigma_{sp}]$. This is again a contradiction by Lemma 4.1 (i), since in $\mathrm{Jord}_\rho(\sigma)$ we have $(2b+1)\_ = 2c_2+1$. This finishes the proof. ■

It remains to note that in case $a+1 \leq 0$, according to [18, Theorems 2.1, 4.1], the induced representation $\delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$ has a discrete series subquotient only in the cases

(i) $[2(-a-1)+1, 2(b+1)+1] \cap \mathrm{Jord}_\rho(\sigma_c) = \emptyset$,

(ii) $2(-a-1)+1, 2(b+1)+1 \notin \mathrm{Jord}_\rho(\sigma_c)$ and $[2(-a-1)+1, 2(b+1)+1] \cap \mathrm{Jord}_\rho(\sigma_c)$ is a singleton.

Note that (ii) is not possible, since the elements of $\mathrm{Jord}_\rho(\sigma_c)$ are consecutive numbers of the same parity and the minimal is equal to 1 or 2. In the case (i), which is equivalent to $-a \geq \alpha+1$, the discrete series subquotients of $\delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$ are obtained by the classification. We denote them σ_3 and σ_4. Clearly,

$$\mathrm{Jord}(\sigma_i) = \{(2(-a-1)+1, \rho), (2(b+1)+1, \rho)\} \cup \mathrm{Jord}(\sigma_c).$$

The epsilon function is defined such that $\epsilon_{\sigma_i}((2(-a-1)+1, \rho), (2(b+1)+1, \rho)) = 1$.

In case $\mathrm{Jord}_\rho(\sigma_c) = \emptyset$, we have $\epsilon_{\sigma_3}(2(-a-1)+1, \rho) = \epsilon_{\sigma_3}(2(b+1)+1, \rho) = 1$ and $\epsilon_{\sigma_4}(2(-a-1)+1, \rho) = \epsilon_{\sigma_4}(2(b+1)+1, \rho) = -1$.

Otherwise, we define $\epsilon_{\sigma_i}((x\_, \rho), (x, \rho)) = -1$, for $x \in \mathrm{Jord}(\sigma_c)$ and $i = 3, 4$, then $\epsilon_{\sigma_3}((2(\alpha-1)+1, \rho), (2(-a-1)+1, \rho)) = 1$ and finally

$$\epsilon_{\sigma_4}((2(\alpha-1)+1, \rho), (2(-a-1)+1, \rho)) = -1.$$

Additionally, if the set $\mathrm{Jord}_\rho(\sigma_c)$ is non-empty and its elements are even numbers, then the minimum of $\mathrm{Jord}_\rho(\sigma_i)$ is equal 2 and we have $\epsilon_{\sigma_i}(2, \rho) = \epsilon_{\sigma_c}(2, \rho) = -1$. The classification of discrete series representations implies that, for $i = 3, 4$, there are two discrete series representations $\sigma_{i,\pm}$ which embed into $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_i$. Clearly, $\mathrm{Jord}(\sigma_{i,\pm}) = \{(2(-a)+1, \rho), (2b+1, \rho)\} \cup \mathrm{Jord}(\sigma_i)$. The epsilon functions are defined such that they are equal to ϵ_{σ_i} on $\mathrm{Jord}(\sigma_i)$, $\epsilon_{\sigma_{i,\pm}}((2(-a)+1, \rho), (2b+1, \rho)) = 1$ and

$\epsilon_{\sigma_{i,\pm}}((2(-a-1)+1,\rho),(2(-a)+1,\rho)) \cdot \epsilon_{\sigma_{i,\pm}}((2b+1,\rho),(2(b+1)+1,\rho)) = 1$. The discrete series representations $\sigma_{i,+}$ and $\sigma_{i,-}$ differentiate because of the definitions of epsilon functions:

$$\epsilon_{\sigma_{i,+}}((2(-a-1)+1,\rho),(2(-a)+1,\rho)) = \epsilon_{\sigma_{i,+}}((2b+1,\rho),(2(b+1)+1,\rho)) = 1,$$

$$\epsilon_{\sigma_{i,-}}((2(-a-1)+1,\rho),(2(-a)+1,\rho)) = \epsilon_{\sigma_{i,-}}((2b+1,\rho),(2(b+1)+1,\rho)) = -1.$$

For $\sigma \in \{\sigma_{i,+}, \sigma_{i,-}\}$, there is the unique irreducible tempered representation τ which embeds into $\delta([\nu^a\rho, \nu^{-a}\rho]) \rtimes \sigma_i$ such that $\sigma \hookrightarrow \delta([\nu^{-a+1}\rho, \nu^b\rho]) \rtimes \tau$. Additionally, we define $\delta([\nu^a\rho, \nu^{-a}\rho]) \rtimes \sigma_i \simeq \tau_{i,+} \oplus \tau_{i,-}$. From Theorem 2.10 (i), it follows that $\tau_{i,+}$ embeds into $\nu^{-a}\rho \times \nu^{-a}\rho \rtimes \lambda_i$ for $\lambda_i \in \mathrm{Irr}(G)$. One can show that $\sigma_{i,+}$ embeds into $\delta([\nu^{-a+1}\rho, \nu^b\rho]) \rtimes \tau_{i,+}$.

Lemma 4.10. *The unique irreducible tempered representation τ such that $\sigma_{i,+}$ embeds into $\delta([\nu^{-a+1}\rho, \nu^b\rho]) \rtimes \tau$ is isomorphic to $\tau_{i,+}$, for $i = 3, 4$.*

Proposition 4.11. *Let $b > -a \geq \alpha+1$. The discrete series representations $\sigma_{3,-}$ and $\sigma_{4,-}$ are the only discrete series subquotients of $\pi_S \rtimes \sigma_c$. Additionally, they are its subrepresentations.*

Proof. From Lemma 2.13 and the embeddings

$$\sigma_{i,\pm} \hookrightarrow \delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c,$$

for $i = 3, 4$, we see that $\sigma_{i,\pm}$ are subrepresentations of the induced representation $\pi_S \rtimes \sigma_c$ or $\delta([\nu^a\rho, \nu^{b+1}\rho]) \times \delta([\nu^{a+1}\rho, \nu^b\rho]) \rtimes \sigma_c$. Note that the following intertwining operators

$$\begin{aligned}
&\delta([\nu^{a+1}\rho, \nu^b\rho]) \times \delta([\nu^a\rho, \nu^{b+1}\rho]) \rtimes \sigma_c \\
&\hookrightarrow \delta([\nu^{-a+1}\rho, \nu^b\rho]) \times \delta([\nu^{a+1}\rho, \nu^{-a}\rho]) \times \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \times \nu^a\rho \rtimes \sigma_c \\
&\simeq \delta([\nu^{-a+1}\rho, \nu^b\rho]) \times \delta([\nu^{a+1}\rho, \nu^{-a}\rho]) \times \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \times \nu^{-a}\rho \rtimes \sigma_c \\
&\simeq \delta([\nu^{-a+1}\rho, \nu^b\rho]) \times \nu^{-a}\rho \times \delta([\nu^{a+1}\rho, \nu^{-a}\rho]) \times \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c \\
&\hookrightarrow \delta([\nu^{-a+1}\rho, \nu^b\rho]) \times \nu^{-a}\rho \times \nu^{-a}\rho \times \delta([\nu^{a+1}\rho, \nu^{-a-1}\rho]) \times \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c
\end{aligned}$$

and Lemma 4.10 imply that $\sigma_{3,-}$ and $\sigma_{4,-}$ are not subrepresentations of the induced representation $\delta([\nu^a\rho, \nu^{b+1}\rho]) \times \delta([\nu^{a+1}\rho, \nu^b\rho]) \rtimes \sigma_c$. Hence, $\sigma_{i,-}$ embeds into $\pi_S \rtimes \sigma_c$, for $i = 3, 4$. By the definition of representations $\sigma_{i,+}$, it follows that $\mu^*(\sigma_{i,+})$ has a constituent of the form $\nu^{b+1}\rho \otimes \pi'$. In this way we see that $\sigma_{i,+}$, for $i = 3, 4$, are not subquotients of the induced representation $\pi_S \rtimes \sigma_c$. ∎

This finishes the analysis of tempered subquotients of $\pi_S \rtimes \sigma_c$ in case $-a \leq b$. To conclude the section, we state the theorem which unifies the obtained results.

Theorem 4.12. *The induced representation*

$$\pi_S \rtimes \sigma_c = L(\delta([\nu^a\rho, \nu^b\rho]), \delta([\nu^{a+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c,$$

such that $0 \leq -a \leq b$, has

(i) *an irreducible tempered subquotient which is not a discrete series representation if and only if $-a = b = \alpha - 1 = 0$ or $-a = b = \alpha = \frac{1}{2}$ or $-a = b \geq \alpha + 1$ or $1 \leq \alpha = -a \leq b$.*

(ii) *a discrete series representation if and only if $a = \alpha - 1 = 0 < b$ or $-a = \alpha = \frac{1}{2} < b$ or $b > -a \geq \alpha + 1$.*

In that case, the irreducible tempered subquotients are subrepresentations of $\pi_S \rtimes \sigma_c$.

If $-a = b$, the irreducible tempered subquotients of $\pi_S \rtimes \sigma_c$ are given by the cases:

- $b = \alpha - 1 = 0$: $\tau_{2,-}^{(0)}$,
- $b = \alpha$: $\tau_{2,-}^{(\frac{1}{2})}$ *for* $\alpha = \frac{1}{2}$, τ_4 *for* $\alpha = 1$, τ_5 *for* $\alpha \geq \frac{3}{2}$,
- $b \geq \alpha + 1$: $\tau_{3,-}, \tau_{4,-}$.

If $-a < b$, the irreducible tempered subquotients of $\pi_S \rtimes \sigma_c$ are given by the cases:

- $-a = \alpha - 1 = 0$: σ_5,
- $-a = \alpha$: σ_6 *for* $\alpha = \frac{1}{2}$, $\tau_{5,-}$ *for* $\alpha \geq 1$,
- $-a \geq \alpha + 1$: $\sigma_{3,-}, \sigma_{4,-}$.

5. Non-tempered subquotients of $\pi_S \rtimes \sigma_c$

In the last section of the paper we give a description of irreducible non-tempered subquotients of $\pi_S \rtimes \sigma_c$. In this way, we complete the description of the composition series of representation $\pi_S \rtimes \sigma_c$.

Let $L(\delta_1, \ldots, \delta_t; \tau)$ denote a subquotient of the induced representation $\pi_S \rtimes \sigma_c$. Then $\mu^*(\pi_S \rtimes \sigma_c)$ has a constituent of the form $\delta_1 \otimes \pi'$, for some $\pi' \in \mathrm{Irr}(G)$. Hence, there is a pair $\big((c_1, c_2), (d_1, d_2)\big) \in \mathrm{Lad}(\pi_S)'$ such that δ_1 is a subquotient of the induced representation (4). Since $e(\delta_1) < 0$, we necessarily have $d_2 = b + 1$. Furthermore, $-a - 1$ and $-a$ differentiate by one, so Lemma 2.3 implies two possibilities:

1. $c_1 = a - 1$. Here δ_1 is a subquotient of $\delta([\nu^{-c_2}\rho, \nu^{-a-1}\rho]) \times \delta([\nu^{d_1+1}\rho, \nu^{b}\rho])$.
2. $c_2 = a$. Note that the inequalities $a - 1 \leq c_1 < c_2$ imply $c_1 = a - 1$. Here δ_1 is isomorphic to $\delta([\nu^{d_1+1}\rho, \nu^{b}\rho])$ and we see that this case is contained in the first one.

Let us now analyze the first, and the only, case. If $d_1 + 1 \leq b$, then there are two possibilities:

- $c_2 = a$. Because of the assumption $e(\delta_1) < 0$, we necessarily have $d_1 + 1 = -b - 1$. For $a = -b - 1$, the obtained constituent of $\mu^*(\pi_S \rtimes \sigma_c)$ equals $\delta([\nu^{-b-1}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{-b}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$.
- $c_2 \geq a + 1$. Here the only option is $d_1 = -a - 1$ and $c_2 = b + 1$. The obtained constituent of $\mu^*(\pi_S \rtimes \sigma_c)$ equals $\delta([\nu^{-b-1}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{a}\rho, \nu^{-a-1}\rho]) \rtimes \sigma_c$.

If $d_1 = b$, the obtained constituents of $\mu^*(\pi_S \rtimes \sigma_c)$ are of the following form:

$$\delta([\nu^{-c_2}\rho, \nu^{-a-1}\rho]) \otimes L(\delta([\nu^{a}\rho, \nu^{b}\rho]), \delta([\nu^{c_2+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c,$$

for a real number c_2 such that $-a \leq c_2 \leq b + 1$. This is the place where we use the results of Section 3. Namely, the tempered parameter τ from $L(\delta_1; \tau)$ is a subquotient of the representation $\pi_x \rtimes \sigma_c$, for a real number x such that $-a \leq x \leq b$.

Firstly, let us analyze the case $b = -a - 1$. From the first item above, we see that the composition series of $\delta([\nu^{-b}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$ determines the candidates in this case.

If $b = -\frac{1}{2}$, the candidates are the following:

$$\begin{cases} L(\nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho; \sigma_c) \text{ and } L(\nu^{-\frac{1}{2}}\rho; \sigma_{(\frac{1}{2})}), & \text{if } \alpha = \frac{1}{2}, \\ L(\nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho; \sigma_c), & \text{otherwise.} \end{cases}$$

If $b \geq 0$, along with the representation $L(\delta([\nu^{-b-1}\rho, \nu^b\rho]), \delta([\nu^{-b-1}\rho, \nu^b\rho]); \sigma_c)$, we have the following candidates:

$$\begin{cases} L(\delta([\nu^{-b-1}\rho, \nu^b\rho]); \tau_{2,+}^{(\alpha-1)}), & \text{if } b = \alpha - 1 = 0, \\ L(\delta([\nu^{-b-1}\rho, \nu^b\rho]); \tau_{2,-}^{(\alpha-1)}), & \text{if } b = \alpha - 1 \geq \frac{1}{2}, \\ L(\delta([\nu^{-b-1}\rho, \nu^b\rho]); \sigma_1) \text{ and } L(\delta([\nu^{-b-1}\rho, \nu^b\rho]); \sigma_2), & \text{if } b \geq \alpha. \end{cases}$$

Here σ_1 and σ_2 denote discrete series representations which embed into the induced representation $\delta([\nu^{-b}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$.

Note that the listed candidates are subquotients of the induced representation $\delta([\nu^{-b-1}\rho, \nu^b\rho]) \times \delta([\nu^{-b}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$. In the Grothendieck group $R(G)$, it equals

$$\pi_S \rtimes \sigma_c + \delta([\nu^{-b-1}\rho, \nu^{b+1}\rho]) \times \delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \sigma_c.$$

Note that the representation $\delta([\nu^{-b-1}\rho, \nu^{b+1}\rho]) \times \delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \sigma_c$ is tempered. Hence, all listed candidates are subquotients of $\pi_S \rtimes \sigma_c$. Additionally, note that all subquotients of $\pi_S \rtimes \sigma_c$ are its subrepresentations. Namely, it is unitary and thus semisimple.

In the rest of the section, we analyze the case $-a \leq b$. We have seen that the following constituents of $\mu^*(\pi_S \rtimes \sigma_c)$ give candidates for non-tempered subquotients of $\pi_S \rtimes \sigma_c$:

1. $\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]) \otimes \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_c$,
2. $\delta([\nu^{-x}\rho, \nu^{-a-1}\rho]) \otimes L(\delta([\nu^a\rho, \nu^b\rho]), \delta([\nu^{x+1}\rho, \nu^{b+1}\rho])) \rtimes \sigma_c$, for $-a \leq x \leq b$,
3. $\delta([\nu^{-b-1}\rho, \nu^b\rho]) \otimes \delta([\nu^a\rho, \nu^{-a-1}\rho]) \rtimes \sigma_c$.

To obtain candidates in the first case, we need to describe the composition series of the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_c$. If $-a = b$, then the representation $\delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \sigma_c$ is

$$\begin{cases} \text{irreducible}, & \text{if } 0 \leq b \leq \alpha - 1, \\ \text{isomorphic to } \tau_{0,+}^{(b)} \oplus \tau_{0,-}^{(b)}, & \text{if } b \geq \alpha. \end{cases}$$

If $-a < b$, along with $L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]); \sigma_c)$, in case $b \geq \alpha$ we also have the following irreducible subquotients:

$$\begin{cases} L(\delta([\nu^{-(\alpha-1)}\rho, \nu^{-a}\rho]); \sigma_{(b)}), & \text{if } 0 \leq -a \leq \alpha - 2, \\ \tau_8, & \text{if } -a = \alpha - 1, \\ \sigma_7 + \sigma_8, & \text{if } -a \geq \alpha. \end{cases}$$

If we denote $\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma_{(b)} \simeq \tau_+ \oplus \tau_-$, then $\tau_8 \simeq \tau_-$, for $\alpha \geq \frac{3}{2}$, and $\tau_8 \simeq \tau_+$, for $\alpha = 1$. The discrete series representations σ_7 and σ_8 are obtained by the classification.

We denote by π_0 any irreducible subquotient of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_c$. Note that, under the corresponding assumptions on α and $-a$, the irreducible representations $L(\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-b}\rho, \nu^{-a}\rho]); \sigma_c)$, $L(\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]); \pi_0)$, for a tempered π_0, and $L(\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-(\alpha-1)}\rho, \nu^{-a}\rho]); \sigma_{(b)})$ are subquotients of

$$\Pi = \delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c.$$

We denote by π_1 any of these irreducible representations.

Lemma 5.1. *The above defined representation π_1 is a subquotient of $\pi_S \rtimes \sigma_c$.*

Proof. Firstly, note that $\pi_S \rtimes \sigma_c$ is a subquotient of Π. Hence it suffices to show that $\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]) \otimes \pi_0$ is a subquotient of $\mu^*(\Pi)$ of the multiplicity one. Here we take for π_0 also the non-tempered irreducible subquotients of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_c$. Namely, we have seen that $\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]) \otimes \pi_0$ is a constituent of $\mu^*(\pi_S \rtimes \sigma_c)$ in the course of determining candidates for subquotients of $\pi_S \rtimes \sigma_c$.

From the embedding $\pi_1 \hookrightarrow \delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]) \rtimes \pi_0$ and the Frobenius reciprocity, it follows that $\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]) \otimes \pi_0$ is also a constituent of $\mu^*(\pi_1)$. In this way, we see that if it appears in $\mu^*(\Pi)$ with the multiplicity one, then π_1 is a subquotient of $\pi_S \rtimes \sigma_c$. One can easily see that this is true since the representation $\nu^{-b-1}\rho$ appears in the cuspidal support of Π with the multiplicity one. Hence, the representation $\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]) \otimes \pi_0$ is a subquotient only of

$$(\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]) \otimes 1) \rtimes \mu^*(\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_c).$$

Now the claim follows from the fact that π_0 is a subquotient of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_c$ of the multiplicity one. ■

In the second case, candidates are determined by the composition series of representation $\pi_x \rtimes \sigma_c$, for $-a \leq x \leq b$. Following the lines of the proof of Lemma 5.1, one can prove that the candidates of the form $L(\delta([\nu^{-x}\rho, \nu^{-a-1}\rho]); \tau)$, for a real number x such that $-a \leq x \leq b$ and a tempered subquotient τ of $\pi_x \rtimes \sigma_c$ described in Theorem 3.8, are subquotients of $\pi_S \rtimes \sigma_c$. We state it in the next lemma.

Lemma 5.2. *The representation $L(\delta([\nu^{-x}\rho, \nu^{-a-1}\rho]); \tau)$ is a subquotient of the induced representation $\pi_S \rtimes \sigma_c$, where τ is an irreducible tempered subquotient of $\pi_x \rtimes \sigma_c$, for a real number x such that $-a \leq x \leq b$.*

It remains to consider the candidates for subquotients of $\pi_S \rtimes \sigma_c$ obtained from the non-tempered subquotients of $\pi_x \rtimes \sigma_c$. From the restrictions of the Langlands classification and structural formula (4), one can see that in this case the only candidates for the subquotients of $\pi_S \rtimes \sigma_c$ are $L(\delta([\nu^a\rho, \nu^{-a-1}\rho]), \delta([\nu^{-b-1}\rho, \nu^b\rho]); \sigma_c)$ and $\pi_0 = L(\delta([\nu^{-(\alpha-1)}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-(\alpha-2)}\rho, \nu^{-a}\rho]); \sigma_{(b,b+1)})$. Note that the first one is the representation that will be studied in the third case. Similarly as in the proof of Lemma 5.1, we obtain the following lemma.

Lemma 5.3. *If $0 \leq -a \leq \alpha - 3$ and $\alpha - 1 \leq b$, then π_0 is a subquotient of $\pi_S \rtimes \sigma_c$.*

In the third case, candidates for the subquotients of $\pi_S \rtimes \sigma_c$ are obtained from the constituents of $\mu^*(\pi_S \rtimes \sigma_c)$ of the form $\delta([\nu^{-b-1}\rho, \nu^b\rho]) \otimes \delta([\nu^a\rho, \nu^{-a-1}\rho]) \rtimes \sigma_c$. In the rest of this section, we denote $L := L(\delta([\nu^{-b-1}\rho, \nu^b\rho]), \delta([\nu^a\rho, \nu^{-a-1}\rho]); \sigma_c)$.

Proposition 5.4. *Assume $-a \geq \alpha$. Then L is a subquotient of $\pi_S \rtimes \sigma_c$.*

Proof. The proof will be an adaptation of the proof of Theorem 3.1 in [19]. To do this, firstly note that $-a \geq \alpha$ implies that the representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_c$ contains two non-equivalent discrete series representations obtained by the classification or two non-equivalent tempered irreducible subrepresentations. More precisely, if $-a < b$ and $\mathrm{Jord}_\rho(\sigma_c) = \emptyset$, then from [18, Theorem 2.3 (ii)] it follows that the representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_c$ has two discrete series subquotients obtained from the classification.

On the other hand, if $\mathrm{Jord}_\rho(\sigma_c) \neq \emptyset$ and $-a \geq \alpha$, the same claim is true because of [18, Theorem 2.1]. If $-a = b$, then the representation $\delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \sigma_c$ is isomorphic to the direct sum of two non-equivalent tempered irreducible representations, according to [26, Theorem 1.1].

In the proof of Theorem 3.1 in [19], it is shown that L is a subquotient of the induced representation $\delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \times \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_c$ of the multiplicity two. Here we assume $a \leq 0$, $-a < b$ and $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_c$ has exactly two discrete series subquotients obtained by the classification. It is easy to check that an analogous argumentation proves this claim about the multiplicity of subquotient L also in the case $-a = b$, where $\delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \sigma_c$ is the direct sum of two non-equivalent irreducible tempered representations. Now the assertion of this proposition follows from the count of the multiplicity of $\delta([\nu^{-b-1}\rho, \nu^b\rho]) \otimes \delta([\nu^a\rho, \nu^{-a-1}\rho]) \rtimes \sigma_c$ in $\mu^*(\Pi)$, where $\Pi = \delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \times \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_c$. Namely, in the Grothendieck group $R(G)$, the representation $\delta([\nu^{a+1}\rho, \nu^{b+1}\rho]) \times \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_c$ equals

$$\pi_S \rtimes \sigma_c + \delta([\nu^{a+1}\rho, \nu^b\rho]) \times \delta([\nu^a\rho, \nu^{b+1}\rho]) \rtimes \sigma_c. \tag{24}$$

Note that $\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]) \otimes 1$ is the only constituent of $M^*(\delta([\nu^{a+1}\rho, \nu^{b+1}\rho]))$ or $M^*(\delta([\nu^a\rho, \nu^b\rho]))$ with $\nu^{-b-1}\rho$ in the cuspidal support. Namely, this follows from the fact that $-b-1$ is the minimal exponent of the multiplicity one in the cuspidal support of Π. Now from the following constituents of $M^*(\delta([\nu^a\rho, \nu^b\rho]))$:

- $\delta([\nu^{-a}\rho, \nu^b\rho]) \otimes \delta([\nu^a\rho, \nu^{-a-1}\rho])$,
- $\nu^{-a}\rho \times \delta([\nu^{-a+1}\rho, \nu^b\rho]) \otimes \delta([\nu^{a+1}\rho, \nu^{-a}\rho])$,

we get that $\delta([\nu^{-b-1}\rho, \nu^b\rho]) \otimes \delta([\nu^a\rho, \nu^{-a-1}\rho]) \rtimes \sigma_c$ is of multiplicity two in $\mu^*(\Pi)$.

We have seen that it is of multiplicity one in $\mu^*(\pi_S \rtimes \sigma_c)$, and as $\delta([\nu^a\rho, \nu^{-a-1}\rho]) \otimes \sigma_c$ is of multiplicity one in $\mu^*(\delta([\nu^a\rho, \nu^{-a-1}\rho]) \rtimes \sigma_c)$, we have the same conclusion for

$$\delta([\nu^{-b-1}\rho, \nu^b\rho]) \otimes \delta([\nu^a\rho, \nu^{-a-1}\rho]) \otimes \sigma_c. \tag{25}$$

The Frobenius reciprocity implies that (25) is a constituent of $r_\beta(L)$, for the appropriate parabolic subgroup P_β. Hence, L is a subquotient of each of summands of (24). ■

Lemma 5.5. *If* $\frac{1}{2} \leq -a \leq \alpha - 1$, *then* L *is not a subquotient of* $\pi_S \rtimes \sigma_c$.

Proof. From the following intertwinning operators

$$\begin{aligned} L &\hookrightarrow \delta([\nu^{-b-1}\rho, \nu^b\rho]) \times \delta([\nu^a\rho, \nu^{-a-1}\rho]) \rtimes \sigma_c \\ &\simeq \delta([\nu^{-b-1}\rho, \nu^b\rho]) \times \delta([\nu^{a+1}\rho, \nu^{-a}\rho]) \rtimes \sigma_c \\ &\simeq \delta([\nu^{a+1}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-b-1}\rho, \nu^b\rho]) \rtimes \sigma_c, \end{aligned}$$

it follows that $\mu^*(L)$ contains a constituent of the form $\nu^{-a}\rho \otimes \pi'$, which is not true for $\mu^*(\pi_S \rtimes \sigma_c)$ in the case $-a < b$. Here we have the first isomorphism because the representation $\delta([\nu^a\rho, \nu^{-a-1}\rho]) \rtimes \sigma_c$ is irreducible, according to [18, Theorem 4.1(i)]. If $-a = b$, then $r_\beta(L)$ contains a constituent of the form $\nu^b\rho \otimes \nu^b\rho \otimes \pi''$ for the corresponding parabolic subgroup P_β, which again is not true for $r_\beta(\pi_S \rtimes \sigma_c)$. Thus we have shown that L is not a subquotient of $\pi_S \rtimes \sigma_c$. ∎

Let us now assume $-a \geq 1$. Along with the representation L, we have candidates for subquotients of $\pi_S \rtimes \sigma_c$ which are determined with the following tempered subquotients of $\delta([\nu^{a+1}\rho, \nu^{-a}\rho]) \rtimes \sigma_c$:

$$\begin{cases} \tau_{21}, & \text{if } -a-1 = \alpha - 1 \geq 0, \\ \sigma_{21}, \sigma_{22}, & \text{if } -a-1 \geq \alpha \geq 1 \text{ or } \alpha \in \{0, \frac{1}{2}\}. \end{cases}$$

Note that this covers both cases when $\mathrm{Jord}_\rho(\sigma_c)$ is empty or not. We denote with τ a tempered subquotient of $\delta([\nu^{a+1}\rho, \nu^{-a}\rho]) \rtimes \sigma_c$ and with π_0 the representation $L(\delta([\nu^{-b-1}\rho, \nu^b\rho]); \tau)$. Similarly as in the proof of Lemma 5.5, we conclude that π_0 is not a subquotient of $\pi_S \rtimes \sigma_c$.

If $-a = \frac{1}{2}$, then for $\alpha = \frac{1}{2}$ we have a candidate $L(\delta([\nu^{-b-1}\rho, \nu^b\rho]); \sigma_{(\frac{1}{2})})$. It can be shown that it is not a subquotient of $\pi_S \rtimes \sigma_c$ in an analogous way as in the case $-a \geq 1$.

If $a = 0$, then $L \simeq L(\delta([\nu^{-b-1}\rho, \nu^b\rho]); \sigma_c)$ is the only candidate for a subquotient. Thanks to Proposition 5.4, it remains to analyze it in the case $\mathrm{Jord}_\rho(\sigma_c) \neq \emptyset$.

If $0 \leq b+1 \leq \alpha - 1$, then $\delta([\nu^{-b-1}\rho, \nu^b\rho]) \rtimes \sigma_c$ is an irreducible representation and we have $L \simeq \delta([\nu^{-b}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c$. Thus $\mu^*(L)$ has a constituent of the form $\nu^{b+1}\rho \otimes \pi'$ for some $\pi' \in \mathrm{Irr}(G)$, which is not true for $\mu^*(\pi_S \rtimes \sigma_c)$. Hence, in this case, L is not a subquotient of $\pi_S \rtimes \sigma_c$.

If $b+1 \geq \alpha$, then we have the following decomposition of the intertwinning operator $j : \delta([\nu^{-b}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c \to \delta([\nu^{-b-1}\rho, \nu^b\rho]) \rtimes \sigma_c$:

$$\begin{aligned} \delta([\nu^{-b}\rho, \nu^{b+1}\rho]) \rtimes \sigma_c &\overset{i}{\hookrightarrow} \delta([\nu\rho, \nu^{b+1}\rho]) \times \delta([\nu^{-b}\rho, \rho]) \rtimes \sigma_c \overset{h}{\to} \\ &\delta([\nu\rho, \nu^{b+1}\rho]) \times \delta([\rho, \nu^b\rho]) \rtimes \sigma_c \overset{g}{\to} \\ &\delta([\rho, \nu^b\rho]) \times \delta([\nu\rho, \nu^{b+1}\rho]) \rtimes \sigma_c \overset{f}{\to} \\ \delta([\nu^{-b-1}\rho, \nu^b\rho]) \rtimes \sigma_c &\hookrightarrow \delta([\rho, \nu^b\rho]) \times \delta([\nu^{-b-1}\rho, \nu^{-1}\rho]) \rtimes \sigma_c. \end{aligned}$$

Therefore, we have $j = f \circ g \circ h \circ i$. Note that $\mathrm{Im}(g) \simeq \pi_S \rtimes \sigma_c$. As L is the unique irreducible subrepresentation of $\delta([\nu^{-b-1}\rho, \nu^b\rho]) \rtimes \sigma_c$ and $\mathrm{Im}(j)$ also embeds into this induced representation, we obtain $L \hookrightarrow \mathrm{Im}(j)$. From the decomposition it follows $L \hookrightarrow \mathrm{Im}(j) \hookrightarrow \mathrm{Im}(f|_{\mathrm{Im}(g)})$. Thus we conclude that the embedding $L \hookrightarrow f(\mathrm{Im}(g))$ implies the existence of an irreducible subquotient π_0 of $\mathrm{Im}(g) \simeq \pi_S \rtimes \sigma_c$ such that $f(\pi_0) \simeq L$. Since π_0 and L are irreducible representations, f induces their isomorphism. Accordingly, L is a subquotient of $\pi_S \rtimes \sigma_c$.

Theorem 5.6. *If $0 \le -a \le b+1$, the induced representation $\pi_S \rtimes \sigma_c$ has the following non-tempered irreducible subquotients by cases:*

- $-a-1=b$: $L(\delta([\nu^{-b-1}\rho, \nu^b\rho]), \delta([\nu^{-b-1}\rho, \nu^b\rho]); \sigma_c)$ *and*
  $L(\nu^{-\frac{1}{2}}\rho; \sigma_{(\frac{1}{2})})$, *if* $b = \alpha - 1 = -\frac{1}{2}$,
  $L(\delta([\nu^{-b-1}\rho, \nu^b\rho]); \tau_{2,+}^{(\alpha-1)})$, *if* $b = \alpha - 1 = 0$,
  $L(\delta([\nu^{-b-1}\rho, \nu^b\rho]); \tau_{2,-}^{(\alpha-1)})$, *if* $b = \alpha - 1 \ge \frac{1}{2}$,
  $L(\delta([\nu^{-b-1}\rho, \nu^b\rho]); \sigma_1)$, $L(\delta([\nu^{-b-1}\rho, \nu^b\rho]); \sigma_2)$, *if* $b \ge \alpha$.
- $-a = b$: $L(\delta([\nu^{-b-1}\rho, \nu^{b-1}\rho]); \delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \sigma_c)$, *if* $0 \le b \le \alpha - 1$,
  $L(\delta([\nu^{-b-1}\rho, \nu^{b-1}\rho]); \tau_{0,\pm}^{(b)})$, $L(\delta([\nu^{-b-1}\rho, \nu^b\rho]), \delta([\nu^{-b}\rho, \nu^{b-1}\rho]); \sigma_c)$, *if* $b \ge \alpha$,
  $L(\delta([\nu^{-b}\rho, \nu^{b-1}\rho]); \tau_{2,-}^{(\alpha-1)})$, *if* $b = \alpha - 1 = 0$,
  $L(\delta([\nu^{-b}\rho, \nu^{b-1}\rho]); \tau_{2,+}^{(\alpha-1)})$, *if* $b = \alpha - 1 \ge \frac{1}{2}$,
  $L(\delta([\nu^{-b-1}\rho, \nu^b\rho]); \sigma_c)$, *if* $b = 0$ *and* $\alpha = 1$.
- $-a < b$: $L(\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-b}\rho, \nu^{-a}\rho]); \sigma_c)$ *and*
  $L(\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-(\alpha-1)}\rho, \nu^{-a}\rho]); \sigma_{(b)})$, *if* $-a \le \alpha - 2$ *and* $b \ge \alpha$,
  $L(\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]); \tau_8)$, *if* $-a = \alpha - 1$,
  $L(\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]); \sigma_7)$, $L(\delta([\nu^{-b-1}\rho, \nu^{-a-1}\rho]); \sigma_8)$,
  $L(\delta([\nu^{-b-1}\rho, \nu^b\rho]), \delta([\nu^a\rho, \nu^{-a-1}\rho]); \sigma_c)$, *if* $-a \ge \alpha$,
  $L(\delta([\nu^{-(\alpha-1)}\rho, \nu^{-a-1}\rho]); \tau_{1,+}^{(0)})$, *if* $-a = \alpha - 2 = 0$,
  $L(\delta([\nu^{-(\alpha-1)}\rho, \nu^{-a-1}\rho]); \tau_{1,-}^{(\alpha-2)})$, *if* $-a = \alpha - 2 \ge \frac{1}{2}$,
  $L(\delta([\nu^{-b-1}\rho, \nu^b\rho]); \sigma_c)$, *if* $a = 0$ *and* $b \ge \alpha - 1$,
  $L(\delta([\nu^{-(\alpha-1)}\rho, \nu^{-a-1}\rho]); \sigma_0)$, *if* $-a \ge \alpha - 1 \ge 1$,
  $L(\delta([\nu^{-(\alpha-1)}\rho, \nu^{-a-1}\rho]), \delta([\nu^{-(\alpha-2)}\rho, \nu^{-a}\rho]); \sigma_{(b,b+1)})$, *if* $-a \le \alpha - 3$ *and* $b \ge \alpha - 1$.

As a direct application of the results of this paper and [4], we get irreducible unitary representations in the end of certain complementary series.

Let $\pi_S^{(b)}$ denote the Speh representation $L(\delta([\nu^{-b-1}\rho, \nu^b\rho]), \delta([\nu^{-b}\rho, \nu^{b+1}\rho]))$, for real number b such that $-\frac{1}{2} \le b < \alpha - 1$. In the beginning of the Section 4, we have shown that $\pi_S^{(b)} \rtimes \sigma_c$ does not have any irreducible tempered subquotient. Moreover, from Theorem 5.6 it follows that the representation $\pi_S^{(b)} \rtimes \sigma_c$ is irreducible. Using the ends of complementary series, we determine unitary irreducible subquotients of representation induced from the essentially Speh and cuspidal irreducible representation. More precisely, let γ be a non-negative real number such that the induced representation

$$\nu^x \pi_S^{(b)} \rtimes \sigma_c \simeq L(\delta([\nu^{-b-1+x}\rho, \nu^{b+x}\rho]), \delta([\nu^{-b+x}\rho, \nu^{b+1+x}\rho])) \rtimes \sigma_c \tag{26}$$

is irreducible for $0 \leq x < \gamma$ and reducible for $x = \gamma$. It is a classical result that the irreducible subquotients of $\nu^\gamma \pi_S^{(b)} \rtimes \sigma_c$ are unitary representations. Our aim is to determine γ. Let us consider $x = \alpha - b - 1$, for which the representation (26) is isomorphic to

$$L(\delta([\nu^{\alpha-2b-2}\rho, \nu^{\alpha-1}\rho]), \delta([\nu^{\alpha-2b-1}\rho, \nu^{\alpha}\rho])) \rtimes \sigma_c.$$

For real numbers x such that $0 \leq x < \alpha - b - 1$ and $-b - 1 + x > 0$, we see that the representation (26) is irreducible from Theorems 1.1 and 1.2 in [10]. For real numbers x such that $0 \leq x < \alpha - b - 1$ and $-b - 1 + x \leq 0$, we see that the representation (26) is irreducible from Theorems 4.12 and 5.6. Therefore, if $\nu^{\alpha-b-1}\pi_S^{(b)} \rtimes \sigma_c$ is reducible representation, then $\gamma = \alpha - b - 1$. Let us define

$$\pi_0 = L(\delta([\nu^{-\alpha}\rho, \nu^{-\alpha+2b+1}\rho]), \delta([\nu^{-\alpha+1}\rho, \nu^{-\alpha+2b+2}\rho]); \sigma_c)$$

and $\pi_1 = L(\delta([\nu^{-(\alpha-1)}\rho, \nu^{-\alpha+2b+1}\rho]), \delta([\nu^{-(\alpha-2)}\rho, \nu^{-\alpha+2b+2}\rho]); \sigma_{(\alpha-1,\alpha)})$.

Corollary 5.7. *For $\alpha - b \in \mathbb{Z}$ and $-\frac{1}{2} \leq b < \alpha - 1$, the induced representation $\nu^{\alpha-b-1}\pi_S^{(b)} \rtimes \sigma_c$ has the following composition series by cases:*

- $\pi_0 + L(\nu^{-1}\rho; \tau_{1,+}^{(0)}) + L(\delta([\nu^{-2}\rho, \nu\rho]); \sigma_c)$, *if* $\alpha - 2 = b = 0$,
- $\pi_0 + L(\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-3}\rho]); \tau_{1,-}^{(\alpha-2)})$, *if* $\alpha - 2 = b \geq \frac{1}{2}$,
- $\pi_0 + \pi_1 + L(\delta([\nu^{-\alpha}\rho, \nu^{\alpha-1}\rho]); \sigma_c)$, *if* $\alpha - 2 = 2b \geq 1$,
- $\pi_0 + \pi_1$, *otherwise.*

Proof. Note that in the case $b \leq \alpha - 2 \leq 2b$, the representation $\nu^{\alpha-b-1}\pi_S^{(b)} \rtimes \sigma_c$ is reducible for every $\alpha - 2$ in $[b, 2b]$. This makes $\nu^{\alpha-b-1}\pi_S^{(b)} \rtimes \sigma_c$ the end of the complementary series for every $-\frac{1}{2} \leq b < \alpha - 1$ and $\alpha \geq 0$. Theorem 5.6 describes irreducible subquotients in case $b \leq \alpha - 2 \leq 2b$, while in case $\alpha - 2 \geq 2b + 1$ subquotients are described in Theorem 4.2 from [4]. Namely, from Theorem 4.12 and [4, Theorem 3.1] we see that $\nu^{\alpha-b-1}\pi_S^{(b)} \rtimes \sigma_c$ does not have any irreducible tempered subquotients. ■

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