Journal of Lie Theory
Volume **33** (2023) 5–28

Karl Heinrich Hofmann and the Structure of Compact Groups and Pro-Lie Groups

Sidney A. Morris

Communicated by K.-H. Neeb

Abstract. This article is dedicated to Karl Heinrich Hofmann on his 90th birthday. The first part of the article records some biographical facts about him. The second part focuses on the research papers and books he published with the author of this article over the last 45 years. These results concern the structure of compact groups and pro-Lie groups.

Mathematics Subject Classification: 22C05, 22E65.

Key Words: Topological group, Lie group, compact group, pro-Lie group, Lie algebra, duality, Pontryagin duality, LCA group.

1. Introduction

It is not possible in this article to cover all the mathematical contributions of Karl Heinrich Hofmann, nor do I feel competent to do so. Rather, I will focus on his research collaboration with me over the last 45 years which has resulted in 31 joint papers, 4 editions of the 1000 page book *The Structure of Compact Groups* and 2 editions of the 700 page book *The Structure of Pro-Lie Groups* (the second due to be published early in 2023). Further, I will endeavour to make the presentation easier to understand by not stating theorems in their full generality as can be easily seen by comparison with the citation for each result. (The citations to *The Structure of Pro-Lie Groups* are to the second edition.)

Karl Heinrich Hofmann was born in Heilbronn on October 3, 1932. His PhD advisor was the Estonian born German mathematician Hellmuth Kneser (1898–1973) and his PhD was awarded by Eberhard-Karls-Universität Tübingen in 1958 for his dissertation "Nichtassoziative topologisch-algebraische Strukturen". The list of his PhD students is:

Hudson, Sigmund, Tulane University 1963
Berglund, John, Tulane University 1967
Eckstein, Frank, Tulane University 1967
Keimel, Klaus, Eberhard-Karls-Universität Tübingen 1967
Lee, Dong, Tulane University 1967
Kahn, (Harold) David, Tulane University 1969
Nummela, Eric, Tulane University 1970
Takahashi, Alonso, Tulane University 1971
Evans, Howard, Tulane University 1972

ISSN 0949–5932 / $2.50

Greene, William, Tulane University 1973
Helmer, Dietrich, Tulane University 1973
LaMartin, William, Tulane University 1973
Varela, Januario, Tulane University 1973
Krauss, Fritz, Tulane University 1974
Yuan, John, Tulane University 1974
Wallace, David, Tulane University 1975
Jones, Lester, Tulane University 1980
Castellano, (Bruno) Michael, Tulane University 1981
Niño-Salcedo, Jaime Tulane University 1981
Hilgert, Joachim, Tulane University 1982
Keith, Verena, Tulane University 1984
Ihringer, Stefan, Technische Universität Darmstadt 1987
Spindler, Karlheinz, Technische Universität Darmstadt 1988
Neeb, Karl-Hermann, Technische Universität Darmstadt 1990
Weiss, Wolfgang, Technische Universität Darmstadt 1990
Dörr, Norbert, Technische Universität Darmstadt 1991
Eggert, Anselm, Technische Universität Darmstadt 1991
Terp, Christian, Technische Universität Darmstadt 1991
Schindler, Werner, Technische Universität Darmstadt 1992
Mittenhuber, Dirk, Technische Universität Darmstadt 1993
Groß, Christian, Technische Universität Darmstadt 1995
May, Angelika, Technische Universität Darmstadt 1995
Schwachhöfer, Martin, Technische Universität Darmstadt 1995
Wüstner, Michael, Technische Universität Darmstadt 1995
Breckner, Brigitte, Technische Universität Darmstadt 1998
Graeff, Robert, Technische Universität Darmstadt 1999
Klein, Ulrike, Technische Universität Darmstadt 2001
Maier, Daniel, Eberhard-Karls-Universität Tübingen 2013

So he has had 38 PhD students and he has 137 mathematical descendants.

Helmuth Kneser assisted the German mathematician Wilhelm Süss (1895–1958) in founding the Mathematical Research Institute of Oberwolfach and served as its second Director from 1958 to 1959. The PhD Advisor of Helmuth Kneser was David Hilbert. David Hilbert was a PhD student of Ferdinand Lindemannn who was in turn a student of Felix Klein who was a student of Julius Plücker and Rudolf Otto Sigismund Lipschitz who was a student of Gustav Peter Lejeune Dirichlet who was a student of Siméon Denis Poisson and Jean-Baptiste Joseph Fourier who were students of Joseph Louis Lagrange, Pierre-Simon Laplace and Leonhard Euler. One cannot fault that mathematical pedigree!

Karl Hofmann was a founding editor over 50 years ago of the journal *Semigroup Forum*. (A history on the founding of the journal appears in https://link.springer.com/article/10.1007/s00233-019-10062-9.) He was instrumental in the founding of the *Journal of Lie Theory* in 1994 as an outgrowth of Seminar Sophus Lie. Seminar Sophus Lie, with the most recent Seminar occurring in the birthplace of Sophus Lie in Nordfjordeid in Norway, is a joint Seminar of a group of German mathematicians interested in the Theory of Lie groups and their wider horizon. It was founded around 1989–90 when, during the Government of the German Democratic Republic in 1989, open contacts between mathematicians in East- and West-Germany became a reality for the first time since 1961.

Karl Hofmann was also a founding editor of *Forum Mathematicum*, which belongs to the top 50 journals in pure and applied mathematics, as measured by citation impact. Early discussions about founding the journal were held in what was then Karl Hofmann's office at Technische Hochschule Darmstadt and involved Manfred Karbe, then mathematics editor of the publisher, de Gruyter. Volume 1 of Forum Mathematicum appeared in 1989. Karl Hofmann is today an editor of the series of Lecture Notes *Research and Exposition in Mathematics* published by Heldermann Verlag.

Karl Hofmann has well over 300 publications, including 5 large coauthored books published in the 21st century. Two of these books, coauthored with Gerhard Gierz, Klaus Keimel, Jimmie D. Lawson, Michael W. Mislove, and Dana S. Scott, have over 2,000 citations, and one other has over 600 citations.

In 2013 he was honoured by the American Mathematical Society in being recognized amongst the Inagural Fellows of the Society.

I now turn to a few words about myself in order to set the context. In 1975 I was elected to the governing body or Council of the Australian Mathematical Society. I therefore attended a Council meeting in May 1976 and at that meeting it was determined that as the Annual Meeting in May 1977 of the Australian Mathematical Society was to be at my university, La Trobe University in Melbourne, I would be the Director of that meeting. It was also decided that for the very first time, the Annual Meeting would bring an international speaker. It was up to me and my co-ordinating committee, to determine who that speaker would be. I asked my committee what eminent scholar we might bring, and Brian Davey suggested Karl Hofmann as a possibility. I knew of Karl Hofmann from his published works, in particular, the book *Elements of Compact Semigroups* [24] which he wrote with Paul Stallings Mostert (1927–2022) and the two volumes *Introduction to Compact Groups Parts I and II*, [10] and [11], which were published by Tulane University. We invited Karl Hofmann to be the first specially invited international speaker at an Annual Meeting of the Australian Mathematical Society, and not only to speak at the Meeting but also to visit Australian universities in Sydney, Adelaide, Canberra, and Perth, spanning a distance of over 3,000 km – and he agreed. So in May 1977 I first met Karl Hofmann.

At that Meeting in 1977, Karl Hofmann spoke in his lecture on the fact that locally compact groups, in general, not just Lie groups, have a Lie algebra and there would be much value in studying these Lie algebras. Karl Hofmann and I have spent about 40 years studying the Lie theory of topological groups. And our two large books and most of our joint papers have been dedicated to the analysis of Lie algebras of locally compact groups and of a much wider class of topological groups.

I was introduced to the study of topological groups while an undergraduate by Ian D. Macdonald, a mathematical descendant of Isaac Newton (1642–1727). And Ian supervised my first research project and publication on varieties of topological groups and free topological groups. Also as an undergraduate I was fortunate to be introduced to category theory by the Australian mathematician Gregory

MAXWELL (MAX) KELLY (1930–2007) and the American mathematician SAUNDERS MAC LANE (1909–2005). The first problem given to me by my PhD Advisor, the Slovak mathematician IGOR KLUVÁNEK (1931–1993), was on free compact abelian groups.

Just before I met Karl Hofmann I published my small, but well cited, monograph *Pontryagin Duality and the Structure of Locally Compact Abelian Groups* [32]. It was, therefore, natural that the first research problem that Karl Hofmann and I studied was "locally compact products and coproducts in categories of topological groups" which resulted in a joint publication in 1977, namely [13].

During Karl Hofmann's visit to Melbourne, I noted his interest in sketching. Over the years this has developed into illustrating books including the book *Proofs from the Book* by Austrian mathematician Martin Aigner (born 1942) and German mathematician GÜNTER MATTHIAS ZIEGLER (born 1963) [1] and of course over 1,000 posters dating back to 1983 for the Technische Universität Darmstadt Colloquium, see tinyurl.com/HofmannColloquiumPosters. One might mention that there is an illustrious history of illustrating mathematics textbooks, for example, LEONARDO DI SER PIERO DA VINCI (1452–1519) illustrated the 16th century book *Summa de arithmetica, geometria, proportioni et proportionalita* (Summary of arithmetic, geometry, proportions and proportionality) by FRA LUCA BARTOLOMEO DE PACIOLI (1447–1517) in which he asserts that a solution to the (general) cubic equation is as impossible as squaring the circle. Watercolours by Karl Hofmann depicting his visits to Australia can be found at https://sidneymorris.net/adelaide.htm and https://sidneymorris.net/ballaratdiary.htm.

Doing joint research with overseas colleagues in the late 1970s was much more tedious than it is today. It was before emails in Australia existed. However by the 1980s this changed and Karl Hofmann and I could even chat online in real time using TALK on Unix. TALK allowed synchronous textual chatting using a split screen.

In 1979 I took a 5 month sabbatical at Tulane University to do research with Karl Hofmann. It was a very enjoyable visit and I saw first hand how very highly regarded Karl Hofmann was by all his colleagues at Tulane and even by the administrative staff. I also met Isolde, Karl's wife, and his two children Georg and Claudia. It was at this time that Karl Hofmann and I started our joint research into compact groups. In particular we started our research into free compact groups and as a first step, free compact abelian groups.

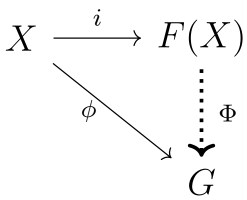
In 1982, forty years ago, while maintaining a relationship with Tulane University which continues until this day, Karl Hofmann returned to Germany and took up a position at Technische Universität Darmstadt (TUD) (then it was called Technische Hochschule Darmstadt). He remained at TUD until his official retirement. In reality he has maintained his relationship with TUD to today.

In 1986, just before the Chernobyl disaster, I began a six week visit to Karl Hofmann at TUD. This was the first of now 11 visits to Darmstadt, each for a few weeks, to work with Karl Hofmann. Over the years Karl Hofmann also visited me in Adelaide in 2000 and twice since in Ballarat, a small city about 120km from Melbourne. These visits allowed us to do joint research and to write books and papers. Our favourite jointly authored book is *The Structure of Compact Groups* which is in its fourth edition, is over 1,000 pages, and is the standard reference on compact groups.

It is a book that Karl Hofmann and I are rightly proud of. And the publisher, de Gruyter, is keen on our writing a fifth edition.

2. Free compact groups

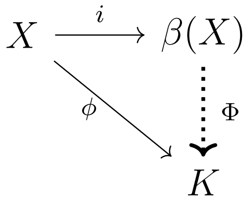
In [40] in 1882 the German mathematician Walther von Dyck (1856–1934) recognized that a certain kind of group which had arisen in hyperbolic geometry had the simplest possible presentation. In [34] in 1917 the Danish mathematician Jakob Nielsen (1890–1959) studied these groups algebraically and called them free groups. For our purposes we define the *free group* $F(X)$ on a set X as the unique group (up to group isomorphism) with the following universal property: there exists an injective mapping $i : X \to F(X)$ such that for each group G and each mapping $\phi\colon X \to G$, there exists a unique homomorphism $\Phi : F(X) \to G$ such that the following diagram commutes.



The *free abelian group* can be similarly defined, except that the free group $F(X)$ is replaced by the free abelian group $A(X)$ and each group G is chosen to be abelian.

The existence and uniqueness of the free group on a set X and the free abelian group on a set X can be proved algebraically. However, the best approach is to use the so-called *Adjoint Functor Theorem*. Before saying a few words about this theorem, let us consider some other examples.

If X is any topological space, then there exists a unique (up to homeomorphism) compact Hausdorff space $\beta(X)$ called the *Stone-Čech compactification* with a similar universal property: there exists a continuous mapping $i : X \to \beta(X)$ such that for every compact Hausdorff space K and continuous mapping of X into S, there exists a unique continuous mapping $\Phi : \beta(X) \to K$ such that the following diagram commutes.



It would appear that the existence of this compactification for Tikhonov (= completely regular Hausdorff) spaces is due to the Russian mathematician Andrey Nikolayevich Tikhonov (1906–1993) in [39] in 1930, and separately in [38] in 1937 to the American mathematician Marshall Harvey Stone (1903–1989) and the Czech mathematician Eduard Čech (1893–1960). The restriction to Tikhonov spaces allows one to insist that the mapping i is a homeomorphism of X onto its image in $\beta(X)$.

In [28] in 1941 the Russian mathematician Andrey Andreyevich Markov (1903–1979) introduced the notion of a free topological group. His proof in 1946 that the free topological group exists on each Tikhonov space was over 100 pages long and included significant information about free topological groups. The Japanese-American mathematician SHIZUO KAKUTANI (1911–2004) in [26] in 1944 produced a very different and short existence proof.

For convenience, in this paper all topological spaces are assumed to be Tikhonov spaces, and all topological groups, and topological vector spaces we consider are assumed to be Hausdorff (and hence all are Tikhonov) spaces.

Category 1	**Category 2**	**Left Adjoint**
Sets	Abelian Groups	Free Abelian Group
Sets	Groups	Free Group
Topological Spaces	Compact Spaces	Stone-Čech compactification
Topological Spaces	Abelian Topological Groups	Free Abelian Topological Group
Topological Spaces	Topological Groups	Free Topological Group
Topological Spaces	Topological Vector Spaces	Free Topological Vector Space
Topological Spaces	Locally Convex Vector Spaces	Free Locally Convex Vector Space
Topological Groups	Compact Groups	Bohr Compactification
Topological Groups	Pro-Lie Groups	Pro-Lieification
Compact Spaces	Compact Abelian Groups	Free Compact Abelian Group
Compact Spaces	Compact Groups	Free Compact Group
Topological Spaces	Abelian Pro-Lie Groups	Free Abelian Pro-Lie Group
Topological Spaces	Pro-Lie Groups	Free Pro-Lie Group

By the Catgeory of Sets, we mean the class of all sets with the morphisms (or arrows) being functions between sets. (For the definition of category, see [22, Theorem A3.1].) The Category of Groups has as its objects the class of all groups and as its morphisms the class of all homomorphisms from one group to another. The Category of all Compact Spaces has as its objects the class of all spaces and as its morphisms all continuous mappings between compact spaces. The Category of Compact Groups has as its objects the class of all compact groups and as its morphisms the class of all continuous homomorphisms between compact groups.

In each one of the rows in our Table we consider the *forgetful functor*, [22, Example A3.19], which maps Category 2 into Catgeory 1; for example, Topological Groups into Topological Spaces. Observe that for each mapping i of any infinite set S into any group G, there is a subgroup H of G such that there exists a mapping I of S into H with $i(s) = I(s)$, for all $s \in S$, with H having the property that its cardinality is not greater than 2^m, where m is the cardinality of S. In particular, the class of all non-isomorphic such H for a given set S is a set rather than a proper class. This condition is known, more generally, as the *Solution Set Condition.*

For the Stone-Čech compactification, the Solution Set Condition is that for every infinite Tikhonov space X and any continuous mapping i of X into a compact Hausdorff space K, there exists a compact Hausdorff subspace C of K and a continuous mapping I of X into C such that $i(x) = I(x)$, for every $x \in X$, and further the cardinality of $C \leq 2^{2^m}$, where m is the cardinality of X. So once again the class of all non-homeomorphic such C is a set rather than a proper class.

The formal definition of the *Solution Set Condition* is given in [22, Definition A3.58]. Further the formal definition of *left adjoint functor* is given in [22, Definition A3.29]. The formal statement of the *Adjoint Functor Theorem* is given in [22, Theorem A3.60]. This Theorem tells us that if the solution set condition is satisfied and the category is "rich enough", specifically it is a complete category (that is, it has arbitrary limits [22, Definition A3.49], and the forgetful functor $\mathcal{F}$ preserves limits, then this functor has a left adjoint functor $\mathcal{U}$.

It is important to note that each of the categories mentioned in our table above is a complete category. For the forgetful functor from the Category of Groups to Sets, the left adjoint factor $\mathcal{U}$ maps X in the category of Sets to $\mathcal{U}(X)$, the free group on the set X. Similarly the left adjoint functor $\mathcal{U}'$ maps K in the category of Compact Spaces to $\mathcal{U}'(K)$, the free compact group on K. For further discussion of the underlying category theory, see [27].

We see that the category theory proves the existence of each left adjoint in the table above. But we need to be cautious in that, while it is almost trivial that the conditions of the Adjoint Functor Theorem are satisfied for the case of Compact Spaces and Compact Groups, one needs to verify that the category of pro-Lie groups is a complete category. By contrast the category of locally compact groups is not a complete category since an infinite product of locally compact groups is only locally compact if all but a finite number of those in the product are compact. So the category of locally compact groups fails to be rich in this sense.

This section was called "Free Compact Groups". Chapters 11 and part of Chapter 8 of [22] are devoted to this topic which was first investigated by Hofmann and Morris in [14, 15, 16].

It is worth noting that for any topological space X, the Bohr compactification $b(A(X))$ of the free abelian topological group on X is the free abelian compact group on X and the Bohr compactification $b(F(X))$ of the free topological group on X is the free compact group on X.

In [28], Markov had asked whether if X and Y are topological spaces such that the free topological groups $F(X)$ and $F(Y)$ are isomorphic as topological groups, are X and Y necessarily homeomorphic. This question was answered in the negative in 1948 by the Russian mathematician MARK IOSIFOVICH GRAEV (1922–2017) in his paper [8]. He gave a simple construction for finding counterexamples. For example if X is the unit interval, then Y can be chosen to be the union of two closed intervals in the shape of the letter T. (We note that this applies to free abelian topological groups too as the free abelian topological group is just the quotient group of the free topological group by its commutator subgroup.) Notwithstanding this, X and Y do share a variety of properties. For example if X is compact metric and has finite dimension n, then Y is also compact metric and has the same dimension as X.

In my PhD study, not knowing of Graev's result, my PhD supervisor asked me the same question as Markov's but for free compact abelian groups. It was a trivial matter for me to show that Graev's example of the unit interval and the T-shaped space also have isomorphic free compact abelian groups. Indeed this is a trivial corollary of the observation above that the Bohr compactification of the free (abelian) topological group is the free (abelian) compact group.

Our first paper on free compact groups yielded a very much stronger structural result for free compact abelian groups:

Theorem 2.1. *The compact abelian group G is a free compact abelian group if and only if G is isomorphic as a topological group to $K \times (\widehat{\mathbb{Q}})^a \times \prod_{p\,\mathrm{prime}} \mathbb{Z}_p^b$, where $\mathbb{Q}$ is the discrete group of rational numbers, K is a compact connected abelian group with dense torsion subgroup, and where a, b are cardinal numbers with $a \geq \max\{2^{\aleph_0}, b, \dim K\}$.*

Corollary 2.2. *If X is a compact (connected) contractible space, then the free compact abelian group $A(X)$ is isomorphic as a topological group to $(\widehat{\mathbb{Q}})^a$, for some cardinal number a.*

Corollary 2.3. *$(\widehat{\mathbb{Q}})^a$, for any infinite cardinal number a, is isomorphic as a topological group to the free abelian compact group on the underlying topological space of the topological group $(\widehat{\mathbb{Q}})^a$.*

Unlike the situation for free topological groups and free abelian topological groups, compact metric spaces of different dimensions such as $[0,1]^n$, $n \in \mathbb{N}$, can have free compact abelian topological groups which are isomorphic as topological groups.

We note also that free compact abelian groups are large in the sense of their weight as a topological space.

Theorem 2.4. *The weight $w(A(X))$ of the free compact abelian group on a compact space X is given by $w(A(X)) = \max((w(X))^{\aleph_0}, 2^{\aleph_0})$.*

Corollary 2.5. *The cardinality m of any infinite free compact abelian group satisfies $m = m^{\aleph_0}$.*

Of course a restriction such as that in Corollary 2.5 is not valid for free or free abelian topological groups.

This is not the place for a full analysis of the structure of free compact groups or even free compact abelian groups as appears in [22, Chapters 11 and 8]. However one startling result is worth mentioning. I say startling, because there is nothing like this for free groups and free abelian groups since the NielsenSchreier Theorem says that any subgroup of a free group is a free group.

Theorem 2.6. [22, Theorem 11.19 and Corollary 11.20] *If X is a contractible space, then the free compact group $F(X)$ is isomorphic as a topological group to the direct product $A(X) \times F(X)'$ of the free compact abelian group on X and the commutator subgroup of $F(X)$.*

While we see very clearly that in contrast to the Nielson-Schreier Theorem for free groups, Theorem 2.6 tells us that a closed subgroup of a free compact group is not necessarily a free compact group and Theorem 2.1 implies that a closed subgroup of a free compact abelian group is not necessarily a free compact abelian group, [22, Corollary 8.72] does tell us that the identity component of a free compact abelian group is a free compact abelian group. Also surprising is [22, Theorem 11.14] that $F(X)_0$, the connected component of the identity of $F(X)$ is a semidirect factor of $F(X)$.

One lesson to be learnt from the study of free compact groups is that to understand the structure of compact groups it is important to study the centre and commutator group.

3. Some history

The text in this section is based on the Preface to the First Edition of [18] and a plenary address I delivered at the First Australasian Mathematical Convention at the University of Canterbury, New Zealand in May 1978 [33].

The concept of a topological group has its roots in the work of the German mathematician (CHRISTIAN) FELIX KLEIN (1849-1925) and the Norwegian mathematician (MARIUS) SOPHUS LIE (1842-1899). In a visit to Paris in 1870, Klein met Lie who had become interested in mathematics only a short time before. The young men were much influenced by the work of the French mathematicians who included in their number (MARIE ENNEMOND) CAMILLE JORDAN (1838–1922). Jordan had just written his treatise on substitution groups and Galois theory of equations. Klein and Lie began to see the importance of group theory. In 1872 Klein became Professor at Erlangen and declared in his inaugural address that one can classify geometries according to properties left invariant under groups of transformations. The study of any classical geometry such as euclidean geometry, affine geometry, projective geometry, etc. may be regarded as an investigation of a particular transformation group. With such transformation groups in mind Lie conceived the concept of *continuous groups* of transformations of manifolds. Of course Lie took differentiability for granted. So while Klein, as a rule concentrated on discontinuous transformation groups, Lie devoted his whole life to the systematic study of continuous transformation groups and their invariants. He demonstrated their central importance as a classifying principle in geometry, mechanics and ordinary and partial differential equations. Lie laid the foundation of the theory named *Lie theory* in honour of its creator. Several mathematicians, likewise prominent in the history of modern mathematics, contributed to its inception in the decades following 1873, which was the year in which Lie started to occupy himself intensively in the study of what he called continuous groups, notably: the German mathematicians FRIEDRICH ENGEL (1861–1941), HERMANN (KLAUS HUGO) WEYL (1885–1955), and WILHELM KARL JOSEPH KILLING (1847–1923), and the French mathematicians ÉLIE (JOSEPH) CARTAN (1869–1951) and (JULES) HENRI POINCARÉ (1854–1912). From the beginning, however, the advance of Lie theory bifurcated into two separate major highways, which is the reason why the words *Lie Theory* mean different things to different people.

The highway that interests us was taken by Killing and Cartan. It led to a study of what soon became known as Lie algebras, of the group and structure theory of Lie groups, and to the geometry of homogeneous spaces. In the 1960s and 1970s there was an encyclopedic attempt by NICOLAS BOURBAKI to summarize what had been achieved, and to the emergence of an immense collection of textbooks at all levels. In 1973 the French mathematician Jean Dieudonné (1906–1992) quipped *"Les groupes de Lie sont devenus le centre des Mathématiques; on ne peut rien faire de sérieux sans eux."* (Lie groups have moved to the centre of mathematics. One cannot seriously undertake anything without them. Gazette des Mathématiciens, Société Mathématique de France, Octobre 1974, p. 77.) By and large, in this line of *Lie Theory* the words meant the structure theory of Lie algebras and Lie groups, and in particular how the latter is based on the former.

The term *Lie group* originally meant 'finite-dimensional Lie group' and most people understand the words in this sense today. The significance of Lie's discoveries was emphasized by the German mathematician David Hilbert (1862–1943) by raising the question in 1900 whether (in later terminology) a locally euclidean topological group is in fact an analytic group in the sense of Lie. This was the fifth of his famous 23 problems which foreshadowed so much of the mathematical creativity of the 20th century. It required half a century of effort on the part of several generations of eminent mathematicians until it was settled in the affirmative. Partial solutions came along as the structure of topological groups was understood better and better: Hermann Weyl and his German student Fritz Peter (18991949) in 1923 laid the foundations of the representation and structure theory of compact groups, and a positive answer to Hilbert's Fifth Problem for compact groups was a consequence, drawn by the Hungarian-American mathematician John von Neumann (1903–1957) in 1932. The Soviet mathematician Lev Semyonovich Pontryagin (1908–1988) and the Dutch mathematician Egbert Rudolf van Kampen (1908–1942) developed in 1932, respectively, 1936, the duality theory of locally compact abelian groups laying the foundations for an abstract harmonic analysis flourishing throughout the second half of the 20th century and providing the central method for attacking the structure theory of compact abelian groups via duality. Again a positive response to Hilbert's question for locally euclidean abelian groups followed in the wash.

One of the most significant and seminal papers in topological group theory was published in 1949 by the Japanese mathematician Kenkichi Iwasawa (1917–1998), some three years before Hilbert's Problem was finally settled by the concerted contribution of the American mathematicians Andrew Mattei Gleason (1921-2008), Dean Montgomery (1909–1992), and Leon Zippin (1905–1995), and the Japanese mathematician Hidehiko Yamabe (1923–1960). It was Iwasawa who clearly recognized for the first time that the structure theory of locally compact groups reduced to that of compact groups and finite-dimensional Lie groups provided one knew that they happen to be approximated by finite-dimensional Lie groups in the sense of projective limits, in other words, if they were pro-Lie groups in our parlance. And this is what Yamabe established in 1953 for all locally compact groups which have a compact factor group modulo their identity component – almost connected locally compact groups as we shall call them. The most influential monograph collecting these results was [29] by Montgomery and Zippin in 1955. The theories of compact groups and of abelian locally compact groups had introduced in the first half of the century classes of groups with an explicit structure theory without the restriction of finite-dimensionality, and in the middle of the century these results opened up an explicit development for numerous results on the structure theory of locally compact groups.

The mathematics discussed in this article is a natural extension of the research begun a century ago (1923 to be precise) by Hermann Weyl and Fritz Peter on the foundations of the representation and structure theory of compact groups.

4. Compact Abelian groups

We begin our study of the structure of general compact groups and pro-Lie groups with compact abelian groups. We shall look at how various forms of connectedness present themselves – and there are some surprises.

The previously-mentioned Pontryagin-van Kampen Duality Theorem is, as shown in [32, 22], key to determining the structure of locally compact abelian groups. The multiplicative group of complex numbers of absolute value 1 with the topology it inherits as a subspace of the euclidean space of complex numbers is known as the *circle group* and is denoted by $\mathbb{T}$. (It is usual to write the group operation in the abelian circle group additively.) Any finite or infinite product of circle groups is known as a *torus*. As a consequence of the Peter-Weyl Theorem [22, Chapter 3] we know:

Theorem 4.1. *Every compact abelian group is isomorphic as a topological group to a closed subgroup of a torus.*

Definition 4.2. Let G be an abelian topological group. The set of continuous homomorphisms from G into $\mathbb{T}$ is denoted by $\widehat{G}$. Such continuous homomorphisms are called *characters* and $\widehat{G}$ is called the *character group* or *dual group* of G, where the group operation is given by $(\phi+\psi)(g)=\phi(g)+\psi(g)$, for $\phi,\psi\in\widehat{G}$. Then $\widehat{G}$ is an abelian topological group if given the compact-open topology.

We now present the Pontryagin-van Kampen Duality Theorem which is [32, Theorem 23] and [22, Theorem 7.63].

Theorem 4.3. (Pontryagin-van Kampen Duality Theorem). *Let G be any locally compact abelian group (LCA-group) and $\widehat{G}$ its dual group and $\widehat{\widehat{G}}$ the dual group of $\widehat{G}$. For fixed $g\in G$, let g' be the function $\widehat{G}\to\mathbb{T}$, given by $g'(\gamma)=\gamma(g)$, for all $\gamma\in\widehat{G}$. If a is the mapping given by $a(g)=g'$ then a is an isomorphism of topological groups G onto $\widehat{\widehat{G}}$.*

Roughly speaking the Pontryagin-van Kampen Duality Theorem says that every LCA-group is the dual group of its dual group. From this we deduce that *every piece of information about G is stored as information about $\widehat{G}$.* In the case of compact abelian groups this is particularly interesting as $\widehat{G}$ has the discrete topology. So *any compact abelian group can be completely described by the purely algebraic properties of its dual group;* for example, for G a compact abelian group:

(a) G is metrizable if and only if $\widehat{G}$ is countable.

(b) G is connected if and only if $\widehat{G}$ is torsion-free if and only if G is divisible. (An abelian group G is said to be *divisible* if for each $g\in G$ and each positive integer n, there exists an element $x\in G$ such that $nx=g$.)

(c) The *dimension*, dim, of G as a topological space equals the torsion-free rank of the group $\widehat{G}$ (that is, the number of elements in a maximal linearly independent subset of $\widehat{G}$).

So questions about compact abelian groups can be reduced to purely algebraic questions about abelian groups. This expresses the beauty and power of duality for compact abelian groups. Of course it is sometimes a challenge to find out how a certain property of a compact abelian group is reflected in a property of its dual group. (One example of this is the property described in The Borel Set Proposition discussed later.)

Later we shall see problems about connected pro-Lie groups solved by examining not their dual groups but their Lie algebras.

Pontryagin Duality can be used to discover the structure of locally abelian groups. See [32, Theorems 25 and 26] and with more detailed information [22, Theorem 7.57].

Theorem 4.4. (Principal Structure Theorem for LCA Groups) *If G is any LCA-group, then G has an open subgroup isomorphic as a topological group to $\mathbb{R}^a \times \mathbb{Z}^b \times C$, where $\mathbb{R}$ is the topological group of all real numbers with the euclidean topology, $\mathbb{Z}$ is the discrete topological group of integers, C is a compact abelian group, and a and b are non-negative integers.*

Corollary 4.5. *Let G be a connected LCA-group. Then G is isomorphic as a topological group to $\mathbb{R}^a \times K$, where a is a non-negative integer and K is a connected compact abelian group.*

The above beautiful structure result, Corollary 4.5, for connected LCA-groups will be one which we see generalized when we reach abelian pro-Lie groups.

Definition 4.6. Let G be an abelian group. The set $\mathrm{Div}(G)$ is defined by

$$\mathrm{Div}(G) = \{g \in G \mid (\forall n \in \mathbb{N})(\exists x \in G)\ n.x = g\}.$$

Each element in $\mathrm{Div}(G)$ is said to be a *divisible element*.

It should be noted that in [22, Theorem A1.32] it is shown that there exist an abelian group G such that the group $\mathrm{Div}(G)$ is not a divisible group. However, the next proposition shows that this pathology does not happen for compact abelian groups.

Proposition 4.7. [22, Proposition 8.2] *If G is a compact abelian group, then $\mathrm{Div}(G)$ is a divisible group, and is the unique largest divisible subgroup of G.*

We have seen that a compact abelian group is connected if and only if it is divisible. More generally we have:

Theorem 4.8. [22, Theorem 8.4] *If G is a compact abelian group G and G_0 is the identity component of G, that is the largest connected subset of G which contains the identity, and G_a is the largest arcwise connected subspace of G containing the identity, then*

$$G_0 = \mathrm{Div}(G) = \overline{G_a}.$$

When we introduce Lie algebras, we shall see the role played by the arc component.

The totally disconnected case is adequately described by the next result. For more information see [41].

Definition 4.9. A compact abelian group is said to be a *compact p-group* if its character group is a p-group.

Theorem 4.10. *A compact abelian group is totally disconnected if and only if it is a direct product of compact p-groups. So an infinite compact abelian totally disconnected group is homeomorphic to a Cantor cube, that is, an infinite product of discrete 2 point spaces.*

We now state a result which should remind you of what we said about free compact abelian groups in Corollary 2.3.

Theorem 4.11. [22, Corollary 8.9] (i) *A compact connected abelian group is torsion-free if and only if it is isomorphic to $(\widehat{\mathbb{Q}})^X$ for some set X.*
(ii) *A compact abelian group G is a torsion group if and only if it is a finite product of totally disconnected compact p-groups G_p each of which has bounded exponent, that is $p^{n(p)}.G_p = \{0\}$ for some natural number $n(p)$.*

Theorem 4.12. [22, Proposition 8.21] *Let G be a compact abelian group and U an arbitrary identity neighbourhood. Then U contains a compact subset homeomorphic to*

$$[-1,1]^{\operatorname{rank}\widehat{G}},$$

where rank refers to torsion-free rank.

Recall that a topological space is said to be *locally connected* if it has a basis of connected open sets. The following theorem gives the structure of locally connected LCA-groups.

Theorem 4.13. [22, Proposition 8.34] *Let G be a locally compact abelian group. Then the following statements are equivalent.*

(1) *G is locally connected.*

(2) *The identity component G_0 is open and locally connected.*

(3) *G is isomorphic to a group $\mathbb{R}^n \times K \times D$ for $n \in \{0,1,2,\dots\}$, K is a connected locally connected compact abelian group, and D is a discrete abelian group.*

Theorem 4.14. [22, Theorem 8.46] *The following statements are equivalent for a compact metric abelian group:*

(1) *G is a torus, that is, isomorphic as a topological group to $\mathbb{T}^X$, for some countable set X.*

(2) *G is arcwise connected.*

(3) *G is connected and locally connected.*

In light of Theorem 4.14 it is reasonable to ask whether every arcwise connected compact group is a torus.

Torus Proposition. *A compact arcwise connected abelian group is a torus.*

For over a decade there were attempts to prove the Torus Proposition. However the Israeli mathematician Saharon Shelah (born 1945) in [36] proved the Torus Proposition is undecidable.

Theorem 4.15. [22, Theorem 8.48] *If ZFC is consistent, then ZFC + Torus Proposition and ZFC+¬Torus Proposition are consistent; i.e. the Torus Proposition is undecidable in ZFC.*

This was a surprising result, but so is the next one:

A compact connected abelian group is completely determined (up to isomorphism of topological groups) by the topology of its underlying space as we shall formulate now. This is in striking contrast with other categories, e.g. that of Banach spaces where all infinite dimensional separable Banach spaces are homeomorphic. (See [2, Corollary 9.1].)

Theorem 4.16. [22, Theorem 8.59] *If compact connected abelian groups G_1 and G_2 are homeomorphic then they are isomorphic as topological groups.*

For example, an abelian topological group homeomorphic to a torus is a torus of the same dimension.

Of course Theorem 4.16 would be false if we deleted connectedness.

We have one more surprising result in this section.

Recall that a *σ-algebra* is a non-empty collection of subsets of a set X which is closed under the operation of forming complements, countable unions and countable intersections. Recall further that a *Borel set* in a topological space is any member of the smallest σ-algebra containing all the open sets in the topology.

The Borel Set Proposition. *In any compact abelian group, the arc components are Borel subsets.*

The Anti-Borel Set Proposition. *There exists a compact abelian group G of weight $w(G) = \aleph_1$ such that G_a is not a Borel set.*

Saharon Shelah proved that the Countability Proposition [37] is undecidable in ZFC. (See [22, Theorem 8.30 and Proposition 8.97].) In [12] Hofmann uses the Shelah theorem to prove the following result:

Theorem 4.17. [22, Theorem 8.99] *There is a model of set theory in which there is a compact abelian group G of weight $\aleph_1 = 2^{\aleph_0}$ such that the arc component factor group is algebraically isomorphic to $\mathbb{Q}$ and hence has arc components which fail to be Borel subsets.*

So there is a model of set theory in which the Generalized Continuum Hypothesis and the Axiom of Choice hold and in which the Anti-Borel Set Proposition holds.

It is an open question whether in a constructible universe the arc components of all compact abelian groups are Borel sets. A proof of this fact would show that the Borel Set Proposition is undecidable in ZFC.

5. Compact groups

Historically, questions about compact groups were answered in each case by seeking a reduction to the case of compact Lie groups. With the publication in 1998 of the book *The Structure of Compact Groups* [22] by Hofmann and Morris this can be avoided by using the structure theorems presented there in Chapters 9 and 10.

This book is in its fourth edition and has grown to over 1,000 pages with each edition including later results. For example, the latest edition includes a fresh and very satisfying approach to the Tannaka-Hochschild Duality Theorem for compact groups [22, Chapter 3]. It shows that there is a precise equivalence between the category of compact groups and the category of weakly complete compactlike real symmetric Hopf algebras. This yields that the category of compact groups is dual to the category of reduced real Hopf algebras.

Having studied compact abelian groups, we shall see in the Borel-Scheerer-Hofmann Splitting Theorem 5.7 that compact abelian groups are not just examples of compact groups but basic ingredients of the structure of all compact groups.

First we need a couple of definitions and preliminary results.

Definition 5.1. If G is a group, then the *commutator subgroup* of G, denoted by G' is the (normal) subgroup generated by the set $\{ghg^{-1}h^{-1} : g, h \in G\}$. The elements $ghg^{-1}h^{-1}$ are called *commutators*.

Theorem 5.2. [22, Theorem 9.2] *Let G be a connected compact group. Then every element of G' is a commutator. Further, the commutator subgroup G' is closed in G and thus is a connected compact group.*

Theorem 5.3. [22, Proposition 9.4] *Let G be a compact connected group. Then $G'' = G'$.*

Definition 5.4. A compact connected group is said to be *semisimple* if $G = G'$.

Corollary 5.5. *The commutator group G' of each compact connected group G is semisimple.*

Definition 5.6. Let G and H be topological groups, and let ι be a homomorphism of H into the group of automorphisms of G. Define a group structure on the set $G \times H$ by putting $(g_1, h_1) \cdot (g_2, h_2) = (g_1\iota(h_1)(g_2), h_1h_2)$.
Further, let $(g, h) \mapsto \eta(h)(g)$ be a continuous map of $G \times H$ onto G. With the product topology and this group structure $G \times H$ is a topological group. It is called the *semidirect product* of G by H that is determined by ι, and is denoted by $G \rtimes_\iota H$.)

One of the main structure theorems for compact groups is the following one.

Theorem 5.7. (The Borel-Scheerer-Hofmann Splitting Theorem) [22, Theorem 9.39] *Let G be a compact connected group. Then G' is a semidirect factor; that is, there is a compact connected abelian subgroup A of G such that*

$$(g, a) \mapsto ga\colon G' \rtimes_\iota A \to G, \qquad \iota(a)(g) = aga^{-1}$$

is an isomorphism of topological groups.
So G is homeomorphic to the product group $G' \times G/G'$.

We have previously described the structure of compact connected abelian groups. So from Theorem 5.7 we see that to understand the structure of a compact connected group we need to examine the structure of its commutator group G', which by Corollary 5.5 is a semisimple group. So our task now is to understand the structure of semisimple groups.

The next theorem tells us that every semisimple compact connected group is "almost" a (finite or infinite) product of simple simply connected compact Lie groups, Recall that the classification of these simple Lie groups is well-known. See for example: https://en.wikipedia.org/wiki/Simple_Lie_group#List.

Theorem 5.8. (Sandwich Theorem for Semisimple Compact Connected Groups) [22, Corollary 9.20]. *Let G be a semisimple compact connected group. Then there is a family $\{S_j \mid j \in J\}$ of simple simply connected compact Lie groups each with finite centre $Z(S_j)$, and natural quotient morphisms $q_j : S_j \to S_j/Z(S)_j$, and surjective morphisms $f\colon \prod_{j\in J} S_j \to G$, and $q : G \to \prod S_j/Z(S_j)$, such that $q \circ f = \prod_{j\in J} q_j$. This is summarized in the following diagram:*

$$\prod_{j\in J} S_j \xrightarrow{f} G \xrightarrow{q} \prod_{j\in J} S_j/Z(S_j),$$

where the kernels of f and q are totally disconnected.

The above theorem leads us to the following more general Sandwich Theorem.

Theorem 5.9. (Sandwich Theorem for Compact Connected Groups) [22, Corollary 9.25]. *Let G be a compact connected group, $Z_0(G)$ the identity component of the centre of G, and $\Delta = Z_0(G) \cap G'$. Then there exist a family $\{S_j \mid j \in J\}$ of simple simply connected compact Lie groups and natural quotient morphisms*

$$f : Z_0(G) \times \prod_{j\in J} S_j \to G \quad \text{and} \quad q : G \to \frac{Z_0(G)}{\Delta} \times \prod_{j\in J} \frac{S_j}{Z(S_j)},$$

where each $Z(S_j)$ is the centre of S_j, and the kernels of f and q are totally disconnected. This is summarized in the following diagram:

$$Z_0(G) \times \prod_{j\in J} S_j \xrightarrow{\quad f \quad} G \xrightarrow{\quad q \quad} \frac{Z_0(G)}{\Delta} \times \prod_{j\in J} \frac{S_j}{Z(S_j)}.$$

We now state a result which describes simply connected compact groups, and tells us that there are no nontrivial abelian ones.

Theorem 5.10. [22, Theorem 9.29] *Each simply connected compact group is isomorphic as a topological group to a product $\prod_{j\in J} S_j$ of a family of simply connected simple compact groups S_j. Further each simply connected compact abelian group is singleton.*

Definition 5.11. A compact connected abelian group is said to be a *pro-torus.* If G is a compact group, then any maximal connected abelian subgroup of G is called a *maximal pro-torus.*

The next theorem records the remarkable fact that maximal pro-tori in compact groups behave exactly like maximal tori do in Lie groups.

Theorem 5.12. (The Maximal Pro-Torus Theorem for Connected Compact groups) [22, Theorem 9.32]. *Let G be a compact connected group. Then the following statements hold.*

(i) *The maximal pro-tori of G are conjugate.*
(ii) *If T is a maximal pro-torus of G, then $G = \bigcup_{g\in G} gTg^{-1}$.*
(iii) *The centre $Z(G)$ is the intersection of all maximal pro-tori.*
(iv) *The centralizer of a connected abelian subgroup of G is connected.*

(v) *Each maximal pro-torus T of G is its own centralizer.*
(vi) *Each maximal pro-torus of G is a maximal abelian subgroup.*

Proposition 5.13. [22, Theorem 9.36] *If G is a semisimple compact connected group, then any maximal pro-torus is a torus, that is a product of circle groups.*

The next result appeared first in the paper [17] called "Weight and c". The name of this paper was apt in that it was 4 years from submission to acceptance of the paper with almost no revision.

Theorem 5.14. (Generating Compact Connected Groups) [22, Theorem 9.38] *Let G be a compact connected group and T a maximal pro-torus. Then there is an element $g \in G$ such that G is topologically generated by $T \cup \{g\}$, that is the smallest subgroup containing $T \cup \{g\}$ is G.*
If the weight of G does not exceed $2^{\aleph_0}$, then G is topologically generated by two suitable elements $g, t \in G$.

Having described the structure of compact connected groups, we now describe how this impacts on the structure of a general compact group.

Theorem 5.15. (Dong Hoon Lee's Supplement Theorem for Compact Groups) [22, Theorem 9.41] *Let G be a compact group and T an arbitrary maximal pro-torus of G. Then there is a compact totally disconnected subgroup D with the following properties:*
(i) $G = G_0D$,
(ii) *$G_0 \cap D$ is normal in G,*
(iii) $G_0 \cap D \subseteq Z(G_0)$,

The next related and powerful theorem exposes the topology of compact groups.

Theorem 5.16. (The Topological Decomposition of Compact Groups) [22, Corollary 10.39] *Each compact group G is homeomorphic to the product space $(G_0)' \times G_0/(G_0)' \times G/G_0$.*

Definition 5.17. A (Hausdorff) topological space is said to be *dyadic* if it is a continuous image of a product of (discrete) two point spaces.

Examples of dyadic spaces are the closed unit interval $[0,1]$ and any of its finite products $[0,1]^n$, where n is a positive integer.

Theorem 5.18. (Dyadicity of Compact Groups) [22, Theorem 10.40] *Each compact group G with infinitely many components is homeomorphic to a product of G_0 and a Cantor cube. Every infinite compact group is dyadic.*

Finally in this section we state an extension of the statement related to Hilbert's 5th problem that a locally euclidean group is a Lie group.

Theorem 5.19. (On Szenthe's Theorem) [22, Theorem 10.80] *Let G be a compact group. Then the following statements are equivalent:*
(1) *G is locally contractible.*
(2) *G is locally connected and finite-dimensional.*
(3) *G is locally euclidean.*
(4) *G is a Lie group.*

In this section we have stated a modest number of significant results on the structure of compact groups. Suffice it to say that Chapters 9 and 10 of [22] alone have about 200 such results.

6. Lie Theory

In this article we have not included any proofs. So while we have stated beautiful results about compact groups from [22], we have not said how they are proved except to mention the usefulness of the Pontryagin-van Kampen Duality in the abelian case.

Another powerful tool is in fact Lie Theory which we shall now introduce. Of course this Lie Theory is the main tool used in [18], a book about pro-Lie groups.

Without further ado we present here one definition of a pro-Lie group. For intuition recall that we could define compact Hausdorff topological spaces simply as *those topological spaces homeomorphic to closed subspaces of products of copies of the unit interval*, compact abelian groups as *those topological groups isomorphic as topological groups to closed subgroups of products of copies of the circle group*, and *compact groups as those topological groups isomorphic as topological groups to closed subgroups of product of compact matrix groups (or if you wish, the Special Orthogonal Groups* SO*(n))*.

Definition 6.1. A topological group is said to be a *pro-Lie group* if it is isomorphic as a topological group to a closed subgroup of a product of finite-dimensional Lie groups.

Remark 6.2. From Definition 6.1 it is immediately clear that a topological group is a pro-Lie group if it is a projective limit of finite-dimensional Lie groups. The converse is perhaps surprising, but is true. (See [18, Theorems 2.37 and 2.38].) This equivalence explains why these topological groups are called pro-Lie groups.

Before stating the key observation about which topological groups are pro-Lie groups, we need one important definition.

Definition 6.3. A topological group G is said to be *almost connected* if the quotient group G/G_0 is compact.

Remark 6.4. Obviously every compact group and every connected group is almost connected. In 1953, Yamabe [42, 43] proved the powerful result that *every almost connected locally compact group is a pro-Lie group.*

Theorem 6.5. *The following topological groups are pro-Lie groups:*

(i) *all finite-dimensional Lie groups;*
(ii) *all finite products and all infinite products of Lie groups;*
(iii) *all finite products and all infinite products of pro-Lie groups;*
(iv) *all compact groups;*
(v) *all locally compact abelian groups;*
(vi) *all finite products and all infinite products of locally compact abelian groups;*
(vii) *all connected and all almost connected locally compact groups.*

We see that infinite products of locally compact non-compact abelian groups are not locally compact, not even $\mathbb{R}^X$ for any infinite set X, but these products are pro-Lie groups. Note also that $\mathbb{Z}^{\mathbb{N}}$ is a pro-Lie group – and the underlying topological space of this topological group is homeomorphic to the space of irrational numbers (which is also homeomorphic to the space of all real (or complex) transcendental numbers [5, Corollary 3.4]).

With each topological group G one can easily associate the topological space $\mathcal{L}(G) = \mathrm{Hom}\,(\mathbb{R}, G)$ of all continuous group homomorphisms from $\mathbb{R}$ into G, endowed with the topology of uniform convergence on compact sets.

We also have the continuous function $\exp\colon \mathcal{L}(G) \to G$ given by $\exp X = X(1)$ and *scalar multiplication* $(r, X) \mapsto r \cdot X : \mathbb{R} \times \mathcal{L}(G) \to \mathcal{L}(G)$ where $(r \cdot X)(s) = X(sr)$.

These concepts are useful when additional properties are satisfied.

Definition 6.6. A topological group G is said to *have a Lie algebra*, if $\mathcal{L}(G)$ has a continuous addition and bracket multiplication making it into a topological Lie algebra in such a fashion that

$$(X+Y)(r) = \lim_{n\to\infty} \left(X\left(\frac{r}{n}\right) Y\left(\frac{r}{n}\right) \right)^n$$

and $$[X,Y](r^2) = \lim_{n\to\infty} \left(X\left(\frac{r}{n}\right) Y\left(\frac{r}{n}\right) X\left(\frac{r}{n}\right)^{-1} Y\left(\frac{r}{n}\right)^{-1} \right)^{n^2}.$$

If G has a Lie algebra, then $\mathcal{L}(G)$ is called *the Lie algebra of* G and $\exp\colon \mathcal{L}(G) \to G$ is called its *exponential function.* If G is a pro-Lie group, $\mathcal{L}(G)$ is said to be a *pro-Lie algebra.*

Clearly a topological group G has a Lie algebra if and only if the identity component G_0 has a Lie algebra and

$$\mathcal{L}(G) = \mathrm{Hom}\,(\mathbb{R}, G) = \mathrm{Hom}\,(\mathbb{R}, G_0) = \mathcal{L}(G_0).$$

The image of the exponential function is contained in G_0. If we believe that $\mathcal{L}(G)$ and the exponential function encapsulate the Lie theory of G, then the identity component G_0 already captures the Lie theory of G.

Theorem 6.7. *Every pro-Lie group G has a Lie algebra $\mathcal{L}(G)$ and the image $\exp \mathcal{L}(G)$ of the exponential function algebraically generates a subgroup which is dense in the identity component G_0 of G.*

It turns out that Lie Theory is exactly what is needed to understand the structure of almost connected pro-Lie groups. The book, [18], on pro-Lie groups does precisely this. Problems about pro-Lie groups are attacked by using their associated pro-Lie algebras. And the results obtained generalize the known structure theory of locally compact groups, and much more. Here we state a small number of the powerful results. One reason that this approach works so well is the simplicity of the topology on $\mathcal{L}(G)$, for any pro-Lie group G.

Definition 6.8. A topological vector space is said to be *weakly complete* if it is isomorphic as a topological vector space to the topological vector space $\mathbb{R}^X$, for some set X.

Proposition 6.9. [18, Proposition 2.7] *Let L be a real topological vector space, and consider it as a commutative topological Lie algebra. Then the following statements are equivalent.*

(i) *L is a pro-Lie algebra.*

(ii) *As a topological vector space, L is weakly complete.*

(iii) *L is isomorphic as a topological vector space and as a topological Lie algebra with zero brackets to a product $\mathbb{R}^X$ for some set X.*

We conclude this section with an obvious question.

The theory of an n-dimensional real Lie group is based on the fact that open subsets of $\mathbb{R}^n$ have a rich differentiable structure that is transported to the group, allowing a differentiable multiplication and inversion in the group. It has been an ongoing effort to replace $\mathbb{R}^n$ by more general, possibly infinite dimensional, topological vector spaces supporting differentiable structures. The theory of Lie groups on differentiable or smooth manifolds modeled on open subsets of locally-convex vector spaces and their real analysis can be found in [7]. One may justifiably ask how the theory of pro-Lie groups and the theory of infinite dimensional differentiable Lie groups in [7] are related. It was shown in [25] that a pro-Lie group is a smooth Lie group in the sense of [7] if and only if it is locally contractible.

7. Abelian pro-Lie groups

As we did in our analysis of compact groups, we shall begin with abelian pro-Lie groups. Our first Theorem 7.1 generalizes well-known results for locally compact abelian groups. (See [32, Theorems 25 and 26].)

Recall that a topological group G is said to be *compactly generated* if it has a compact subset K such that the smallest subgroup containing K is G. The subset of G which consists of all elements which are contained in a compact subset of G is denoted by $\mathrm{comp}(G)$. Often $\mathrm{comp}(G)$ is not a compact set.

Theorem 7.1. (Vector Group Splitting Theorem for Abelian Pro-Lie Groups) [18, Theorems 4.12 and 4.22] *If G is an abelian pro-Lie group then it has closed subgroups V and H, such that V is isomorphic as a topological group to the weakly complete vector group $\mathbb{R}^I$, for some set I, the identity component H_0 of H is compact, and $(v,h) \mapsto v+h\colon V \times H \to G$ is an isomorphism of topological groups. Further,*

(i) *if G is an almost connected abelian pro-Lie group, then G is isomorphic as a topological group to the direct product of a weakly complete vector group $\mathbb{R}^I$ and a compact abelian group;*

(ii) *If G is compactly generated abelian pro-Lie group, then G is isomporphic as a topological group to the direct product of $\mathbb{R}^n$ and a closed subgroup H which has its identity component H_0 compact, where n is a non-negative integer.*

Before leaving our discussion of abelian pro-Lie groups, it is appropriate to mention duality. We have already mentioned that if G is a locally compact abelian group, then it is *reflexive* in the sense that the natural map of G into its double dual $\widehat{\widehat{G}}$ is an isomorphism of topological groups.

The following theorem extends the results of [3].

Theorem 7.2. [18, Theorem 4.38] *Every almost connected abelian pro-Lie group is reflexive.*

8. Pro-Lie groups

As a rule the structural results we prove about pro-Lie groups are obtained by first proving the related result for pro-Lie algebras. It is not surprising that in moving from the abelian case we must travel through the solvable and nilpotent cases.

Definition 8.1. A pro-Lie algebra is said to be *semisimple* if it is a product of finite-dimensional simple Lie algebras. A connected pro-Lie group is said to be *semisimple* if its Lie algebra is semisimple.

As a generalisation of what we saw in Theorem 5.8 for compact groups, we have:

Theorem 8.2. (Sandwich Theorem for Semisimple Pro-Lie Groups) [18, Theorem 9.29] *Let G be a connected semisimple pro-Lie group. Then there exist simply connected simple Lie groups S_j, $j \in J$, such that*

$$\prod_{j\in J} S_j \xrightarrow{f} G \xrightarrow{q} \prod_{j\in J} S_j/Z(S_j),$$

where the composition $q \circ f$ is just the quotient morphism obtained by passing to the quotient $S_j \to S_j/Z(S_j)$ in each factor.

Definition 8.3. A pro-Lie algebra $\mathfrak{g}$ is said to be *prosolvable* if every finite-dimensional quotient algebra of $\mathfrak{g}$ is solvable. It is said to be *pronilpotent* if every finite-dimensional quotient algebra of $\mathfrak{g}$ is nilpotent.

Definition 8.4. Let $\mathfrak{g}$ be a topological Lie algebra. The ideal $\mathfrak{r}(\mathfrak{g})$ in $\mathfrak{g}$ is said to be its *radical* if it is the unique largest prosolvable ideal in $\mathfrak{g}$.

Theorem 8.5. [18, Theorem 6.49] *Every pro-Lie algebra $\mathfrak{g}$ has a radical $\mathfrak{r}(\mathfrak{g})$; the radical is closed, and the factor algebra $\mathfrak{g}/\mathfrak{r}(\mathfrak{g})$ is semisimple, that is, is a product of finite-dimensional simple Lie algebras. A pro-Lie algebra $\mathfrak{g}$ is semisimple if and only if its radical $\mathfrak{r}(\mathfrak{g})$ vanishes.*

Theorem 8.6. [18, Theorem 6.53] *A pro-Lie algebra $\mathfrak{g}$ is the semidirect sum of its radical $\mathfrak{r}(\mathfrak{g})$ and a closed semisimple subalgebra of $\mathfrak{g}$.*

Theorem 8.7. (Structure Theorem for Simply Connected Pro-Lie Groups) [18, Theorems 7.14 and 7.15] *Let G be a simply connected pro-Lie group with Lie algebra $\mathfrak{g}$. Then:*

(i) *G is the semidirect product $R \rtimes_I S$, where R is the closed normal subgroup R whose Lie algebra $\mathcal{L}(R)$ is the radical $\mathfrak{r}(\mathfrak{g})$ and S is a closed subgroup of G.*

(ii) *There is a family of simply connected simple Lie groups S_j, $j \in J$, such that S is isomorphic as a topological group to $\prod_{j\in J} S_j$.*

(iii) *R is homeomorphic to $\mathbb{R}^J$ for some set J.*

Theorem 8.8. [20, Theorem 8.6] *Let G be an almost connected pro-Lie group. Then there is a set J such that G is homeomorphic to $\mathbb{R}^J \times C$ for a maximal compact subgroup C, and to $\mathbb{R}^J \times C_0 \times (C/C_0)$.*

Theorem 8.9. [20, Corollary 8.9] *Each almost connected pro-Lie group G contains a compact connected semisimple subgroup S and a compact connected Abelian subgroup A, such that for suitable sets I and J, the topological group G is homeomorphic to the topological group $\mathbb{R}^I \times S \times A \times \Delta$, where:*

$$\Delta = \begin{cases} \mathbb{Z}(n), & \text{if } G \text{ has finitely many components}, \\ \mathbb{Z}(2)^J, & \text{otherwise.} \end{cases}$$

This result has several corollaries important for the topology of pro-Lie groups.

Corollary 8.10. *The space underlying an almost connected pro-Lie group is a Baire space.*

This follows from Theorem 8.9 and Oxtoby's results in [35].

Corollary 8.11. *Every almost connected pro-Lie group is homotopy equivalent to a compact group.*

Indeed as $\mathbb{R}^I$ is homotopy equivalent to a singleton, *the algebraic topology of each almost connected pro-Lie group is the same as that of a compact group.*

Corollary 8.12. *An almost connected pro-Lie group is locally compact if and only if, in* Theorem 8.9*, the set I is finite.*

Corollary 8.13. *Every finite-dimensional almost connected pro-Lie group is locally compact.*

Theorem 8.14. [18, Corollary 11.55] *Each connected pro-Lie group G contains maximal compact connected subgroups and these maximal compact connected subgroups are conjugate in G under inner automorphisms. Further, each compact connected subgroup is contained in a maximal compact connected subgroup.*
The above statements remain true if "subgroup" is replaced everywhere by "abelian subgroup".

In [18] there is a large number of theorems on the structure of pro-Lie groups. Even when the statements are generalisations of well-known results for locally compact groups, the proofs are very often quite technical.

Hopefully the few results presented here give you some appreciation of the material on the structure of pro-Lie groups.

Acknowledgments. The author gratefully acknowledges the referee and the Managing Editor whose comments and criticisms improved the presentation of this paper.

References

[1] M. Aigner, G. M. Ziegler: *Proofs from The Book*, with illustrations by K. H. Hofmann, Springer, Berlin (2018).

[2] C. Bessaga, A. Pelczyński: *Selected Topics in Infinite Dimensional Topology*, Mathematical Monographs 58, Polish Scientific Publishers, Warsaw (1975).

[3] R. Brown, P. J. Higgins, S. A. Morris: *Countable products and sums of lines and circles: their closed subgroups, quotients and duality properties*, Math. Proc. Cambridge Phil. Soc. 78 (1975) 19–32.

[4] E. Čech: *On bicompact spaces*, Annals of Mathematics 38/4 (1937) 823–844.

[5] T. P. Chalebgwa, S. A. Morris: *A continuous homomorphism of a thin set onto a fat set*, Bull. Australian Math. Soc. 106/3 (2022) 500–503.

[6] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott: *Continuous Lattices and Domains*, Cambridge University Press, Cambridge (2003).

[7] H. Glöckner, K.-H. Neeb: *Infinite Dimensional Lie Groups: General Theory and Main Examples*, Graduate Texts in Mathematics 935, Springer, New York (2015).

[8] M. I. Graev: *Free topological groups (Russian)*, Izv. Akad. Nauk SSSR, Ser. Mat. 12 (1948) 279–321. English translation in Amer. Math. Soc. Translations 35 (1951). Reprinted in Amer. Math. Soc. Translations (l) 8 (1962) 305–364.

[9] W. Herfort, K. H. Hofmann, F. G. Russo: *Periodic Locally Compact Groups*, Studies in Mathematics 71, De Gruyter, Berlin (2019).

[10] K. H. Hofmann: *Introduction to the Theory of Compact Groups I*, Tulane University, New Orleans (1966–67).

[11] K. H. Hofmann: *Introduction to the Theory of Compact Groups II*, Tulane University, New Orleans (1968–69).

[12] K. H. Hofmann: *Arc components in locally compact groups are Borel sets*, Bull. Austral. Math. Soc. 65 (2002) 1–8.

[13] K. H. Hofmann, S. A. Morris: *Locally compact products and coproducts in categories of topological groups*, Bull. Austral. Math. Soc. 17 (1977) 401–417.

[14] K. H. Hofmann, S. A. Morris: *Free compact groups. I: Free compact abelian groups*, Topology Appl. 23 (1986) 101–102.

[15] K. H. Hofmann, S. A. Morris: *Free compact groups. II: The center*, Topology Appl. 28 (1988) 215–231.

[16] K. H. Hofmann, S. A. Morris: *Free compact groups. III: Free semisimple compact groups*, in: *Categorical Topology and its Applications to Analysis*, J. Adamek et al. (eds.), World Scientific Publishers, Singapore (1989) 208–219.

[17] K. H. Hofmann, S. A. Morris: *Weight and c*, J. Pure Applied Algebra 68 (1990) 181–194.

[18] K. H. Hofmann, S. A. Morris: *The Lie Theory of Connected Pro-Lie Groups: A Structure Theory for Pro-Lie Algebras, and Connected Locally Compact Groups*, European Mathematical Society, Zurich (2007); 2nd ed.: *The Structure of Pro-Lie Groups*, European Mathematical Society, Zurich (2023).

[19] K. H. Hofmann, S. A. Morris: *Contributions to the structure of connected pro-Lie groups*, Topology Proc. 33 (2009) 225–237.

[20] K. H. Hofmann, S. A. Morris: *The structure of almost connected pro-Lie groups*, J. Lie Theory 21 (2011) 347–383.

[21] K. H. Hofmann, S. A. Morris: *Pro-Lie groups: a survey with open problems*, Axioms 4 (2015) 294–312.

[22] K. H. Hofmann, S. A. Morris: *The Structure of Compact Groups: A Primer for the Student – A Handbook for the Expert*, 4th ed., De Gruyter, Berlin (2020).

[23] K. H. Hofmann, S. A. Morris: *Advances in the theory of compact groups and pro-Lie groups in the last quarter century*, Axioms 10 (2021) 190.

[24] K. H. Hofmann, P. S. Mostert: *Elements of Compact Semigroups*, C. E. Merrill Books, Columbus (1966).

[25] K. H. Hofmann, K.-H. Neeb: *Pro-Lie groups as infinite dimensional Lie groups*, Math. Proc. Cambridge Phil. Soc. 146 (2008) 351–378.

[26] S. Kakutani: *Free topological groups and infinite direct product topological groups*, Proc. Imp. Acad. Tokyo 20 (1944) 595–598.

[27] S. Maclane: *Categories for the Working Mathematician*, Springer, New York (1971).

[28] A. A. Markov: *On free topological groups (Russian)*, C. R. (Doklady) Acad. Sci. URSS, (N.S.) 31 (1941) 299–301. Izv. Akad. Nauk SSSR, Ser. Math. 9 (1945), 3–64. English translation in Amer. Math. Soc. Translations 30 (1950) 11–88; reprinted in Amer. Math. Soc. Translations (l) 8 (1962) 195–273.

[29] D. Montgomery, L. Zippin: *Topological Transformation Groups*, Interscience Publisher, New York (1955).

[30] S. A. Morris: *Varieties of topological groups*, Bull. Austral. Math. Soc. 1 (1969) 145–160.

[31] S. A. Morris: *Free compact abelian group*, Mat. Cas. Slov. Akad. Vied 22 (1972) 141–147.

[32] S. A. Morris: *Pontryagin Duality and the Structure of Locally Compact Abelian Groups*, Cambridge University Press, Cambridge (1977).

[33] S. A. Morris: *Duality and structure of locally compact abelian groups ... for the layman*, Math. Chronicle 8 (1979) 39–56.

[34] J. Nielsen: *Die Isomorphismen der allgemeinen unendlichen Gruppe mit zwei Erzeugenden*, Math. Annalen. 78 (1917) 385–397.

[35] J. C. Oxtoby: *Cartesian products of Baire spaces*, Fundamenta Math. 49 (1961) 157–166.

[36] S. Shelah: *Infinite abelian groups, Whitehead problem, and some constructions*, Israel J. Math. 18 (1974) 243–256.

[37] S. Shelah: *The consistency of* $Ext(G, \mathbb{Z})$, Israel J. Math. 39 (1981) 74–82.

[38] M. H. Stone: *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. 41 (1937) 375–481.

[39] A. Tychonoff: *Über die topologische Erweiterung von Räumen*, Math. Annalen 102 (1930) 544–561.

[40] W. von Dyck: *Gruppentheoretische Studien*, Math. Annalen 20 (1882) 1–44.

[41] J. S. Wilson: *Profinite Groups*, Oxford University Press, Oxford (1998).

[42] H. Yamabe: *On the conjecture of Iwasawa and Gleason*, Ann. of Math. (2) 58 (1953) 48–54.

[43] H. Yamabe: *Generalization of a theorem of Gleason*, Ann. of Math. (2) 58 (1953) 351–365.

Sidney A. Morris, (1) School of Engineering, IT and Physical Sciences, Federation University, Ballarat, Victoria, Australia; (2) Department of Mathematical and Physical Sciences, La Trobe University, Melbourne, Victoria, Australia;
Morris.Sidney@gmail.com, ORCID: 0000-0002-0361-576X.

Received September 21, 2022
and in final form September 30, 2022