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Engel Marginal Subgroups in Pro-Lie Groups

Yanga Bavuma and Francesco G. Russo

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Abstract. The position of the n-Engel marginal and verbal subgroups is placed between appropriate terms of the (upper or lower) central series of an abstract group, when n is small enough. In case of compact (Hausdorff) groups, the presence of a topological structure allows us to consider larger values of n, again controlling very well the position of these subgroups with respect to the terms of the central series. We observe a similar behaviour in the wider context of pro-Lie groups.

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1. Introduction and statement of the main result

An element x of a group G is a *right Engel element* if for each $y \in G$ there exists a positive integer n (depending on y) such that the *right n-Engel commutator word*

$$e_n(x, y) = [x, y, y, y, ..., y] = [x,_n y] \tag{1}$$

is equal to one. The set of all *right n-Engel elements* of G is

$$R_n(G) = \{x \in G \mid [x,_n y] = 1 \ \forall y \in G\} \tag{2}$$

and by analogy we may introduce the set of all *left n-Engel elements* of G

$$L_n(G) = \{x \in G \mid [y,_n x] = 1 \ \forall y \in G\}, \tag{3}$$

replacing the role of x with that of y. In general $R_n(G) \neq L_n(G)$; actually $R_n(G)$ and $L_n(G)$ are just sets, so they do not necessarily possess the structure of subgroups of G (see [1]). When this happens and we have $G = R_n(G) = L_n(G)$, we say that G is an *n-Engel group*, or briefly an *Engel group*. The theory of Engel groups is well established, but there are several open problems which make it interesting, especially regarding the area of topological group theory. In fact the presence of a topological structure can help to detect conditions of nilpotence. A classical line of research in abstract group theory was initiated by Baer [3, 4, 5, 6] and contributions were given by Gruenberg [13], Gupta [15], Havas [15], Ivanov [26], Heineken [18, 19], Juhász [7, 8, 27], Kappe [28, 29], Merzljakov [33, 34].

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Along with the right n-Engel commutator word, one can introduce the corresponding n-Engel verbal and marginal subgroups

$$E_n(G) = \langle [x,_n y] \mid \forall x, y \in G \rangle; \tag{4}$$

$$E_n^*(G) = \{a \in G \mid [x,_n y] = [ax,_n y] = [x,_n ay] \ \forall x, y \in G\}. \tag{5}$$

Marginal and verbal subgroups were studied originally by Hall [16] and Turner-Smith [44] for commutator words, but it is more recent their study for Engel words [2, 36, 37, 40]. Note that (4) and (5) are characteristic in G (see [38, 40]) and dual in the sense of [40, Theorems 1.1, 1.2].

The motivation of the present paper is related to an improvement of the results in [2, 36, 37, 40], considering a class of topological (Hausdorff) groups, namely the *pro-Lie groups*, largely studied by Hofmann and Morris [22]. Regarding the notion of compact Lie group, see [23, Corollary 2.40, Definition 2.41], but for that of noncompact Lie groups, see [22, Definition 2.1, Proposition 2.2].

Definition 1.1 (See [22], Definition 3.25)**.** A topological (Hausdorff) group G is a pro-Lie group if it is complete and if every identity neighborhood of G contains a closed normal subgroup N such that G/N is a Lie group. In particular, a locally compact group G is a pro-Lie group if $G = \lim_{i \in I} G_i$ is the projective limit of Lie groups $G_i = G/K_i$ with a descending family $(K_i)_{i \in I}$ of closed compact normal subgroups K_i of G satisfying $\bigcap_{i \in I} K_i = 1$. ■

The reader can refer to [22, Chapter 3] for details on the notions in Definition 1.1, or to [24]. Note that a topological group G is said to be *almost connected* if the factor group G/G_0 of G modulo the connected component G_0 of the identity is compact. See also [22, Definitions A, B, C, Chapter 3].

Remark 1.2. Note that we will always assume in the present note that topological groups satisfy the axiom of separation of Hausdorff. Moreover we will work mainly with locally compact pro-Lie groups. In particular, almost connected locally compact groups are captured by Definition 1.1. ■

The general term of the *lower central series* of a group G will be denoted by $\gamma_i(G)$, where $i \in I$ and I is an arbitrary set of indices, and the general term of the *upper central series* of G by $Z_i(G)$. While $Z_i(G)$ is always a closed subgroup of a pro-Lie group G, this is no longer true for $\gamma_i(G)$, which should be replaced then by $\overline{\gamma_i(G)}$. For details see [23] and [22, Chapter 10].

When we have groups possessing central series, several authors investigated theorems of embeddings of peculiar subgroups among consecutive terms of the central series. In [40, Theorem 2.3 (a), Corollary 2.8] the first condition of embedding of margins of Engel words in central series of abstract groups was found:

$$Z_2(G) \subseteq E_2^*(G) \subseteq Z_3(G), \tag{6}$$

then [2, Theorem 1.1 (i)] shows

$$\gamma_4(G) \subseteq E_2(G) \subseteq \gamma_3(G). \tag{7}$$

The case of n-Engel words of length $n \geq 3$ is discussed in [2, 36, 37], but for large values of $n \geq 5$ the presence of a compact topology on G helps, since otherwise pathologies may appear such as the example of Golod [12]. Looking for generalizations of the conditions (6) and (7) for large n and out from the category of compact groups is an open problem at the moment.

Recall from [22, Appendix 2, Definition A2.5, Corollary A2.9] that a *weakly complete topological vector space* V is a real vector space of the form $V = \mathbb{R}^J$ with J arbitrary set of indices and $\mathbb{R}^J$ cartesian sum of $|J|$-copies of $\mathbb{R}$ (note that the symbol $\mathbb{R}^{(J)}$ denotes the direct sum of $|J|$-copies of $\mathbb{R}$, and in general $\mathbb{R}^{(J)}$ is strictly contained in $\mathbb{R}^J$). Note also that $\mathbb{R}^{\mathbb{N}}$ is a pro-Lie group which is not locally compact. Now if G is a pro-Lie group, a *weakly complete vector subgroup* H of G is an abelian subgroup H of G of the form $H = \mathbb{R}^J$. Note also that a topological group G is said to be *topologically finitely generated* if $G = \overline{\langle x_1, x_2, \ldots, x_d \rangle}$, where d is a positive integer. Moreover we say that a topological space X is σ*-compact* if X is a countable union of compact subspaces. We have all that we need, in order to formulate our main result.

Theorem 1.3. *Let G be a locally compact pro-Lie group possessing a weakly complete closed normal vector subgroup N such that G/N is compact.*

(i) *If G/N is abelian, then $E_n(G)$ is abelian and $E_n(G) \subseteq \overline{\gamma_{n+1}(G)} \cap N$ for all $n \geq 1$. Moreover if $E_n(G)$ is closed, then it is open, σ-compact and $N/E_n(G)$ is connected abelian;*

(ii) *if $E^*_n(G)$ is closed, then $E^*_n(G)$ is an open σ-compact subgroup containing $Z_n(G)$. Moreover if $E^*_n(G)$ is topologically finitely generated, then $E^*_n(G)$ is compact nilpotent.*

Section 2 contains information from the literature which is mentioned in the bibliography and results which will be used in Section 3 for the proof of Theorem 1.3. Section 3 is indeed devoted to prove our main result. Examples of groups are reported in Section 4, showing the degree of generalization which we obtain with respect to the previous results on Engel marginal and verbal subgroups in compact groups. A few comments of historical nature are placed along the examples of Section 4, justifying applications of Theorem 1.3.

2. Preliminaries and facts from the literature

First of all we note from [22, Example 3.38] that there are locally compact groups which are not pro-Lie groups: the special linear group $\mathrm{Sl}(2, \mathbb{Q}_p)$ of (2×2)-matrices on the field $\mathbb{Q}_p$ of p-adic rationals is a p-adic Lie group, and it is a totally disconnected locally compact group which is not a pro-Lie group (p prime). In fact $\mathrm{Sl}(2, \mathbb{Q}_p)$ has no nontrivial normal subgroups except the identity matrix and its negative. On the other hand, pro-Lie groups generalize very well the usual category of compact groups (with corresponding morphisms, see [23]) and contain the almost connected locally compact groups studied by Yamabe [45, 46].

Secondly, we point out that the conditions of embeddings of the form of (6) and (7) for pro-Lie groups here are interesting for large n in a noncompact context, since Medvedev [32] showed the following result:

Proposition 2.1 (See [32], Medvedev's Theorem)**.** *Engel compact groups are locally nilpotent.*

From [22, Definition 5.6 (ii)], a topological group G is said to be *compactly generated* if there is a compact generating set for G, i.e.: there is a compact subset K of G such that $\langle K \rangle = G$. Also from [22, Definition 5.6 (iii)], G is said to be *compactly topologically generated*, if there is a compact subset K of G such that $\overline{\langle K \rangle} = G$. These two notions are in general different, but they can coincide in some circumstances (for instance, when the group is discrete). In particular, topologically finitely generated groups are compactly topologically generated, because a finite set of topological generators is of course compact. A few more notions should be recalled from [22, Chapter 5]: first of all, a topological space X is a *Polish space* if it is completely metrizable and second countable.

Lemma 2.2. *The following statements are true:*

(i) $\mathbb{R}^{\mathbb{N}}$ *and* $\mathbb{Z}^{\mathbb{N}}$ *are Polish pro-Lie groups;*

(ii) *Every almost connected locally compact group is compactly generated;*

(iii) *Every compactly generated topological group is* σ*-compact;*

(iv) *A* σ*-compact Polish group is locally compact;*

(v) *A topologically finitely generated pro-Lie group is* σ*-compact.*

Proof. (i) is clear from definitions and from the fact that countable products of Polish spaces are Polish spaces. Now (ii), (iii) and (iv) follow from [22, Remark 5.22]. Finally (v) follows from definitions and (iii). ∎

It is also useful to clarify here that weakly complete closed normal vector subgroups in Theorem 1.3 are finite dimensional, because the whole group is locally compact:

Lemma 2.3 (See [22], Lemma 5.29)**.** *For a weakly complete topological vector space* W*, the following statements are equivalent:*

(i) W *is* σ*-compact;*

(ii) W *is locally compact;*

(iii) W *is finite-dimensional;*

(iv) W *is compactly generated.*

Another result of topological nature should be recalled here: every locally compact, or completely metrizable space, is a Baire space, see [23, Appendix 1, Exercise EA1.21]. In particular, this is true for locally compact pro-Lie groups and we report below a formulation which will be useful in Lemma 3.1 later on.

Lemma 2.4 (Baire Category Theorem)**.** *If* G *is a locally compact pro-Lie group and if there is a countable collection of closed subsets* $(F_n)_{n\in\mathbb{N}}$ *of* G *such that we have* $G = \bigcup_{n\in\mathbb{N}} F_n$*, then* F_n *has nonempty interior for some* $n \in \mathbb{N}$*.*

Proof. This follows from [11, Chapter XI, Theorem 10.3] for arbitrary locally compact spaces (in fact the structure of group is not involved). ∎

The following lemma describes a symmetry between $Z_i(G)$ and $\gamma_i(G)$ for compact groups.

Lemma 2.5 (See [37], Theorem 1.1). *Let G be a compact group.*

(i) *If $E_n(G)$ is closed and $G/E_n(G)$ is topologically generated by r elements, then there exist a positive integer $c(n,r)$ depending only on n and r such that $\overline{\gamma_{c(n,r)+1}(G)} \subseteq E_n(G) \subseteq \overline{\gamma_{n+1}(G)}$.*

(ii) *If $E_n^*(G)$ is a closed subgroup of G and topologically generated by s elements, then there exists a positive integer $c(n,s)$ depending only on n and s such that $Z_n(G) \subseteq E_n^*(G) \subseteq Z_{c(n,s)}(G)$.*

In particular, the result above ensures conditions for which both $E_n(G)$ and $E_n^*(G)$ are nilpotent subgroups in a compact groups. Apart from the exact placement of these subgroups between the central series of G, the main information is related to their nilpotence.

Corollary 2.6. *Let G be a compact group.*

(i) *If $E_n(G)$ is closed and $G/E_n(G)$ is topologically finitely generated, then $E_n(G)$ is nilpotent.*

(ii) *If $E_n^*(G)$ is closed and topologically finitely generated, then it is nilpotent.*

Proof. This follows from Lemma 2.5. ■

Zorn [47] showed that every finite Engel group is nilpotent, and, among the main consequences of Proposition 2.1 Medvedev proved a strong generalisation of Zorn's Theorem, showing that a compact Engel group that is topologically finitely generated is nilpotent. Another useful result which we use in the main proofs is the following.

Lemma 2.7 (See [43], Theorem, p.143). *If G is a topologically finitely generated compact group and H a closed normal subgroup of G such that $H \subseteq R_n(G)$, then $H \subseteq Z_m(G)$ for some $m \geq 1$.*

After we have illustrated some of the reasons of our investigation, we note from [38] that the *verbal mapping*, related to $e_n(x,y) = e_n(g_1,g_2)$, is

$$e_n : (g_1,g_2) \in G \times G \mapsto e_n(g_1,g_2) = \prod_{j=1}^{s} g_{i_j}^{\varepsilon_j} \in G, \tag{8}$$

where $\varepsilon_j \in \{\pm 1\}$, $j \in \{1,\dots,s\}$, s is the length of $e_n(g_1,g_2)$ and $i_1,\dots,i_s \in \{1,2\}$. The set of *left n-Engel values* of G is

$$\{E_n\}(G) = \{e_n(x,y) \mid x,y \in G\} = \{[x,_n y] \mid x,y \in G\} \tag{9}$$

and we have $E_n(G) = \langle \{E_n\}(G) \rangle$. We say that $e_n(x,y)$ has *finite width m* if

$$E_n(G) = \{g_1 g_2 \dots g_m \mid g_1,g_2,\dots,g_m \in \{E_n\}(G)\}. \tag{10}$$

Instead of $e_n(g_1,g_2)$, we may have an arbitrary word $w = w(g_1,\dots,g_n)$, then we consider the set G_w of its values in G, the verbal subgroup $w(G) = \langle G_w \rangle$ and the set

$$G_w^{*m} = \{g_1 g_2 \dots g_m \mid g_1,g_2,\dots,g_m \in G_w\}, \tag{11}$$

so we may formulate the notion of finite width for an arbitrary word w on G. We say that G is *verbally elliptic* if each word on G has finite width. The specific interest, which is related to $e_n(x,y)$, originates from the fact that $e_n(x,y)$ could not have finite width in an infinite group G.

Proposition 2.8 (See [38], Proposition 4.1.2)**.** *Let G be a profinite group. Then $w(G)$ is closed in G if and only if w has finite width in G.*

A topological group, which is isomorphic to a closed subgroup of the general linear group $\mathrm{Gl}(n, \mathbb{Z}_p)$ with coefficients in the group of the p-adic integers $\mathbb{Z}_p$ (for some n), is called *compact p-adic analytic*, see [38, p. 106]. We cannot assume for free that the marginal and the verbal subgroups are always closed (in particular $E_n(G)$ and $E^*_n(G)$), but this is possible for compact p-adic analytic groups.

Proposition 2.9 (See [38], Theorem 5.4.1)**.** *Let G be a compact p-adic analytic group. Then G is verbally elliptic.*

Of course, there are problems to detect verbally elliptic groups which do not belong to the class of compact p-adic analytic groups, therefore one cannot control the notion of finite width in general for arbitrary words.

3. Proofs of the main result

Since the results are clear enough in the case of compact p-groups, and especially for compact p-adic analytic groups, now we are going to recall some well known facts in the construction of a locally compact pro-Lie group of the following type

$$G = N \rtimes C, \text{ where } N \simeq \mathbb{R}^n \text{ and } C \text{ is a compact group.} \tag{12}$$

This will play a crucial role in the proofs of the main results of the present paper. From the definition of semidirect product, we can write uniquely two arbitrary elements $g_1 = n_1c_1$ and $g_2 = n_2c_2$ of G as products of elements $n_1, n_2 \in N$ and $c_1, c_2 \in C$; in particular, if C is abelian, then we have

$$\begin{aligned}[g_1, g_2] &= [n_1c_1, n_2c_2] = [n_1, n_2c_2]^{c_1}\ [c_1, n_2c_2] = [n_1, n_2c_2]^{c_1}\ [c_1, c_2]\ [c_1, n_2]^{c_2} \\ &= [n_1, n_2c_2]^{c_1}\ [c_1, n_2]^{c_2} = ([n_1, c_2]\ [n_1, n_2]^{c_2})^{c_1}\ [c_1, n_2]^{c_2} = [n_1, c_2]^{c_1}\ [c_1, n_2]^{c_2} \\ &= [n_1^{c_1}, c_2^{c_1}]\ [c_1^{c_2}, n_2^{c_2}] = [n_1^{c_1}, c_2]\ [c_1, n_2^{c_2}] \end{aligned} \tag{13}$$

and this shows that
$$[g_1, g_2] = [n_1^{c_1}, c_2][c_1, n_2^{c_2}]. \tag{14}$$

Note that we used the fact that C is abelian in (13), in fact (14) is false when C is nonabelian. If in addition $g_1 \in Z(G)$ (or alternatively $g_2 \in Z(G)$), then $1 = [g_1, g_2]$, i.e. $[n_1^{c_1}, c_2]^{-1} = [c_1, n_2^{c_2}]$ and so $[c_2, n_1^{c_1}] = [c_1, n_2^{c_2}]$, hence

$$Z(G) = \{g_1 = n_1c_1 \in NC \mid [c_2, n_1^{c_1}] = [c_1, n_2^{c_2}] \ \ \forall g_2 = n_2c_2 \in NC\}. \tag{15}$$

At the level of closed commutator subgroups, (14) may be rewritten as

$$\overline{[G,G]} = \overline{[N^C, C]\ [C, N^C]} = \overline{[N^C, C]\ [N^C, C]} = \overline{[N^C, C]} = \overline{[N, C]}, \tag{16}$$

where the normality of N in G is used.

Lemma 3.1. *In a locally compact pro-Lie group G, where N is a weakly complete closed normal vector subgroup and G/N is compact abelian, we have that $E_n(G)$ is abelian and $E_n(G) \subseteq \overline{\gamma_{n+1}(G)} \cap N$ for all $n \geq 1$. Moreover if $E_n(G)$ is closed, then $E_n(G)$ is an open σ-compact subgroup such that $N/E_n(G)$ is abelian connected.*

Proof. From [22, Theorem 11.15] and Lemma 2.3 we may assume $G = N \rtimes C$ as in (12). The upper inclusion $E_n(G) \subseteq \gamma_{n+1}(G) \subseteq \overline{\gamma_{n+1}(G)}$ follows easily from the definitions, because a commutator $[x,_n y] \in E_n(G)$ belongs of course to

$$\gamma_{n+1}(G) = \langle [x_1, x_2, \ldots, x_{n+1}] \mid x_1, x_2, \ldots, x_{n+1} \in G \rangle \tag{17}$$

and so to $\overline{\gamma_{n+1}(G)}$. Then one has from the definitions also

$$\ldots \overline{\gamma_{n+1}(G)} \subseteq \overline{\gamma_n(G)} \subseteq \ldots \subseteq \overline{\gamma_2(G)}. \tag{18}$$

Note also that from (16) we have $E_n(G) \subseteq \overline{[N, C]} \subseteq \overline{N} = N$, because each elements of $[N, C]$ is of the form $n^{-1}n^c \in N$ and N closed normal in G. Therefore $E_n(G) \subseteq \overline{\gamma_{n+1}(G)} \cap N \subseteq N$. Note also that $E_n(G)$ is abelian because $[E_n(G), E_n(G)] \subseteq [N, N] = 1$. Form the quotient group $N/E_n(G)$, since $E_n(G)$ is closed by assumption. This is a continuous epimorphic image of the connected abelian group N, so it is connected abelian as well.

Since $N \simeq \mathbb{R}^n$, it is compactly generated by Lemma 2.3, so there exists some compact set M such that $\langle M \rangle = N$. Then $G = NC = \langle M, C \rangle = \langle M \cup C \rangle$ is compactly generated. Since $E_n(G)$ is a closed subgroup of $N \simeq \mathbb{R}^n$, then it is σ-compact and locally compact. This allows us to write $E_n(G) = \bigcup_{k \in \mathbb{N}} F_k$, where F_k are closed subsets of $E_n(G)$, and $E_n(G)$ is a Baire space, so there are some F_k with nontrivial interior by Lemma 2.4. From [23, Proposition A4.25], $E_n(G)$ has nontrivial interior, so it is open. ■

It is also important to mention that the *nilradical* $N(P)$ of an arbitrary pro-Lie group P is the largest connected *transfinitely topologically nilpotent* normal subgroup of P, see [22, Definitions 10.5, 10.40]. Its existence is guaranteed by [22, Theorem 10.42] in a pro-Lie group. Moreover $N(P)$ turns out to be a closed connected characteristic subgroup of P, see [22, Lemma 10.41]. Note that for discrete groups the notion of being transfinitely topologically nilpotent in [22, Definitions 10.5] is exactly the notion of being *hypercentral* in Dixmier [10] and McLain [31], that is, abstract groups in which $\gamma_i(P)$ reaches the trivial subgroup after $|I|$-steps.

Corollary 3.2. *A locally compact pro-Lie group G, where N is a weakly complete closed normal vector subgroup and G/N compact abelian, has $E_n(G)$ abelian and contained in $N(G)$.*

Proof. From Lemma 3.1, $E_n(G)$ is abelian and $E_n(G) \subseteq N \subseteq N(G)$. ■

We can say something similar for the n-Engel marginal subgroups.

Lemma 3.3. *In a locally compact pro-Lie group G, where N is a weakly complete closed normal vector subgroup and G/N compact, if $E^*_n(G)$ is closed, then it is an open σ-compact subgroup containing $Z_n(G)$. Moreover if $E^*_n(G)$ is topologically finitely generated, then $E^*_n(G)$ is compact nilpotent.*

Proof. From [22, Theorem 11.15] and Lemma 2.3 we may assume $G = N \rtimes C$ as in (12). For all $n \geq 1$ note that

$$\overline{Z_n(G)} = Z_n(G) = \{g \in G \mid [g, x_1, x_2, \ldots, x_n] = 1 \ \forall x_1, x_2, \ldots, x_n \in G\} \tag{19}$$

and the lower inclusion $Z_n(G) \subseteq E^*_n(G) \subseteq \overline{E^*_n(G)}$ follows from the definitions. In order to conclude that $E^*_n(G)$ is an open σ-compact subgroup, we may apply the same argument which has been applied to $E_n(G)$ in the proof of Lemma 3.1.

Here $E^*_n(G)$ is a locally compact σ-compact n-Engel subgroup of G. Now if $E^*_n(G)$ is topologically finitely generated, then it is an n-Engel topologically finitely generated compact group and so nilpotent by Lemma 2.7. The result follows. ■

The reader can observe that Lemmas 3.1 and 3.3 are quite asymmetric, in the sense that the assumption that G/N is compact abelian in Lemma 3.1 is fundamental, where this is not the case in Lemma 3.3. In addition, we are dealing with some σ-compact groups in these two lemmas:

Remark 3.4. In Lemmas 3.1 and 3.3, the group G is σ-compact. In fact in both cases G turns out to be a compactly generated locally compact group, so it is σ-compact by Lemma 2.2 (iii). ■

We go ahead to discuss the structure of a semidirect product in a topological group G, considering an element $x \in G$ and the continuous inner automorphism $\varphi_x : g \in G \mapsto \varphi_x(g) = g^x \in G$ of G. Note that

$$\mathrm{Inn}(G) = \{\varphi_x \mid x \in G \text{ and } \varphi_x \text{ is a continuous inner automorphism}\} \tag{20}$$

is a closed normal subgroup of the continuous automorphism group $\mathrm{Aut}(G)$ of G.

Lemma 3.5. *If $G = N \rtimes C$ is a topological group, where N is a closed abelian normal subgroup of G, C is a compact abelian subgroup and $\varphi_c \in \mathrm{Inn}(G)$ satisfies $\varphi_c(N) \subseteq C$ for some $c \in C$, then G is compact abelian.*

Proof. Consider $c \in C$ and $\varphi_c \in \mathrm{Inn}(G)$. Since N is fixed by the elements of $\mathrm{Inn}(G)$, $\varphi_c(N) = N \subseteq N \cap C = 1$. Then we note that (13), (14) and (16) are still valid in the present context, and we find that $\overline{[G,G]} = \overline{[N,C]} = 1$, hence $G = C$. ■

Note that Lemma 3.5 basically gives a condition for which $E_n(G)$ and $E^*_n(G)$ are abelian, and this is in the spirit of Lemma 2.5 and Corollary 2.6.
We have all that we need for the proof of the main result.

Proof of Theorem 1.3. From [22, Theorem 11.15] and Lemma 2.3, we may restrict our attention to groups of the type (12). The result follows from Lemmas 3.1 and 3.3. ■

4. Examples and constructions

We end with a series of instructive examples, adapted from [22, Chapter 14, Examples 14.26, 14.27, 14.28, 14.29], or present in [24] in different contexts; they show the limits of our results, but at the same time some generalizations from previous results in the literature, regarding the marginal and verbal n-Engel subgroups in the case of locally compact noncompact groups.

Example 4.1. Let $L = \mathbb{R}$ be the usual additive group of the real numbers (endowed with its usual topology). Then $P = L^{\mathbb{Z}}$ of $|\mathbb{Z}|$-copies of L is a pro-Lie group, but not a locally compact pro-Lie group. Let $\alpha : \mathbb{Z} \to \mathrm{Aut}(P)$ be the representation given by $\alpha(n)((x_m)_{m\in\mathbb{Z}}) = (x_{m-n})_{m\in\mathbb{Z}}$. If the additive group of the integers $\mathbb{Z}$ acts automorphically on P via $n \cdot p = \alpha(n)(p)$, the only $\mathbb{Z}$-invariant identity neighbourhood of P is P itself. We may form the semidirect product

$$G = P \rtimes_\alpha \mathbb{Z}, \tag{21}$$

which has a series of interesting properties, in connection with Theorem 1.3.

First of all, G is a semidirect product of two pro-Lie groups, namely P, which is not locally compact, and $\mathbb{Z}$, which is locally compact. The group (21) is not a locally compact pro-Lie group. Moreover $L^{\mathbb{Z}} \times \{0\}$ is the unique smallest normal open subgroup. Of course (21) is metabelian, and we note that also (12) is a metabelian group when C is abelian, but in Theorem 1.3 we always require that groups of the type (12) are locally compact pro-Lie groups. Therefore (21) shows that among metabelian groups there are examples which are not pro-Lie groups. Kappe [28] describes completely marginal subgroups of Engel words in metabelian (discrete) infinite groups when $n \leq 4$. On the other hand, (21) requires a specific analysis for the computation of $E^*_n(G)$ and $E_n(G)$ when $n \geq 5$, and even if metabelian, this cannot be deduced from the results in [9, 13, 15, 17, 28, 29, 30, 35, 39, 41, 42, 43].

Now consider L to be a compact Lie group in (21), i.e.: $L = \mathbb{R}/\mathbb{Z} = \mathbb{T}$ is the abelian torus of dimension one. If $\mathbb{Z}$ acts on $P = \mathbb{T}^{\mathbb{Z}}$ with the shift action σ, then $P \rtimes_\sigma \mathbb{Z}$ turns out to be a locally compact group, but still it is not a pro-Lie group. We noted before that also $\mathrm{Sl}(2, \mathbb{Q}_p)$ is a locally compact group which is not a pro-Lie group, but $P \rtimes_\sigma \mathbb{Z}$ is an example of a semidirect product of two locally compact pro-Lie groups that is not a pro-Lie group. ■

Of course, the following example shows a first generalization of Theorem 1.3.

Example 4.2. Consider a locally compact pro-Lie group G as in (12). If N is trivial, then $G = C$ is a compact group (not necessarily abelian). This case can be realized choosing $\mathbb{R}^0 = N = 1$. Here Lemma 2.5 and 2.6 apply, and Theorem 1.3 as well. ■

The original idea of considering (12) comes from the case of the usual dihedral groups, but, again, one has to be very careful regarding dihedral groups which are pro-Lie groups. The following example helps in this sense.

Example 4.3. Let D be a discrete abelian group, and A a compact nontrivial group of automorphisms of D. Then the compact group $A^{\mathbb{N}}$ acts automorphically on the discrete group $V = D^{(\mathbb{N})}$ via $(a_n)_{n\in\mathbb{N}} \cdot (d_n)_{n\in\mathbb{N}} = (a_n d_n)_{n\in\mathbb{N}}$. Set

$$G = V \rtimes A^{\mathbb{N}}. \tag{22}$$

The group (22) is not a pro-Lie group, but if A is abelian, then G is metabelian and has a closed normal Lie subgroup $H = V \times 1$ such that $G/H \simeq A^{\mathbb{N}}$. Now look at Lemma 3.5 and at a topological group S, which can be written as $S = N \rtimes C$, where N is a closed abelian normal subgroup of S and C a compact abelian subgroup of S and $\varphi_c \in \mathrm{Inn}(S)$ with $c \in C$ such that $\varphi_c(N) \subseteq C$. In this situation S is compact abelian, so a pro-Lie group, and S is a semidirect product as G.

It is indeed interesting to note that the analogy in terms of structure between (12) and (22) does not imply necessarily that we have a pro-Lie group when we deal with the semidirect product of two pro-Lie groups. In fact (12) is requested to be a pro-Lie group a priori, but (22) is not a pro-Lie group, so the consideration of (22) shows that even in the case of metabelian groups, one needs to check the presence of a weakly complete closed normal vector subgroup, and this is a relevant assumption in Theorem 1.3, not guaranteed a priori. ■

We end with another example which is also metabelian but different from the description of the main results of this paper. It shows the difficulty of working with Engel groups in contexts which are broader than the compact groups.

Example 4.4. Consider the real Heisenberg group

$$\mathrm{Heis}(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\} \tag{23}$$

represented as group of matrices in the general linear group $\mathrm{Gl}(3, \mathbb{R})$ of dimension 3 on $\mathbb{R}$. This is a locally compact noncompact nilpotent Lie group, see details in [22, Examples 10.44 and 14.39]. Most of the structural properties of this group (and general structural information on metabelian Lie groups) can be found in [14, 20, 21, 25, 23, 22]. In particular, $\mathrm{Heis}(\mathbb{R})$ is a locally compact pro-Lie group. It is well known that $\mathrm{Heis}(\mathbb{R})$ has one-dimensional center

$$Z(\mathrm{Heis}(\mathbb{R})) = \overline{[\mathrm{Heis}(\mathbb{R}), \mathrm{Heis}(\mathbb{R})]} = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; c \in \mathbb{R} \right\} \tag{24}$$

which is isomorphic to $(\mathbb{R}, +)$ as topological group and the central quotient

$$\mathrm{Heis}(\mathbb{R})/Z(\mathrm{Heis}(\mathbb{R})) \simeq \mathbb{R}^2 \tag{25}$$

is isomorphic to 2 copies of $(\mathbb{R}, +)$ (as topological group). It is also known that (23) may be written as semidirect product in the following form

$$\mathrm{Heis}(\mathbb{R}) = A \rtimes B \simeq \mathbb{R}^2 \rtimes \mathbb{R}, \tag{26}$$

where
$$A = Z(\mathrm{Heis}(\mathbb{R})) \oplus \overline{\left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a \in \mathbb{R} \right\}} \tag{27}$$

and
$$B = \overline{\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; b \in \mathbb{R} \right\}}. \tag{28}$$

We provide an explicit computation for $E_n(\mathrm{Heis}(\mathbb{R}))$ and $E_n^*(\mathrm{Heis}(\mathbb{R}))$. Looking at the definition of Engel marginal subgroup (5), we note that $Z_n(\mathrm{Heis}(\mathbb{R})) \subseteq E_n^*(\mathrm{Heis}(\mathbb{R}))$ for all $n \geq 2$, but here $Z_2(\mathrm{Heis}(\mathbb{R})) = Z_n(\mathrm{Heis}(\mathbb{R})) = \mathrm{Heis}(\mathbb{R})$, hence $E_n^*(\mathrm{Heis}(\mathbb{R})) = \mathrm{Heis}(\mathbb{R})$ for all $n \geq 2$. On the other hand, again from (4) we note that $E_n(\mathrm{Heis}(\mathbb{R})) \subseteq \overline{[\mathrm{Heis}(\mathbb{R}), \mathrm{Heis}(\mathbb{R})]} = Z(\mathrm{Heis}(\mathbb{R}))$ for all $n \geq 2$. It is instructive to note that we cannot apply Theorem 1.3 since (26) is not a decomposition of a pro-Lie group by a weakly complete closed normal vector subgroup with compact quotient. Moreover (23) possesses a normal vector subgroup, namely its center, but does not split on it. See [22, Lemma 11.13, Corollary 11.14] for conditions of splitting. Therefore the computation of marginal and verbal n-Engel subgroups has been made directly here. ∎

Despite the fact that Theorem 1.3 cannot be applied, it is interesting to note that it is still true when we look at Example 4.4. In fact identifying $G = \mathrm{Heis}(\mathbb{R})$ in Theorem 1.3 and $N = A$ in (27), the quotient group $\mathrm{Heis}(\mathbb{R})/N$ is locally compact abelian and

$$\begin{aligned} E_n(\mathrm{Heis}(\mathbb{R})) &\subseteq \gamma_2(\mathrm{Heis}(\mathbb{R})) \cap N = \gamma_2(\mathrm{Heis}(\mathbb{R})) = \overline{[\mathrm{Heis}(\mathbb{R}), \mathrm{Heis}(\mathbb{R})]} \\ &= Z(\mathrm{Heis}(\mathbb{R})). \end{aligned} \tag{29}$$

Then the conclusions of Theorem 1.3 (i) are true, even if $\mathrm{Heis}(\mathbb{R})/N$ is noncompact abelian in this situation. About the conclusions of Theorem 1.3 (ii), these are also true, since $E^*_n(\mathrm{Heis}(\mathbb{R})) = \mathrm{Heis}(\mathbb{R})$ is open σ-compact noncompact.

Conjecture 4.5. Evidences such as Example 4.4 suggest that Theorem 1.3 might be still valid if we weaken the hypothesis of the presence of a compact quotient G/N via a weakly complete closed normal vector subgroup N in a locally compact pro-Lie group G.

On the other hand, we do not know what happens to Theorem 1.3 if G is a pro-Lie group in general. In other words, while we know that it is possible to place $E_n(G)$ and $E^*_n(G)$ between terms of the lower (resp. upper) central series of G, when G is a compact group (see Lemma 2.5), or a locally compact pro-Lie group (see Theorem 1.3), the question remains open in its generality in the category of pro-Lie groups and for topological groups which are not locally compact.

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Yanga Bavuma, Department of Mathematical Sciences, University of South Africa, Pretoria, South Africa; yangabavuma@gmail.com.

Francesco G. Russo, (1) Department of Mathematics and Applied Mathematics, University of Cape Town, South Africa; (2) Department of Mathematics and Applied Mathematics, University of the Western Cape, Bellville, South Africa; francescog.russo@yahoo.com.

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