Journal of Lie Theory
Volume **33** (2023) 195–215
© 2023 Heldermann Verlag

Central Extensions of Restricted Affine Nilpotent Lie Algebras $n_+(A_1^{(1)})(p)$

Tyler J. Evans and Alice Fialowski

Communicated by K.-H. Neeb

Abstract. Consider the maximal nilpotent subalgebra $n_+(A_1^{(1)})$ of the simplest affine algebra $A_1^{(1)}$ which is one of the $\mathbb{N}$-graded Lie algebras with minimal number of generators. We show that truncated versions of this algebra in positive characteristic admit the structure of a family of restricted Lie algebras. We compute the ordinary and restricted 1- and 2-cohomology spaces with trivial coefficients by giving bases. With these we explicitly describe the restricted 1-dimensional central extensions.

Mathematics Subject Classification: 17B50, 17B56,17B67, 17B70.

Key Words: Restricted Lie algebra, cohomology, central extension, affine Lie algebra.

1. Introduction

$\mathbb{N}$-graded Lie algebras over a field $\mathbb{F}$ are a special important class of infinite dimensional Lie algebras. By $\mathbb{N}$-graded we mean that the Lie algebra $\mathfrak{g}$ is the direct sum of subspaces $\mathfrak{g}_i, i \in \mathbb{N}$, such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. Such algebras obviously must have at least two generators. A special class of these algebras are Lie algebras of maximal class where $\dim \mathfrak{g}_1 = \dim \mathfrak{g}_2 = 1$, $\dim \mathfrak{g}_i \leq 1$ for $i \geq 3$ and $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$ for $i \geq 1$.

These are also called filiform Lie algebras, and it is known (see [8]) that exactly three of them are $\mathbb{N}$-graded. If we denote the basis elements by $e_i, i \in \mathbb{N}$, then these three are the maximal nilpotent subalgebra L_1 of the Witt algebra with nonzero brackets $[e_i, e_j] = (j-i)e_{i+j}$, the algebra $\mathfrak{m}_0$ with nonzero brackets $[e_1, e_i] = e_{i+1}$ for $i \geq 2$, and the Lie algebra $\mathfrak{m}_2$ with nonzero brackets $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$ and $[e_2, e_j] = e_{j+2}$ for all $j \geq 3$.

The filiform algebras $L_1, \mathfrak{m}_0$ and $\mathfrak{m}_2$ are well-studied, and some of their important invariants, such as 1- and 2-cohomology spaces with trivial and adjoint coefficients, are computed. Finite dimensional versions of these algebras are also studied in characteristic $p > 0$. Namely, in [3, 4, 5] we considered restricted algebras of these types, and computed the ordinary and restricted 1- and 2-cohomology spaces with trivial coefficients.

There are other types of $\mathbb{N}$-graded Lie algebras with the minimal number of generators e_1 and e_2. One of these is the maximal nilpotent part $n_+(A_1^{(1)})$ of the simplest

ISSN 0949–5932 / $2.50

affine Lie algebra $A_1^{(1)}$ (see [14]) with Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

which has a basis $\{e_i \mid i \in \mathbb{N}\}$ with brackets

$$[e_i, e_j] = \begin{cases} e_{i+j} & \text{if } j - i \equiv 1 \pmod 3; \\ 0 & \text{if } j - i \equiv 0 \pmod 3; \\ -e_{i+j} & \text{if } j - i \equiv -1 \pmod 3. \end{cases}$$

The other $\mathbb{N}$-graded nilpotent affine Lie algebra is the maximal nilpotent subalgebra $n_+(BA_2)$ of BA_2 (see [14]), which is defined by the Cartan matrix

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}.$$

It has a basis $\{e_i \mid i \in \mathbb{N}\}$ with brackets $[e_i, e_j] = b_{ij} e_{i+j}$ where the coefficients b_{ij} depend on the residue obtained when dividing i and j by 8, according to the rule

$$b_{ij} + b_{i'j'} = 0 \text{ if } i + i' \text{ and } j + j' \text{ are divisible by } 8.$$

The Table below gives the numbers b_{ij} (the others are determined from $b_{ij} = -b_{ji}$ and $b_{ij} + b_{8-i,8-j} = 0$).

$j \pmod 8$	$i \pmod 8$						
	1	2	3	4	5	6	7
0	1	−2	−1	0	1	2	−1
1	0	1	−1	3	−2	0	1
2	−1	0	0	0	1	−1	0
3	1	0	0	3	1	1	−2

In addition to these five $\mathbb{N}$-graded algebras with two generators there is a family of Lie algebras with countably many parameters $\lambda_{4k} \in \mathbb{FP}^1$, see [8].

The cohomology with trivial coefficients of the affine subalgebras $n_+(A_1^{(1)})$ and $n_+(BA_1)$ are computed, but nothing is known about restricted cohomology, even though it gives an important invariant of those algebras.

As in the case of the three filiform algebras with two generators, it is interesting to consider whether there is a restricted Lie algebra structure on the truncated algebras $n_+(A_1^{(1)})(p)$ where $p > 0$ is a prime, and to describe their central extensions, as we did for the three filiform cases [3, 4, 5].

In this paper we concentrate on the algebras $n_+(A_1^{(1)})(p)$. The goal of the paper is to classify the restricted one-dimensional central extensions of these restricted Lie algebras. To do this, we compute the ordinary and restricted 1- and 2-cohomology spaces with trivial coefficients.

The structure of the paper is as follows. In Section 2 we define finite dimensional truncated versions of the algebra $n_+(A_1^{(1)})$ in characteristic $p > 0$, and show that they admit the structure of (a family of) restricted Lie algebras. Explicit formulas for the Lie brackets and $[p]$-operators are given. In Section 3 we use the Chevalley-Eilenberg cochain complex to explicitly describe bases for the ordinary 1- and 2-cohomology spaces with trivial coefficients. In Section 4 we use Hochschild's six-term exact sequence and the partial complex in [6] to compute the restricted 2-cohomology spaces with trivial coefficients, again with descriptions of the bases. Finally, in Section 5 we explicitly describe the restricted one-dimensional central extensions corresponding to the non-trivial restricted 2-cocycles.

Acknowledgements. The authors are grateful for the referee's valuable suggestions as they greatly improved the exposition.

2. Restricted Lie Algebra structures

Let p be a prime and $\mathbb{F}$ a field of characteristic p. For $i,j \in \mathbb{N}$, define

$$a_{i,j} = \begin{cases} -1 & \text{if } j-i \equiv -1 \pmod 3; \\ 0 & \text{if } j-i \equiv 0 \pmod 3; \\ 1 & \text{if } j-i \equiv 1 \pmod 3). \end{cases} \tag{1}$$

Throughout the paper, we will let $[m]_3$ denote the congruence class of an integer m modulo 3 so $a_{i,j} = [j-i]_3$. Let $\mathfrak{g} = n_+(A_1^{(1)}) = \bigoplus_{i=1}^{\infty} \mathbb{F}e_i$ with bracket $[e_i, e_j] = a_{i,j} e_{i+j}$, $\mathfrak{i} = (e_{p+1})$ the ideal generated by e_{p+1} and

$$\mathfrak{g}(p) = n_+(A_1^{(1)})(p) = n_+(A_1^{(1)})/\mathfrak{i}.$$

Then $\mathfrak{g}(p)$ is a finite dimensional, $\mathbb{N}$-graded Lie algebra with basis $\{e_1, \ldots, e_p\}$. The k-th graded component of $\mathfrak{g}(p)$ is $\mathfrak{g}_k(p) = \mathbb{F}e_k$ for $1 \leq k \leq p$, and $\mathfrak{g}_k(p) = 0$ for $k > p$. If $\alpha_i, \beta_i \in \mathbb{F}$, $g = \sum_{i=1}^{p} \alpha_i e_i$ and $h = \sum_{i=1}^{p} \beta_i e_i$, then

$$[g,h] = \sum_{i=1}^{(p-1)/2} \sum_{j=i+1}^{p-i} a_{i,j}(\alpha_i\beta_j - \beta_i\alpha_j)e_{i+j}. \tag{2}$$

For $g_1, \ldots, g_n \in \mathfrak{g}(p)$, denote the n-fold bracket by

$$[[\cdots[[g_1, g_2], g_3]\cdots], g_n] := [g_1, g_2, \ldots, g_n].$$

Note that $[g_1, g_2, \ldots, g_p] = 0$ for all $g_1, \ldots, g_p \in \mathfrak{g}(p)$ so for all $g, h \in \mathfrak{g}(p)$, we have

$$(\operatorname{ad} g)^p(h) = [h, \underbrace{g, \ldots, g}_{p}] = 0.$$

If, for each $1 \leq k \leq p$, we choose an element $e_k^{[p]}$ in the center of $\mathfrak{g}(p)$, we have

$$\operatorname{ad} e_k^{[p]} = 0 = (\operatorname{ad} e_k)^p,$$

so the operator $-^{[p]} : \mathfrak{g}(p) \to \mathfrak{g}(p)$ gives $\mathfrak{g}(p)$ the structure of a restricted Lie algebra [11, 15]. Since p-fold brackets are 0 in $\mathfrak{g}(p)$, we have $(g+h)^{[p]} = g^{[p]} + h^{[p]}$ for all $g, h \in \mathfrak{g}(p)$. The center $Z(\mathfrak{g}(p))$ is given by

$$Z(\mathfrak{g}(p)) = \begin{cases} \mathbb{F}e_p & \text{if } p \not\equiv 2 \pmod 3; \\ \mathbb{F}e_{p-1} \oplus \mathbb{F}e_p & \text{if } p \equiv 2 \pmod 3). \end{cases}$$

If $p \not\equiv 2 \pmod 3$, then choosing $e_k^{[p]} \in Z(\mathfrak{g}(p))$ is equivalent to choosing $\lambda_k \in \mathbb{F}$ and setting $e_k^{[p]} = \lambda_k e_p$, hence $\lambda \in \mathbb{F}^p$ determines a restricted Lie algebra structure on $\mathfrak{g}(p)$ which we denote by $\mathfrak{g}^\lambda(p)$. Conversely, if $-^{[p]} : \mathfrak{g}(p) \to \mathfrak{g}(p)$ is any restricted Lie algebra structure on $\mathfrak{g}(p)$, then $\operatorname{ad} e_k^{[p]} = (\operatorname{ad} e_k)^p = 0$ for all $1 \leq k \leq p$ so that $e_k^{[p]} = \lambda_k e_p$ for some $\lambda_k \in \mathbb{F}$. For any such restricted Lie algebra structure, if $g = \sum_{i=1}^p \alpha_i e_i$, then

$$g^{[p]} = \left(\sum_{i=1}^{p} \alpha_i^p \lambda_i \right) e_p. \tag{3}$$

Likewise, if $p \equiv 2 \pmod 3$, the restricted Lie algebra structures on $\mathfrak{g}(p)$ correspond to pairs of elements $\mu, \lambda \in \mathbb{F}^p$ where $e_k^{[p]} = \mu_k e_{p-1} + \lambda_k e_p$. Denote this restricted Lie algebra by $\mathfrak{g}^{\mu,\lambda}(p)$. If $g = \sum_{i=1}^p \alpha_i e_i$, then

$$g^{[p]} = \left(\sum_{i=1}^{p} \alpha_i^p \mu_i \right) e_{p-1} + \left(\sum_{i=1}^{p} \alpha_i^p \lambda_i \right) e_p. \tag{4}$$

If $p = 2$, then $n_+(A_1^{(1)})(2) = \mathfrak{m}_0(2)$, and if $p = 3$, $n_+(A_1^{(1)})(3) = \mathfrak{m}_0(3) = \mathfrak{m}_2(3) = L_1(3)$ and hence the possible restricted structures are the same. These filiform algebras were already studied in [3, 4] so everywhere below we assume $p \geq 5$.

3. Ordinary cohomology

Our primary interest is in classifying (restricted) one-dimensional central extensions, so we describe only the (graded) Chevalley-Eilenberg cochain spaces $C^q = C^q(\mathfrak{g}(p)) = C^q(\mathfrak{g}(p), \mathbb{F})$ for $q = 0, 1, 2, 3$ and differentials $d^q : C^q(\mathfrak{g}(p)) \to C^{q+1}(\mathfrak{g}(p))$ for $q = 0, 1, 2$ (for more details on the Chevalley-Eilenberg cochain complex we refer the reader to [1] or [9]). Set $C^0 = \mathbb{F}$ and $C^q = (\wedge^q \mathfrak{g}(p))^*$ for $q = 1, 2, 3$. We will use the following bases, ordered lexicographically, throughout the paper.

$$\begin{aligned}
&C^0 : \{1\} \\
&C^1 : \{e^k \mid 1 \leq k \leq p\} \\
&C^2 : \{e^{i,j} \mid 1 \leq i < j \leq p\} \\
&C^3 : \{e^{u,v,w} \mid 1 \leq u < v < w \leq p\}
\end{aligned}$$

Here e^k, $e^{i,j}$ and $e^{u,v,w}$ denote the dual vectors of the basis vectors $e_k \in \mathfrak{g}(p)$, $e_{i,j} = e_i \wedge e_j \in \wedge^2 \mathfrak{g}(p)$ and $e_{u,v,w} = e_u \wedge e_v \wedge e_w \in \wedge^3 \mathfrak{g}(p)$, respectively. The differentials $d^q : C^q \to C^{q+1}$ are defined for $\psi \in C^1$, $\phi \in C^2$ and $g, h, f \in \mathfrak{g}$ by

$$\begin{aligned}
&d^0 : C^0 \to C^1, \;\; d^0 = 0 \\
&d^1 : C^1 \to C^2, \;\; d^1(\psi)(g \wedge h) = \psi([g, h]) \\
&d^2 : C^2 \to C^3, \;\; d^2(\phi)(g \wedge h \wedge f) = \phi([g, h] \wedge f) - \phi([g, f] \wedge h) + \phi([h, f] \wedge g).
\end{aligned}$$

The cochain spaces $C^q(\mathfrak{g}(p))$ are graded, and

$$\begin{aligned}
&C_k^1(\mathfrak{g}(p)) = \operatorname{span}\{e^k \mid 1 \leq k \leq p\} \\
&C_k^2(\mathfrak{g}(p)) = \operatorname{span}\{e^{i,j} \mid 1 \leq i < j \leq p, i + j = k, 3 \leq k \leq 2p - 1\} \\
&C_k^3(\mathfrak{g}(p)) = \operatorname{span}\{e^{u,v,w} \mid 1 \leq u < v < w \leq p, u + v + w = k, 6 \leq k \leq 3p - 3\}.
\end{aligned}$$

We have $\dim(C_k^1) = 1$ for $1 \le k \le p$, $\dim(C_k^2) = s(k)$ for $3 \le k \le p+1$ and $\dim(C_k^2) = \dim(C_{2p+2-k}^2)$ for $p+2 \le k \le 2p-1$, where the map $s : \mathbb{N} \to \mathbb{N}$ is defined by

$$s(k) = \begin{cases} \frac{k}{2} - 1 & \text{if } k \text{ is even;} \\ \frac{k-1}{2} & \text{if } k \text{ is odd.} \end{cases}$$

The differentials $d_k^q : C_k^q \to C_k^{q+1}$ are graded maps so $H^q = \bigoplus_k H_k^q$ for $q = 1, 2$. Note that $d_1^1 = d_2^1 = 0$ and $d_3^2 = d_4^2 = d_5^2 = 0$. If $3 \le k \le p$, direct computation shows that

$$d_k^1(e^k) = \sum_{i=1}^{s(k)} a_{i,k-i} e^{i,k-i}. \tag{5}$$

The expression (5) together with $d_1^1 = d_2^1 = 0$ immediately gives the following

Theorem 3.1. *A basis for* $\ker d^1$ *is* $\{e^1, e^2\}$, *hence* $\dim H^1(\mathfrak{g}(p)) = 2$, *and the classes of* $\{e^1, e^2\}$ *form a basis.*

Remark 3.2. It is possible to simultaneously determine both the ordinary and restricted 1-cohomology without reference to any complex. In this Remark, we will abuse notation and write $\mathfrak{g}(p)$ in place of $\mathfrak{g}^\lambda(p)$ or $\mathfrak{g}^{\mu,\lambda}(p)$. Since we are assuming $p \ge 5$, the bracket formula (2) shows $[\mathfrak{g}(p), \mathfrak{g}(p)] = \bigoplus_{i \ge 3} \mathbb{F} e_i$ and hence

$$\dim_{\mathbb{F}}(\mathfrak{g}(p)/[\mathfrak{g}(p), \mathfrak{g}(p)]) = 2.$$

Moreover, the $[p]$-operator formulas (3) and (4) give $\langle \mathfrak{g}(p)^{[p]} \rangle_{\mathbb{F}} \subseteq [\mathfrak{g}(p), \mathfrak{g}(p)]$. For any restricted Lie algebra $\mathfrak{g}$, [9, (1.4.2)] and [10, Theorem 2.1] respectively state that

$$H^1(\mathfrak{g}, \mathbb{F}) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* \text{ and } H_*^1(\mathfrak{g}, \mathbb{F}) = (\mathfrak{g}/([\mathfrak{g}, \mathfrak{g}] + \langle \mathfrak{g}^{[p]} \rangle_{\mathbb{F}}))^*.$$

It follows that $H^1(\mathfrak{g}(p), \mathbb{F}) = H_*^1(\mathfrak{g}(p), \mathbb{F})$ and the classes of $\{e^1, e^2\}$ form a basis. ■

The situation for H_k^2 is more complicated. If $6 \le k \le 2p-1$, $e^{i,j} \in C_k^2(\mathfrak{g}(p))$ (so $i + j = k$), then the formula for the differential d^2 and the restricted bracket (2) gives

$$\begin{aligned} d_k^2(e^{i,j}) = &\sum_{n=1}^{s(i)} a_{n,i-n} e^{n,i-n,k-i} - \sum_{n=k-2i+1}^{s(k-i)} a_{n,k-i-n} e^{n,k-i-n,i} \\ &+ \sum_{n=1}^{i-1} a_{n,k-i-n} e^{n,i,k-i-n} - \sum_{n=i+1}^{p-2} a_{n,k-i-n} e^{i,n,k-i-n}. \end{aligned} \tag{6}$$

We will use the following notation to describe index ranges in our computations:

$$M(p,k) = \begin{cases} 1 & \text{if } 3 \le k \le p+1; \\ k-p & \text{if } p+2 \le k \le 2p-1, \end{cases}$$

$$G(p,k) = \begin{cases} 1 & \text{if } 6 \le k \le 2p; \\ 1+k-2p & \text{if } 2p+1 \le k \le 3p-3, \end{cases}$$

$$F(p,k) = \begin{cases} \frac{k}{3} - 1 & \text{if } k \equiv 0 \pmod 3; \\ \frac{k-1}{3} - 1 & \text{if } k \equiv 1 \pmod 3; \\ \frac{k-2}{3} - 1 & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

If $e^{u,v,k-v-u} \in C^3_k$, then $G(p,k) \le u \le F(p,k)$ and $u+1 \le v \le s(k-u)$. If

$$d^2_k\left(\sum_{\substack{1\le i<j\le p\\ i+j=k}} \alpha_{i,j}e^{ij}\right) = \sum_{\substack{1\le i<j\le p\\ i+j=k}} \alpha_{i,j} \sum_{\substack{1\le u<v<w\le p\\ u+v+w=k}} A^{ij}_{uvw}e^{u,v,w} = 0,$$

then the term $e^{u,v,k-v-u}$ gives the "*basic equation*"

$$\boxed{\underbrace{-\alpha_{u,k-u}a_{v,k-v-u}}_{k-u\le p} + \underbrace{\alpha_{v,k-v}a_{u,k-v-u}}_{k-v\le p} + \underbrace{\underbrace{\alpha_{u+v,k-v-u}a_{u,v}}_{u+v<k-v-u\le p} - \underbrace{\alpha_{k-v-u,u+v}a_{u,v}}_{k-v-u<u+v\le p}}_{\text{never both non-zero}} = 0.} \qquad (7)$$

The kernel of d^2_k is the space of solutions to the "basic equation" system (7), where $G(p,k) \le u \le F(p,k)$ and $u+1 \le v \le s(k-u)$. For $3 \le k \le 2p-1$, define the cochains

$$\phi_k = \sum_{i=M(p,k)}^{s(k)} a_{i,k-i}e^{i,k-i}. \qquad (8)$$

The next Theorem is the main result of this section.

Theorem 3.3. *A basis for* $\ker d^2$ *is* $\{e^{1,4}, e^{2,5}, \phi_3, \phi_4, ..., \phi_{p+1}\}$, *so* $\dim\ker d^2 = p+1$. *Moreover,* $\phi_k = d^1(e^k)$ *for* $3 \le k \le p$, *hence* $\dim H^2(\mathfrak{g}(p)) = 3$ *and the classes of* $\{e^{1,4}, e^{2,5}, \phi_{p+1}\}$ *form a basis.*

We have already remarked that the cochain spaces and differentials are graded, so Theorem 3.3 follows from

Theorem 3.4. *If* $k \ge p+2$, *then* $\ker d^2_k = 0$. *If* $k < p+2$, *then* $\ker d^2_k = \{\phi_k\}$ *unless* $k=5$ *or* $k=7$. *For all* $p \ge 5$, $\ker d^2_5 = \{e^{1,4}, \phi_5\}$, *and* $\ker d^2_7 = \{e^{2,5}\}$ *when* $p=5$ *and* $\ker d^2_7 = \{e^{2,5}, \phi_7\}$ *when* $p>5$. *Moreover,* $\phi_k = d^1(e^k)$ *for* $3 \le k \le p$, *hence* $H^2_k(\mathfrak{g}(p)) = 0$ *unless* $k = 5, 7, p+1$. *Bases for* $H^2_5(\mathfrak{g}(p))$, $H^2_7(\mathfrak{g}(p))$ *and* $H^2_{p+1}(\mathfrak{g}(p))$ *consist of the classes of* $\{e^{1,4}\}$, $\{e^{2,5}\}$ *and* $\{\phi_{p+1}\}$, *respectively.*

Proof. We have already noted that $d^2_3 = d^2_4 = d^2_5 = 0$, and hence $\{e^{1,2}\}$, $\{e^{1,3}\}$ and $\{e^{1,4}, e^{1,5}\}$ form bases for $\ker d^2_k$ for $k = 3,4,5$, respectively. A direct computation using formula (6) for d^2 gives a basis for $\ker d^2_6$ which is $\{\phi_6\}$, a basis for $\ker d^2_7$ is $\{e^{2,5}\}$ when $p=5$, and a basis for $\ker d^2_7$ is $\{e^{2,5}, \phi_7\}$ when $p>5$. For the remainder of the proof we assume $p \ge 5$ and $k \ge 8$. The proof is combinatorial and consists of three parts:

$$\text{(i)}\quad k > p+2; \qquad \text{(ii)}\quad k = p+2; \qquad \text{(iii)}\quad k < p+2.$$

In each part, we will treat the cases $k = 6t + r$ where $0 < t$ and $0 \le r < 6$ separately, and only use the *basic equations* (7) with $u=1$ or $u=2$. We have included examples for $p = 23$ in the Appendix to illustrate the computations.

(i) $k > p+2 \Longrightarrow \ker d^2_k = 0$.

If $u=1$ and $v = i-1$, then (7) reduces to

$$\begin{array}{ll} \alpha_{k-p,p}[r-p+1]_3 = 0 & (i = k-p) \\ \alpha_{i-1,k-i+1}[r-i-1]_3 + \alpha_{i,k-i}[i+1]_3 = 0 & (k-p+1 \le i \le s(k)), \end{array} \qquad (9)$$

and if $u = 2$ and $v = i-2$, then (7) reduces to

$$\begin{aligned} &\alpha_{k-p,p}[r-p-1]_3 = 0 && (i = k-p) \\ &\alpha_{k-p+1,p-1}[r-p]_3 = 0 && (i = k-p+1) \\ &\alpha_{i-2,k-i+2}[r-i+1]_3 + \alpha_{i,k-i}[i-1]_3 = 0 && (k-p+2 \le i \le s(k)). \end{aligned} \tag{10}$$

Suppose $[r]_3 = 1$. If $[i]_3 = 0$, then (9) shows $\alpha_{i,k-i} = 0$. If $[i]_3 = 1$, then $[i-1]_3 = 0$ so (9) again gives $\alpha_{i,k-i} = 0$. If $[i]_3 = -1$, then (10) shows $\alpha_{i,k-i} = 0$.

Suppose that $[r]_3 = -1$. If $[i]_3 = 1$, then (9) gives $\alpha_{i,k-i} = 0$. If $[i]_3 = 0$, then (9) shows $\alpha_{i-1,k-i+1} + \alpha_{i,k-i} = 0$ and (10) gives $\alpha_{i,k-i} = 0$, hence $\alpha_{i-1,k-i+1} = 0$ as well. (If $r = 5$, then $[s(k)]_3 = -1 \ne 0$, but $[s(k)-2]_3 = 0$ so again (10) implies $\alpha_{s(k),k-s(k)} = 0$.)

Finally suppose $[r]_3 = 0$. If $[i]_3 = 0$, then (9) applied to i and $i+1$ gives

$$\alpha_{i-1,k-i+1} = \alpha_{i,k-i} = \alpha_{i+1,k-i-1}. \tag{11}$$

Moreover, using (10) and (11), we see that if $k-p+2 \le i$ and $\alpha_{i-2,k-i+2} = 0$, then $\alpha_{i-1,k-i+1} = \alpha_{i,k-i} = \alpha_{i+1,k-i-1} = 0$. If $[p]_3 = -1$, then (9) shows $\alpha_{k-p,p} = 0$ and (10) gives $\alpha_{k-p+1,p-1} = 0$, and hence $\alpha_{i,k-i} = 0$ for all i by the previous remark. If $[p]_3 = 1$, then (10) shows $\alpha_{k-p,p} = 0$ so again, $\alpha_{i,k-i} = 0$ for all i. This completes the proof of part (i).

(ii) $k = p+2 \Longrightarrow \ker d_k^2 = 0$

Since $p \ge 5$ is a prime, $p+2 = k = 6t+r$ implies $r = 1$ or $r = 3$. Moreover, $r = 1$ if and only if $[p]_3 = -1$, and $r = 3$ if and only if $[p]_3 = 1$.

Suppose that $r = 1$. If $u = 1$ and $v = i-1$, then the *basic equation* (7) still reduces to (9). If $u = 2$ and $v = i-2$, then (7) reduces to

$$\begin{gathered} -\alpha_{2,p}[i]_3 + \alpha_{i-2,k-i+2}[r-i+1]_3 + \alpha_{i,k-i}[i-1]_3 = 0 \\ (k-p+3 \le i \le s(k)) \end{gathered} \tag{12}$$

In particular, taking $i = 5$ and $i = 7$, we have, respectively,

$$\alpha_{2,p} + \alpha_{5,p-3} = 0 \quad \text{and} \quad -\alpha_{2,p} + \alpha_{5,p-3} = 0.$$

It follows that $\alpha_{2,p} = 0$, and hence (12) reduces to (10), and the argument for $[r]_3 = 1$ in part (i) shows that $\alpha_{i,k-i} = 0$ for all i.

Suppose $r = 3$. If $u = 1$ and $v = i-1$, then the *basic equation* (7) again reduces to (9), and applying (9) to i and $i+1$ for $[i]_3 = 0$, we have

$$\alpha_{i-1,k-i+1} = \alpha_{i,k-i} = \alpha_{i+1,k-i-1}. \tag{13}$$

Let $A_m = \alpha_{3m,k-3m}$ for $1 \le m \le t$. Now, we use (12) for all $[i]_3 = 1$ to get a system of equations

$$\begin{aligned} -A_1 - A_1 + A_2 &= 0 \\ -A_1 - A_m + A_{m+1} &= 0 \quad (2 \le m \le t-1) \\ -A_1 - A_t - A_t &= 0. \end{aligned} \tag{14}$$

These equations imply $A_t = -tA_1$ and $(2t-1)A_1 = 0$ so that $A_1 = A_t = 0$. From this it follows that $A_m = 0$ for all m, and hence (13) gives $\alpha_{i,k-i} = 0$ for all i. This completes the proof of part (ii).

(iii) $k < p+2 \Longrightarrow \ker d_k^2 = \mathrm{span}_{\mathbb{F}}\{\phi_k\}$

Recall $\dim C_k^2(\mathfrak{g}(p)) = s(k)$ for $k < p+2$. A direct calculation shows that if $k < p+2$, then the element ϕ_k defined in (8) is a solution of the *basic equation* (7), so $\phi_k \in \ker(d_k^2)$. We complete the proof by showing that the set

$$\{d_k^2(e^{i,k-i}) \mid 1 \le i \le s(k)-1\} \tag{15}$$

is linearly independent.

If $u = 1$ and $v = i$ satisfies $2 \le i \le s(k)-1$, then (7) reduces to

$$-\alpha_{1,k-1}[r+i-1]_3 + \alpha_{i,k-i}[r-i+1]_3 + \alpha_{i+1,k-i-1}[i-1]_3 = 0. \tag{16}$$

If $u = 2$ and $v = i$ satisfies $3 \le i \le s(k)-2$, then (7) reduces to

$$-\alpha_{2,k-2}[r+i+1]_3 + \alpha_{i,k-i}[r-i-1]_3 + \alpha_{i+2,k-i-2}[i-2]_3 = 0. \tag{17}$$

We begin by showing that if $[r]_3 = 1$ or 2, then $\alpha_{1,k-1} = \alpha_{2,k-2} = 0$ and hence (16) and (17) reduce to (9) and (10), respectively. The proof in (i) then shows $\alpha_{i,k-i} = 0$ for $2 \le i \le s(k)-1$ and the set (15) is linearly independent.

The following Table lists selected values of v for each value of r and the resulting simplified equations from (7). Since we are considering only linear combinations of the elements of the set (15), these equations show $\alpha_{1,k-1} = \alpha_{2,k-2} = 0$ by inspection.

$u=1$		
$r=1$	$v=s(k)-1$	$\alpha_{1,k-1}+\alpha_{s(k),k-s(k)}=0$
$r=2,4$	$v=s(k)$	$-\alpha_{1,k-1}+\alpha_{s(k),k-s(k)}[r/2-1]_3=0$
$r=5$	$v=s(k)-2$ $v=s(k)-1$	$-\alpha_{1,k-1}-\alpha_{s(k)-1,k-s(k)+1}=0$ $\alpha_{1,k-1}-\alpha_{s(k)-1,k-s(k)+1}=0$
$u=2$		
$r=1$	$v=s(k)-3$ $v=s(k)-1$	$\alpha_{2,k-2}+\alpha_{s(k)-1,k-s(k)+1}=0$ $-\alpha_{2,k-2}+\alpha_{s(k)-1,k-s(k)+1}=0$
$r=2$	$v=s(k)-2$	$-\alpha_{2,k-2}+\alpha_{s(k),k-s(k)}=0$
$r=4$	$v=s(k)-1$	$-\alpha_{2,k-2}=0$
$r=5$	$v=s(k)-1$	$-\alpha_{2,k-2}+\alpha_{s(k),k-s(k)}=0$

If $r = 0$, then selecting $u = 1$ and $v = s(k)$ shows $\alpha_{1,k-1} = 0$ as above. Using this, (16) reduces to (9), and applying (9) to i and $i+1$ for $[i]_3 = 0$, we again have

$$\alpha_{i-1,k-i+1} = \alpha_{i,k-i} = \alpha_{i+1,k-i-1}. \tag{18}$$

Let $A_m = \alpha_{3m,k-3m}$ for $1 \le m \le t-1$. Letting $u = 2$ and $v = i$ $(3 \le i \le s(k)-2)$ in the *basic equation* (7), we have

$$-\alpha_{2,k-2}[i+1]_3 - \alpha_{i,k-i}[i+1]_3 + \alpha_{i+2,k-i-2}[i+1]_3 = 0. \tag{19}$$

Using (19) for all $[i]_3 = 0$, we get a system of equations

$$\begin{aligned} -A_1 - A_m + A_{m+1} &= 0 \quad (1 \le m \le t-2) \\ -A_1 - A_{t-1} &= 0. \end{aligned}$$

These equations imply $tA_1 = 0$ and hence $A_m = 0$ for all m. This, together with (18), implies $\alpha_{i,k-i} = 0$ for all $1 \le i \le s(k) - 1$.

Finally, assume $r = 3$. Letting $u = 1$ and $u = 2$, respectively, with $v = i = s(k) - 1$ in (7), we have

$$\begin{aligned} \alpha_{1,k-1} + \alpha_{s(k)-1,k-s(k)+1} - \alpha_{s(k),k-s(k)} &= 0 \\ -\alpha_{2,k-2} - \alpha_{s(k)-1,k-s(k)+1} - \alpha_{s(k),k-s(k)} &= 0 \end{aligned}$$

We are considering linear combinations of elements of (15), hence these equations show that

$$\alpha_{1,k-1} = -\alpha_{s(k)-1,k-s(k)+1} = \alpha_{2,k-2}. \tag{20}$$

Now, for $u = 1$ and $u = 2$ respectively, we have

$$\begin{aligned} \alpha_{i+1,k-i-1}[i-1]_3 &= \alpha_{1,k-1}[i-1]_3 + \alpha_{i,k-i}[i-1]_3 \ (2 \le i \le s(k) - 1) \\ \alpha_{i+2,k-i-}[i+1]_3 &= \alpha_{2,k-2}[i+1]_3 + \alpha_{i,k-i}[i+1]_3 \ (3 \le i \le s(k) - 2) \end{aligned} \tag{21}$$

Together, (20), (21) and an induction shows that

$$\alpha_{i,k-i} = c(i)\alpha_{1,k-1}, 2 \le i \le s(k) - 1.$$

Moreover, $c(i) \le i$ for all i, hence, in particular,

$$c(s(k) - 1) \le s(k) - 1 = 3t < 6t + 1 \le p - 1.$$

This implies that

$$\alpha_{s(k)-1,k-s(k)+1} = -\alpha_{1,k-1} \text{ and } \alpha_{s(k)-1,k-s(k)+1} = c(s(k) - 1)\alpha_{1,k-1}$$

with $c(s(k) - 1) \neq -1$ in $\mathbb{F}$. This is a contradiction, unless $\alpha_{1,k-1} = 0$, so (15) is linear independent and the proof is complete. ■

Remark 3.5. It is interesting to note that $\dim H^2(n_+(A_1^{(1)})(p)) = 3$ is independent of the prime p, just as for the filiform algebras $\mathfrak{m}_2(p)$ [3]. On the other hand, $\dim H^2(\mathfrak{m}_0(p))$ depends on p [4]. It would be interesting to see whether or not the dimension of the 2-cohomology for the other affine algebras $n_+(BA_2)(p)$ depends on p. Of course, in all cases, the dimension of the ordinary 1-cohomology spaces is 2 as all of these algebras are 2-generated. ■

4. Restricted cohomology

For general information on restricted cohomology of restricted Lie algebras, we refer the reader to [7, 13, 12]. We have already computed the restricted cohomology spaces $H^1_*(\mathfrak{g}^\lambda(p); \mathbb{F})$ and $H^1_*(\mathfrak{g}^{\mu,\lambda}(p); \mathbb{F})$ (see Remark 3.2). We wish to explicitly describe bases for the restricted cohomology spaces $H^2_*(\mathfrak{g}^\lambda(p); \mathbb{F})$ and $H^2_*(\mathfrak{g}^{\mu,\lambda}(p); \mathbb{F})$. We will use the partial cochain complex described in [6] along with Hochschild's six term exact sequence linking ordinary and restricted 1- and 2-cohomology spaces [10, p. 575]. For completeness and to establish notation, we briefly describe the partial complex in [6].

Where no confusion can arise, for notational simplicity we denote by C^q_* both the spaces $C^q_*(\mathfrak{g}^\lambda(p)) = C^q_*(\mathfrak{g}^\lambda(p); \mathbb{F})$ and $C^q_*(\mathfrak{g}^{\mu,\lambda}(p)) = C^q_*(\mathfrak{g}^{\mu,\lambda}(p); \mathbb{F})$. We have only the restricted cochain spaces C^q_* for $q = 0, 1, 2, 3$ and the restricted differentials

$d_*^q : C_*^q \to C_*^{q+1}$ for $q = 0, 1, 2$. The computations of the restricted cohomology spaces $H_*^2(\mathfrak{g}^\lambda(p); \mathbb{F})$ and $H_*^2(\mathfrak{g}^{\mu,\lambda}(p); \mathbb{F})$ will be carried out in separate subsections.

Given $\phi \in C^2$, a map $\omega : \mathfrak{g}(p) \to \mathbb{F}$ is *ϕ-compatible* (in [6], the authors use the term ω *has the $*$-property with respect to* ϕ) if for all $g, h \in \mathfrak{g}(p)$ and all $\alpha \in \mathbb{F}$, $\omega(\alpha g) = \alpha^p \omega(g)$ and

$$\omega(g+h) = \omega(g) + \omega(h) + \sum_{\substack{g_i = g \text{ or } h \\ g_1 = g, g_2 = h}} \frac{1}{\#(g)} \phi([g_1, g_2, g_3, \ldots, g_{p-1}] \wedge g_p). \tag{22}$$

Remark 4.1. A map $\omega\colon \mathfrak{g}(p) \to \mathbb{F}$ is 0-compatible if and only if ω is an element of the space $\mathrm{Hom}_{\mathrm{Fr}}(\mathfrak{g}(p), \mathbb{F})$ of *Frobenius homomorphisms* from $\mathfrak{g}(p)$ to $\mathbb{F}$. That is

$$\mathrm{Hom}_{\mathrm{Fr}}(\mathfrak{g}(p), \mathbb{F}) = \{f\colon \mathfrak{g}(p) \to \mathbb{F} \mid f(\alpha x + \beta y) = \alpha^p f(x) + \beta^p f(y)\}$$

for all $\alpha, \beta \in \mathbb{F}$ and all $x, y \in \mathfrak{g}(p)$. ■

If $\zeta \in C^3$, then a map $\eta : \mathfrak{g}(p) \times \mathfrak{g}(p) \to \mathbb{F}$ is *ζ-compatible* (in [6], η *has the $**$-property with respect to* ζ) if for all $\alpha \in \mathbb{F}$ and all $g, h, h_1, h_2 \in \mathfrak{g}(p)$, $\eta(\cdot, h)$ is linear in the first coordinate, $\eta(g, \alpha h) = \alpha^p \eta(g, h)$ and

$$\eta(g, h_1 + h_2) = \eta(g, h_1) + \eta(g, h_2) - \sum_{\substack{l_1, \ldots, l_p = 1 \text{or} 2 \\ l_1 = 1,\ l_2 = 2}} \frac{1}{\#\{l_i = 1\}} \zeta(g \wedge [h_{l_1}, \cdots, h_{l_{p-1}}] \wedge h_{l_p}).$$

Define the restricted cochain spaces as $C_*^0 = C^0$, $C_*^1 = C^1$,

$$C_*^2 = \{(\phi, \omega) \mid \phi \in C^2, \omega : \mathfrak{g} \to \mathbb{F} \text{ is } \phi\text{-compatible}\}$$

$$C_*^3 = \{(\zeta, \eta) \mid \zeta \in C^3, \eta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F} \text{ is } \zeta\text{-compatible}\}.$$

If $\phi \in C^2$, we can assign values $\omega(e_k)$ arbitrarily to elements e_k of a basis $\{e_1, \ldots, e_p\}$ for $\mathfrak{g}(p)$, set $\omega(\alpha e_k) = \alpha^p \omega(e_k)$ for all $\alpha \in \mathbb{F}$ and use (22) to determine a unique ϕ-compatible map $\omega : \mathfrak{g}(p) \to \mathbb{F}$ [4]. In particular, we can define $\widetilde{\phi}(e_k) = 0$ for all k and use (22) to determine a unique ϕ-compatible map $\widetilde{\phi} : \mathfrak{g}(p) \to \mathbb{F}$. Note that, in general, $\widetilde{\phi} \neq 0$ but $\widetilde{\phi}(0) = 0$.

Lemma 4.2. *If $\phi_1, \phi_2 \in C^2$ and $\alpha \in \mathbb{F}$, then $\widetilde{(\alpha\phi_1 + \phi_2)} = \alpha\widetilde{\phi_1} + \widetilde{\phi_2}$.*

Proof. An easy computation shows $(\alpha\widetilde{\phi_1} + \widetilde{\phi_2})(e_k) = 0$ for all k, and $\alpha\widetilde{\phi_1} + \widetilde{\phi_2}$ satisfies the compatibility condition (22) with $\phi = \alpha\phi_1 + \phi_2$. The result now follows from uniqueness. ■

For $1 \le k \le p$, define the mapping $\overline{e}^k\colon \mathfrak{g}(p) \to \mathbb{F}$ by $\overline{e}^k\left(\sum_{n=1}^p \alpha_n e_n\right) = \alpha_k^p$. The set $\{\overline{e}^k \mid 1 \le k \le p\}$ is a basis for $\mathrm{Hom}_{\mathrm{Fr}}(\mathfrak{g}(p), \mathbb{F})$. It is shown in [6] that

$$\dim C_*^2 = \binom{p+1}{2} = \binom{p}{2} + p,$$

hence it follows that

$$\{(e^{i,j}, \widetilde{e^{i,j}}) \mid 1 \le i < j \le p\} \cup \{(0, \overline{e}^k) \mid 1 \le k \le p\}$$

is a basis for C_*^2. We use this basis in all computations that follow.

Remark 4.3. The $(p-1)$-fold bracket in (22) always gives a multiple of e_p. For $p \geq 5$, if $k < p+1$, then ϕ_k is identically zero on $e_p \wedge \mathfrak{g}^\lambda(p)$ (or $e_p \wedge \mathfrak{g}^{\mu,\lambda}(p)$), and hence $\widetilde{\phi_k} = 0$ because $\widetilde{\phi_k}(e_i) = 0$ for all i. Likewise $\widetilde{e^{1,4}} = 0$ and $\widetilde{e^{2,5}} = 0$, unless $p = 5$. The restriction of ϕ_{p+1} to $e_p \wedge \mathfrak{g}^\lambda(p)$ (or $e_p \wedge \mathfrak{g}^{\mu,\lambda}(p)$) is equal to $e^{1,p}$, hence $\widetilde{\phi_{p+1}} = \widetilde{e^{1,p}}$. We can then compute, using (22):

$$\begin{aligned}
\widetilde{e^{2,5}}\left(\sum_{i=1}^{p} \alpha_i e_i\right) &= \frac{1}{2}\alpha_1^3\alpha_2^2 \quad (p=5),\\
\widetilde{\phi_{p+1}}\left(\sum_{i=1}^{p} \alpha_i e_i\right) &= \widetilde{e^{1,p}}\left(\sum_{i=1}^{p} \alpha_i e_i\right) = \alpha_1^{p-1}\alpha_2.
\end{aligned}$$ ■

For $\psi \in C^1_*$, define the map $\operatorname{ind}^1(\psi)\colon \mathfrak{g}(p) \to \mathbb{F}$ by $\operatorname{ind}^1(\psi)(g) = \psi(g^{[p]})$.
The mapping $\operatorname{ind}^1(\psi)$ is $d^1(\psi)$-compatible for all $\psi \in C^1_*$, and the differential $d^1_*\colon C^1_* \to C^2_*$ is defined by $d^1_*(\psi) = (d^1(\psi), \operatorname{ind}^1(\psi))$. For $(\phi,\omega) \in C^2_*$, define the map $\operatorname{ind}^2(\phi,\omega) : \mathfrak{g}(p) \times \mathfrak{g}(p) \to \mathbb{F}$ by the formula

$$\operatorname{ind}^2(\phi,\omega)(g,h) = \phi(g \wedge h^{[p]}).$$

The mapping $\operatorname{ind}^2(\phi,\omega)$ is $d^2(\phi)$-compatible for all $\phi \in C^2$, and the differential $d^2_*\colon C^2_* \to C^3_*$ is defined by $d^2_*(\phi,\omega) = (d^2(\phi), \operatorname{ind}^2(\phi,\omega))$ (see [6] for details). Note that (with trivial coefficients) if ω_1 and ω_2 are both ϕ-compatible, then $\operatorname{ind}^2(\phi,\omega_1) = \operatorname{ind}^2(\phi,\omega_2)$.

Lemma 4.4. *If $(\phi,\omega) \in C^2_*$ and $\phi = d^1(\psi)$ with $\psi \in C^1$, then $(\phi, \operatorname{ind}^1(\psi)) \in C^2_*$ and $\operatorname{ind}^2(\phi,\omega) = \operatorname{ind}^2(\phi, \operatorname{ind}^1(\psi))$.*

Proof. We know that $\operatorname{ind}^1(\psi)$ is $d^1(\psi)$-compatible for all $\psi \in C^1 = C^1_*$ [6]. If $\phi = d^1(\psi)$, then $(\phi, \operatorname{ind}^1(\psi)) = (d^1(\psi), \operatorname{ind}^1(\psi)) \in C^2_*$, and ind^2 depends only on ϕ by the last sentence in the previous paragraph. ■

If $\mathfrak{g}$ is any restricted Lie algebra and M any restricted $\mathfrak{g}$-module, there is a six-term exact sequence [10, p. 575]:

$$\begin{aligned}
0 \longrightarrow H^1_*(\mathfrak{g},M) \to H^1(\mathfrak{g},M) \longrightarrow \operatorname{Hom}_{\mathrm{Fr}}(\mathfrak{g}, M^{\mathfrak{g}}) \longrightarrow \\
\longrightarrow H^2_*(\mathfrak{g},M) \to H^2(\mathfrak{g},M) \to \operatorname{Hom}_{\mathrm{Fr}}(\mathfrak{g}, H^1(\mathfrak{g},M))
\end{aligned} \tag{23}$$

If we let $\mathfrak{g} = \mathfrak{g}^\lambda(p)$ or $\mathfrak{g}^{\mu,\lambda}(p)$ and $M = \mathbb{F}$, the map

$$\Delta : H^2(\mathfrak{g},M) \to \operatorname{Hom}_{\mathrm{Fr}}(\mathfrak{g}, H^1(\mathfrak{g},M))$$

in (23) is given by

$$\Delta_\phi(g)\cdot h = \phi(g \wedge h^{[p]}) = \operatorname{ind}^2(\phi,\omega)(g,h)$$

where $\phi \in C^2(\mathfrak{g})$ and $g,h \in \mathfrak{g}$ [16]. Remark 3.2 shows that the map $H^1_*(\mathfrak{g}) \to H^1(\mathfrak{g})$ is an isomorphism so the sequence (23) decouples to the exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathrm{Fr}}(\mathfrak{g},\mathbb{F}) \longrightarrow H^2_*(\mathfrak{g}) \longrightarrow H^2(\mathfrak{g}) \xrightarrow{\Delta} \operatorname{Hom}_{\mathrm{Fr}}(\mathfrak{g}, H^1(\mathfrak{g})). \tag{24}$$

The map $\operatorname{Hom}_{\mathrm{Fr}}(\mathfrak{g},\mathbb{F}) \to H^2_*(\mathfrak{g})$ sends ω to the class of $(0,\omega)$ (see Remark 4.1 above). The image of this map is a p-dimensional subspace of $H^2_*(\mathfrak{g})$ spanned by the classes $(0,\overline{e}^k)$. This space corresponds to the space of restricted one-dimensional central extensions that split as ordinary Lie algebra extensions.

We will use the following notation throughout the remainder of this section: $g=\sum \alpha_i e_i$, $h=\sum \beta_i e_i$, $\psi=\sum \gamma_i e^i$ and $\phi=\sum \sigma_{i,j}e^{i,j}$.

Restricted cohomology for $p \not\equiv 2 \pmod 3$.

Assume p is a prime and $p \not\equiv 2 \pmod 3$ so $p>5$. Using the $[p]$-operator (3) and the definition of the map ind^2 we have

$$\operatorname{ind}^2(\phi,\omega)(g,h)=\left(\sum_{i=1}^{p}\beta_i^p\lambda_i\right)\left(\sum_{i=1}^{p-1}\sigma_{i,p}\alpha_i\right). \tag{25}$$

Theorem 4.5. *If $p>5$ and $p\not\equiv 2 \pmod 3$, then*

$$H^2_*(\mathfrak{g}^\lambda(p)) = \operatorname{Hom}_{\mathrm{Fr}}(\mathfrak{g}(p),\mathbb{F})\oplus H^2(\mathfrak{g}(p)),$$

and the classes of $\{(0,\overline{e}^1),\ldots,(0,\overline{e}^p),\ (e^{1,4},\widetilde{e^{1,4}}),\ (e^{2,5},\widetilde{e^{2,5}}),\ (\phi_{p+1},\widetilde{\phi_{p+1}})\}$ *form a basis.*

Proof. If the map $\Delta=0$, the sequence (24) reduces to the short exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathrm{Fr}}(\mathfrak{g}^\lambda(p),\mathbb{F}) \longrightarrow H^2_*(\mathfrak{g}^\lambda(p)) \longrightarrow H^2(\mathfrak{g}(p)) \longrightarrow 0, \tag{26}$$

and the Theorem follows immediately from this and Theorem 3.3.

If $\lambda=0$, then (25) shows that $\operatorname{ind}^2(\phi,\omega)=0$ for all $(\phi,\omega)\in C^2_*$ so the map $\Delta=0$. For any n such that $\lambda_n\neq 0$, if $(\phi,\omega)\in C^2_*$, then (25) shows

$$\operatorname{ind}^2(\phi,\omega)(e_m,e_n)=\lambda_n\sigma_{m,p}$$

for all $1\le m\le p-1$. By Theorem 3.3, if $\phi=\sum\sigma_{i,j}e^{i,j}$ is an ordinary cocycle, then $\sigma_{2,p}=\cdots=\sigma_{p-1,p}=0$ $(p>5)$. Moreover, the coefficient of $e^{1,p}$ in ϕ_{p+1} is $\sigma_{1,p}=a_{1,p}=p-1 \pmod 3 = 0 \pmod 3$, showing $\Delta=0$ in this case as well. ∎

Restricted cohomology for $p\equiv 2 \pmod 3$.

Assume $p\ge 5$ is prime and $p\equiv 2\pmod 3$. Using the p-operator formula (4) and the definitions of the map ind^2, we have

$$\begin{aligned}\operatorname{ind}^2(\phi,\omega)(g,h)=&\left(\sum_{i=1}^{p}\beta_i^p\mu_i\right)\left(\sum_{i=1}^{p-2}\sigma_{i,p}\alpha_i-\sigma_{p-1,p}\alpha_p\right)\\&+\left(\sum_{i=1}^{p}\beta_i^p\lambda_i\right)\left(\sum_{i=1}^{p-1}\sigma_{i,p}\alpha_i\right).\end{aligned} \tag{27}$$

Theorem 4.6. *If $p\ge 5$, $p\equiv 2\pmod 3$ and $\mu+\lambda=0$, then*

$$H^2_*(\mathfrak{g}^{\mu,\lambda}(p)) = \operatorname{Hom}_{\mathrm{Fr}}(\mathfrak{g}(p),\mathbb{F})\oplus H^2(\mathfrak{g}(p)).$$

Proof. If $(\phi,\omega)\in C^2_*$, then (27) shows for $1\le n\le p$

$$\operatorname{ind}^2(\phi,\omega)(e_m,e_n)=\begin{cases}(\mu_n+\lambda_n)\sigma_{m,p} & \text{if } 1\le m\le p-2\\ (-\mu_n+\lambda_n)\sigma_{m,p} & \text{if } m=p-1\\ 0 & m=p\end{cases} \tag{28}$$

Since $\sigma_{p-1,p}=0$ in any cocycle $\phi=\sum\sigma_{i,j}e^{i,j}$, (28) shows if $\mu+\lambda=0$, then the map $\Delta=0$. ■

If $p\equiv 2\,(\mathrm{mod}\ 3)$ and $\mu+\lambda\neq 0$, then the map $\Delta\neq 0$ and the sequence (24) does not give a short exact sequence (26).

Theorem 4.7. *If $p>5$, $p\equiv 2\,(\mathrm{mod}\ 3)$ and $\mu+\lambda\neq 0$, then it follows $\dim H^2_*(\mathfrak{g}^{\mu,\lambda}(p))=p+2$ and the classes of*

$$\{(e^{1,4},\widetilde{e^{1,4}}),(e^{2,5},\widetilde{e^{2,5}}),(0,\overline{e}^1),\ldots,(0,\overline{e}^p)\}$$

form a basis. If $p=5$, then $\dim H^2_*(\mathfrak{g}^{\mu,\lambda}(5))=6$ and the classes of

$$\{(e^{1,4},\widetilde{e^{1,4}}),(0,\overline{e}^1),\ldots,(0,\overline{e}^5)\}$$

form a basis.

Proof. Suppose $\mu+\lambda\neq 0$. An element $(\phi,\omega)\in C^2_*$ is a restricted cocycle if and only if $\phi\in C^2$ is an ordinary cocycle and $\operatorname{ind}^2(\phi,\omega)=0$. If ϕ is an ordinary cocycle, then (28) shows that $\operatorname{ind}^2(\phi,\omega)=0$ if and only if $\sigma_{m,p}=0$ for all $1\le m\le p-2$ ($\sigma_{p-1,p}=0$ if ϕ is a cocycle). For $p>5$, Theorem 3.3 shows that if $d^2(\phi)=0$, then $\sigma_{m,p}=0$ for $2\le m\le p-1$. Moreover for ϕ_{p+1}, $\sigma_{1,p}=a_{1,p}=p-1\not\equiv 0\,(\mathrm{mod}\ 3)$, and hence $d^2_*(\phi_{p+1},\widetilde{\phi_{p+1}})\neq(0,0)$. For any other ordinary cocycle ϕ, we have $\sigma_{1,p}=0$. Therefore the kernel of d^2_* is

$$\{(e^{1,4},\widetilde{e^{1,4}}),(e^{2,5},\widetilde{e^{2,5}}),(\phi_3,\widetilde{\phi_3}),\ldots,(\phi_p,\widetilde{\phi_p}),(0,\overline{e}^1),\ldots,(0,\overline{e}^p)\}.$$

If $p=5$, then $\phi=e^{2,5}$ is a cocycle with $\sigma_{2,5}=1\neq 0$ so that $d^2_*(e^{2,5},\widetilde{e^{2,5}})\neq(0,0)$. For any other ordinary cocycle ϕ, we have $\sigma_{2,5}=0$ so the kernel in the case $p=5$ is the set

$$\{(e^{1,4},\widetilde{e^{1,4}}),(\phi_3,\widetilde{\phi_3}),(\phi_4,\widetilde{\phi_4}),(\phi_5,\widetilde{\phi_5}),(0,\overline{e}^1),\ldots,(0,\overline{e}^5\}.$$

If $3\le k\le p$, then $\phi_k=d^1(e^k)$, and Lemma 4.4 shows that we can replace $\widetilde{\phi_k}$ with $\operatorname{ind}^1(e^k)$, so $d^1_*(e^k)=(\phi_k,\operatorname{ind}^1(e^k))$. If $p>5$, it follows that $\dim H^2_*(\mathfrak{g}^{\mu,\lambda}(p))=p+2$ and the classes of

$$\{(e^{1,4},\widetilde{e^{1,4}}),(e^{2,5},\widetilde{e^{2,5}}),(0,\overline{e}^1),\ldots,(0,\overline{e}^p)\}$$

form a basis. If $p=5$, we lose the class $(e^{2,5},\widetilde{e^{2,5}})$ and $\dim H^2_*(\mathfrak{g}^{\mu,\lambda}(5))=6$. ■

5. One-dimensional central extensions

Classes of one-dimensional central extensions $E=\mathfrak{g}\oplus\mathbb{F}c$ of an ordinary Lie algebra $\mathfrak{g}$ (that is, classes of short exact sequences $0\to\mathbb{F}\to E\to\mathfrak{g}\to 0$) are characterized by the cohomology space $H^2(\mathfrak{g})$ [9, 1.4.6], and restricted one-dimensional central extensions of a restricted Lie algebra $\mathfrak{g}$ with $c^{[p]} = 0$ are characterized by the restricted cohomology space $H^2_*(\mathfrak{g})$ [6, Corollary 4]. If $(\phi, \omega) \in C^2_*(\mathfrak{g})$ is a restricted 2-cocycle, then the corresponding restricted one-dimensional central extension $E = \mathfrak{g} \oplus \mathbb{F}c$ has Lie bracket and $[p]$-operator defined by

$$[g,h] = [g,h]_{\mathfrak{g}} + \phi(g \wedge h)c\,; \quad [g,c] = 0\,; \quad g^{[p]} = p^{[p]_{\mathfrak{g}}} + \omega(g)c\,; \quad c^{[p]} = 0, \tag{29}$$

where $[\cdot,\cdot]_{\mathfrak{g}}$ and $\cdot^{[p]_{\mathfrak{g}}}$ denote the Lie bracket and $[p]$-operator in $\mathfrak{g}$, respectively. We can use (29) to explicitly describe the restricted one-dimensional central extensions corresponding to the restricted cocycles in Theorems 4.5, 4.6 and 4.7. For the rest of this section, let $g = \sum \alpha_i e_i$ and $h = \sum \beta_i e_i$ denote two arbitrary elements of $\mathfrak{g}^{\lambda}(p)$ or $\mathfrak{g}^{\mu,\lambda}(p)$.

Just as in [2, 3, 4], if $E_k = g^{\lambda}(p) \oplus \mathbb{F}c$ denotes the one-dimensional restricted central extension of $\mathfrak{g}^{\lambda}(p)$ (or $\mathfrak{g}^{\mu,\lambda}(p)$) determined by the cohomology class of the restricted cocycle $(0, \overline{e}^k)$, then (29) gives the bracket and $[p]$-operation in E_k:

$$[g,h] = [g,h]_{\mathfrak{g}^{\lambda}(p)}\,; \quad [g,c] = 0\,; \quad g^{[p]} = g^{[p]_{\mathfrak{g}^{\lambda}(p)}} + \alpha_k^p c\,; \quad c^{[p]} = 0.$$

The central extensions E_k form a basis for the p-dimensional space of restricted one-dimensional central extensions that split as ordinary Lie algebra extensions.

For the restricted cocycles $(e^{1,4}, \widetilde{e^{1,4}}), (e^{2,5}, \widetilde{e^{2,5}})$ and $(\phi_{p+1}, \widetilde{\phi_{p+1}})$, we summarize the corresponding restricted one-dimensional central extensions E in the following Table (see Remark 4.3). Everywhere in the Table below we omit the brackets $[g,c] = 0$ and $[p]$-operation $c^{[p]} = 0$ for brevity.

Table 1: Restricted one-dimensional central extensions
with $p \equiv 2 \pmod 3$ and $\mu + \lambda = 0$

$p > 5$	
$(e^{1,4}, 0)$	$[g,h] = [g,h]_{\mathfrak{g}^{\mu,\lambda}(p)} + (\alpha_1\beta_4 - \alpha_4\beta_1)c$ $g^{[p]} = g^{[p]_{\mathfrak{g}^{\mu,\lambda}(p)}}$
$(e^{2,5}, 0)$	$[g,h] = [g,h]_{\mathfrak{g}^{\mu,\lambda}(p)} + (\alpha_2\beta_5 - \alpha_5\beta_2)c$ $g^{[p]} = g^{[p]_{\mathfrak{g}^{\mu,\lambda}(p)}}$
$(\phi_{p+1}, \widetilde{\phi_{p+1}})$	$[g,h] = [g,h]_{\mathfrak{g}^{\mu,\lambda}(p)} + \left(\sum_{i=1}^{s(p+1)} a_{i,p+1-i}(\alpha_i\beta_{p+1-i} - \alpha_{p+1-i}\beta_i)\right)c$ $g^{[p]} = g^{[p]_{\mathfrak{g}^{\mu,\lambda}(p)}} + \alpha_1^{p-1}\alpha_2 c$

If $p = 5$, the extensions $(e^{1,4}, 0)$ and $(\phi_{p+1}, \widetilde{\phi_{p+1}})$ are as in Table 1, but the map $\widetilde{e^{2,5}} \neq 0$ and the $[p]$-operator for the extension $(e^{2,5}, \widetilde{e^{2,5}})$ is given by

$$g^{[p]} = g^{[p]_{\mathfrak{g}^{\mu,\lambda}(p)}} + \frac{1}{2}\alpha_1^3\alpha_2^2.$$

The bracket for $(e^{2,5}, \widetilde{e^{2,5}})$ is unchanged.

If $\mu + \lambda \neq 0$, then Theorem 4.7 states that $(\phi_{p+1}, \widetilde{\phi_{p+1}})$ is not a cocycle, and neither is $(e^{2,5}, \widetilde{e^{2,5}})$ if $p = 5$. The other two (or one if $p = 5$) restricted one-dimensional extensions in Table 1 are unchanged.

If $p \not\equiv 2 \pmod 3$, then $p > 5$ and Theorem 4.6 implies $H^2_*(\mathfrak{g}^{\mu,\lambda}(p)) = H^2_*(\mathfrak{g}^{\lambda}(p))$, and hence the restricted one-dimensional central extensions are the same as those in Table 1 with $p > 5$ and $\mathfrak{g}^{\mu,\lambda}(p)$ replaced with $\mathfrak{g}^{\lambda}(p)$.

Appendix to the proof of Theorem 3.4

The examples below illustrate the computations in the proof of Theorem 3.3 for the prime $p = 23$ and values of $k = 6t + r$ for $r = 0, 1, 2, 3, 4, 5$. The examples are listed in the order they appear in parts (i), (ii) and (iii) of the proof. In part (ii), the example for $r = 3$ uses $p = 19$ because necessarily $p \equiv 1 \pmod 3$ in this case. In each matrix, only the rows for $u = 1$ and $u = 2$ are shown. The circles indicate entries for which there is exactly one non-zero entry in the row which shows the corresponding coefficient $\alpha_{i,k-i} = 0$. The boxes indicate terms from the simplified equations from (12).

(i) $([r]_3 = 1)$ $k = 28 = 6 \cdot 4 + 4$ with $r = 4$. Table A1

	$e^{5,23}$	$e^{6,22}$	$e^{7,21}$	$e^{8,20}$	$e^{9,19}$	$e^{10,18}$	$e^{11,17}$	$e^{12,16}$	$e^{13,15}$
$e^{1,4,23}$	0								
$e^{1,5,22}$	0	1							
$e^{1,6,21}$		$\boxed{-1}$	$\boxed{-1}$						
$e^{1,7,20}$			1	0					
$e^{1,8,19}$				0	1				
$e^{1,9,18}$					$\boxed{-1}$	$\boxed{-1}$			
$e^{1,10,17}$						1	0		
$e^{1,11,16}$							0	1	
$e^{1,12,15}$								$\boxed{-1}$	$\boxed{-1}$
$e^{1,13,14}$									1
$e^{2,3,23}$	1								
$e^{2,4,22}$		-1							
$e^{2,5,21}$	1		0						
$e^{2,6,20}$		0		1					
$e^{2,7,19}$			-1		-1				
$e^{2,8,18}$				1		0			
$e^{2,9,17}$					0		1		
$e^{2,10,16}$						-1		-1	
$e^{2,11,15}$							1		0
$e^{2,12,14}$								0	
$\vdots$									

(i) $([r]_3 = -1)$ $k = 29 = 6 \cdot 4 + 5$ with $r = 5$. Tabel A2

	$e^{6,23}$	$e^{7,22}$	$e^{8,21}$	$e^{9,20}$	$e^{10,19}$	$e^{11,18}$	$e^{12,17}$	$e^{13,16}$	$e^{14,15}$
$e^{1,5,23}$	1								
$e^{1,6,22}$	0	-1							
$e^{1,7,21}$		-1	0						
$e^{1,8,20}$			$\boxed{1}$	$\boxed{1}$					
$e^{1,9,19}$				0	-1				
$e^{1,10,18}$					-1	0			
$e^{1,11,17}$						$\boxed{1}$	$\boxed{1}$		
$e^{1,12,16}$							0	-1	
$e^{1,13,15}$								-1	0
$e^{2,4,23}$	-1								
$e^{2,5,22}$		0							
$e^{2,6,21}$	1		1						
$e^{2,7,20}$		0		-1					
$e^{2,8,19}$			-1		0				
$e^{2,9,18}$				1		1			
$e^{2,10,17}$					0		-1		
$e^{2,11,16}$						-1		0	
$e^{2,12,15}$							$\boxed{1}$		$\boxed{1}$
$e^{2,13,14}$								0	1
$\vdots$									

(i) $([r]_3 = 0$ and $[p]_3 = -1)$ $k = 30 = 6 \cdot 5 + 0$ with $r = 0$. Table A3

	$e^{7,23}$	$e^{8,22}$	$e^{9,21}$	$e^{10,20}$	$e^{11,19}$	$e^{12,18}$	$e^{13,17}$	$e^{14,16}$
$e^{1,6,23}$	-1							
$e^{1,7,22}$	0	0						
$e^{1,8,21}$		$\boxed{-1}$	$\boxed{1}$					
$e^{1,9,20}$			$\boxed{1}$	$\boxed{-1}$				
$e^{1,10,19}$				0	0			
$e^{1,11,18}$					$\boxed{-1}$	$\boxed{1}$		
$e^{1,12,17}$						$\boxed{1}$	$\boxed{-1}$	
$e^{1,13,16}$							0	0
$e^{1,14,15}$								-1
$e^{2,5,23}$	0							
$e^{2,6,22}$		1						
$e^{2,7,21}$	$\boxed{1}$		$\boxed{-1}$					
$e^{2,8,20}$		0		0				
$e^{2,9,19}$			-1		1			
$e^{2,10,18}$				$\boxed{1}$		$\boxed{-1}$		
$e^{2,11,17}$					0		0	
$e^{2,12,16}$						-1		1
$e^{2,13,15}$							1	
$\vdots$								

(ii) $(r = 1 \text{ so } [p]_3 = -1)$ $k = 25 = 6 \cdot 4 + 1$. Table A4

	$e^{2,23}$	$e^{3,22}$	$e^{4,21}$	$e^{5,20}$	$e^{6,19}$	$e^{7,18}$	$e^{8,17}$	$e^{9,16}$	$e^{10,15}$	$e^{11,14}$	$e^{12,13}$
$e^{1,2,22}$	0	1									
$e^{1,3,21}$		-1	-1								
$e^{1,4,20}$			1	0							
$e^{1,5,19}$				0	1						
$e^{1,6,18}$					-1	-1					
$e^{1,7,17}$						1	0				
$e^{1,8,16}$							0	1			
$e^{1,9,15}$								-1	-1		
$e^{1,10,14}$									1	0	
$e^{1,11,13}$										0	1
$e^{2,3,20}$	$\boxed{1}$	$\boxed{0}$		$\boxed{1}$							
$e^{2,4,19}$	0		-1		-1						
$e^{2,5,18}$	$\boxed{-1}$			$\boxed{1}$		0					
$e^{2,6,17}$	1				0		1				
$e^{2,7,16}$	0					0		-1			
$e^{2,8,15}$	-1						1		0		
$e^{2,9,14}$	1							0		1	
$e^{2,10,13}$	0								-1		-1
$e^{2,11,12}$	-1									1	0
$\vdots$											

(ii) $(r = 3 \text{ so } [p]_3 = 1,\ p = 19)$, $k = 21 = 6 \cdot 3 + 3$. Table A5

	A_1			A_2			A_3		
	$e^{2,19}$	$e^{3,18}$	$e^{4,17}$	$e^{5,16}$	$e^{6,15}$	$e^{7,14}$	$e^{8,13}$	$e^{9,12}$	$e^{10,11}$
$e^{1,2,18}$	$\boxed{-1}$	$\boxed{1}$							
$e^{1,3,17}$		$\boxed{1}$	$\boxed{-1}$						
$e^{1,4,16}$			0	0					
$e^{1,5,15}$				$\boxed{-1}$	$\boxed{1}$				
$e^{1,6,14}$					$\boxed{1}$	$\boxed{-1}$			
$e^{1,7,13}$						0	0		
$e^{1,8,12}$							$\boxed{-1}$	$\boxed{1}$	
$e^{1,9,11}$								$\boxed{1}$	$\boxed{-1}$
$e^{2,3,16}$	$\boxed{-1}$	$\boxed{-1}$		$\boxed{1}$					
$e^{2,4,15}$	1		1		-1				
$e^{2,5,14}$	0			0		0			
$e^{2,6,13}$	$\boxed{-1}$				$\boxed{-1}$		$\boxed{1}$		
$e^{2,7,12}$	1					1		-1	
$e^{2,8,11}$	0						0		0
$e^{2,9,10}$	$\boxed{-1}$							$\boxed{-1}$	$\boxed{-1}$
$\vdots$									

(iii) $(r=1)$ $k = 19 = 6\cdot 3+1$. Table A6

	$e^{1,18}$	$e^{2,17}$	$e^{3,16}$	$e^{4,15}$	$e^{5,14}$	$e^{6,13}$	$e^{7,12}$	$e^{8,11}$	$e^{9,10}$
$e^{1,2,16}$	1	0	1						
$e^{1,3,15}$	0		-1	-1					
$e^{1,4,14}$	-1			1	0				
$e^{1,5,13}$	1				0	1			
$e^{1,6,12}$	0					-1	-1		
$e^{1,7,11}$	-1						1	0	
$e^{1,8,10}$	1							0	1
$e^{2,3,14}$		1	0		1				
$e^{2,4,13}$		0		-1		-1			
$e^{2,5,12}$		-1			1		0		
$e^{2,6,11}$		$\boxed{1}$				$\boxed{0}$		$\boxed{1}$	
$e^{2,7,10}$		0					-1		-1
$e^{2,8,9}$		$\boxed{-1}$						$\boxed{1}$	0
$\vdots$									

(iii) $(r=2)$ $k = 20 = 6\cdot 3+2$. Table A7

	$e^{1,19}$	$e^{2,18}$	$e^{3,17}$	$e^{4,16}$	$e^{5,15}$	$e^{6,14}$	$e^{7,13}$	$e^{8,12}$	$e^{9,11}$
$e^{1,2,17}$	0	1	1						
$e^{1,3,16}$	-1		0	-1					
$e^{1,4,15}$	1			-1	0				
$e^{1,5,14}$	0				1	1			
$e^{1,6,13}$	-1					0	-1		
$e^{1,7,12}$	1						-1	0	
$e^{1,8,11}$	0							1	1
$e^{1,9,10}$	-1								0
$e^{2,3,15}$		0	1		1				
$e^{2,4,14}$		-1		0		-1			
$e^{2,5,13}$		1			-1		0		
$e^{2,6,12}$		0				1		1	
$e^{2,7,11}$		-1					0		-1
$e^{2,8,10}$		1						-1	
$\vdots$									

(iii) $(r=4)$ $k=22=6\cdot 3+4$. Table A8

	$e^{1,21}$	$e^{2,20}$	$e^{3,19}$	$e^{4,18}$	$e^{5,17}$	$e^{6,16}$	$e^{7,15}$	$e^{8,14}$	$e^{9,13}$	$e^{10,12}$
$e^{1,2,19}$	1	0	1							
$e^{1,3,18}$	0		-1	-1						
$e^{1,4,17}$	-1			1	0					
$e^{1,5,16}$	1				0	1				
$e^{1,6,15}$	0					-1	-1			
$e^{1,7,14}$	-1						1	0		
$e^{1,8,13}$	1							0	1	
$e^{1,9,12}$	0								-1	-1
$e^{1,10,11}$	-1									1
$e^{2,3,17}$		1	0		1					
$e^{2,4,16}$		0		-1		-1				
$e^{2,5,15}$		-1			1		0			
$e^{2,6,14}$		1				0		1		
$e^{2,7,13}$		0					-1		-1	
$e^{2,8,12}$		-1						1		0
$e^{2,9,11}$		1							0	
$\vdots$										

(iii) $(r=5)$ $k=23=6\cdot 3+5$. Table A9

	$e^{1,22}$	$e^{2,21}$	$e^{3,20}$	$e^{4,19}$	$e^{5,18}$	$e^{6,17}$	$e^{7,16}$	$e^{8,15}$	$e^{9,14}$	$e^{10,13}$	$e^{11,12}$
$e^{1,2,20}$	0	1	1								
$e^{1,3,19}$	-1		0	-1							
$e^{1,4,18}$	1			-1	0						
$e^{1,5,17}$	0				1	1					
$e^{1,6,16}$	-1					0	1				
$e^{1,7,15}$	1						-1	0			
$e^{1,8,14}$	0							1	1		
$e^{1,9,13}$	$\boxed{-1}$								$\boxed{0}$	$\boxed{-1}$	
$e^{1,10,12}$	$\boxed{1}$									$\boxed{-1}$	0
$e^{2,3,18}$		0	1		1						
$e^{2,4,17}$		-1		0		1					
$e^{2,5,16}$		1			-1		0				
$e^{2,6,15}$		0				1		1			
$e^{2,7,14}$		-1					0		-1		
$e^{2,8,13}$		1						-1		0	
$e^{2,9,12}$		0							1		1
$e^{2,10,11}$		-1								0	1
$\vdots$											

(iii) $(r = 0)$ $k = 24 = 6 \cdot 4 + 0$. Table A10

		A_1			A_2			A_3			
	$e^{1,23}$	$e^{2,22}$	$e^{3,21}$	$e^{4,20}$	$e^{5,19}$	$e^{6,18}$	$e^{7,17}$	$e^{8,16}$	$e^{9,15}$	$e^{10,14}$	$e^{11,13}$
$e^{1,2,21}$	-1	$\boxed{-1}$	$\boxed{1}$								
$e^{1,3,20}$	1		$\boxed{1}$	$\boxed{-1}$							
$e^{1,4,19}$	0			0	0						
$e^{1,5,18}$	-1				$\boxed{-1}$	$\boxed{1}$					
$e^{1,6,17}$	1					$\boxed{1}$	$\boxed{-1}$				
$e^{1,7,16}$	0						0	0			
$e^{1,8,15}$	-1							$\boxed{-1}$	$\boxed{1}$		
$e^{1,9,14}$	1								$\boxed{1}$	$\boxed{-1}$	
$e^{1,10,13}$	0									0	0
$e^{1,11,12}$	-1										-1
$e^{2,3,19}$		$\boxed{-1}$	$\boxed{-1}$		$\boxed{1}$						
$e^{2,4,18}$		1		1		-1					
$e^{2,5,17}$		0			0		0				
$e^{2,6,16}$		$\boxed{-1}$				$\boxed{-1}$		$\boxed{1}$			
$e^{2,7,15}$		1					1		-1		
$e^{2,8,14}$		0						0		0	
$e^{2,9,13}$		$\boxed{-1}$							$\boxed{-1}$		1
$e^{2,10,12}$		1								1	
$\vdots$											

(iii) $(r = 3)$ $k = 21 = 6 \cdot 3 + 3$. Table A11

Here the rows consisting only of boxed entries indicate $\alpha_{1,k-1} = -\alpha_{s(k)-1,k-s(k)+1} = \alpha_{2,k-2}$ as in (20), and the single boxed entries indicate the terms $\alpha_{i+1,k-i-1}$ and $\alpha_{i+2,k-i-2}$ from (21).

	$e^{1,20}$	$e^{2,19}$	$e^{3,18}$	$e^{4,17}$	$e^{5,16}$	$e^{6,15}$	$e^{7,14}$	$e^{8,13}$	$e^{9,12}$	$e^{10,11}$
$e^{1,2,18}$	-1	-1	$\boxed{1}$							
$e^{1,3,17}$	1		1	$\boxed{-1}$						
$e^{1,4,16}$	0			0	0					
$e^{1,5,15}$	-1				-1	$\boxed{1}$				
$e^{1,6,14}$	1					1	$\boxed{-1}$			
$e^{1,7,13}$	0						0	0		
$e^{1,8,12}$	-1							-1	$\boxed{1}$	
$e^{1,9,11}$	$\boxed{1}$								$\boxed{1}$	-1
$e^{2,3,16}$		-1	-1		$\boxed{1}$					
$e^{2,4,15}$		1		1		-1				
$e^{2,5,14}$		0			0		0			
$e^{2,6,13}$		-1				-1		$\boxed{1}$		
$e^{2,7,12}$		1					1		-1	
$e^{2,8,11}$		0						0		0
$e^{2,9,10}$		$\boxed{-1}$							$\boxed{\,-1\,}$	-1
$\vdots$										

References

[1] C. Chevalley, S. Eilenberg: *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. 63 (1948) 84–124.

[2] T. J. Evans, A. Fialowski: *Restricted one-dimensional central extensions of restricted simple Lie algebras*, Lin. Algebra Appl. 513 (2017) 96–102.

[3] T. J. Evans, A. Fialowski: *Cohomology of restricted filiform Lie algebras* $\mathfrak{m}_2^\lambda(p)$, SIGMA 15 (2019), art. no. 95, 11 p.

[4] T. J. Evans, A. Fialowski: *Restricted one-dimensional central extensions of the restricted filiform Lie algebras* $\mathfrak{m}_0^\lambda(p)$, Linear Algebra Appl. 565 (2019) 244–257.

[5] T. J. Evans, A. Fialowski, M. Penkava: *Restricted cohomology of modular Witt algebras*, Proc. Amer. Math. Soc. 144 (2016) 1877–1886.

[6] T. J. Evans, D. B. Fuchs: *A complex for the cohomology of restricted Lie algebras*, J. Fixed Point Theory Appl. 3/1 (2008) 159–179.

[7] J. Feldvoss: *On the cohomology of restricted Lie algebras*, Comm. Algebra 19/10 (1991) 2865–2906.

[8] A. Fialowski: *On the classification of graded Lie algebras with two generators*, Moscow Univ. Math. Bull. 38/2 (1983) 76–79.

[9] D. B. Fuchs: *Cohomology of Infinite Dimensional Lie Algebras*, Consultants Bureau, New York (1986).

[10] G. Hochschild: *Cohomology of restricted Lie algebras*, Amer. J. Math. 76/3 (1954) 555–580.

[11] N. Jacobson: *Lie Algebras*, John Wiley, New York (1962).

[12] J. C. Jantzen: *Representations of Algebraic Groups*, 2nd ed., Mathematical Surveys and Monographs 107, American Mathematical Society, Providence (2003).

[13] J. C. Jantzen: *Restricted Lie algebra cohomology*, in: *Algebraic Groups Utrecht*, Lecture Notes in Mathematics 1986, A. M. Cohen et al. (eds.), Springer, Berlin (2006) 91–108.

[14] V. G. Kac: *Infinite-Dimensional Lie Algebras*, Cambridge University Press, Cambridge (1985).

[15] H. Strade, R. Farnsteiner: *Modular Lie Algebras and their Representations*, Monographs and Textbooks in Pure and Applied Mathematics 116, Marcel Dekker, New York (1988).

[16] F. Viviani: *Restricted inifinitesimal deformations of restricted simple Lie algebras*, J. Algebra Appl. 11/5 (2012), art. no. 1250091, 19 p.

Tyler J. Evans, Department of Mathematics, California State Polytechnic University - Humboldt, Arcata, U.S.A.; evans@humboldt.edu.

Alice Fialowski, Faculty of Informatics, Eötvös Loránd University, Budapest, Hungary; fialowski@inf.elte.hu, alice.fialowski@gmail.com.

Received August 8, 2022
and in final form September 22, 2022