Journal of Lie Theory
Volume **33** (2023) 29–77

Askey-Wilson Polynomials and Branching Laws

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Communicated by K.-H. Neeb

Abstract. Connection coefficient formulas for special functions describe change of basis matrices under a parameter change, for bases formed by the special functions. Such formulas are related to branching questions in representation theory. The Askey-Wilson polynomials are one of the most general 1-variable special functions. Our main results are connection coefficient formulas for shifting one of the parameters of the nonsymmetric Askey-Wilson polynomials. We also show how one of these results can be used to re-prove an old result of Askey and Wilson in the symmetric case. The method of proof combines establishing a simpler special case of shifting one parameter by a factor of q with using a co-cycle condition property of the transition matrices involved. Supporting computations use the Noumi representation and are based on simple formulas for how some basic Hecke algebra elements act on natural 'almost symmetric' Laurent polynomials.

Mathematics Subject Classification: 33D67, 22E47, 33D45, 17B37.

Key Words: Connection coefficients, branching, DAHA, Askey-Wilson polynomials, spherical functions.

1. Introduction

The Askey scheme [8] is a hierarchy of hypergeometric and q-hypergeometric orthogonal poynomials, which includes many families of classical polynomials, including the Hermite, Laguerre, Gegenbauer, and Jacobi polynomials. Each family depends on a number of auxiliary parameters, and lower families in the hierarchy can be obtained from higher families by a suitable specialization of parameters.

If $\mathcal{P} = \{P_0, P_1, \ldots\}$ and $\mathcal{P}' = \{P'_0, P'_1, \ldots\}$ are two sequences of polynomials satisfying $\deg(P_n) = \deg(P'_n) = n$, then one has an expansion of the form

$$P_n = \sum\nolimits_{m \le n} c_{m,n} P'_m. \tag{1}$$

(The coefficients $c_{m,n}$ depend on the ordered bases $\mathcal{P}$ and $\mathcal{P}'$ and so might be more fully denoted by $c^{\mathcal{P},\mathcal{P}'}_{m,n}$. However to keep our notation more spare, we usually make clear the bases by context and omit the superscripts.)

Now suppose $\mathcal{P}$ and $\mathcal{P}'$ are from the *same* family in the Askey-Wilson scheme and differ in only *one* auxiliary parameter. In this case one can sometimes obtain an explicit formula for the "connection' coefficients" $c_{m,n}$ in (1) in terms of Pochammer symbols and their q-analogs, which are defined as follows:

ISSN 0949–5932 / \$2.50

$$(a)_n = \prod_{k=0}^{n-1}(a+k), \quad (a_1,\dots,a_r)_n = \prod_{i=1}^{r}(a_i)_n, \tag{2}$$

$$(a|q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a_1,\dots,a_r|q)_n = \prod_{i=1}^{r}(a_i|q)_n. \tag{3}$$

For example, one has the following classical result:

Theorem 1.1. (3.40 in [2]) *If $\mathcal{P} = \{P_n^{(\gamma,\beta)}(x)\}$ and $\mathcal{P}' = \{P_n^{(\alpha,\beta)}(x)\}$ are two sequences of Jacobi polynomials differing in one parameter, then one has*

$$P_n^{(\gamma,\beta)}(x) = \sum_{k=0}^{n} \frac{(\beta+k+1)_{n-k}(\gamma-\alpha)_{n-k}(\beta+\gamma+n+1)_k(2k+\alpha+\beta+1)}{(\alpha+\beta+k+1)_{n+1}(n-k)!} P_k^{(\alpha,\beta)}(x). \tag{4}$$

Similarly equation (3.42) in [2] tells us for Gegenbauer polynomials

$$C_n^{\lambda}(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(\lambda-\nu)_k(\lambda)_{n-k}(n-2k+\nu)}{(\nu)_{n-k+1}k!} C_{n-2k}^{\nu}(x). \tag{5}$$

(Equations (3.40) and (3.42) in [2] and elsewhere are often expressed in terms of Gamma functions, but the identity

$$\Gamma(k+\alpha) = (\alpha)_k\Gamma(\alpha) \qquad \text{for } k \in \mathbb{N}$$

readily leads to formulas (4) and (5).)

A similar result holds for the Askey-Wilson polynomials $P_n(z;a,b,c,d|q)$, which are a 4 parameter q-hypergeometric family at the top of the Askey hierarchy.

Theorem 1.2. ([3], Askey-Wilson) *If $\mathcal{P} = \{P_n(z;a,b,c,d|q)\} = \{P_n\}$ and $\mathcal{P}' = \{P_n(z;e,b,c,d|q)\} = \{P'_n\}$ are two sequences of Askey-Wilson polynomials differing in one parameter, then one has*

$$P_n = \sum_{m\le n} c_{m,n}(a,e;b,c,d)P'_m \qquad \textit{where} \tag{6}$$

$$c_{m,n}(a,e;b,c,d) = \frac{(q^{n-m+1}|q)_m(bcq^m, bdq^m, cdq^m, ae^{-1}|q)_{n-m}}{(q|q)_m(abcdq^{n+m-1}, bcdeq^{2m}|q)_{n-m}} e^{n-m}. \tag{7}$$

Here we follow the notation of [20] and regard the Askey-Wilson polynomials as Laurent polynomials in z, symmetric under the inversion $z \mapsto z^{-1}$. This is related to the ordinary polynomial variable x of Askey-Wilson [3] by $2x = z + z^{-1}$. Our normalization of P_n as a monic Laurent polynomial in z is also different from [3].

As explained in [20], the Askey-Wilson polynomials admit nonsymmetric analogs $E_r = E_r(z;a,b,c,d|q)$, defined for any integer r, which are eigenfunctions of certain q-difference operators. While $P_n = P_n(z;a,b,c,d|q)$ can be obtained from $E_{\pm n}$ by a suitable symmetrization operator, the E_r themselves are not symmetric under $z \mapsto z^{-1}$. Also, while the P_n depend symmetrically on a, b, c and d, the E_r only have the symmetries $a \leftrightarrow b$ and $c \leftrightarrow d$.

We define the "zig-zag" order $\prec$ on integers as follows:

$$0 \prec -1 \prec 1 \prec -2 \prec 2 \prec \cdots . \tag{8}$$

We say a Laurent polynomial $F(z)$ has *degree* an integer r if there is a constant $c \neq 0$ such that $F(z) - cz^r$ is in the span of $\{z^s, s \prec r\}$. Such a Laurent polynomial is called *monic* if the leading coefficient c is 1. If $\mathcal{F} = \{F_0, F_{-1}, F_1, \ldots\}$ and $\mathcal{F}' = \{F'_0, F'_{-1}, F'_1, \ldots\}$ are two families of Laurent polynomials satisfying the condition $\deg(F_r) = \deg(F'_r) = r$, then one can again consider connection coefficients (e.g. denoted by $b_{r,s}$) such that

$$F_s = \sum\nolimits_{r \preceq s} b_{r,s} F'_r. \tag{9}$$

The nonsymmetric Askey-Wilson polynomials satisfy the degree condition, and our main results are formulas of this form giving the connection coefficients for two sequences of such polynomials differing in one parameter. In view of the $a \leftrightarrow b$ and $c \leftrightarrow d$ symmetries, there are only two distinct cases to consider: (1) the parameter a is replaced by e, say, and (2) the parameter c is replaced by g, say.

The first case is a change of basis relationship from basis $\{E_r(z; a, b, c, d|q)\}$ to basis $\{E_r(z; e, b, c, d|q)\}$. We often refer to the corresponding matrix transforming components relative to these bases as a *transition matrix.*

Theorem 1.3. *Let $\mathcal{F} = \{E_r(z; a, b, c, d|q)\} = \{E_r\}$ and $\mathcal{F}' = \{E_r(z; e, b, c, d|q)\} = \{E'_r\}$ be sequences of nonsymmetric Askey-Wilson polynomials. Then*

$$E_s = \sum_{r \preceq s} d_{r,s} c_{|r|,|s|} E'_r$$

where $c_{m,n}$ is the symmetric connection coefficient $c_{m,n}(a, e; b, c, d)$ as in (7), and

$$d_{r,s} = \begin{cases} \dfrac{q^{s-r}(abcdq^{s+r-1}|q)_1}{(abcdq^{2s-1}|q)_1} & \text{if } r \geq 0, s \geq 0 \\ \dfrac{(q^{-(r+s)}|q)_1}{(q^{-s}, cdq^{-(s+1)}|q)_1} & \text{if } r \geq 0, s < 0 \\ \dfrac{bcdeq^{s-(r+1)}(q^{-r}, cdq^{-(r+1)}, ae^{-1}q^{s+r}|q)_1}{(abcdq^{2s-1}, bcdeq^{-(2r+1)}|q)_1} & \text{if } r < 0, s \geq 0 \\ \dfrac{(q^{-r}, cdq^{-(r+1)}, bcdeq^{-(r+s+1)}|q)_1}{(q^{-s}, cdq^{-(s+1)}, bcdeq^{-(2r+1)}|q)_1} & \text{if } r < 0, s < 0 \end{cases}$$

Remark 1.4. It is worth noting that if one were to replace $E_r(z; a, b, c, d|q)$ above by $\psi(r)E_r(z; a, b, c, d|q)$ then there is some simplification in the resultant $d_{r,s}$ formulas when $\psi(r)$ is defined to be

$$\psi(r) = \begin{cases} 1 & \text{if } r \geq 0 \\ (1 - q^{-r})(1 - cdq^{-(r+1)}) & \text{if } r < 0. \end{cases} \tag{10}$$

(This corresponds to a change of normalization of the E_r for $r < 0$.) ■

We now consider the case where the parameter c is replaced by g.

Theorem 1.5. *Let $\mathcal{F} = \{E_r(z; a, b, c, d|q)\} = \{E_r\}$ and $\mathcal{F}'' = \{E_r(z; a, b, g, d|q)\} = \{E''_r\}$ be sequences of nonsymmetric Askey-Wilson polynomials. Then*

$$E_s = \sum_{r \preceq s} d''_{r,s} c''_{|r|,|s|} E''_r$$

where

(1) $c''_{m,n}$ *is the symmetric connection coefficient* $c_{m,n}(c, g; a, b, d)$

(2) $d''_{r,s}$ *is* $d^c_{r,s}(c,g;a,b,d)$ *and*

$$d^c_{r,s}(c,g;a,b,d) = \begin{cases} \dfrac{(abq^s, abcdq^{r+s-1}|q)_1}{(abq^r, abcdq^{2s-1}|q)_1} & \text{if } r \geq 0, s \geq 0 \\ -\dfrac{abq^r(q^{-r-s}|q)_1}{(q^{-s}, abq^r|q)_1} & \text{if } r \geq 0, s < 0 \\ -\dfrac{dq^{-r-1}g(q^{-r}, abq^s, cg^{-1}q^{s+r}|q)_1}{(abcdq^{2s-1}, abdgq^{-2r-1}|q)_1} & \text{if } r < 0, s \geq 0 \\ \dfrac{(q^{-r}, abdgq^{-r-s-1}|q)_1}{(q^{-s}, abdgq^{-2r-1}|q)_1} & \text{if } r < 0, s < 0. \end{cases}$$

Acknowledgement. The research of SS was partially supported by NSF grants DMS-1939600 and 2001537, as well as Simons Foundation grant 509766. The research of BS was partially supported by Simons Foundation grant 89086.

Motivation and applications

For certain values of the parameters, formulas such as (4) in Theorem 1.1 and (5) can be interpreted as describing aspects of branching theorems for spherical representations of compact classical Lie groups. Spherical representations of compact groups are isomorphic to irreducible subrepresentations of

$$L^2(G/K) = \oplus_\mu W_\mu$$

for a compact symmetric space G/K. Here G acts on the left. (See [7] and chapters 1 and 2 of [23].) Each W_μ contains a K-bi-invariant function $f_\mu \in C^\infty(K,\ G/K)$. In representation theory f_μ is referred to as a spherical function for G/K, but sometimes in the literature the term zonal spherical function is used. Denoting by T the one dimensional maximal torus of a rank 1 symmetric space, we have $G/K = KT$ by the maximal torus theorem for compact symmetric spaces and thus we can consider f_μ as a function on T. As described on page 59 of [6] (also e.g. page 65 of [1], chapter 3 of [23], or Table 1 in [4]), Gegenbauer and Jacobi polynomials can be viewed (up to scaling) as spherical functions on the rank 1 symmetric spaces S^n and $\mathbb{C}P^n$,

These observations extend to the other compact simply connected rank 1 symmetric spaces: the quaternionic projective space and the Cayley plane. The classical polynomial parameter values associated with these geometric examples are also presented in these references. For the sphere S^n, the parameter ν in the Gegenbauer polynomial $C^\nu_k(x)$ is $\frac{n-1}{2}$. For the complex projective space $\mathbb{C}P^n$, the parameters in the Jacobi polynomial $P^{(\alpha,\beta)}_k$ are given by $\alpha = n-1$ and $\beta = 0$.

Recall how branching of spherical representations from G to G' can give rise to the connection coefficient formulas (5) and (4). The $G' \subset G$ situations will be $SO(2n) \subset SO(2n+1)$ and $SU(n-1) \subset SU(n)$. We start with a spherical representation τ of G, realized as a subrepresentation of $L^2(G/K)$. We then want to consider $G' \subset G$ with an associated rank 1 symmetric space G'/K'. Since they are both rank 1 symmetric spaces, their maximal tori in the sense of symmetric spaces may be identified, so we may view

$$G'/K' = K'T.$$

Let f_τ denote the spherical function associated to τ. When we restrict τ to G' we obtain $\oplus\tau'_\alpha$. For each summand τ'_α we have the corresponding spherical function f'_α and a relationship

$$f_\tau(x) = \sum c_\alpha f'_\alpha(x)$$

where the c_α are non-negative constants. (Since spherical functions with respect to G/K restricted to T are multiples of matrix elements $< v, \tau(g)v >$ for v a K-invariant vector of τ, the restricted functions $f'_\alpha(x)$ are spherical functions with respect to G'/K'.)

For $SO(2n) \subset SO(2n+1)$ and $SU(n-1) \subset SU(n)$, this translates to the formulas (5) and (4) for Gegenbauer and Jacobi functions respectively. So c_α becomes a connection coefficient in a formula like (5) and (4).

The connection coefficient relations (5) and (4) can also be viewed as being related to calculating integrals of a product of two such special polynomials with different parameters; as such one might for example use Rodriguez' formula and integration by parts to determine them.

In this paper we shall generalize these classical relations by purely combinatorial methods. Although there could also be relations to deformations of algebraic structures such as double affine Hecke algebras via their generators and eigenfunctions, and also possibly to the theory of spherical functions for certain quantum symmetric spaces, we shall not go into these.

Structure of the proof of Theorem 1.3

One natural approach to proving this theorem would be to use the vector-valued reformulation of the E_r in [12], which extends earlier results in [11].

Our proof here uses an interesting alternative. We start with proving the special case of a and e differing by a factor of q. This is a q-shift analogous to a shift by 1 in one of the classical integer parameters. Since transition functions describing change of basis matrices satisfy a natural co-cycle condition, we can establish the case of a and e differing by an integral power of q by showing that our asserted expressions for the transition functions also satisfy the co-cycle condition. And then, by observing that everything involved is given by rational functions agreeing at infinitely many values, we obtain the theorem for arbitrary a and e.

The co-cycle condition in detail and proof plan A

The partition of powers of z into non-negative vs. negative (corresponding to the same notions on the root system C_1 sitting inside the affine root system $\tilde{C}_1$) gives a direct sum decomposition of Laurent polynomials

$$\mathcal{R} = \mathcal{R}^0 \oplus \mathcal{R}^1$$

where we can choose ordered bases

$$E_0, E_1, E_2, \ldots \text{ for } \mathcal{R}^0 \quad \text{and} \quad E_{-1}, E_{-2}, \ldots \text{ for } \mathcal{R}^1.$$

We view connection coefficient relations like those given in Theorem 1.3 as describing a change of basis in the space of Laurent polynomials, perhaps truncated in degree, so as to reference a finite dimensional subspace.

Unless otherwise specified, we will henceforth treat the common parameters b, c, d, and q as unchanged and drop them from the argument lists of the E_r (and associated connection coefficients.) For example $E_n(a)$ is a shorthand for $E_n(z; a, b, c, d|q)$.

We use $\boldsymbol{\mathcal{T}}(a,e)$ for the 'true' transition matrix from components relative to the $\{E_r(a)\}$ basis to components relative the $\{E_r(e)\}$ basis. The notation $T(a,e)$ will refer to the transition matrix specified by the formulas in Theorem 1.3. Thus proving Theorem 1.3 amounts to showing $\boldsymbol{\mathcal{T}}(a,e) = T(a,e)$.

As described fully in Appendix A, this block decomposition and choice of ordered bases corresponds to $T(a,e)$ having the block decomposition

$$T = \begin{bmatrix} T^{00} & T^{01} \\ T^{10} & T^{11} \end{bmatrix}$$

with

$$T^{00} = \begin{bmatrix} \tau_{0,0} & \tau_{0,1} & \tau_{0,2} & \tau_{0,3} & \cdots \\ 0 & \tau_{1,1} & \tau_{1,2} & \tau_{1,3} & \cdots \\ 0 & 0 & \tau_{2,2} & \tau_{2,3} & \cdots \\ 0 & 0 & 0 & \ddots & \vdots \end{bmatrix} \qquad T^{01} = \begin{bmatrix} \sigma_{0,-1} & \sigma_{0,-2} & \sigma_{0,-3} & \sigma_{0,-4} & \cdots \\ 0 & \sigma_{1,-2} & \sigma_{1,-3} & \sigma_{1,-4} & \cdots \\ 0 & 0 & \sigma_{2,-3} & \sigma_{2,-4} & \cdots \\ 0 & 0 & 0 & \ddots & \vdots \end{bmatrix}$$

$$T^{10} = \begin{bmatrix} 0 & \sigma_{-1,1} & \sigma_{-1,2} & \sigma_{-1,3} & \cdots \\ 0 & 0 & \sigma_{-2,2} & \sigma_{-2,3} & \cdots \\ 0 & 0 & 0 & \sigma_{-3,3} & \cdots \\ 0 & 0 & 0 & \ddots & \vdots \end{bmatrix} \qquad T^{11} = \begin{bmatrix} \tau_{-1,-1} & \tau_{-1,-2} & \tau_{-1,-3} & \tau_{-1,-4} & \cdots \\ 0 & \tau_{-2,-2} & \tau_{-2,-3} & \tau_{-2,-4} & \cdots \\ 0 & 0 & \tau_{-3,-3} & \tau_{-3,-4} & \cdots \\ 0 & 0 & 0 & \ddots & \vdots \end{bmatrix}.$$

Here $\tau_{r,s}$ and $\sigma_{r,s}$ (zero unless $r \preceq s$) are the products of the c's and d's defined by

$$\tau_{r,s} = d_{r,s}c_{|r|,|s|} \qquad \text{if } (r \geq 0 \text{ and } s \geq 0) \text{ or } (r < 0 \text{ and } s < 0) \tag{11}$$

$$\sigma_{r,s} = d_{r,s}c_{|r|,|s|} \qquad \text{if } (r \geq 0 \text{ and } s < 0) \text{ or } (r < 0 \text{ and } s > 0). \tag{12}$$

We think of this transition matrix as acting on the left on column vectors of components relative to one basis and producing a column vector of components relative to the other basis.

The 'true' transition function $\boldsymbol{\mathcal{T}}(a,e)$ satisfies the co-cycle condition

$$\boldsymbol{\mathcal{T}}(a,e) = \boldsymbol{\mathcal{T}}(f,e)\boldsymbol{\mathcal{T}}(a,f) \tag{13}$$

since each side describes a valid way to go from a-coordinates to e-coordinates. In particular this means a 'discrete' co-cycle condition; namely for any non-negative integer p :

$$\boldsymbol{\mathcal{T}}(a,aq^{p+1}) = \boldsymbol{\mathcal{T}}(aq^p,aq^{p+1})\boldsymbol{\mathcal{T}}(a,aq^p). \tag{14}$$

Our proof of Theorem 1.3 has three steps which we refer to as

$$\text{(PROOF PLAN A)} \tag{15}$$

1. Show that the entries of both $T(a,e)$ and $\boldsymbol{\mathcal{T}}(a,e)$ are rational functions of e with coefficients in the filed $\mathbb{Q}(a,b,c,d,q)$.

2. Show that $\boldsymbol{\mathcal{T}}(a,aq) = T(a,aq)$.

3. Show that T also satisfies the discrete co-cycle condition

$$T(a,aq^{p+1}) = T(aq^p,aq^{p+1})T(a,aq^p)$$

for any $p \in \mathbb{N}$.

Since both $T(a,a)$ and $\boldsymbol{\mathcal{T}}(a,a)$ are the identity, and equation (14) says $\boldsymbol{\mathcal{T}}$ satisfies the discrete co-cycle condition, it is immediate from parts 2 and 3 above that $\boldsymbol{\mathcal{T}}(a,e)$ and $T(a,e)$ agree whenever $e = aq^p$ for non-negative p. Now using part 1 of Proof Plan A, we see that each entry of the two matrices is a rational function of e agreeing with the other at infinitely many points. So they must agree $\big($as rational functions with coefficients in $\mathbb{Q}(a,b,c,d,q)\big)$ for all e.

Structure of the paper

We have already stated our main results above, explained how these kind of results can relate to branching theorems in representation theory, and described the basic approach of the proof.

The heart of the proof of Theorem 1.3 are the last two steps of (15), Proof Plan A. These are carried out in Sections 5 and 6. We have written out the verification of these two steps in a detailed step-by-step way, and so these two sections constitute about one third of the main body of our paper.

Section 2 briefly recalls the basic double affine Hecke algebra (DAHA) and Noumi representation points of view about the nonsymmetric Askey-Wilson polynomials. Much of what we use is based on the approach of [19], which was specialized to the one variable case in [20], and further enhanced (including some notational adjustments) in [22]. At certain points, we need a little more detail than was recorded in the statements of the theorems proved there, and so explain how those come from this earlier work.

The zig-zag order leads to a filtration of the Laurent polynomials and Section 3 exploits some aspects of this. Some of the results are conveniently expressed in terms of what we call 'almost symmetric' basis elements. These are based on Laurent polynomials which are either symmetric or skew-symmetric under one of the involutions $z \mapsto z^{-1}$ or $z \mapsto qz^{-1}$.

The recursive description of the $\{E_r\}$, equations (26), plays a key role. It immediately gives us the first step of (15), Proof Plan A. The filtration properties of this 'zig-zag recursion' allow us to obtain explicit formulas for the three highest zig-zag degree terms of each E_r. The coefficients of these terms with respect to the appropriate almost symmetric basis elements are determined in the later parts of Section 3. Our determination of the last of these, Theorem 3.11, is a little involved, and so we have also included a detailed step-by-step presentation of the argument.

Properties of the filtration allow us to quickly express $\boldsymbol{\mathcal{T}}(a,aq)$ in terms of the three highest zig-zag degree terms of the E_r. Thus the later parts of Section 3 are exactly what we need to carry out the second step of (15), Proof Plan A, in Section 5.

It is natural to compare our nonsymmetric Askey-Wilson connection coefficient results (Theorems 1.3 and 1.5) to the corresponding long known symmetric case, Theorem 1.2. That is why those former statements are in terms of products like $d_{r,s}c_{|r|,|s|}$. However, for carrying out the proof here, it is cumbersome to be constantly writing out these products and so an alternate notation for these products is introduced in Section 4.

Because of the natural 2×2 block structure of our transition matrices, verifying the co-cycle condition in Section 6 has four somewhat similar pieces. It also turns out that those verifications can be carried out more simply by first simplifying some ratios, and that is done at the beginning of Section 6.

Using our result, we also give a re-proof of Askey and Wilson's Theorem 1.2 in Section 7 of our paper. Here the $d_{r,s}c_{|r|,|s|}$ representation of the connection coefficients in Theorem 1.3 greatly simplifies the exposition.

While the considerations are very elementary, making explicit how our conventions lead to the precise matrices we use is important for being able to verify correctness of our arguments. This is done in Section 8, Appendix A.

Section 9, Appendix B, a summary table, includes a few DAHA related formulas that we do not use in this paper, but which are of the same nature as ones we fully justify in the main body.

Since the technique is so close to the one we fully describe for Theorem 1.3, we omit the details of the proof of Theorem 1.5.

We mention one striking feature, however. In the proof of the discrete co-cycle condition for the shift-c case (Propositions 6.4, 6.5, 6.6, and 6.7 below in the shift-a case), the polynomials p_2, whose vanishing in the last quarter of the proofs we are demonstrating, can in a natural way be chosen to be *identical* to the p_2 of the shift-a case.

2. Preliminaries

First we recall the double affine Hecke algebra (DAHA) point of view and the Noumi representation on Laurent polynomials.

Let $\mathcal{R}$ denote the Laurent polynomials in one variable z with coefficients in a field such as $\mathbb{Q}(a,b,c,d,q)$.

Let $\mathbb{F}$ be the field $\mathbb{Q}(q^{\frac{1}{2}}, t_0^{\frac{1}{2}}, t_1^{\frac{1}{2}}, u_0^{\frac{1}{2}}, u_1^{\frac{1}{2}})$. As described in [19] and [20], the Noumi representation [16] is a faithful representation of a double affine Hecke algebra (DAHA) $\mathcal{H}$ with coefficients in $\mathbb{F}$. Here $\mathcal{H}$ is the $\mathbb{F}$-algebra with generators T_0, T_1, U_0, U_1 and relations

$$T_0 \sim t_0,\ T_1 \sim t_1,\ U_0 \sim u_0,\ U_1 \sim u_1, \text{ and } T_1T_0U_0U_1 = q^{-\frac{1}{2}}$$

where the meaning of $F \sim f$ is $F - F^{-1} = f^{\frac{1}{2}} - f^{-\frac{1}{2}}$. The scalars $q, a, b, c, d, t_0, t_1, u_0, u_1$ are related by

$$\begin{array}{llll} t_0 = -\frac{cd}{q} & t_1 = -ab & u_0 = -\frac{c}{d} & u_1 = -\frac{a}{b} \\ a = t_1^{\frac{1}{2}}u_1^{\frac{1}{2}} & b = -t_1^{\frac{1}{2}}u_1^{-\frac{1}{2}} & c = q^{\frac{1}{2}}t_0^{\frac{1}{2}}u_0^{\frac{1}{2}} & d = -q^{\frac{1}{2}}t_0^{\frac{1}{2}}u_0^{-\frac{1}{2}} \end{array} \tag{16}$$

For our purposes, we mostly want to reason about Laurent polynomials with *rational* coefficients $\mathbb{Q}(a,b,c,d,q)$. So we work with elements $\widetilde{T}_0, \widetilde{T}_1, \widetilde{U}_0,$ and $\widetilde{U}_1$ (part of this described in [22]) that are multiples of the usual $T_0, T_1, U_0,$ and U_1:

$$\widetilde{T}_0 = t_0^{\frac{1}{2}}T_0 \qquad \widetilde{T}_1 = t_1^{\frac{1}{2}}T_1 \qquad \widetilde{U}_0 = (qt_0)^{\frac{1}{2}}U_0. \qquad \widetilde{U}_1 = (t_1)^{\frac{1}{2}}U_1.$$

These adjusted elements now satisfy the product relation $\widetilde{T}_1\widetilde{T}_0\widetilde{U}_0\widetilde{U}_1 = t_0t_1$ and the usual nicely symmetric DAHA inversion formulas, e.g.

$$T_1^{-1} = T_1 - t_1^{\frac{1}{2}} + t_1^{-\frac{1}{2}},$$

translate (as in [22]) to ones like

$$\left(\widetilde{T}_1 + 1\right)\left(\widetilde{T}_1 - t_1\right) = 0; \qquad \text{i.e. } \widetilde{T}_1^{-1} = \frac{\widetilde{T}_1}{t_1} - 1 + \frac{1}{t_1}.$$

With these adjustments, the Noumi representation of the DAHA $\mathcal{H}$ on a Laurent polynomial $f \in \mathcal{R}$ looks like:

$$\left[\widetilde{T}_0 f\right](z) = t_0 f(z) + \frac{(z-c)(z-d)}{(z^2-q)}\left[f\big(\tfrac{q}{z}\big) - f(z)\right] \tag{17}$$

$$\left[\widetilde{T}_1 f\right](z) = t_1 f(z) + \frac{(1-az)(1-bz)}{(1-z^2)}\left[f\big(\tfrac{1}{z}\big) - f(z)\right] \tag{18}$$

$$\widetilde{U}_0 f = t_0 \widetilde{T}_0^{-1} X f = \Big(\widetilde{T}_0 X + (1-t_0)X\Big) f \tag{19}$$

since $t_0\widetilde{T}_0^{-1} = \widetilde{T}_0 + (1-t_0)$ and we use the multiplication operator

$$[Xf](z) = z\big(f(z)\big).$$

We call the operator X because these DAHA preliminaries are often expressed in terms of a *Laurent series* variable x while we consistently use z for the Laurent variable and its related $x = \frac{1}{2}(z + z^{-1})$ for the ordinary polynomial variable.

The nonsymmetric Askey-Wilson polynomials are eigenvectors of

$$\widetilde{Y} = \widetilde{T}_1 \widetilde{T}_0 \text{ (as well as of } Y = T_1 T_0)$$

with $\widetilde{Y} E_r = \widetilde{\mu}_r E_r$ and $\qquad \widetilde{\mu}_r = \begin{cases} (t_0 t_1) q^r & \text{if } r \geq 0 \\ q^r & \text{if } r < 0. \end{cases}$

$\big(\widetilde{\mu}_r = (t_0 t_1)^{\frac{1}{2}} q^{\overline{r}}\big)$, where $\qquad q^{\overline{r}} = \begin{cases} (t_0 t_1)^{\frac{1}{2}} q^r & \text{if } r \geq 0 \\ (t_0 t_1)^{-\frac{1}{2}} q^r & \text{if } r < 0 \end{cases} \qquad (20)$

is the notation used in [20] for the eigenvalues of $Y = T_1 T_0$.

The definition of the E_r on page 278 of [19] is via creation operators $\mathcal{S}_0$ and $\mathcal{S}_1$. Up to normalization, for $n \geq 0$, $\mathcal{S}_0$ takes E_n to $E_{-(n+1)}$ and $\mathcal{S}_1$ takes $E_{-(n+1)}$ to E_{n+1}. This is the original recursive description of the nonsymmetric Askey-Wilson polynomials.

Theorem 4.1 on page 402 of [20] makes this (in arbitrary rank) more explicit. The proof of that theorem introduces a variant $\mathcal{S}'_0$ of $\mathcal{S}_0$ which also takes $E_{-(n+1)}$ to a multiple of E_{n+1}. It is not explicitly noted in the theorem, but the proof makes clear that the formulas stated are simply the result of applying the operators $\mathcal{S}'_0$ and $\mathcal{S}_1$.

Using our usual $\widetilde{U_0}$ and $\widetilde{T}_1$, the corresponding creation operators are

$$\widetilde{\mathcal{S}'}_0 = [\widetilde{Y}, \widetilde{U}_0] \qquad \widetilde{\mathcal{S}}_1 = [\widetilde{T}_1, \widetilde{Y}]$$

with $\widetilde{\mathcal{S}'}_0 = t_0(qt_1)^{\frac{1}{2}}\mathcal{S}'_0$ and $\widetilde{\mathcal{S}}_1 = (t_0)^{\frac{1}{2}} t_1 \mathcal{S}_1$.

The normalization condition on E_r is that the highest zig-zag degree term z^r has coefficient 1; we refer to this as E_r being zig-zag monic. We introduce the following $\widetilde{\zeta}_{i,r}$ notation for the explicit rescaling factors (r of any sign):

$$\widetilde{\mathcal{S}'}_0 E_r = \widetilde{\zeta'}_{0,-(r+1)} E_{-(r+1)} \qquad \widetilde{\mathcal{S}}_1 E_{-(r+1)} = \widetilde{\zeta}_{1,r+1} E_{r+1}.$$

For $n \geq 0$ we will determine $\widetilde{\zeta'}_{0,-(n+1)}$ and $\widetilde{\zeta}_{1,n}$ below in Proposition 3.7.

In those terms, Theorem 4.1 of [20] for the rank 1 case translates to:

$$E_{-(n+1)} = \left[\widetilde{\zeta'}_{0,-(n+1)}\right]^{-1} \widetilde{\mathcal{S}'}_0 E_n = t_0 \left[\widetilde{\zeta'}_{0,-(n+1)}\right]^{-1} \left[\left(\frac{\widetilde{a}_{-(n+1)}}{t_0}\right)\widetilde{U}_0 + \widetilde{b}_{-(n+1)}\right] E_n \quad (21)$$

$$E_n = \left[\widetilde{\zeta}_{1,n}\right]^{-1} \widetilde{\mathcal{S}}_1 E_{-n} = t_1 \left[\widetilde{\zeta}_{1,n}\right]^{-1} \left[\left(\frac{\widetilde{c}_n}{t_1}\right)\widetilde{T}_1 + \widetilde{d}_n\right] E_{-n} \quad (n \neq 0) \quad (22)$$

where

$$\frac{\widetilde{a}_n}{t_0} = \begin{cases} \dfrac{(abcdq^{2n}|q)_1}{cdq^n} & \text{if } n \geq 0 \\ -\dfrac{1 - abcdq^{-2(n+1)}}{cdq^{-(n+1)}} & \text{if } n < 0 \end{cases}$$

$$\widetilde{b}_n = \begin{cases} abcdq^n\left(\dfrac{1}{c} + \dfrac{1}{d}\right) - (a+b) & \text{if } n \geq 0 \\ q^{-(n+1)}\left(\dfrac{1}{c} + \dfrac{1}{d}\right) - (a+b) & \text{if } n < 0 \end{cases}$$

$$\frac{\widetilde{c}_n}{t_1} = \begin{cases} 0 & \text{if } n = 0 \\ -\dfrac{(abcdq^{2n-1}|q)_1}{abq^n} & \text{if } n > 0 \\ \dfrac{1 - abcdq^{-2n-1}}{abq^{-n}} & \text{if } n < 0 \end{cases}$$

$$\widetilde{d}_n = \begin{cases} \dfrac{q - abcd}{q} & \text{if } n = 0 \\ -\dfrac{(abq^n|q)_1 + \left(abcdq^{n-1}|q\right)_1}{abq^n} & \text{if } n > 0 \\ \dfrac{-cdq^{-n}(ab+1) + (cd+q)}{q} & \text{if } n < 0 \end{cases}$$

Theorem 1.2 in [20] stated this, but as mentioned in [22], has some typos. The formulas in equations (21) and (22) hold for any sign of n above, but we emphasize, in this paper, the $n \geq 0$ cases because they, together with $E_0 = 1$, give a straightforward way to determine the E_r inductively, for r increasing in the zig-zag order sense. And it is easier to prove, as we do later in Proposition 3.7, the (also simpler) formulas for $\widetilde{\zeta'}_{0,-(n+1)}$ and $\widetilde{\zeta}_{1,-n}$ in those $n \geq 0$ cases.

Comment on Some Awkward Looking Factors

$\frac{\widetilde{a}_n}{t_0}$ and $\frac{\widetilde{c}_n}{t_1}$ as well as the t_i factors in relating $\mathcal{S'}_0$ (respectively $\mathcal{S}_1$) to $\frac{\widetilde{a}_n}{t_0} + \widetilde{b}_n$ (respectively $\frac{\widetilde{c}_n}{t_1} + \widetilde{d}_n$) arise to make $\widetilde{a}_n$ and $\widetilde{c}_n$ differences of eigenvalues of $\widetilde{Y}$ just as a_n and c_n are differences of eigenvalues of Y; e.g. $\widetilde{c}_n = \widetilde{\mu}_{-n} - \widetilde{\mu}_n$ in analogy to $c_n = q^{\overline{-n}} - q^{\overline{n}}$ (correcting a typo on page 397 of [20].)

More details on the translation from [20]

The starting point is Theorem 4.2 of [20] using the value $n = 1$ (1 variable case) there. We now review the elements in the proof and application of this theorem.

Besides DAHA identity manipulation, the proof is based on three things:

1. The relations among eigenvalues of $Y = T_1 T_0$ that correspond to the intertwining identities

$$Y\mathcal{S}_1 = \mathcal{S}_1 Y^{-1} \qquad Y\mathcal{S}_0 = q^{-1}\mathcal{S}_0 Y^{-1} \text{ where } \mathcal{S}_0 = q^{\frac{1}{2}} Y \mathcal{S'}_0. \quad (23)$$

2. The creation operator definition, up to normalization, in [19] of the E_r.
3. The fact that the creation operators $\mathcal{S}_0$ and $\mathcal{S}_1$ have squares which, when restricted to their natural invariant 2-dimensional subspaces, are multiples of the identity.

The eigenvalue relations may either be viewed:

1. As consequences of the formula (20) (above) for $q^{\overline{n}}$, originally given in [20].
2. Or as applications of Theorem 5.1 in [19]. In particular, in Theorem 5.1, one can choose $\tilde{\nu} = 1 + 0\delta \in \mathbb{Z} \times \mathbb{Z}\delta$ satisfying

$$s_0(1) = -1 - \delta \qquad s_1(1) = -1.$$

The statements about $\mathcal{S}_0^2$ and $\mathcal{S}_1^2$ are Corollary 5.2 for $n = 1$ in [19]. Here one has to keep in mind that the E_r are eigenvectors of $Y = T_1 T_0$. The following table summarizes many relationships between the original objects in [19], [20] and our notation here.

$\widetilde{T}_0 = t_0^{\frac{1}{2}} T_0$	$\widetilde{T}_1 = t_1^{\frac{1}{2}} T_1$	$\widetilde{Y} = (t_0 t_1)^{\frac{1}{2}} Y$
$U_0 = q^{-\frac{1}{2}} T_0^{-1} X$	$\widetilde{U}_0 = t_0 \widetilde{T}_0^{-1} X$	$\widetilde{U}_0 = (q t_0)^{\frac{1}{2}} U_0$
$U_1 = (T_1 X)^{-1}$	$\widetilde{U}_1 = t_1 (\widetilde{T}_1 X)^{-1}$	$\widetilde{U}_1 = t_1^{\frac{1}{2}} U_1$
$T_1 T_0 U_0 U_1 = q^{-\frac{1}{2}}$	$\widetilde{T}_1 \widetilde{T}_0 \widetilde{U}_0 \widetilde{U}_1 = t_0 t_1$	
$\mathcal{S}_1 = [T_1, Y]$	$\widetilde{\mathcal{S}}_1 = [\widetilde{T}_1, \widetilde{Y}]$	$\widetilde{\mathcal{S}}_1 = t_0^{\frac{1}{2}} t_1 \mathcal{S}_1$
$\mathcal{S}'_0 = [Y, U_0]$	$\widetilde{\mathcal{S}}'_0 = [\widetilde{Y}, \widetilde{U}_0]$	$\widetilde{\mathcal{S}}'_0 = t_0 (q t_1)^{\frac{1}{2}} \mathcal{S}'_0$
		$\mathcal{S}'_0 E_{-(n+1)} = (a_n U_0 + b_n)\, E_{-(n+1)}$
		$\mathcal{S}_1 E_{-n} = (c_n T_1 + d_n)\, E_{-n}$
$\mathcal{S}_0 = [Y, U_1^{-1}]$	$\widetilde{\mathcal{S}}_0 = [\widetilde{Y}, \widetilde{U}_1^{-1}]$	$\widetilde{\mathcal{S}}_0 = (t_0)^{\frac{1}{2}} \mathcal{S}_0$
$\dfrac{\widetilde{a}_n}{t_0} = \left(\dfrac{t_1}{t_0}\right)^{\frac{1}{2}} a_n$	$\widetilde{b}_n = (q t_1)^{\frac{1}{2}} b_n$	$\dfrac{\widetilde{a}_n}{t_0} \widetilde{U}_0 + \widetilde{b}_n = (q t_1)^{\frac{1}{2}} (a_n U_0 + b_n)$
$\dfrac{\widetilde{c}_n}{t_1} = \left(\dfrac{t_0}{t_1}\right)^{\frac{1}{2}} c_n$	$\widetilde{d}_n = t_0^{\frac{1}{2}} d_n$	$\dfrac{\widetilde{c}_n}{t_1} \widetilde{T}_1 + \widetilde{d}_n = t_0^{\frac{1}{2}} (c_n T_1 + d_n)$

To get the a_n and b_n (as in the above table) from Theorem 4.2 in [20] (and then the asserted $\widetilde{a}_n$ and $\widetilde{b}_n$), use $\lambda = \lambda_1 = n$, $\mu = \mu_1 = -(n+1)$. Here the b_n comes from the statement about c_0 in Theorem 4.2.
To get the c_n and d_n (as in the above table) from Theorem 4.2 in [20] (and then the asserted $\widetilde{c}_n$ and $\widetilde{d}_n$), use $\lambda = \lambda_1 = n$, $\mu = \mu_1 = -n$. Here the d_n comes from the statement about (a different) c_n for $n = 1$ in Theorem 4.2.
Expressing everything in terms of a, b, c, d, q, and n (of any sign) via equations (16) gives the asserted formulas for $\widetilde{a}_n$, $\widetilde{b}_n$, $\widetilde{c}_n$, and $\widetilde{d}_n$.

3. Almost symmetric bases and their applications

For any integer n, let $\mathcal{R}_n$ denote the elements of the Laurent polynomials $\mathcal{R}$ of zig-zag degree at most n in the zig-zag order. This gives rise to a filtration

$$\mathcal{R}_0 \subset \mathcal{R}_{-1} \subset \mathcal{R}_1 \subset \mathcal{R}_{-2} \subset \mathcal{R}_2 \subset \dots$$

with both E_n and z^n projecting to the same nonzero generator of the 1-dimensionsal $\mathcal{R}_n/\mathcal{R}_{n-}$. Here $n-$ denotes the predecessor of n in the zig-zag order.
In thinking about the operators $\widetilde{T}_i$, it is natural to relate them to skew-symmetrization. If we denote the involutions by $s_i : \mathcal{R} \to \mathcal{R}$, namely

$$[s_1(f)](z) = f\left(z^{-1}\right) \qquad [s_0(f)](z) = f\left(qz^{-1}\right)$$

and the corresponding skew (respectively q-skew) symmetrizations by

$$\Lambda_1(f) = \frac{1}{2}(1-s_1)(f) \qquad \text{(respectively } \Lambda_0(f) = \frac{1}{2}(1-s_0)(f))$$

then Λ_1 (respectively Λ_0) has eigenfunctions which we denote for $n \geq 0$ as follows:
Eigenvalue $+1$: $f_n = (z+z^{-1})^n$ (resp. $f_{n\_q} = (z+qz^{-1})^n$.)
Eigenvalue -1: $g_n = (z-z^{-1})(z+z^{-1})^{n-1}$ (resp. $g_{n\_q} = (z-qz^{-1})(z+qz^{-1})^{n-1}$.)
Unfortunately neither f_{n+1} nor g_{n+1} belong to $\mathcal{R}_{-(n+1)}$, but their difference does. So for $n \geq 0$ we define

$$h_{n+1} = z^{-1}(z+z^{-1})^n \qquad \text{(respectively } h_{n+1,\_q} = qz^{-1}(z+qz^{-1})^n.)$$

Now $\{f_0, h_1, f_1, h_2, f_2, \ldots\}$ and $\{f_{0\_q}, h_{1\_q}, f_{1\_q}, h_{2\_q}, f_{2\_q}, \ldots\}$ both form bases compatible with the filtration

$$\mathcal{R}_0 \subset \mathcal{R}_{-1} \subset \mathcal{R}_1 \subset \ldots\ .$$

We refer to these as the *almost symmetric* and *almost q-symmetric* bases respectively. The operator $\widetilde{T}_0$ involves division by z^2-q. Easy argument shows that for any Laurent polynomial f, its q-skew symmetrization $\Lambda_0(f)$ is divisible by z^2-q. To understand the operators $\widetilde{T}_0, \widetilde{U}_0$ and $\widetilde{T}_1$ more fully, we define Laurent polynomials f_{skew}, $f_{\text{q\_skew}}$, $f_{\text{skew\_rdcd}}$, $f_{\text{q\_skew\_rdcd}}$, f_{sym}, and $f_{\text{q\_sym}}$, for any Laurent polynomial f by:

$$\begin{aligned} f_{\text{skew}} &= \Lambda_1(f) & f_{\text{q\_skew}} &= \Lambda_0(f) \\ f_{\text{skew}}(z) &= (z^2-1)\left[f_{\text{skew\_rdcd}}(z)\right] & f_{\text{q\_skew}}(z) &= (z^2-q)\left[f_{\text{q\_skew\_rdcd}}(z)\right] \\ f_{\text{sym}} &= \tfrac{1}{2}(1+s_1)(f) & f_{\text{q\_sym}} &= \tfrac{1}{2}(1+s_0)(f)\,. \end{aligned}$$

Thus
$$f = f_{\text{sym}} + f_{\text{skew}} = f_{\text{q\_sym}} + f_{\text{\_skew}}.$$

In terms of these we have:

Proposition 3.1.

$$\begin{aligned} \widetilde{U}_0 f &= \left(\frac{(c+d)z-cd}{z}\right) f - 2q\left(\frac{(z-c)(z-d)}{z}\right) f_{q\_skew\_rdcd} \\ &= -2qzf_{q\_skew\_rdcd} + \left[c+d+\left(\frac{cd}{2q}\right)\left(z-\frac{q}{z}\right)\right. \\ &\quad \left. -\left(\frac{cd}{2q}\right)\left(z+\frac{q}{z}\right)\right](f+2qf_{q\_skew\_rdcd}) \end{aligned} \tag{24}$$

$$\begin{aligned} \widetilde{T}_1 f &= -abf + 2\left[(1-az)(1-bz)\right] f_{skew\_rdcd} \\ &= -abf + \left[-2(a+b)+(ab+1)\left(z+\frac{1}{z}\right)+(ab-1)\left(z-\frac{1}{z}\right)\right](zf_{skew\_rdcd}) \end{aligned} \tag{25}$$

Proof. To establish (24), using the definitions (17) and (19), we start with

$$\left(\widetilde{U}_0 f\right)(z) = \left(\widetilde{T}_0 + (1-t_0)\right)\left(z\,[f(z)]\right)$$
$$= (1-t_0)z\,[f(z)] + t_0\left(z\,[f(z)]\right) + \left(\frac{(z-c)(z-d)}{z^2-q}\right)\left(\left(\frac{q}{z}\right)\left[f\left(\frac{q}{z}\right)\right] - z\,[f(z)]\right)$$

Since $\dfrac{1}{z} - \dfrac{z}{z^2-q} = -\dfrac{q}{z(z^2-q)}$, this gives

$$(\widetilde{U}_0 f)(z) = zf(z) - (z-c)(z-d)\left[\frac{1}{z}f\left(\frac{q}{z}\right) - \frac{z}{z^2-q}\left(f\left(\frac{q}{z}\right) - f(z)\right)\right]$$
$$= z\left[f_{\text{q\_sym}}(z) + f_{\text{q\_skew}}(z)\right]$$
$$- (z-c)(z-d)\left[\frac{1}{z}\left(f_{\text{q\_sym}}(z) - f_{\text{q\_skew}}(z)\right) - \frac{z}{z^2-q}\left(-2f_{\text{q\_skew}}(z)\right)\right]$$
$$= z\left[f_{\text{q\_sym}}(z) + f_{\text{q\_skew}}(z)\right]$$
$$- \frac{(z-c)(z-d)}{z}\left[\left(f_{\text{q\_sym}}(z) - f_{\text{q\_skew}}(z)\right) + \frac{2z^2}{z^2-q}\left(f_{\text{q\_skew}}(z)\right)\right]$$
$$= z\left[f_{\text{q\_sym}}(z) + f_{\text{q\_skew}}(z)\right] - \frac{(z-c)(z-d)}{z}\left[f_{\text{q\_sym}}(z) + \frac{z^2+q}{z^2-q}\left(f_{\text{q\_skew}}(z)\right)\right]$$
$$= z\left[f_{\text{q\_sym}}(z) + f_{\text{q\_skew}}(z)\right]$$
$$- \frac{(z-c)(z-d)}{z}\left[f_{\text{q\_sym}}(z) + f_{\text{q\_skew}}(z) + \frac{2q}{z^2-q}\left(f_{\text{q\_skew}}(z)\right)\right]$$
$$= \left(z - \frac{(z-c)(z-d)}{z}\right)f - \left(2q\frac{(z-c)(z-d)}{z}\right)f_{\text{q\_skew\_rdcd}}(z)$$
$$= \left(\frac{(c+d)z-cd}{z}\right)f - \left(2q\frac{(z-c)(z-d)}{z}\right)f_{\text{q\_skew\_rdcd}}(z)$$

The proof of the second form of (24) comes from further observing

$$\frac{(c+d)z-cd}{z} = c+d+\left(\frac{cd}{2q}\right)\left(z-\frac{q}{z}\right) - \left(\frac{cd}{2q}\right)\left(z+\frac{q}{z}\right)$$

and

$$\frac{(z-c)(z-d)}{z} = z - \left(\frac{(c+d)z-cd}{z}\right).$$

To establish the first form of (25), using the definition (18), we start with

$$\left(\widetilde{T}_1 f\right)(z) = t_1\,[f(z)] + \left(\frac{(1-az)(1-bz)}{(1-z^2)}\right)\left(f\big(\frac{1}{z}\big) - f(z)\right)$$
$$= -ab\,[f(z)] + \left(\frac{(1-az)(1-bz)}{(1-z^2)}\right)\left[-2f_{\text{\_skew}}(z)\right]$$
$$= -ab\,[f(z)] + \left(\frac{(1-az)(1-bz)}{(1-z^2)}\right)\left(-2(z^2-1)\right)\left[f_{\text{\_skew\_rdcd}}(z)\right]$$
$$= -ab\,[f(z)] + 2(1-az)(1-bz)\left[f_{\text{\_skew\_rdcd}}(z)\right]$$

which is the first form of (25). To get the second form, we use

$$2\,\frac{(1-az)(1-bz)}{z} = (ab+1)\big(z+\frac{1}{z}\big) + (ab-1)\big(z-\frac{1}{z}\big) - 2(a+b). \qquad \blacksquare$$

An immediate corollary is that $\widetilde{U}_0$ and $\widetilde{T}_1$ behave very nicely on our filtered bases:

Corollary 3.2. $\widetilde{U}_0(f_{n\_q}) = -\left(\frac{cd}{q}\right) h_{n+1,\_q} + (c+d) f_{n\_q}, \quad \widetilde{U}_0(h_{n\_q}) = q f_{n-1,\_q},$

$$\widetilde{T}_1(f_n) = -ab f_n\,, \quad \widetilde{T}_1(h_n) = -ab f_n - h_n + (a+b) f_{n-1}\,.$$

Proof. (1) For $f = f_{n\_\mathrm{q}} = \left(z + \frac{q}{z}\right)^n$, we have $f_{\mathrm{q\_skew\_rdcd}} = 0$. So

$$\widetilde{U}_0(f) = ((c+d) + \left(\frac{cd}{2q}\right)\left(z - \frac{q}{z}\right) - \left(\frac{cd}{2q}\right)\left(z + \frac{q}{z}\right)) f_{n\_\mathrm{q}}$$
$$= \left((c+d) - \left(\frac{2q}{z}\right)\left(\frac{cd}{2q}\right)\right) f_{n\_\mathrm{q}} = (c+d) f_{n\_\mathrm{q}} - \left(\frac{cd}{q}\right) h_{n+1,\_\mathrm{q}}.$$

(2) For $f = h_{n\_\mathrm{q}} = \left(\frac{q}{z}\right)\left(z + \frac{q}{z}\right)^{n-1}$,

$$f_{\mathrm{q\_skew}} = \left(\frac{1}{2}\right)\left(\frac{q}{z} - z\right)\left(z + \frac{q}{z}\right)^{n-1} = -\left(\frac{1}{2z}\right)(z^2 - q)\left(z + \frac{q}{z}\right)^{n-1}$$

So $\quad f_{\mathrm{q\_skew\_rdcd}} = -\left(\frac{1}{2z}\right)\left(z + \frac{q}{z}\right)^{n-1} = -\left(\frac{1}{2q}\right) h_{n\_\mathrm{q}} = -\frac{f}{2q}.$

Thus $f + 2q f_{\mathrm{q\_skew\_rdcd}} = 0$ and

$$z f_{\mathrm{q\_skew\_rdcd}} = -\left(\frac{1}{2}\right)\left(z + \frac{q}{z}\right)^{n-1} = -\left(\frac{1}{2}\right) f_{n-1,\_\mathrm{q}}.$$

This gives us $\quad \widetilde{U}_0(h_{n\_\mathrm{q}}) = -2q\big(-\frac{1}{2} f_{n-1,\_\mathrm{q}}\big) + 0 = q f_{n-1,\_\mathrm{q}}\,.$

(3) For $f = f_n = \big(z + \frac{1}{z}\big)^n$, we have $f_{\mathrm{skew\_rdcd}} = 0$. So by (25), $\widetilde{T}_1 f_n = -ab f_n\,.$

(4) For $f = h_n = \big(\frac{1}{z}\big)\big(z + \frac{1}{z}\big)^n$,

$$f_{\mathrm{skew}} = \left(\frac{1}{2}\right)\left(\frac{1}{z} - z\right)\left(z + \frac{1}{z}\right)^{n-1} = -\left(\frac{1}{2z}\right)(z^2 - 1)\left(z + \frac{1}{z}\right)^{n-1}.$$

So $f_{\mathrm{skew\_rdcd}} = -\left(\frac{1}{2z}\right)\left(z + \frac{1}{z}\right)^{n-1} = -\frac{h_n}{2}$ and $z f_{\mathrm{skew\_rdcd}} = -\frac{f_{n-1}}{2}\,.$

Thus by (25),

$$\widetilde{T}_1 h_n = -(ab) h_n + \Big\{ -2(a+b) + 2ab\left(z + \frac{1}{z}\right) - 2(ab-1)\left(\frac{1}{z}\right)\Big\}\left(-\frac{f_{n-1}}{2}\right)$$
$$= -(ab) h_n + (a+b) f_{n-1} - (ab) f_n + (ab-1) h_n$$
$$= -(ab) f_n - h_n + (a+b) f_{n-1}. \qquad \blacksquare$$

Remark 3.3. Corollary 3.2 shows, for $n \geq 0$:

(1) $\widetilde{U}_0$ maps $\mathcal{R}_n$ to $\mathcal{R}_{-(n+1)}$ and $\mathcal{R}_{-(n+1)}$ to $\mathcal{R}_{-(n+1)}$.

(2) $\widetilde{T}_1$ maps $\mathcal{R}_{-(n+1)}$ to $\mathcal{R}_{n+1}$ and $\mathcal{R}_n$ to $\mathcal{R}_n$. $\blacksquare$

Because of Corollary 3.2 and the recursion (21) and (22), we will need to convert between the almost symmetric bases in low zig-zag co-degree.

Proposition 3.4.

$$f_n = f_{n\_q} + (q^{-n} - 1)h_{n\_q} \ mod\ \mathcal{R}_{-(n-1)} \qquad h_n = q^{-n}h_{n\_q} \ mod\ \mathcal{R}_{-(n-1)}$$

$$h_{n+1\_q} = q^{n+1}h_{n+1} \ mod\ \mathcal{R}_{n-1} \qquad f_{n\_q} = f_n + (q^n - 1)h_n \ mod\ \mathcal{R}_{n-1}$$

Proof.

$$h_n = z^{-1}(z + z^{-1})^{n-1} = z^{-n} = q^{-n}h_{n\_\mathrm{q}} \bmod \mathcal{R}_{-(n-1)}$$

$$\begin{aligned} f_n &= (z + z^{-1})^n = z^n + z^{-n} = z^n + q^n z^{-n} + (1 - q^n)z^{-n} \\ &= z^n + q^n z^{-n} + (q^{-n} - 1)q^n z^{-n} \\ &= f_{n\_\mathrm{q}} + (q^{-n} - 1)h_{n\_\mathrm{q}} \bmod \mathcal{R}_{-(n-1)} \end{aligned}$$

$$h_{n+1\_\mathrm{q}} = qz^{-1}(z + qz^{-1})^n = q^{n+1}z^{-(n+1)} = q^{n+1}h_{n+1} \bmod \mathcal{R}_{n-1}$$

$$\begin{aligned} f_{n+1\_\mathrm{q}} &= (z + qz^{-1})^{n+1} = z^{n+1} + q^{n+1}z^{-(n+1)} \\ &= z^{n+1} + z^{-(n+1)} + (q^{n+1} - 1)z^{-(n+1)} \\ &= f_{n+1} + (q^{n+1} - 1)h_{n+1} \bmod \mathcal{R}_{n-1}. \end{aligned}$$

■

The almost symmetric bases allow us to make the rescalings involved in the recursive computation of the E_n explicit. This also makes the rationality of the formulas transparent.

Since the recursion in equations (21) and (22) is in terms of $\widetilde{U}_0$ and $\widetilde{T}_1$, Corollary 3.2 makes an expansion of the E_r in terms of the almost symmetric bases natural. For $n \geq 0$, we will use the notation (remembering e.g. $h_{m+1} \in \mathcal{R}_{-(m+1)}$):

$$\begin{aligned} E_n &= \sum_{m=0}^{n} \lambda_{m,n} f_m + \sum_{m=0}^{n-1} \mu_{-(m+1),n} h_{m+1} \\ E_n &= \sum_{m=0}^{n} \lambda_{m,n\_\mathrm{q}} f_{m\_\mathrm{q}} + \sum_{m=0}^{n-1} \mu_{-(m+1),n} h_{m+1,\_\mathrm{q}} \\ E_{-(n+1)} &= \sum_{m=0}^{n} \lambda_{-(m+1),-(n+1)} h_{m+1} + \sum_{m=0}^{n} \mu_{m,-(n+1)} f_m \\ E_{-(n+1)} &= \sum_{m=0}^{n} \lambda_{-(m+1),-(n+1)\_\mathrm{q}} h_{m+1\_\mathrm{q}} + \sum_{m=0}^{n} \mu_{m,-(n+1)\_\mathrm{q}} f_{m\_\mathrm{q}}. \end{aligned}$$

Using Corollary 3.4, we quickly see

Proposition 3.5. *For* $n \geq 0$:

$$\begin{aligned} \lambda_{-(n+1),-(n+1)} &= \lambda_{-(n+1),-(n+1)\_q} q^{n+1} \\ \mu_{n,-(n+1)} &= \mu_{n,-(n+1)\_q} \\ \lambda_{-n,-(n+1)} &= \mu_{n,-(n+1)\_q}(q^n - 1) + \lambda_{-n,-(n+1)\_q} q^n \ (n \geq 1) \\ \lambda_{n,n\_q} &= \lambda_{n,n} \\ \mu_{-(n+1),n+1\_q} &= \lambda_{n+1,n+1}(q^{-(n+1)} - 1) + \mu_{-(n+1),n+1} q^{-(n+1)} \\ \lambda_{n,n+1\_q} &= \lambda_{n,n+1}. \end{aligned}$$

Proof. Modulo $\mathcal{R}_{n-1}$:

$$\begin{aligned}
E_{-(n+1)} &= \lambda_{-(n+1),-(n+1)_q} h_{n+1,_q} + \mu_{n,-(n+1)_q} f_{n,_q} + \lambda_{-n,-(n+1)_q} h_{n,_q} \\
&= \lambda_{-(n+1),-(n+1)_q}\left[q^{n+1} h_{n+1}\right] + \mu_{n,-(n+1)_q}\left[f_n + (q^n - 1) h_n\right] \\
&\quad + \lambda_{-n,-(n+1)_q}\left[q^n h_n\right] \\
&= \left[\lambda_{-(n+1),-(n+1)_q} q^{n+1}\right] h_{n+1} + \left[\mu_{n,-(n+1)_q}\right] f_n \\
&\quad + \left[\mu_{n,-(n+1)_q}(q^n - 1) + \lambda_{-n,-(n+1)_q} q^n\right] h_n
\end{aligned}$$

from which we read off the first three statements.

Modulo $\mathcal{R}_{-(n-1)}$:

$$\begin{aligned}
E_n &= \lambda_{n,n} f_n + \mu_{-n,n} h_n + \lambda_{n-1,n} f_{n-1} \\
&= \lambda_{n,n}\left[f_{n_q} + (q^{-n} - 1) h_{n_q}\right] + \mu_{-n,n}\left[q^{-n} h_{n_q}\right] + \lambda_{n-1,n}\left[f_{n-1,_q}\right] \\
&= \left[\lambda_{n,n}\right] f_{n_q} + \left[\lambda_{n,n}(q^{-n} - 1) + \mu_{-n,n} q^{-n}\right] h_{n_q} + \left[\lambda_{n-1,n}\right] f_{n-1,_q}
\end{aligned}$$

from which we read off the last three statements. ■

The zig-zag monic condition on the E_r immediately implies

Proposition 3.6. *For any $n \geq 0$:*

$$\lambda_{n,n} = 1, \quad \lambda_{n,n\_q} = 1, \quad \lambda_{-(n+1),-(n+1)} = 1, \quad \lambda_{-(n+1),-(n+1)\_q} = q^{-(n+1)}.$$

For our purpose, we need the explicit scaling factors arising in the zig-zag increasing cases of the creation operators acting on E_r. We use the following notation for the exact coefficients:

$$\begin{array}{lll}
(\mathrm{E}_{\text{negative}} \text{ case}) & E_{-(n+1)} = & \left(\hat{a}_{-(n+1)} \widetilde{U}_0 + \hat{b}_{-(n+1)}\right) E_n \\
(\mathrm{E}_{\text{positive}} \text{ case}) & E_n = & \left(\hat{c}_n \widetilde{T}_1 + \hat{d}_n\right) E_{-n}
\end{array} \tag{26}$$

Proposition 3.7. *For $n \geq 0$:*

$$\begin{aligned}
\hat{a}_{-(n+1)} &= -\frac{1}{cdq^n} & \hat{c}_n &= -\frac{1}{ab} \\
\widetilde{\zeta}'_{0,-(n+1)} &= -cdq^{-1}(abcdq^{2n}|q)_1 & \widetilde{\zeta}_{1,n} &= -abq^{-n}(abcdq^{2n-1}|q)_1 \\
\hat{b}_{-(n+1)} &= \frac{(c+d) - cdq^n(a+b)}{cdq^n(abcdq^{2n}|q)_1} & \hat{d}_n &= -\frac{(abq^n|q)_1 + ab(cdq^{n-1}|q)_1}{ab(1 - abcdq^{2n-1})}.
\end{aligned}$$

Proof. We use the combination of the creation operator point of view with Corollary 3.2.

(1) For the $\hat{c}_n$ and $\widetilde{\zeta}_{1,n}$ determination, we start with

$$E_n = (\hat{c}_n \widetilde{T}_1 + \hat{d}_n) E_{-n} = \hat{c}_n \widetilde{T}_1 h_n \bmod \mathcal{R}_{-n} = -ab\hat{c}_n f_n \bmod \mathcal{R}_{-n}$$

Since $E_n = f_n \bmod \mathcal{R}_{-n}$, this gives the asserted formula for $\hat{c}_n$.

This also means $\widetilde{T}_1 E_{-n} = [\hat{c}_n]^{-1} E_n \bmod E_{-n}$.

Then
$$\begin{aligned}
\widetilde{\zeta}_{1,n} E_n = \widetilde{\mathcal{S}}_1 E_{-n} = [\widetilde{T}_1, \widetilde{Y}] E_{-n} &= (\widetilde{\mu}_{-n} - \widetilde{\mu}_n)\left(\widetilde{T}_1 E_{-n}\right) \bmod E_{-n} \\
&= (\widetilde{\mu}_{-n} - \widetilde{\mu}_n)\,[\hat{c}_n]^{-1} E_n
\end{aligned}$$

gives the $\widetilde{\zeta}_{1,n}$ determination.

(2) For the $\hat{a}_{-(n+1)}$ and $\widetilde{\zeta'}_{0,-(n+1)}$ determination, we start with

$$\begin{aligned} E_{-(n+1)} &= \left(\hat{a}_{-(n+1)}\widetilde{U}_0 + \hat{b}_{-(n+1)}\right)E_n = \hat{a}_{-(n+1)}\widetilde{U}_0 f_{n\_\mathrm{q}} \bmod E_n \\ &= \hat{a}_{-(n+1)}\left(-\frac{cd}{q}\right) h_{n+1\_\mathrm{q}} \bmod \mathcal{R}_n = -cdq^n \hat{a}_{-(n+1)} h_{n+1} \bmod \mathcal{R}_n. \end{aligned}$$

Since $E_{-(n+1)} = h_{n+1} \bmod \mathcal{R}_n$, this gives the asserted formula for $\hat{a}_{-(n+1)}$. This also means

$$\widetilde{U}_0 E_n = \left[\hat{a}_{-(n+1)}\right]^{-1} E_{-(n+1)} \bmod E_n.$$

Then

$$\begin{aligned} \widetilde{\zeta'}_{0,-(n+1)} E_{-(n+1)} &= \widetilde{\mathcal{S}'}_0 E_n = [\widetilde{Y}, \widetilde{U}_0]\, E_n = \left(\widetilde{\mu}_{-(n+1)} - \widetilde{\mu}_n\right)\left(\widetilde{U}_0 E_n\right) \bmod E_n \\ &= \left(\widetilde{\mu}_{-(n+1)} - \widetilde{\mu}_n\right)\left[\hat{a}_{-(n+1)}\right]^{-1} E_{-(n+1)} \end{aligned}$$

gives the $\widetilde{\zeta'}_{0,-(n+1)}$ determination.

(3) The $\hat{b}_{-(n+1)}$ and $\hat{d}_n$ formulas come from equations (21) and (22) together with the previously translated formulas from [20] for $\widetilde{b}_{-(n+1)}$ and $\widetilde{d}_n$. ■

In this paper, we only use the zig-zag increasing cases above and so have included just the proofs of those. However Appendix B includes a table with the zig–zag decreasing cases as well. (Formulas (4.11) and (4.12) of [11] could be appealed to since they are equivalent to the determination of the $\hat{c}_r, \hat{d}_r$ for any sign of r.) We mention the simplified forms of those others as well because they may be of interest. They may be established, e.g. in the $\hat{a}_n, \hat{b}_n$ case, either using Corollary 5.2 of [19] about $\mathcal{S}_i^2$ or by noting that the relations

$$E_{-(n+1)} = \left(\hat{a}_{-(n+1)}\widetilde{U_0} + \hat{b}_{-(n+1)}\right)E_n \qquad\qquad E_n = \left(\hat{a}_n\widetilde{U_0} + \hat{b}_n\right)E_{-(n+1)}$$

are inverse to each other. (Corollary 3.2 also clarifies what happens in low zig-zag co-degree.)

If using Corollary 5.2 of [19], one might first confirm, by thinking about the DAHA relation, that $t_i^{\frac{1}{2}}$ and $u_i^{\frac{1}{2}}$ are mapped by the involution ϵ of that paper to what one might guess from its action on T_i and U_i. That then makes immediate what $\epsilon(a)$ and $\epsilon(c)$ are. Then the automorphism property, together with $t_0 = -cdq^{-1}$ and $t_1 = -ab$ gives the needed $\epsilon(b)$ and $\epsilon(d)$.

An immediate corollary of Proposition 3.7 and recursion relations (26) is the $\boldsymbol{\mathcal{T}}(a,e)$ part of the first step of (15), Proof Plan A :

Corollary 3.8. *The entries of both $T(a,e)$ and $\boldsymbol{\mathcal{T}}(a,e)$ are rational functions of e with coefficients in the filed $\mathbb{Q}(a,b,c,d,q)$.*

(Rationality of entries of $T(a,e)$ is immediate from the formulas written down in theorem 1.3, since $T(a,e)$ just refers to the matrix given by the formulas (37), perse.)

We now work out the details of the low zig-zag co-degree expansion of the nonsymmetric polynomials in terms of the almost symmetric bases.

The recursive relations (26) with exact scaling factors implies the following for the low zig-zag co-degree almost symmetric basis coefficients:

Proposition 3.9. *For* $n \geq 0$:

$$
\begin{aligned}
\lambda_{n+1,n+1} &= \lambda_{-(n+1),-(n+1)} \left[-ab\hat{c}_{n+1}\right] \\
\lambda_{-(n+1),-(n+1)\_q} &= \lambda_{nn\_q} \left[\Big(-\frac{cd}{q}\Big)\hat{a}_{-(n+1)}\right] \\
\mu_{-(n+1),n+1} &= \lambda_{-(n+1),-(n+1)} \left[-\hat{c}_{n+1} + \hat{d}_{n+1}\right] \\
\mu_{n,-(n+1)\_q} &= \lambda_{nn\_q} \left[(c+d)\,\hat{a}_{-(n+1)} + \hat{b}_{-(n+1)}\right] \\
\lambda_{n,n+1} &= \lambda_{-(n+1),-(n+1)} \left[(a+b)\hat{c}_{n+1}\right] + \mu_{n,-(n+1)} \left[(-ab)\hat{c}_{n+1} + \hat{d}_{n+1}\right] \\
&\quad + \lambda_{-n,-(n+1)} \left[-ab\hat{c}_{n+1}\right] \\
\lambda_{-n,-(n+1)\_q} &= \mu_{-n,n\_q}\hat{b}_{-(n+1)} + \lambda_{n-1,n\_q} \left[\Big(-\frac{cd}{q}\Big)\hat{a}_{-(n+1)}\right]
\end{aligned}
$$

Proof. For the first 2 assertions on the left and the next to last one, recall that our standard notation is:

$$
\begin{aligned}
E_{n+1} &= \lambda_{n+1,n+1} f_{n+1} + \mu_{-(n+1),n+1} h_{n+1} + \lambda_{n,n+1} f_n \\
E_{-(n+1)} &= \lambda_{-(n+1),-(n+1)} h_{n+1} + \mu_{n,-(n+1)} f_n + \lambda_{-n,-(n+1)} h_n.
\end{aligned}
$$

So

$$
\begin{aligned}
&\big(\hat{c}_{n+1}\widetilde{T}_1 + \hat{d}_{n+1}\big)E_{-(n+1)} \bmod \mathcal{R}_{-n} \\
&\quad = \lambda_{-(n+1),-(n+1)} \left[\hat{c}_{n+1}\left[-abf_{n+1} - h_{n+1} + (a+b)f_n\right] + \hat{d}_{n+1}h_{n+1}\right] \\
&\qquad + \mu_{n,-(n+1)} \left[\hat{c}_{n+1}\left[-abf_n\right] + \hat{d}_{n+1}f_n\right] + \lambda_{-n,-(n+1)} \left[\hat{c}_{n+1}\left[-abf_n\right]\right] \\
&\quad = f_{n+1}\left[\lambda_{-(n+1),-(n+1)}\left[-ab\hat{c}_{n+1}\right]\right] \\
&\qquad + h_{n+1}\left[\lambda_{-(n+1),-(n+1)}\left[-\hat{c}_{n+1} + \hat{d}_{n+1}\right]\right] \\
&\qquad + f_n\Big[\lambda_{-(n+1),-(n+1)}\left[(a+b)\hat{c}_{n+1}\right] \\
&\qquad + \mu_{n,-(n+1)}\left[-ab\hat{c}_{n+1} + \hat{d}_{n+1}\right] + \lambda_{-n,-(n+1)}\left[-ab\hat{c}_{n+1}\right]\Big]
\end{aligned}
$$

from which we can read off the three results.
For the other 3 assertions, start with our standard notation of:

$$
\begin{aligned}
E_{-(n+1)} &= \lambda_{-(n+1),-(n+1)\_\mathrm{q}} h_{n+1\_\mathrm{q}} + \mu_{n,-(n+1)\_\mathrm{q}} f_{n\_\mathrm{q}} + \lambda_{-n,-(n+1)\_\mathrm{q}} h_{n\_\mathrm{q}} \\
E_n &= \lambda_{n,n\_\mathrm{q}} f_{n\_\mathrm{q}} + \mu_{-n,n\_\mathrm{q}} h_{n\_\mathrm{q}} + \lambda_{n-1,n\_\mathrm{q}} f_{n-1,\_\mathrm{q}}.
\end{aligned}
$$

So

$$
\begin{aligned}
&\big(\hat{a}_{-(n+1)}\widetilde{U}_0 + \hat{b}_{-(n+1)}\big)E_n \bmod \mathcal{R}_{n-1} \\
&= \lambda_{n,n\_\mathrm{q}}\left[\hat{a}_{-(n+1)}\left[\Big(-\frac{cd}{q}\Big)h_{n+1\_\mathrm{q}} + (c+d)f_{n\_\mathrm{q}}\right] + \hat{b}_{-(n+1)}f_{n\_\mathrm{q}}\right] \\
&\quad + \mu_{-n,n\_\mathrm{q}}\left[\hat{a}_{-(n+1)}\left[qf_{n-1\_\mathrm{q}}\right] + \hat{b}_{-(n+1)}h_{n\_\mathrm{q}}\right] + \lambda_{n-1,n\_\mathrm{q}}\left[\hat{a}_{-(n+1)}\left[\Big(-\frac{cd}{q}\Big)h_{n\_\mathrm{q}}\right]\right] \\
&= h_{n+1\_\mathrm{q}}\left[\lambda_{n,n\_\mathrm{q}}\left[\Big(-\frac{cd}{q}\Big)\hat{a}_{-(n+1)}\right]\right] + f_{n\_\mathrm{q}}\left[\lambda_{n,n\_\mathrm{q}}\left[(c+d)\hat{a}_{-(n+1)} + \hat{b}_{-(n+1)}\right]\right] \\
&\quad + h_{n\_\mathrm{q}}\left[\mu_{-n,n\_\mathrm{q}}\left[\hat{b}_{-(n+1)}\right] + \lambda_{n-1,n\_\mathrm{q}}\left[\Big(-\frac{cd}{q}\Big)\hat{a}_{-(n+1)}\right]\right]
\end{aligned}
$$

from which we can read off the other three results. ∎

Proposition 3.10. *For $n \geq 0$:*

$$\mu_{n,-(n+1)} = \frac{abq^n(c+d)-(a+b)}{(abcdq^{2n}|q)_1} = \mu_{n,-(n+1)\_q} \tag{27}$$

$$\mu_{-(n+1),n+1} = -\frac{(q^{n+1},cdq^n|q)_1}{(abcdq^{2n+1}|q)_1} \tag{28}$$

$$\mu_{-(n+1),n+1\_q} = \frac{cd(q^{n+1},abq^{n+1}|q)_1}{q(abcdq^{2n+1}|q)_1}. \tag{29}$$

Proof. By Proposition 3.9 we have

$$\begin{aligned}\mu_{n,-(n+1)\_\mathrm{q}} &= \lambda_{n,n\_\mathrm{q}}\left[(c+d)\,\hat{a}_{-(n+1)}+\hat{b}_{-(n+1)}\right]\\ &= (c+d)\left(-\frac{1}{cdq^n}\right)+\frac{(c+d)-cdq^n(a+b)}{cdq^n(abcdq^{2n}|q)_1}\\ &= \frac{(c+d)abcdq^{2n}-cdq^n(a+b)}{cdq^n(abcdq^{2n}|q)_1},\end{aligned} \tag{30}$$

which equals (27) above. And by Proposition 3.5 this is also the value of $\mu_{n,-(n+1)}$. Similarly

$$\begin{aligned}\mu_{-(n+1),n+1} &= \lambda_{-(n+1),-(n+1)}\left[-\hat{c}_{n+1}+\hat{d}_{n+1}\right]\\ &= -\left(-\frac{1}{ab}\right)+\left(-\frac{(abq^{n+1}|q)_1+ab(cdq^n|q)_1}{ab(1-abcdq^{2n+1})}\right)\\ &= \frac{-abcdq^{2n+1}+abq^{n+1}-ab+abcdq^n}{ab(1-abcdq^{2n+1})} = -\frac{(1-q^{n+1})(1-cdq^n)}{(1-abcdq^{2n+1})}\end{aligned} \tag{31}$$

as in (28) above. And by Proposition 3.5

$$\begin{aligned}\mu_{-(n+1),n+1\_\mathrm{q}} &= \lambda_{n+1,n+1}(q^{-(n+1)}-1)+\mu_{-(n+1),n+1}q^{-(n+1)}\\ &= (q^{-(n+1)}-1)+q^{-(n+1)}\left(-\frac{(1-q^{n+1})(1-cdq^n)}{(1-abcdq^{2n+1})}\right)\\ &= \left(\frac{1-q^{n+1}}{q^{n+1}(1-abcdq^{2n+1})}\right)\left(-abcdq^{2n+1}+cdq^n\right),\end{aligned}$$

which is equal to (29) above. ■

Theorem 3.11. *The zig-zag co-degree 2 coefficients of the $E_r(a)$ are given, for $n \geq 1$ by:*

$$\lambda_{-n,-(n+1)\_q} = -\frac{(c+d)(q^n,abq^n|q)_1+(a+b)(q^n,cdq^n|q)_1}{q^n(q,abcdq^{2n}|q)_1}$$

$$\lambda_{-n,-(n+1)} = -\frac{(c+d)(q^n,abq^{n+1}|q)_1+q(a+b)(q^n,cdq^{n-1}|q)_1}{(q,abcdq^{2n}|q)_1}$$

$$\lambda_{n-1,n} = \lambda_{n-1,n\_q} = -\frac{(c+d)(q^n,abq^n|q)_1+q(a+b)(q^n,cdq^{n-1}|q)_1}{(q,abcdq^{2n-1}|q)_1}.$$

Proof. We prove these $\lambda_{s,r}$ and λ_{s,r_q} formulas by induction on zig-zag degree r. Note in the case $r=-1$, $\lambda_{-0,-1}$ is not even defined (there is a constant term $\mu_{0,-1}$ in E_{-1}), so we can treat it as 0 by convention; we are not assuming anything from this proposition in proving the $r=1$ case and so view this case as the start of the zig-zag induction.

Assuming correctness for $r = -(n+1)$, we establish the $r = n+1 \geq 1$ case: (The first case to prove is $s = 0,\ r = 1$).

$$\begin{aligned}
\lambda_{n,n+1} &= \lambda_{-(n+1),-(n+1)}\left[(a+b)\hat{c}_{n+1}\right] + \mu_{n,-(n+1)}\left[(-ab)\hat{c}_{n+1} + \hat{d}_{n+1}\right] \\
&\quad + \lambda_{-n,-(n+1)}\left[-ab\hat{c}_{n+1}\right] \\
&= (1)\left[(a+b)\left[-\frac{1}{ab}\right]\right] + \left(\frac{abq^n(c+d)-(a+b)}{(abcdq^{2n}|q)_1}\right)\cdot \\
&\quad \cdot \left\{\left[(-ab)\left[-\frac{1}{ab}\right]\right] - \frac{(abq^{n+1}|q)_1 + ab(cdq^n|q)_1}{ab(1-abcdq^{2n+1})}\right\} \\
&\quad + \left(-\frac{(c+d)(q^n, abq^{n+1}|q)_1 + q(a+b)(q^n, cdq^{n-1}|q)_1}{(q, abcdq^{2n}|q)_1}\right)\left[(-ab)\left[-\frac{1}{ab}\right]\right] \qquad (32)
\end{aligned}$$

The top line of the right hand side of the last equals sign combines to

$$\begin{aligned}
&\left(-\frac{1}{ab(abcdq^{2n}, abcdq^{2n+1}|q)_1}\right)\Big\{(a+b)(1-abcdq^{2n})(1-abcdq^{2n+1}) \\
&\quad + [abq^n(c+d)-(a+b)]\big\{-ab(1-abcdq^{2n+1}) + (1-abq^{n+1}) + ab(1-cdq^n)\big\}\Big\} \\
&= \left(-\frac{1}{ab(abcdq^{2n}, abcdq^{2n+1}|q)_1}\right)\Big\{(a+b)(1-abcdq^{2n})(1-abcdq^{2n+1}) \\
&\quad + [abq^n(c+d)-(a+b)]\big\{(1-abq^{n+1})(1-abcdq^n)\big\}\Big\} \qquad (33) \\
&= \left(-\frac{1}{ab(abcdq^{2n}, abcdq^{2n+1}|q)_1}\right)\Big\{(a+b)\big\{(1-abcdq^{2n})(1-abcdq^{2n+1}) \\
&\quad - (1-abq^{n+1})(1-abcdq^n)\big\} + (c+d)\big\{abq^n(1-abq^{n+1})(1-abcdq^n)\big\}\Big\}. \qquad (34)
\end{aligned}$$

For $n = 0$ (the start of the induction), this is all there is for $\lambda_{0,1}$, and we note this simplifies to the asserted formula. Continuing for $n \geq 1$, combining equation (34) with the bottom line of (32) gives:

$$\begin{aligned}
&\left(-\frac{1}{ab(q, abcdq^{2n}, abcdq^{2n+1}|q)_1}\right)\Big\{(a+b)\big\{(1-q)(1-abcdq^{2n})(1-abcdq^{2n+1}) \qquad (35) \\
&\quad - (1-q)(1-abq^{n+1})(1-abcdq^n) + abq(1-abcdq^{2n+1})(q^n, cdq^{n-1}|q)_1)\big\} \\
&\quad + (c+d)\big\{(1-q)abq^n(1-abq^{n+1})(1-abcdq^n) + ab(1-abcdq^{2n+1})(q^n, abq^{n+1}|q)_1)\big\}\Big\}.
\end{aligned}$$

We view (35) as $\left(-\frac{1}{ab(q, abcdq^{2n}, abcdq^{2n+1}|q)_1}\right)\Big\{(a+b)p_1 + (c+d)p_2\Big\}$

where p_1 and p_2 are polynomials. We will finish this step in the inductive proof of the $\lambda_{n,n+1}$ formula in the theorem (with n replaced by $n+1$) by showing

$$p_1 = abq(1-abcdq^{2n})\left[(q^{n+1}, cdq^n|q)_1\right] \qquad p_2 = ab(1-abcdq^{2n})\left[(q^{n+1}, abq^{n+1}|q)_1\right].$$

To verify the p_2 claim:

$$\begin{aligned}
p_2 &= (1-q)abq^n(1-abq^{n+1})(1-abcdq^n) + ab(1-abcdq^{2n+1})(q^n, abq^{n+1}|q)_1) \\
&= (abq^{n+1}|q)_1\big\{(1-q)abq^n(1-abcdq^n) + ab(1-abcdq^{2n+1})(1-q^n)\big\} \\
&\qquad \text{(because the } abq^n \text{ and } a^2b^2cdq^{2n+1} \text{ terms cancel)} \\
&= (abq^{n+1}|q)_1\big\{-abq^{n+1} - a^2b^2cdq^{2n} + ab + a^2b^2cdq^{3n+1}\big\}
\end{aligned}$$

$$= ab(abq^{n+1}|q)_1\big\{(1-q^{n+1})(1-abcdq^{2n})\big\}$$

as asserted. The p_1 simplification is a little more involved:

$$\begin{aligned}
p_1 &= (1-q)(1-abcdq^{2n})(1-abcdq^{2n+1}) - (1-q)(1-abq^{n+1})(1-abcdq^{n}) \\
&\quad + abq(1-abcdq^{2n+1})(q^n, cdq^{n-1}|q)_1) \\
&= (1-q)(1-abcdq^{2n})(1-abcdq^{2n+1}) \\
&\quad - (1-q)(1-abq^{n+1})(1-abcdq^{2n}+abcdq^{2n}-abcdq^{n}) \\
&\quad + abq(1-abcdq^{2n}+abcdq^{2n}-abcdq^{2n+1})(q^n, cdq^{n-1}|q)_1) \\
&= (1-abcdq^{2n})\big\{(1-q)(1-abcdq^{2n+1}) - (1-q)(1-abq^{n+1}) \\
&\quad + abq(1-q^n)(1-cdq^{n-1})\big\} - (1-q)(1-abq^{n+1})\,[abcdq^n(q^n-1)] \\
&\quad + \left[a^2b^2cdq^{2n+1}(1-q)\right](1-q^n)(1-cdq^{n-1}) \\
&= (1-abcdq^{2n})\big\{(1-q)abq^{n+1}(1-cdq^n) + abq(1-q^n)(1-cdq^{n-1})\big\} \\
&\quad + (1-q)abcdq^n(1-q^n)\big\{(1-abq^{n+1}) + abq^{n+1}(1-cdq^{n-1})\big\} \\
&= (1-abcdq^{2n})\big\{(1-q)abq^{n+1}(1-cdq^n) \\
&\quad + abq(1-q^n)(1-cdq^{n-1}) + abcdq^n(1-q)(1-q^n)\big\} \\
&= (1-abcdq^{2n})\big\{(1-q)abq^{n+1}(1-cdq^n) + abq(1-q^n)\big[(1-cdq^{n-1}) + cdq^{n-1}(1-q)\big]\big\} \\
&= (1-abcdq^{2n})(1-cdq^n)\big\{(1-q)abq^{n+1} + abq(1-q^n)\big\} \\
&= abq(1-abcdq^{2n})(1-cdq^n)\big\{1-q^{n+1}\big\}
\end{aligned}$$

as claimed. This finishes showing that the the zig-zag degree $-(n+1)$ case implies the asserted $\lambda_{n,n+1}$ formula.

We have $\lambda_{n,n+1} = \lambda_{n,n+1\_\mathrm{q}}$ by Proposition 3.5 finishing induction starting as well as going from $r = -(n+1)$ to $r = n+1$.

Assuming correctness for $r = n \geq 1$, we now establish the $r = -(n+1)$ case:

By Proposition 3.9

$$\begin{aligned}
\lambda_{-n,-(n+1)_\mathrm{q}} &= \mu_{-n,n_\mathrm{q}}\hat{b}_{-(n+1)} + \lambda_{n-1,n_\mathrm{q}}\left[\left(-\frac{cd}{q}\right)\hat{a}_{-(n+1)}\right] \\
&= \left(\frac{cd(q^n, abq^n|q)_1}{q(abcdq^{2n-1}|q)_1}\right)\left[\frac{(c+d)-cdq^n(a+b)}{cdq^n(abcdq^{2n}|q)_1}\right] \\
&\quad + \left(-\frac{(c+d)(q^n, abq^n|q)_1 + q(a+b)(q^n, cdq^{n-1}|q)_1}{(q, abcdq^{2n-1}|q)_1}\right)\left[\left(-\frac{cd}{q}\right)\left(-\frac{1}{cdq^n}\right)\right] \\
&= \left(\frac{(q^n|q)_1}{q^{n+1}(q, abcdq^{2n-1}, abcdq^{2n}|q)_1}\right)\Big\{(1-q)(abq^n|q)_1\left[(c+d)-cdq^n(a+b)\right] \\
&\quad - (1-abcdq^{2n})\left[(c+d)(abq^n|q)_1 + q(a+b)(cdq^{n-1}|q)_1\right]\Big\} \\
&= \left(\frac{(q^n|q)_1}{q^{n+1}(q, abcdq^{2n-1}, abcdq^{2n}|q)_1}\right)\Big\{(c+d)(abq^n|q)_1\left[abcdq^{2n}-q\right] \\
&\quad + (a+b)\left[-(1-q)(1-abq^n)(cdq^n) - q(1-abcdq^{2n})(1-cdq^{n-1})\right]\Big\} \\
&= \left(\frac{(q^n|q)_1}{q^{n+1}(q, abcdq^{2n-1}, abcdq^{2n}|q)_1}\right)\Big\{(c+d)(abq^n|q)_1\left[-q(abcdq^{2n-1}|q)_1\right] \\
&\quad + (a+b)\left[abcdq^{2n} + cdq^{n+1} - q - abc^2d^2q^{3n}\right]\Big\}
\end{aligned}$$

$$= \left(\frac{(q^n|q)_1}{q^{n+1}(q, abcdq^{2n-1}, abcdq^{2n}|q)_1}\right)\Big\{(c+d)(abq^n|q)_1\left[-q(abcdq^{2n-1}|q)_1\right]$$
$$+ (a+b)\left[abcdq^{2n}(cdq^n|q)_1 - q(cdq^n|q)_1\right]\Big\}$$
$$= -\left(\frac{(q^n|q)_1}{q^n(q, abcdq^{2n}|q)_1}\right)\Big\{(c+d)(abq^n|q)_1 + (a+b)(cdq^n|q)_1\Big\}$$

as asserted for the $r = -(n+1)$ case. Now that we have $\lambda_{-n,-(n+1)\_\mathrm{q}}$,

$$\lambda_{-n,-(n+1)} = \mu_{n,-(n+1)\_\mathrm{q}}(q^n - 1) + \lambda_{-n,-(n+1)\_\mathrm{q}}q^n = (q^n - 1)\left(\frac{abq^n(c+d) - (a+b)}{(abcdq^{2n}|q)_1}\right)$$
$$+ q^n\left(-\frac{(c+d)(q^n, abq^n|q)_1 + (a+b)(q^n, cdq^n|q)_1}{q^n(q, abcdq^{2n}|q)_1}\right)$$
$$= -\left(\frac{1-q^n}{(q, abcdq^{2n}|q)_1}\right)\big\{abq^n(c+d)(1-q) - (a+b)(1-q) + (c+d)(1-abq^n)$$
$$+ (a+b)(1-cdq^n)\big\}$$
$$= -\left(\frac{1-q^n}{(q, abcdq^{2n}|q)_1}\right)\big\{(c+d)(1-abq^{n+1}) + (a+b)(q-cdq^n)\big\}.$$

which agrees with the asserted $\lambda_{-n,-(n+1)}$. ■

The formulas (37) for $T(a, aq)$ involve factors $(ae^{-1}|q)_r$. When $e = aq$,, these vanish for $r \geq 2$. Consequently the decomposition

$$T(a, aq) = \begin{bmatrix} T^{00}(a, aq) & T^{01}(a, aq) \\ T^{10}(a, aq) & T^{11}(a, aq) \end{bmatrix}$$

specializes to

$$T^{00}(a, aq) = \begin{bmatrix} \tau_{0,0}(a, aq) & \tau_{0,1}(a, aq) & 0 & 0 \\ 0 & \tau_{1,1}(a, aq) & \tau_{1,2}(a, aq) & \ddots \\ 0 & \ddots & \ddots & \ddots \end{bmatrix}$$

$$T^{01}(a, aq) = \begin{bmatrix} \sigma_{0,-1}(a, aq) & 0 & 0 \\ 0 & \sigma_{1,-2}(a, aq) & 0 \\ 0 & \ddots & \ddots \end{bmatrix}$$

$$T^{10}(a, aq) = \begin{bmatrix} 0 & \sigma_{-1,1}(a, aq) & 0 & 0 \\ 0 & 0 & \sigma_{-2,2}(a, aq) & \ddots \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$

$$T^{11}(a, aq) = \begin{bmatrix} \tau_{-1,-1}(a, aq) & \tau_{-1,-2}(a, aq) & 0 & 0 \\ 0 & \tau_{-2,-2}(a, aq) & \tau_{-2,-3}(a, aq) & \ddots \\ 0 & 0 & \tau_{-3,-3}(a, aq) & \ddots \end{bmatrix}.$$

This has strong implications for the form of the discrete co-cycle condition (14) as well, which we come back to in Section 6.

4. Combining the $d_{r,s}c_{|r|,|s|}$ products into the τ and σ

The most combinatorially involved part of our proof of Theorem 1.3 involves co-cycle condition verification.

To facilitate its formulation and the clarity of correctness of our arguments, it is useful to combine the $d_{rs}c_{|r|,|s|}$ expressions into more distinctive and mnemonic expressions. As mentioned earlier, equations (11) and (12) give the main notation we use, namely

$$\tau_{r,s} = d_{r,s}c_{|r|,|s|} \qquad \text{if } (r \geq 0 \text{ and } s \geq 0) \text{ or } (r < 0 \text{ and } s < 0)$$

$$\sigma_{r,s} = d_{r,s}c_{|r|,|s|} \qquad \text{if } (r \geq 0 \text{ and } s < 0) \text{ or } (r < 0 \text{ and } s \geq 0).$$

In this way, the sign portion of the $r \preceq s$ relation leads to the natural 2×2 block matrix form of the transition matrices. (This is where the 4 cases of the d_{rs} come from.) Explicitly this means we are using the notation:

For $n \geq 0$

$$\begin{aligned} E_n(z;a,b,c,d|q) &= \sum_{m=0}^{n} [\tau_{m,n}(a,e;b,c,d|q)]\, E_m(z;e,b,c,d|q) \\ &\quad + \sum_{m=0}^{n-1} \left[\sigma_{-(m+1),n}(a,e;b,c,d|q)\right] E_{-(m+1)}(z;e,b,c,d|q) \\ E_{-(n+1)}(z;a,b,c,d|q) &= \sum_{m=0}^{n} \left[\tau_{-(m+1),-(n+1)}(a,e;b,c,d|q)\right] E_{-(m+1)}(z;e,b,c,d|q) \\ &\quad + \sum_{m=0}^{n} \left[\sigma_{m,-(n+1)}(a,e;b,c,d|q)\right] E_m(z;e,b,c,d|q) \end{aligned} \tag{36}$$

where for $k, n \geq 0$

$$\begin{aligned} \tau_{k,n}(a,e;b,c,d|q) &= \frac{(q^{n-k+1}|q)_k (eq)^{n-k} (bcq^k, bdq^k, cdq^k, ae^{-1}|q)_{n-k}}{(q|q)_k (abcdq^{n+k}, bcdeq^{2k}|q)_{n-k}} \\ \sigma_{k,-(n+1)}(a,e;b,c,d|q) &= \frac{(q^{n-k+1}|q)_k e^{n+1-k} (bcq^k, bdq^k, ae^{-1}|q)_{n+1-k} (cdq^k|q)_{n-k}}{(q|q)_k (abcdq^{n+k}, bcdeq^{2k}|q)_{n+1-k}} \\ \sigma_{-(k+1),n}(a,e;b,c,d|q) &= \frac{(q^{n-k}|q)_{k+1} (bcq^{k+1}, bdq^{k+1}|q)_{n-k-1} (cdq^k, ae^{-1}|q)_{n-k}}{(q|q)_k (abcdq^{n+k}, bcdeq^{2k+1}|q)_{n-k}} \\ &\qquad \times bcde^{n-k} q^{n+k} \\ \tau_{-(k+1),-(n+1)}(a,e;b,c,d|q) &= \frac{(q^{n-k+1}|q)_k e^{n-k} (bcq^{k+1}, bdq^{k+1}, cdq^k, ae^{-1}|q)_{n-k}}{(q|q)_k (abcdq^{n+k+1}, bcdeq^{2k+1}|q)_{n-k}}. \end{aligned} \tag{37}$$

(The verification below is of τ, σ given previous definitions of c_{rs}, d_{rs}.)

Proof. For $k, n \geq 0$,

(1) $\tau_{k,n}(a,e;b,c,d|q) = d_{k,n}c_{k,n}$

$$= \left\{ \frac{q^{n-k}(abcdq^{n+k-1}|q)_1}{(abcdq^{2n-1}|q)_1} \right\} \left\{ \frac{e^{n-k}(q^{n-k+1}|q)_k (bcq^k, bdq^k, cdq^k, ae^{-1}|q)_{n-k}}{(q|q)_k (abcdq^{n+k-1}, bcdeq^{2k}|q)_{n-k}} \right\}$$

$$= \frac{(q^{n-k+1}|q)_k (eq)^{n-k} (bcq^k, bdq^k, cdq^k, ae^{-1}|q)_{n-k}}{(q|q)_k (abcdq^{n+k}, bcdeq^{2k}|q)_{n-k}}$$

(2) $\sigma_{k,-(n+1)}(a,e;b,c,d|q) = d_{k,-(n+1)}c_{k,n+1}$

$$= \left\{ \frac{(q^{n-k+1}|q)_1}{(q^{n+1}, cdq^n|q)_1} \right\} \left\{ \frac{e^{n-k+1}(q^{n-k+2}|q)_k(bcq^k, bdq^k, cdq^k, ae^{-1}|q)_{n-k+1}}{(q|q)_k(abcdq^{n+k}, bcdeq^{2k}|q)_{n-k+1}} \right\}$$

$$= \frac{(q^{n-k+1}|q)_k e^{n+1-k}(bcq^k, bdq^k, ae^{-1}|q)_{n+1-k}(cdq^k|q)_{n-k}}{(q|q)_k(abcdq^{n+k}, bcdeq^{2k}|q)_{n+1-k}}$$

(3) $\sigma_{-(k+1),n}(a,e;b,c,d|q) = d_{-(k+1),n}c_{k+1,n}$

$$= \left\{ \frac{bcdeq^{n+k}(q^{k+1}, cdq^k, ae^{-1}q^{n-k-1}|q)_1}{(abcdq^{2n-1}, bcdeq^{2k+1}|q)_1} \right\}$$

$$\cdot \left\{ \frac{e^{n-k-1}(q^{n-k}|q)_{k+1}(bcq^{k+1}, bdq^{k+1}, cdq^{k+1}, ae^{-1}|q)_{n-k-1}}{(q|q)_{k+1}(abcdq^{n+k}, bcdeq^{2(k+1)}|q)_{n-k-1}} \right\}$$

$$= \frac{(q^{n-k}|q)_{k+1}bcde^{n-k}q^{n+k}(bcq^{k+1}, bdq^{k+1}|q)_{n-k-1}(cdq^k, ae^{-1}|q)_{n-k}}{(q|q)_k(abcdq^{n+k}, bcdeq^{2k+1}|q)_{n-k}}$$

(4) $\tau_{-(k+1),-(n+1)}(a,e;b,c,d|q) = d_{-(k+1),-(n+1)}c_{k+1,n+1}$

$$= \left\{ \frac{(q^{k+1}, cdq^k, bcdeq^{n+k+1}|q)_1}{(q^{n+1}, cdq^n, bcdeq^{2k+1}|q)_1} \right\}$$

$$\cdot \left\{ \frac{e^{n-k}(q^{n-k+1}|q)_{k+1}(bcq^{k+1}, bdq^{k+1}, cdq^{k+1}, ae^{-1}|q)_{n-k}}{(q|q)_{k+1}(abcdq^{n+k+1}, bcdeq^{2(k+1)}|q)_{n-k}} \right\}$$

$$= \frac{(q^{n-k+1}|q)_k e^{n-k}(bcq^{k+1}, bdq^{k+1}, cdq^k, ae^{-1}|q)_{n-k}}{(q|q)_k(abcdq^{n+k+1}, bcdeq^{2k+1}|q)_{n-k}}. \qquad \blacksquare$$

5. The true $\mathcal{T}(a, aq)$ matches the $T(a, aq)$ of Theorem 1.3

Here we use the notation

$$\Delta_{ae} f = f(e) - f(a)$$

for the change in values of a function f as one moves from argument a to argument e. Since the $\{\lambda_{r,s}, \mu_{r,s}\}$ are also entries in a change of basis relationship, the true transition functions $\mathcal{T}(a, aq)$ may be expressed in terms of them. However, in the low zig-zag co-degree cases, the transition matrix entries may also be conveniently determined by a successive substitution argument.

Proposition 5.1.

$$\mathcal{T}^{00}_{n-1,n}(a,e) = \tau_{n-1,n} = -\Delta_{ae}\lambda_{n-1,n} + [\mu_{n-1,-n}(e)]\,\Delta_{ae}\mu_{-n,n} \qquad (38)$$

$$\mathcal{T}^{01}_{n,n}(a,e) = \sigma_{n,-(n+1)} = -\Delta_{ae}\mu_{n,-(n+1)} \qquad (39)$$

$$\mathcal{T}^{10}_{n,n+1}(a,e) = \sigma_{-(n+1),n+1} = -\Delta_{ae}\mu_{-(n+1),n+1} \qquad (40)$$

$$\mathcal{T}^{11}_{n-1,n}(a,e) = \tau_{-n,-(n+1)} = -\Delta_{ae}\lambda_{-n,-(n+1)} + [\mu_{-n,n}(e)]\,\Delta_{ae}\mu_{n,-(n+1)}. \qquad (41)$$

Proof. We write down the proof of (38) and (40), the other two being similar. Recall $\lambda_{r,r} = 1$ for all parameters and any sign of r.

Then modulo $\mathcal{R}_{-(n-1)}$:

$$E_{n-1}(e) = [\lambda_{n-1,n-1}(e)]\, f_{n-1} \;\Rightarrow\; f_{n-1} = E_{n-1}(e)$$

$$\begin{aligned} E_{-n}(e) &= [\lambda_{-n,-n}(e)]\, h_n + [\mu_{n-1,-n}(e) f_{n-1}(e)] \\ &\Rightarrow\; h_n = E_{-n}(e) - [\mu_{n-1,-n}(e)]\, E_{n-1}(e) \end{aligned}$$

$$\begin{aligned} f_n &= E_n(e) - [\mu_{-n,n}(e)]\, h_n - [\lambda_{n-1,n}(e)]\, f_{n-1} \\ &= E_n(e) - [\mu_{-n,n}(e)] \big(E_{-n}(e) - [\mu_{n-1,-n}(e)]\, E_{n-1}(e)\big) - [\lambda_{n-1,n}(e)]\, E_{n-1}(e) \end{aligned}$$

So

$$\begin{aligned} E_n(a) &= f_n + [\mu_{-n,n}(a)]\, h_n + [\lambda_{n-1,n}(a)]\, f_{n-1} \\ &= E_n(e) - [\mu_{-n,n}(e)] \big(E_{-n}(e) - [\mu_{n-1,-n}(e)]\, E_{n-1}(e)\big) - [\lambda_{n-1,n}(e)]\, E_{n-1}(e) \\ &\quad + [\mu_{-n,n}(a)] \big(E_{-n}(e) - [\mu_{n-1,-n}(e)]\, E_{n-1}(e)\big) + [\lambda_{n-1,n}(a)]\, E_{n-1}(e) \end{aligned}$$

Combining terms and comparing coefficients of $E_{-n}(e)$ (for (40)) and $E_{n-1}(e)$ (for (38)) with the definition

$$E_n(a) = \tau_{n,n} E_n(e) + \sigma_{-n,n} E_{-n}(e) + \tau_{n-1,n} E_{n-1}(e) \quad \mod \mathcal{R}_{-(n-1)}$$

gives the asserted formulas. ■

The zig-zag co-degree 1 formulas (39) and (40) above have an obvious linearity based on

$$\Delta_{uw} f = \Delta_{uv} f + \Delta_{vw} f \text{ since } \big(f(w) - f(u)\big) = \big(f(v) - f(u)\big) + \big(f(w) - f(v)\big).$$

This immediately implies that the zig-zag co-degree 1 matrix entries of T satisfy what is required by the discrete co-cycle condition. We shall need these special cases in the next section, so we record them in the corollary below.

Corollary 5.2. *For any $a, e,$ and f,*

$$\begin{aligned} \sigma_{n,-(n+1)}(a,e) &= \sigma_{n,-(n+1)}(f,e) + \sigma_{n,-(n+1)}(a,f). \\ \sigma_{-(n+1),n+1}(a,e) &= \sigma_{-(n+1),n+1}(f,e) + \sigma_{-(n+1),n+1}(a,f). \end{aligned}$$

Proposition 5.3. *In the special case of $e = aq$, the zig-zag co-degree 1 transition functions satisfy:*

(1) $$\boldsymbol{\mathcal{T}}^{01}_{n,n}(a,aq) = \sigma_{n,-(n+1)}(a,aq) = \frac{aq(bcq^n, bdq^n, q^{-1}|q)_1}{(abcdq^{2n}|q)_1 (abcdq^{2n+1}|q)_1}$$

(2) $$\boldsymbol{\mathcal{T}}^{10}_{n,n+1}(a,aq) = \sigma_{-(n+1),n+1}(a,aq) = \frac{abcdq^{2(n+1)}(q^{n+1}, cdq^n, q^{-1}|q)_1}{(abcdq^{2n+1}|q)_1 (abcdq^{2(n+1)}|q)_1}$$

Proof. (1) By Proposition 5.1: $\boldsymbol{\mathcal{T}}^{01}_{n,n}(a,aq) = \sigma_{n,-(n+1)}(a,aq) = -\Delta_{ae}\mu_{n,-(n+1)}$.

So $$\begin{aligned} \boldsymbol{\mathcal{T}}^{01}_{nn}(a,aq) = -\Big\{ &\Big\{ -\frac{[([aq]\, bq^n|q)_1 - 1]\,(c+d) + ([aq]+b)}{([aq]\, bcdq^{2n}|q)_1} \Big\} \\ &- \Big\{ -\frac{[(abq^n|q)_1 - 1]\,(c+d) + (a+b)}{(abcdq^{2n}|q)_1} \Big\} \Big\} \end{aligned}$$

$$= \Big\{ \frac{1}{(abcdq^{2n}|q)_1 (abcdq^{2n+1}|q)_1} \Big\} \cdot \big\{ (abcdq^{2n}|q)_1 \left[-abq^{n+1}(c+d) + (aq+b) \right] \\ - (abcdq^{2n+1}|q)_1 \left[-abq^n(c+d) + (a+b) \right] \big\}.$$

Comparing with the statement of the proposition and keeping in mind that

$$q(q^{-1}|q)_1 = -(q|q)_1,$$

we see that we need to verify equality of the numerators, both of which are polynomials; i.e. the vanishing of

$$(abcdq^{2n}|q)_1 \left[-abq^{n+1}(c+d) + (aq+b) \right] \\ - (abcdq^{2n+1}|q)_1 \left[-abq^n(c+d) + (a+b) \right] \\ + a(bcq^n|q)_1 (bdq^n|q)_1 (q|q)_1 = 0$$

Introducing an additional variable u (intuitively, replacing q^n), it suffices to show that the *linear* in q polynomial

$$(abcdu^2|q)_1 \left[-abuq(c+d) + (aq+b) \right] \\ - (abcdu^2q|q)_1 \left[-abu(c+d) + (a+b) \right] + a(bcu|q)_1 (bdu|q)_1 (q|q)_1 = 0$$

vanishes for all a, b, c, d, u.

(a) For $q = 0$, we have

$$b(abcdu^2|q)_1 - \left[-abu(c+d) + (a+b) \right] + a(bcu|q)_1 (bdu|q)_1 = 0$$

(b) For $q = 1$, we have

$$(abcdu^2|q)_1 \left[-abu(c+d) + (a+b) \right] - (abcdu^2|q)_1 \left[-abu(c+d) + (a+b) \right] = 0$$

So, since the 1 variable polynomial in q with coefficients in $\mathbb{Q}(a,b,c,d,u)$ is at most degree 1 and vanishes at two distinct values of q, it must be identically zero and the required identity has been established.

(2) By Proposition 5.1 we have

$$\boldsymbol{\mathcal{T}}^{10}_{n,n+1}(a, aq) = \sigma_{-(n+1),n+1}(a, aq) = -\Delta_{ae}\mu_{-(n+1),n+1}.$$

So $$\boldsymbol{\mathcal{T}}^{10}_{n,n+1}(a, aq) = -\Big\{ \Big\{ -\frac{(cdq^n|q)_1 (q^{n+1}|q)_1}{([aq]\, bcdq^{2n+1}|q)_1} \Big\} - \Big\{ -\frac{(cdq^n|q)_1 (q^{n+1}|q)_1}{(abcdq^{2n+1}|q)_1} \Big\} \Big\}$$

$$= \frac{(cdq^n|q)_1 (q^{n+1}|q)_1}{(abcdq^{2n+1}|q)_1 (abcdq^{2(n+1)}|q)_1} \Big\{ (abcdq^{2n+1}|q)_1 - (abcdq^{2(n+1)}|q)_1 \Big\}$$

$$= \frac{(cdq^n|q)_1 (q^{n+1}|q)_1 \big\{ -abcdq^{2n+1}(q|q)_1 \big\}}{(abcdq^{2n+1}|q)_1 (abcdq^{2(n+1)}|q)_1} = \frac{(cdq^n|q)_1 (q^{n+1}|q)_1 \big\{ abcdq^{2(n+1)}(q^{-1}|q)_1 \big\}}{(abcdq^{2n+1}|q)_1 (abcdq^{2(n+1)}|q)_1}.$$

This completes the proof. ■

Proposition 5.4. *In the special case of $e = aq$, the zig-zag co-degree 2 transition functions satisfy:*

(1) $$\mathcal{T}^{00}_{n-1,n}(a,aq) = \tau_{n-1,n} = -\frac{aq(q^n, bcq^{n-1}, bdq^{n-1}, cdq^{n-1}|q)_1}{\left[(abcdq^{2n-1}|q)_1\right]^2}$$

(2) $$\mathcal{T}^{11}_{n-1,n}(a,aq) = \tau_{-n,-(n+1)} = -\frac{a(q^n, bcq^n, bdq^n, cdq^{n-1}|q)_1}{\left[(abcdq^{2n}|q)_1\right]^2}$$

Proof. (1) By Proposition 5.1: $\mathcal{T}^{00}_{n-1,n}(a,aq) = -\Delta_{ae}\lambda_{n-1,n} + [\mu_{n-1,-n}(e)]\,\Delta_{ae}\mu_{-n,n}$. So

$$\begin{aligned}
&\mathcal{T}^{00}_{n-1,n}(a,aq)\\
&= -\Big\{\Big\{-\frac{(q^n|q)_1}{(q|q)_1([aq]\,bcdq^{2n-1}|q)_1}\left[(c+d)([aq]\,bq^n|q)_1 + q([aq]+b)(cdq^{n-1}|q)_1\right]\Big\}\\
&\quad -\Big\{-\frac{(q^n|q)_1}{(q|q)_1(abcdq^{2n-1}|q)_1}\left[(c+d)(abq^n|q)_1 + q(a+b)(cdq^{n-1}|q)_1\right]\Big\}\Big\}\\
&\quad +\Big\{-\frac{\left[([aq]\,bq^{n-1}|q)_1 - 1\right](c+d) + ([aq]+b)}{([aq]\,bcdq^{2(n-1)}|q)_1}\Big\}\\
&\quad \cdot\Big\{\Big\{-\frac{(cdq^{n-1}|q)_1(q^n|q)_1}{([aq]\,bcdq^{2n-1}|q)_1}\Big\} - \Big\{-\frac{(cdq^{n-1}|q)_1(q^n|q)_1}{(abcdq^{2n-1}|q)_1}\Big\}\Big\}
\end{aligned}$$

Using the notation $\left[a,b,c,\dots\right] = (a|q)_1(b|q)_1(c|q)_1\dots$, the abbreviation variables $u = q^{n-1}$, $y = abcd$ and multiplying by

$$\frac{(q,q)_1\left[(yq^{2n-1}|q)_1\right]^2(yq^{2n}|q)_1}{(q^n|q)_1} = \frac{\left[q, yu^2q, yu^2q, yu^2q^2\right]}{(uq|q)_1}$$

it suffices to show the vanishing of

$$\begin{aligned}
p_1 &= \Big\{aq\left[q, yu^2q^2, bcu, bdu, cdu\right]\Big\}\\
&\quad + \Big\{\left[(yu^2q|q)_1\right]^2\left[(c+d)(abuq^2|q)_1 + q(aq+b)(cdu|q)_1\right]\Big\}\\
&\quad - \Big\{(yu^2q|q)_1(yu^2q^2|q)_1\left[(c+d)(abuq|q)_1 + q(a+b)(cdu|q)_1\right]\Big\}\\
&\quad + \Big\{(q,q)_1(yu^2q|q)_1\Big[\left[(abuq|q)_1 - 1\right](c+d) + (aq+b)\Big](cdu|q)_1\Big\}\\
&\quad - \Big\{(q,q)_1(yu^2q^2|q)_1\Big[\left[(abuq|q)_1 - 1\right](c+d) + (aq+b)\Big](cdu|q)_1\Big\}\\
&= \Big\{aq\left[q, yu^2q^2, bcu, bdu, cdu\right]\Big\}\\
&\quad + (yu^2q|q)_1\Big\{(c+d)\big\{(yu^2q|q)_1(abuq^2|q)_1 - (yu^2q^2|q)_1(abuq|q)_1\big\}\\
&\quad + (cdu|q)_1\Big\{aq\big\{(yu^2q|q)_1(q) - (yu^2q^2|q)_1(1)\big\} + bq\big\{(yu^2q|q)_1 - (yu^2q^2|q)_1\big\}\Big\}\Big\}\\
&\quad + [q, cdu]\,\big\{(-abuq)(c+d) + (aq+b)\big\}\big\{(yu^2q|q)_1 - (yu^2q^2|q)_1\big\}
\end{aligned}$$

Note (keeping in mind, e.g., $(yu^2q)(q) - (yu^2q^2)(1) = 0$) that

$$\begin{aligned}
&(yu^2q|q)_1(q) - (yu^2q^2|q)_1(1) = q - 1 = -(q|q)_1\\
&(yu^2q|q)_1 - (yu^2q^2|q)_1 = -yu^2q(q|q)_1
\end{aligned}$$

$$\begin{aligned}(yu^2q|q)_1(abuq^2|q)_1 - (yu^2q^2|q)_1(abuq|q)_1 \\ = -yu^2q - abuq^2 + yu^2q^2 + abuq = uq(q|q)_1(ab-yu) \\ = uq(q|q)_1(ab - abcdu) = abuq(q|q)_1(cdu|q)_1 \end{aligned} \tag{42}$$

So p_1 is also equal to

$$\begin{aligned} p_2 = &\Big\{aq\big[q, yu^2q^2, bcu, bdu, cdu\big]\Big\} \\ &+ (yu^2q|q)_1\Big\{(c+d)\big\{abuq(q|q)_1(cdu|q)_1\big\} \\ &+ (cdu|q)_1\Big\{aq\big\{-(q|q)_1\big\} + bq\big\{-yu^2q(q|q)_1\big\}\Big\}\Big\} \\ &+ [q, cdu]\,\big\{\,(-abuq)\,(c+d) + (aq+b)\big\}\big\{-yu^2q(q|q)_1\big\} \end{aligned}$$

Multiplying p_2 by $[q(q|q)_1\;(cdu|q)_1]^{-1}$ we are reduced to showing the vanishing of

$$\begin{aligned} p_3 = &\big\{a\big[yu^2q^2, bcu, bdu\big]\big\} + (yu^2q|q)_1\big\{(c+d)\big\{abu\big\} \\ &+ \big\{a\big\{-1\big\} + b\big\{-yu^2q\big\}\big\}\big\} + \big\{\,(-abuq)\,(c+d) + (aq+b)\big\}\big\{-yu^2(q|q)_1\big\} \\ = &\,a\big[abcdu^2q^2, bcu, bdu\big] + a(abcdu^2q|q)_1\Big\{bu(c+d) - \big(1 + b^2cdu^2q\big)\Big\} \\ &- abcdu^2(q|q)_1\big\{\,(-abuq)\,(c+d) + (aq+b)\big\} \\ = &\,a\big[abcdu^2q^2, bcu, bdu\big] + a(abcdu^2q|q)_1\Big\{-(1-bcu)(1-bdu) + b^2cdu^2(1-q)\Big\} \\ &- abcdu^2(q|q)_1\big\{aq(1-bcu)(1-bdu) + b(1-abcdu^2q)\big\} \\ = &\,a\big[bcu, bdu\big]\big\{(abcdu^2q^2|q)_1 - (abcdu^2q|q)_1 - abcdu^2q(q|q)_1\big\} \\ &+ ab^2cdu^2\,\big[q, abcdu^2q\big]\,\big\{1-1\big\} \\ = &\,a\big[bcu, bdu\big]\big\{(abcdu^2q(q|q)_1 - abcdu^2q(q|q)_1\big\} = 0, \end{aligned}$$

thus proving the first formula.

(2) By Proposition 5.1 we have

$$\mathcal{T}^{11}_{n-1,n}(a, aq) = -\Delta_{ae}\lambda_{-n,-(n+1)} + [\mu_{-n,n}(e)]\,\Delta_{ae}\mu_{n,-(n+1)}.$$

So

$$\begin{aligned} &\mathcal{T}^{11}_{n-1,n}(a, aq) \\ &= -\Big\{\Big\{-\frac{(q^n|q)_1}{(q|q)_1([aq]\,bcdq^{2n}|q)_1}\,\big[(c+d)([aq]\,bq^{n+1}|q)_1 + q([aq]+b)(cdq^{n-1}|q)_1\big]\Big\} \\ &\quad - \Big\{-\frac{(q^n|q)_1}{(q|q)_1(abcdq^{2n}|q)_1}\,\big[(c+d)(abq^{n+1}|q)_1 + q(a+b)(cdq^{n-1}|q)_1\big]\Big\}\Big\} \\ &\quad + \Big\{-\frac{(cdq^{n-1}|q)_1(q^n|q)_1}{([aq]\,bcdq^{2n-1}|q)_1}\Big\}\cdot\Big\{\Big\{-\frac{[([aq]\,bq^n|q)_1 - 1]\,(c+d) + ([aq]+b)}{([aq]\,bcdq^{2n}|q)_1}\Big\} \\ &\quad - \Big\{-\frac{[(abq^n|q)_1 - 1]\,(c+d) + (a+b)}{(abcdq^{2n}|q)_1}\Big\}\Big\}. \end{aligned}$$

Using the notation $\big[a, b, c, \dots\big] = (a|q)_1(b|q)_1(c|q)_1\dots$, the abbreviation variables $u = q^n, y = abcd$ and multiplying by

$$\frac{(q,q)_1\,\big[(yu^2|q)_1\big]^2\,(yu^2q|q)_1}{(u|q)_1} = \frac{\big[q, yu^2, yu^2, yu^2q\big]}{(u|q)_1},$$

it suffices to show the vanishing of

$$
\begin{aligned}
p_1 &= \Big\{a\left[q, yu^2q, bcu, bdu\right](cdq^{n-1}|q)_1\Big\} \\
&\quad + \Big\{\left[(yu^2, yu^2\right]\big\{(c+d)(abuq^2|q)_1 + q(aq+b)(cdq^{n-1}|q)_1\big\}\Big\} \\
&\quad - \Big\{\left[yu^2, yu^2q\right]\big\{(c+d)(abuq|q)_1 + q(a+b)(cdq^{n-1}|q)_1\big\}\Big\} \\
&\quad + \Big\{\left[q, yu^2\right](cdq^{n-1}|q)_1\Big\{-abuq(c+d)+aq+b\Big\} \\
&\quad - \left[q, yu^2q\right](cdq^{n-1}|q)_1\Big\{-abu(c+d)+(a+b)\Big\}\Big\} \\
&= \Big\{a\left[q, yu^2q, bcu, bdu\right](cdq^{n-1}|q)_1\Big\} \\
&\quad + (yu^2|q)_1\Big\{(c+d)\big\{(yu^2|q)_1(abuq^2|q)_1 - (yu^2q|q)_1(abuq|q)_1\big\} \\
&\quad + (cdq^{n-1}|q)_1\Big\{aq\big\{(yu^2|q)_1(q) - (yu^2q|q)_1(1)\big\} + bq\big\{(yu^2|q)_1 - (yu^2q|q)_1\big\}\Big\}\Big\} \\
&\quad + \left[q, cdq^{n-1}\right]\Big\{a(1-bu(c+d))\big\{(yu^2|q)_1(q) - (yu^2q|q)_1(1)\big\} \\
&\quad + b\big\{(yu^2|q)_1 - (yu^2q|q)_1\big\}\Big\}
\end{aligned}
$$

Note (keeping in mind, e.g., $(yu^2)(q) - (yu^2q)(1) = 0$) that

$$
\begin{aligned}
&(yu^2|q)_1(q) - (yu^2q|q)_1(1) = q - 1 = -(q|q)_1 \\
&(yu^2|q)_1 - (yu^2q|q)_1 = -yu^2(q|q)_1 \\
&(yu^2|q)_1(abuq^2|q)_1 - (yu^2q|q)_1(abuq|q)_1 \\
&\qquad = -yu^2 - abuq^2 + yu^2q + abuq = u(q|q)_1(abq - yu) \\
&\qquad = abu(q|q)_1(q - cdu) = abuq(q|q)_1(1 - cdq^{n-1}) \\
&\qquad = abuq(q|q)_1(cdq^{n-1}|q)_1
\end{aligned} \tag{43}
$$

So p_1 is also equal to

$$
\begin{aligned}
p_2 &= \Big\{a\left[q, yu^2q, bcu, bdu\right](cdq^{n-1}|q)_1\Big\} \\
&\quad + (yu^2|q)_1\Big\{(c+d)\big\{abuq(q|q)_1(cduq^{n-1}|q)_1\big\} \\
&\quad + (cdq^{n-1}|q)_1\Big\{aq\big\{-(q|q)_1\big\} + bq\big\{-yu^2(q|q)_1\big\}\Big\}\Big\} \\
&\quad + \left[q, cdq^{n-1}\right]\Big\{a(1-bu(c+d))\big\{-(q|q)_1\big\} + b\big\{-yu^2(q|q)_1\big\}\Big\}
\end{aligned}
$$

So, upon multiplying by $\left[(q|q)_1(cdq^{n-1}|q)_1\right]^{-1}$ we are reduced to showing the vanishing of

$$
\begin{aligned}
p_3 &= \Big\{a\left[yu^2q, bcu, bdu\right]\Big\} + (yu^2|q)_1\Big\{(c+d)\big\{abuq\big\} \\
&\quad + \Big\{aq\big\{-1\big\} + bq\big\{-yu^2\big\}\Big\}\Big\} \\
&\quad + (q|q)_1\Big\{a(1-bu(c+d))\big\{-1\big\} + b\big\{-yu^2\big\}\Big\} \\
&= a\left[abcdu^2q, bcu, bdu\right] + q(abcdu^2|q)_1\Big\{abu(c+d) - a(1+b^2cdu^2)\Big\} \\
&\quad + a(q|q)_1\Big\{-(1-bu(c+d)) - b^2cdu^2\Big\}
\end{aligned}
$$

$$= a\left[abcdu^2q, bcu, bdu\right] - aq\left[abcdu^2, bcu, bdu\right] - a(q|q)_1\left[bcu, bdu\right]$$
$$= a\left[bcu, bdu\right]\left\{(abcdu^2q|q)_1 - q(abcdu^2|q)_1 - (q|q)_1\right\}$$
$$= a\left[bcu, bdu\right]\left\{(q|q)_1 - (q|q)_1\right\} = 0.$$

Thus the proposition proof is complete. ■

Propositions 5.3 and 5.4 complete the proof of the second step of (15) of Proof Plan A.

6. Proof of the discrete co-cycle identity for T

The block form of the equation

$$T(aq^p, aq^{p+1})T(a, aq^p) = T(a, aq^{p+1})$$

is

$$\begin{bmatrix} T^{00}(aq^p, aq^{p+1} & T^{01}(aq^p, aq^{p+1} \\ T^{10}(aq^p, aq^{p+1} & T^{11}(aq^p, aq^{p+1} \end{bmatrix}\begin{bmatrix} T^{00}(a, aq^p) & T^{01}(a, aq^p) \\ T^{10}(a, aq^p) & T^{11}(a, aq^p) \end{bmatrix}$$
$$= \begin{bmatrix} T^{00}(a, aq^{p+1}) & T^{01}(a, aq^{p+1}) \\ T^{10}(a, aq^{p+1}) & T^{11}(a, aq^{p+1}) \end{bmatrix}$$

So we have the four matrix equations:

$$T^{00}(a, aq^{p+1}) = \left[T^{00}(aq^p, aq^{p+1})\right]\left[T^{00}(a, aq^p)\right] + \left[T^{01}(aq^p, aq^{p+1})\right]\left[T^{10}(a, aq^p)\right]$$
$$T^{01}(a, aq^{p+1}) = \left[T^{00}(aq^p, aq^{p+1})\right]\left[T^{01}(a, aq^p)\right] + \left[T^{01}(aq^p, aq^{p+1})\right]\left[T^{11}(a, aq^p)\right]$$
$$T^{10}(a, aq^{p+1}) = \left[T^{10}(aq^p, aq^{p+1})\right]\left[T^{00}(a, aq^p)\right] + \left[T^{11}(aq^p, aq^{p+1})\right]\left[T^{10}(a, aq^p)\right]$$
$$T^{11}(a, aq^{p+1}) = \left[T^{10}(aq^p, aq^{p+1})\right]\left[T^{01}(a, aq^p)\right] + \left[T^{11}(aq^p, aq^{p+1})\right]\left[T^{11}(a, aq^p)\right]$$

Keep in mind, as described at the end of Section 3, that the entries of each $T^{ij}(aq^p, aq^{p+1})$ are zero except (possibly) on the diagonal and the superdiagonal.

In terms of τ and σ, these equations are:

$$\tau_{k,n}(a, aq^{p+1}) = \left[\tau_{k,k}(aq^p, aq^{p+1})\right]\left[\tau_{k,n}(a, aq^p)\right] + \left[\tau_{k,k+1}(aq^p, aq^{p+1})\right] \cdot \left\{\left[\tau_{k+1,n}(a, aq^p)\right]\right\} + \left[\sigma_{k,-(k+1)}(aq^p, aq^{p+1})\right]\left[\sigma_{-(k+1),n}(a, aq^p)\right] \tag{44}$$

$$\sigma_{k,-(n+1)}(a, aq^{p+1}) = \left[\tau_{k,k}(aq^p, aq^{p+1})\right]\left[\sigma_{k,-(n+1)}(a, aq^p)\right] + \left[\tau_{k,k+1}(aq^p, aq^{p+1})\right] \cdot \left\{\left[\sigma_{k+1,-(n+1))}(a, aq^p)\right]\right\} + \left[\sigma_{k,-(k+1)}(aq^p, aq^{p+1})\right]\left[\tau_{-(k+1),-(n+1)}(a, aq^p)\right] \tag{45}$$

$$\sigma_{-(k+1),n}(a, aq^{p+1}) = \left[\sigma_{-(k+1),k+1}(aq^p, aq^{p+1})\right]\left[\tau_{k+1,n}(a, aq^p)\right] + \left[\tau_{-(k+1),-(k+1)}(aq^p, aq^{p+1})\right] \cdot \left\{\left[\sigma_{-(k+1),n}(a, aq^p)\right]\right\} + \left[\tau_{-(k+1),-(k+2)}(aq^p, aq^{p+1})\right]\left[\sigma_{-(k+2),n}(a, aq^p)\right] \tag{46}$$

$$\tau_{-(k+1),-(n+1)}(a, aq^{p+1}) = \left[\sigma_{-(k+1),k+1}(aq^p, aq^{p+1})\right]\left[\sigma_{k+1,-(n+1)}(a, aq^p)\right] + \left[\tau_{-(k+1),-(k+1)}(aq^p, aq^{p+1})\right]\left[\tau_{-(k+1),-(n+1)}(a, aq^p)\right] + \left[\tau_{-(k+1),-(k+2)}(aq^p, aq^{p+1})\right]\left[\tau_{-(k+2),-(n+1)}(a, aq^p)\right] \tag{47}$$

In the proofs of these identites, we will often reduce them to the vanishing of a polynomial. To further that, the following easily proven identities will often be used:

Lemma 6.1.
$$q^d(q^e|q)_1 = (q^{d+e}|q)_1 - (q^d|q)_1$$
$$q^d(q^{f-g}|q)_1 = (q^{d+f-g}|q)_1 - (q^d|q)_1$$
$$(q^{f-g}|q)_1 = q^{-g}\left\{(q^f|q)_1 - (q^g|q)_1\right\}$$
$$(q^{-p}|q)_1 = -q^{-p}(q^p|q)_1$$

It is feasible to directly check the identities (44), (45), (46) and (47) since there is a great deal of cancellation. However the readability of the quantities involved is enhanced by formulating some simplification lemmas for ratios which may be interpreted as appearing in the identities.

It may be helpful motivationally to note that in both Lemmas 6.2 and 6.3, the first index of the transition coefficient in the denominator is always the zig-zag successor of that of the numerator. And in Lemma 6.2, there is a further difference in the powers of q between numerator and denominator; aq^{p+1} in the numerator vs. aq^p in the denominator.

Lemma 6.2. *For $k, n \geq 0$*

(1)
$$\frac{\tau_{k,n}(a, aq^{p+1})}{\sigma_{-(k+1),n}(a, aq^p)} = \frac{q^{n-k}(bcq^k, bdq^k, q^{-(p+1)}|q)_1}{bcdq^{2k}(q^{n-k}, q^{n-k-p-1}|q)_1},$$

(2)
$$\frac{\sigma_{k,-(n+1)}(a, aq^{p+1})}{\tau_{-(k+1),-(n+1)}(a, aq^p)} = \frac{aq^{n-k+p+1}(bcq^k, bdq^k, q^{-(p+1)}|q)_1}{(abcdq^{n+k}, abcdq^{n+k+p+1}|q)_1},$$

(3)
$$\frac{\sigma_{-(k+1),n}(a, aq^{p+1})}{\tau_{k+1,n}(a, aq^p)} = \frac{abcdq^{n+k+p+1}(q^{k+1}, cdq^k, q^{-(p+1)}|q)_1}{(abcdq^{n+k}, abcdq^{n+k+p+1}|q)_1},$$

(4)
$$\frac{\tau_{-(k+1),-(n+1)}(a, aq^{p+1})}{\sigma_{k+1,-(n+1)}(a, aq^p)} = \frac{q^{n-k}(q^{k+1}, cdq^k, q^{-(p+1)}|q)_1}{(q^{n-k}, q^{n-k-p-1}|q)_1}.$$

Proof.

(1)
$$\Bigg(\left\{(q^{n-k+1}|q)_k[aq^{p+1}]^{n-k}q^{n-k}(bcq^k|q)_{n-k}\right\} \cdot \left\{\frac{(bdq^k|q)_{n-k}(cdq^k|q)_{n-k}(q^{-(p+1)}|q)_{n-k}}{(q|q)_k(abcdq^{n+k}|q)_{n-k}(bcd[aq^{p+1}]q^{2k}|q)_{n-k}}\right\}\Bigg)$$
$$\cdot \Bigg(\left\{\frac{1}{(q^{n-k}|q)_{k+1}bcd[aq^p]^{n-k}q^{n+k}(bcq^{k+1}|q)_{n-k-1}}\right\} \cdot \left\{\frac{(q|q)_k(abcdq^{n+k}|q)_{n-k}(bcd[aq^p]q^{2k+1}|q)_{n-k}}{(bdq^{k+1}|q)_{n-k-1}(cdq^k|q)_{n-k}(q^{-p}|q)_{n-k}}\right\}\Bigg)$$
$$= \frac{q^{n-k}(bcq^k, bdq^k, q^{-(p+1)}|q)_1}{bcdq^{2k}(q^{n-k}, q^{n-k-p-1}|q)_1}.$$

(2)
$$\Bigg(\left\{(q^{n-k+1}|q)_k\left[aq^{p+1}\right]^{n+1-k}(bcq^k|q)_{n+1-k}\right\} \cdot \left\{\frac{(bdq^k|q)_{n+1-k}(cdq^k|q)_{n-k}(q^{-(p+1)}|q)_{n+1-k}}{(q|q)_k(abcdq^{n+k}|q)_{n+1-k}(bcd\left[aq^{p+1}\right]q^{2k}|q)_{n+1-k}}\right\}\Bigg)$$
$$\cdot \Bigg(\left\{\frac{1}{(q^{n-k+1}|q)_k\left[aq^p\right]^{n-k}}\right\}\left\{\frac{(q|q)_k(abcdq^{n+k+1}|q)_{n-k}(bcd\left[aq^p\right]q^{2k+1}|q)_{n-k}}{(bcq^{k+1}|q)_{n-k}(bdq^{k+1}|q)_{n-k}(cdq^k|q)_{n-k}(q^{-p}|q)_{n-k}}\right\}\Bigg)$$
$$= \frac{aq^{n-k+p+1}(bcq^k, bdq^k, q^{-(p+1)}|q)_1}{(abcdq^{n+k}, abcdq^{n+k+p+1}|q)_1}.$$

$$(3)\quad \Big(\Big\{ (q^{n-k}|q)_{k+1} bcd \left[aq^{p+1}\right]^{n-k} q^{n+k} (bcq^{k+1}|q)_{n-k-1} \Big\} \cdot \Big\{ \frac{(bdq^{k+1}|q)_{n-k-1}(cdq^k|q)_{n-k}(q^{-(p+1)}|q)_{n-k}}{(q|q)_k(abcdq^{n+k}|q)_{n-k}(bcd\,[aq^{p+1}]\,q^{2k+1}|q)_{n-k}} \Big\} \Big)$$
$$\cdot \Big(\Big\{ \frac{1}{(q^{n-k}|q)_{k+1}\,[aq^p]^{n-k-1}\, q^{n-k-1}} \Big\} \cdot \Big\{ \frac{(q|q)_{k+1}(abcdq^{n+k+1}|q)_{n-k-1}(bcd\,[aq^p]\,q^{2(k+1)}|q)_{n-k-1}}{(bcq^{k+1}|q)_{n-k-1}(bdq^{k+1}|q)_{n-k-1}(cdq^{k+1}|q)_{n-k-1}(q^{-p}|q)_{n-k-1}} \Big\} \Big)$$
$$= \frac{abcdq^{n+k+p+1}(q^{k+1}, cdq^k, q^{-(p+1)}|q)_1}{(abcdq^{n+k}, abcdq^{n+k+p+1}|q)_1}.$$

$$(4)\quad \Big(\Big\{ (q^{n-k+1}|q)_k \left[aq^{p+1}\right]^{n-k} \Big\} \cdot \Big\{ \frac{(bcq^{k+1}|q)_{n-k}(bdq^{k+1}|q)_{n-k}(cdq^k|q)_{n-k}(q^{-(p+1)}|q)_{n-k}}{(q|q)_k(abcdq^{n+k+1}|q)_{n-k}(bcd\,[aq^{p+1}]\,q^{2k+1}|q)_{n-k}} \Big\} \Big)$$
$$\cdot \Big(\Big\{ \frac{1}{(q^{n-k}|q)_{k+1}\,[aq^p]^{n-k}} \Big\} \Big\{ \frac{(q|q)_{k+1}(abcdq^{n+k+1}|q)_{n-k}(bcd\,[aq^p]\,q^{2(k+1)}|q)_{n-k}}{(bcq^{k+1}|q)_{n-k}(bdq^{k+1}|q)_{n-k}(cdq^{k+1}|q)_{n-k-1}(q^{-p}|q)_{n-k}} \Big\} \Big)$$
$$= \frac{q^{n-k}(q^{k+1}, cdq^k, q^{-(p+1)}|q)_1}{(q^{n-k}, q^{n-k-p-1}|q)_1}.$$

This completes the proof. ■

Lemma 6.3. *For* $k, n \geq 0$ *:*

$$(1)\quad \frac{\tau_{k,n}(a, aq^p)}{\sigma_{-(k+1),n}(a, aq^p)} = \frac{(bcq^k, bdq^k, abcdq^{n+k+p}|q)_1}{bcdq^{2k}(q^{n-k}, abcdq^{2k+p}|q)_1},$$

$$(2)\quad \frac{\sigma_{k,-(n+1)}(a, aq^p)}{\tau_{-(k+1),-(n+1)}(a, aq^p)} = \frac{aq^p(bcq^k, bdq^k, q^{n-k-p}|q)_1}{(abcdq^{n+k}, abcdq^{2k+p}|q)_1},$$

$$(3)\quad \frac{\sigma_{-(k+1),n}(a, aq^p)}{\tau_{k+1,n}(a, aq^p)} = \frac{abcdq^{2k+p+1}(q^{k+1}, q^{n-k-p-1}, cdq^k|q)_1}{(abcdq^{n+k}, abcdq^{2k+p+1}|q)_1},$$

$$(4)\quad \frac{\tau_{-(k+1),-(n+1)}(a, aq^p)}{\sigma_{k+1,-(n+1)}(a, aq^p)} = \frac{(q^{k+1}, cdq^k, abcdq^{n+k+p+1}|q)_1}{(q^{n-k}, abcdq^{2k+p+1}|q)_1}.$$

Proof.

$$(1)\quad \Big(\Big\{ (q^{n-k+1}|q)_k [aq^p]^{n-k} q^{n-k} \Big\} \cdot \Big\{ \frac{(bcq^k|q)_{n-k}(bdq^k|q)_{n-k}(cdq^k|q)_{n-k}(q^{-p}|q)_{n-k}}{(q|q)_k(abcdq^{n+k}|q)_{n-k}(bcd[aq^p]q^{2k}|q)_{n-k}} \Big\} \Big)$$
$$\cdot \Big(\Big\{ \frac{1}{(q^{n-k}|q)_{k+1} bcd[aq^p]^{n-k} q^{n+k}} \Big\} \cdot \Big\{ \frac{(q|q)_k(abcdq^{n+k}|q)_{n-k}(bcd[aq^p]q^{2k+1}|q)_{n-k}}{(bcq^{k+1}|q)_{n-k-1}(bdq^{k+1}|q)_{n-k-1}(cdq^k|q)_{n-k}(q^{-p}|q)_{n-k}} \Big\} \Big)$$
$$= \frac{(bcq^k, bdq^k, abcdq^{n+k+p}|q)_1}{bcdq^{2k}(q^{n-k}, abcdq^{2k+p}|q)_1}.$$

(2)
$$\left(\left\{(q^{n-k+1}|q)_k\,[aq^p]^{n+1-k}\right\}\cdot\frac{(bcq^k|q)_{n+1-k}(bdq^k|q)_{n+1-k}(cdq^k|q)_{n-k}(q^{-p}|q)_{n+1-k}}{(q|q)_k(abcdq^{n+k}|q)_{n+1-k}(bcd\,[aq^p]\,q^{2k}|q)_{n+1-k}}\right)$$
$$\cdot\left(\left\{\frac{1}{(q^{n-k+1}|q)_k\,[aq^p]^{n-k}\,(bcq^{k+1}|q)_{n-k}}\right\}\cdot\left\{\frac{(q|q)_k(abcdq^{n+k+1}|q)_{n-k}(bcd\,[aq^p]\,q^{2k+1}|q)_{n-k}}{(bdq^{k+1}|q)_{n-k}(cdq^k|q)_{n-k}(q^{-p}|q)_{n-k}}\right\}\right)$$
$$=\frac{aq^p(bcq^k,bdq^k,q^{n-k-p}|q)_1}{(abcdq^{n+k},abcdq^{2k+p}|q)_1}\,.$$

(3)
$$\left(\left\{(q^{n-k}|q)_{k+1}bcd\,[aq^p]^{n-k}\,q^{n+k}\right\}\cdot\frac{(bcq^{k+1}|q)_{n-k-1}(bdq^{k+1}|q)_{n-k-1}(cdq^k|q)_{n-k}(q^{-p}|q)_{n-k}}{(q|q)_k(abcdq^{n+k}|q)_{n-k}(bcd\,[aq^p]\,q^{2k+1}|q)_{n-k}}\right)$$
$$\cdot\left(\left\{\frac{1}{(q^{n-k}|q)_{k+1}\,[aq^p]^{n-k-1}\,q^{n-k-1}}\right\}\cdot\left\{\frac{(q|q)_{k+1}(abcdq^{n+k+1}|q)_{n-k-1}(bcd\,[aq^p]\,q^{2(k+1)}|q)_{n-k-1}}{(bcq^{k+1}|q)_{n-k-1}(bdq^{k+1}|q)_{n-k-1}(cdq^{k+1}|q)_{n-k-1}(q^{-p}|q)_{n-k-1}}\right\}\right)$$
$$=\frac{abcdq^{2k+p+1}(q^{k+1},q^{n-k-p-1},cdq^k|q)_1}{(abcdq^{n+k},abcdq^{2k+p+1}|q)_1}\,.$$

(4)
$$\left(\left\{(q^{n-k+1}|q)_k\,[aq^p]^{n-k}\right\}\cdot\left\{\frac{(bcq^{k+1}|q)_{n-k}(bdq^{k+1}|q)_{n-k}(cdq^k|q)_{n-k}(q^{-p}|q)_{n-k}}{(q|q)_k(abcdq^{n+k+1}|q)_{n-k}(bcd\,[aq^p]\,q^{2k+1}|q)_{n-k}}\right\}\right)$$
$$\cdot\left(\left\{\frac{1}{(q^{n-k}|q)_{k+1}\,[aq^p]^{n-k}}\right\}\cdot\frac{(q|q)_{k+1}(abcdq^{n+k+1}|q)_{n-k}(bcd\,[aq^p]\,q^{2(k+1)}|q)_{n-k}}{(bcq^{k+1}|q)_{n-k}(bdq^{k+1}|q)_{n-k}(cdq^{k+1}|q)_{n-k-1}(q^{-p}|q)_{n-k}}\right)$$
$$=\frac{(q^{k+1},cdq^k,abcdq^{n+k+p+1}|q)_1}{(q^{n-k},abcdq^{2k+p+1}|q)_1}.$$
■

The T^{00} identity

Proposition 6.4. *When* $0\le k\le n-1$

$$\tau_{k,n}(a,aq^{p+1})=\left[\tau_{k,k}(aq^p,aq^{p+1})\right]\left[\tau_{k,n}(a,aq^p)\right]+\left[\tau_{k,k+1}(aq^p,aq^{p+1})\right]\left[\tau_{k+1,n}(a,aq^p)\right]$$
$$+\left[\sigma_{k,-(k+1)}(aq^p,aq^{p+1})\right]\left[\sigma_{-(k+1),n}(a,aq^p)\right].$$

And $$\tau_{n,n}(a,aq^{p+1})=\left[\tau_{n,n}(aq^p,aq^{p+1})\right]\left[\tau_{n,n}(a,aq^p)\right].$$

Proof. The second identity just says $1=1\cdot 1$.
For the first, upon dividing by $\sigma_{-(k+1),n}(a,aq^p)$, we see it is sufficient to prove

$$\frac{\tau_{k,n}(a,aq^{p+1})}{\sigma_{-(k+1),n}(a,aq^p)}=\left\{\tau_{k,k}(aq^p,aq^{p+1})\right\}\left\{\frac{\tau_{k,n}(a,aq^p)}{\sigma_{-(k+1),n}(a,aq^p)}\right\}$$
$$+\left\{\tau_{k,k+1}(aq^p,aq^{p+1})\right\}\left\{\frac{\tau_{k+1,n}(a,aq^p)}{\sigma_{-(k+1),n}(a,aq^p)}\right\}+\left\{\sigma_{k,-(k+1)}(aq^p,aq^{p+1})\right\}.$$

Using Lemmas 6.2 and 6.3, this means we need to show

$$\begin{aligned}\left\{\frac{q^{n-k}(bcq^k,bdq^k,q^{-(p+1)}|q)_1}{bcdq^{2k}(q^{n-k},q^{n-k-p-1}|q)_1}\right\} &= \{1\}\,\frac{(bcq^k,bdq^k,abcdq^{n+k+p}|q)_1}{bcdq^{2k}(q^{n-k},abcdq^{2k+p}|q)_1}\\ &+\left\{\frac{(q^2|q)_k[aq^{p+1}]q(bcq^k|q)_1(bdq^k|q)_1(cdq^k|q)_1(q^{-1}|q)_1}{(q|q)_k([aq^p]bcdq^{2k+1}|q)_1(bcd[aq^{p+1}]q^{2k}|q)_1}\right\}\\ &\cdot\left\{\frac{(abcdq^{n+k},abcdq^{2k+p+1}|q)_1}{abcdq^{2k+p+1}(q^{k+1},q^{n-k-p-1},cdq^k|q)_1}\right\}\\ &+\left\{\frac{(q|q)_k[aq^{p+1}](bcq^k|q)_1(bdq^k|q)_1(q^{-1}|q)_1}{(q|q)_k([aq^p]bcdq^{2k}|q)_1(bcd[aq^{p+1}]q^{2k}|q)_1}\right\}\cdot 1 \qquad (48)\end{aligned}$$

Multiplying (48) by

$$\frac{bcdq^{3k+p+1}(q,q^{n-k},abcdq^{2k+p},abcdq^{2k+p+1},q^{n-k-p-1}|q)_1}{(bcq^k,bdq^k|q)_1}$$

we see it is sufficient to show the vanishing of the polynomial p_1 below. (To arrive at the final form of p_1, we use Lemma 6.1 above to simplify the following expressions:

$$(q^{-1}|q)_1,\ (q^{-(p+1)}|q)_1,\ (q^{n-k}|q)_1,\ \text{and}\ (q^{n-k-p-1}|q)_1.\ )$$

We will eventually reduce this identity to the vanishing of a 1-variable polynomial in q with coefficients in the field $\mathbb{Q}(a,b,c,d,y,u,v,w)$ with the property that when

$$y = abcd \quad u = q^n \quad v = q^k \quad w = q^p$$

we obtain a unit in the coefficient field times the difference between the two sides of equation (48) above. So, effectively, we can use the variables y, u, v, w as abbreviations for these expressions.

$$\begin{aligned}p_1 = &-\left\{(yq^{2k+p}|q)_1(yq^{2k+p+1}|q)_1(q|q)_1q^{n+p+1}(q^{-(p+1)}|q)_1\right\}\\ &+\{1\}\left\{(yq^{n+k+p}|q)_1(yq^{2k+p+1}|q)_1q^{k+p+1}(q|q)_1(q^{n-k-p-1}|q)_1\right\}\\ &+\left\{(yq^{n+k}|q)_1(yq^{2k+p}|q)_1(q^{n-k}|q)_1q^{k+p+2}(q^{-1}|q)_1\right\}\\ &+\left\{(q^{n-k}|q)_1y(q|q)_1q^{3k+2p+2}(q^{-1}|q)_1(q^{n-k-p-1}|q)_1\right\} \qquad (49)\\ = &-\left\{(yq^{2k+p}|q)_1(yq^{2k+p+1}|q)_1(q|q)_1q^{n+p+1}\left[-q^{-(p+1)}(q^{p+1}|q)_1\right]\right\}\\ &+\{1\}\left\{(yq^{n+k+p}|q)_1(yq^{2k+p+1}|q)_1(q|q)_1q^{k+p+1}\right.\\ &\qquad\left.\left[q^{-(k+p+1)}\left[(q^n|q)_1-(q^{k+p+1}|q)_1\right]\right]\right\}\\ &+\left\{(yq^{n+k}|q)_1(yq^{2k+p}|q)_1q^{k+p+2}\left[q^{-k}\left[(q^n|q)_1-(q^k|q)_1\right]\right]\left[-q^{-1}(q|q)_1\right]\right\}\\ &+\left\{\left[q^{-k}\left[(q^n|q)_1-(q^k|q)_1\right]\right](q|q)_1yq^{3k+2p+2}\left[-q^{-1}(q|q)_1\right]\right.\\ &\cdot\left.\left[q^{-(k+p+1)}\left[(q^n|q)_1-(q^{k+p+1}|q)_1\right]\right]\right\}\\ = &\left\{(yq^{2k+p}|q)_1(yq^{2k+p+1}|q)_1(q|q)_1q^n(q^{p+1}|q)_1\right\}\\ &+\{1\}\left\{(yq^{n+k+p}|q)_1(yq^{2k+p+1}|q)_1(q|q)_1\left[(q^n|q)_1-(q^{k+p+1}|q)_1\right]\right\}\\ &-\left\{(yq^{n+k}|q)_1(yq^{2k+p}|q)_1q^{p+1}\left[(q^n|q)_1-(q^k|q)_1\right](q|q)_1\right\}\\ &-\left\{\left[(q^n|q)_1-(q^k|q)_1\right](q|q)_1yq^{k+p}\left[(q|q)_1\right]\left[(q^n|q)_1-(q^{k+p+1}|q)_1\right]\right\}\end{aligned}$$

Using the notation $\big[a, b, c, \dots\big] = (a|q)_1(b|q)_1(c|q)_1\dots$, and multiplying by $[(q|q)_1]^{-1}$ it suffices to show the vanishing of

$$\begin{aligned} p_2 = {} & u[yv^2w, yv^2wq, wq] + (vwq-u)[yuvw, yv^2wq] \\ & - wq(v-u)[yuv, yv^2w] - yvw(v-u)(vwq-u)(q|q)_1 \end{aligned}$$

This expression may be interpreted as a one variable polynomial of degree at most 2 in q with coefficients in the field $\mathbb{Q}(y, u, v, w)$. The coefficient of q^2 is

$$u(1-yv^2w)(-yv^2w)(-w) + (vw)(1-yuvw)(-yv^2w) - yvw(v-u)(vw)(-1) = 0.$$

So the polynomial p_2 is of degree at most 1 in q. Evaluating

at $q = 0$: $\quad p_2(0) = u(1-yv^2w) - u(1-yuvw) - yvw(v-u)(-u) = 0;$

at $q = u(vw)^{-1}$: Note at this point $yv^2wq = yuv$ and $wq = uv^{-1}$. So

$$p_2\big(u(vw)^{-1}\big) = u[yv^2w, yuv](1-uv^{-1}) - uv^{-1}(v-u)[yv^2w, yuv] = 0.$$

Thus the polynomial p_2 is 0 and the proposition is proven. ∎

The T^{01} identity

Proposition 6.5. *When* $0 \le k \le n-1$

$$\begin{aligned} \sigma_{k,-(n+1)}(a, aq^{p+1}) = {} & \big[\tau_{k,k}(aq^p, aq^{p+1})\big]\big[\sigma_{k,-(n+1)}(a, aq^p)\big] \\ & + \big[\tau_{k,k+1}(aq^p, aq^{p+1})\big]\big[\sigma_{k+1,-(n+1))}(a, aq^p)\big] \\ & + \big[\sigma_{k,-(k+1)}(aq^p, aq^{p+1})\big]\big[\tau_{-(k+1),-(n+1)}(a, aq^p)\big]. \end{aligned} \tag{50}$$

And

$$\begin{aligned} \sigma_{n,-(n+1)}(a, aq^{p+1}) = {} & \big[\tau_{n,n}(aq^p, aq^{p+1})\big]\big[\sigma_{n,-(n+1)}(a, aq^p)\big] \\ & + \big[\sigma_{n,-(n+1)}(aq^p, aq^{p+1})\big]\big[\tau_{-(n+1),-(n+1)}(a, aq^p)\big]. \end{aligned} \tag{51}$$

Proof. The second is immediate from Corollary 5.2 together with the observation that $\tau_{r,r} = 1$ for any sign of r.

For the first, upon dividing by $\tau_{-(k+1),-(n+1)}(a, aq^p)$, we see it is sufficient to prove

$$\begin{aligned} \frac{\sigma_{k,-(n+1)}(a, aq^{p+1})}{\tau_{-(k+1),-(n+1)}(a, aq^p)} = {} & \big\{\tau_{kk}(aq^p, aq^{p+1})\big\}\left\{\frac{\sigma_{k,-(n+1)}(a, aq^p)}{\tau_{-(k+1),-(n+1)}(a, aq^p)}\right\} \\ + \big\{\tau_{k,k+1}(aq^p, aq^{p+1})\big\}\left\{\frac{\sigma_{k+1,-(n+1))}(a, aq^p)}{\tau_{-(k+1),-(n+1)}(a, aq^p)}\right\} & + \big\{\sigma_{k,-(k+1)}(aq^p, aq^{p+1})\big\} \cdot 1. \end{aligned}$$

That means we need to show

$$\begin{aligned} \frac{aq^{n-k+p+1}(bcq^k, bdq^k, q^{-(p+1)}|q)_1}{(abcdq^{n+k}, abcdq^{n+k+p+1}|q)_1} = {} & \{1\}\left\{\frac{aq^p(bcq^k, bdq^k, q^{n-k-p}|q)_1}{(abcdq^{n+k}, abcdq^{2k+p}|q)_1}\right\} \\ & + \left\{\frac{(q^2|q)_k\,\big[aq^{p+1}\big]\,q(bcq^k|q)_1(bdq^k|q)_1(cdq^k|q)_1(q^{-1}|q)_1}{(q|q)_k([aq^p]\,bcdq^{2k+1}|q)_1(bcd\,[aq^{p+1}]\,q^{2k}|q)_1}\right\} \\ & \cdot\left\{\frac{(q^{n-k}, abcdq^{2k+p+1}|q)_1}{(q^{k+1}, cdq^k, abcdq^{n+k+p+1}|q)_1}\right\} \\ & + \left\{\frac{(q|q)_k\,\big[aq^{p+1}\big]\,(bcq^k|q)_1(bdq^k|q)_1(q^{-1}|q)_1}{(q|q)_k([aq^p]\,bcdq^{2k}|q)_1(bcd\,[aq^{p+1}]\,q^{2k}|q)_1}\right\} \cdot 1 \end{aligned} \tag{52}$$

We will eventually reduce this identity to the vanishing of a 1-variable polynomial in y with coefficients in the field $\mathbb{Q}(a,b,c,d,u,v,w,q)$ with the property that when

$$y = abcd, \quad u = q^n, \quad v = q^k, \quad w = q^p,$$

we obtain a unit in the coefficient field times the difference between the two sides of equation (52) above. So, effectively, we can use the variables y,u,v,w as abbreviations for the above expressions. Multiplying (52) by

$$\frac{q^k(abcdq^{n+k}, abcdq^{n+k+p+1}, abcdq^{2k+p}, abcdq^{2k+p+1}|q)_1}{a(bcq^k, bdq^k|q)_1} \tag{53}$$

we see it is sufficient to show the vanishing of the polynomial p_1 below. We use y as an abbreviation for $abcd$. (To arrive at the final form of p_1, we use Lemma 6.1 above to simplify the following expressions:

$$(q^{-1}|q)_1,\ (q^{-p}|q)_1,\ (q^{-(p+1)}|q)_1,\ (q^{n-k}|q)_1, \text{ and } (q^{n-k-p}|q)_1\big).$$

$$\begin{aligned}
p_1 = &-\{(yq^{2k+p}|q)_1(yq^{2k+p+1}|q)_1 q^{n+p+1}(q^{-(p+1)}|q)_1\} \\
&+\{1\}\{(yq^{n+k+p+1}|q)_1(yq^{2k+p+1}|q)_1 q^{k+p}(q^{n-k-p}|q)_1\} \\
&+\left\{\frac{(q^{n-k}|q)_1(yq^{n+k}|q)_1(yq^{2k+p}|q)_1 q^{k+p+2}(q^{-1}|q)_1}{(q|q)_1}\right\} \\
&+\{(yq^{n+k}|q)_1(yq^{n+k+p+1}|q)_1 q^{k+p+1}(q^{-1}|q)_1\} \\
= &-\Big\{(yq^{2k+p}|q)_1(yq^{2k+p+1}|q)_1 q^{n+p+1}\Big\{-q^{-(p+1)}(q^{p+1}|q)_1\Big\}\Big\} \\
&+\{1\}\{(yq^{n+k+p+1}|q)_1(yq^{2k+p+1}|q)_1 q^{k+p}\{q^{-(k+p)}\left[(q^n|q)_1-(q^{k+p}|q)_1\right]\}\} \\
&+\frac{\{\{q^{-k}\left[(q^n|q)_1-(q^k|q)_1\right]\}(yq^{n+k}|q)_1(yq^{2k+p}|q)_1 q^{k+p+2}\left[-q^{-1}(q|q)_1\right]\}}{(q|q)_1} \\
&+\{(yq^{n+k}|q)_1(yq^{n+k+p}|q)_1 q^{k+p+1}\left[-q^{-1}(q|q)_1\right]\} \\
= &\Big\{(yq^{2k+p}|q)_1(yq^{2k+p+1}|q)_1 q^n\Big\{(q^{p+1}|q)_1\Big\}\Big\} \\
&+\{1\}\{(yq^{n+k+p+1}|q)_1(yq^{2k+p+1}|q)_1\{\left[(q^n|q)_1-(q^{k+p}|q)_1\right]\}\} \\
&-\{(yq^{n+k}|q)_1(yq^{2k+p}|q)_1 q^{p+1}\}\{\left[(q^n|q)_1-(q^k|q)_1\right]\} \\
&-\{(yq^{n+k}|q)_1(yq^{n+k+p+1}|q)_1 q^{k+p}\left[(q|q)_1\right]\}
\end{aligned}$$

Using the notation $\big[a,b,c,\dots\big] = (a|q)_1(b|q)_1(c|q)_1\dots$, setting $y = abcd$, $u = q^n$, $v = q^k$, and $w = q^p$, we see it is sufficient to show the vanishing of

$$\begin{aligned}
p_2 = u\left[yv^2w, yv^2wq, wq\right] + (vw-u)\left[yuvwq, yv^2wq\right] \\
- wq(v-u)\left[yuv, yv^2w\right] - vw\left[yuv, yuvwq, q\right]
\end{aligned}$$

his expression may be interpreted as a one variable polynomial of degree at most 2 in y with coefficients in the field $\mathbb{Q}(u,v,w,q)$. Evaluating

at $y = 0$: $p_2(0) = u(1-wq) + (vw-u) - wq(v-u) - vw(1-q) = 0;$

at $y = (v^2wq)^{-1}$: When $y = (v^2wq)^{-1}$, note $yv^2w = q^{-1}$, $yuv = u(vwq)^{-1}$, and $yuvwq = uv^{-1}$. So

$$\begin{aligned} p_2\big((v^2wq)^{-1}\big) &= -wq(v-u)\left[u(vwq)^{-1}, q^{-1}\right] - vw\left[u(vwq)^{-1}, uv^{-1}, q\right] \\ &= (1-u(vwq)^{-1})\big\{-wq(v-u)\left(-q^{-1}(1-q)\right) - vw(1-uv^{-1})(1-q)\big\} \\ &= w(1-u(vwq)^{-1})(1-q)\big\{v-u-v(1-uv^{-1})\big\} = 0\,; \end{aligned}$$

at $y = (uv)^{-1}$: When $y = (uv)^{-1}$, note $yv^2w = u^{-1}vw$, $yv^2wq = u^{-1}vwq$, and $yuvwq = wq$. So

$$p_2\big((uv)^{-1}\big) = (1-u^{-1}vwq)(1-wq)\big\{u(1-u^{-1}vw) + vw - u\big\} = 0$$

Thus the polynomial p_2 is 0 and the proposition is proven. ■

The T^{10} identity

Proposition 6.6. *When* $0 \le k \le n-2$

$$\begin{aligned} \sigma_{-(k+1),n}(a, aq^{p+1}) = &\left[\sigma_{-(k+1),k+1}(aq^p, aq^{p+1})\right]\left[\tau_{k+1,n}(a, aq^p)\right] \\ &+ \left[\tau_{-(k+1),-(k+1)}(aq^p, aq^{p+1})\right]\left[\sigma_{-(k+1),n}(a, aq^p)\right] \\ &+ \left[\tau_{-(k+1),-(k+2)}(aq^p, aq^{p+1})\right]\left[\sigma_{-(k+2),n}(a, aq^p)\right] \end{aligned} \tag{54}$$

And

$$\begin{aligned} \sigma_{-n,n}(a, aq^{p+1}) = &\left[\sigma_{-n,n}(aq^p, aq^{p+1})\right]\left[\tau_{n,n}(a, aq^p)\right] \\ &+ \left[\tau_{-n,-n}(aq^p, aq^{p+1})\right]\left[\sigma_{-n,n}(a, aq^p)\right]. \end{aligned}$$

Proof. The second is immediate from Corollary 5.2 together with the observation that $\tau_{r,r} = 1$ for any sign of r.

For the first, upon dividing by $\tau_{k+1,n}(a, aq^p)$, we see it is sufficient to prove

$$\begin{aligned} \frac{\sigma_{-(k+1),n}(a, aq^{p+1})}{\tau_{k+1,n}(a, aq^p)} = &\left\{\sigma_{-(k+1),k+1}(aq^p, aq^{p+1})\right\} \cdot 1 \\ &+ \left\{\tau_{-(k+1),-(k+1)}(aq^p, aq^{p+1})\right\}\left\{\frac{\sigma_{-(k+1),n}(a, aq^p)}{\tau_{k+1,n}(a, aq^p)}\right\} \\ &+ \left\{\tau_{-(k+1),-(k+2)}(aq^p, aq^{p+1})\right\}\left\{\frac{\sigma_{-(k+2),n}(a, aq^p)}{\tau_{k+1,n}(a, aq^p)}\right\}. \end{aligned}$$

That means we need to show

$$\begin{aligned} &\frac{abcdq^{n+k+p+1}(q^{k+1}, cdq^k, q^{-(p+1)}|q)_1}{(abcdq^{n+k}, abcdq^{n+k+p+1}|q)_1} \\ &= \left\{\frac{(q|q)_{k+1}bcd\left[aq^{p+1}\right]q^{2k+1}(cdq^k|q)_1(q^{-1}|q)_1}{(q|q)_k([aq^p]\,bcdq^{2k+1}|q)_1(bcd\left[aq^{p+1}\right]q^{2k+1}|q)_1}\right\} \cdot 1 \\ &\quad + \{1\} \cdot \left\{\frac{abcdq^{2k+p+1}(q^{k+1}, q^{n-k-p-1}, cdq^k|q)_1}{(abcdq^{n+k}, abcdq^{2k+p+1}|q)_1}\right\} \\ &\quad + \left\{\frac{(q^2|q)_k\left[aq^{p+1}\right](bcq^{k+1}|q)_1(bdq^{k+1}|q)_1(cdq^k|q)_1(q^{-1}|q)_1}{(q|q)_k([aq^p]\,bcdq^{2(k+1)}|q)_1(bcd\left[aq^{p+1}\right]q^{2k+1}|q)_1}\right\} \\ &\quad \cdot \left\{\frac{bcdq^{2(k+1)}(q^{n-k-1}, abcdq^{2k+p+2}|q)_1}{(bcq^{k+1}, bdq^{k+1}, abcdq^{n+k+p+1}|q)_1}\right\}. \end{aligned} \tag{55}$$

We will eventually reduce this identity to the vanishing of a 1-variable polynomial in y with coefficients in the field $\mathbb{Q}(a,b,c,d,u,v,w,q)$ with the property that when

$$y = abcd \quad u = q^n \quad v = q^k \quad w = q^p$$

we obtain a unit in the coefficient field times the difference between the two sides of equation (55) above. So, effectively, we can use the variables y,u,v,w as abbreviations for the above expressions. Multiplying (55) by

$$\frac{(q, abcdq^{n+k}, abcdq^{n+k+p+1}, abcdq^{2k+p+1}, abcdq^{2k+p+2}|q)_1}{abcd(q^{k+1}, cdq^k|q)_1}$$

we see it is sufficient to show the vanishing of the polynomial p_1 below. (To arrive at the final form of p_1, we use Lemma 6.1 above to simplify the following expressions:

$$(q^{-1}|q)_1, \quad (q^{-(p+1)}|q)_1, \quad \text{and} \quad (q^{n-k-p-1}|q)_1.)$$

$$\begin{aligned}
p_1 = &- \left\{(yq^{2k+p+1}|q)_1(yq^{2k+p+2}|q)_1(q|q)_1 q^{n+k+p+1}(q^{-(p+1)}|q)_1\right\}\\
&+ \left\{(yq^{n+k}|q)_1(yq^{n+k+p+1}|q)_1(q|q)_1 q^{2k+p+2}(q^{-1}|q)_1\right\}\\
&+ \{1\}\cdot\left\{(yq^{n+k+p+1}|q)_1(yq^{2k+p+2}|q)_1(q|q)_1 q^{2k+p+1}(q^{n-k-p-1}|q)_1\right\}\\
&+ \left\{(yq^{n+k}|q)_1(yq^{2k+p+1}|q)_1(q^{n-k-1}|q)_1 q^{2k+p+3}(q^{-1}|q)_1\right\}\\
= &- \left\{(yq^{2k+p+1}|q)_1(yq^{2k+p+2}|q)_1(q|q)_1 q^{n+k+p+1}\left[-q^{-(p+1)}(q^{p+1}|q)_1\right]\right\}\\
&+ \left\{(yq^{n+k}|q)_1(yq^{n+k+p+1}|q)_1(q|q)_1 q^{2k+p+2}\left[-q^{-1}(q|q)_1\right]\right\}\\
&+ \{1\}\cdot\left\{(yq^{n+k+p+1}|q)_1(yq^{2k+p+2}|q)_1(q|q)_1 q^{2k+p+1}\right.\\
&\qquad\qquad \left.\left[q^{-(k+p+1)}\left[(q^n|q)_1 - (q^{k+p+1}|q)_1\right]\right]\right\}\\
&+ \left\{(yq^{n+k}|q)_1(yq^{2k+p+1}|q)_1\left[q^{-(k+1)}\left[(q^n|q)_1 - (q^{k+1}|q)_1\right]\right] q^{2k+p+3}\left[-q^{-1}(q|q)_1\right]\right\}\\
= &\left\{(yq^{2k+p+1}|q)_1(yq^{2k+p+2}|q)_1(q|q)_1 q^{n+k}(q^{p+1}|q)_1\right\}\\
&- \left\{(yq^{n+k}|q)_1(yq^{n+k+p+1}|q)_1\left[(q|q)_1\right]^2 q^{2k+p+1}\right\}\\
&+ \{1\}\cdot\left\{(yq^{n+k+p+1}|q)_1(yq^{2k+p+2}|q)_1(q|q)_1 q^k\left[\left[(q^n|q)_1 - (q^{k+p+1}|q)_1\right]\right]\right\}\\
&- \left\{(yq^{n+k}|q)_1(yq^{2k+p+1}|q)_1\left[\left[(q^n|q)_1 - (q^{k+1}|q)_1\right]\right] q^{k+p+1}(q|q)_1\right\}
\end{aligned}$$

Using the notation $[a,b,c,\dots] = (a|q)_1(b|q)_1(c|q)_1\dots$, recalling our abbreviation variables u,v,w, and multiplying by $v^{-1}(q|q)^{-1}$ it suffices to show the vanishing of

$$\begin{aligned}
p_2 = &\, u[yv^2wq, yv^2wq^2, wq] - vwq[yuv, yuvwq, q]\\
&+ (vwq-u)[yuvwq, yv^2wq^2] - wq(vq-u)[yuv, yv^2wq]
\end{aligned}$$

This expression may be interpreted as a one variable polynomial of degree at most 2 in y with coefficients in the field $\mathbb{Q}(u,v,w,q)$. Evaluating

at $y=0$: $p_2(0) = u(1-wq) - vwq(1-q) + (vwq-u) - wq(vq-u) = 0$;

at $y=(v^2wq)^{-1}$: Note that $yv^2wq^2 = q$, $yuvwq = uv^{-1}$ and $yuv = u(vwq)^{-1}$. So

$$\begin{aligned}
p_2\big((v^2wq)^{-1}\big) &= (1-uv^{-1})\big\{-vwq(1-u(vwq)^{-1})(1-q) + (vwq-u)(1-q)\big\}\\
&= (1-uv^{-1})(1-q)(vwq-u)\big\{-1+1\big\} = 0;
\end{aligned}$$

at $y=(uvwq)^{-1}$: Note that $yv^2wq=u^{-1}v$, $yv^2wq^2=u^{-1}vq$ and $yuv=(wq)^{-1}$. So

$$\begin{aligned} p_2\big((uvwq)^{-1}\big) &= (1-u^{-1}v)\big\{u(1-u^{-1}vq)(1-wq)-wq(vq-u)(1-(wq)^{-1}\big\} \\ &= (1-u^{-1}v)(vq-u)(1-wq)\big\{-1+1\big\}=0. \end{aligned}$$

Thus p_2 being of degree at most 2 and vanishing at 3 points implies p_2 is identically 0 and the proposition proof is complete. ■

The T^{11} Identity

Proposition 6.7. *When* $0\le k\le n-1$

$$\begin{aligned} \tau_{-(k+1),-(n+1)}(a,aq^{p+1}) = &\big[\sigma_{-(k+1),k+1}(aq^p,aq^{p+1})\big]\big[\sigma_{k+1,-(n+1)}(a,aq^p)\big] \\ &+\big[\tau_{-(k+1),-(k+1)}(aq^p,aq^{p+1})\big]\big[\tau_{-(k+1),-(n+1)}(a,aq^p)\big] \\ &+\big[\tau_{-(k+1),-(k+2)}(aq^p,aq^{p+1})\big]\big[\tau_{-(k+2),-(n+1)}(a,aq^p)\big]. \quad (56) \end{aligned}$$

And $\tau_{-(n+1),-(n+1)}(a,aq^{p+1})=\big[\tau_{-(n+1),-(n+1)}(aq^p,aq^{p+1})\big]\big[\tau_{-(n+1),-(n+1)}(a,aq^p)\big]$.

Proof. The second identity just says $1=1\cdot 1$.

For the first, upon dividing by $\sigma_{k+1,-(n+1)}(a,aq^p)$, we see it is sufficient to prove

$$\begin{aligned} \frac{\tau_{-(k+1),-(n+1)}(a,aq^{p+1})}{\sigma_{k+1,-(n+1)}(a,aq^p)} &= \big\{\sigma_{-(k+1),k+1}(aq^p,aq^{p+1})\big\}\cdot 1 \\ &+\big\{\tau_{-(k+1),-(k+1)}(aq^p,aq^{p+1})\big\}\left\{\frac{\tau_{-(k+1),-(n+1)}(a,aq^p)}{\sigma_{k+1,-(n+1)}(a,aq^p)}\right\} \\ &+\big\{\tau_{-(k+1),-(k+2)}(aq^p,aq^{p+1})\big\}\left\{\frac{\tau_{-(k+2),-(n+1)}(a,aq^p)}{\sigma_{k+1,-(n+1)}(a,aq^p)}\right\}. \end{aligned}$$

That means we need to show

$$\begin{aligned} &\frac{q^{n-k}(q^{k+1},cdq^k,q^{-(p+1)}|q)_1}{(q^{n-k},q^{n-k-p-1}|q)_1} \\ &\quad=\left\{\frac{(q|q)_{k+1}bcd\left[aq^{p+1}\right]q^{2k+1}(cdq^k|q)_1(q^{-1}|q)_1}{(q|q)_k(\left[aq^p\right]bcdq^{2k+1}|q)_1(bcd\left[aq^{p+1}\right]q^{2k+1}|q)_1}\right\}\cdot 1 \\ &\quad+\{1\}\left\{\frac{(q^{k+1},cdq^k,abcdq^{n+k+p+1}|q)_1}{(q^{n-k},abcdq^{2k+p+1}|q)_1}\right\} \\ &\quad+\left\{\frac{(q^2|q)_k\left[aq^{p+1}\right](bcq^{k+1}|q)_1(bdq^{k+1}|q)_1(cdq^k|q)_1(q^{-1}|q)_1}{(q|q)_k(\left[aq^p\right]bcdq^{2(k+1)}|q)_1(bcd\left[aq^{p+1}\right]q^{2k+1}|q)_1}\right\} \\ &\quad\cdot\left\{\frac{(abcdq^{n+k+1},abcdq^{2k+p+2}|q)_1}{aq^p(bcq^{k+1},bdq^{k+1},q^{n-k-p-1}|q)_1}\right\} \qquad (57) \end{aligned}$$

Multiplying (57) by

$$\frac{q^{k+p+1}(q,q^{n-k},q^{n-k-p-1},abcdq^{2k+p+1},abcdq^{2k+p+2}|q)_1}{(q^{k+1},cdq^k|q)_1}$$

we see it is sufficient to show the vanishing of the polynomial p_1 below.

(To arrive at the final form of p_1, we use Lemma 6.1 above to simplify the following expressions:

$$(q^{-1}|q)_1, \quad (q^{-p}|q)_1, \quad (q^{-(p+1)}|q)_1, \quad \text{and} \quad q^{n-k-p-1}|q)_1.)$$

We will eventually reduce this identity to the vanishing of a 1-variable polynomial in y with coefficients in the field $\mathbb{Q}(a,b,c,d,u,v,w,q)$ with the property that when

$$y = abcd, \quad u = q^n, \quad v = q^k \quad w = q^p,$$

we obtain a unit in the coefficient field times the difference between the two sides of equation (57) above. So, effectively, we can use the variables y, u, v, w as abbreviations for the above expressions.

$$\begin{aligned}
p_1 = &- \left\{(yq^{2k+p+1}|q)_1(yq^{2k+p+2}|q)_1(q|q)_1 q^{n+p+1}(q^{-(p+1)}|q)_1\right\} \\
&+ \left\{(q^{n-k}|q)_1(q|q)_1 yq^{3k+2p+3}(q^{-1}|q)_1(q^{n-k-p-1}|q)_1\right\} \\
&+ \{1\}\left\{(yq^{n+k+p+1}|q)_1(yq^{2k+p+2}|q)_1(q|q)_1 q^{k+p+1}(q^{n-k-p-1}|q)_1\right\} \\
&+ \left\{(yq^{n+k+1}|q)_1(yq^{2k+p+1}|q)_1(q^{n-k}|q)_1(q^{-1}|q)_1 q^{k+p+2}\right\} \\
= &-\left\{(yq^{2k+p+1}|q)_1(yq^{2k+p+2}|q)_1(q|q)_1 q^{n+p+1}\left[-q^{-(p+1)}(q^{p+1}|q)_1\right]\right. \\
&+ \left\{\left[q^{-k}\left[(q^n|q)_1 - (q^k|q)_1\right]\right](q|q)_1 yq^{3k+2p+3}\left[-q^{-1}(q|q)_1\right]\right. \\
&\cdot\left[q^{-k-p-1}\left[(q^n|q)_1 - (q^{k+p+1}|q)_1\right]\right]\Big\} \\
&+ \{1\}\left\{(yq^{n+k+p+1}|q)_1(yq^{2k+p+2}|q)_1(q|q)_1 q^{k+p+1}\right. \\
&\qquad\qquad \left[q^{-k-p-1}\left[(q^n|q)_1 - (q^{k+p+1}|q)_1\right]\right]\Big\} \\
&+ \left\{(yq^{n+k+1}|q)_1(yq^{2k+p+1}|q)_1\left[q^{-k}\left[(q^n|q)_1 - (q^k|q)_1\right]\right]\left[-q^{-1}(q|q)_1\right] q^{k+p+2}\right\} \\
= &\left\{(yq^{2k+p+1}|q)_1(yq^{2k+p+2}|q)_1(q|q)_1 q^n(q^{p+1}|q)_1\right\} \\
&- \left\{\left[(q^n|q)_1 - (q^k|q)_1\right] yq^{k+p+1}\left[(q|q)_1\right]^2\left[(q^n|q)_1 - (q^{k+p+1}|q)_1\right]\right\} \\
&+ \{1\}\left\{(yq^{n+k+p+1}|q)_1(yq^{2k+p+2}|q)_1(q|q)_1\left[(q^n|q)_1 - (q^{k+p+1}|q)_1\right]\right\} \\
&- \left\{(yq^{n+k+1}|q)_1(yq^{2k+p+1}|q)_1\left[(q^n|q)_1 - (q^k|q)_1\right](q|q)_1 q^{p+1}\right\}
\end{aligned}$$

Using the notation $\left[a,b,c,\dots\right] = (a|q)_1(b|q)_1(c|q)_1\dots$, recalling our abbreviation variables u, v, w, and multiplying by $\left[(q|q)_1\right]^{-1}$ it suffices to show the vanishing of

$$\begin{aligned}
p_2 = &\ u[yv^2wq, yv^2wq^2, wq] - yvwq(v-u)(vwq-u)(q|q)_1 \\
&+ (vwq-u)[yuvwq, yv^2wq^2] - wq(v-u)[yuvq, yv^2wq]
\end{aligned}$$

This expression may be interpreted as a one variable polynomial of degree at most 2 in y with coefficients in the field $\mathbb{Q}(u,v,w,q)$. Evaluating

at $y=0$: $p_2(0) = u(1-wq) + (vwq-u) - wq(v-u) = 0$.

at $y = (v^2wq)^{-1}$: Note that $yv^2wq^2 = q$, $yvwq = v^{-1}$, and $yuvwq = uv^{-1}$. So

$$\begin{aligned}
p_2\big((v^2wq)^{-1}\big) &= (vwq-u)\big\{-v^{-1}(v-u)(1-q) + (1-uv^{-1})(1-q)\big\} \\
&= (vwq-u)(1-uv^{-1})(1-q)\big\{-1+1\big\} = 0\,;
\end{aligned}$$

at $y=(v^2wq^2)^{-1}$: Note that $yv^2wq=q^{-1}$, $yvwq=(vq)^{-1}$, and $yuvq=u(vwq)^{-1}$. So

$$\begin{aligned} p_2\big((v^2wq^2)^{-1}\big) &= (v-u)\big\{-(vq)^{-1}(vwq-u)(1-q)-wq(1-u(vwq)^{-1})(1-q^{-1})\big\} \\ &= (v-u)(1-q)\big\{-(w-u(vq)^{-1})+(w-u(vq)^{-1})\big\}=0. \end{aligned}$$

Thus p_2 being of degree 2 and vanishing at 3 points implies p_2 is identically 0 and the proposition proof is complete. ■

The combination of Propositions 3.8, 5.3, 5.4, 6.4, 6.5, 6.6, and 6.7 finishes the proof of all 3 steps of (15), Proof Plan A, and so completes the proof of Theorem 1.3.

7. Reproof of the Askey-Wilson Result Using the Nonsymmetric Version

Having given a direct proof of the shift-a connection coefficient formula Theorem 1.3, we now show that it can be used to give another proof of the original Askey-Wilson result Theorem 1.2.

For this purpose, consistent with $x=\cos\theta, z=e^{i\theta}$, so $2x=(z+z^{-1})$, we view the n'th Askey-Wilson polynomial P_n as a zig-zag monic Laurent polynomial in z. So as an ordinary polynomial in x, the leading coefficient would be 2^n.

As earlier for the E_r, we will shorten $P_n(z;a,b,c,d|q)$ to $P_n(a)$.

First note that the usual DAHA relation on T_1

$$T_1-T_1^{-1}=t_1^{\frac{1}{2}}-t_1^{-\frac{1}{2}}$$

translates to the quadratic relation

$$\Big(T_1+t_1^{-\frac{1}{2}}\Big)\Big(T_1-t_1^{\frac{1}{2}}\Big)=0. \tag{58}$$

showing that the possible eigenvalues of T_1 are $-t_1^{-\frac{1}{2}}$ and $t_1^{\frac{1}{2}}$. Define

$$L_1=T_1+t_1^{-\frac{1}{2}} \quad\text{and}\quad L_2=-T_1+t_1^{\frac{1}{2}}. \tag{59}$$

Definition 1.2 in [20] defined the Askey-Wilson P_n (respectively Q_n) as normalized multiples of L_1E_n (respectively L_2E_n) for $n\geq 0$. Theorem 1.3 there (referring to [19]) pointed out that this definition of P_n agrees with the usual P_n.

It was pointed out in Theorem 1.3 of [20] that, up to a normalizing factor, $P_{|n|}$ is $(T_1+t_1^{-\frac{1}{2}})E_n$ for $n\geq 0$.

A similar fact holds when E_n is replaced by E_{-n} for $n>0$. The reason for this is that by the recursion (21) and (22), (originally proved in [20]) for $n>0$, the 2-dimensional subspace spanned by E_{-n} and E_n is invariant under T_1. So L_1+L_2 is a multiple of the identity on these 2-dimensional subspaces and easy DAHA calculations show:

Lemma 7.1. *Up to scalar factors, the operators L_1 and L_2 are algebraically orthogonal projections onto 1-dimensional subspaces of the 2-dimensional subspace (for $n\geq 0$) spanned by $E_{-(n+1)}$ and E_{n+1}. In fact*

(1) $L_1L_2=L_2L_1=0$, (2) $L_1^2=(t_1^{\frac{1}{2}}+t_1^{-\frac{1}{2}})L_1$, (3) $L_2^2=(t_1^{\frac{1}{2}}+t_1^{-\frac{1}{2}})L_2$.

Remark 7.2. By algebraically orthogonal projections, we are referring to endomorphisms π_1 and $\pi_2 = \text{Identity} - \pi_1$ satisfying $\pi_1^2 = \pi_1$. This of course also implies $\pi_2^2 = \pi_2$ and $\pi_1\pi_2 = \pi_2\pi_1 = 0$. These projections can also be related to the natural inner product specified in Definition 1.4 of [20] but we omit the details here. ■

Proof of Lemma 7.1. (1) By the basic DAHA relation we have

$$L_1 = T_1 + t_1^{-\frac{1}{2}} = T_1^{-1} + t_1^{\frac{1}{2}}.$$

Then
$$L_1L_2 = \left(T_1^{-1} + t_1^{\frac{1}{2}}\right)\left(-T_1 + t_1^{\frac{1}{2}}\right) = -1 - t_1^{\frac{1}{2}}\left(T_1 - T_1^{-1}\right) + t_1$$
$$= -1 - t_1^{\frac{1}{2}}\left(t_1^{\frac{1}{2}} - t_1^{-\frac{1}{2}}\right) + t_1 = 0.$$

And $L_2L_1 = L_1L_2$.

(2) $L_1^2 = L_1(L_1 + L_2) = \left(t_1^{\frac{1}{2}} + t_1^{-\frac{1}{2}}\right)L_1$.

(3) $L_2^2 = (L_1 + L_2)L_2 = \left(t_1^{\frac{1}{2}} + t_1^{-\frac{1}{2}}\right)L_2$. ■

Consequently, for $n > 0$, P_n is also a multiple of $\left(T_1 + t_1^{-\frac{1}{2}}\right)E_{-n}$. (The operator L_1, up to constant factors, is sometimes referred to as *Hecke-symmetrization.*) Translating to the operator $\widetilde{T}_1 = t_1^{\frac{1}{2}}T_1$ which we have mostly been using (and which avoids square roots), we see that P_n for $n > 0$ is a multiple of $(\widetilde{T}_1 + 1)E_{-n}$.

With the zig-zag monic normalization conventions, Proposition 3.7 and equation (26), give us more explicitly for $n \geq 0$

$$\widetilde{T}_1E_{-(n+1)} = [\hat{c}_{n+1}]^{-1}\left\{E_{n+1} - \hat{d}_{n+1}E_{-(n+1)}\right\}. \tag{60}$$

where $\hat{c}_{n+1} = -\frac{1}{ab} = t_1^{-1}$, $\hat{d}_{n+1} = -\frac{(abq^{n+1}|q)_1 + ab(cdq^n|q)_1}{ab(1 - abcdq^{2n+1})}$.

Proposition 7.3. *For $n \geq 0$: $P_{n+1} = t_1^{-1}(\widetilde{T}_1 + 1)E_{-(n+1)}$. (Also $P_0 = E_0 = 1$.)*
More explicitly: $P_{n+1}(a) = E_{n+1}(a) + [\gamma_{n+1}(a)]\,E_{-(n+1)}(a)$,
where $\gamma_{n+1}(a)$ (really a function of $a, b, c, d,$ and q) is the scalar defined by

$$\gamma_{n+1}(a) = \frac{(q^{n+1}, cdq^n|q)_1}{(abcdq^{2n+1}|q)_1}. \tag{61}$$

Proof. We've already argued $t_1^{-1}(\widetilde{T}_1 + 1)E_{-(n+1)}$ is a multiple of P_{n+1}. Equation (60) implies $t_1^{-1}[\hat{c}_{n+1}]^{-1} = 1$. And $P_{n+1} = E_{n+1} = f_{n+1} \bmod \mathcal{R}_{-(n+1)}$ while $E_{-(n+1)} \in \mathcal{R}_{-(n+1)}$. So $t_1^{-1}(\widetilde{T}_1 + 1)E_{-(n+1)}$ gives P_{n+1} exactly.

Remembering $t_1\hat{c}_{n+1} = 1$ and plugging the formulas of (60) into the equation $P_{n+1} = t_1^{-1}(\widetilde{T}_1 + 1)E_{-(n+1)}$ gives us

$$P_{n+1} = E_{n+1} + \left(\hat{c}_{n+1} - \hat{d}_{n+1}\right)E_{-(n+1)}.$$

(Since our normalization convention is that both P_{n+1} and E_{n+1} are zig-zag monic, the coefficient 1 above in front of E_{n+1} was known in advance.) Our definition of $\gamma_{n+1}(a)$ is just a simplification of $\hat{c}_{n+1} - \hat{d}_{n+1}$ as demonstrated in the easy Lemma 7.4 below. ■

Lemma 7.4. *For $n \geq 0$:* $\hat{c}_{n+1} - \hat{d}_{n+1} = \dfrac{(q^{n+1}, cdq^n|q)_1}{(abcdq^{2n+1}|q)_1}$.

This quantity is the $\gamma_{n+1}(a)$ (really $\gamma_{n+1}(a,b,c,d|q)$) defined in Proposition 7.3.

Proof.

$$\begin{aligned}\hat{c}_{n+1} - \hat{d}_{n+1} &= -\frac{1}{ab} + \frac{(abq^{n+1}|q)_1 + ab(cdq^n|q)_1}{ab(1-abcdq^{2n+1})} \\ &= -\left(\frac{1}{ab(1-abcdq^{2n+1})}\right)\left\{1 - abcdq^{2n+1} + (abq^{n+1}-1) + ab(cdq^n - 1)\right\} \\ &= \frac{(1-q^{n+1})(1-cdq^n)}{1-abcdq^{2n+1}}.\end{aligned}$$ ■

Now that we know

$$\begin{aligned}P_{n+1}(a) &= E_{n+1}(a) + [\gamma_{n+1}(a)]\left[E_{-(n+1)}(a)\right] \\ P_{n+1}(e) &= E_{n+1}(e) + [\gamma_{n+1}(e)]\left[E_{-(n+1)}(e)\right],\end{aligned}$$

a natural way to obtain the Askey-Wilson connection coefficient relation Theorem 1.2 is:

(PROOF PLAN B) (62)

1. Start with the top line for $P_{n+1}(a)$.
2. Apply Theorem 1.3 to express each of $E_{\pm(n+1)}(a)$ in terms of the $E_r(e)$. (Here r can be of any sign.)
3. Show that the combinations of $E_{\pm(m+1)}(e)$ (for $m \geq 0$) which result are in fact the $c_{m+1,n+1}P_{m+1}(e)$ of the Askey-Wilson result. (As well as the $E_0(e)$ coefficient matching $c_{0,n+1}$.)

We will now show that *PROOF PLAN B* can be carried out to prove the Askey-Wilson result. Recall our generic notation for the E connection coefficient relations. For $n \geq 0$:

$$E_n(a) = \sum_{m=0}^{n} [\tau_{m,n}]\, E_m(e) + \sum_{m=0}^{n-1} \left[\sigma_{-(m+1),n}\right] E_{-(m+1)}(e). \tag{63}$$

$$E_{-(n+1)}(a) = \sum_{m=0}^{n} \left[\tau_{-(m+1),-(n+1)}\right] E_{-(m+1)}(e) + \sum_{m=0}^{n} \left[\sigma_{m,-(n+1)}\right] E_m(e). \tag{64}$$

Our original formulation of Theorem 1.3, introduced the notation $d_{r,s}$ which is related to the (variously subscripted) τ, σ by:

$$d_{r,s} = \begin{cases} \tau_{r,s}/c_{|r|,|s|} & \text{if } (r \geq 0 \text{ and } s \geq 0) \text{ or } (r < 0 \text{ and } s < 0) \\ \sigma_{r,s}/c_{|r|,|s|} & \text{if } (r \geq 0 \text{ and } s < 0) \text{ or } (r < 0 \text{ and } s > 0). \end{cases} \tag{65}$$

We return to that notation now and write:

$$E_n(a) = \sum_{m=0}^{n} [d_{m,n}c_{m,n}]\, E_m(e) + \sum_{m=0}^{n-1} \left[d_{-(m+1),n}c_{m+1,n}\right] E_{-(m+1)}(e). \tag{66}$$

$$\begin{aligned}E_{-(n+1)}(a) = &\sum_{m=0}^{n} \left[d_{-(m+1),-(n+1)}c_{m+1,n+1}\right] E_{-(m+1)}(e) \\ &+ \sum_{m=0}^{n} \left[d_{m,-(n+1)}c_{m,n+1}\right] E_m(e).\end{aligned} \tag{67}$$

Plugging these two into $P_{n+1}(a) = E_{n+1}(a) + [\gamma_{n+1}(a)]\left[E_{-(n+1)}(a)\right]$ and considering the coefficients of $E_{\pm(m+1)}(e)$ which result makes clear the relevance of the following two propositions.

Proposition 7.5. *For* $0 \le m \le n$

$$d_{m+1,n+1} + \gamma_{n+1}(a) d_{m+1,-(n+1)} = 1. \tag{68}$$

Proof. $d_{m+1,n+1} + \gamma_{n+1}(a) d_{m+1,-(n+1)}$

$$= \left\{ \frac{q^{n-m}(abcdq^{n+m+1}|q)_1}{(abcdq^{2n+1}|q)_1} \right\} + \left\{ \frac{(q^{n+1}, cdq^n|q)_1}{(abcdq^{2n+1}|q)_1} \right\} \left\{ \frac{(q^{n-m}|q)_1}{(q^{n+1}, cdq^n|q)_1} \right\}$$
$$\cdot \left(\frac{1}{(abcdq^{2n+1}|q)_1} \right) \left\{ q^{n-m} - abcdq^{2n+1} + 1 - q^{n-m} \right\} = 1. \qquad \blacksquare$$

Proposition 7.6. *For* $0 \le m \le n$

$$d_{-(m+1),n+1} + \gamma_{n+1}(a) d_{-(m+1),-(n+1)} = \gamma_{m+1}(e). \tag{69}$$

Proof. $d_{-(m+1),n+1} + \gamma_{n+1}(a) d_{-(m+1),-(n+1)} =$

$$= \left\{ \frac{bcdeq^{n+m+1}(q^{m+1}, cdq^m, ae^{-1}q^{n-m}|q)_1}{(abcdq^{2n+1}, bcdeq^{2m+1}|q)_1} \right\}$$
$$+ \left\{ \frac{(q^{n+1}, cdq^n|q)_1}{(abcdq^{2n+1}|q)_1} \right\} \left\{ \frac{(q^{m+1}, cdq^m, bcdeq^{n+m+1}|q)_1}{(q^{n+1}, cdq^n, bcdeq^{2m+1}|q)_1} \right\}$$
$$= \left\{ \frac{(q^{m+1}, cdq^m|q)_1}{(abcdq^{2n+1}, bcdeq^{2m+1}|q)_1} \right\} \left\{ bcdeq^{n+m+1}(ae^{-1}q^{n-m}|q)_1 + (bcdeq^{n+m+1}|q)_1 \right\}$$
$$= \left\{ \frac{(q^{m+1}, cdq^m|q)_1}{(abcdq^{2n+1}, bcdeq^{2m+1}|q)_1} \right\} \left\{ bcdeq^{n+m+1} - abcdq^{2n+1} + 1 - bcdeq^{n+m+1} \right\}$$
$$= \frac{(q^{m+1}, cdq^m|q)_1}{(bcdeq^{2m+1}|q)_1} = \gamma_{m+1}(e)$$

as asserted. $\blacksquare$

Now we prove the Askey-Wilson connection coefficient result:

Proof of Theorem 1.2. The case of $n = 0$ checks since P_0 is 1, independent of parameters, and $c_{0,0} = 1$.

For $n \ge 0$, using equations (66) and (67) as well as Propositions 7.5 and 7.6

$$P_{n+1}(a) = E_{n+1}(a) + [\gamma_{n+1}(a)] E_{-(n+1)}(a)$$
$$= \sum_{m=0}^{n+1} [d_{m,n+1} c_{m,n+1}] E_m(e) + \sum_{m=0}^{n} \left[d_{-(m+1),n+1} c_{m+1,n+1}\right] E_{-(m+1)}(e)$$
$$+ [\gamma_{n+1}(a)] \left\{ \sum_{m=0}^{n} \left[d_{-(m+1),-(n+1)} c_{m+1,n+1}\right] E_{-(m+1)}(e) + \sum_{m=0}^{n} \left[d_{m,-(n+1)} c_{m,n+1}\right] E_m(e) \right\}$$
$$= c_{0,n+1} \left\{ d_{0,n+1} E_0(e) + \gamma_{n+1}(a) d_{0,-(n+1)} \right\} E_0(e)$$
$$+ \sum_{m=1}^{n} c_{m,n+1} \left\{ d_{m,n+1} + \gamma_{n+1}(a) d_{m,-(n+1)} \right\} E_m(e) + 1 \cdot E_{n+1}(e)$$
$$+ \sum_{m=0}^{n} c_{m+1,n+1} \left\{ d_{-(m+1),n+1} + \gamma_{n+1}(a) d_{-(m+1),-(n+1)} \right\} E_{-(m+1)}(e)$$

$$
\begin{aligned}
&= \text{(by Propositions 7.5 and 7.6)}\\
&= c_{0,n+1}\cdot 1\cdot E_0(e)+\sum_{m=0}^{n-1} c_{m+1,n+1}\cdot 1\cdot E_{m+1}(e)+c_{n+1,n+1}E_{n+1}(e)\\
&+\sum_{m=0}^{n-1} c_{m+1,n+1}\cdot\gamma_{m+1}(e)\cdot E_{-(m+1)}(e)+c_{n+1,n+1}\cdot\gamma_{n+1}(e)\cdot E_{-(m+1)}(e)\\
&=\sum_{m=0}^{n+1} c_{m,n+1}P_m
\end{aligned}
$$

completing the re-proof of the Askey-Wilson Theorem 1.2. ■

Remark 7.7. Instead of proving Lemma 7.1 and Proposition 7.3, we could have based those aspects of our proof on the discussion in Section 3 of [11]. In particular, that reference clearly explains how, for $n > 0$, using the alternate choice

$$D = \widetilde{Y} + q^{-1}abcd\widetilde{Y}^{-1} \quad (\text{with } \widetilde{Y} = \widetilde{T}_1\widetilde{T}_0)$$

for second order operator with Askey-Wilson polynomials P_n as eigenfunctions, the corresponding eigenspace of D is 4 dimensional and spanned by the polynomials P_n, Q_n, E_n, and E_{-n}. Moreover D commutes with both $\widetilde{T}_1$ and $\widetilde{T}_0$, with P_n and Q_n being eigenfunctions of $\widetilde{T}_1$. The respective eigenvalues are t_1 and -1. Exact formulas expressing $E_{\pm n}$ as linear combinations of P_n and the eigenvalue -1 eigenfunction $Q_n^\dagger$ of $\widetilde{T}_0$ are also written down there. ■

8. Appendix A on Change of Basis Conventions

Let $\mathcal{B} = \{e_\alpha\}_{\alpha\in\mathcal{I}}$ be an ordered basis of a vector space $\mathcal{V}$ where the index set $\mathcal{I} = 0, 1, 2, \ldots$. For $x \in \mathcal{V}$, we represent the linear combination $x = \sum_{\alpha\in\mathcal{I}} v^\alpha e_\alpha$ by the column vector

$$[x]_{\mathcal{B}} = \begin{bmatrix} v^0 \\ v^1 \\ \vdots \end{bmatrix}.$$

If $\overline{\mathcal{B}} = \{\bar{e}_\beta\}_{\beta\in\mathcal{I}}$ is another ordered basis, then

$$[x]_{\overline{\mathcal{B}}} = \begin{bmatrix} \bar{v}^0 \\ \bar{v}^1 \\ \vdots \end{bmatrix}.$$

is the column vector corresponding to the linear combination $x = \sum_{\beta\in\mathcal{I}} \bar{v}^\beta \bar{e}_\beta$. Then a change of basis relationship $e_\alpha = \sum_{\beta\in\mathcal{I}} T_{\beta\alpha}\bar{e}_\beta$ corresponds to

$$\begin{bmatrix} \bar{v}^0 \\ \bar{v}^1 \\ \vdots \end{bmatrix} = T \begin{bmatrix} v^0 \\ v^1 \\ \vdots \end{bmatrix}$$

where the row β column α entry of T is $T_{\beta\alpha}$.

This is easily confirmed by noting

$$x = \sum_{\alpha} v^{\alpha} e_{\alpha} = \sum_{\alpha,\beta} v^{\alpha} T_{\beta\alpha} \bar{e}_{\beta} = \sum_{\beta} \Big(\sum_{\alpha} T_{\beta\alpha} v^{\alpha} \Big) \bar{e}_{\beta} = \sum_{\beta} \bar{v}^{\beta} \bar{e}_{\beta} = x.$$

In the case

$$\begin{aligned} n \geq 0 : E_n(z; a, b, c, d|q) &= \\ &= \sum_{m=0}^{n} \tau_{m,n} E_m(z; e, b, c, d|q) + \sum_{m=0}^{n-1} \sigma_{-(m+1),n} E_{-(m+1)}(z; e, b, c, d|q) \\ n \geq 0 : E_{-(n+1)}(z; a, b, c, d|q) &= \\ &= \sum_{m=0}^{n} \tau_{-(m+1),-(n+1)} E_{-(m+1)}(z; e, b, c, d|q) + \sum_{m=0}^{n} \sigma_{m,-(n+1)} E_m(z; e, b, c, d|q) \end{aligned}$$

we are viewing $E_0, E_1, \ldots$ as an ordered basis for $\mathcal{R}^0$ and $E_{-1}, E_{-2}, \ldots$ as an ordered basis for $\mathcal{R}^1$

Note that E_{-1} is in the initial position (index 0) of the second list, in conformity with viewing it as $E_{-(n+1)}$ for $n = 0$.

Thinking of parameter e as giving rise to the $\bar{e}_{\beta}$ basis and parameter a the e_{α} basis, then our generic change of basis formula

$$e_{\alpha} = \sum_{\beta} T_{\beta\alpha} \bar{e}_{\beta}$$

becomes, in block form,
$$T = \begin{bmatrix} T^{00} & T^{01} \\ T^{10} & T^{11} \end{bmatrix}$$

so that *for column vectors of (blocks of length n or $n+1$) components* relative to these bases

$$\begin{bmatrix} \bar{v}^0 \\ \bar{v}^1 \end{bmatrix} = T \begin{bmatrix} v^0 \\ v^1 \end{bmatrix} = \begin{bmatrix} T^{00} & T^{01} \\ T^{10} & T^{11} \end{bmatrix} \begin{bmatrix} v^0 \\ v^1 \end{bmatrix}.$$

Thus for $n \geq 0$

$$E_n(z; a, b, c, d|q) = \sum_{m=0}^{n} \tau_{m,n} E_m(z; e, b, c, d|q) + \sum_{m=0}^{n-1} \sigma_{-(m+1),n} E_{-(m+1)}(z; e, b, c, d|q)$$

corresponds to

$$E_n(z; a, b, c, d|q) = \sum_{m=0}^{n} T^{00}_{m,n} E_m(z; e, b, c, d|q) + \sum_{m=0}^{n-1} T^{10}_{m,n} E_{-(m+1)}(z; e, b, c, d|q).$$

Note the second matrix entry above really is $T^{10}_{m,n}$ (rather than $T^{01}_{m,n}$) in line with the general property (of our conventions) that the first $n+1$ *columns* of T are expressing the decomposition of the first $n+1$ vectors $E_k(z; a, b, c, d|q)$ $(0 \leq k \leq n)$ as linear combinations of the ordered basis

$$\begin{aligned} &E_0(z; e, b, c, d|q),\ E_1(z; e, b, c, d|q), \ldots, \\ &\qquad E_n(z; e, b, c, d|q), E_{-1}(z; e, b, c, d|q), \ldots, E_{-n}(z; e, b, c, d|q). \end{aligned}$$

Similarly

$$E_{-(n+1)}(z;a,b,c,d|q) = \sum_{m=0}^{n} \tau_{-(m+1),-(n+1)} E_{-(m+1)}(z;e,b,c,d|q) + \sum_{m=0}^{n} \sigma_{m,-(n+1)} E_m(z;e,b,c,d|q)$$

corresponds to

$$E_{-(n+1)}(z;a,b,c,d|q) = = \sum_{m=0}^{n} T^{11}_{m,n} E_{-(m+1)}(z;e,b,c,d|q) + \sum_{m=0}^{n} T^{01}_{m,n} E_m(z;e,b,c,d|q)$$

These tell us that the row index m and column index n entries of the four matrices are given by

$$T^{00}_{m,n} = \tau_{m,n} \qquad T^{01}_{m,n} = \sigma_{m,-(n+1)}$$
$$T^{10}_{m,n} = \sigma_{-(m+1),n} \qquad T^{11}_{m,n} = \tau_{-(m+1),-(n+1)}.$$

9. Appendix B

In this paper, we only use the zig-zag increasing cases in the following table and so have included just the proofs of those. We mention the others, which we have also proven, because they may be of interest.

Table 1: Summary of $\hat{a}_m, \hat{b}_m, \hat{c}_m, \hat{d}_m, \quad m = n \text{ or } -(n+1),\ n \geq 0.$

$n \geq 0$ (*Non-Negative Cases*)	$-(n+1) < 0$ (*Negative Cases*)
$\hat{a}_n = \frac{q^n\left[(abcdq^{2n}\|q)_1\right]^2}{(acq^n, bcq^n, adq^n, bdq^n\|q)_1}$	$\hat{a}_{-(n+1)} = -\frac{1}{cdq^n}$
$\hat{b}_n = [cdq^{2n}(abcdq^{2n}\|q)_1] \cdot \frac{[ab(c+d)q^n - (a+b)]}{(acq^n, bcq^n, adq^n, bdq^n\|q)_1}$	$\hat{b}_{-(n+1)} = \frac{(c+d) - cdq^n(a+b)}{cdq^n(abcdq^{2n}\|q)_1}$
$\hat{c}_n = -\frac{1}{ab}$	$\hat{c}_{-(n+1)} = \frac{\left[(abcdq^{2n+1}\|q)_1\right]^2}{(q^{n+1}, abq^{n+1}, cdq^n, abcdq^n\|q)_1}$
$\hat{d}_n = \frac{(abq^n - 1) + ab(cdq^{n-1} - 1)}{ab(1 - abcdq^{2n-1})}$	$\hat{d}_{-(n+1)} = [abq^n(abcdq^{2n+1}\|q)_1] \cdot \frac{\left[q(abcdq^n\|q)_1 + cd(q^{n+1}\|q)_1\right]}{(q^{n+1}, abq^{n+1}, cdq^n, abcdq^n\|q)_1}$
$\widetilde{\mu}_n = t_0 t_1 q^n = abcdq^{n-1}$	$\widetilde{\mu}_{-(n+1)} = q^{-(n+1)}$
$\widetilde{\zeta}_{0n} = [\hat{a}_n]^{-1}\left[\widetilde{\mu}_n - \widetilde{\mu}_{-(n+1)}\right] = -\frac{(acq^n, bcq^n, adq^n, bdq^n\|q)_1}{q^{2n+1}(abcdq^{2n}\|q)_1}$	$\widetilde{\zeta}_{0,-(n+1)} = \left[\hat{a}_{-(n+1)}\right]^{-1}\left[\widetilde{\mu}_{-(n+1)} - \widetilde{\mu}_n\right] = -cdq^{-1}(abcdq^{2n}\|q)_1$
$\widetilde{\zeta}_{1n} = [\hat{c}_n]^{-1}\left[\widetilde{\mu}_{-n} - \widetilde{\mu}_n\right]$ $-abq^{-n}(abcdq^{2n-1}\|q)_1$	$\widetilde{\zeta}_{1,-(n+1)} = \left[\hat{c}_{-(n+1)}\right]^{-1}\left[\widetilde{\mu}_{n+1} - \widetilde{\mu}_{-(n+1)}\right] = -\frac{(q^{n+1}, abq^{n+1}, cdq^n, abcdq^n\|q)_1}{q^{n+1}(abcdq^{2n+1}\|q)_1}$

For any signs of n, k :

$$
\begin{aligned}
E_{-(n+1)} &= \hat{a}_{-(n+1)}\widetilde{U}_0\Big(E_n\Big) + \hat{b}_{-(n+1)}E_n \\
E_n &= \hat{a}_n\widetilde{U}_0\Big(E_{-(n+1)}\Big) + \hat{b}_nE_{-(n+1)} \\
E_{-n} &= \hat{c}_{-n}\widetilde{T}_1\Big(E_n\Big) + \hat{d}_{-n}E_n \\
E_n &= \hat{c}_n\widetilde{T}_1\Big(E_{-n}\Big) + \hat{d}_nE_{-n}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{U}_0\Big(E_k\Big) &= \Big[\hat{a}_{-(k+1)}\Big]^{-1}\Big[E_{-(k+1)} - \hat{b}_{-(k+1)}E_k\Big] \\
\widetilde{T}_1\Big(E_k\Big) &= \big[\hat{c}_{-k}\big]^{-1}\Big[E_{-k} - \hat{d}_{-k}E_k\Big] \quad (k \neq 0)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{U}_0\Big(E_{-(k+1)}\Big) &= \big[\hat{a}_k\big]^{-1}\Big[E_k - \hat{b}_kE_{-(k+1)}\Big] \\
\widetilde{T}_1\Big(E_{-(k+1)}\Big) &= \big[\hat{c}_{k+1}\big]^{-1}\Big[E_{k+1} - \hat{d}_{k+1}E_{-(k+1)}\Big] \quad (k \neq -1)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathcal{S}'}_0E_n &= \widetilde{\zeta}_{0,-(n+1)}E_{-(n+1)} & \widetilde{\mathcal{S}'}_0E_{-(n+1)} &= \widetilde{\zeta}_{0,n}E_n \\
\widetilde{\mathcal{S}}_1E_n &= \widetilde{\zeta}_{1,-n}E_{-n} & \widetilde{\mathcal{S}}_1E_{-n} &= \widetilde{\zeta}_{1n}E_n
\end{aligned}
$$

Since the multiplication operator $X = t_0\widetilde{T}_1^{-1}\widetilde{Y}\widetilde{U}_0$, the above imply general 4-dimensional invariant subspaces for X and X^{-1}. Simplifications of the matrix entries of these rel the $\{E_r\}$ arise. The simplified expressions are mostly products and quotients of q-Pochhammer symbols, with an occasional monomial or sum of 2 symbols factor.

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Received August 17, 2022
and in final form November 15, 2022