Journal of Lie Theory
Volume **33** (2023) 133–148
© 2023 Heldermann Verlag

Unitarizable Vector-Valued Holomorphic Discrete Series and the Laplace Transform : an Example

Jean-Louis Clerc

Communicated by K.-H. Neeb

Abstract. For T_Ω a Hermitian symmetric tube-type domain, a family $(\pi_\mu)_{\mu\in\mathbb{C}}$ of holomorphic vector-valued representations is studied. The corresponding Wallach set is determined. The main tool is a realization of the representations as weighted L^2-spaces on the cone Ω through the Laplace transform.

Mathematics Subject Classification: 22E46, 32M15, 44A10

Key Words: Tube-type domains, Euclidean Jordan algebras, holomorphic discrete series, weighted Bergman spaces, Laplace transform, Wallach set.

1. Introduction

The holomorphic discrete series has been studied intensively after its introduction by Harish Chandra. In particular the analytic continuation of the series and the possible unitarizable representations beyond the discrete series is a difficult problem, solved in full generality in the 80's (see [3], [6]). Earlier, for scalar-valued cases, the description of the *Wallach set* [9] was an important result. For tube-type domains in particular, the Laplace transform offers explicit and nice realizations of the singular representations (see [4], [8]). There were a few attempts to study the vector-valued cases by a similar approach (see [1], [2], [5], [7]), but far from being conclusive. The present work addresses these questions on a modest but typical example.

The theory of tube-type domains which are Hermitian symmetric spaces is intimately connected to the theory of Euclidean Jordan algebras, as developed in [4]. To any (say) simple Euclidean Jordan algebra J is associated a symmetric cone Ω, and the tube-domain is $J\oplus i\Omega$. The group G of bi-holomorphic automorphisms has a family of generators which are easily described in the framework of Jordan algebras. The maximal compact subgroup U of G is a real form of the *structure group* of $\mathbb{J}$, the complexification of J. Hence among the irreducible representations of U one can single out those which are irreducible sub-representations of the natural action of U on the polynomial algebra on $\mathbb{J}$. The simplest component is the space of degree one homogeneous polynomials (i.e. linear forms). This allows to define a family of holomorphic representations $(\pi_\mu), \mu\in\mathbb{C}$ (in fact projective representations, which can be realized as representations of the universal covering of G). For μ real-valued and large enough, the representations are unitary and realized on *weighted Bergman spaces*. Their reproducing kernels $\mathcal{Q}_\mu$ are easy to determine and it is a natural

ISSN 0949–5932 / \$2.50 © Heldermann Verlag

question to ask for which values of μ is $\mathcal{Q}_\mu$ still positive-definite, or said in different terms to determine the *Wallach set* corresponding to this family of unitarizable vector-valued holomorphic representations. Simple Euclidean Jordan algebras are characterized by three numbers n, r, d which satisfy the relation $n = r + \frac{r(r-1)}{2}d$. The main theorem of the present article can be formulated as follows [1]

Theorem 1.1. *The kernel $\mathcal{Q}_\mu$ is positive-definite if and only if μ belongs to*

$$\left\{2\,\frac{d}{2}\,, 3\,\frac{d}{2}\,, \cdots, (r-1)\,\frac{d}{2}\right\} \bigcup \left[r\,\frac{d}{2}, +\infty\right).$$

In order to investigate other cases, the obstacle is in the complexity of the decomposition of the chosen representation of L, when restricted to the subgroup K of automorphisms of the Jordan algebra J. In the present example, there are only two components and it is possible to handle the case. The investigation through the Laplace transform gives interesting insights in the problem, and our results gives some motivation for considering more examples.

2. Euclidean Jordan algebra and tube-type domains

Let J a simple Euclidean Jordan algebra. Our main reference on the subject is [4] and we usually follow their notation. In particular, a simple Euclidean Jordan algebra is characterized by three integers : its dimension n, its *rank* r and an integer d, which satisfy

$$n = r + \frac{r(r-1)}{2}\,d\,.$$

The standard inner product on J is given by

$$x, y \in J, \qquad (x|y) = \operatorname{tr}(xy),$$

where tr is used for the *trace* function on J. Let L be the neutral component of the *structure group* of J. Then L is a reductive group, and $K = L \cap O(J)$ is a maximal compact subgroup of L which is also the stabilizer in L of the unit element e and the neutral component of the automorphisms group of J.

Denote by det the *determinant* of the Jordan algebra J. Recall that det is a homogeneous polynomial of degree r on J. Let $\chi : L \longrightarrow \mathbb{R}^+$ be the character on L which satisfies the following identity,

$$\det \ell x = \chi(\ell) \det x\ ,$$

for $\ell \in L$ and any $x \in J$. Let P be the quadratic representation of J, and recall that for any $x \in \Omega$

$$P(x) \in L \quad \text{and} \quad \chi\big(P(x)\big) = (\det x)^2\,. \tag{1}$$

Another useful formula, valid for any $\ell \in L$ is $\operatorname{Det} \ell = \chi(\ell)^{n/r}$.

Let Ω be the positive cone of J. The measure $d^*x = (\det x)^{-\frac{n}{r}}dx$ is invariant under the action of L.

[1] The case where $r = 1$ corresponds to the complex half-line and has only scalar representations, in the case $r = 2$, the set is equal to $[d, +\infty)$

Introduce the Γ*-function* of the cone as the integral

$$\Gamma_\Omega(\lambda) = \int_\Omega e^{-\operatorname{tr} x} \det(x)^\lambda \, d^*x \, .$$

The integral converges absolutely for $\Re(\lambda) > (r-1)\frac{d}{2} = \frac{n}{r} - 1$ and

$$\Gamma_\Omega(\lambda) = (2\pi)^{\frac{n-r}{2}} \Gamma(\lambda)\Gamma\left(\lambda - \frac{d}{2}\right) \dots \Gamma\left(\lambda - (r-1)\frac{d}{2}\right) .$$

For $\alpha \in \mathbb{C}, \Re(\alpha) > (r-1)\frac{d}{2}$, recall the *Riesz integral* given for a Schwarz function $\varphi \in \mathcal{S}(J)$ by

$$T_\alpha(\varphi) = \frac{1}{\Gamma_\Omega(\alpha)} \int_\Omega \varphi(x)(\det x)^\alpha d^*x \, . \tag{2}$$

Proposition 2.1. *For φ in $\mathcal{S}(J)$, $T_\alpha(\varphi)$ admits an analytic continuation as an entire function of α. Moreover, the analytic continuation T_α is a tempered distribution on J for all $\alpha \in \mathbb{C}$ and satisfies*

$$T_\alpha(\varphi \circ \ell^{-1}) = \chi(\ell)^\alpha T_\alpha(\varphi) \, . \tag{3}$$

The closure $\overline{\Omega}$ contains $r+1$ orbits under the action of L, namely

$$\overline{\Omega} = \bigsqcup_{k=0}^{r} \Omega^{(k)} \ ,$$

where $\Omega^{(k)}$ is the set of elements of rank k in $\overline{\Omega}$. Notice that $\Omega^{(0)} = \{0\}$ and $\Omega^{(r)} = \Omega$.

Proposition 2.2. *For $\alpha = k\frac{d}{2}, 0 \le k \le r-1$, the distribution T_α is a positive measure (from now on denoted by ν_k) supported by $\overline{\Omega^{(k)}}$. The measure ν_k is quasi-invariant by L and satisfies*

$$\nu_k(\ell \, . \,) = \chi(\ell)^{k\frac{d}{2}} \, \nu_k(\, . \,) \, . \tag{4}$$

Let $c \neq 0$ be an idempotent of J. Then the *Pierce decomposition* of J with respect to c is given by

$$J = J(c,1) \oplus J(c,1/2) \oplus J(c,0) \ ,$$

where for $\alpha = 0, 1/2, 1, \quad J(c,\alpha) = \{x \in J, \ cx = \alpha x\}$.

A *Jordan frame* of J is a collection $(c_1, c_2, \dots, c_r)$ of strongly orthogonal primitive idempotents which satisfy $e = c_1 + c_2 + \dots + c_r$.

To each Jordan frame corresponds a *Peirce decomposition*

$$J = \bigoplus_{1 \le i \le j \le r} J_{ij},$$

where for $1 \le i \le r$ and $i+1 \le j \le r$,

$$J_{ii} = J(c_i, 1) = \mathbb{R}c_i, \qquad J_{ij} = J(c_i, 1/2) \cap J(c_j, 1/2) \, .$$

The spaces J_{ij} for $1 \le i < j \le r$ have the same dimension d.

Let $\mathbb{J}$ be the complexification of J. Extending the Jordan algebra structure of J to $\mathbb{J}$ in a $\mathbb{C}$-linear way, $\mathbb{J}$ becomes a complex simple Jordan algebra. Let $\mathbb{L}$ be the connected component of the structure group of $\mathbb{J}$, which is a reductive group. Extend the Euclidean inner product of J to a Hilbertian product on $\mathbb{J}$ and let $U(\mathbb{J})$ be the corresponding unitary group. Then

$$U = \mathbb{L} \cap U(\mathbb{J}) \tag{5}$$

is a connected maximal compact subgroup of $\mathbb{L}$. Notice that L and U are two real forms of $\mathbb{L}$.

Let $T_\Omega = \{z = x + iy \in \mathbb{J}, y \in \Omega\}$ be the corresponding *tube domain*. Let G be the neutral component of the group of biholomorphic diffeomorphisms of T_Ω. The group G is generated by

- the group L, after complex extension to $\mathbb{J}$ of its elements
- the group of translations $N = \{t_u : z \mapsto z + u, u \in J\}$
- ι the *inversion* $z \longmapsto -z^{-1}$.

The group G acts transitively on T_Ω and the space T_Ω, equipped with the Bergman metric, is a non compact Hermitian symmetric space.

Let $g \in G$ and let $z \in T_\Omega$. Then the complex differential $Dg(z)$ is an element of the complex structure group $\mathbb{L}$ of the complex Jordan algebra $\mathbb{J}$. For the generators of G, notice that for $z \in T_\Omega$

- $D\ell(z) = \ell$ for $\ell \in L$.
- $Dt_u(z) = \mathrm{id}$ for $u \in J$
- $D\iota(z) = P\left(z^{-1}\right) = P(z)^{-1}$ for the inversion ι.

For $g \in G$ and $z \in T_\Omega$, let $J(g, z) = Dg(z)$ be the differential of the map g at z. An important result is that

$$J(g, z) \ \in \ \mathbb{L}\ ,$$

as can be verified on the generators of G and extended to G by using the chain rule.

Let $\widetilde{U}$ be the stabilizer of the origin $ie \in T_\Omega$. Then $\widetilde{U}$ is a maximal compact subgroup of G. To describe this subgroup, it is convenient to refer to the *bounded realization* of T_Ω. Let $|\,.\,|$ be the *spectral norm* on $\mathbb{J}$ and let D be the corresponding open unit ball

$$D = \{w \in \mathbb{J}, \quad |w| < 1\}\,.$$

Let $G(D)^0$ be the neutral component of the group of biholomorphic diffeomorphisms of D. Then the stabilizer of $0 \in D$ in $G(D)^0$ is equal to U as defined by (5).

Define the *Cayley transform* to be the rational map c on $\mathbb{J}$ given by

$$c(w) = i(e + w)(e - w)^{-1}\,.$$

Then c is well-defined for $w \in D$, its image $c(w)$ belongs to T_Ω and c yields a biholomorphic diffeomorphism from D into T_Ω. Now for $u \in U$, define

$$\widetilde{u} = c \circ u \circ c^{-1}\,. \tag{6}$$

Then $u \in \widetilde{U}$ and the map $u \longmapsto \widetilde{u}$ is a isomorphism of U to $\widetilde{U}$. Notice that

$$D\widetilde{u}(ie) = u\,. \tag{7}$$

This is obtained by the chain rule applied to (6) and the fact that $Dc(0) = 2i\,\mathrm{id}_{\mathbb{J}}$.

3. Vector-valued holomorphic representations and Laplace transform

Let (σ, V_σ) be a finite dimensional (holomorphic) irreducible representation of $\mathbb{L}$ (equivalently of U or L) and choose an inner product $(\,.\,,\,.\,)_{V_\sigma}$ on V_σ which is invariant under the action of U. Let $\mathcal{O}_\sigma$ be the Montel space of V_σ-valued holomorphic functions on T_Ω. The following formula defines a representation of G on $\mathcal{O}_\sigma$:

$$\pi_\sigma(g)F(z) = \sigma\big(J(g^{-1}, z)\big)^{-1} F\big(g^{-1}(z)\big)\,. \tag{8}$$

For $F, G \in \mathcal{O}_\sigma$ let $(F,G)_\sigma = \displaystyle\int_{T_\Omega} \big(\sigma(P(y)^{-1}F(z), G(z)\big)_{V_\sigma}\, d_*z,$

where $d_*z = (\det y)^{-\frac{2n}{r}}\, dx\, dy$ is the G-invariant measure on T_Ω, and let

$$\mathcal{H}_\sigma = \{F \in \mathcal{O}_\sigma, (F,F)_\sigma < +\infty\}\,.$$

When this space is not reduced to $\{0\}$, it is a Hilbert space, stable by the action of G and π_σ yields a unitary representation of G. Then the evaluation map at any $z \in T_\Omega$

$$\mathcal{H}_\sigma \ni F \longmapsto F(z) \in V_\sigma$$

is easily shown to be continuous. Define for $z, w \in T_\Omega$ the *reproducing kernel* of $\mathcal{H}_\sigma$ by

$$\mathcal{Q}_\sigma(z,w) = E_z E_w^*\,.$$

Proposition 3.1. *Assume that* $\mathcal{H}_\sigma \neq \{0\}$.

Then $\mathcal{Q}_\sigma(z,w) = c\,\sigma\left(P\left(\dfrac{z-\overline{w}}{2i}\right)\right)$ *for some constant* $c > 0$.

See [1] for a proof. In the sequel, by redefining the inner product on $\mathcal{H}_\sigma$, we assume that the constant c is equal to 1. Denote by $\mathrm{Herm}^+(V_\sigma)$ the space of positive semi-definite operators on V_σ.

Proposition 3.2. *Suppose that* $\mathcal{H}_\sigma \neq \{0\}$. *There exists a unique* $\mathrm{Herm}^+(V_\sigma)$*-valued measure* dR_σ *on* $\overline{\Omega}$ *such that*

$$\mathcal{Q}_\sigma(z,w) = \int_{\overline{\Omega}} e^{-(\frac{z-\overline{w}}{2i}\,|\,v)} dR_\sigma(v)\,. \tag{9}$$

Moreover, the measure dR_σ *satisfies*

$$\forall \ell \in L, \qquad dR_\sigma(\ell\,.\,) = \sigma(\ell)^{*-1}\, dR_\sigma(\,.\,)\,\sigma(\ell)^{-1}\ , \tag{10}$$

$$\int_{\overline{\Omega}} e^{-\operatorname{tr} v} dR_\sigma(v) = \mathrm{Id}_{V_\sigma}\ . \tag{11}$$

For a proof see [1].

Let $\mathcal{L}_\sigma$ be the space of measurable functions $f : \overline{\Omega} \longrightarrow V_\sigma$ which satisfy

$$\int_{\overline{\Omega}} \big(dR_\sigma(2v)f(v), f(v)\big)_{V_\sigma} < +\infty\,. \tag{12}$$

After identifying two functions which are equal dR_σ-a.e., $\mathcal{L}_\sigma$ becomes a Hilbert space for the inner product

$$(f,g)_{\mathcal{L}_\sigma} = \int_{\overline{\Omega}} \big(dR_\sigma(2v)f(v), g(v)\big)_{V_\sigma}\,. \tag{13}$$

Define the (modified) *Laplace transform* $\mathcal{F}_\sigma$ by

$$\mathcal{F}_\sigma f(z) = \int_{\overline{\Omega}} e^{i\,(z\,|\,v)} dR_\sigma(2v) f(v)\,. \tag{14}$$

Proposition 3.3. *Assume that* $\mathcal{H}_\sigma \neq \{0\}$. *Then the Laplace transform* $\mathcal{F}_\sigma$ *yields an isometry from* $\mathcal{L}_\sigma$ *onto* $\mathcal{H}_\sigma$.

For the proof, see [1], Theorem 3.5.

The results presented so far can be summarized as follows: to each representation σ such that $\mathcal{H}_\sigma$ is not reduced to $\{0\}$, there corresponds an $\mathrm{Herm}^+(V_\sigma)$-valued measure dR_σ on $\overline{\Omega}$ which satisfies (10) and (11). In some sense, there is a converse construction, starting form the measure dR_σ and constructing the Hilbert space $\mathcal{H}_\sigma$.

Proposition 3.4. *Let* σ *be a holomorphic irreducible representation of* $\mathbb{L}$ *and assume that there exists an* $\mathrm{Herm}^+(V_\sigma)$*-valued measure* dR_σ *on* $\overline{\Omega}$ *which satisfies* (10) *and* (11). *Define the space* $\mathcal{L}_\sigma$ *by the condition* (12) *and let* $\widetilde{\mathcal{H}}_\sigma$ *be the image of* $\mathcal{L}_\sigma$ *by the Laplace transform* (14). *Then* $\widetilde{\mathcal{H}}_\sigma$, *equipped with the inner product given by*

$$(\mathcal{F}_\sigma f, \mathcal{F}_\sigma g) = (f, g)_{\mathcal{L}_\sigma}$$

is a Hilbert space of holomorphic functions in T_Ω, *which admits the reproducing kernel given by*

$$\mathcal{Q}_\sigma(z,w) = \int_{\overline{\Omega}} e^{-(\frac{z-\overline{w}}{2i}\,|\,v)} dR_\sigma(v)\,.$$

The space $\widetilde{\mathcal{H}}_\sigma$ *is stable under the action of* G *given by* (8) *and* $(\pi_\sigma, \widetilde{\mathcal{H}}_\sigma)$ *is an irreducible unitary representation of* G.

For the proof see again [1], Theorem 3.5.

Remark 3.5. For $z, w \in T_\Omega$, we have $\Re\left(\frac{z-\overline{w}}{2i}\right) = \frac{\Im(z)+\Im(w)}{2} \in \Omega$, and on the diagonal $\Re\left(\frac{z-\overline{z}}{2i}\right) = \Im(z) \in \Omega$. Hence $\det\left(\frac{z-\overline{w}}{2i}\right) \neq 0$ on $T_\Omega \times T_\Omega$, and there exists a unique determination of $\log\det\left(\frac{z-\overline{w}}{2i}\right)$ on $T_\Omega \times T_\Omega$ which coincides on the diagonal $\{w = z\}$ with $\ln(\det(\Im z))$. With this choice, define for $\mu \in \mathbb{C}$,

$$\det\left(\frac{z-\overline{w}}{2i}\right)^\mu = e^{\mu \log\left(\det\left(\frac{z-\overline{w}}{2i}\right)\right)}, \quad \text{and let} \quad \sigma_\mu(\ell) = \chi(\ell)^{-\frac{\mu}{2}}\sigma(\ell)\,.$$

This makes sense for $\ell \in L$ and defines a representation of L, which can also be considered as a representation of the universal covering of $\mathbb{L}$. It may be used to define a holomorphic representation π_{σ_μ} of the universal covering of G still using formula (8). Thanks to (1), the corresponding reproducing kernel is given by

$$\mathcal{Q}_{\sigma,\mu}(z,w) = \det\left(\frac{z-\overline{w}}{2i}\right)^{-\mu} \mathcal{Q}_\sigma(z,w)\,.$$

This opens the possibility of studying the existence of dR_σ by using techniques of *analytic continuation.*

4. Polynomial representations of L and the associated holomorphic discrete series

Let $\mathcal{P}$ be the space of holomorphic polynomials on $\mathbb{J}$. The group $\mathbb{L}$ acts naturally on $\mathcal{P}$ by

$$\pi(\ell)p\,(z) = p(\ell^{-1}z)\ ,$$

for $p \in \mathcal{P}$ and $\ell \in \mathbb{L}$. It may also be regarded as a representation of U or of L, and clearly the decomposition of $\mathcal{P}$ into invariant minimal subspaces is the same for the three different points of view.

Fix a Jordan frame $(c_1, c_2, \ldots, c_r)$ of J. For $1 \leq k \leq r$ let

$$e_k = c_1 + c_2 + \cdots = c_k, \qquad J_k = J(e_k, 1)\,.$$

Then J_k is simple Jordan algebra with neutral element e_k. Notice that for any $k, 0 \leq k \leq r$, we have $\Omega_k = L\, e_k$.

Let det_k be its determinant and let p_k be the orthogonal projector of J unto J_k. We define $\Delta_k(x) = \det_k(p_k x)$ as the *k-th principal minor.*

A multiindex $\boldsymbol{m} = (m_1, m_2, \ldots, m_r)$ where $m_j \in \mathbb{Z}, 1 \leq j \leq r$ is said to be *positive* if $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$. The set of positive multiindices is denoted by $\boldsymbol{\mathcal{M}}_+$.

For $\boldsymbol{m} \in \boldsymbol{\mathcal{M}}_+$ let $\qquad \Delta_{\boldsymbol{m}}(x) = \Delta_1(x)^{m_1 - m_2} \Delta_2(x)^{m_2 - m_3} \ldots \Delta_r(x)^{m_r}$

and let $\mathcal{P}_{\boldsymbol{m}}$ be the subspace of $\mathcal{P}$ generated by $\{\pi(\ell)\Delta_{\boldsymbol{m}},\ \ell \in L\}$.

Proposition 4.1. *The subspaces $\mathcal{P}_{\boldsymbol{m}}$ are mutually inequivalent irreducible subspaces under the action of L and $\mathcal{P}$ is the direct orthogonal sum*

$$\mathcal{P} = \bigoplus_{\boldsymbol{m} \in \boldsymbol{\mathcal{M}}_+} \mathcal{P}_{\boldsymbol{m}}\,.$$

Denote by $\pi_{\boldsymbol{m}}$ the restriction of π to the subspace $\mathcal{P}_{\boldsymbol{m}}$ will be denoted by $\pi_{\boldsymbol{m}}$.

Let $\boldsymbol{m} = (m_1, m_2, \ldots, m_r) \in \boldsymbol{\mathcal{M}}_+$ and let $\mathcal{O}_{\boldsymbol{m}} = \mathcal{O}(T_\Omega, \mathcal{P}_{\boldsymbol{m}}) \simeq \mathcal{O}(T_\Omega) \otimes \mathcal{P}_{\boldsymbol{m}}$ be the space of $\mathcal{P}_{\boldsymbol{m}}$-valued holomorphic functions on T_Ω. Then G acts on $\mathcal{O}_{\boldsymbol{m}}$ by

$$\pi_{\boldsymbol{m}}(g)F(z) = \sigma_{\boldsymbol{m}}\big(J(g^{-1}, z)\big)^{-1} F\big(g^{-1}(z)\big)\,.$$

Form the corresponding Hilbert space $\mathcal{H}_{\sigma_{\boldsymbol{m}}}$ and the corresponding representation $\pi_{\sigma_{\boldsymbol{m}}}$ of G on $\mathcal{H}_{\sigma_{\boldsymbol{m}}}$. Let further

$$\mathcal{Q}_{\boldsymbol{m}}(z, w) = \sigma_{\boldsymbol{m}}\left(P\left(\frac{z - \overline{w}}{2i}\right)\right).$$

When $\mathcal{H}_{\boldsymbol{m}} \neq \{0\}$, this is (up to a positive scalar) the reproducing kernel of $\mathcal{H}_{\boldsymbol{m}}$, and there exists a $\mathrm{Herm}^+(\mathcal{P}_{\boldsymbol{m}})$-valued measure $dR_{\boldsymbol{m}}$, supported in $\overline{\Omega}$, such that (9) is satisfied. The question we address is to calculate $dR_{\boldsymbol{m}}$, using the two properties (10) and (11) which allow in principle to calculate a formal solution. It remains to test the positivity of the solution to conclude.

In this case, the remark can be exploited as follows. Let $\mu \in \mathbb{N}$, and consider the map $I_\mu : \mathcal{P} \to \mathcal{P}$ given by

$$(I_\mu\, p\,)(z) = (\det z)^\mu p(z)$$

and define the representation $\sigma_{\boldsymbol{m},\mu}$ of $\mathbb{L}$ on $\mathcal{P}_{\boldsymbol{m}}$ given by

$$\sigma_{\boldsymbol{m},\mu}(\ell)\, p = \chi(\ell)^{-\mu}\ (p \circ \ell^{-1})\,.$$

Proposition 4.2. *The operator I_μ maps $\mathcal{P}_{\boldsymbol{m}}$ onto $\mathcal{P}_{\boldsymbol{m}+\mu}$ and intertwines $\sigma_{\boldsymbol{m},\mu}$ and $\sigma_{\boldsymbol{m}+\mu}$.*

Proof. Now first we note that

$$(I_\mu\,\Delta_{\boldsymbol{m}})(z) = \Delta_1(x)^{m_1-m_2}\dots\Delta_r^{m_r+\mu}(z)$$
$$= \Delta_{m_1+\mu,m_2+\mu,\dots,m_{r-1}+\mu,m_r+\mu}(z) = \Delta_{\boldsymbol{m}+\mu}(z) \in \mathcal{P}_{\boldsymbol{m}+\mu}\,.$$

Next for $p \in \mathcal{P}_{\boldsymbol{m}}$

$$I_\mu(\sigma_{\boldsymbol{m},\mu}(\ell)p)(z) = \chi(\ell)^{-\mu}p(\ell^{-1}z)\Delta_r^\mu(z) = p(\ell^{-1}z)\Delta_r^\mu(\ell^{-1}z) = (I_\mu p)(\ell^{-1}z)\,.$$

Combining both formulas,

$$I_\mu(\sigma_{\boldsymbol{m},\mu}(\ell)\Delta_{\boldsymbol{m}}) = \Delta_{\boldsymbol{m}+\mu}\circ\ell^{-1} = \sigma_{\boldsymbol{m}+\mu(\ell)}I_\mu\Delta_{\boldsymbol{m}}\,. \qquad \blacksquare$$

As observed previously, this can be extended for μ a complex number, by considering the universal covering of $\mathbb{L}$.

The strategy is to consider a multiindex $\boldsymbol{m}$ with last index $m_r = 0$ and study the family of kernels $\mathcal{Q}_{\sigma_{\boldsymbol{m}+\mu}}, \mu \in \mathbb{C}$ by analytic continuation.

The scalar case (i.e. $\boldsymbol{m} = (m, m, \dots, m)$) is treated in [4] ch. XIII and the study leads to the so-called *Wallach set* (see [9] and [8] for original proofs).

For $x \in \Omega$, $\Delta_j(x) > 0$. For $\boldsymbol{s} = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r$, write

$$\Delta_{\boldsymbol{s}}(x) = \Delta_1(x)^{s_1-s_2}\Delta_2(x)^{s_2-s_3}\dots\Delta_r(x)^{s_r}.$$

The integral is absolutely convergent for $\Re s_j > (j-1)\frac{d}{2}, j = 1, 2, \dots, r$ and can be meromorphically continued to $\mathbb{C}^r$. Moreover, see [4] Ch. VII, we have

$$\Gamma_\Omega(\boldsymbol{s}) = (2\pi)^{\frac{n-r}{2}}\prod_{j=1}^{r}\Gamma\left(s_j - (j-1)\frac{d}{2}\right)\,. \tag{15}$$

5. The case of $\mathcal{P}_{1,0,\dots,0}$

In this section, we achieve the computation of dR_μ for the case corresponding to the family of indices $\boldsymbol{m} = (1, 0, \dots, 0) + \mu, \mu \in \mathbb{C}$. [2]

The space $\mathcal{P}_{1,0,\dots,0}$ is the dual space $\mathbb{J}'$ of complex linear forms on $\mathbb{J}$. Using the duality (over $\mathbb{C}$) given by

$$(x, y) \longmapsto \operatorname{tr} xy$$

it is convenient to identify $\mathbb{J}'$ with $\mathbb{J}$. The corresponding representation σ of $\mathbb{L}$ is then given by

$$\sigma(\ell)v = \ell^{t^{-1}}v\,.$$

The corresponding U-invariant Hilbertian inner product is given by

$$(x|y) = \operatorname{tr} x\overline{y}\,.$$

The following (folklore) lemma will be useful later.

[2] The case where J is of rank 1, i.e. $J = \mathbb{R}$, is excluded as there are only scalar scalar valued cases.

Lemma 5.1. *Let J be a simple Euclidean Jordan algebra of rank $r \geq 2$. Let Q be a Hermitian operator on $\mathbb{J}$ which is invariant under K. Then there exists $\alpha, \beta \in \mathbb{R}$ such that for any $v, w \in \mathbb{J}$: $Qv = \alpha v + \beta (v|e)\, e$.*

Proof. Let $q(v,w) = (Qv, w)$ be the associated Hermitian form on $\mathbb{J}$. Let $(c_1, c_2, \dots, c_r)$ be a Jordan frame of J. Let $\mathbb{A} = \bigoplus_{j=1}^r \mathbb{C}c_j$ and let $q_{\mathbb{A}}$ be the restriction of q to $\mathbb{A}$. Given a permutation σ of $\{1, 2, \dots, r\}$ there exists an element k_σ of K such that $k_\sigma c_j = c_{\sigma(j)}$. Hence $q_{\mathbb{A}}$ is invariant under $\mathfrak{S}_r$. The space $\mathbb{A} \simeq \mathbb{C}^r$ decomposes under the action of $\mathfrak{S}_r$ as

$$\mathbb{A} = \mathbb{C}e \oplus \mathbb{A}_0 = \mathbb{C}e \bigoplus \left\{ a = \sum_{j=1}^r a_j c_j,\ \sum_{j=1}^r a_j = 0 \right\} ,$$

and $\mathbb{A}_0$ is irreducible under the action of $\mathfrak{S}_r$. Hence the space of $\mathfrak{S}_r$-invariant Hermitian forms on $\mathbb{A}$ is of dimension 2. The two Hermitian forms $\sum_{j=1}^r a_j \overline{b}_j$ and $(\sum_{j=1}^r a_j)(\sum_{j=1}^r \overline{b}_j)$ on $\mathbb{C}^r$ are $\mathfrak{S}_r$-invariant and they are linearly independent. Hence there exists two real numbers α, β such that

$$q_{\mathbb{A}}(\sum_{j=1}^r a_j c_j, \sum_{j=1}^r b_j c_j) = \alpha \left(\sum_{j=1}^r a_j \overline{b}_j \right) + \beta \left(\sum_{j=1}^r a_j \right) \left(\sum_{j=1}^r \overline{b}_j \right) .$$

As a consequence, the two Hermitian forms q and $\alpha \operatorname{tr}(x\overline{x}) + \beta \operatorname{tr}(x) \operatorname{tr}(\overline{x})$ on $\mathbb{J}$ are invariant by K and coincide on $\mathbb{A}$. By the spectral theorem for J, they coincide on J and hence also on $\mathbb{J}$. The conclusion follows. ■

Recall that σ_μ is the representation of L given by $\sigma_\mu(\ell) = \chi(l)^{-\frac{\mu}{2}} \sigma(\ell)$.
The measure dR_μ on $\overline{\Omega}$ should satisfy

$$\text{for all } \ell \in L, \qquad dR_\mu(\ell .) = \chi(\ell)^\mu \ell \, dR_\mu(.)\, \ell^t \tag{16}$$

and

$$\int_{\overline{\Omega}} e^{-\operatorname{tr} v} dR_\mu(v) = \operatorname{Id}_{\mathbb{J}} . \tag{17}$$

Theorem 5.2. *Let $\Re\mu > (r-1)\frac{d}{2}$. Let dR_μ be the $Herm(\mathbb{J})$-valued measure on Ω defined by*

$$dR_\mu(y) = \frac{1}{\mu(\mu+1)(\mu - \frac{d}{2})\Gamma_\Omega(\mu)} \left(\mu P(y) - \frac{d}{2} p_y \right) (\det y)^\mu d^*y \tag{18}$$

where for $y \in J$, p_y is the operator defined by $p_y v = (v|y)y$. Then

$$\int_\Omega e^{-\operatorname{tr} y}\, dR_\mu(y) = \operatorname{Id}_{\mathbb{J}} .$$

Before giving a proof of the theorem, we present a heuristic approach which led us to the formula. Assume that the measure dR_μ is absolutely continuous with respect to the L-invariant measure $(\det y)^{-\frac{n}{r}} dy$ on Ω, and let

$$dR_\mu(y) = r_\mu(y) (\det y)^{-\frac{n}{r}} dy$$

be its expression, where r_μ is now a $\operatorname{Herm}(\mathbb{J})^+$-valued function on Ω.

The function r_μ has to satisfy

$$\forall \ell \in L, \forall x \in \Omega, \qquad r_\mu(\ell x) = \chi(l)^\mu \, l \, r_\mu(x) \, l^t \tag{19}$$

and

$$\int_\Omega e^{-\operatorname{tr} x} r_\mu(x)\, dx^* = \mathrm{id}_{\mathbb{J}} \ . \tag{20}$$

Lemma 5.3. *Let r_μ a* $\mathrm{Herm}(\mathbb{J})$*-valued function on Ω which satisfies* (19)*. Then there exist two real numbers $\alpha(\mu), \beta(\mu)$ such that*

$$r_\mu(y) = (\det y)^\mu \big(\alpha(\mu) P(y) + \beta(\mu)\, p_y\big)\, . \tag{21}$$

Proof. Letting $\ell \in K$ and $v = e$, (19) implies that for any $k \in K$

$$k\, r_\mu(e) = r_\mu(e)\, k\, .$$

Now use Lemma 5.1 to conclude that there exists two real numbers $\alpha(\mu)$ and $\beta(\mu)$ such that

$$r_\mu(e) v = \alpha(\mu) v + \beta(\mu) \operatorname{tr}(v) e\, .$$

Further, let $y \in \Omega$ and use again (19) with $x = e$ and $\ell = P(y^{1/2})$ to get

$$r_\mu(y) = (\det y)^\mu \big(\alpha(\mu) P(y) + \beta(\mu)\, p_y\big)\, . \qquad \blacksquare$$

To prove Theorem 5.2, it remains to compute $\alpha(\mu)$ and $\beta(\mu)$, so that (20) is satisfied. Consider the four following functions on J

$$\operatorname{Tr} P(x), \quad \operatorname{Tr} p_x, \quad (P(x)e|e), \quad (p_x(e)|e)\, .$$

They are homogeneous polynomials of degree 2 and they are invariant by K. The space $\mathcal{P}_2$ of homogeneous polynomials of degree 2 decomposes under the action of L as

$$\mathcal{P}_2 = \mathcal{P}_{2,0,\dots,0} \oplus \mathcal{P}_{1,1,0,\dots,0}\ ,$$

as a consequence of Proposition 4.1. Now each space $\mathcal{P}_{\boldsymbol{m}}$ contains a unique (up to a scalar) K-invariant vector (see [4] Proposition XI.3.1), so that the four polynomials are linear combination of the two K-invariant polynomials in $\mathcal{P}_{2,0}$ and $\mathcal{P}_{1,1}$.

Lemma 5.4. *Consider the following polynomials*

$$p_{2,0}(x) = \frac{d}{2} (\operatorname{tr} x)^2 + \operatorname{tr}(x^2)\ , \quad p_{1,1}(x) = (\operatorname{tr} x)^2 - \operatorname{tr}(x^2)\, .$$

Then $p_{2,0}$ (resp. $p_{1,1}$) is the unique (up to a scalar) K-invariant element in $\mathcal{P}_{2,0,\dots,0}$ (resp. $\mathcal{P}_{1,1,0,\dots,0}$).

For a proof, see [4] ch XI, Exercice 2.

Lemma 5.5. *Let J be a simple Euclidean Jordan algebra of rank $r \geq 2$. Then the following identities hold*

$$\operatorname{Tr} P(x) = \frac{2}{d+2} \left(p_{2,0}(x) \ + \ \frac{d^2}{4}\, p_{1,1}(x)\right) \tag{22}$$

$$(P(x)e|e) = \operatorname{Tr} p_x = \frac{2}{d+2}\big(p_{2,0}(x)\big) - \frac{d}{2}\big(p_{1,1}(x)\big) \tag{23}$$

$$(p_x e|e) = \frac{2}{d+2}\big(p_{2,0}(x) + p_{1,1}(x)\big)\, . \tag{24}$$

Proof. First observe that

$$(P(x)e|e) = (x^2|e) = \operatorname{tr}(x^2), \quad \operatorname{Tr} p_x = (x|x) = \operatorname{tr}(x^2),$$

and
$$(p_x e|e) = (x|e)(x|e) = (\operatorname{tr} x)^2 .$$

Hence (23) and (24) express the change of basis from $\{\operatorname{tr}(x^2), (\operatorname{tr} x)^2\}$ to $\{p_{2,0}, p_{1,1}\}$. For (22), it suffices to evaluate both sides on $x = e$ and on $x = c$ a primitive idempotent (e and c are not proportional, as $\operatorname{rank} J \geq 2$). Now

$$\operatorname{tr}(e) = \operatorname{tr}(e^2) = r, \quad \operatorname{Tr} P(e) = n, \quad \operatorname{tr}(c) = \operatorname{tr}(c^2) = 1, \quad \operatorname{Tr} P(c) = 1$$

and the verification follows by elementary computation. ■

Lemma 5.6. *For* $\Re\mu > (r-1)\frac{d}{2}$,

$$I_{2,0} = \int_\Omega e^{-\operatorname{tr} x} p_{2,0}(x)(\det x)^\mu d^*x = r\Big(1 + \frac{rd}{2}\Big)\mu(\mu+1)\Gamma_\Omega(\mu) \tag{25}$$

$$I_{1,1} = \int_\Omega e^{-\operatorname{tr} x} p_{1,1}(x)(\det x)^\mu d^*x = r(r-1)\,\mu\,(\mu - \frac{d}{2})\Gamma_\Omega(\mu)\,. \tag{26}$$

Proof. Let $p \in \mathcal{P}_{\boldsymbol{m}}$. For $\Re\mu > (r-1)\frac{d}{2}$,

$$\int_\Omega e^{-\operatorname{tr} y} p(y)(\det y)^\mu\, d^*x = \Gamma_\Omega(\boldsymbol{m} + \mu)\, p(e)\,. \tag{27}$$

See [4] Lemma XI.2.3. Apply now this formula and for $\boldsymbol{m} = (2, 0, \ldots, 0)$

$$\int_\Omega e^{-\operatorname{tr} x} p_{2,0}(x)\,(\det x)^\mu\, d^*x =$$

$$= (2\pi)^{\frac{n-r}{2}} r\left(1 + \frac{rd}{2}\right)\Gamma(2+\mu)\prod_{j=2}^{r}\Gamma\left(\mu - (j-1)\frac{d}{2}\right) = r\left(1 + \frac{rd}{2}\right)(\mu+1)\,\mu\,\Gamma_\Omega(\mu)$$

and for $\boldsymbol{m} = (1, 1, 0, \ldots, 0)$ to obtain

$$\int_\Omega e^{-\operatorname{tr} x} p_{1,1}(\det x)^\mu\, d^*x =$$

$$= (2\pi)^{\frac{n-r}{2}} r(r-1)\Gamma(\mu+1)\Gamma\left(\mu + 1 - \frac{d}{2}\right)\prod_{j=3}^{r}\Gamma\left(\mu - (j-1)\frac{d}{2}\right)$$

$$= r\,(r-1)\,\mu\left(\mu - \frac{d}{2}\right)\Gamma_\Omega(\mu)\,.$$ ■

Proposition 5.7. *For* $\Re\mu > (r-1)\frac{d}{2}$

$$\int_\Omega e^{-\operatorname{tr} x}\operatorname{Tr}(P(x))(\det x)^\mu d^*x =$$

$$= \frac{2}{d+2}\Gamma_\Omega(\mu) r\mu\Big((1 + \frac{rd}{2})(\mu+1) + \frac{d^2}{4}(r-1)(\mu - \frac{d}{2})\Big) \tag{28}$$

$$\int_\Omega e^{-\operatorname{tr} x}\operatorname{Tr}(p_x)(\det x)^\mu d^*x =$$

$$= \frac{2}{d+2}\Gamma_\Omega(\mu) r\mu\Big((1 + \frac{rd}{2})(\mu+1) - \frac{d}{2}(r-1)(\mu - \frac{d}{2})\Big) \tag{29}$$

$$\int_\Omega e^{-\operatorname{tr} x}(P(x)e|e)(\det x)^\mu d^*x = $$
$$= \frac{2}{d+2}\Gamma_\Omega(\mu) r\mu\Big((1+\frac{rd}{2})(\mu+1) - \frac{d}{2}(r-1)(\mu-\frac{d}{2})\Big) \tag{30}$$
$$\int_\Omega e^{-\operatorname{tr} x}(p_x e|e)(\det x)^\mu d^*x = $$
$$= \frac{2}{d+2}\,\Gamma_\Omega(\mu) r\mu\Big((1+\frac{rd}{2})(\mu+1) + (r-1)(\mu-\frac{d}{2})\Big)\,. \tag{31}$$

Proof. Combine Lemma 5.5 and Lemma 5.6 to get the results. ■

As $\operatorname{Tr}(\mathrm{Id}_{\mathbb{J}}) = n$ and $(\mathrm{Id}_{\mathbb{J}}\, e, e) = \operatorname{tr} e = r$, $\alpha(\mu), \beta(\mu)$ have to satisfy

$$\begin{pmatrix} (2+rd)(\mu+1) + (r-1)\frac{d^2}{2}(\mu-\frac{d}{2}) & (2+rd)(\mu+1) - (r-1)d(\mu-\frac{d}{2}) \\ (2+rd)(\mu+1) - d(r-1)(\mu-\frac{d}{2}) & (2+rd)(\mu+1) + 2(r-1)(\mu-\frac{d}{2}) \end{pmatrix} \begin{pmatrix} \alpha(\mu) \\ \beta(\mu) \end{pmatrix}$$
$$= \frac{d+2}{r\mu\Gamma_\Omega(\mu)} \begin{pmatrix} n \\ r \end{pmatrix}.$$

Now observe that the four coefficients of the matrix vanish identically for $d=-2$. Hence the system can be rewritten as

$$\begin{pmatrix} \left(\frac{(r-1)d}{2}+1\right)\mu - (r-1)\frac{d^2}{4} + \frac{(r-1)d}{2} + 1 & \mu + \frac{(r-1)d}{2} + 1 \\ \mu + \frac{(r-1)d}{2} + 1 & r\mu + 1 \end{pmatrix} \begin{pmatrix} \alpha(\mu) \\ \beta(\mu) \end{pmatrix}$$
$$= \frac{1}{r\mu\Gamma_\Omega(\mu)} \begin{pmatrix} n \\ r \end{pmatrix}.$$

The determinant of the matrix is a polynomial of degree 2 in μ, and it is easily seen that it vanishes for $\mu = -1$ and $\mu = \frac{d}{2}$. Next, the coefficient of degree 2 is equal to

$$\begin{vmatrix} \left(\frac{(r-1)d}{2}+1\right) & 1 \\ 1 & r \end{vmatrix} = n-1\,.$$

Hence the determinant of the matrix of the system is equal to

$$(n-1)(\mu+1)\left(\mu - \frac{d}{2}\right).$$

The solutions of the system are given by

$$\begin{pmatrix} \alpha(\mu) \\ \beta(\mu) \end{pmatrix} = \frac{1}{r(n-1)\mu(\mu+1)(\mu-\frac{d}{2})\Gamma_\Omega(\mu)}$$
$$\begin{pmatrix} r\mu+1 & -\mu - \frac{(r-1)d}{2} - 1 \\ -\mu - \frac{(r-1)d}{2} - 1 & \left(\frac{(r-1)d}{2}+1\right)\mu - (r-1)\frac{d^2}{4} + \frac{(r-1)d}{2} + 1 \end{pmatrix} \begin{pmatrix} n \\ r \end{pmatrix}.$$

Notice that $n(r\mu+1)-r\mu-(r(r-1)\frac{d}{2}+r)=(n-1)r\mu$, whereas

$$-n\mu-\frac{(r-1)d}{2}n-n+n\mu-r(r-1)\frac{d^2}{4}+n=-(r-1)\frac{d}{2}\left(n+r\frac{d}{2}\right)$$
$$=-\frac{d}{2}\left(n(r-1)+r(r-1)\frac{d}{2}\right)=-\frac{d}{2}\left(n(r-1)+(n-r)\right)=-\frac{d}{2}(n-1)r\,.$$

Hence
$$\begin{pmatrix}\alpha(\mu)\\ \beta(\mu)\end{pmatrix}=\frac{1}{\mu(\mu+1)(\mu-\frac{d}{2})\Gamma_\Omega(\mu)}\begin{pmatrix}\mu\\ -\frac{d}{2}\end{pmatrix}.$$

Now, following (21) set

$$r_\mu(x)=\frac{1}{\mu(\mu+1)(\mu)-\frac{d}{2})\Gamma_\Omega(\mu)}\left(\mu P(x)-\frac{d}{2}p_x\right)(\det x)^\mu$$

and this achieves the proof of Theorem 5.2.

We now discuss for which values of μ is r_μ $\mathrm{Herm}^+(\mathbb{J})$-valued.

Lemma 5.8. *Let $y\in\overline{\Omega}$. Then for any $v\in\mathbb{J}$*

$$\big(p_y\,v|v\big)\le r\big(P(y)v|v\big)\,. \tag{32}$$

Moreover, if $y\in\Omega$ and $v\neq 0$, then equality in (32) holds if and only if $y=te, t\in\mathbb{R}^+$ and $v=\lambda e,\lambda\in\mathbb{C}^*$.

Proof. We may assume that $y\neq 0$. The space $\mathbb{J}$ splits as $\mathbb{J}=\mathbb{C}y\oplus(y)^\perp$. Both subspaces are invariant under p_y. Moreover $P(y)$ preserves $\mathbb{C}y$ and as it is a selfadjoint operator, it also preserves $(y)^\perp$. Hence to prove the inequality (32), it is enough to prove the inequality separately on both subspaces. The inequality is trivial on $(y)^\perp$, so that it is enough to prove it on $\mathbb{C}y$. Now

$$\big(p_y\,y|y\big)=(y|y)^2,\qquad \big(P(y)\,y|y\big)=(y^3|y)=(y^2|y^2)\,.$$

By the Cauchy-Schwarz inequality,

$$(y|y)=(y^2|e)\le\big(y^2|y^2\big)^{\frac{1}{2}}(e|e)^{\frac{1}{2}}, \tag{33}$$

so that
$$(y|y)^2\le r(y^2|y^2)$$

and hence (32) is also satisfied on $\mathbb{C}y$.

Assume now that $y\in\Omega$. Then as $P(y)$ is positive-definite, the inequality (32) is strict on $(y)^\perp$. So equality can occur only if $v=\mu y$ for some $\mu\in\mathbb{C},\mu\neq 0$. Conversely, equality in (32) implies

$$(y|y)^2=r(y^2|y^2),$$

hence corresponds to the case of equality in the Cauchy-Schwarz inequality (33) and implies $y=te$ for some $t>0$. This completes the proof. ∎

The next proposition is just an immediate consequence of the previous lemma.

Corollary 5.9. *Let $\mu\in\mathbb{R}$. For $y\in\overline{\Omega}$, the operator $r_\mu(y)$ is positive semi-definite if and only if $\mu\ge\frac{rd}{2}$.*

6. The singular cases

As shown in [1], it remains to look to the cases $\mu = k\frac{d}{2}, 0 \leq k \leq r-1$. If $k=0$, i.e. for the orbit $\mathcal{O}^{(0)} = \{0\}$, there is clearly no solution to the condition (16). The next section will treat the case $k=1$. We examine in this section the case where $2 \leq k \leq r-1$. The next lemma is a generalization of Lemma 5.8.

Lemma 6.1. *Let $1 \leq k \leq r$ and let $y \in \Omega^{(k)}$. Then for all $v \in \mathbb{J}$*

$$\big(p_y\, v \,|\, v\big) \leq k\big(P(y)v\,|v\big)\,. \tag{34}$$

Proof. Let c be an idempotent of rank k. Decompose $\mathbb{V}$ as $\mathbb{C}c \oplus (c)^\perp$. Both subspaces are invariant by p_c and by $P(c)$. Hence it suffices to verify (34) separately on each subspace. The inequality being trivial on $(c)^\perp$, it suffices to verify it on $\mathbb{C}c$. So let $v = \lambda c$, with $\lambda \in \mathbb{C}$. Then

$$\big(p_c(v)|v\big) = |\lambda|^2 (c|c), \qquad \big(P(c)v|v\big) = |\lambda|^2\,.$$

As $(c,c) = k$, (34) is valid for $y = c$. Now let y in $\Omega^{(k)}$. There exists an idempotent c of rank k and some $\ell \in L$ such that $y = \ell c$. Now $p_y = \ell^* p_c \ell$ and $P(y) = \ell^* P(c)\ell$ and (34) follows by using the result obtained for c when applied to $w = \ell v$. ■

Theorem 6.2. *Let $2 \leq k \leq r-1$. Let*

$$r_k(y) = \frac{4}{k(k-1)(kd+2)d}\,(kP(y) - p_y)\,.$$

Then

$$\int_{\Omega^{(k)}} e^{-\operatorname{tr} y} r_k(y) d\nu_k(y) = \mathrm{Id}_{\mathbb{J}}\,. \tag{35}$$

Proof. Rewrite Theorem 5.2 using the Riesz integrals (cf (2)), extended to operator-valued functions to obtain

$$\mathrm{Id}_{\mathbb{J}} = \frac{1}{\mu(\mu+1)\big(\mu - \frac{d}{2}\big)} \left(T_\mu(y)\,,\, \mu P(y) - \frac{d}{2}\,p_y\right)\,. \tag{36}$$

The statement is valid for $\Re(\mu)$ large enough, and both sides can be continued analytically. By analytic continuation we see that the two sides coincide at $\mu = k\frac{d}{2}$ for $k = 2, \ldots, r-1$, and, as $d\nu_k = T_{k\frac{d}{2}}$, (35) is just an equivalent formulation. ■

The positivity of $r_k(y)$ for $y \in \mathcal{O}^{(1)}$ follows from Lemma 6.1 and hence the values $\mu = k\dfrac{d}{2}$, $2 \leq k \leq r-1$ belong to the Wallach set.

7. The case $\mu = \frac{d}{2}$

The last case concerns the case where $\mu = \frac{d}{2}$. In fact, the factor $(\mu - \frac{d}{2})$ in the denominator of the right hand expression of (36) leads to an indetermination, as for c a primitive (i.e. $\operatorname{rank}(c) = 1$) idempotent,

$$p_c(v) = P(c)v = (v,c)\,c\ ,$$

and hence the expression $\mu P(y) - \frac{d}{2} p_y$ vanishes for $\mu = \frac{d}{2}$ identically on $\Omega^{(1)}$.

Proposition 7.1. *There is no* $\mathrm{Herm}^+(\mathbb{J})$ *-valued measure supported on* $\overline{\Omega^{(1)}}$ *such that conditions* (16) *and* (17) *are satisfied.*

Before giving the proof, a few elementary lemmas are needed.

Lemma 7.2. *Let* c *be an idempotent in* J. *For* $s > 0$, *let* $\ell_s = P(c + s(e - c))$. *Then* $\ell_s = \mathrm{id}$ *on* $\mathbb{J}(c, 1)$, $\ell_s = s\,\mathrm{id}$ *on* $\mathbb{J}(c, \frac{1}{2})$ $\ell_s = s^2\,\mathrm{id}$ *on* $\mathbb{J}(c, 0)$.

Proof. By using the definition of the quadratic operator P

$$P\big(s + (e - c)\big) = P(c) + 2s\big((L(c)L(e - c) + L(e - c)L(c)\big) + s^2 L(e - c)^2$$

and the verification of the lemma is a routine calculation. ■

Lemma 7.3. *Let* c *be a primitive idempotent of* J. *Let* L^c *be the stabilizer of* c *in* L. *Let* $S \in \mathrm{Herm}^+(\mathbb{J})$ *and assume that*

$$\forall \ell \in L^c, \qquad S = \ell\, S\, \ell^t\,. \tag{37}$$

Then there exists a scalar $\alpha \in [0, +\infty)$ *such that* $S = \alpha P(c)$.

Proof. Recall first the formula $P(\ell x) = \ell P(x) \ell^t$ for any element x and $\ell \in L$. Hence $P(c)$ satisfies (37). Now assume that $S \in \mathrm{Herm}^+(\mathbb{J})$ satisfies (37). For $s > 0$, $\ell_s = P\big(c + s(e - c)\big)$ belongs to L^c and is symmetric. Hence condition (38) implies that $S = \ell_s\, S\, \ell_s$ for any $s > 0$. Let $v \in \mathbb{J}(c, \frac{1}{2})$. Then by Lemma 7.2

$$\ell_s\, S\, v = \frac{1}{s}\, S\, v$$

which is clearly impossible unless $Sv = 0$. Hence S vanishes on $J(c, \frac{1}{2})$. A similar argument shows that S vanishes on $J(c, 0)$. As c is primitive $J(c, 1) = \mathbb{R}c_1$ and $Sc_1 = \ell_s S \ell_s c_1 = \ell_s S c_1$, which forces $Sc_1 \in \mathbb{C}c_1$. The conclusion follows. ■

Now we are in condition to prove Proposition 7.1. Assume there exists a measure, which we denote by dR_1 for simplicity, satisfying both conditions. Then there would exist a $\mathrm{Herm}^+(\mathbb{J})$-valued function r_1 on $\Omega^{(1)}$ such that $dR_1(x) = r_1(x)\, d\nu_1(x)$ satisfying, for $x \in \Omega^{(1)}$

$$r_1(\ell x) = \ell\, r_1(x)\, \ell^t \tag{38}$$

and

$$\int_{\overline{\Omega}_1} e^{-\operatorname{tr} x} r_1(x)\, d\nu_k(x) = \mathrm{Id}_{\mathbb{J}}\,. \tag{39}$$

Let c be a primitive idempotent and let $S = r_1(c)$. Then S satisfies the conditions of Lemma 7.3 and hence there exists $\alpha \in [0, +\infty)$ such that $S = \alpha P(c)$. Now all primitive idempotents are conjugate by some element of K and S commutes with K (a consequence of (16)). Hence the constant α does not depend on c. Any element x of $\Omega^{(1)}$ is a multiple of a primitive idempotent and hence satisfies $x = |x|c$ for some primitive idempotent c. In turn, this implies that

$$r_1(x) = |x|^2 r_1(c) = \alpha |x|^2 P(c) = \alpha P(x)\,.$$

Hence $\quad \operatorname{Tr} r_1(x) = \alpha|x|^2, \qquad (r_1(x)e|e) = \alpha(x^2|e) = \alpha|x|^2\,.$

As $\operatorname{Tr} r_1(x) = (r_1(x)e|e)$ for any $x \in \Omega^{(1)}$, the integral on the left hand side of (39), call it Σ, satisfies $\operatorname{Tr}(\Sigma) = (\Sigma e, e)$. As $\operatorname{Tr}\operatorname{Id}_{\mathbb{J}} = n$ and $(\operatorname{Id} e|e) = r$, Σ cannot be equal to $\operatorname{Id}_{\mathbb{J}}$. This achieves the proof of Proposition 7.1.

The proof of Theorem 1.1 is now achieved.

References

[1] J.-L. Clerc: *Laplace transform and unitary highest weight modules*, J. Lie Theory 5 (1995) 225–240.

[2] H. Ding: *Weighted Laplace transforms and Bessel functions on Hermitian symmetric domains*, Trans. Amer. Math. Soc. 351 (1999) 4204–4243.

[3] T. Enright, R. Howe, N. Wallach: *A classification of unitary highest weight modules*, in: *Representation Theory of Reductive Groups*, Park City, Progress in Mathematics 40, Birkhäuser, Boston (1982) 97–143.

[4] J. Faraut, A. Korànyi: *Analysis on Symmetric Cones*, Oxford Science Publications, Clarendon Press, Oxford (1994).

[5] J. Hilgert, K.-H. Neeb: *Vector valued Riesz distributions on Euclidean Jordan algebras*, J. Geom. Analysis 11 (2001) 43–75

[6] H.-P. Jacobsen: *Hermitian symmetric spaces and their unitary representations*, J. Funct. Analysis 52 (1983) 384–412.

[7] R. Kunze: *Positive definite operator-valued kernels and unitary representations*, in: Proc. Conf. Irvine, California, Academic Press, London (1966) 235–247.

[8] H. Rossi, M. Vergne: *Analytic continuation of the holomorphic discrete series of a semisimple Lie group*, Acta Math. 136 (1976) 1–59.

[9] N. Wallach: *The analytic continuation of the discrete series, I, II*, Trans. Amer. Math. Soc. 251 (1979) 1–17, 18–37.

Jean-Louis Clerc, Université de Lorraine, 54000 Nancy, France; jean-louis.clerc@univ-lorraine.fr.

Received June 14, 2022
and in final form June 23, 2022