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L-Iwasawa Decomposition of the Generalized Lorentz Group

Edgar N. Reyes

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Abstract. Let $n \geq 2$. Let $O(1,n)$ be the generalized Lorentz Lie group, and let $\mathfrak{so}(1,n)$ be its Lie algebra. Let $L = diag(1,-1,I_{n-1})$ be a diagonal matrix. We state a sufficient condition that if satisfied by $G \in O(1,n)$ then there exists $t \in \mathbb{R}$, $k \in O(1,n)$, $V_1, Y \in \mathfrak{so}(1,n)$ such that $LkL^{-1} = k$, $V_1 \neq 0$, $LV_1L^{-1} = -V_1$, $[V_1, Y] = Y$, and $G = ke^{tV_1}e^Y$.

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1. Introduction

Let $\mathbb{R}^{n\times m}$ denote the set of n-by-m real matrices. Let $n \geq 2$, $M_n(\mathbb{R}) = \mathbb{R}^{n\times n}$, and let I_n denote the n-by-n identity matrix. Let $GL_n(\mathbb{R})$ be the multiplicative group of nonsingular matrices in $M_n(\mathbb{R})$. Let $J = 1 \oplus (-I_n) \in GL_{n+1}(\mathbb{R})$ be a diagonal matrix. For $X \in M_n(\mathbb{R})$, let X^T denote the transpose of X. We consider the generalized Lorentz group

$$O(1,n) = \{G \in GL_{n+1}(\mathbb{R}) : G^T JG = J\}.$$

For brevity, we say $O(1,n)$ is a Lorentz group. Let $O(n)$ denote the multiplicative group of real orthogonal matrices $B \in GL_n(\mathbb{R})$ satisfying $B^T B = I_n$. The Lie algebra $\mathfrak{so}(1,n)$ of the Lie group $O(1,n)$ satisfies

$$\mathfrak{so}(1,n) = \{X \in M_{n+1}(\mathbb{R}) : X^T J + JX = 0\}.$$

Let $w \in \mathbb{R}^{n+1}$ satisfy $w^T Jw \neq 0$. Let $S_w = I_{n+1} - 2(w^T Jw)^{-1} ww^T J$. The matrix S_w is a J-Householder matrix satisfying $S_w^{-1} = S_w = S_{tw} \in O(1,n)$ for all $t \neq 0$.

We associate an involution $\rho_w : O(1,n) \to O(1,n)$ by letting $\rho_w(G) = S_w G S_w^{-1}$, $G \in O(1,n)$. The differential $d\rho_w$ of ρ_w at I_{n+1} is a Lie algebra involution

$$d\rho_w : \mathfrak{so}(1,n) \to \mathfrak{so}(1,n)$$

satisfying $d\rho_w(X) = S_w X S_w^{-1}$, $X \in \mathfrak{so}(1,n)$. The eigenspaces of $d\rho_w$, and the subgroup of fixed points of ρ_w are denoted as follows.

1. $\mathfrak{p}_w = \{X \in \mathfrak{so}(1,n) : d\rho_w(X) = -X\}$
2. $\mathfrak{K}_w = \{X \in \mathfrak{so}(1,n) : d\rho_w(X) = X\}$
3. $K_w = \{k \in O(1,n) : \rho_w(k) = k\}$

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Let $v \in \mathbb{R}^{n+1}$ satisfy $v^T J v = w^T J w \neq 0$. Choose a J-Householder matrix $D \in O(1,n)$ satisfying $Dv = -w$ or $Dv = w$ [2]. In any case, $DS_v D^{-1} = S_{Dv} = S_w$. Let $\psi : O(1,n) \to O(1,n)$ be an inner-automorphism satisfying

$$\psi(G) = DGD^{-1},\ G \in O(1,n).$$

The differential $d\psi$ of ψ at I_{n+1} satisfies $d\psi(X) = DXD^{-1}$, $X \in \mathfrak{so}(1,n)$. Note, $\psi \circ \rho_w = \rho_v \circ \psi$. We find $k \in K_w$ iff $\rho_w(k) = k$ iff $\psi(k) = \rho_v(\psi(k))$ iff $\psi(k) \in K_v$. Thus, $\psi(K_w) = K_v$. Likewise, $X \in \mathfrak{K}_w$ iff $d\rho_w(X) = X$ iff $d\psi(X) = d\rho_v(d\psi(X))$ iff $d\psi(X) \in \mathfrak{K}_v$. Then $d\psi(\mathfrak{K}_w) = \mathfrak{K}_v$. Similarly, $d\psi(\mathfrak{p}_w) = \mathfrak{p}_v$.

Lemma 1.1. *Let $v, w \in \mathbb{R}^{n+1}$ satisfy $v^T J v = w^T J w \neq 0$. Then we have direct sums of subspaces $\mathfrak{so}(1,n) = \mathfrak{K}_v \oplus \mathfrak{p}_v$ and $\mathfrak{so}(1,n) = \mathfrak{K}_w \oplus \mathfrak{p}_w$. Also, there exists an inner-automorphism ψ of $O(1,n)$ satisfying $\psi(K_w) = K_v$, $d\psi(\mathfrak{K}_w) = \mathfrak{K}_v$, and $d\psi(\mathfrak{p}_w) = \mathfrak{p}_v$.*

Suppose $u \in \mathbb{R}^{n+1}$ satisfies $u^T J u > 0$. Without loss of generality, we suppose $u = (1, 0, \ldots, 0)^T \in \mathbb{R}^{n+1}$. In this case, $u^T J u = 1$ and $S_u = I_{n+1} - 2uu^T J = -J$. For $G \in O(1,n)$, we find $\rho_u(G) = JGJ = G^{-T}$. Then $d\rho_u(X) = -X^T$, $X \in \mathfrak{so}(1,n)$. Note, $d\rho_u$ is a Cartan involution of $\mathfrak{so}(1,n)$. In fact, $\mathfrak{p}_u$ consists of the symmetric matrices in $\mathfrak{so}(1,n)$. Also, $\mathfrak{K}_u$ consists of the skew-symmetric matrices in $\mathfrak{so}(1,n)$. The subgroup K_u consists of orthogonal matrices in $O(1,n)$ expressible in block form $\varepsilon \oplus M$ where $\varepsilon \in \{\pm 1\}$, $M \in O(n)$.

Moreover, there is a bijection $\phi_u : \mathfrak{p}_u \times K_u \to O(1,n)$ where $\phi_u(X,k) = e^X k$, $X \in \mathfrak{p}_u$, $k \in K_u$. A Cartan involution induces an Iwasawa decomposition of $\mathfrak{so}(1,n)$, and an Iwasawa decomposition of the connected component of $O(1,n)$. We briefly describe the Iwasawa decompositions. Consider the block matrix

$$V_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 0_{n-1} \in \mathfrak{so}(1,n).$$

Let $ad(V_1) : \mathfrak{so}(1,n) \to \mathfrak{so}(1,n)$ be a function where $ad(V_1)(X) = [V_1, X] \equiv V_1 X - X V_1$, $X \in \mathfrak{so}(1,n)$. It is well-known that the eigenvalues of $ad(V_1)$ are 0 and ± 1. Denote the 1-eigenspace of $ad(V_1)$ by

$$\mathfrak{so}(1,n)_1 = \{X \in \mathfrak{so}(1,n) : ad(V_1)(X) = X\}.$$

Moroever, $\mathfrak{so}(1,n)_1$ is an abelian Lie subalgebra of $\mathfrak{so}(1,n)$. Let $\mathfrak{h}_{\mathfrak{p}_u} = \{tV_1 : t \in \mathbb{R}\}$. The direct sum of Lie subalgebras

$$\mathfrak{so}(1,n) = \mathfrak{K}_u \oplus \mathfrak{h}_{\mathfrak{p}_u} \oplus \mathfrak{so}(1,n)_1$$

is an Iwasawa decomposition of $\mathfrak{so}(1,n)$. Let A and N be the connected Lie subgroups of $O(1,n)$ with Lie algebras $\mathfrak{h}_{\mathfrak{p}_u}$ and $\mathfrak{so}(1,n)_1$, respectively. Then

$$O(1,n) = K_u AN$$

is an Iwasawa decomposition for $O(1,n)$. For each $G \in O(1,n)$, there is a unique triple $(k, t, W) \in K_u \times \mathbb{R} \times \mathfrak{so}(1,n)_1$ satisfying $G = ke^{tV_1}e^W$. The mapping $K_u \times A \times N \to O(1,n)$ satisfying $(k, a, n) \mapsto kan$ is a continuous bijection [1], [4], [5].

We are motivated to investigate Iwasawa-like decompositions of $O(1,n)$ that are induced by non-Cartan involutions $d\rho_w$ where $w^TJw<0$. Similar to the construction of the Iwasawa decomposition, we analyze the subset $K_wA_wN_w$ of $O(1,n)$ where K_w, A_w, and N_w are Lie subgroups of $O(1,n)$ that we briefly describe. That is, K_w is the subgroup of fixed points in $O(1,n)$ of ρ_w; however, K_w is not compact unlike K_u which is compact since $d\rho_u$ is a Cartan involution. Secondly, let $\mathfrak{h}_{\mathfrak{p}_w}$ be a maximal subspace of $\mathfrak{p}_w$ such that $\mathfrak{h}_{\mathfrak{p}_w}$ is an abelian Lie subalgebra of $\mathfrak{so}(1,n)$. In the next section, assuming $w=0\oplus e_1$, we show $\mathfrak{h}_{\mathfrak{p}_w}$ is a 1-dimensional subspace spanned by V_1. In our above notation, A_w is the connected abelian Lie subgroup with Lie algebra $\mathfrak{h}_{\mathfrak{p}_w}$. Moreover, we show the 1-eigenspace, denoted by $\mathfrak{so}(1,n)_1$, of $ad(V_1)$ is an abelian Lie subalgebra of $\mathfrak{so}(1,n)$. In the Iwasawa-like decomposition, N_w is the connected abelian Lie subgroup with Lie subalgebra $\mathfrak{n}\equiv\mathfrak{so}(1,n)_1$.

In this paper, we study the case when $w^TJw<0$. Let $e_1=(1,0,\dots,0)\in\mathbb{R}^n$, and let $L=diag(1,-1,I_{n-1})$ be a diagonal matrix. Without loss of generality, we assume and set for the remainder of the paper the following

$$w=0\oplus e_1\in\mathbb{R}^{n+1}.$$

Then $w^TJw=-1$ and $S_w=L$. For conciseness, we suppress w, and write

1. $\mathfrak{K}=\mathfrak{K}_w$
2. $\mathfrak{p}=\mathfrak{p}_w$, and
3. $K=K_w$.

Let $\mathfrak{h}_{\mathfrak{p}}$ be a maximal subspace of $\mathfrak{p}$ such that $\mathfrak{h}_{\mathfrak{p}}$ is an abelian Lie algebra. We choose $\mathfrak{h}_{\mathfrak{p}}=\mathfrak{h}_{\mathfrak{p}_u}=\{tV_1:t\in\mathbb{R}\}$. In Section 2, we show the following is a direct sum of subspaces

$$\mathfrak{so}(1,n)=\mathfrak{K}\oplus\mathfrak{h}_{\mathfrak{p}}\oplus\mathfrak{so}(1,n)_1$$

which we call an L-Iwasawa decomposition of $\mathfrak{so}(1,n)$. Let G_{ij} denote the (i,j)-entry of $G\in O(1,n)$. In the next lemma, we describe the subgroup K of fixed points of ρ_w.

Lemma 1.2. *Let $G\in O(1,n)$, $n\geq 2$. Then $G\in K$ if and only if $G_{i,2}=0$ for all $i\neq 2$.*

In Section 2, we study the connected Lie subgroup S of $O(1,n)$ with Lie algebra $\mathfrak{h}_{\mathfrak{p}}\oplus\mathfrak{so}(1,n)_1$. We show $S=AN$ is a solvable semi-direct product group. In Section 3, we state a sufficient condition for a matrix $G\in O(1,n)$ to satisfy to ensure $G\in KAN$. In such a case, we say G has an L-Iwasawa decomposition. For the development of Section 3, we quote the following theorem [3].

Theorem 1.3. *Let $v\in\mathbb{R}^{n+1}$ satisfy $v^TJv=-1$. Let $\phi_v:\mathfrak{p}_v\times K_v\to O(1,n)$ be defined by $\phi_v(X,k)=e^Xk$, $X\in\mathfrak{p}_v$, $k\in K_v$. Then ϕ_v is surjective. However, ϕ_v is not injective.*

2. Preliminaries

Let $\mathfrak{so}(m)$ denote the Lie algebra of m-by-m real skew-symmetric matrices.

Lemma 2.1. *Let $n \geq 2$, and let $X \in M_{n+1}(\mathbb{R})$. Then $X \in \mathfrak{so}(1,n)$ if and only if there exist $a \in \mathbb{R}$, $b_1, b_2 \in \mathbb{R}^{n-1}$, and $D \in \mathfrak{so}(n-1)$ such that*

$$X = \left[\begin{array}{cc|c} 0 & a & b_1^T \\ a & 0 & b_2^T \\ \hline b_1 & -b_2 & D \end{array}\right]. \tag{1}$$

Let $X \in \mathfrak{so}(1,n)$.

(1) *$X \in \mathfrak{p}$ if and only if X has the form in (1) where $b_1 = 0$ and $D = 0$.*

(2) *$X \in \mathfrak{K}$ if and only if X has the form in (1) where $a = 0$ and $b_2 = 0$.*

We call a subspace $\mathfrak{a}$ of $\mathfrak{p}$ an abelian subspace if $[X, Y] = 0$ for all $X, Y \in \mathfrak{a}$.

Lemma 2.2. *A maximal abelian subspace of $\mathfrak{p}$ is 1-dimensional.*

Proof. Let $a_1, a_2 \in \mathbb{R}$, and let $b_2, d_2 \in \mathbb{R}^{n-1}$.

Let $X_1 = \left[\begin{array}{cc|c} 0 & a_1 & 0 \\ a_1 & 0 & b_2^T \\ \hline 0 & -b_2 & 0 \end{array}\right] \neq 0$, and let X_1 lie in a maximal abelian subspace of $\mathfrak{p}$.

Suppose $X_2 = \left[\begin{array}{cc|c} 0 & a_2 & 0 \\ a_2 & 0 & d_2^T \\ \hline 0 & -d_2 & 0 \end{array}\right] \in \mathfrak{p}$ and $[X_1, X_2] = 0$. We find

$$[X_1, X_2] = \left[\begin{array}{cc|c} 0 & 0 & a_1 d_2^T - a_2 b_2^T \\ 0 & 0 & 0 \\ \hline a_1 d_2 - a_2 b_2 & 0 & d_2 b_2^T - b_2 d_2^T \end{array}\right].$$

If $a_1 \neq 0$, we find $X_2 = \frac{a_2}{a_1} X$. If $a_1 = 0$, then $b_2 \neq 0$. In particular, one of the entries of b_2 is nonzero, say, the ith entry $b_{2,i} \neq 0$ for some $i \in \{1, \ldots, n-1\}$. Thus, $X_2 = \frac{d_{2,i}}{b_{2,i}} X_1$. Hence, the maximal abelian subspace of $\mathfrak{p}$ containing X_1 is 1-dimensional. ∎

Let $V_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 0_{n-1} \in \mathfrak{p}$. Let $X \in \mathfrak{so}(1,n)$ be as in Lemma 2.1, and let

$$X' = \left[\begin{array}{cc|c} 0 & 0 & b_2^T \\ 0 & 0 & b_1^T \\ \hline b_2 & -b_1 & 0_{n-1} \end{array}\right].$$

Then $ad(V_1)X = X'$. For $\lambda \in \mathbb{R}$, let $\mathfrak{so}(1,n)_\lambda = \{X \in \mathfrak{so}(1,n) : [V_1, X] = \lambda X\}$, the eigenspace of $ad(V_1)$ corresponding to eigenvalue λ.

Lemma 2.3. *The eigenvalues of $ad(V_1)$ are $0, 1, -1$, and*

$$\mathfrak{so}(1,n) = \mathfrak{so}(1,n)_0 \oplus \mathfrak{so}(1,n)_1 \oplus \mathfrak{so}(1,n)_{-1}$$

is a direct sum of eigenspaces where

(1) $\mathfrak{so}(1,n)_0 = \left\{ \left[\begin{array}{cc|c} 0 & a & 0^T \\ a & 0 & 0^T \\ \hline 0 & 0 & D \end{array}\right] : a \in \mathbb{R}, D \in \mathfrak{o}(n-1) \right\}$

(2) $\mathfrak{so}(1,n)_1 = \left\{ \left[\begin{array}{cc|c} 0 & 0 & b^T \\ 0 & 0 & b^T \\ \hline b & -b & 0_{n-1} \end{array}\right] : b \in \mathbb{R}^{n-1} \right\}$

(3) $\mathfrak{so}(1,n)_{-1} = \left\{ \left[\begin{array}{cc|c} 0 & 0 & b^T \\ 0 & 0 & -b^T \\ \hline b & b & 0_{n-1} \end{array} \right] : b \in \mathbb{R}^{n-1} \right\}$

Proof. Recall, $ad(V_1)X = X'$. Clearly, the sets in the right side of statements 1-3 are contained in $\mathfrak{so}(1,n)_0, \mathfrak{so}(1,n)_1, \mathfrak{so}(1,n)_{-1}$, respectively. Since $\mathfrak{so}(1,n) = \mathfrak{K} \oplus \mathfrak{p}$, the sum of the sets in the right side of statements 1-3 is $\mathfrak{so}(1,n)$. Moreover, the sum is a direct sum. The proof of the lemma is complete. ∎

Let $\mathfrak{n} = \mathfrak{so}(1,n)_1$. Let $\mathfrak{m} = \{x \in \mathfrak{K} : [x,h] = 0,\ \forall h \in \mathfrak{h}_\mathfrak{p}\}$, i.e, the centralizer of $\mathfrak{h}_\mathfrak{p}$ in $\mathfrak{K}$. Recall, $\mathfrak{h}_\mathfrak{p} = \{tV_1 : t \in \mathbb{R}\}$.

Corollary 2.4. *In each of the following, we have a direct sum of subspaces.*

(1) $\mathfrak{so}(1,n) = \mathfrak{K} \oplus \mathfrak{h}_\mathfrak{p} \oplus \mathfrak{n}$

(2) $\mathfrak{so}(1,n)_0 = \mathfrak{m} \oplus \mathfrak{h}_\mathfrak{p}$

Proof. Recall, $\mathfrak{so}(1,n) = \mathfrak{K} \oplus \mathfrak{p}$. Then the first part of the corollary follows from Lemma 2.3. The second part follows since $\mathfrak{h}_\mathfrak{p}$ is a maximal abelian subspace of $\mathfrak{p}$. ∎

Consider the following direct sum of subspaces

$$\mathfrak{s} = \mathfrak{h}_\mathfrak{p} \oplus \mathfrak{n}. \tag{2}$$

Notice, $\mathfrak{n}$ is an abelian Lie algebra. Now, let $\mathcal{D}\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}]$ denote the derived Lie algebra of $\mathfrak{s}$. That is, $\mathcal{D}\mathfrak{s}$ is the linear span of $[X,Y]$ where $X, Y \in \mathfrak{s}$. In particular, if $t_1, t_2 \in \mathbb{R}$, and $X_1, X_2 \in \mathfrak{n}$, then $[t_1V_1 + X_1, t_2V_1 + X_2] = t_1X_2 - t_2X_1 \in \mathfrak{n}$. Then $\mathcal{D}\mathfrak{s} = \mathfrak{n}$. Since $\mathfrak{n}$ is an abelian Lie algebra, the second derived algebra of $\mathfrak{s}$ is $\mathcal{D}^2\mathfrak{s} = \mathcal{D}\mathfrak{n} = [\mathfrak{n}, \mathfrak{n}] = 0$. Thus, $\mathfrak{s}$ is a solvable Lie algebra. Note, $\mathfrak{s}$ is not a nilpotent Lie algebra.

Lemma 2.5. $\mathfrak{n}$ *is an abelian Lie algebra, and* $\mathfrak{s}$ *is a solvable Lie algebra.*

Recall A and N are the connected Lie subgroups of $O(1,n)$ with Lie algebras $\mathfrak{h}_\mathfrak{p}$ and $\mathfrak{n}$, respectively. Since $\mathfrak{h}_\mathfrak{p}$ and $\mathfrak{n}$ are abelian Lie algebras, we find $A = \exp(\mathfrak{h}_\mathfrak{p})$, and $N = \exp(\mathfrak{n})$. Clearly, $AN = \{an : a \in A, n \in N\}$ is a connected subset of $O(1,n)$. For each integer $k \geq 1$, let

$$\exp(\mathfrak{s})^k = \left\{ \prod_{i=1}^{k} \exp(a_i) : a_i \in \mathfrak{s} \right\}.$$

Let S be the connected Lie subgroup of $O(1,n)$ with Lie algebra $\mathfrak{s}$ satisfying (2). We find $S = \prod_{k=1}^{\infty} \exp(\mathfrak{s})^k$. Applying (2), we find $AN \subseteq S$. We show $S = AN$ in Corollary 2.10 below.

Definition 2.6. For $t \in \mathbb{R}$ and $t \neq 0$, let $\alpha_1(t) = \dfrac{e^t - 1}{t}$, $\alpha_2(t) = \dfrac{1 - e^{-t}}{t}$, and let $\alpha(t) = \dfrac{\cosh(t) - 1}{t^2}$.

Lemma 2.7. *If* $t \in \mathbb{R} \setminus \{0\}$, *then* $\alpha_1(t)\alpha_2(t) = 2\alpha(t)$ *and* $\dfrac{\alpha_1(t)}{\alpha_2(t)} = e^t$. *There is a removable discontinuity at* $t = 0$ *in* $\alpha_1(t)$, $\alpha_2(t)$, *and* $\alpha(t)$, *and we can set* $\alpha_1(0) = 1 = \alpha_2(0)$, *and* $\alpha(0) = \frac{1}{2}$.

Let $b \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$. We define the following matrices that we will adopt for the remainder of this section.

1. Let $W = \left[\begin{array}{cc|c} 0 & 0 & b^T \\ 0 & 0 & b^T \\ \hline b & -b & 0_{n-1} \end{array}\right] \in \mathfrak{n}$

2. Let $Z = tV_1 + W = \left[\begin{array}{cc|c} 0 & t & b^T \\ t & 0 & b^T \\ \hline b & -b & 0_{n-1} \end{array}\right] \in \mathfrak{s}$

The next lemma follows from a proof by induction.

Lemma 2.8. *For all integers $k \geq 1$, we find*

1. $Z^{2k-1} = \left[\begin{array}{cc|c} 0 & t^{2k-1} & t^{2(k-1)}b^T \\ t^{2k-1} & 0 & t^{2(k-1)}b^T \\ \hline t^{2(k-1)}b & -t^{2(k-1)}b & 0_{n-1} \end{array}\right]$

2. $Z^{2k} = \left[\begin{array}{cc|c} t^{2(k-1)}(t^2 + b^Tb) & -t^{2(k-1)}b^Tb & t^{2k-1}b^T \\ t^{2(k-1)}b^Tb & t^{2(k-1)}(t^2 - b^Tb) & t^{2k-1}b^T \\ \hline -t^{2k-1}b & t^{2k-1}b & 0_{n-1} \end{array}\right].$

Lemma 2.9.

$$\exp(Z) = \left[\begin{array}{cc|c} \cosh(t) + \alpha(t)b^Tb & \sinh(t) - \alpha(t)b^Tb & \alpha_1(t)b^T \\ \sinh(t) + \alpha(t)b^Tb & \cosh(t) - \alpha(t)b^Tb & \alpha_1(t)b^T \\ \hline \alpha_2(t)b & -\alpha_2(t)b & I_{n-1} \end{array}\right]$$

Proof. We know $\exp(Z) = \sum_{k=0}^{\infty} Z^k/k!$. Let $\exp(Z)_{2,1}$ denote the $(2,1)$-block entry of $\exp(Z)$. Applying Lemma 2.8, we obtain

$$\exp(Z)_{2,1} = \left(\sum_{k=1}^{\infty} \frac{t^{2(k-1)}}{(2k-1)!} - \sum_{k=1}^{\infty} \frac{t^{2k-1}}{(2k)!}\right)[b, -b]$$

Applying the trigonometric hyperbolic functions, we obtain

$$\exp(Z)_{2,1} = \frac{1}{t}\left(\sinh(t) - \cosh(t) + 1\right)[b, -b] = \frac{1}{t}\left(1 - e^{-t}\right)[b, -b].$$

This proves $\exp(Z)_{2,1} = \alpha_2(t)\,[b, -b]$ for all $t \in \mathbb{R}$.

Let $\exp(Z)_{1,1}$ denote the $(1,1)$-block entry of $\exp(Z)$. Applying Lemma 2.8, we find $\exp(Z)_{1,1} - I_2$, is equal to the 2-by-2 matrix below.

$$\begin{bmatrix} (t^2 + b^Tb)\sum_{k=1}^{\infty} \frac{t^{2(k-1)}}{(2k)!} & -(b^Tb)\sum_{k=1}^{\infty} \frac{t^{2(k-1)}}{(2k)!} + \sum_{k=1}^{\infty} \frac{t^{2k-1}}{(2k-1)!} \\ (b^Tb)\sum_{k=1}^{\infty} \frac{t^{2(k-1)}}{(2k)!} + \sum_{k=1}^{\infty} \frac{t^{2k-1}}{(2k-1)!} & (t^2 - b^Tb)\sum_{k=1}^{\infty} \frac{t^{2(k-1)}}{(2k)!} \end{bmatrix}.$$

Notice, $t^2\alpha(t) = \cosh(t) - 1$ and $\sum_{k=1}^{\infty} \frac{t^{2(k-1)}}{(2k)!} = \alpha(t)$. Then

$$\exp(Z)_{1,1} = \begin{bmatrix} \cosh(t) + \alpha(t)b^Tb & \sinh(t) - \alpha(t)b^Tb \\ \sinh(t) + \alpha(t)b^Tb & \cosh(t) - \alpha(t)b^Tb \end{bmatrix} \in M_2(\mathbb{R}).$$

Similarly, the $(1,2)$-block entry of $\exp(Z)$ can be shown to satisfy

$$\exp(Z)_{1,2} = \alpha_1(t)[b,b]^T \in M_{2,n-1}(\mathbb{R});$$

however, we omit its proof. Clearly, the $(2,2)$-block entry satisfies $\exp(Z)_{2,2} = I_{n-1}$. This completes the proof of the lemma. ■

Applying Lemma 2.9, if $t \in \mathbb{R}$, we obtain

$$\exp(tV_1) = \left[\begin{array}{cc|c} \cosh(t) & \sinh(t) & 0^T \\ \sinh(t) & \cosh(t) & 0^T \\ \hline 0 & 0 & I_{n-1} \end{array}\right] \in A$$

and

$$\exp(W) = \left[\begin{array}{cc|c} 1+\frac{1}{2}b^Tb & -\frac{1}{2}b^Tb & b^T \\ \frac{1}{2}b^Tb & 1-\frac{1}{2}b^Tb & b^T \\ \hline b & -b & I_{n-1} \end{array}\right] \in N.$$

Then A and N are closed subgroups of $O(1,n)$.

Corollary 2.10. *Then* $\exp(tV_1 + W) = \exp(tV_1)\exp(\alpha_2(t)W)$. *In particular,* $S = AN$.

Proof. The corollary is true if $t = 0$. We assume $t \neq 0$. Applying Lemma 2.9, we find $\exp(tV_1)\exp(\alpha_2(t)W)$ is equal to

$$\left[\begin{array}{cc|c} \cosh(t) & \sinh(t) & 0^T \\ \sinh(t) & \cosh(t) & 0^T \\ \hline 0 & 0 & I_{n-1} \end{array}\right] \left[\begin{array}{cc|c} 1+\frac{1}{2}\alpha_2(t)^2b^Tb & -\frac{1}{2}\alpha_2(t)^2b^Tb & \alpha_2(t)b^T \\ \frac{1}{2}\alpha_2(t)^2b^Tb & 1-\frac{1}{2}\alpha_2(t)^2b^Tb & \alpha_2(t)b^T \\ \hline \alpha_2(t)b & -\alpha_2(t)b & I_{n-1} \end{array}\right].$$

Recall, $\cosh(t) + \sinh(t) = e^t$, $e^t\alpha_2(t) = \alpha_1(t)$, and $\alpha_1(t)\alpha_2(t) = 2\alpha(t)$. By multiplying the above matrices, we obtain the matrix for $\exp(tV_1 + W)$ in Lemma 2.9. This proves the first part of the corollary.
Recall, $W \in \mathfrak{n}$, $[V_1, W] = W$, and $N = \exp(\mathfrak{n})$. We find A normalizes N since

$$e^{tV_1}e^We^{-tV_1} = e^{Ad(e^{tV_1})(W)} = e^{e^{t ad V_1}(W)} = e^{e^tW} \in N.$$

In particular, $AN = NA$, AN is a group, N is a normal subgroup of AN, and AN is a semi-direct product group. The first part of the corollary implies $\exp(\mathfrak{s}) \subseteq AN$. Since AN is a subgroup, we find $S = \prod_{i=1}^k \exp(\mathfrak{s})^k \subseteq AN$. Earlier, we have seen $AN \subseteq S$. Hence, $S = AN$. ■

Corollary 2.11. *The restriction of* $\exp$ *to* $\mathfrak{s}$ *is injective.*

Proof. For $i \in \{1,2\}$, let $t_i \in \mathbb{R}$, $b_i \in \mathbb{R}^{n-1}$, and let

$$W_i = \left[\begin{array}{cc|c} 0 & 0 & b_i^T \\ 0 & 0 & b_i^T \\ \hline b_i & -b_i & 0_{n-1} \end{array}\right] \in \mathfrak{n}.$$

Suppose $\exp(t_1V_1 + W_1) = \exp(t_2V_1 + W_2)$. Applying Corollary 2.10, we obtain $\exp(t_1V_1)\exp(\alpha_2(t_1)W_1) = \exp(t_2V_1)\exp(\alpha_2(t_2)W_2)$. Since $\mathfrak{a}$ and $\mathfrak{n}$ are abelian Lie algebras, we obtain $\exp((t_1 - t_2)V_1) = \exp(\alpha_2(t_2)W_2 - \alpha_2(t_1)W_1)$. Let $t = t_1 - t_2$. Let $b = \alpha_2(t_2)b_2 - \alpha_2(t_1)b_1 \in \mathbb{R}^{n-1}$, and let W be the corresponding matrix defined in statement 1 after Lemma 2.7. Then $\exp(tV_1) = \exp(W)$. Applying Lemma 2.9, we find

$$\left[\begin{array}{cc|c} \cosh(t) & \sinh(t) & 0^T \\ \sinh(t) & \cosh(t) & 0^T \\ \hline 0 & 0 & I_{n-1} \end{array}\right] = \left[\begin{array}{cc|c} 1+\frac{1}{2}b^Tb & -\frac{1}{2}b^Tb & b^T \\ \frac{1}{2}b^Tb & 1-\frac{1}{2}b^Tb & b^T \\ \hline b & -b & I_{n-1} \end{array}\right].$$

Comparing the corresponding entries of the above matrices, we find $b = 0$. Then $\sinh(t) = 0$. Since $t \in \mathbb{R}$, we find $t = 0$ and $t_1 = t_2$. Then $\alpha_2(t_1) = \alpha_2(t_2)$. Since $b = 0$, we find $\alpha_2(t_2)b_2 = \alpha_2(t_1)b_1 = \alpha_2(t_2)b_1$. Since $\alpha_2(t_2) \neq 0$ for all $t_2 \in \mathbb{R}$, we obtain $b_1 = b_2$. Hence, $W_1 = W_2$. This proves the corollary. ∎

Corollary 2.12. *The exponential mapping is a bijection form $\mathfrak{s}$ onto S.*

Proof. Due to Corollary 2.11, we only need to show exp is surjective from $\mathfrak{s}$ onto S. Let $s \in S$. Since $S = AN$, choose $a \in A$, and $n \in N$ such that $s = an$. Recall, $\exp(\mathfrak{h}_\mathfrak{p}) = A$ and $\exp(\mathfrak{n}) = N$. Choose $t \in \mathbb{R}$ and $W \in \mathfrak{n}$ such that $a = \exp(tV_1)$ and $n = \exp(W)$. Then $s = \exp(tV_1)\exp(W)$. Note, $\alpha_2(t)^{-1}W \in \mathfrak{n}$. Applying Corollary 2.10, we find $\exp(tV_1 + \alpha_2(t)^{-1}W) = \exp(tV_1)\exp(W) = s$. Hence, exp is surjective from $\mathfrak{s}$ onto S. ∎

3. Main result

Let $n \geq 2$, and let $G \in O(1,n)$. Applying Theorem 1.3, we find $G^{-1} = e^X k$ for some $X \in \mathfrak{p}$ and $k \in K$. In [3], e^X is constructed as follows.

Lemma 3.1. *Let $x_1, \dots, x_n \in \mathbb{R}$, and let*

$$X = \left[\begin{array}{c|cccc} 0 & x_1 & 0 & \cdots & 0 \\ \hline x_1 & 0 & x_2 & \cdots & x_n \\ 0 & -x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -x_n & 0 & \cdots & 0 \end{array}\right] \in \mathfrak{p}.$$

Then

$$X^2 = \left[\begin{array}{c|cccc} x_1^2 & 0 & x_1x_2 & \cdots & x_1x_n \\ \hline 0 & x_1^2 - \sum_{k=2}^n x_k^2 & 0 & \cdots & 0 \\ -x_1x_2 & 0 & -x_2^2 & \cdots & -x_2x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_1x_n & 0 & -x_2x_n & \cdots & -x_n^2 \end{array}\right]. \tag{1}$$

$$e^X = I_{n+1} + \frac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}X + \frac{\cosh(\sqrt{\beta}) - 1}{\beta}X^2 \tag{2}$$

where $\beta = x_1^2 - x^Tx$ and $x = (x_2, \dots, x_n)$.

For conciseness, we adopt the notation $s = \dfrac{\sinh(\sqrt{\beta})}{\sqrt{\beta}}$.

Suppose
$$\xi \equiv sx_1 - \cosh(\sqrt{\beta}) \neq 0. \tag{3}$$

Let $t \in \mathbb{R}$, $d \in \mathbb{R}^{n-1}$, and let

$$Y = \left[\begin{array}{cc|c} 0 & t & d^T \\ t & 0 & d^T \\ \hline d & -d & 0_{n-1} \end{array}\right] \in \mathfrak{s}.$$

Applying Lemma 2.9, we find

$$e^Y = \left[\begin{array}{cc|c} \cosh(t) + \alpha(t)d^Td & \sinh(t) - \alpha(t)d^Td & \alpha_1(t)d^T \\ \sinh(t) + \alpha(t)d^Td & \cosh(t) - \alpha(t)d^Td & \alpha_1(t)d^T \\ \hline \alpha_2(t)d & -\alpha_2(t)d & I_{n-1} \end{array}\right].$$

Let $Z = e^Y e^X$. Let $z_{i,j}$ denote the (i,j)-entry of Z where $1 \leq i,j \leq n+1$. Let $d^T = (d_1, \dots, d_{n-1})$.

Lemma 3.2. *Let* $3 \leq i \leq n+1$.

(1) $z_{12} = (\cosh(t) + \alpha(t)d^Td)sx_1 + (\sinh(t) - \alpha(t)d^Td)\cosh(\sqrt{\beta}) - \alpha_1(t)sd^Tx$

(2) $z_{i,2} = \alpha_2(t)d_{i-2}\xi - sx_{i-1}$

Proof. Using matrix multiplication, we obtain the above entries of Z in the lemma. ■

Next, we rewrite z_{12} as follows.

$$\begin{aligned} z_{12} &= \cosh(t)sx_1 + \alpha(t)d^Td\left(sx_1 - \cosh(\sqrt{\beta})\right) - \alpha_1(t)sd^Tx + \sinh(t)\cosh(\sqrt{\beta}) \\ &= \cosh(t)sx_1 + \alpha(t)d^Td\xi - \alpha_1(t)sd^Tx + \sinh(t)\cosh(\sqrt{\beta}). \end{aligned}$$

Recall, $\xi \neq 0$. Notice, for any $t \in \mathbb{R}$ and $3 \leq i \leq n+1$, we find $z_{i,2} = 0$ if and only if $d_{i-2} = \frac{s}{\alpha_2(t)\xi}x_{i-1}$. In the next several steps, we choose some value $t = t_0$ such that $z_{12} = 0$. Consequently, we will have then found some $Y \in \mathfrak{s}$ such that $Z \in K$, by Lemma 1.2. Now, we substitute $d = \frac{s}{\alpha_2(t)\xi}x$ into the expression for z_{12}. Then

$$\begin{aligned} z_{12} &= \cosh(t)sx_1 + \alpha(t)\left(\frac{s}{\alpha_2(t)\xi}\right)^2 x^Tx\xi - \frac{\alpha_1(t)s^2}{\alpha_2(t)\xi}x^Tx + \sinh(t)\cosh(\sqrt{\beta}) \\ &= \cosh(t)sx_1 + \left(\frac{\alpha(t)}{\alpha_2(t)^2} - \frac{\alpha_1(t)}{\alpha_2(t)}\right)\frac{s^2x^Tx}{\xi} + \sinh(t)\cosh(\sqrt{\beta}) \\ &= \cosh(t)sx_1 - \frac{e^t}{2}\frac{s^2x^Tx}{\xi} + \sinh(t)\cosh(\sqrt{\beta}). \end{aligned}$$

Rewriting, we obtain

$$\begin{aligned} 2\xi z_{12} &= 2\left[\cosh(t)sx_1 + \sinh(t)\cosh(\sqrt{\beta})\right]\xi - e^ts^2x^Tx \\ &= \left[(e^t + e^{-t})sx_1 + (e^t - e^{-t})\cosh(\sqrt{\beta})\right]\xi - e^ts^2x^Tx. \end{aligned}$$

Applying the definition of ξ in (3), we obtain

$$\begin{aligned} 2\xi z_{12} &= e^{-t}(sx_1)^2 + (e^t - e^{-t})sx_1\cosh(\sqrt{\beta}) \\ &\quad - \left[(e^t + e^{-t})sx_1 + (e^t - e^{-t})\cosh(\sqrt{\beta})\right]\cosh(\sqrt{\beta}) + e^ts^2(x_1^2 - x^Tx) \\ &= e^{-t}(sx_1)^2 - e^{-t}sx_1\cosh(\sqrt{\beta}) \\ &\quad - \left[e^{-t}sx_1 + (e^t - e^{-t})\cosh(\sqrt{\beta})\right]\cosh(\sqrt{\beta}) + e^ts^2(x_1^2 - x^Tx). \end{aligned}$$

However, $s^2(x_1^2 - x^T x) = \cosh^2(\beta) - 1$. Then

$$\begin{aligned} 2\xi z_{12} &= e^{-t}(sx_1)^2 - e^{-t} sx_1 \cosh(\sqrt{\beta}) \\ &\quad - \left[e^{-t} sx_1 + (e^t - e^{-t})\cosh(\sqrt{\beta})\right]\cosh(\sqrt{\beta}) + e^t(\cosh^2(\beta) - 1) \\ &= e^{-t}(sx_1)^2 - e^{-t} sx_1 \cosh(\sqrt{\beta}) \\ &\quad - \left[(e^{-t}) sx_1 - e^{-t}\cosh(\sqrt{\beta})\right]\cosh(\sqrt{\beta}) - e^t \\ &= e^{-t}\left[(sx_1 - \cosh(\sqrt{\beta}))^2 - e^{2t}\right]. \end{aligned}$$

Then $2\xi z_{12} = e^{-t}\left[\xi^2 - e^{2t}\right]$. Choose $t_0 \in \mathbb{R}$ such that $\xi^2 - e^{2t_0} = 0$. Thus, if $t = t_0$ and $d = \frac{s}{\alpha_2(t_0)\xi} x$, then $z_{12} = 0$ and $z_{i2} = 0$ for all $i \in \{3, \ldots, n+1\}$. Consequently, $Z \in K$ and $e^Y G^{-1} = e^Y e^X k = Zk \in K$. Since $Y \in \mathfrak{s}$, we find $G \in KS = KAN$.

Theorem 3.3. *Let $n \geq 2$. Let $G \in O(1,n)$, and suppose $G^{-1} = e^X k$ for some $X \in \mathfrak{p}$, $k \in K$. If $\xi \neq 0$, then $G \in KAN$.*

Finally, we claim there exists $G \in O(1,n)$ such that $G \notin KAN$. Choose $x_1 = 1$ and $x = (x_2, \ldots, x_n)^T \in \mathbb{R}^{n-1}$ such that $x_1 - x^T x = 0$. With the chosen x_1 and x, construct the matrix $X \in \mathfrak{p}$ in the beginning of Section 3. Let $G = e^{-X}$. Notice, for any $Y \in \mathfrak{s}$, we find $\xi = 0$. Since $x^T x = 1$, there exists $x_{i-1} \neq 0$ for some $i \in \{3, \ldots, n+1\}$. Thus, $z_{i,2} = -x_{i-1} \neq 0$. In particular, $e^Y e^X \notin K$ for all $Y \in \mathfrak{s}$. Then $G \notin KAN$.

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Edgar N. Reyes, Department of Mathematics, Southeastern Louisiana University, Hammond, Louisiana, U.S.A.; ereyes@selu.edu.

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