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Weakly Associative and Symmetric Leibniz Algebras

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Abstract. We study a special class of weakly associative algebras: the symmetric Leibniz algebras. We describe the structure of the commutative and skew-symmetric algebras associated with the polarization-depolarization principle. We also give a structure theorem for the symmetric Leibniz algebras and we describe the low dimensional classification. We finally study formal deformations in the context of deformation quantization.

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1. Introduction

The notion of weakly associative algebra was introduced in [10] in order to broadening the notion of deformation quantization of Poisson algebras. In fact, a deformation quantization of a Poisson algebra $(A, \bullet, [\ ,\ ])$ is given by a formal deformation of the associative commutative algebra $(A, \bullet)$ whose linear term (the term of degree 1 in the series expansion of the formal deformations) satisfies the Lie-admissible identity and determines by taking the skew-symmetric part the Lie bracket $[\ ,\ ]$. Recall that $(A, \bullet, [\ ,\ ])$ is a *Poisson algebra* if

1. $(A, \bullet)$ is a associative commutative algebra,
2. $(A, [\ ,\ ])$ is a Lie algebra,
3. these two multiplications on A are related by the Leibniz Identity

$$[X \bullet Y, Z] = X \bullet [Y, Z] + [X, Z] \bullet Y$$

for all $X, Y, Z \in A$.

In this context, a formal deformation $(A_t, \bullet_t)$ of $(A, \bullet)$ is a formal associative algebra and the set of all these deformations are governed by the deformation cohomology which coincides in this case with the Hochschild cohomology of $(A, \bullet)$. It has been shown in [10] Theorem 10 that a Poisson algebra $(A, \bullet, [\ ,\ ])$ can be obtained from a formal deformation of the associative commutative algebra $(A, \bullet)$ but in a larger category of algebras namely the weakly associative one. This class of algebras contains in particular Lie algebras, associative algebras and commutative algebras.

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We show that the *symmetric Leibniz algebras*, that is algebras $(A,*)$ satisfying the pair of identities

$$\begin{cases} X*(Y*Z) = (X*Y)*Z + Y*(X*Z), \\ (Y*Z)*X = (Y*X)*Z + Y*(Z*X) \end{cases}$$

for all $X,Y,Z \in A$, are also *weakly associative*. We use this property to describe these algebras which leads to a structure theorem (Theorem 6.3). As an application we reprove some results of [1].

In Section 1 we recall the definition and properties of weakly associative algebras, the polarization-depolarization principle which permits to consider these algebras as nonassociative Poisson algebras. Then in Section 2 we first study the properties of symmetric Leibniz as one multiplication nonassociative algebra: they are weakly associative algebras. But we also study symmetric Leibniz algebras considered as Poisson algebra after polarization and show in Section 3 that the associative commutative product $\bullet$ is 2-step nilpotent and the Lie bracket $[\ ,\ ]$ has the property that $[X, Y\bullet Z] = 0$ and $X\bullet[Y,Z] = 0$ for all vectors X,Y,Z. Then in Section 4 we give a classification of symmetric Leibniz algebras for the small dimensional cases. We finally investigate formal deformations of symmetric Leinbiz algebras and determine which Poisson algebras are obtained by deformation quantization.

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2. Weakly associative algebras

Let $\mathbb{K}$ be a field of characteristic 0 and $(A,*)$ a nonassociative $\mathbb{K}$-algebra. Recall that an algebra $(A,*)$ is a nonassociative algebra if it is a $\mathbb{K}$-vector space with a bilinear binary multiplication $*$ which may be associative or not. The *associator* of the algebra $(A,*)$ is the trilinear map $\mathcal{A}_*$ defined for all $X,Y,Z \in A$ by

$$\mathcal{A}_*(X,Y,Z) = X*(Y*Z) - (X*Y)*Z.$$

Definition 2.1. ([10] Definition 1) A $\mathbb{K}$-algebra $(A,*)$ is called *weakly associative* if its associator $\mathcal{A}_*$ satisfies, for all $X,Y,Z \in A$,

$$\mathcal{A}_*(X,Y,Z) + \mathcal{A}_*(Y,Z,X) - \mathcal{A}_*(Y,X,Z) = 0. \qquad \blacksquare$$

To simplify, we shall denote by $\mathcal{WA}_*(X,Y,Z)$ the trilinear map

$$\mathcal{WA}_*(X,Y,Z) = \mathcal{A}_*(X,Y,Z) + \mathcal{A}_*(Y,Z,X) - \mathcal{A}_*(Y,X,Z)$$

and a $\mathbb{K}$-algebra $(A,*)$ is weakly associative if and only if, for all $X,Y,Z \in A$,

$$\mathcal{WA}_*(X,Y,Z) = 0.$$

Examples 2.2. ([10] Section 2.1)

(1) Any associative algebra is weakly associative.

(2) Any Abelian (i.e. commutative) algebra is weakly associative.

(3) Any Lie algebra is weakly associative. In fact, if $(\mathfrak{g},[\ ,\ ])$ is a Lie $\mathbb{K}$-algebra, the associator of the Lie bracket satisfies, for all $X,Y,Z \in A$,

$$\mathcal{A}_{[,]}(X,Y,Z) = [X,[Y,Z]] - [[X,Y],Z].$$

From the Jacobi identity we get

$$\mathcal{A}_{[,]}(X,Y,Z) = [\,[Z,X],Y].$$

Then $$\mathcal{WA}_{[,]}(X,Y,Z) = [\,[Z,X],Y] + [\,[X,Y],Z] + [\,[Y,Z],X]$$

so from the Jacobi identity $\mathcal{WA}_{[,]}(X,Y,Z) = 0$.

(4) In Section 5, we will show another important example of weakly associative algebras given by the class of *symmetric Leibniz algebras*. ∎

We can also recall a characterization of weakly associative algebra

Proposition 2.3. ([10], Theorem 2) *Let $(A,*)$ be a nonassociative $\mathbb{K}$-algebra. Then for all $X \in A$, the endomorphism $L_X - R_X$ defined by*

$$(L_X - R_X)(Y) = X * Y - Y * X$$

is a derivation of $(A,*)$ if and only if $(A,*)$ is weakly associative.

3. Polarization-depolarization of $\mathbb{K}$-algebras

The polarization technique consists in representing a given one-operation $\mathbb{K}$-algebra $(A,*)$ without particular symmetry as an algebra with two operations, one commutative and the other skew-symmetric. Explicitly we will decompose the multiplication $*$ of the $\mathbb{K}$-algebra $(A,*)$ using:

(1) $X \bullet Y = \frac{1}{2}(X * Y + Y * X)$ its symmetric part,

(2) $[X,Y] = \frac{1}{2}(X * Y - Y * X)$ its skew-symmetric part,

for $X, Y \in A$. The triplet $(A, \bullet, [\ ,\ ])$ will be referred to as the *polarization* of $(A,*)$.

Conversely, starting with an algebra $(A, \bullet, [\ ,\ ])$ with a commutative nonassociative product $\bullet$ and a skew-symmetric multiplication $[\ ,\ ]$, we obtain an algebra $(A,*)$ with only one nonassociative multiplication defined by $X * Y = X \bullet Y + [X,Y]$. The algebra $(A,*)$ is called the *depolarization* of the algebra $(A, \bullet, [\ ,\ ])$.

The polarization-depolarization technique of [7] then makes a link between Poisson algebras and some nonassociative algebras and permits to present a Poisson algebra as a nonassociative algebra $(A,*)$ satisfying

$$3(x*y)*z - 3x*(y*z) = (x*z)*y + (y*z)*x - (y*x)*z - (z*x)*y.$$

Other applications of this technique are found in [3].

The polarization technique also permits to study weakly associative algebras as *nonassociative Poisson algebras.*

Definition 3.1. Let $(A, \bullet, [\,,\,])$ be a triple where A is a $\mathbb{K}$-vector space with two multiplications $\bullet$ and $[\,,\,]$. We say that $(A, \bullet, [\,,\,])$ is a *nonassociative Poisson algebra* if

(1) $(A, \bullet)$ is a nonassociative commutative algebra,

(2) $(A, [\ ,\ ])$ is a Lie algebra,

(3) the Leibniz identity between $[\,,\,]$ and $\bullet$ is satisfied, that is, for all $X, Y, Z \in A$

$$[X \bullet Y, Z] = X \bullet [Y,Z] + [X,Z] \bullet Y.$$ ∎

As the terminology suggests, nonassociative Poisson algebras generalize Poisson algebras by relaxing the associativity condition on the underlying commutative algebra.

The relation between Poisson algebras and weakly associative algebras is summarized in the following result:

Theorem 3.2. *Let* $(A, \bullet, [\ ,\ ])$ *be a nonassociative Poisson algebra.*
Consider on A *the third multiplication* $X * Y = X \bullet Y + [X, Y]$*.*
Then the algebra $(A, *)$*, that is the depolarization of* $(A, \bullet, [\ ,\ ])$*, is weakly associative. Conversely, if* $(A, *)$ *is a weakly associative algebra, then its polarization* $A(\bullet, [\ ,\ ])$ *is a nonassociative Poisson algebra.*

Proof. See [10], Theorem 8 and 9. ∎

4. Polarisation of a Lie-admissible algebra

Let $(A, *)$ be a Lie-admissible algebra, that is the bracket

$$[X, Y] = X * Y - Y * X$$

is a Lie bracket. Recall the following result:

Lemma 4.1. *Consider* $(A, \bullet, [\ ,\ ])$ *the polarization of* $(A, *)$*. The following relations are equivalent:*

(1) $\text{Leibniz}([\ ,\ ], *):\ [X * Y, Z] - X * [Y, Z] - [X, Z] * Y = 0$ *for all* $X, Y, Z \in A$*,*

(2) $\text{Leibniz}([\ ,\ ], \bullet):\ [X \bullet Y, Z] - X \bullet [Y, Z] - [X, Z] \bullet Y = 0$ *for all* $X, Y, Z \in A$*.*

Notice also that in order to obtain the classical notion of Poisson algebra, the multiplication $\bullet$ has also to be associative. A direct computation gives

$$\begin{aligned} 4\mathcal{A}_\bullet(X, Y, Z) = {} & X * (Y * Z + Z * Y) + (Y * Z + Z * Y) * X \\ & - (X * Y + Y * X) * Z - Z * (X * Y + Y * X). \end{aligned}$$

Denote by $\mathcal{B}_*$ the trilinear map associated to the multiplication $*$ defined by

$$\mathcal{B}_*(X, Y, Z) = X * (Y * Z) + (Y * Z) * X.$$

Then $\quad 4\mathcal{A}_\bullet(X, Y, Z) = \mathcal{B}_*(X, Y, Z) + \mathcal{B}_*(X, Z, Y) - \mathcal{B}_*(Z, X, Y) - \mathcal{B}_*(Z, Y, X)$

and therefore a sufficient condition for the associativity of $\bullet$ is that $\mathcal{B}_*(X, Y, Z) = 0$.

Theorem 4.2. *Let* $(A, *)$ *be a Lie-admissible algebra,* $(A, \bullet, [\ ,\ ])$ *its polarization and* $\mathcal{B}_*$ *the trilinear map defined by*

$$\mathcal{B}_*(X, Y, Z) = X * (Y * Z) + (Y * Z) * X$$

for all $X, Y, Z \in A$*. If the map* $\mathcal{B}_*$ *is trivial the algebra* $(A, \bullet)$ *is associative.*

5. Symmetric Leibniz algebras

A *symmetric Leibniz algebra* is an algebra $(A, *)$ such that for all $X, Y, Z \in A$, we have

$$\begin{cases} X * (Y * Z) = (X * Y) * Z + Y * (X * Z), \\ (Y * Z) * X = (Y * X) * Z + Y * (Z * X). \end{cases} \tag{1}$$

Denote by $\mathcal{A}_*$ the associator of the multiplication $*$, that is

$$\mathcal{A}_*(X,Y,Z) = X*(Y*Z) - (X*Y)*Z$$

for all $X,Y,Z \in A$, the defining identities (1) become

$$\mathcal{A}_*(X,Y,Z) = Y*(X*Z) \text{ and } \mathcal{A}_*(Y,Z,X) = -(Y*X)*Z.$$

We deduce that $(A,*)$ is a symmetric Leibniz algebra if and only if

$$\mathcal{A}_*(X,Y,Z) = Y*(X*Z) \text{ and } \mathcal{A}_*(X,Y,Z) = -(X*Z)*Y,$$

or equivalently

$$\mathcal{A}_*(X,Y,Z) = Y*(X*Z) \text{ and } \mathcal{B}_*(X,Y,Z) = 0.$$

In particular we deduce

$$\mathcal{A}_*(X,Y,Z) + \mathcal{A}_*(Y,Z,X) = Y*(X*Z) - (Y*X)*Z = \mathcal{A}_*(Y,X,Z)$$

that is
$$\mathcal{A}_*(X,Y,Z) + \mathcal{A}_*(Y,Z,X) - \mathcal{A}_*(Y,X,Z) = 0$$

and A is a weakly associative algebra.

Proposition 5.1. *Any symmetric Leibniz algebra is weakly associative.*

Since any symmetric Leibniz algebra $(A,*)$ is weakly associative, it is also Lie-admissible [5] and its polarized algebra $(A,\bullet,[\ ,\ ])$ is a nonassociative Poisson algebra. But we also have that $\mathcal{B}_*(X,Y,Z) = 0$ for all $X,Y,Z \in A$ so we obtain by using Theorem 4.2 the following result:

Proposition 5.2. *Let $(A,*)$ be a symmetric Leibniz algebra and $A(\bullet,[\ ,\ ])$ its polarization. We have the following properties:*

(1) $(A,*)$ *is weakly associative,*

(2) $(A,\bullet)$ *is commutative associative algebra,*

(3) $A(\bullet,[\ ,\ ])$ *is a Poisson algebra.*

So a Poisson algebra can be naturally obtained from a symmetric Leibniz algebra by polarization. This result is already proved in [2].

Remark 5.3. In [6], we have studied some classes of nonassociative algebras in terms of action of the symmetric group Σ_3: the identites of nonassociativity are defined by the action of the symmetric group Σ_3 on the associator. In this list we find the symmetric Leibniz algebras. ∎

6. Structure of symmetric Leibniz algebras

Recall that a symmetric Leibniz algebra is an algebra $(A,*)$ which satisfies the identities

$$\begin{cases} X*(Y*Z) = (X*Y)*Z + Y*(X*Z), \\ (Y*Z)*X = (Y*X)*Z + Y*(Z*X). \end{cases}$$

In particular it satisfies the following relation

$$\mathcal{B}_*(Y,X,Z) = Y*(X*Z) + (X*Z)*Y = 0.$$

Using the polarization, this equation reads

$$[Y,[X,Z]] + [Y, X\bullet Z] + Y\bullet[X,Z] + Y\bullet(X\bullet Z) + [[X,Z],Y]$$
$$+[X\bullet Z, Y] + [X,Z]\bullet Y + (X\bullet Z)\bullet Y = 0$$

which is equivalent to

$$Y\bullet[X,Z] + Y\bullet(X\bullet Z) + [X,Z]\bullet Y + (X\bullet Z)\bullet Y = 0$$

or

$$Y\bullet[X,Z] + Y\bullet(X\bullet Z) = 0. \tag{2}$$

We deduce that we also have $Y\bullet[Z,X] + Y\bullet(Z\bullet X) = 0$.

If we add these two identities we obtain:

Proposition 6.1. *Let $(A,*)$ be a symmetric Leibniz algebra and $(A,\bullet,[\ ,\ ])$ its polarization. Then $\bullet$ is an associative commutative multiplication which satisfies*

$$X\bullet(Y\bullet Z) = 0 \ \text{ for all } X,Y,Z \in A.$$

Hence the associative commutative algebra $(A,\bullet)$ is two-step nilpotent.

Denote by $A^2_\bullet$ the subalgebra of $(A,\bullet)$ generated by the products $X\bullet Y$ when $X,Y \in A$ and by $A^1_\bullet$ a vector subspace of A isomorphic to $A/A^2_\bullet$ then we have the grading

$$A = A^1_\bullet \oplus A^2_\bullet$$

with

$$A^1_\bullet\bullet A^1_\bullet = A^2_\bullet, \quad A^1_\bullet\bullet A^2_\bullet = A^2_\bullet\bullet A^2_\bullet = 0.$$

Now let us look at the properties of the Lie bracket associated with the polarization process of the symmetric Leibniz multiplication. Let $[A,A]$ be the derived Lie subalgebra of the Lie algebra $(A,[\ ,\ ])$ that is the Lie subalgebra generated by $[X,Y]$ for all $X,Y \in A$. Equation (2) says that $Y\bullet[X,Z] + Y\bullet(X\bullet Z) = 0$ for all $X,Y,Z \in A$. Since $Y\bullet(X\bullet Z) = 0$, we deduce

Proposition 6.2. *Let $(A,*)$ be a symmetric Leibniz algebra and $(A,\bullet,[\ ,\ ])$ its polarization. Then $[A,A]$ is contained in the center of the associative algebra $(A,\bullet)$.*

Since for all $X,Y,Z \in A$ we have $X\bullet(Y\bullet Z) = 0$ and $X\bullet[Y,Z] = 0$, the relation $\mathcal{A}_*(X,Y,Z) - Y*(X*Z) = 0$ is equivalent to

$$[X, Y\bullet Z] - [X\bullet Y, Z] - [Y, X\bullet Z] = 0.$$

Then we also have $[X, Z\bullet Y] - [X\bullet Z, Y] - [Z, X\bullet Y] = 0$.

Since $\bullet$ is commutative, by adding these relations we obtain that

$$[X, Y\bullet Z] = 0 \ \text{ for all } X,Y,Z \in A.$$

Theorem 6.3. *Let $(A, *)$ be a $\mathbb{K}$-algebra and $(A, \bullet, [\ ,\ ])$ its polarized algebra. Then $(A, *)$ is a symmetric Leibniz algebra if and only if the following conditions are satisfied:*

(1) $X \bullet (Y \bullet Z) = 0$ *for all* $X, Y, Z \in A$
(*that is* $(A, \bullet)$ *is an associative commutative* 2*-step nilpotent algebra*),

(2) $[\ ,\ ]$ *is a Lie bracket with the property that*

$$[X, Y \bullet Z] = 0 \text{ and } X \bullet [Y, Z] = 0 \text{ for all } X, Y, Z \in A.$$

Proof. We have to prove only the converse implication. By hypothesis, $\bullet$ is commutative and $[\ ,\ ]$ is a Lie bracket satisfy therefore

$$X \bullet (Y \bullet Z) = 0, \quad [X, Y \bullet Z] = 0, \quad X \bullet [Y, Z] = 0$$

for all $X, Y, Z \in A$. Then

$$\begin{aligned} 2(\mathcal{A}_*(X, Y, Z) - Y * (X * Z)) &= X \bullet (Y \bullet Z) - (X \bullet Y) \bullet Z + X \bullet [Y, Z] + [X, Y] \bullet Z \\ &+ [X, Y \bullet Z] - [X \bullet Y, Z] + [X, [Y, Z]] + [Z, [X, Y]] - Y \bullet (X \bullet Z) - Y \bullet [X, Z] \\ &- [Y, X \bullet Z] + [Y, [Z, X]] = [X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0. \end{aligned}$$

The second identity of symmetric Leibniz algebras is shown in the same way. ∎

Consequence. We have already proven that the polarization of a symmetric Leibniz algebra $(A, *)$ has a Poisson algebra structure (Proposition 5.2). Moreover, Theorem 6.3 shows that the Leibniz identity

$$[X \bullet Y, Z] - X \bullet [Y, Z] - [X, Z] \bullet Y = 0$$

is trivially satisfied because each term vanishes. Notice also that from Lemma 4.1 the pair $(*, [\ ,\ ])$ satisfies also the Leibniz identity, that is

$$[X * Y, Z] - X * [Y, Z] - [X, Z] * Y = 0$$

but $(A, *, [\ ,\ ])$ is not a Poisson algebra because the multiplication $*$ is not associative. It is a nonassociative Poisson algebra (see Definition 3.1).

7. Finite dimensional cases and structure constants

From now on we consider that the field $\mathbb{K}$ is of characteristic different from 2 or 3. Let $(A, *)$ be a symmetric Leibniz $\mathbb{K}$-algebra and $(A, \bullet, [\ ,\ ])$ its polarized Poisson algebra. Consider the subalgebra of $(A, \bullet)$:

$$Z_\bullet(A) = \{X \in A \,/\, \forall Y \in A,\ X \bullet Y\}.$$

This subalgebra contains $A_\bullet^2$. Assume moreover that A is finite dimensional and consider a basis $\{u_1, \cdots, u_r, v_1, \cdots, v_p, w_1, \cdots, w_q\}$ of A satisfying

(1) $\{w_1, \cdots, w_q\}$ is a basis of $A_\bullet^2$

(2) $\{v_1, \cdots, v_p, w_1, \cdots, w_q\}$ a basis of $Z_\bullet(A)$.

The nontrivial structure constants associated with this basis are given by

$$u_i \bullet u_j = \sum_{k=1}^{q} A_{i,j}^k w_k \quad \text{with} \quad A_{i,j}^k = A_{j,i}^k.$$

Since $A^2_\bullet$ is contained in the center of $(A,[\ ,\ ])$ and $A^2_{[,]}$ is contained in $Z_\bullet(A)$, where $A^2_{[,]}$ is the derived Lie subalgebra of $(A,[\ ,\ ])$, we have

$$[u_i, w_j] = 0, \quad 1 \leq i \leq r,\ 1 \leq j \leq q,$$
$$[v_i, w_j] = 0, \quad 1 \leq i \leq p,\ 1 \leq j \leq q$$

and

$$[u_i, u_j] = \sum_{k=1}^{p} C^k_{i,j} v_k + \sum_{l=1}^{q} D^l_{i,j} w_l,$$
$$[u_i, v_j] = \sum_{k=1}^{p} E^k_{i,j} v_k + \sum_{l=1}^{q} F^l_{i,j} w_l,$$
$$[v_i, v_j] = \sum_{k=1}^{p} G^k_{i,j} v_k + \sum_{l=1}^{q} H^l_{i,j} w_l$$

satisfying the Jacobi identities.

Example 7.1. Consider $\dim A = n$ and $\dim A^2_\bullet = 1$. In this case the structure constants associated with the basis $\{u_1, v_1, \cdots, v_{n-2}, w_1\}$ of $(A,\bullet)$ are given by the only non-trivial product $u_1 \bullet u_1 = w_1$.

We define Lie algebra structures $(A,[\ ,\ ])$ as follows:

- We consider on the $\mathbb{K}$-vector space $\mathfrak{g} = \mathbb{K}\{v_1, \cdots, v_{n-2}\}$ any $(n-2)$-dimensional Lie algebra structure.
- Let θ be a 2-form on $\mathfrak{g}$. We consider a central extension $\mathfrak{g}_1 = \mathfrak{g} \oplus \mathbb{K}\{w_1\}$ of $\mathfrak{g}$ associated with θ, that is the bracket of $\mathfrak{g}_1$ is defined from the bracket of $\mathfrak{g}$ and by the relation
$$[v_i, v_j]_{\mathfrak{g}_1} = [v_i, v_j]_{\mathfrak{g}} + \theta(v_i, v_j) w_1 \quad \text{for } i, j = 1, \cdots, n-2.$$
- Let g be a derivation of $\mathfrak{g}_1$ satisfying $g(w_1) = 0$. We consider an extension of $\mathfrak{g}_1$ by the derivation g, that is if $A = \mathbb{K}\{u_1\} \oplus \mathfrak{g}_1$, the bracket of A is that of $\mathfrak{g}_1$ and $[u_1, X] = g(X)$ for all $X \in \mathfrak{g}_1$.

The Lie structures obtained on A satisfy the required conditions. ■

Let us give examples of such a construction:

(1) If $\dim \mathfrak{g} = 1$ that is $\mathfrak{g} = \mathbb{K}\{v_1\}$, the Lie algebra $\mathfrak{g}_1 = \{v_1, w_1\}$ is Abelian. Thus the Lie bracket on $(A,[\ ,\ ])$ is given by $[u_1, v_1] = av_1 + bw_1$.

In this case we obtain the 3-dimensional symmetric Leibniz algebras whose multiplication is defined by

$$\begin{cases} u_1 * u_1 = w_1, \\ u_1 * v_1 = -v_1 * u_1 = av_1 + bw_1. \end{cases} \tag{3}$$

(2) If $\dim \mathfrak{g} = 2$ then $\mathfrak{g} = \mathbb{K}\{v_1, v_2\}$ is Abelian. Thus $\mathfrak{g}_1 = \mathbb{K}\{v_1, v_2, w_1\}$ is isomorphic to the Heisenberg algebra defined by $[v_1, v_2] = w_1$ as soon as $\theta \neq 0$. Considering a general derivation g of $\mathfrak{g}_1$ the Lie bracket of $(A,[\ ,\ ])$ is:

$$[u_1, v_1] = a_1 v_1 + b_1 v_2 + c_1 w_1,\ [u_1, v_2] = a_2 v_1 - a_1 v_2 + c_2 w_1,\ [v_1, v_2] = w_1$$

and we obtain the following 4-dimensional symmetric Leibniz algebras

$$\begin{cases} u_1 * u_1 = w_1, \\ u_1 * v_1 = -v_1 * u_1 = a_1 v_1 + b_1 v_2 + c_1 w_1, \\ u_1 * v_2 = -v_2 * u_1 = a_2 v_1 - a_1 v_2 + c_2 w_1, \\ v_1 * v_2 = -v_2 * v_1 = w_1. \end{cases}$$

Particular case: the four-dimensional oscillator Lie algebra. This case corresponds to $a_2 = -b_1 = 1$ and the other parameters equal to zero. Then the Lie bracket of $(A, [\ ,\ ])$ is given by

$$[u_1, v_1] = v_2, \ [u_1, v_2] = -v_1, \ [v_1, v_2] = w_1.$$

This Lie algebra is usually called the oscillator Lie algebra. It is a linear Lie algebra whose elements are the matrices

$$\begin{pmatrix} 0 & -z & y & 2t \\ 0 & 0 & -x & y \\ 0 & x & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding symmetric Leibniz algebra is given by

$$\begin{cases} u_1 * u_1 = w_1, \\ u_1 * v_1 = -v_1 * u_1 = -v_2, \\ u_1 * v_2 = -v_2 * u_1 = v_1, \\ v_1 * v_2 = -v_2 * v_1 = w_1. \end{cases}$$

We find again the result proved in [1] that the oscillator Lie algebra can be endowed with a symmetric Leibniz algebra structure and with a Poisson algebra structure.

8. Classifications

The classifications in small dimensions of the associative commutative 2-step nilpotent algebras have already been established in [4, 8]. Recall these results:

1. $\dim A = 2$
   - (a) $A^2_\bullet = 0$, that is $A = \mathbb{K}\{v_1, v_2\}$ and $v_1 \bullet v_2 = 0$.
   - (b) $A = \mathbb{K}\{u_1, w_1\}$ and $u_1 \bullet u_1 = w_1$.
2. $\dim A = 3$ (From now on, we do not write the decomposable algebras).
   - (a) $A = \mathbb{K}\{u_1, u_2, w_1\}$ and $u_1 \bullet u_1 = w_1, \ u_2 \bullet u_2 = w_1$.
   - (b) $A = \mathbb{K}\{u_1, u_2, w_1\}$ and $u_1 \bullet u_1 = w_1, \ u_2 \bullet u_2 = -w_1$.
   - (c) $A = \mathbb{K}\{u_1, u_2, w_1\}$ and $u_1 \bullet u_2 = u_2 \bullet u_1 = w_1$.

   These three cases are isomorphic if $\mathbb{K}$ is algebraically closed.
3. $\dim A = 4$ ($\mathbb{K}$ is algebraically closed)
   - (a) $A = \mathbb{K}\{u_1, u_2, u_3\} \oplus \mathbb{K}\{w_1\}$ and $u_1 \bullet u_1 = w_1, \ u_2 \bullet u_2 = w_1, \ u_3 \bullet u_3 = w_1$.
   - (b) $A = \mathbb{K}\{u_1, u_2\} \oplus \mathbb{K}\{w_1, w_2\}$ and $u_1 \bullet u_1 = w_1, \ \ u_1 \bullet u_2 = u_2 \bullet u_1 = w_2$.
   - (c) $A = \mathbb{K}\{u_1, u_2\} \oplus \mathbb{K}\{w_1, w_2\}$ and $u_1 \bullet u_1 = w_1, \ u_2 \bullet u_2 = w_1,$
     $u_1 \bullet u_2 = u_2 \bullet u_1 = w_2$.

Using this list we describe, when $\mathbb{K}$ is algebraically closed, the symmetric Leibniz algebras in dimension less than or equal to 4.

1. $\dim A = 2$.

1. $A_\bullet^2 = 0$, that is $A = \mathbb{K}\{v_1, v_2\}$ and $v_1 \bullet v_2 = 0$. In this case $[v_1, v_2] = av_1 + bv_2$ and we obtain two classes of symmetric Leibniz algebras: the trivial one, that is, $v_i * v_j = 0$ for all $i, j \in \{1, 2\}$, and a second one defined by the relation $v_1 * v_2 = -v_2 * v_1 = v_2$.
2. $A = \mathbb{K}\{u_1, w_1\}$ and $u_1 \bullet u_1 = w_1$. Its center is 1-dimensional and coincides with $A_\bullet^2 = \mathbb{K}\{w_1\}$. We deduce that $[u_1, w_1] = 0$ and the associated symmetric Leibniz algebra is given by

$$\begin{cases} u_1 * u_1 = w_1, \\ u_1 * w_1 = w_1 * u_1 = 0. \end{cases}$$

2. $\dim A = 3$.

1. Assume that $\dim A = 3$ and the multiplication $\bullet$ is given by

$$u_1 \bullet u_1 = u_2 \bullet u_2 = w_1.$$

The Lie bracket satisfies $[u_i, w_1] = 0$. We define $[u_1, u_2] = \alpha w_1$. We obtain the symmetric Leibniz algebra

$$\begin{cases} u_1 * u_1 = u_2 * u_2 = w_1 \\ u_1 * u_2 = -u_2 * u_1 = \alpha w_1. \end{cases}$$

Remark 8.1. For $\mathbb{K} = \mathbb{R}$ we obtain two non isomorphic classes symmetric Leibniz algebras:

(a)
$$\begin{cases} u_1 * u_1 = u_2 * u_2 = w_1 \\ u_1 * u_2 = -u_2 * u_1 = \alpha w_1. \end{cases}$$

(b)
$$\begin{cases} u_1 * u_1 = u_2 * u_2 = 0 \\ u_1 * u_2 = (1 + \alpha) w_1 \\ u_2 * u_1 = (1 - \alpha) w_1. \end{cases}$$

2. Assume that $(A, \bullet)$ is decomposable with a non trivial product $\bullet$. In consequence $\{u_1, v_1, w_1\}$ is the basis of A and we obtain the following symmetric Leibniz algebra

$$\begin{cases} u_1 * u_1 = w_1, \\ u_1 * v_1 = -v_1 * u_1 = av_1 + bw_1, \end{cases}$$

that is the algebra construct in the previous section (see Equations (3)).

3. $\dim A = 4$.

1. $A = \mathbb{K}\{u_1, u_2, u_3\} \oplus \mathbb{K}\{w_1\}$ and

$$u_1 \bullet u_1 = w_1, \ u_2 \bullet u_2 = w_1, \ u_3 \bullet u_3 = w_1.$$

In this case, we have $[u_i, u_j] = \alpha_{i,j} w_1, \ \ 1 \le i < j \le 3$.

The Lie algebra $(A, [\ ,\ ])$ is a nilpotent Lie algebra isomorphic to the direct sum of the 3-dimensional Heisenberg algebra with a 1-dimensional Abelian Lie algebra.

This gives the following symmetric Leibniz algebras:

$$\begin{cases} u_1 * u_1 = u_2 * u_2 = u_3 * u_3 = w_1, \\ u_1 * u_2 = -u_2 * u_1 = \alpha_{1,2} w_1, \\ u_1 * u_3 = -u_3 * u_1 = \alpha_{1,3} w_1, \\ u_2 * u_3 = -u_3 * u_2 = \alpha_{2,3} w_1. \end{cases}$$

2. $A = \mathbb{K}\{u_1, u_2\} \oplus \mathbb{K}\{w_1, w_2\}$ and

$$u_1 \bullet u_1 = w_1, \quad u_1 \bullet u_2 = u_2 \bullet u_1 = w_2.$$

In this case, the non trivial Lie bracket is $[u_1, u_2] = \alpha w_1 + \beta w_2$, that is $(A, [\ ,\ ])$ is the Heisenberg algebra or the Abelian Lie algebra. We deduce the corresponding symmetric Leibniz algebras:

$$\begin{cases} u_1 * u_1 = w_1, \\ u_1 * u_2 = \alpha w_1 + (\beta + 1) w_2, \\ u_2 * u_1 = -\alpha w_1 - (\beta - 1) w_2. \end{cases}$$

3. $A = \mathbb{K}\{u_1, u_2\} \oplus \mathbb{K}\{w_1, w_2\}$ and

$$u_1 \bullet u_1 = w_1, \quad u_2 \bullet u_2 = w_1, \quad u_1 \bullet u_2 = u_2 \bullet u_1 = w_2$$

In this case, we also have $[u_1, u_2] = \alpha w_1 + \beta w_2$ and the corresponding symmetric Leibniz algebra is

$$\begin{cases} u_1 * u_1 = u_2 * u_2 = w_1, \\ u_1 * u_2 = \alpha w_1 + (\beta + 1) w_2, \\ u_2 * u_1 = -\alpha w_1 - (\beta - 1) w_2. \end{cases}$$

4. Assume that $(A, \bullet)$ is decomposable. The first case is when $(A, \bullet)$ is trivial that is $A = \mathbb{K}\{v_1, v_2, v_3, v_4\}$ and $v_i \bullet v_j = 0$. In this case $(A, [\ ,\ ])$ is any 4-dimensional Lie algebra and the symmetric Leibniz algebra coincides with this Lie algebra and the product $*$ is then skew-symmetric.

5. $A = \mathbb{K}\{u_1\} \oplus \mathbb{K}\{v_1, v_2\} \oplus \mathbb{K}\{w_1\}$.

   In this case $u_1 \bullet u_1 = w_1$ and we have

$$[u_1, v_j] = \sum_{k=1}^{2} E_{1,j}^k v_k + F_{1,j} w_1, \quad j = 1, 2$$

$$[v_1, v_2] = \sum_{k=1}^{2} G_{1,2}^k v_k + H_{1,2} w_1.$$

   This case has also been studied in a previous example. In particular, we find the 4-dimensional oscillator Lie algebra.

6. $A = \mathbb{K}\{u_1, u_2\} \oplus \mathbb{K}\{v_1\} \oplus \mathbb{K}\{w_1\}$.

   In this case $u_1 \bullet u_1 = u_2 \bullet u_2 = w_1$ and the Lie bracket reads:

$$[u_1, u_2] = C_{1,2} v_1 + D_{1,2} w_1,$$

$$[u_i, v_1] = E_{i,1} v_1 + F_{i,1} w_1, \quad i = 1, 2$$

   with the Jacobi condition $F_{1,1} E_{2,1} - F_{2,1} E_{1,1} = 0$.

Every symmetric Leibniz algebra $A=\mathbb{K}\{u_1,u_2\}\oplus\mathbb{K}\{v_1\}\oplus\mathbb{K}\{w_1\}$ is of the form below:

$$\begin{cases} u_1*u_1=u_2*u_2=w_1, \\ u_1*u_2=-u_2*u_1=C_{1,2}v_1+D_{1,2}w_1 \\ u_1*v_1=-v_1*u_1=E_{1,1}v_1+F_{1,1}w_1, \\ u_2*v_1=-v_1*u_2=E_{2,1}v_1+F_{2,1}w_1. \end{cases}$$

with $$F_{1,1}E_{2,1}-F_{2,1}E_{1,1}=0.$$

9. Deformation quantization of Poisson algebras in a symmetric Leibniz formal deformation

As it was recalled in the introduction, formal deformations of commutative weakly associative algebras give a construction of nonassociative Poisson algebras. An interesting case corresponds to a formal deformation of an associative commutative algebra in the category of weakly associative algebras. This gives a Poisson algebra and enlarges the spectrum of deformation quantization. Moreover the class of weakly associative algebras is the only one which permits to construct such a Poisson algebra. In this section we investigate the formal deformation of commutative symmetric Leibniz algebras in the class of symmetric Leibniz algebras.

Notation

Consider φ a bilinear map from A to A. We denote by ψ_φ and ρ_φ the bilinear maps defined by

$$\begin{cases} \psi_\varphi(X,Y)=\frac{1}{2}(\varphi(X,Y)-\varphi(Y,X)) \\ \rho_\varphi(X,Y)=\frac{1}{2}(\varphi(X,Y)+\varphi(Y,X)), \end{cases}$$

for every $X,Y\in A$ which correspond to the polarization of φ.

Definition 9.1. Let $(A,*)$ be a commutative symmetric Leibniz algebra. A *symmetric Leibniz formal deformation* $(A[[t]],*_t)$ of $(A,*)$ is given by a symmetric Leibniz formal product on $A[[t]]$ the algebra of formal series in t with coefficients in A. It can be represented by a formal series

$$X*_tY=X*Y+\sum_{i\geq 1}t^i\varphi_i(X,Y)$$

for all $X,Y\in A$, where φ_i are bilinear maps in A and which satisfies

$$\begin{cases} X*_t(Y*_tZ)-(X*_tY)*_tZ-Y*_t(X*_tZ)=0, \\ Y*_t(Z*_tX)-(Y*_tZ)*_tX+(Y*_tX)*_tZ=0. \end{cases}$$ ■

This system of equations implies

- at order 0 that $*$ is a symmetric Leibniz product,
- at order 1 that the linear bilinear map φ_1 satisfies

$$\begin{cases} \delta\varphi_1^{(1)}(X,Y,Z) & =\varphi_1(X,Y*Z)-\varphi_1(X*Y,Z)-\varphi_1(Y,X*Z) \\ & \quad +X*\varphi_1(Y,Z)-\varphi_1(X,Y)*Z-Y*\varphi_1(X,Z)=0 \\ \delta\varphi_1^{(2)}(X,Y,Z) & =\varphi_1(Y,Z*X)-\varphi_1(Y*Z,X)+\varphi_1(Y*X,Z) \\ & \quad +Y*\varphi_1(Z,X)-\varphi_1(Y,Z)*X+\varphi_1(Y,X)*Z=0, \end{cases}$$

where $\delta^{(1)}$ (respectively $\delta^{(2)}$) corresponds to the deformation cohomology operator of the right-symmetric (respectively left-symmetric Leibniz algebras).

- at order 2 that the linear bilinear map φ_2 satisfies

$$\begin{cases} \mathcal{A}_{\varphi_1}(X,Y,Z) - \varphi_1(Y,\varphi_1(X,Z)) + \delta\varphi_2^{(1)}(X,Y,Z) = 0 \\ \mathcal{A}_{\varphi_1}(Y,Z,X) + \varphi_1(\varphi_1(Y,X),Z) + \delta\varphi_2^{(2)}(X,Y,Z) = 0 \end{cases}$$

which is equivalent, since $*$ is assumed to be commutative, to

$$\begin{cases} \mathcal{A}_{\varphi_1}(X,Y,Z) + \mathcal{A}_{\varphi_1}(Y,Z,X) - \mathcal{A}_{\varphi_1}(Y,X,Z) + \delta\varphi_2^{(1)}(X,Y,Z) \\ \qquad\qquad +\delta\varphi_2^{(2)}(X,Y,Z) = 0 \\ \varphi_1(\varphi_1(X,Z),Y) + \varphi_1(Y,\varphi_1(X,Z)) + \delta\varphi_2^{(2)}(Z,X,Y) - \delta\varphi_2^{(1)}(X,Y,Z) = 0. \end{cases}$$

The map $\delta\varphi_2 = \delta\varphi_2^{(1)} + \delta\varphi_2^{(2)}$ satisfies

$$\begin{cases} \delta\varphi_2(X,Y,Z) = & \varphi_2(X,Y*Z) - \varphi_2(X,Y)*Z - Y*\varphi_2(X,Z) - \varphi_2(Y*Z,X) \\ & +Y*\varphi_2(Z,X) + \varphi_2(Y,X)*Z \\ \qquad = & 2\psi_{\varphi_2}(X,Y*Z) - 2\psi_{\varphi_2}(X,Y)*Z - 2Y*\psi_{\varphi_2}(X,Z) \end{cases}$$

where $2\psi_{\varphi_2}(U,V) = \varphi_2(U,V) - \varphi_2(V,U)$ for $U,V \in A$. Then

$$\delta\varphi_2(X,Y,Z) - \delta\varphi_2(X,Z,Y) = 0.$$

We deduce that

$$\mathcal{A}_{\varphi_1}(X,Y,Z) + \mathcal{A}_{\varphi_1}(Y,Z,X) - \mathcal{A}_{\varphi_1}(Y,X,Z) - \mathcal{A}_{\varphi_1}(X,Z,Y) \\ -\mathcal{A}_{\varphi_1}(Z,Y,X) + \mathcal{A}_{\varphi_1}(Z,X,Y) = 0$$

which is equivalent to say that φ_1 is Lie-admissible or equivalently that ψ_{φ_1} is a Lie bracket.

Lemma 9.2. *Let $(A[[t]],*_t)$ be a formal deformation of the commutative symmetric Leibniz algebra $(A,*)$. The algebra (A,ψ_{φ_1}) is a Lie algebra.*

Since $\delta\varphi_1^{(1)} = \delta\varphi_1^{(2)} = 0$ we deduce also from this computation that

$$\frac{\delta\varphi_1(X,Y,Z)}{2} = \psi_{\varphi_1}(X,Y*Z) - \psi_{\varphi_1}(X,Y)*Z - Y*\psi_{\varphi_1}(X,Z) = 0. \tag{4}$$

Then the Lie bracket ψ_{φ_1} and the commutative nonassociative (symmetric Leibniz) product $*$ satisfy the Leibniz rule. We deduce

Lemma 9.3. *Let $(A[[t]],*_t)$ be a formal deformation of the commutative symmetric Leibniz algebra $(A,*)$. The algebra $(A,*,\psi_{\varphi_1})$ is a nonassociative Poisson algebra.*

So we find the same result obtained for the weakly associative algebras in the symmetric Leibniz algebras' context. But in this case, we shall show that the pair

$(*, \psi_{\varphi_1})$ satisfy the relations of Theorem 6.3 which are more restricted than the Leibniz rule. In fact the algebra $(A[[t]], *_t)$ is a commutative symmetric Leibniz algebra. Consider $(A[[t]], \bullet_t, [\, ,\,]_t)$ its polarization. The polarization of $(A, *)$ is $(A, \bullet, [\, ,\,]) = (A, *, 0)$ because $*$ is commutative which implies that $\bullet = *$ and $[\, ,\,] = 0$. Moreover $[\, ,\,]_t$ is a formal deformation of $[\, ,\,]$ that is

$$[\, ,\,]_t = 0 + t\psi_{\varphi_1} + \cdots$$

and $\bullet_t$ a formal deformation of $\bullet$ that is

$$\bullet_t = * + t\rho_{\varphi_1} + \cdots .$$

From Theorem 6.3 we have $X \bullet_t [Y, Z]_t = 0, \quad [X, Y \bullet_t Z]_t = 0$ for all $X, Y, Z \in A$. At order 1 we obtain

$$\rho_{\varphi_1}(X, [Y, Z]) + X \bullet \psi_{\varphi_1}(Y, Z) = 0,\ \psi_{\varphi_1}(X, Y \bullet Z) + [X, \rho_{\varphi_1}(Y, Z)] = 0.$$

Since $[\, ,\,] = 0$ and $\bullet = *$, these equations read

$$X * \psi_{\varphi_1}(Y, Z) = 0,\ \psi_{\varphi_1}(X, Y * Z) = 0. \tag{5}$$

Theorem 9.4. *Let $(A, *)$ be a commutative symmetric Leibniz algebra and $(A[[t]], *_t = * + \sum_{i \geq 1} t^i \varphi_i)$ a symmetric Leibniz formal deformation of $(A, *)$. Consider the skew-symmetric map ψ_{φ_1} given by $\psi_{\varphi_1}(X, Y) = \frac{1}{2}(\varphi_1(X, Y) - \varphi_1(Y, X))$. The triple $(A, *, \psi_{\varphi_1})$ is a nonassociative Poisson algebra and the Lie bracket ψ_{φ_1} satisfies*

$$X * \psi_{\varphi_1}(Y, Z) = 0,\ \psi_{\varphi_1}(X, Y * Z) = 0.$$

for all $X, Y, Z \in A$.

Thus from Theorem 6.3, we have:

Theorem 9.5. *Let $(A, *)$ be a commutative symmetric Leibniz algebra and $(A[[t]], *_t = * + \sum_{i \geq 1} t^i \varphi_i)$ a symmetric Leibniz formal deformation of $(A, *)$. The depolarization $(A, *_{\varphi_1})$ of $(A, *, \psi_{\varphi_1})$ is a Leibniz symmetric algebra.*

Remark 9.6. (1) An associative commutative algebra is always weakly associative ([10] see examples Section 1.1). But it is a symmetric Leibniz algebra if and only if it is 2-step nilpotent. Then Theorem 9.4 permits to construct from an associative commutative 2-step nilpotent algebra a classical Poisson algebra via a formal symmetric Leibniz deformation process.

(2) When we consider a deformation quantization of a Poisson algebra, that is when the Poisson algebra comes from a deformation of an associative commutative algebra, contrary of the previous theorem, we have no other identities on the Lie bracket than those coming from the Jacobi and Leibniz identities. In fact $\delta_H \varphi_1 = 0$ implies in this case

$$\begin{cases} \psi(XY, Z) - X\psi(Y, Z) - \psi(X, Z)Y = 0, \\ X\rho(Y, Z) - \rho(XY, Z) + \rho(X, YZ) - \rho(X, Y)Z = \delta_H \rho(X, Y, Z) = 0. \end{cases} \tag{6}$$

Denote by $\mathcal{L}(\nu,\mu)$ the default of satisfying the Leibniz identity between the multiplications ν and μ, that is

$$\mathcal{L}(\nu,\mu)(X,Y,Z)=\mu(\nu(X,Y),Z)-\nu(X,\mu(Y,Z))-\nu(\mu(X,Z),Y)),$$

System (6) is equivalent to

$$\begin{cases}\mathcal{L}(*,\psi)(X,Y,Z)=0,\\ -\mathcal{L}(*,\rho)(X,Y,Z)+\mathcal{L}(*,\rho)(Y,Z,X)=0.\end{cases}$$

For example for the algebra $A=\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ of $\mathcal{C}^\infty$ fonctions with values in $\mathbb{R}$ we have $\psi=0$ and $\varphi_1=\rho$. Thus $\varphi_1(1,f)=af$ where $a=\varphi_1(1,1)$. A particular case corresponds to $\mathcal{L}(*,\rho)=0$. In this case $a=0$. Another example but in the finite dimension case is the following: Consider the 2-dimensional algebra $(A,*)$ given in a basis $\{e_1,e_2\}$ by

$$e_i*e_i=e_i,\ i=1,2,\quad e_1*e_2=e_2*e_1=e_2.$$

It is a commutative associative algebra. From [9] Section 2.4, $\psi=0$ and if φ_1 is the linear term of a formal deformation of $*$, then $\varphi_1=\rho_1$ and we have

$$\varphi_1(e_1,e_1)=a,\ \ \varphi_1(e_1,e_2)=\varphi_1(e_2,e_1)=ae_2,\ \varphi_1(e_2,e_2)=ae_2$$

that is $\varphi_1=a*$. This last relation shows that for every formal deformation of $*$ the first term φ_1 which is always a 2-cocycle for the deformation cohomology of $*$ is also a 2-coboundary. Then any deformation is trivial that is, isomorphic to $*$ and we say that $*$ is rigid.

(3) However for a weakly associative formal deformation of the associative commutative multiplication $*$, that is

$$(A[[t]],*_t=*+\sum_{i=1}^{\infty}t^i\varphi_i)$$

is a formal weakly associative algebra, the algebra $(A,*,\psi_{\varphi_1})$ is also a Poisson algebra [10]. Denote by $\delta^2_{WA}\varphi_1$ the coboundary operator of the deformation cohomlology of weakly associative cohomology, that is,

$$\delta^2_{WA}\varphi_1=\delta^2_H\varphi_1(X,Y,Z)+\delta^2_H\varphi_1(Y,Z,X)-\delta^2_H\varphi_1(Y,X,Z)$$

with

$$\delta^2_H\varphi_1=X*\varphi_1(Y,Z)-\varphi_1(X*Y,Z)+\varphi_1(X,Y*Z)-\varphi_1(X,Y)*Z.$$

The condition $\delta^2_{WA}\varphi_1=0$ is equivalent to $\mathcal{L}(*,\psi_{\varphi_1})=0$ because $\delta^2_{WA}\rho_{\varphi_1}=0$ (it is the case for every commutative multiplication).

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