

Harish-Chandra Highest Weight Representations of Semisimple Lie Algebras and Lie Groups*

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Abstract. In this expository paper we describe the theory of Harish-Chandra highest weight representations and their explicit geometric realizations.

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1. Introduction

The theory initiated by Harish-Chandra in 1956, with his three seminal papers in the American Journal of Mathematics [20, 21, 22] on representations of real semisimple Lie groups, in the so called *holomorphic discrete series representations*, influenced researchers for the next decades and is still a source of inspiration for many. He constructed these modules first infinitesimally and then globally, in the space of sections on holomorphic bundles on symmetric spaces and laid the foundation for further investigation of unitary representations and harmonic analysis of real semisimple Lie groups ([13], [28], [39]). His research also paved the road to the successes of geometric quantization (orbit method) and models (see [16], [17], [30], [33] and refs. therein) and later on to the SUSY generalizations [5], [6], [28] and refs. therein. Despite his untimely death, his mathematical vision had a great impact on generations of mathematicians; for a thorough walk through his beautiful mathematical achievements, see recollections by Howe [26], Langlands [35] and Varadarajan et al. [44], [46].

In the present work we want to give a self contained exposition of the theory of the infinitesimal and global realizations of the highest weight Harish-Chandra modules, elucidating Harish-Chandra arguments as in [20, 21, 22]. Despite this material is known, we believe it is of some interest to have a concise, but complete account of this beautiful theory, elucidating the original arguments by Harish-Chandra, which do not, in general, appear in more modern textbooks. Furthermore, recently new interest sparked about the super generalization of this theory (see [2], [5], [6], [15], [28], [37], [45] and refs. therein), thus making our concise treatment for the ordinary setting, important to pursue with more confidence the super generalizations. It must

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also be noted that one very effective way to understand a Lie supergroup is via the super Harish-Chandra pairs (SHCP) formalism, which actually leads to an equivalence of categories between SHCP and Lie supergroups. Indeed, understanding in depth the original Harish-Chandra theory, via ordinary Harish-Chandra pairs, is the first step to reach a full understanding of the category of supergroups and their representations.

The first step (Sec. 2) in Harish-Chandra theory is the *infinitesimal theory*, that is the study of Harish-Chandra representations for the pair $(\mathfrak{g}, \mathfrak{k})$, where \mathfrak{g} is a semisimple Lie algebra and \mathfrak{k} the complexification of the maximal compact subalgebra of a chosen real form $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ (real Cartan decomposition). Notice that the index 0 always denotes real forms and when we drop it we mean the complexification. Such a pair $(\mathfrak{g}, \mathfrak{k})$ is called an *Harish-Chandra pair* (HC), and it is uniquely determined by a real form \mathfrak{g}_0 of \mathfrak{g} (see Sec. 2.2).

By definition, a Harish-Chandra representation of the HC pair $(\mathfrak{g}, \mathfrak{k})$, is a representation of \mathfrak{g} in which \mathfrak{k} acts *finitely*, that is the sum of irreducible representations of \mathfrak{k} with the same character is finite dimensional (see (2)). The construction of the universal highest weight Harish-Chandra representation U^λ , $\lambda \in \mathfrak{h}^*$ (\mathfrak{h} a Cartan subalgebra of both \mathfrak{g} and \mathfrak{k}) is based on the existence of an *admissible system* for \mathfrak{g} . These are positive systems in which the adjoint representation of \mathfrak{k} on \mathfrak{p} stabilizes \mathfrak{p}^\pm , the sum of the positive (negative) non compact root spaces. The existence of such systems is equivalent to the existence of a non zero center \mathfrak{c} for \mathfrak{k} (Prop. 2.19) and to a natural invariant complex structure on the real symmetric space G_0/K_0 , $\mathfrak{g}_0 = \text{Lie}(G_0)$, $\mathfrak{k}_0 = \text{Lie}(K_0)$ (Rem. 2.20). The dimension of the center is linked to the number of non equivalent invariant complex structures on G_0/K_0 (see [1]). The main result for this part is Thm. 2.9. Then, based on our treatment of admissible systems, we proceed with the construction of the infinitesimal universal highest weight HC module U^λ and we give a sufficient condition for its irreducibility (Thm. 2.11).

For this part we are following the original treatment by Harish-Chandra [20], where the existence of such systems is instrumental to the realization of the representations of the real group in the space of sections of a line bundle on the homogeneous space. We believe that, though the arguments appear at times involved they elucidate better the representation theory aspects of the property of total positivity. Furthermore, the representation theory aspect of admissibility, suggested generalization to the affine and super setting [7, 9], besides having interesting applications to the classification of complex structures on flag manifolds [1] (see also Rem. 2.18). The original Harish-Chandra arguments do not make use of orthogonality with respect to the Cartan-Killing form as subsequent treatments do [34], [31], Ch. VII, Sec. 9. This allows for a more natural generalization, whenever the Cartan-Killing form is degenerate or not positive definite.

Once the infinitesimal theory is fully elucidated, we proceed (Sec. 3), following [20], to the geometric realization of the global Harish-Chandra representation of the real supergroup G_0 in the space of holomorphic sections of a line bundle on the symmetric superspace $G_0/K_0 \subset G/B$, B the borel subgroup corresponding to the fixed admissible system. These infinite dimensional representations of G_0 are the global counterparts of the infinitesimal highest weight representations constructed previously. We proceed as follows. First we consider the Fréchet space of sections of

the complex line bundle L on the quotient G/B , associated with the infinitesimal character λ (see Def. 4). Then, on a neighbourhood U of the origin, we define a left ℓ and a right ∂ action of $\mathcal{U}(\mathfrak{g})$ on $L(U)$ and establish a duality between a subrepresentation of $L(U)$ (as left $\mathcal{U}(\mathfrak{g})$ module) and the infinitesimal HC representations studied in Sec. 2 (Thm. 3.12). Choosing $U = G_0B$, we can prove the main result of this part, namely obtain the highest weight HC representation inside $L(G_0S)$, provided however that $L(G_0S) \neq 0$ (Thm. 3.14). Then we give conditions (Thm. 3.18) for which this occurs. In order to do so, we define the *Harish-Chandra decomposition* $P^-KP^+ \cong P^KP^+ \subset G$ open, $\mathfrak{p}^\pm = \text{Lie}(P^\pm)$, where P^\pm are abelian subgroups (see Sec. 3.6 and Lemma 3.15). This is a remarkable result by itself, based on the peculiar properties of admissible systems; $\Omega = P^-KP^+$ is called the *Harish-Chandra big cell*.

Our paper is organized as follows. In Sec. 2 we discuss infinitesimal representations of Harish-Chandra pairs and the concept of admissible system. For this part, we assume the reader to have basic knowledge of the theory of semisimple Lie algebras and their representations, for example Ch. I–III in [27]. In Sec. 3 we construct the global versions of the representations described in the previous section. We assume the reader to know the basic theory of Lie groups as in [43] Ch. 2.

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2. Theory on the Lie algebra

2.1. Highest weight modules

Let \mathfrak{g} be a semisimple Lie algebra over \mathbb{C} . In the theory of finite dimensional representations of \mathfrak{g} , a fundamental role is played by the highest weight modules. These are defined with respect to the choice of a Cartan subalgebra (CSA) \mathfrak{h} of \mathfrak{g} and a positive system P of roots of $(\mathfrak{g}, \mathfrak{k})$. They are parametrized by their highest weights, namely the elements $\lambda \in \mathfrak{h}^*$. The universal highest weight modules are known as Verma modules and are infinite dimensional. The irreducible highest weight modules are uniquely determined by their highest weights and are the unique irreducible quotients of the Verma modules. The irreducible modules are finite dimensional if and only if the highest weight is dominant integral, and one obtains all irreducible finite dimensional representations of \mathfrak{g} in this manner. These representations lift to the simply connected complex group G corresponding to \mathfrak{g} . For the basic theory of highest weight modules see [43], Ch. 4, [27], Ch. II and III.

2.2. Harish-Chandra pairs and Harish-Chandra (HC) modules

In order to study the theory of representations for real semisimple Lie groups, it is necessary to work in a more structured context. A *Harish-Chandra pair* over a field F of characteristic 0 is a pair $(\mathfrak{g}, \mathfrak{k})$, where:

- (i) \mathfrak{g} is semisimple over F and \mathfrak{k} is the set of fixed points of an involutive automorphism θ of \mathfrak{g} ;
- (ii) \mathfrak{k} is reductive in \mathfrak{g} .

This implies that \mathfrak{k} is reductive and

$$\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{c}, \quad \mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}], \quad \mathfrak{c} = \text{center of } \mathfrak{k}$$

(see to [31], Ch. VI and [25], Ch. III, Sec. 7 for the complete theory of Cartan involutions and their complexifications). In what follows we work over \mathbb{C} . Since $\theta^2 = 1$, \mathfrak{g} is the direct sum of the subspaces where $\theta = \pm 1$. We write \mathfrak{p} for the eigenspace of θ for the eigenvalue -1 , \mathfrak{k} by, its very definition, being the eigenspace of eigenvalue 1. Since the Cartan-Killing form is invariant under all automorphisms of \mathfrak{g} we have:

$$\mathfrak{p} = \mathfrak{k}^\perp, \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where \perp refers to the Cartan-Killing form. The restrictions of this form to \mathfrak{k} and \mathfrak{p} are therefore both non-degenerate. The fact that θ is an involutive automorphism implies that

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \quad (1)$$

These properties of $(\mathfrak{g}, \mathfrak{k})$ lead to calling the pairs $(\mathfrak{g}, \mathfrak{g}_0)$, in the theory of Lie super algebras and Lie supergroups, *super Harish-Chandra pairs* [4], [5] (\mathfrak{g} a Lie superalgebra, \mathfrak{g}_0 its even part). Nevertheless in this note, we limit ourselves to the ordinary theory of Lie algebras, and we shall use, as customary, the suffix 0 to indicate a real form of a Lie algebra or a Lie group. Let G_0 be a real connected semisimple group having finite center with Lie algebra \mathfrak{g}_0 . Let K_0 be a maximal compact subgroup of G_0 with Lie algebra $\mathfrak{k}_0 \subset \mathfrak{g}_0$. Then we say that $(\mathfrak{g}_0, \mathfrak{k}_0)$ is a *Harish-Chandra pair (HC pair) over \mathbb{R}* ; its complexification $(\mathfrak{g}, \mathfrak{k})$ is a HC pair over \mathbb{C} . In this case, there is a unique (up to conjugation) involution of \mathfrak{g}_0 such that \mathfrak{k}_0 is its set of fixed points, called the *Cartan involution* (see [31], Ch. VI, Cor. 6.18, 6.19).

However, it must be noted that there are involutions of \mathfrak{g}_0 which are not Cartan involutions, which also can account for HC pairs. For example, for $\mathfrak{g}_0 = \mathfrak{sl}(n, \mathbb{R})$, we can take θ to be $X \mapsto -FX^tF$, where F is the matrix:

$$F = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \quad n = p + q, \quad p, q \leq 1.$$

Then $\mathfrak{k}_0 = \mathfrak{so}(p, q, \mathbb{R})$ and $(\mathfrak{g}_0, \mathfrak{k}_0)$ is also a HC pair over \mathbb{R} , but in this case \mathfrak{k}_0 corresponds to a non-compact subgroup of G_0 . The HC pairs arising from (G_0, K_0) , K_0 compact, are our main interest, although much of the theory can be formulated in the more general context of HC pairs over \mathbb{C} . When we write $(\mathfrak{g}, \mathfrak{k})$, it means the HC pair is defined over \mathbb{C} ; the suffix 0 indicates that it is defined over \mathbb{R} .

For a Harish-Chandra pair $(\mathfrak{g}, \mathfrak{k})$, we have the class of $(\mathfrak{g}, \mathfrak{k})$ -modules, which are \mathfrak{g} -modules V such that V splits as an algebraic direct sum of finite dimensional irreducible \mathfrak{k} -modules. In this case we can write

$$V = \bigoplus_{\theta \in E(\mathfrak{k})} V_\theta$$

where $E(\mathfrak{k})$ is the set of equivalence classes of irreducible finite dimensional representations of \mathfrak{k} and V_θ is the span of all \mathfrak{k} -irreducible subspaces of V belonging to the class θ . The module V is called a *Harish-Chandra module* if:

$$\dim(V_\theta) < \infty \quad \text{for all } \theta. \quad (2)$$

The significance of this class of modules can be seen from the following remark, please see [42], Ch. 5, for more details. We shall not pursue these more general questions in this note.

Remark 2.1. Suppose that the Harish-Chandra pair $(\mathfrak{g}, \mathfrak{k})$ arises from a pair (G_0, K_0) , i.e. by complexifying the Lie algebras. Then, irreducible $(\mathfrak{g}, \mathfrak{k})$ -modules, which are HC-modules, are the objects of interest from the point of view of representations of the group G_0 because of the following fact: if H is an irreducible Banach space representation of G_0 and V is the set of K_0 -finite vectors (see below), then V is an irreducible HC-module (Thm. 3, Ch. 5 in [42]). The converse is also true: every irreducible HC-module arises in this manner. The generic HC module is not of the highest weight type, but under certain circumstances highest weight modules are also HC modules and it is the purpose of these notes to investigate these closely.

2.3. Highest weight $(\mathfrak{g}, \mathfrak{k})$ -modules when rank of \mathfrak{g} =rank of \mathfrak{k}

If V is a \mathfrak{g} -module and $\mathfrak{a} \subset \mathfrak{g}$ is a Lie subalgebra, a vector $v \in V$ is \mathfrak{a} -finite, if $v \in W$ for some finite dimensional \mathfrak{a} -stable subspace W . Let $V[\mathfrak{a}]$ be the subspace of all \mathfrak{a} -finite vectors. Then $V[\mathfrak{a}]$ is a \mathfrak{g} -submodule of V ; this follows from the easily proved fact that if W is finite dimensional and \mathfrak{a} -stable, then $\mathfrak{g}[W]$ is again finite dimensional and \mathfrak{a} -stable. Let us now assume that:

$$(\mathfrak{g}, \mathfrak{k}) \text{ is a HC pair and } \text{rk } \mathfrak{g} = \text{rk } \mathfrak{k}.$$

Then we can choose a Cartan subalgebra (CSA) \mathfrak{h} so that

$$\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$$

and \mathfrak{h} will be a CSA of both \mathfrak{k} and \mathfrak{g} . We fix a positive system P of roots for $(\mathfrak{g}, \mathfrak{h})$ and write $\alpha > 0$ interchangeably with $\alpha \in P$. We are interested in highest weight modules (with respect to P) which are also HC modules.

Lemma 2.2. *If U is a highest weight module with a highest weight vector u , the following are equivalent:*

- (a) $\dim(\mathcal{U}(\mathfrak{k})u) < \infty$.
- (b) U is $(\mathfrak{g}, \mathfrak{k})$ -module.
- (c) U is a HC module.

If these are satisfied, $\mathcal{U}(\mathfrak{k})u$ is an irreducible \mathfrak{k} -module.

Proof. Since \mathfrak{k} is not semisimple, one must be a little careful. Let \mathfrak{c} denote the center of \mathfrak{k} . For instance, a finite dimensional \mathfrak{k} -module is fully reducible if and only if the action of \mathfrak{c} is completely reducible. In the present case, since $\mathfrak{c} \subset \mathfrak{h}$ and U is a direct sum of weight spaces, it follows that \mathfrak{c} acts semisimply on U . In particular, any finite dimensional submodule for \mathfrak{k} is fully reducible. If u is \mathfrak{k} -finite, then $u \in U[\mathfrak{k}]$. As $U[\mathfrak{k}]$ is a \mathfrak{g} -module, we see that, $U[\mathfrak{k}] = U$. So (a) \implies (b).

(c) \implies (a) trivially.

If we assume (b), we must prove that the spaces U_θ are finite dimensional. If not, and if μ is a weight of θ , then μ occurs with infinite multiplicity, so that $\dim U[\mu] = \infty$, a contradiction. So (b) \implies (c). Now $\mathcal{U}(\mathfrak{k})u$ is a highest weight module for \mathfrak{k} of finite dimension on which \mathfrak{c} acts through scalars, namely, $Cw = \lambda(C)w$ where

$C \in \mathfrak{c}, w \in \mathcal{U}(\mathfrak{k})u$ and λ is the highest weight. Hence $\mathcal{U}(\mathfrak{k})u = \mathcal{U}(\mathfrak{k}')u$, ($\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$), and so it is a finite dimensional highest weight module for the semisimple algebra \mathfrak{k}' . Hence it is irreducible. ■

Since $\mathfrak{h} \subset \mathfrak{k}$, both \mathfrak{k} and \mathfrak{p} are stable under $\text{ad } \mathfrak{h}$, and as the root spaces are one-dimensional, each root space \mathfrak{g}_α is contained either in \mathfrak{k} or \mathfrak{p} ; we then refer to α as a *compact* or *non-compact* root respectively.

Write P_k, P_n for the set of compact and non-compact roots in P . If $\alpha, \beta, \alpha + \beta$ are roots, then the relation $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ implies the following: if α, β are both compact or both non-compact, then $\alpha + \beta$ is compact, while if one of them is compact and the other non-compact, then $\alpha + \beta$ is non-compact; this is a straightforward consequence of (1). We are interested in determining the highest weight modules for $(\mathfrak{g}, \mathfrak{h}, P)$ which are $(\mathfrak{g}, \mathfrak{k})$ -modules. Let V be one of such modules, of highest weight $\lambda \in \mathfrak{h}^*$ and $v \in V$ a highest weight vector. Since $\mathcal{U}(\mathfrak{k})v$ is a finite dimensional highest weight module for $(\mathfrak{k}, \mathfrak{h}, P_k)$, it follows that the highest weight λ must be dominant integral for P_k , i.e., we must have

$$\lambda(H_\alpha) \in \mathbb{Z}_{\geq 0}, \quad (\alpha \in P_k)$$

where H_α are defined in [31], Ch. II, Sec. 4.

We shall now show that λ must satisfy additional conditions.

Let \mathfrak{p}^\pm be the span of the root spaces corresponding to the roots $\beta \in \pm P_n$. Neither of these is in general stable under the adjoint action of \mathfrak{k} , but they may admit non-zero subspaces stable under $\text{ad } \mathfrak{k}$; since $\mathfrak{h} \subset \mathfrak{k}$, such subspaces are spans of root spaces for non-compact roots. A root β is said to be *totally positive* if \mathfrak{g}_β is contained in a subspace \mathfrak{m} of \mathfrak{p}^+ stable under \mathfrak{k} ; if $\mathfrak{m} = \bigoplus_{\gamma \in R} \mathfrak{g}_\gamma$, then $\beta \in R$ and all roots in R are also totally positive. Negatives of totally positive roots are called *totally negative*. Their behavior is similar to the totally positive roots because of the fact that there is an automorphism of \mathfrak{g} that is $-\text{id}$ on \mathfrak{h} . Such an automorphism will take \mathfrak{g}_α to $\mathfrak{g}_{-\alpha}$ for all roots α , in particular preserving \mathfrak{k} and \mathfrak{p} . We write P_t for the set of totally positive roots and

$$\mathfrak{p}_t^\pm = \bigoplus_{\beta \in P_t} \mathfrak{g}_{\pm\beta}$$

Obviously \mathfrak{p}_t is the largest $\text{ad } \mathfrak{k}$ -stable subspace of \mathfrak{p} ; it may be 0. We say that the positive system P is *admissible* if

$$\mathfrak{p}_t^\pm \neq 0, \quad \text{i.e.,} \quad \pm P_t \neq \emptyset.$$

For example in A_2 , with root system $\Delta = \pm\{\alpha, \beta, \alpha + \beta\}$, and compact roots $\pm\alpha$, we have that $P = \{\alpha, \beta, \alpha + \beta\}$ is admissible ($P_k = \{\alpha\}$, $P_n = \{\beta, \alpha + \beta\}$), while $P' = \{\alpha + \beta, -\beta, \alpha\}$ is not admissible ($P'_k = \{\alpha\}$, $P'_n = \{\alpha + \beta, -\beta\}$), see [31], Ch. VII, [34] and also [7], [9] for generalizations.

Our aim is to prove the following theorem, which reveals the significance of total positivity for the problem of constructing infinite dimensional highest weight $(\mathfrak{g}, \mathfrak{k})$ -modules. In our treatment we shall follow quite closely the original exposition by Harish-Chandra, [20], Sec. 3, 4, 5.

Theorem 2.3. *Let V be a non-zero highest weight $(\mathfrak{g}, \mathfrak{k})$ -module with respect to a positive system P of roots of $(\mathfrak{g}, \mathfrak{h})$, of highest weight λ . Then $\lambda(H_\gamma) \in \mathbb{Z}_{\geq 0}$ for all positive roots γ which are not totally positive. In particular, if P is not admissible and V is irreducible, then V is finite dimensional.*

The proof of this theorem is quite delicate and depends on the following lemmas. Before stating them we need some preparation.

Let v be a highest weight vector. For any positive root γ let $\mathfrak{a}_\gamma = \mathfrak{h} \oplus \mathfrak{g}_\gamma \oplus \mathfrak{g}_{-\gamma}$. Let Q be the set of all positive roots such that v is \mathfrak{a}_γ -finite and let W_Q be the subgroup of the Weyl group generated by the reflections s_γ for $\gamma \in Q$. Let us also write W_k for the Weyl group of \mathfrak{k} . Since the vectors $w \in V$, such that w is \mathfrak{a}_γ -finite, form a \mathfrak{g} -module, Q is also the set of all roots $\gamma > 0$ such that v is \mathfrak{a}_γ -finite. It is known that v is \mathfrak{a}_γ -finite if and only if $X^r v = 0$ for $r \gg 0$ ($X \in \mathfrak{g}_\gamma$), and that, in this case, $\lambda(H_\gamma) \in \mathbb{Z}_{\geq 0}$. If $\gamma \in Q$, we can split V as a direct sum of finite dimensional irreducible \mathfrak{a}_γ -modules from which we conclude that the set of weights of V is stable under W_Q . Since v is \mathfrak{k} -finite, $P_k \subset Q$ and so $W_k \subset W_Q$. Suppose β is totally positive. Then, there is a subset R of P such that R contains β and $\bigoplus_{\gamma \in R} \mathfrak{g}_\gamma$ is stable under the adjoint action of \mathfrak{k} . Hence R is stable under W_k , showing that $s\beta > 0$ and in fact totally positive for all $s \in W_k$. The lemma below goes in the other direction.

Lemma 2.4. *Let γ be a positive non-compact root. If $s\gamma < 0$ for some $s \in W_Q$, then $\gamma \in Q$.*

Proof. The set of weights of V is stable under W_Q and so $s\lambda$ is a weight of V . Hence

$$\lambda - s\lambda = \sum_{1 \leq i \leq \ell} n_i \beta_i, \quad (n_i \text{ are integers } \geq 0)$$

where $\{\beta_1, \dots, \beta_\ell\}$ is the set of simple roots in P . We shall show that $X_{-\gamma}^p v = 0$ if $p > \max_i n_i$. Suppose for some integer $p \geq 1$, we have $X_{-\gamma}^p v \neq 0$. Then $\lambda - p\gamma$ is a weight of V . Write $s\gamma = -\beta$ where $\beta > 0$. Then $s\lambda + p\beta$ is a weight of V so that $\lambda - s\lambda - p\beta = \sum_{1 \leq i \leq \ell} m_i \beta_i$ the $m_i \in \mathbb{Z}_{\geq 0}$. Then $p\beta = \sum_{1 \leq i \leq \ell} (n_i - m_i) \beta_i$; as $\beta > 0$, we can also write $\beta = \sum_i k_i \beta_i$ where the $k_i \in \mathbb{Z}_{\geq 0}$ $k_{i_0} > 0$ for some i_0 . We have $pk_{i_0} = n_{i_0} - m_{i_0} \leq n_{i_0}$ giving $p \leq pk_{i_0} \leq n_{i_0}$. ■

Lemma 2.5. *If γ is a positive root that vanishes on the center \mathfrak{c} of \mathfrak{k} , then $s\gamma < 0$ for some $s \in W_k$ and hence $\gamma \in Q$. If γ is totally positive, it cannot vanish on \mathfrak{c} and all $s\gamma > 0$, even totally positive, for $s \in W_k$. In particular $\mathfrak{c} \neq 0$ when totally positive roots exist.*

Proof. Let $\delta = \sum_{s \in W_k} s\gamma$ (δ need not be a root). Since \mathfrak{c} is fixed elementwise by W_k , we see that all $s\gamma$ vanish on \mathfrak{c} and so δ must be 0 on \mathfrak{c} . On the other hand, δ is fixed by all elements of W_k and so $\delta(H_\theta) = 0$, for all $\theta \in P_k$. So δ must be 0 since the H_θ and \mathfrak{c} span \mathfrak{h} . But then a sum of positive roots cannot be 0 and so $s\gamma$ must be < 0 for some $s \in W_k$. By Lemma 2.4, we conclude that $\gamma \in Q$. If γ is totally positive, then \mathfrak{g}_γ is contained in a sum \mathfrak{m} of root spaces contained in \mathfrak{p}^+ and stable under \mathfrak{k} , so that all the root spaces $\mathfrak{g}_{s\gamma}$ are contained in \mathfrak{m} ; this shows that all $s\gamma > 0$. The previous argument shows that γ cannot be 0 on \mathfrak{c} . ■

Now we go to the proof of Theorem 2.3.

Proof. Let v be a highest weight vector. We want to prove that if $\gamma > 0$ is not in Q , then γ is totally positive. Since we know that V is \mathfrak{k} -finite, we have $P_k \subset Q$ and so we may assume that γ is a non-compact positive root. Let \mathfrak{m} be the minimal ad \mathfrak{k} -stable subspace of \mathfrak{p} containing \mathfrak{g}_γ . Now \mathfrak{m} is spanned by the non-compact root spaces, and as these are all one-dimensional, it follows from the complete reducibility of the action of \mathfrak{k} on \mathfrak{p} that \mathfrak{m} is irreducible. Let β_0 and β_1 be the highest and lowest roots belonging to \mathfrak{m} . If $\beta_1 > 0$ all the roots belonging to \mathfrak{m} will be > 0 , showing that γ must be totally positive. Hence it suffices to prove that $\beta_1 > 0$. Suppose to the contrary that $\beta_1 < 0$. Now there is an element $s \in W_k$ such that $s\beta_0 = \beta_1 < 0$. So by Lemma 2.4, we know that $\beta_0 \in Q$. Let β be a maximal positive root belonging to \mathfrak{m} , but which is not in Q . Clearly $\beta \neq \beta_0$. Then there is a positive compact root α such that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta} \neq 0$ and so $\alpha + \beta$ is a root, which is positive. Moreover $\alpha + \beta \in Q$. Clearly $\alpha + \beta$ is positive non-compact. Let $\theta = \beta + (\alpha + \beta) = 2\beta + \alpha$. We claim that θ is not a root. If it were a root, it must be compact, and so θ and α are both compact roots. Hence they both vanish on \mathfrak{c} , from which we infer that $\beta = (1/2)(\theta - \alpha)$ must also vanish on \mathfrak{c} . By Lemma 2.5 we have that $\beta \in Q$, a contradiction. Hence $\beta + (\alpha + \beta)$ is not a root but $\beta - (\alpha + \beta) = -\alpha$ is a root. Let $t \geq 1$ be the largest integer such that $\beta - t(\alpha + \beta)$ is a root. Then $s_{\alpha+\beta}(\beta - t(\alpha + \beta)) = \beta$ or, equivalently,

$$s_{\alpha+\beta}\beta = \beta - t(\alpha + \beta) = -\alpha - (t-1)(\alpha + \beta) < 0$$

showing that $s_{\alpha+\beta}\beta < 0$. But $\alpha + \beta \in Q$ and so, by Lemma 2.4, $\beta \in Q$, a contradiction. This finishes the proof of the theorem. \blacksquare

2.4. Structure of the set of totally positive roots

Recall that P_n denotes the set of positive non compact roots in the positive system $P = P_k \cup P_n$, while θ is the Cartan involution of the complex semisimple Lie algebra \mathfrak{g} . The basic result we want to prove is the following.

Theorem 2.6. *Let $\mathfrak{g}_t = \mathfrak{k}_t \oplus \mathfrak{p}_t$, where $\mathfrak{k}_t = [\mathfrak{p}_t, \mathfrak{p}_t]$, $\mathfrak{g}_1 = \mathfrak{g}_t^\perp$.*

Then $\mathfrak{g}_t, \mathfrak{g}_1$ are ideals of \mathfrak{g} which are θ -stable, and $\mathfrak{g} = \mathfrak{g}_t \oplus \mathfrak{g}_1$.

Moreover P_t is precisely the set of positive non-compact roots of \mathfrak{g}_t and $P_n \setminus P_t$ is precisely the set of positive non-compact roots of \mathfrak{g}_1 .

Before its proof we need a lemma.

Lemma 2.7. *Let the notation be as above. Let $\mathfrak{q}_1, \mathfrak{q}_2$ be two subspaces of \mathfrak{p} stable under ad \mathfrak{k} . Suppose that $\mathfrak{q}_1 \perp \mathfrak{q}_2$. Then $[\mathfrak{q}_1, \mathfrak{q}_2] = 0$.*

Proof. Let B be the Cartan-Killing form of \mathfrak{g} . If $X \in \mathfrak{k}, Y \in \mathfrak{q}_1, Z \in \mathfrak{q}_2$, then $B(X, [Y, Z]) = B([X, Y], Z) = 0$ because $[X, Y] \in \mathfrak{q}_1, Z \in \mathfrak{q}_2$. \blacksquare

Now we go to the proof of Theorem 2.6.

Proof. Since \mathfrak{p}_t is stable under ad \mathfrak{k} , it is immediate that \mathfrak{g}_t is stable under ad \mathfrak{k} . It is also obvious that \mathfrak{g}_t is stable under \mathfrak{p}_t . We must show that it is stable under \mathfrak{p} . Let $\mathfrak{q} = \mathfrak{p}_t^\perp$. Then \mathfrak{q} is also stable under ad \mathfrak{k} . By Lemma 2.4 we have $[\mathfrak{q}, \mathfrak{p}_t] = 0$, hence also $[\mathfrak{q}, \mathfrak{k}_t] = 0$, since $\mathfrak{k}_t = [\mathfrak{p}_t, \mathfrak{p}_t]$. So $[\mathfrak{q}, \mathfrak{g}_t] = 0$. To prove that \mathfrak{g}_t is stable under ad \mathfrak{p}_t is thus enough to show that $\mathfrak{p} = \mathfrak{p}_t \oplus \mathfrak{q}$, or equivalently, $\mathfrak{q} \cap \mathfrak{p}_t = 0$.

If \mathfrak{g}_β belongs to this intersection, then from $\mathfrak{g}_\beta \in \mathfrak{p}_t$, we get $\beta = c\gamma$ where $c = \pm 1$ and $\gamma \in P_t$. On the other hand, as $\mathfrak{g}_\beta \in \mathfrak{q}$, \mathfrak{g}_β is orthogonal to $\mathfrak{g}_{\pm\gamma}$. Hence $\mathfrak{g}_\beta \perp \mathfrak{g}_{\pm\beta}$. This is impossible. Thus we have proved that \mathfrak{g}_t is an ideal. It is clearly θ -stable and so \mathfrak{g}_1 is stable under θ and is the ideal complementary to \mathfrak{g}_t . The remaining assertions are now obvious. ■

Remark 2.8. The condition $\mathfrak{k}_t = [\mathfrak{p}_t, \mathfrak{p}_t]$ means that \mathfrak{g}_t has no ideal factors which are “compact”.

Because of this result, we can direct our attention to the case when $P_n = P_t$, i.e., $\mathfrak{g} = \mathfrak{g}_t$; hence all non compact positive roots are totally positive. Since $P_n = P_t$, we know that \mathfrak{p}_t^+ is stable under $\text{ad } \mathfrak{k}$ and multiplicity free (as the root spaces are one dimensional). Hence we have a unique (up to ordering) decomposition

$$\mathfrak{p}_t^+ = \bigoplus_{1 \leq i \leq s} \mathfrak{p}_i \quad (1 \leq i \leq s)$$

into irreducible modules for \mathfrak{k} . Define \mathfrak{p}_i^- so that the roots belonging \mathfrak{p}_i^- are the negatives of the roots belonging to \mathfrak{p}_i^+ . Let β_i ($1 \leq i \leq s$) be the lowest root of \mathfrak{p}_i^+ . We now state a result summarizing some key properties of admissible systems and totally positive roots. For more details see [20], Sec. 4, Lemma 10, Lemma 12.

Theorem 2.9. *Let $P_n = P_t$ and let $\alpha_1, \dots, \alpha_r$ be the simple roots of P_k . Then, $\{\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s\}$ is the set of simple roots of P . Moreover we have*

- (a) $r + s = rk \mathfrak{h}$.
- (b) $s = \dim(\mathfrak{c})$ where \mathfrak{c} is the center of \mathfrak{k} , and the restrictions of the β_i to \mathfrak{c} are linearly independent.
- (c) If β is a non-compact positive root, there is exactly one i ($1 \leq i \leq s$) and integers $m_j \geq 0$ such that $\beta = \beta_i + m_1\alpha_1 + \dots + m_r\alpha_r$, where i, m_j are uniquely determined by β . In particular $[\mathfrak{p}^+, \mathfrak{p}^+] = 0$.
- (d) \mathfrak{g} decomposes as the sum of s ideals $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ which are θ -stable. Each of these have the property that all non-compact roots are totally positive, $[\mathfrak{p}_i, \mathfrak{p}_i] = \mathfrak{k}_i$, and the dimension of the center of \mathfrak{k}_i is 1. In particular each \mathfrak{g}_i is simple.
- (e) The integer s is also the number of irreducible components of \mathfrak{p}_t^+ as a \mathfrak{k} -module. In particular, \mathfrak{g} is simple if and only if \mathfrak{p}_t^+ is irreducible.

Proof. Let β_i be the lowest root (i.e. weight) of \mathfrak{p}_i^+ . Then, for any non-compact root β , there exists a unique i such that $\mathfrak{g}_\beta \subset \mathfrak{p}_i^+$, and \mathfrak{g}_β can be reached by applying positive compact root vectors to \mathfrak{g}_{β_i} . Hence, $\beta = \beta_i + m_1\alpha_1 + \dots + m_r\alpha_r$, where the m_j are integers ≥ 0 . Let $S = \{\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s\}$. Then, every positive root is a non-negative integral linear combination of elements of S . So $r + s \geq \ell$, where ℓ is the rank of \mathfrak{g} . On the other hand, suppose $H \in \mathfrak{h}$ is an element such that all elements of S vanish at H . Then, H must centralize \mathfrak{k} and \mathfrak{p} , hence also \mathfrak{g} . So $H = 0$, showing that the elements of S are linearly independent. The same argument shows also that the restrictions to \mathfrak{c} of the β_i are linearly independent. So S is the set of simple roots in P and i and the m_j are unique. The formula for the non-compact roots shows that the sum of two elements of P_t is never a root. The proofs of (a), (b), (c) are now clear.

We now take up (d) and (e). Let $\mathfrak{p}_i = \mathfrak{p}_i^+ \oplus \mathfrak{p}_i^-$. Clearly the \mathfrak{p}_i are mutually orthogonal and $\mathfrak{p} = \bigoplus_i \mathfrak{p}_i$. Let $\mathfrak{k}_i = [\mathfrak{p}_i, \mathfrak{p}_i]$ and $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$. Then \mathfrak{g}_i is stable under θ as well as $\text{ad } \mathfrak{k}$ and $\text{ad } \mathfrak{p}_i$. We now claim that for $i \neq j$, $[\mathfrak{p}_i, \mathfrak{p}_j] = 0$. Certainly $\mathfrak{p}_i \perp \mathfrak{p}_j$, because, if $\beta_r \in \mathfrak{p}_r$, then $\beta_i \pm \beta_j \neq 0$ proving the claim in view of Lemma 2.7. Hence \mathfrak{g}_i is stable under all $\text{ad } \mathfrak{p}_j$ ($j \neq i$), hence under $\text{ad } \mathfrak{p}$. So the \mathfrak{g}_i are ideals in \mathfrak{g} . The results of (a)-(c) now give (d) and (e). \blacksquare

We also have the following result we use in the sequel.

Proposition 2.10. *Let the notation be as above.*

- (a) *If β is a totally positive root, then any root of the form $\gamma = \beta + \sigma$, where σ is an integral linear combination of the compact roots is totally positive and lies in the irreducible $\text{ad } \mathfrak{k}$ -module generated by \mathfrak{g}_β .*
- (b) *If γ is totally positive and α is a compact root, the α -chain containing γ has length ≤ 3 .*
- (c) *If \mathfrak{g} is simple then either no root is totally positive or all positive roots are totally positive.*
- (d) *If $\mathfrak{g} = \mathfrak{g}_t$ is simple, then P and $P^- = P_k \cup -P_n$ are the only two admissible positive systems containing P_k a fixed positive system for \mathfrak{k} . Otherwise, if \mathfrak{g}_i , ($1 \leq i \leq s$) are the simple ideals of \mathfrak{g} , and \mathfrak{p}_i^\pm are as defined earlier, then the number of admissible possible systems containing P_k is 2^s . A non-compact root belongs, to exactly one of \mathfrak{p}_i^\pm .*

Proof. (a) We have the ideal decomposition $\mathfrak{g} = \mathfrak{g}_t \oplus \mathfrak{g}_1$ and β is a root of \mathfrak{g}_t . If γ is a root of \mathfrak{g}_1 , it will vanish on $\mathfrak{c} \cup \mathfrak{g}_t$ and so β will also vanish there, contradicting total positivity. Thus γ is also a root of \mathfrak{g}_t . Let us write σ (σ^+) with or without suffixes for integral (positive integral) linear combinations of the simple compact roots of \mathfrak{g}_t . Then, using the notation of Theorem 2.9, we have $\beta = \beta_i + \sigma_1^+$ so that $\gamma = \beta_i + \sigma_2$ on the one hand and $\gamma = \pm(\beta_j + \sigma_3^+)$ on the other hand, depending on whether γ is positive or negative. Hence $\beta_i = \pm(\beta_j + \sigma_3^+)$. Restricting to the center of \mathfrak{k}_t and remembering that the restrictions of the β_m are linearly independent, we see that we have to take the plus sign and $j = i$. The conclusions of (a) now follow at once.

(b) We can take the α -chain to be $\{\gamma - p\alpha\}$, ($p = 0, 1, \dots, k$) where $\gamma - k\alpha = s_\alpha\gamma$ so that $k = \gamma(H_\alpha)$. It is a question of proving that $\gamma(H_\alpha) \leq 2$. Suppose $\gamma(H_\alpha) \geq 3$. Then $\alpha(H_\gamma) > 0$ and so ≥ 1 , showing that $m = \alpha(H_\gamma)\gamma(H_\alpha) \geq 3$. Consider the root $\beta = s_\gamma s_\alpha \gamma = (m-1)\gamma - \gamma(H_\alpha)\alpha$. Since $m-1 \geq 2$ this contradicts (c) of Theorem 2.9.

(c) If \mathfrak{g} is simple, then $\mathfrak{g} = \mathfrak{g}_t$ or $\mathfrak{g} = \mathfrak{g}_1$.

(d) Let P' be an admissible positive system. Then all roots of P'_n belong to \mathfrak{g}_t . We may assume that $\mathfrak{g} = \mathfrak{g}_t$ and is simple. If P'_t contains an element from P_n^\pm , then by the irreducibility of \mathfrak{p}^\pm under \mathfrak{k} we see that P'_t must contain all of P_n^\pm . Thus P^\pm are the only admissible positive systems containing P_k . In the general case, let $\epsilon = (\epsilon_i)$ be an s -tuple of signs ± 1 and let $\mathfrak{q}^\epsilon = \bigoplus \mathfrak{p}_i^{\epsilon_i}$. Let P^ϵ be the set of roots belonging to \mathfrak{q}^ϵ . We claim that P^ϵ is an admissible positive system. To prove that it is a positive system, it is enough to find a point in \mathfrak{h} at which all the elements of P^ϵ are > 0 . Write $\beta_i^+ = \beta_i$ for the lowest root in \mathfrak{p}_i^+ ; if γ_i is the highest root of

\mathfrak{p}_i^+ , then $\beta_i = -\gamma_i$ is the lowest root of \mathfrak{p}_i , and it is a question of finding a point of \mathfrak{h} at which all of P_k and all $\beta_i^{\epsilon_i}$ are > 0 . Since the β_i are linearly independent when restricted to \mathfrak{c} we can find a $C \in \mathfrak{c}$ such that $\beta_i^{\epsilon_i}(C) > 0$ for all i . On the other hand, we can find a U in the span \mathfrak{h}' of the H_α for the compact roots such that $\alpha(U) > 0$ for all $\alpha \in P_k$. Then, for $H = C + \eta U$ for sufficiently small $\eta > 0$ has the property that $\beta(H) > 0$ for all $\beta = \alpha \in P_k$, $\beta = \beta_i^{\epsilon_i}$. Obviously there are no other admissible positive systems containing P_k . ■

2.5. Harish-Chandra homomorphism

Let ζ be the center of $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}[0]$ the subalgebra of $\mathcal{U}(\mathfrak{g})$ commuting with \mathfrak{h} . More generally, for any $\mu \in \mathfrak{h}$, let $\mathcal{U}[\mu]$ be the subspace of $\mathcal{U}(\mathfrak{g})$ given by

$$\mathcal{U}[\mu] = \{a \in \mathcal{U}(\mathfrak{g}) \mid [H, a] = \mu(H)a, \text{ for all } H \in \mathfrak{h}\}$$

Then $\mathcal{U}[0]$ is a subalgebra, $\zeta \subset \mathcal{U}[0]$, and $(\mathcal{U}[\mu])$ is a grading of $\mathcal{U}(\mathfrak{g})$; moreover $\mathcal{U}[\mu] \neq 0$ if and only if μ is in the \mathbb{Z} -span of the roots (the root lattice). If $\gamma_1, \dots, \gamma_t$ is an enumeration of the positive roots and (H_i) is a basis for \mathfrak{h} , then elements of $\mathcal{U}[0]$ are linear combinations of

$$X_{-\gamma_1}^{n_1} \dots H_1^{c_1} \dots X_{\gamma_1}^{p_1} \dots \quad \text{with} \quad (p_1 - n_1)\gamma_1 + \dots = 0.$$

It is then clear that every term occurring in such a linear combination must necessarily have some $p_i > 0$ except those that are just monomials in the H_i alone. So for any $u \in \mathcal{U}[0]$ we have an element $\beta_P(u) = \beta(u) \in \mathcal{U}(\mathfrak{h})$ such that

$$u \equiv \beta(u) \pmod{\mathcal{P}}, \quad \mathcal{P} := \mathcal{U}(\mathfrak{g})\mathfrak{g}_\gamma, \quad \gamma \in P \tag{3}$$

The action of u on the Verma module V_λ must leave the weight spaces stable since it commutes with \mathfrak{h} , and so applying it to the highest weight vector v_λ we see that $uv_\lambda = \beta(u)(\lambda)v_\lambda$ where we are identifying $\mathcal{U}(\mathfrak{h})$ with the algebra of all polynomials on \mathfrak{h}^* so that $\beta(u)(\lambda)$ makes sense. It follows from this that, if $u \in \mathcal{U}(\mathfrak{h}) \cap \mathcal{P}$, then $u(\lambda) = 0$ for all λ and so $u = 0$, i.e., $\mathcal{U}(\mathfrak{h}) \cap P = 0$. Hence $\beta(u)$ is uniquely determined by the equation (3), and the map $u \mapsto \beta(u)$ is a homomorphism of $\mathcal{U}[0]$ onto $\mathcal{U}(\mathfrak{h})$. Since $\zeta \subset \mathcal{U}[0]$, we thus have a homomorphism of ζ into $\mathcal{U}(\mathfrak{h})$. This is the *Harish-Chandra homomorphism* (see [31], Ch. VII). Harish-Chandra proved that β is an isomorphism of ζ onto the algebra of all elements of $\mathcal{U}(\mathfrak{h})$ invariant under a certain (affine) action of the Weyl group W on \mathfrak{h} . More precisely, let $\delta = \delta_P = (1/2) \sum_{\alpha \in P} \alpha$, and for $s \in W$ let $s_A \lambda = s(\lambda + \delta) - \delta$. Then $s \mapsto s_A$ is an (affine) action of W on \mathfrak{h}^* , and β is an isomorphism of ζ with $\mathcal{U}(\mathfrak{h})^W$. For more details on this key highly non trivial fact see [31], Ch. V, Sec. 5, Thm. 5.44. As a side remark, we notice that in the Lie superalgebra theory the Harish-Chandra homomorphism is defined similarly, but it will not be an isomorphism: this is due to the presence of *isotropic roots* that is roots with length zero (see [36], Ch. 13.1).

Now, for $z \in \zeta$,
$$zv_\lambda = \beta(z)(\lambda)v_\lambda$$

but since z commutes with the actions of all elements of $\mathcal{U}(\mathfrak{g})$ and so, as v_λ is cyclic for V_λ , we find that $zv = \beta(z)(\lambda)v$ on all $v \in V_\lambda$. Thus

$$z = \chi_\lambda(z)I, \quad \text{on } V_\lambda, \quad \chi_\lambda(z) := \beta(z)(\lambda).$$

It follows from this that, if $\lambda, \mu \in \mathfrak{h}^*$ are such that $\chi_\lambda(z) = \chi_\mu(z)$ for all $z \in \zeta$, then for some $s \in W$, we must have $s_A \lambda = \mu$. For more details on this key point see [31], Ch. V, Sec. 5, Thm. 5.62.

A consequence of this is the following: if U_1, U_2 are highest weight modules with highest weights μ_1, μ_2 respectively, and if there is a non zero morphism $U_2 \rightarrow U_1$, then there is an element $s \in W$ such that $\mu_2 + \delta = s(\mu_1 + \delta)$. Indeed, if $z \in \zeta$, then z acts as $\chi_{\mu_2}(z)$ on U_2 and by $\chi_{\mu_1}(z)$ on U_1 and these two numbers must be the same for all $z \in \zeta$. This gives the required result. The Harish-Chandra homomorphism on ζ depends on the choice of the positive system P . Let

$$\gamma(z)(\lambda) = \beta_P(z)(\lambda - \delta_P) \quad (z \in \zeta).$$

It is then easy to show that $\gamma(z)$ is independent of P , and that γ is an isomorphism of ζ with the algebra $\mathcal{U}(\mathfrak{h})^W$ of all elements of $\mathcal{U}(\mathfrak{h})$ invariant under the usual linear action of W on \mathfrak{h} (see also [31], Ch. VII).

2.6. Converse to Theorem 2.3

We want to construct highest weight $(\mathfrak{g}, \mathfrak{h})$ -modules when λ satisfies the condition of Theorem 2.3. In view of the splitting $\mathfrak{g} = \mathfrak{g}_t \times \mathfrak{g}_1$, it is enough to consider the case of \mathfrak{g}_t , since we can tensor with finite dimensional modules for \mathfrak{g}_1 .

Let $\lambda \in \mathfrak{h}^*$ be such that $\lambda(H_\alpha)$ is an integer ≥ 0 for all $\alpha \in P_k$. Let $F = F_\lambda$ be the irreducible finite dimensional module for \mathfrak{k} of highest weight λ . Note that $\lambda(H_\beta)$ can be arbitrary for positive non-compact roots β . Write $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$. Recall that $[\mathfrak{k}, \mathfrak{p}^+] \subset \mathfrak{p}^+$ and so we can turn F into a left \mathfrak{q} -module by letting \mathfrak{p}^+ act trivially.

Define $U^\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} F$ and view U^λ as a $\mathcal{U}(\mathfrak{g})$ -module by left action:

$$a(b \otimes f) = ab \otimes f. \quad \text{Let } \delta = (1/2) \sum_{\gamma \in P} \gamma.$$

Theorem 2.11. *U^λ is the universal HC module of highest weight λ . If*

$$(\lambda + \delta)(H_\gamma) \quad \text{is real and } \leq 0$$

for all $\gamma \in P_n$, then U^λ is irreducible.

Proof. Let $M \subset U^\lambda$ be a nonzero submodule of U^λ . Since the weights of M are $\leq \lambda$ we can choose a maximal one, say μ ; if u is a corresponding weight vector, $X_\gamma u = 0$ for all $\gamma \in P$ and so $\mathcal{U}(\mathfrak{g})u$ is a highest weight module of highest weight μ . From the properties of the infinitesimal character and the Harish-Chandra homomorphism this implies that $\mu + \delta = s(\lambda + \delta)$ for some $s \in W$. The condition on λ can be rewritten as $(\lambda + \delta)(H_\gamma) \geq 0$ for all $\gamma \in P = P_k \cup (-P_n)$. Let s_0 be the element of W_k that takes P_k to $-P_k$. Then $s_0 P = -P_k \cup P_n$ and so $P^- = -s_0 P$ is also a positive system. Let $\{\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s\}$ be the simple system of P and let $\gamma_j = s_0 \beta_j$. Then the γ_j are also in P_t and in fact γ_j is the highest root of the \mathfrak{k} -module \mathfrak{p}_j^+ of which β_j is the lowest root. Hence $\gamma_j = \beta_j + \sigma_j$ where σ_j is a sum of positive compact roots.

We claim that $\{\alpha_1, \dots, \alpha_r, -\gamma_1, \dots, -\gamma_s\}$ is the simple system of P^- . Let $\gamma \in P_n$. Then $s_0 \gamma = b_1 \alpha_1 + \dots + b_r \alpha_r + e_1 \beta_1 + \dots + e_s \beta_s$ where the b_i, e_j are integers ≥ 0 . Hence applying $-s_0$, we get $-\gamma = b'_1 \alpha_1 + \dots + b'_r \alpha_r - e_1 \gamma_1 - \dots - e_s \gamma_s$ where the b'_j are integers ≥ 0 . This proves the claim.

Since $(\lambda + \delta)(H_\gamma) \geq 0$ for all $\gamma \in P^-$ we can write

$$(\lambda + \delta) - s(\lambda + \delta) = \sum_{1 \leq i \leq r} c_i \alpha_i - \sum_{1 \leq j \leq s} d_j \gamma_j$$

where $c_i, d_j \geq 0$. Thus

$$\lambda - \mu = (\lambda + \delta) - s(\lambda + \delta) = \sum_{1 \leq i \leq r} c_i \alpha_i - \sum_{1 \leq j \leq s} d_j \gamma_j$$

But

$$\lambda - \mu = \sum_{1 \leq i \leq r} a_i \alpha_i + \sum_{1 \leq j \leq s} b_j \beta_j$$

where the a_i, b_j are integers ≥ 0 . Hence, writing $\gamma_j = \beta_j + \sigma_j$ as above and restricting to \mathfrak{c} we get

$$\sum_{1 \leq j \leq s} (d_j + f_j)(\beta_j)|_{\mathfrak{c}} = 0$$

Since the $(\beta_j)|_{\mathfrak{c}}$ are linearly independent by (b) of Theorem 2.9 we have $d_j + f_j = 0$ for all j , and hence, as the d_j, f_j are ≥ 0 , we must have $d_j = f_j = 0$, for all j . Hence

$$\lambda - \mu = \sum_{1 \leq i \leq r} a_i \alpha_i$$

where the a_i are integers ≥ 0 .

Now u is a linear combination of $X_{-\gamma_1} \dots X_{-\gamma_m} v$ where each γ_j is in $\{\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s\}$ and

$$\lambda - \mu = \gamma_1 + \dots + \gamma_m.$$

Writing each γ_j as a linear combination of α_i ($1 \leq i \leq r$) and the β_j ($1 \leq j \leq s$) with integer coefficients ≥ 0 , and noting that $\lambda - \mu$ does not involve the β_j , we conclude that each γ_j does not involve any β_j . In other words, $u \in \mathcal{U}(\mathfrak{k})v$. But then u must be a multiple of v , showing that $M = U^\lambda$. This proves that U^λ is already irreducible. ■

We shall now study the structure of U^λ as a \mathfrak{g} -module for arbitrary λ with $\lambda(H_\alpha)$ an integer ≥ 0 , for all $\alpha \in P_k$. For this we need a standard lemma.

Lemma 2.12. *Let g be a field and A, B algebras over g . Suppose $B \subset A$, A is a free right B -module, F a left B -module, and $V = A \otimes_B F$. If (a_i) is a free basis for A as a right B -module, and $L = \sum_i g \cdot a_i$, then the map taking $l \otimes_g f$ to $l \otimes_B f$ is a linear isomorphism of $L \otimes_g F$ with V .*

Proof. This is standard but we give a proof. All symbols \otimes without any suffix mean tensor products over the field g . Let (b_j) be a g -basis for B with $b_0 = 1$. Then V is a quotient of $A \otimes F$ by the span S of elements of the form $ab \otimes f - a \otimes bf$ where $a \in A, b \in B, f \in F$. Let (f_k) be a g -basis for F . We assert that S is spanned by $a_i b_j \otimes f_k - a_i \otimes b_j f_k$. Indeed, S is spanned by $a_i b_j b \otimes f_k - a_i b_j \otimes b f_k$. Now

$$a_i b_j b \otimes f_k - a_i b_j \otimes b f_k = (a_i b_j b \otimes f_k - a_i \otimes b_j b f_k) - (a_i b_j \otimes b f_k - a_i \otimes b_j b f_k).$$

Expressing $b_j b$ in the terms of the first group as a linear combination of the b_r and the $b f_k$ of the second group in terms of the f_l , we see that our assertion is proved. Note that we only need the terms with $j \neq 0$ as $a_i b_j \otimes f_k - a_i \otimes b_j f_k = 0$ for $j = 0$.

Since the map $L \otimes F \rightarrow V$ is obviously surjective, it is enough to show that the linear span of the $a_i \otimes f_k$ has 0 intersection with S . Suppose $\sum_{i,k} D_{ik} a_i \otimes f_k \in S$. Then we can write

$$\sum_{i,j,k,j \neq 0} C_{ijk} (a_i b_j \otimes f_k - a_i \otimes b_j f_k) = \sum_{i,k} D_{ik} a_i \otimes f_k.$$

Since $b_j f_k$ is a linear combination of the f_r it follows that

$$\sum_{i,j,k,j \neq 0} C_{ijk} a_i b_j \otimes f_k = \sum E_{ir} a_i \otimes f_r.$$

This means that $C_{ijk} = 0$ for all i, j, k with $j \neq 0$, hence that $\sum_{i,k} D_{ik} a_i \otimes f_k = 0$. This proves the lemma. ■

We regard $\mathcal{U}(\mathfrak{p}^-) \otimes F$ as a $\mathcal{U}(\mathfrak{p}^-)$ -module by $a, b \otimes f \mapsto ab \otimes f$. Since is stable under $\text{ad } \mathfrak{k}$ we may view $\mathcal{U}(\mathfrak{p}^-) \otimes F$ as a \mathfrak{k} -module also.

Corollary 2.13. *The map $\phi: a \otimes f \mapsto a \otimes U(\mathfrak{q})f$ is a linear isomorphism of $\mathcal{U}(\mathfrak{p}^-) \otimes F$ with U^λ that intertwines the actions of $\mathcal{U}(\mathfrak{p}^-)$ and $\mathcal{U}(\mathfrak{k})$. In particular, U^λ is a free $\mathcal{U}(\mathfrak{p}^-)$ -module with basis $1 \otimes U(\mathfrak{q})f_j$ where (f_j) is a basis for F .*

Proof. Since $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{q}$ it follows that $a \otimes b \mapsto ab$ is a linear isomorphism of $\mathcal{U}(\mathfrak{p}^-) \otimes \mathcal{U}(\mathfrak{q})$ with $\mathcal{U}(\mathfrak{g})$. It is clear from this that $\mathcal{U}(\mathfrak{g})$ is a free right $\mathcal{U}(\mathfrak{q})$ -module, and that any basis of $\mathcal{U}(\mathfrak{p}^-)$ is a free right $\mathcal{U}(\mathfrak{q})$ -basis for $\mathcal{U}(\mathfrak{g})$.

Lemma 2.12 now applies and shows that ϕ is an isomorphism. It obviously commutes with the action of $\mathcal{U}(\mathfrak{p}^-)$. The verification of the commutativity with respect to \mathfrak{k} is also straightforward. ■

Remark 2.14. This gives a formula for the multiplicity for the weight spaces $U^\lambda[\mu]$ of U^λ . Let $\lambda_0 = \lambda$ and $\lambda_i, (0 \leq i \leq r)$ be the weights of F with k_i as the multiplicity of λ_i . Let $\gamma_1, \dots, \gamma_q$ be the distinct totally positive roots. For any linear function ν on \mathfrak{h} let $N(\nu)$ be the number of distinct ways of writing $\nu = m_1 \gamma_1 + \dots + m_q \gamma_q$ where the m_j are integers ≥ 0 . Then

$$\dim U^\lambda[\mu] = \sum_{0 \leq i \leq r} k_i N(\mu_i - \lambda).$$

Remark 2.15. There is a criterion for the Verma module V^λ to be irreducible, namely that

$$(\lambda + \delta)(H_\gamma) \in / \{1, 2, \dots\} \quad \text{for all } \gamma \in P.$$

This is due to M. Dufflo [11] and it is a variant of the similar condition for the spherical principal series for a complex group to be irreducible, due to K. R. Parthasarathy, R. Ranga Rao and V. S. Varadarajan [38].

2.7. Totally positive roots, real HC pairs, and complex geometry

In practice the HC pairs arise by complexification of real HC pairs. Let $(\mathfrak{g}_0, \mathfrak{k}_0)$ be a real HC pair, \mathfrak{g}_0 simple. We assume that G_0 is a connected real Lie group with Lie algebra \mathfrak{g}_0 and K_0 is the analytic subgroup defined by \mathfrak{k}_0 . We also assume that:

1. $\text{Ad } K_0$ is the maximal compact subgroup of $\text{Ad } G_0$;
2. G_0 and K_0 have the same rank: $\mathfrak{h}_0 \subset \mathfrak{k}_0 \subset \mathfrak{g}_0$, where \mathfrak{h}_0 is a CSA for both.

Let A_0 be the Cartan subgroup (CSG) of \mathfrak{h}_0 in G_0 so that it is centralizer of \mathfrak{h}_0 in G_0 . So $\text{Ad } A_0$ is compact and the roots of $(\mathfrak{g}, \mathfrak{k})$ are the eigencharacters of $\text{Ad } A_0$. Thus all roots are pure imaginary on \mathfrak{h}_0 . Now, \mathfrak{g}_0 being a real form of \mathfrak{g} , there is a conjugation $X \rightarrow X^{\text{conj}}$ on \mathfrak{g} , which is an antilinear bijection of \mathfrak{g} with itself preserving brackets, with \mathfrak{g}_0 as the set of its fixed points. Since it is the identity on \mathfrak{h}_0 , it follows by conjugating $[H, X_\beta] = \beta(H)X_\beta$ that $\mathfrak{g}_\beta^{\text{conj}} = \mathfrak{g}_{-\beta}$. In the above setting $(\mathfrak{g}_0, \mathfrak{k}_0)$ is a HC pair but K_0 need not be semisimple, i.e., the center \mathfrak{c}_0 may be $\neq 0$. If \mathfrak{k}_0 has zero center, the group K_0 is compact even when G_0 is the simply connected group corresponding to \mathfrak{g}_0 and G_0 has finite center.

Lemma 2.16. *We have $[\mathfrak{p}_0, \mathfrak{p}_0] = \mathfrak{k}_0$ and \mathfrak{p}_0 is irreducible as a \mathfrak{k}_0 -module. Moreover the H_β for noncompact roots β span $i\mathfrak{h}_0$ over \mathbb{R} .*

Proof. The first statement follows from the fact that $[\mathfrak{p}_0, \mathfrak{p}_0] \oplus \mathfrak{p}_0$ is a nonzero ideal in \mathfrak{g}_0 and so has to be \mathfrak{g}_0 . Indeed, since \mathfrak{p}_0 is stable under \mathfrak{k}_0 , so is $[\mathfrak{p}_0, \mathfrak{p}_0]$, so that stability under \mathfrak{k}_0 is clear; the stability under \mathfrak{p}_0 is obvious. Since $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$ is spanned by $[X_\beta, X_\gamma]$ for β, γ noncompact, it follows that \mathfrak{h} is spanned by the $[X_\beta, X_{-\beta}]$ for noncompact β which proves the last statement. Note that as all roots are pure imaginary on \mathfrak{h}_0 , $H_\gamma \in i\mathfrak{h}_0$ for all roots γ . The irreducibility of \mathfrak{p}_0 is however less trivial. The Killing form is positive definite on \mathfrak{p}_0 and negative definite on \mathfrak{k}_0 , and is \mathfrak{g}_0 invariant. We write $\mathfrak{p}_0 = \bigoplus_{1 \leq j \leq r} \mathfrak{p}_j$ where the \mathfrak{p}_j are mutually orthogonal and irreducible for \mathfrak{k}_0 . Let $\mathfrak{k}_j := [\mathfrak{p}_j, \mathfrak{p}_j]$, $\mathfrak{g}_j := \mathfrak{k}_j \oplus \mathfrak{p}_j$. We shall prove that the \mathfrak{g}_j are ideals in \mathfrak{g}_0 and that $\mathfrak{g}_0 = \bigoplus_{1 \leq j \leq r} \mathfrak{g}_j$. If this is so, then $j = 1$ and we are done. Fix j, k with $1 \leq j, k \leq r$ and $j \neq k$. We shall prove the following in succession.

(A) $[\mathfrak{p}_j, \mathfrak{p}_k] = 0, [\mathfrak{k}_j, \mathfrak{p}_k] = 0, [\mathfrak{k}_j, \mathfrak{k}_k] = 0, j \neq k$.

Let $X \in \mathfrak{k}_0, Y_i \in \mathfrak{p}_i, i = j, k$. Then $B(X, [Y_j, Y_k]) = B([X, Y_j], Y_k) = 0$ as $[X, Y_j] \in \mathfrak{p}_j$. Since this is true for all $X \in \mathfrak{k}_0$, we must have $[Y_j, Y_k] = 0$. If $Y_j, Z_j \in \mathfrak{p}_j, Y_k \in \mathfrak{p}_k$ and $X = [Y_j, Z_j]$, then $[X, Y_k] = -[[Z_j, Y_k], Y_j] - [[Y_k, Y_j], Z_j] = 0$ by the previous result.

If $Y_j, Z_j \in \mathfrak{p}_j, X_k \in \mathfrak{k}_k$, then

$$[[Y_j, Z_j], X_k] = -[[Z_j, X_k], Y_j] - [[X_k, Y_j], Z_j] = 0$$

by the previous result.

(B) $\mathfrak{k}_j \perp \mathfrak{k}_k$.

Let $Y_j, Z_j \in \mathfrak{p}_j, X_k \in \mathfrak{k}_k$. Then $B([Y_j, Z_j], X_k) = -B(Z_j, [Y_j, X_k]) = 0$ since $[Y_j, X_k] = 0$. So $\mathfrak{g}_1, \dots, \mathfrak{g}_r$ are mutually orthogonal and $[\mathfrak{g}_j, \mathfrak{g}_k] = 0$ for $j \neq k$. Since B is separately definite on \mathfrak{k}_0 and \mathfrak{p}_0 , the \mathfrak{k}_j and \mathfrak{p}_j are linearly independent among themselves, from which it follows easily that the \mathfrak{g}_j are linearly independent. We now claim that the \mathfrak{g}_j are subalgebras. Since

$$[\mathfrak{k}_j, \mathfrak{p}_j] \subset [\mathfrak{k}, \mathfrak{p}_j] \subset \mathfrak{p}_j$$

and $[\mathfrak{p}_j, \mathfrak{p}_j] \subset \mathfrak{k}_j$, it is enough to verify that $[\mathfrak{k}_j, \mathfrak{k}_j] \subset \mathfrak{k}_j$; but $[\mathfrak{k}_0, \mathfrak{p}_j] \subset \mathfrak{p}_j$ so that $[\mathfrak{k}_0, \mathfrak{k}_j] \subset \mathfrak{k}_j$, and $[\mathfrak{k}_j, \mathfrak{k}_j] \subset [\mathfrak{k}_0, \mathfrak{k}_j] \subset \mathfrak{k}_j$. As $[\mathfrak{p}_0, \mathfrak{p}_0] = \mathfrak{k}_0, [\mathfrak{p}_j, \mathfrak{p}_k] = 0 (j \neq k)$, we have

$\mathfrak{k}_0 = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r$, hence $\mathfrak{g}_0 = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$. The \mathfrak{g}_j are subalgebras and $[\mathfrak{g}_j, \mathfrak{g}_k] = 0$ for $j \neq k$. Hence the \mathfrak{g}_j are ideals. ■

Lemma 2.17. *Let \mathfrak{g}_0 be simple. Then \mathfrak{c}_0 is either 0 or has dimension 1. In the latter case we can find a $J \in \mathfrak{c}_0$, unique up to a sign, such that $\text{ad}(J)^2 = -I$ on \mathfrak{p}_0 . Moreover $\beta(iJ) = \pm 1$ for all non compact roots β .*

Proof. Suppose $\mathfrak{c}_0 \neq 0$. Since \mathfrak{c}_0 is compact, the action of \mathfrak{c}_0 on \mathfrak{p}_0 is completely reducible with pure imaginary eigenvalues. The eigenvalues are of the form 0 or $\pm im_j$ where m_j are real non zero linear functions on \mathfrak{h}_0 . The eigenspaces are stable under \mathfrak{k}_0 and so \mathfrak{p}_0 is a direct sum of \mathfrak{p}_j where the \mathfrak{p}_j are stable under \mathfrak{k}_0 and the eigenvalues on $\mathbb{C} \cdot \mathfrak{p}_j$ are either 0 or $\pm im_j$. By Lemma 2.16, there is only one of the \mathfrak{p}_j . If 0 is an eigenvalue, then \mathfrak{c}_0 centralizes \mathfrak{p}_0 , hence \mathfrak{g}_0 . This is not possible and so the eigenvalues of \mathfrak{c}_0 are $\pm im$. If $\dim(\mathfrak{c}_0) > 1$ we can find a nonzero $H \in \mathfrak{c}_0$ such that $m(H) = 0$ so that H centralizes \mathfrak{g}_0 , which is not possible. So $\dim(\mathfrak{c}_0) = 1$. The existence of J and its uniqueness are now obvious. Since J has eigenvalues $\pm i$ on \mathfrak{p} , and \mathfrak{p} is spanned by the X_β for non compact roots β , we have $\beta(iJ) = \pm 1$ for all β . ■

Remark 2.18. The tangent space to $S_0 := G_0/K_0$ at $\bar{1}$, the image of 1 in the canonical map $G_0 \rightarrow G_0/K_0$, is isomorphic to \mathfrak{p}_0 as a K_0 module. Thus $\pm J \in \text{End}(\mathfrak{p}_0)$ and satisfies $(\pm J)^2 = -I$ and is fixed under the adjoint action of K_0 on \mathfrak{p}_0 , hence defines K_0 -invariant sections $\pm J$ of the endomorphism bundle of the tangent bundle of S_0 , satisfying $(\pm J)^2 = -I$. Thus we have defined two canonical almost complex structures on S_0 . We shall see later that these are actually integrable and so define G_0 -invariant complex structures on S_0 .

We also notice that the two canonical complex structures are in bijection with the admissible simple systems of \mathfrak{g} ; this fact is true more in general, for a non necessarily maximal compact K_0 , as detailed in [1] and refs. therein. In this case, in fact, the complex structures on G_0/K_0 are in bijection with simple systems of generalized root systems, as studied by Kostant in [34] and later on in [9], [10].

We now state a result, which is a refinement of Theorem 2.9 in this context.

Proposition 2.19. *Let the notation be as above.*

- (1) *If $\mathfrak{c}_0 = 0$ there are no admissible positive systems.*
- (2) *If $\dim(\mathfrak{c}_0) = 1$, there are exactly two admissible positive systems containing a given positive system P_k of compact roots. For a fixed choice of J these are $P^\pm = P_k \cup \pm Q$ where Q is the P set of non compact roots taking the value 1 at iJ . The subspaces $\mathfrak{p}^\pm := \sum_{\beta \in \pm Q} \mathfrak{g}_\beta$ are stable under \mathfrak{k} , abelian, and irreducible.*
- (3) *Finally, if $\{\alpha_1, \dots, \alpha_{\ell-1}\}$ are the simple roots in P_k , we can find a unique non compact root β such that $\{\alpha_1, \dots, \alpha_{\ell-1}, \beta\}$ is the set of simple roots of P^+ , $\{\alpha_1, \dots, \alpha_{\ell-1}, -\beta\}$ being then the simple roots in P^- .*

Proof. (1) Assume first that $\mathfrak{c}_0 = 0$. Let P be a positive system and let $R \subset P$ be a non empty set such that $L = \sum_{\beta \in R} \mathfrak{g}_\beta$ is \mathfrak{k} -stable. If W_k is the subgroup of the Weyl group generated by the compact roots, it is then clear that R , the set of weights for \mathfrak{k} in L , is stable under W_k . Fix a $\beta \in R$ and let $\lambda = \sum_{s \in W_k} s \cdot \beta$. Then

λ is invariant under W_k , and the condition $s_\alpha \lambda = \lambda$ for all compact roots α gives $\lambda(H_\alpha) = 0$ for all compact roots α , so that $\lambda = 0$, since the H_α span \mathfrak{h} . This is impossible since all the $s \cdot \beta$ are > 0 and so their sum cannot be 0.

(2) Let us now suppose that $\dim(\mathfrak{c}_0) = 1$. If α is a compact root and $\beta \in \pm P_n$ it is immediate that $\beta + \alpha$ and β have the same value ± 1 at iJ . If $\beta + \alpha$ is a root, it must be non compact and so is in $\pm P_n$. Hence $[X_\alpha, X_\beta]$ is either 0 or in \mathfrak{p}^\pm , showing that \mathfrak{p}^\pm are stable under \mathfrak{k} . If β, γ are both in $\pm P_n$, $\beta + \gamma$ takes the value ± 2 at iJ and so is not a root, showing that \mathfrak{p}^\pm are abelian. Now $\mathfrak{g}_\beta^{\text{conj}} = \mathfrak{g} - \beta$ and so $\mathfrak{p}^- = (\mathfrak{p}^+)^{\text{conj}}$. So, if there is a proper subspace \mathfrak{q} of \mathfrak{p}^+ stable under \mathfrak{k} , then $(\mathfrak{q} + \mathfrak{q}^{\text{conj}}) \cap \mathfrak{p}_0$ is a proper subspace of \mathfrak{p}_0 stable under \mathfrak{k}_0 , which is not possible since \mathfrak{p}_0 is irreducible. So \mathfrak{p}^\pm are irreducible.

(3) If the positive system P' contains P_k and is admissible, there is a non empty subset $R \subset P_n$ such that $L := \sum_{\beta \in R} \mathfrak{g}_\beta$ is stable under \mathfrak{k} . If $\beta \in R$, then $\beta \in P^\pm$ and so $L \cap \mathfrak{p}^\pm \neq 0$ showing that L contains one of \mathfrak{p}^\pm , hence equal to one of them. Hence $P' = P^\pm$. To find the simple roots of \mathfrak{p}^+ , let β be the lowest weight (relative to P_k) for \mathfrak{p}^+ as a \mathfrak{k} -module. If γ is a non compact root in P_n , it is then clear that $\gamma - \beta$ is a sum of positive compact roots and so $\{\alpha_1, \dots, \alpha_{\ell-1}, \beta\}$ is the simple system for \mathfrak{p}^+ ; changing β to $-\beta$, we get the simple system for P^- . The uniqueness of β is obvious. ■

Remark 2.20. The relation $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^- = \mathfrak{p}^+ \oplus (\mathfrak{p}^+)^{\text{conj}}$ shows that the action of K_0 on \mathfrak{p} splits as $\sigma \oplus \sigma^{\text{conj}}$ where σ is the irreducible action of K_0 on \mathfrak{p}^+ . If $L \in \text{End}(\mathfrak{p}_0)$ commutes with K_0 , then L is a scalar λ on \mathfrak{p}^+ and λ^{conj} on \mathfrak{p}^- . If now in addition we have $L^2 = -I$ it is immediate that $\lambda = \pm i$. Thus $L = \pm J$. This shows that the only G_0 -invariant almost complex structures on $S_0 = G_0/K_0$ are those defined by $\pm J$. This also shows that the type of the \mathbb{R} -representation of K_0 on \mathfrak{p}_0 is \mathbb{R} , namely, that the commutator is inside $\mathbb{C}_\mathbb{R}$, the complex numbers viewed as a real algebra.

2.8. Unitarity

We extend the homomorphism $\beta : \mathcal{U}[0] \rightarrow \mathcal{U}(\mathfrak{h})$ constructed in Sec. 2.5 to a linear map $\mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{h})$ by making it 0 on $\mathcal{U}[q]$ for $q \neq 0$. Let V be a \mathfrak{g} -module. Then V is said to be *unitary* if there is a positive definite hermitian product $(,)$ for V such that

$$(Xu, v) + (u, Xv) = 0, \quad (u, v \in V, X \in \mathfrak{g}_0)$$

It is a theorem of Harish-Chandra that if V is a unitary HC module and H is the completion of V in the norm $\|x\| = +\sqrt{(x, x)}$, and if the \mathfrak{k}_0 -action on V lifts to a K_0 -action, there is a unitary representation of G_0 in H such that V is the space of K_0 -finite vectors of H as a \mathfrak{g} -module (see [42], Ch. 5, Thm. 20, p. 121).

To discuss unitarity it is convenient to define the adjoint operation directly on $\mathcal{U}(\mathfrak{g})$. The map $X \mapsto -X$ extends to an antiautomorphism of $\mathcal{U}(\mathfrak{g}_0)$; it can then be extended to a conjugate linear antiautomorphism of $\mathcal{U}(\mathfrak{g})$. It is denoted by $a \mapsto a^*$ and has the following properties:

- (i) $a^{**} = a$;
- (ii) $(ab)^* = b^*a^*$;
- (iii) a^* is conjugate linear in a ;
- (iv) $X^* = -X$ for all $X \in \mathfrak{g}_0$.

It is uniquely determined by these requirements. The unitarity condition is now

$$(au, v) = (u, a^*v), \quad (a \in \mathcal{U}(\mathfrak{g}), u, v \in V).$$

Lemma 2.21. *We can choose the root vectors $X_\gamma \in \mathfrak{g}_\gamma$ in such a way that $[X_\gamma, X_{-\gamma}] = H_\gamma$ and*

$$X_\gamma^* = \begin{cases} X_{-\gamma}, & \gamma \text{ compact} \\ -X_{-\gamma}, & \gamma \text{ non compact} \end{cases} \quad X_\gamma^{\text{conj}} = \begin{cases} -X_{-\gamma}, & \gamma \text{ compact} \\ X_{-\gamma}, & \gamma \text{ non compact} \end{cases}$$

where conj is the conjugation of \mathfrak{g} defined by \mathfrak{g}_0 .

Proof. Let $0 \neq X_\gamma \in \mathfrak{g}_\gamma$ be arbitrary to start with, but satisfying $[X_\gamma, X_{-\gamma}] = H_\gamma$. The relation $[H, X_\gamma] = \gamma(H)X_\gamma$ gives, on applying $*$, $[H, X_\gamma^*] = -\gamma(H)X_\gamma^*$, so that $X_\gamma^* = c(\gamma)X_{-\gamma}$. As $*$ is involutive we get $c(\gamma)^{\text{conj}}c(-\gamma) = 1$, where conj denotes complex conjugation. On the other hand, from the standard theory, $[X_\gamma, X_{-\gamma}] = B(X_\gamma, X_{-\gamma})H'_\gamma$ where B is the Killing form and H'_γ is the image of γ under the isomorphism $\mathfrak{h}^* \cong \mathfrak{h}$ induced by B . Hence $H_\gamma = B(X_\gamma, X_{-\gamma})H'_\gamma$. As $H_\gamma = cH'_\gamma$ where $c > 0$, we see that $B(X_\gamma, X_{-\gamma})$ is real and > 0 . On the other hand, we claim that if $X \neq 0$ is in \mathfrak{g} , $B(X, X^*)$ is > 0 or < 0 according as $X \in \mathfrak{k}$ or $X \in \mathfrak{p}$. To see this, write $X = Y + iZ$ where $Y, Z \in \mathfrak{g}_0$; then

$$B(X, X^*) = B(Y + iZ, -Y + iZ) = -B(Y, Y) - B(Z, Z)$$

which proves our claim (since $Y, Z \in \mathfrak{k}_0$ or \mathfrak{p}_0 according as $X \in \mathfrak{k}$ or \mathfrak{p}). This proves the claim. But now $B(X_\gamma, X_\gamma^*) = c(\gamma)B(X_\gamma, X_{-\gamma})$, so that, from our earlier remark we infer that $c(\gamma)$ is > 0 or < 0 according as γ is compact or non compact. In any case $c(\gamma)$ is real and so $c(\gamma)c(-\gamma) = 1$. Write $X'_\gamma = |c(\gamma)|^{-1/2}X_\gamma$. Then $[X'_\gamma, X_{-\gamma}] = H_\gamma$ still. On the other hand,

$$\begin{aligned} (X'_\gamma)^* &= |c(\gamma)|^{-1/2}X_\gamma^* = |c(\gamma)|^{-1/2}c(\gamma)|c(-\gamma)|^{1/2}X'_{-\gamma} \\ &= c(\gamma)|c(\gamma)|^{-1}X'_{-\gamma} = \text{sgn}c(\gamma)X'_{-\gamma} \end{aligned}$$

This proves the first assertion. The second is immediate from $X_\gamma^{\text{conj}} = -X_\gamma^*$, by the very definitions. ■

Let $\beta : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{h})$ be the Harish-Chandra homomorphism as in Sec. 2.5. We state here a key result of Harish-Chandra, sketching the proof, see [20], Sec. 7, Lemma 18 for further details.

Theorem 2.22. *Let $\lambda \in \mathfrak{h}^*$ and let π_λ be the irreducible highest weight module of highest weight λ with respect to an admissible system. The π_λ is unitary if and only if $\beta(a^*a)(\lambda) \geq 0$ for all $a \in \mathcal{U}(\mathfrak{g})$. In particular it is necessary that:*

$$\begin{aligned} \lambda(H_\gamma) &\in \mathbb{Z}_{\geq 0}, \quad \text{for compact positive } \gamma, \\ \lambda(H_\gamma) &\leq 0 \quad (\text{real}) \quad \text{for non compact positive } \gamma. \end{aligned}$$

Proof. Assume that π_λ acting on V is unitary and write v for the highest weight vector. Since, for $H \in \mathfrak{h}_0$,

$$\lambda(H)(v, v) = (Hv, v) = -(v, Hv) = -\lambda(H)^{\text{conj}}(v, v),$$

it follows that λ is pure imaginary on \mathfrak{h}_0 . Hence all weights are pure imaginary on \mathfrak{h}_0 (as the roots are so). A similar argument shows that the weight spaces are mutually orthogonal. By definition of β it follows that for $c \in \mathcal{U}(\mathfrak{g})$, we have $(cv, v) = \beta(c)(\lambda)(v, v)$. In particular,

$$0 \leq (av, av) = \beta(a^*a)(\lambda)(v, v)$$

from which we get the necessary part.

With X_γ as in Lemma 2.21, we have $X_{-\gamma}^*X_{-\gamma} = \pm X_\gamma X_{-\gamma}$ according as γ is compact or not. Hence

$$\beta(X_{-\gamma}X_{-\gamma}) = \beta(\pm X_\gamma X_{-\gamma}) = \pm H_\gamma$$

so that $\lambda(H_\gamma)$ is ≥ 0 or ≤ 0 according as γ is compact or not. For the sufficiency of the condition for unitarity we define $(a, b) = \beta(b^*a)(\lambda)$ for $a, b \in \mathcal{U}(\mathfrak{g})$. Then (a, b) is sesqui-linear and (a, a) is real and ≥ 0 for all a . Hence (a, b) is Hermitian symmetric and $(a, a) \geq 0$ for all a . Let R be the radical of this Hermitian form: $a \in R$ if and only if $(a, a) = 0$ or equivalently $(a, b) = 0$ for all $b \in \mathcal{U}(\mathfrak{g})$. Moreover $(ca, b) = \beta(b^*ca) = (a, c^*b)$. We claim that R has the following properties.

- (i) R is a left ideal.
- (ii) X_γ ($\gamma > 0$), $H - \lambda(H)$ ($H \in \mathfrak{h}_0$) are in R .

Let us prove these assertions. Since $(1, 1) = 1$ we have $1 \notin R$. If $a \in R$ and $c \in \mathcal{U}(\mathfrak{g})$, $(ca, b) = (a, c^*b) = 0$ for all b and so $ca \in R$, proving (i). For $\gamma > 0$, $(X_\gamma, b) = \beta(b^*X_\gamma) = 0$ for all b and so $X_\gamma \in R$.

$$(H - \lambda(H), H - \lambda(H)) = (-H + \lambda(H))(H - \lambda(H))(\lambda) = 0,$$

proving (ii). At this stage we know that $W = \mathcal{U}(\mathfrak{g})/R$ is a $\mathcal{U}(\mathfrak{g})$ -module of highest weight λ and 1 as the corresponding weight vector, and that W is unitary. We claim that W is irreducible, hence isomorphic to π_λ . Suppose $W' \neq W$ is a submodule. Then W'^\perp is also a submodule. Since the weight spaces are mutually orthogonal and finite dimensional it follows that $W = W' \oplus (W')^\perp$. Since $W' \neq W$, we must have $1 \notin W'$. As

$$W[\lambda] = W'[\lambda] \oplus (W')^\perp[\lambda],$$

we must have $1 \in W'^\perp$. Hence $(W')^\perp = W$, showing that $W' = 0$. ■

Observation 2.23. These remarks imply that the only irreducible finite dimensional unitary module is the trivial representation. Suppose π_λ is finite dimensional and unitary. Then $\lambda(H_\gamma) \geq 0$ for all roots $\gamma > 0$. By Theorem 2.22, we then conclude that $\lambda(H_\gamma) = 0$ for all non compact roots γ . By Lemma 2.16 $\lambda = 0$, hence π_λ is the trivial one dimensional representation.

Remark 2.24. The importance of total positivity is elucidated by the Cor. 1 in [20], Sec.7, here stated in our setting as part of our Thm.2.22. In fact, in Cor. 1 Harish-Chandra gives the following necessary conditions for π_λ to be unitary:

- (1) $\lambda(H_\alpha) \in \mathbb{Z}_{\geq 0}$, α compact;
- (2) $\lambda(H_\beta) = 0$ non compact positive, but not totally positive;
- (3) $\lambda(H_\beta) < 0$ (real) non compact positive, but totally positive.

In our setting, since we assume the positive system to be admissible, non compact positive roots are necessarily totally positive. However, Harish-Chandra considers a more general situation, and indeed the Theorem 2.22 could be proven, with a bit more effort, to accomodate this more general setting. This was later on generalized by Enright, who considers in [12] non highest weight representations, where there exists non totally positive non compact roots (see also next remark).

Remark 2.25. By using global methods Harish-Chandra proved that when $\dim(\mathfrak{c}_0) = 1$ and $(\lambda + \delta)(H_\gamma) \leq 0$ for all non compact positive roots γ , then the module U^λ of Theorem 2.9 is unitary. The full set of unitary highest weight modules was later on determined by Enright, Howe, and Wallach [13], (see also [32], [39] and refs. therein). A generalization to the super setting is due to Jakobsen [28] (see also [5], [6] and refs. therein).

3. Representations of the group

3.1. Geometry

The objective now is to construct the representations of the group G_0 that correspond to the highest weight HC modules constructed in Sec. 2. Let $\mathfrak{g}_0 = \text{Lie}(G_0)$ and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition of \mathfrak{g}_0 ([31], Ch. VI, Sec. 2, p. 360), unique up to conjugation. It is obtained via Cartan involution θ_0 , $\theta_0|_{\mathfrak{k}_0} = \text{id}$, $\theta_0|_{\mathfrak{p}_0} = -\text{id}$, which also unique up to conjugation ([31], Ch. VI, Sec. 2, Cor. 6.19). \mathfrak{k}_0 is a maximal compact subalgebra of \mathfrak{g}_0 ; all such are conjugate in \mathfrak{g}_0 .

We shall eventually assume that $\text{rk}(\mathfrak{g}_0) = \text{rk}(\mathfrak{k}_0)$ and that the center \mathfrak{c}_0 of \mathfrak{k}_0 has $\dim(\mathfrak{c}_0) = 1$. But initially we drop the condition on the center of \mathfrak{k}_0 .

Lemma 3.1. *Let M be a Lie group (real or complex), A_1, A_2 Lie subgroups with $\text{Lie}(A_1) + \text{Lie}(A_2) = \text{Lie}(M)$. Then A_1A_2 is open in M . If $A_1 \cap A_2 = \{1\}$, then A_i are closed and $a_1, a_2 \mapsto a_1a_2$ is an analytic diffeomorphism of $A_1 \times A_2$ with A_1A_2 .*

Proof. Let $\mathfrak{a}_i = \text{Lie}(A_i)$, $\mathfrak{m} = \text{Lie}(M)$. If f is the map $a_1, a_2 \mapsto a_1a_2$ of $A_1 \times A_2$ into M , then df is submersive at $(1, 1)$ and hence everywhere since f intertwines the actions $(b_1, b_2): (a_1, a_2) \mapsto (b_1a_1, a_2b_2)$ and $(b_1, b_2): x \mapsto b_1xb_2$. Thus $A_1A_2 = f(A_1 \times A_2)$ is open in M . For the second part, note that when $A_1 \cap A_2 = \{1\}$, f is bijective and \mathfrak{m} is the direct sum $\mathfrak{a}_1 \oplus \mathfrak{a}_2$ so that df is also bijective; f is thus a diffeomorphism. So the A_i are closed in A_1A_2 , i.e., locally closed. This means that they are open in their closures, and being subgroups, are therefore closed. ■

Let G be a complex semisimple Lie group, $G_0 \subset G$ a connected real form. Let $R_0 \subset G_0$ a closed subgroup and let

$$\text{Lie}(G_0) = \mathfrak{g}_0, \quad \text{Lie}(G) = \mathfrak{g} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0, \quad \text{Lie}(R_0) = \mathfrak{r}_0.$$

Lemma 3.2. *Suppose there exists a complex Lie subalgebra $\mathfrak{q} \subset \mathfrak{g}$ with $\mathfrak{g}_0 + \mathfrak{q} = \mathfrak{g}$, $\mathfrak{g}_0 \cap \mathfrak{q} = \mathfrak{r}_0$. Assume that the subgroup $Q \subset G$ defined by \mathfrak{q} is closed. Then*

- (a) G_0Q is open in G .
- (b) If $R_1 = Q \cap G_0$, then $G_0/R_1 \cong G_0Q/Q$.
- (c) G_0/R_0 has a G_0 -invariant complex structure.

Proof. (a) follows from Lemma 3.1. The map $g, q \mapsto gq$ of $G_0 \times Q \rightarrow G$ induces a G_0 -equivariant map of $G_0/R_1 \cong G_0Q/Q$ with bijective differential. The complex structure on G_0/R_1 is the pull back from G_0Q/Q . But R_0 and R_1 have the same Lie algebra, namely \mathfrak{r}_0 , and so $G_0/R_0 \rightarrow G_0/R_1$ is a covering map. So the complex structure on G_0/R_1 can be pulled back to G_0/R_0 . ■

From now on we assume that \mathfrak{g}_0 is semisimple and use notation of Sec. 2. We take G to be simply connected. At first we assume only that \mathfrak{g}_0 and \mathfrak{k}_0 have the same rank and that $\mathfrak{h}_0 \subset \mathfrak{k}_0$ is a CSA for both \mathfrak{g}_0 and \mathfrak{k}_0 . Fix a positive system P of roots for $(\mathfrak{g}, \mathfrak{k})$. Define

$$\mathfrak{b}^\pm = \mathfrak{h} \oplus \sum_{\alpha \in \pm P} \mathfrak{g}_\alpha, \quad \mathfrak{n}^\pm = \bigoplus_{\alpha \in \pm P} \mathfrak{g}_\alpha.$$

Let B^\pm, N^\pm be the subgroups of G defined by $\mathfrak{b}^\pm, \mathfrak{n}^\pm$. We will drop the suffix “+” when clear from the context. Let A (resp. A_0) be the subgroup of G (resp. G_0) defined by \mathfrak{h} (resp. \mathfrak{h}_0). It follows from Lemma 3.2 that $\Gamma = N^-B^+ = N^-AN^+$ is open in G (the big Bruhat cell).

Define $\mathfrak{u} := \mathfrak{k}_0 \oplus i\mathfrak{p}_0$. Then \mathfrak{u} is a compact form of \mathfrak{g} ([31], Ch. VI, Sec. 1, proof of Thm. 6.11, Sec. 3 Cor. 6.22 and p. 360). Notice $\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u}$ and \mathfrak{u} is unique up to conjugation ([31], Ch. VI, Sec. 3, Cor. 6.22).

The corresponding subgroup U ($\mathfrak{u} = \text{Lie}(U)$) of G is compact and simply connected. The subgroup A_0 defined by \mathfrak{h}_0 is a torus and is a maximal torus of U as well as G_0 .

Lemma 3.3. *If \mathfrak{m}_0 is a real form of \mathfrak{g} containing \mathfrak{h}_0 , we have $\mathfrak{m}_0 + \mathfrak{b}^+ = \mathfrak{g}$. In particular M_0B^\pm is open in G , M_0 being the group defined by \mathfrak{m}_0 .*

Proof. If we denote the conjugation of \mathfrak{g} with respect to \mathfrak{m}_0 by $X \mapsto X^\sim$, then

$$\mathfrak{g}^\sim_\beta = \mathfrak{g}_{-\beta} \quad \text{and} \quad X_{-\beta} = X_{\beta^\sim} = (X_\beta + X_{\beta^\sim}) - X_\beta \in \mathfrak{m}_0 + \mathfrak{n},$$

showing that $\mathfrak{n}^- \subset \mathfrak{m}_0 + \mathfrak{n}^+$. So $\mathfrak{m}_0 + \mathfrak{b}^+$ contains $\mathfrak{g} = \mathfrak{n}^+ + \mathfrak{h} + \mathfrak{n}^-$. The second assertion now follow from Lemma 3.2. ■

Corollary 3.4. *The conjugation of \mathfrak{g} with respect to \mathfrak{m}_0 takes \mathfrak{g}_β to $\mathfrak{g}_{-\beta}$ for all roots β .*

Proof. Follows from the previous proof. ■

If we regard \mathfrak{g} as a Lie algebra over the reals, it is semisimple and $\mathfrak{g} = \mathfrak{u} + i\mathfrak{u}$ is its Cartan decomposition ([31], p. 360). Then $\mathfrak{g} = \mathfrak{u} + i\mathfrak{h}_0 + \mathfrak{n}^\pm$ are its Iwasawa decompositions ([31], Ch. VI, Sec. 4, Prop. 6.43 where $\mathfrak{u} = \mathfrak{k}$). In our setting, Iwasawa decomposition is easily proven by a dimension counting argument. Thus $G = UA_{\mathbb{R}}N^\pm$ are the global Iwasawa decompositions of G ([31], Ch. VII, Sec. 2, Prop. 7.31), where we write $A_{\mathbb{R}}$ for the subgroup defined by $i\mathfrak{h}_0$. The exponential maps:

$$\exp : \mathfrak{n}^\pm \rightarrow N^\pm, \quad i\mathfrak{h} \rightarrow A_{\mathbb{R}}$$

are analytic diffeomorphisms (resp. complex and real). This is not difficult to see: $N^+ \cong \prod_{\alpha > 0} N_\alpha$, N_α one parameter subgroups generated by X_α root vector of α and the exponential map (for matrix groups) is a finite sum for nilpotent subgroups. For $i\mathfrak{h}$ it is immediate: choose \mathfrak{h} as diagonal matrices in \mathfrak{g} .

Also A normalizes N^\pm . We write conj for the conjugation of \mathfrak{g} and G with respect to G_0 and \mathfrak{g}_0 respectively.

Lemma 3.5. *We have $N^- \cap B^+ = \{1\}$, $A \cap N^+ = \{1\}$. In particular the map*

$$\psi : N^- \times A \times N^+ \longrightarrow G, \quad \psi(n^-, h, n^+) = n^- h n^+$$

is an analytic diffeomorphism onto the big Bruhat cell Γ , and A , N^\pm are closed.

Proof. If $\exp Z \in B^+$ for some $Z \in \mathfrak{n}^-$, then $e^{\text{ad}(Z)}$ normalizes \mathfrak{b}^+ , hence $[Z, \mathfrak{b}^+] \subset \mathfrak{b}^+$ which is impossible unless $Z = 0$. If $\exp Z \in A$ for some $Z \in \mathfrak{n}^+$, $e^{\text{ad}(Z)}$ is semisimple and unipotent, hence $\text{ad}(Z) = 0$, or $Z = 0$. ■

Lemma 3.6. *We have $G_0 \cap B^\pm = A_0$ and $G_0 B^\pm$ are open in G while $G = UB^+$. Moreover $G_0/A_0 \cong G_0 B^\pm/B^\pm$ acquires a G_0 -invariant complex structure. Finally, N^- may be viewed as a section for Γ/B^+ , and the left action of A on Γ/B^+ is given by $h, nB^+ \mapsto (hnh^{-1})B^+$.*

Proof. Let $a = hn^+ \in G_0$ where $h \in A$, $n^+ \in B^+$. Then $a = h^{\text{conj}} n^-$ where $n^- = n^{\text{conj}} \in N^-$. So $hn^+ = h^{\text{conj}} n^-$ giving $(h^{\text{conj}})^{-1} hn^+ = n^-$. Thus $n^- = n^+ = 1$, $h = h^{\text{conj}}$ which give $a \in A_0$. The rest follow from Lemmas 3.2 and 3.3. Since B^+ is closed and U is compact, UB^+ is closed, and as it is also open, it is G . The last statement is trivial. ■

3.2. Holomorphically induced sheaves and group representations

The characters of A_0 extend uniquely to holomorphic characters of A . Since G is simply connected, these are precisely of the form

$$\chi_\lambda : \exp H \mapsto e^{\lambda(H)}$$

where λ is integral, i.e., $\lambda(H_\gamma)$ is an integer for all roots γ . We write χ_λ again for the character $hn^+ \mapsto \chi_\lambda(h)$ of B^+ . Throughout what follows, λ will be a fixed holomorphic character of A and we shall suppress mentioning λ whenever there is no confusion. Let π be the natural map $G \longrightarrow G/B^+ =: X$. For fixed $\chi = \chi_\lambda$ and any open set $E \subset X$ we define $L(E)$ to be the linear space of all functions f on $\pi^{-1}(E)$ that are holomorphic and satisfy:

$$f(ub) = f(u)\chi(b) \quad \text{for all } b \in B, u \in \pi^{-1}(E) \quad (4)$$

(see [21] Sec. 4,5, p. 12–14, where $L(E)$ is denoted by \mathcal{H}_ξ , $\xi = \chi = e^\lambda$ in our notation, for $E = G_0 B$).

By abuse of notation we also write sometimes $L(\pi^{-1}(E))$ for $L(E)$. Then $L : E \longrightarrow L(E)$ is a sheaf on X . For any E , $L(E)$ is a Fréchet space in the topology of uniform convergence on compact sets. This sheaf may be naturally interpreted as the sheaf of sections of a holomorphic line bundle on X canonically associated to χ . Let G_0 be a connected real form of G . Write $S = G_0 B^+$. The restriction of L to S/B^+ (which is open in X) is a sheaf on S/B^+ . The group G_0 acts naturally from the left on the space $L(S/B^+)$ of global sections of this sheaf. It is easy to verify that this action gives a representation of G_0 on $L(S/B^+)$. Our goal is to study this representation by analyzing the action of A_0 . Note that at this stage we are not asserting that this space is even $\neq 0$. For the definition of a representation of a locally compact group on a Fréchet space, or more generally, a complete locally convex space, see [24] p. 5–14 (for basic facts on Fréchet spaces see [41]). In our case

the holomorphy implies that all vectors are C^∞ , that is the action of the group on them is smooth; we do not use this fact however.

Let F be a Fréchet space, H a Lie group, and $K \subset H$ a compact subgroup of H . In our applications we will have $H = G_0$ and $K = A_0$ or K_0 . Let R a representation of H in F . For any irreducible character τ of K of degree $d(\tau)$, we define the operator

$$P(\tau) = d(\tau) \int_K \tau(k)^{\text{conj}} R(k) dk \quad \int_K dk = 1.$$

Then $P(\tau)$ is a continuous projection $F \rightarrow F(\tau)$ where $F(\tau) = P(\tau)F$. $F(\tau)$ is a closed subspace of F . We have $P(\tau)P(\tau') = 0$, ($\tau \neq \tau'$). The $P(\tau)$ commute with the K -action, and further, commute with any continuous endomorphism E of F that commutes with K . It is easy to show that $F(\tau)$ is the algebraic linear span of all finite dimensional subspaces of F which are stable under K and on which K acts irreducibly according to a representation with character τ . The $F(\tau)$ are linearly independent (see [42]).

We say that the representation R in a vector space V is K -finite, if each $F(\tau)$ is finite dimensional. In this case we write $F^0 = F_K^0 = \sum_\tau F(\tau)$. Let F^∞ be the subspace of C^∞ vectors $v \in V$ for R , that is vectors for which the map:

$$x \mapsto R(x)v \quad \text{is } C^\infty, \forall x \in G$$

Then, F^∞ is dense in F (Cor. 1 in [24], p. 7). Thus, $F^\infty(\tau) := P(\tau)F^\infty$ is dense in $F(\tau)$. For any $f \in F$ let $f_\tau = P(\tau)f$. Then, $\sum_\tau f_\tau$ is called the *Fourier series* of f . We have that, when $f \in F^\infty$,

$$f = \sum_\tau f_\tau, \quad (f \in F^\infty)$$

where the series converges absolutely ([24], Sec. 3, Lemma 5).

We now need some standard results; we prefer to give the proofs to keep our paper self contained for our Fréchet representation setting, making an ad hoc reference not readily available.

Lemma 3.7. *Suppose $F_0 \subset F$ is a dense subspace, with $F_0 = \sum_\tau L_\tau$ where the sum is algebraic and all L_τ are finite dimensional with $L_\tau \subset F(\tau)$. Then $L_\tau = F(\tau)$ for all τ , $F^0 = \sum_\tau F(\tau)$, and $F^0 \subset F^\infty$. Suppose $L \subset K$ is a compact subgroup and F is L -finite; then F is K -finite and $F_K^0 = F_L^0$.*

Proof. Clearly $L_\tau = P(\tau)F_0$ is dense in $P(\tau)F = F(\tau)$. But L_τ is closed in F because it is finite dimensional, so that $F(\tau) = L_\tau$. Since F^∞ is dense in F , the same argument shows that $F^\infty(\tau)$ is dense in $F(\tau)$, hence $F^\infty(\tau) = F(\tau)$. In particular $F_0 = \sum_\tau F(\tau) \subset F^\infty$. For the last assertion let s be an irreducible representation of K with character τ . The restriction of s to L will contain irreducible representation (inequivalent) t_1, \dots, t_r of L with characters $\sigma_1, \dots, \sigma_r$ respectively. It is then clear that $F_K(\tau) \subset \sum_i F_L(\sigma_i)$. This proves that $F_K(\tau)$ is finite dimensional, hence that F is K -finite, and that $F_K^0 \subset F_L^0$. To prove equality here, let us consider $F_L(\sigma)$ where σ is the character of an irreducible representation t of L . We know that $F_L(\sigma) \subset F_K^\infty$ and so each $f \in F_L(\sigma)$ has an absolutely convergent expansion $f = \sum_\tau f_\tau$ where the τ vary over the irreducible characters of K . It is enough to show that $P(\tau)F_L(\sigma) = 0$ for almost all τ ; for then by the

Fourier expansion, $F_L(\sigma)$ is contained in the sum of finitely many $F(\tau)$ and so is contained in F_K^0 . Indeed, suppose for some τ we have $P(\tau)F_L(\sigma) \neq 0$. Since $P(\tau)$ commutes with K , hence with L , this means that $P(\tau)F_L(\sigma) \subset F(\tau) \cap F_L(\sigma)$. Actually we have equality here; for, if $v \in F(\tau) \cap F_L(\sigma)$, then $P(\tau)v = v$. Let $S(\tau) = F(\tau) \cap F_L(\sigma)$. Then $S(\tau) \neq 0$, and, when τ varies over the set of characters for which $P(\tau)F_L(\sigma) \neq 0$, the $S(\tau)$ are non-zero linear subspaces of $F_L(\sigma)$ which are linearly independent (as the $F(\tau)$ are linearly independent). Hence the number of such τ cannot exceed $\dim F_L(\sigma)$. ■

3.3. Action of $\mathcal{U}(\mathfrak{g})$ on holomorphic sections

For any (second countable) complex manifold M , write $\text{Hol}(M)$ for the algebra of holomorphic functions on M equipped with the topology of uniform convergence on compact sets. Then $\text{Hol}(M)$ is a Fréchet space [41]. If H is a locally compact group acting on M via holomorphic diffeomorphisms of M , then the corresponding natural action of H on $\text{Hol}(M)$ is a representation (see [24] Lemma 1). Let

$$\Gamma = N^- B^+$$

be the big Bruhat cell. We begin by studying $F = L(\Gamma)$. Although G does not act on F , $\mathcal{U}(\mathfrak{g})$ does, from the left, as explained below. For any $Z \in \mathfrak{g}$ we have the right invariant vector field $\partial_r(Z)$ on G . Then for any open $W \subset G$ and any $f \in \text{Hol}(W)$,

$$-(\partial_r(Z))f(u) = \left(\frac{d}{dt} \right)_{t=0} f(\exp(-tZ)u) \quad (u \in W).$$

([24] p. 7). The map $Z \rightarrow -\partial_r(Z)$ is a Lie homomorphism of \mathfrak{g} into the Lie algebra of holomorphic vector fields on W . So it extends to a representation ℓ of $\mathcal{U}(\mathfrak{g})$ on $\text{Hol}(W)$. We refer to this as the left action of $\mathcal{U}(\mathfrak{g})$ on $\text{Hol}(W)$. Similarly there is an action from the right using the elements of \mathfrak{g} viewed as left invariant vector fields:

$$(\partial(Z))f(u) = \left(\frac{d}{dt} \right)_{t=0} f(u \exp(tZ)) \quad (u \in W).$$

For $u = X_1 X_2 \dots X_r \in \mathcal{U}(\mathfrak{g})$, $X_i \in \mathfrak{g}$, we have

$$(\partial(u)f)(g) = \left((\partial^r f / \partial t_1 \dots \partial t_r)(g \exp(t_1 X_1) \dots \exp(t_r X_r)) \right)_0$$

where $(\dots)_0$ means the derivative is calculated at $t_1 = \dots = t_r = 0$. In contrast to this is

$$(\ell(u)f)(g) = \left((\partial^r f / \partial t_1 \dots \partial t_r)(\exp(-t_r X_r) \dots \exp(-t_1 X_1)g) \right)_0.$$

Clearly ∂ and ℓ commute with each other. We shall determine ℓ explicitly on $\text{Hol}(N^- A N^+)$ (Theorem 3.12). For $g \in \text{Hol}(N^-)$ we write \tilde{g} for the unique element of F whose restriction to N^- is g , so that $\tilde{g}(nb) = g(n)\chi(b)$ for $n \in N^-$, $b \in B^+$. The correspondence $g \leftrightarrow \tilde{g}$ is a linear topological isomorphism between $\text{Hol}(N^-)$ and F , the inverse being the restriction map from F to $\text{Hol}(N^-)$. Since $\exp : \mathfrak{n}^- \rightarrow N^-$ is an analytic diffeomorphism, we can introduce global coordinates $(t_\alpha)_{\alpha \in P}$ on N^- by

$$t_\alpha(n) = y_\alpha, \quad n = \exp \left(\sum_{\alpha \in P} y_\alpha X_{-\alpha} \right)$$

We have $t_\alpha(a^{-1}na) = t_\alpha(n)\chi_\alpha(a)$, $(a \in A_0, n \in N^-)$.

The polynomials in the t_α form a dense subalgebra \mathcal{P} of $\text{Hol}(N^-)$ and \mathcal{P}^\sim is the corresponding dense subalgebra of F . If i_a ($a \in A_0$) is the map $n \mapsto ana^{-1}$ of N^- , it is immediate that

$$\ell_a g^\sim = \chi(a)^{-1}(i_a g)^\sim, \quad (\ell_a g^\sim = g^\sim \circ \ell_a^{-1}, i_a g = g \circ i_a^{-1}).$$

For $r = (r_\alpha)$ let $t^r = \prod t_\alpha^{r_\alpha}$. We have

$$i_\alpha t^r = \prod (t_\alpha \circ i_\alpha^{-1})^{r_\alpha} = \chi_{r^*} t_r \quad \left(r^* = \sum_\alpha r_\alpha \alpha \right).$$

Then, we obtain the following explicit decompositions of \mathcal{P} and \mathcal{P}^\sim under A_0 :

$$\mathcal{P} = \bigoplus_r \mathbb{C}.t^r, \quad i_\alpha t^r = \chi_{r^*} t_r, \quad \mathcal{P}_\lambda^\sim = \bigoplus_r \mathbb{C}.(t^r)^\sim, \quad \ell_a(t_r)^\sim = \chi_{-\lambda+r^*}(t_r)^\sim.$$

The elements of \mathcal{P} are polynomials in the coordinates t_α . One can describe \mathcal{P} also intrinsically. If we identify N^- with \mathfrak{n}^- via the exponential map, then \mathcal{P} is simply the space of polynomials on the vector space \mathfrak{n}^- . By the Baker-Campbell-Hausdorff (BCH) formula the multiplication in $\mathfrak{n}^- \cong N^-$ is given by

$$X \cdot Y = p(X, Y), \quad X^{-1} = -X \tag{5}$$

where p is a polynomial map $\mathfrak{n}^- \times \mathfrak{n}^- \rightarrow \mathfrak{n}^-$. We can also write this as

$$n_1 n_2 = m(n_1, n_2), \quad n^{-1} = i(n) \tag{6}$$

where m, i are polynomial maps $N^- \times N^- \rightarrow N^-$ and $N^- \rightarrow N^-$ respectively. There is also another way of thinking about this. G may be viewed as a complex affine algebraic group in a unique manner (the regular functions on G form the algebra generated by the matrix elements of holomorphic finite dimensional representations of G); N^- is an algebraic subgroup of G whose underlying variety is the vector space \mathfrak{n}^- . The polynomials on N^- are thus the regular functions on N^- , denoted as $\mathcal{O}_{N^-}(N^-)$. Using either point of view, one can conclude that the adjoint representation Ad of N^- on \mathfrak{g} is rational and so its matrix elements are polynomials on N^- . Let D^+ be the semi group in \mathfrak{h}^* generated by the positive roots, namely, the set of elements of the form $\sum_i m_i \alpha_i$ where the m_i are integers ≥ 0 and the α_i are the simple roots in P . Then

$$\mathcal{P} = \bigoplus_{d \in D^+} \mathcal{P}_d, \quad \mathcal{P}_d = \bigoplus_{r^*=d} \mathbb{C}.t^r$$

while
$$\mathcal{P} = \bigoplus_{d \in D^+} \mathcal{P}_d^\sim, \quad \mathcal{P}_d^\sim = \bigoplus_{r^*=d} \mathbb{C}.(t^r)^\sim.$$

We now state a lemma, which is also found in [21] Sec.4, Lemma 7 and 8 and discussion preceding those.

Lemma 3.8. *We have $F(\tau) \neq 0$ if and only if $\tau = \chi_{-\lambda} + d$ for some $d \in D^+$. Moreover*

$$\dim F(\lambda + d) = \# \left\{ r = (r_\alpha) \mid r_\alpha \geq 0, \sum_\alpha r_\alpha \alpha = d \right\} \tag{7}$$

Proof. Since \mathcal{P} is dense in $\text{Hol}(N^-)$, we see that \mathcal{P}_λ^\sim is dense in F_λ ($F = L(\Gamma)$, in (4) with $E = \Gamma$). The first statement now follows from Lemma 3.7. The second statement is obvious. ■

We have used the fact that the algebra $\text{Pol}(V)$ polynomials on a finite dimensional complex vector space V form a dense subspace of $\text{Hol}(V)$. This is in general not true for arbitrary open subdomains of V ; such domains where this is true are called Runge domains.

For our purpose it is enough to know the following.

Lemma 3.9. *Let V be a vector space with a compact torus T acting on it. Suppose that, for any character τ of T , the subspace of $\text{Pol}(V)$ of all p such that $p(t^{-1}v) = \tau(t)p(v)$ for all $v \in V$, is finite dimensional. Then, any open connected subset of V , which is T -invariant and contains the origin, is a Runge domain.*

Proof. We may assume that $V = \mathbb{C}^N$ with T -action

$$t, (z_1, \dots, z_N) \mapsto (f_1(t)z_1, \dots, f_N(t)z_N)$$

where the f_j are characters of T . Let U be an open connected subset of \mathbb{C}^N containing the origin and stable under T . The action of T induces an action on $\text{Hol}(U)$ which is a representation. It is enough to prove that the closure of $\text{Pol}(\mathbb{C}^N)$ contains $\text{Hol}(U)^\infty$. Since the Fourier series of any f in $\text{Hol}(U)^\infty$ converges to f , it is enough to show that any eigenfunction of T in $\text{Hol}(U)$ is a polynomial. Suppose $g \neq 0$ is in $\text{Hol}(U)$ such that $g(t^{-1}(u)) = f(t)g(u)$ for all $t \in T$ and $u \in U$, f being a character of T . Since $0 \in U$, we can expand g as a power series $g(u) = \sum_r c_r u^r$ where we write $r = (r_i)$ $1 \leq i \leq N$, $u = u_1 \dots u_N$, valid in a polydisk. Then $c_r f^r = c_r g$ whenever $c_r \neq 0$. So only the r with $f = f^r$ appear in the expansion of g . We claim that there are only finitely many such r ; once this claim is proven we are done, because g is a linear combination of the monomials u_r with $f^r = f$, hence g is a polynomial. To prove the claim, note that all such u_r are eigenfunctions for T for the eigencharacter f , and by assumption, there are only finitely many of these. ■

Recall that $\Gamma = N^- B^+$ and that

$$anb = (n^a)ab, \quad (n^a = ana^{-1}, a \in A_0)$$

Thus, A_0 -invariant open connected subsets U of N^- correspond via $U \leftrightarrow U^\sim = UB^+$ to connected open sets $U^\sim = A_0 U^\sim B^+$. From the remark above we know that any such U is a Runge domain. Let $\Gamma^1 = A_0 \Gamma B^+$ be connected and open in G , $\Gamma^2 = (\Gamma^1 \cap \Gamma)^0$ where the superscript 0 refers to the connected component containing 1. Clearly $\Gamma^2 = U_2 B^+$ where U_2 is a connected A_0 -invariant open subset of N^- containing 1. By what we saw above, U_2 is a Runge domain in N^- . Let

$$F^1 = L(\Gamma^1), \quad F^2 = L(\Gamma^2), \quad (\Gamma^1 = A_0 \Gamma B^+, \Gamma^2 = (\Gamma^1 \cap \Gamma)^0)$$

Lemma 3.10. *Let notation be as above. Then the restriction map $F^1 \rightarrow F^2$ is a continuous injection and $F^1(\tau)$ maps into $F^2(\tau)$ for all characters τ of A_0 . Moreover $F^2(\tau) = F(\tau)$.*

Proof. The first statement is just the principle of analytic continuation. The second follows from the fact that the polynomials are dense in $\text{Hol}(U_2)$. ■

Lemma 3.11. *The left action of $\mathcal{U}(\mathfrak{g})$ on $L(W)$ for any open $W \subset \Gamma$ leaves \mathcal{P}^\sim invariant.*

Proof. Let $Z \in \mathfrak{g}$. Then, for fixed $n \in N^-$ and $t \in \mathbb{C}$ sufficiently small, we have $\exp(-tZ^{n^{-1}})n \in N^-B^+$ and so $\exp(-tZ^{n^{-1}})nb \in N^-B^+$ for all $b \in B^+$. Write

$$\exp(-tZ)nb = n \exp(-tZ^{n^{-1}})b, \quad \exp(-tZn) = \nu(n, t)\beta(n, t) \tag{8}$$

with $\nu(n, t) \in N^-, \nu(n, 0) = 1, \beta(n, t) \in B^+, \beta(n, 0) = 1$.

Hence, for any $f \in \mathcal{P}$, we have

$$(\exp(tZ) \cdot f^\sim)(nb) = f^\sim(\exp(-tZ)nb) = f^\sim(n\nu(n, t)\beta(n, t)b)$$

resulting in $(\exp(tZ) \cdot f^\sim)(nb) = f(n\nu(n, t))\chi(\beta(n, t))\chi(b)$.

We now differentiate with respect to t at $t = 0$. Let

$$V(n) = (d/dt)_{t=0}\nu(n, t) \in \mathfrak{n}^-, \quad W(n) = (d/dt)_{t=0}\beta(n, t) \in \mathfrak{b}^+.$$

Then $(Z \cdot f^\sim)(nb) = (V(n)f)(n)\chi(b) + f(n)d\chi(W(n))\chi(b)$.

To determine $V(n)$ and $W(n)$, we differentiate (8) at $t = 0$ to get

$$-Z^{n^{-1}} = V(n) + W(n)$$

so that $V(n)$ (resp. $W(n)$) is the projection of $-Z^{n^{-1}}$ on \mathfrak{n}^- (resp. \mathfrak{b}^+) corresponding to the direct sum $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}^+$. We have seen already that the adjoint action of N^- on \mathfrak{g} is rational. Hence if $(Y_j), (H_i), (X_k)$ are bases of $\mathfrak{n}^-, \mathfrak{h}, \mathfrak{n}^+$ respectively, then

$$V(n) = \sum_j f_j(n)Y_j, \quad W(n) = \sum_i h_i(n)H_i + \sum_k g_k(n)X_k$$

where f_j, h_i, g_k are polynomial functions on N^- . Hence

$$(Z \cdot f^\sim)(nb) = g^\sim(nb)$$

where $g(n) = \sum_j f_j(n)(Y_j f)(n) + \sum_i \lambda(H_i)h_i(n)$.

From (5) and (6) we know that the action of Y_j on f is by a polynomial differential operator of degree 1. Hence we get a polynomial differential operator of degree 1, say D_Z , such that

$$Z \cdot f^\sim = g^\sim, \quad g = D_Z f.$$

But $D_Z f \in \mathcal{P}$ for all $f \in \mathcal{P}$. This proves that for any $f \in \mathcal{P}$, $Z \cdot f^\sim \in \mathcal{P}^\sim$. ■

3.4. Pairings of $U(\mathfrak{g})$ modules of holomorphic sections

In the previous section we have defined two actions of $\mathcal{U}(\mathfrak{g})$ on $L(\Gamma)$. Now we want to show that the submodule $\mathcal{P} \subset L(\Gamma)$ consisting of polynomials is in perfect duality with the highest weight representation given by the Verma module $V_\lambda = \mathcal{U}(\mathfrak{g})/M_\lambda$ (see Thm. 3.12). Notice that, because of Lemma 3.11, such representation is independent from the chosen open set; as we shall see in the sequel, we need to replace Γ with G_0B , in order to obtain representations of G_0 in the space $L(G_0B)$. For the basic theory regarding Verma modules, see [31], Ch. V, Sec. 3, Prop. 5.11.

Let M_1, M_2 be two modules for \mathfrak{g} . By a \mathfrak{g} -pairing between them we mean a bilinear form $\langle \cdot, \cdot \rangle$ on $M_1 \times M_2$ with the property that

$$\langle Xm_1, m_2 \rangle = \langle m_1, -Xm_2 \rangle \quad (m_i \in M_i, X \in \mathfrak{g}).$$

Since the M_i are modules for $\mathcal{U}(\mathfrak{g})$ this implies that

$$\langle X_1 \dots X_r m_1, m_2 \rangle = \langle m_1, (-1)^r X_r \dots X_1 m_2 \rangle \quad (m_i \in M_i, X_j \in \mathfrak{g}).$$

The map $X \rightarrow -X$ of \mathfrak{g} is an involutive anti-automorphism of \mathfrak{g} . It extends uniquely to an involutive anti-automorphism $u \rightarrow u^T$ of $\mathcal{U}(\mathfrak{g})$. The \mathfrak{g} -pairing requirement is equivalent to

$$\langle um_1, m_2 \rangle = \langle m_1, u^T m_2 \rangle (m_i \in M_i, u \in \mathcal{U}(\mathfrak{g})).$$

We refer to this as a $\mathcal{U}(\mathfrak{g})$ -pairing also. The pairing is said to be *non-singular* if $\langle m_1, m_2 \rangle = 0$ for all m_2 (resp. for all m_1) implies that $m_1 = 0$ (resp. $m_2 = 0$).

The following result, though not fully spelled out as we state it, is contained in [21] Sec. 4 and 5.

Theorem 3.12. *There is a non-singular $\mathcal{U}(\mathfrak{g})$ -pairing between \mathcal{P}^\sim and the Verma module V_λ . Moreover every non-zero submodule of \mathcal{P}^\sim contains the element 1^\sim corresponding to the constant function $1 \in \mathcal{P}$. In particular, the submodule \mathcal{I}^\sim of \mathcal{P}^\sim generated by 1^\sim is irreducible and is the unique irreducible submodule of \mathcal{P}^\sim . Finally, \mathcal{I}^\sim is the unique irreducible module of lowest weight $-\lambda$.*

Proof. It is clear that $(\ell(u)f)(1) = (\partial(u^T)f)(1)$; this is seen by taking $u = X_1 \dots X_r, X_i \in \mathfrak{g}$. We now define, for $u \in \mathcal{U}(\mathfrak{g}), f \in \mathcal{P}^\sim$,

$$\langle f, u \rangle = (\partial(u)f)(1) = (\ell(u^T)f)(1) \quad (9)$$

Then, for $c \in \mathcal{U}(\mathfrak{g})$,

$$\begin{aligned} \langle \ell(c)f, u \rangle &= (\partial(u)\ell(c)f)(1) = (\ell(c)\partial(u)f)(1) = (\partial(c^T)\partial(u)f)(1) \\ &= (\partial(c^T u)f)(1) = \langle f, c^T u \rangle. \end{aligned}$$

We thus have a \mathfrak{g} -pairing between \mathcal{P}^\sim and $\mathcal{U}(\mathfrak{g})$ where the latter is regarded as a $\mathcal{U}(\mathfrak{g})$ -module under left multiplication. For $f \in \mathcal{P}^\sim$ we have, for all $w \in \Gamma$,

$$f(w \exp tX_\alpha) = f(w) \quad (\alpha > 0), \quad f(w \exp tH) = e^{t\lambda(H)} f(w)$$

from which we get $\partial(Z)f = 0$ ($Z = X_\alpha, H - \lambda(H)$).

If M_λ is the left ideal generated by the X_α ($\alpha > 0$) and $H - \lambda(H)$ for $H \in \mathfrak{h}$, then we have

$$\langle f, u \rangle = 0 \quad (u \in M_\lambda).$$

Hence $\langle \cdot, \cdot \rangle$ defines a \mathfrak{g} -pairing between \mathcal{P}^\sim and $\mathcal{U}(\mathfrak{g})/M_\lambda = V_\lambda$, the Verma module. We shall now show that this is a non-singular pairing. If $\langle f, u \rangle = 0$ for all $u \in \mathcal{U}(\mathfrak{g})$, then $(\partial(u)f)(1) = 0$ for all $u \in \mathcal{U}(\mathfrak{g})$ which implies that $f = 0$. Conversely, suppose that $\langle f, u \rangle = 0$ for all $f \in \mathcal{P}^\sim$. We wish to prove that $u \in M_\lambda$.

We can write u as $u = v + \mu$ where μ is in the enveloping algebra of N^- and $v \in M_\lambda$. Since $(\partial(v)f)(1) = 0$, we have $(\partial(\mu)f)(1) = 0$ for all $f \in \mathcal{P}^\sim$. This means that $(\partial(\mu)f)(1) = 0$ for all polynomials g on N^- . It is elementary to show that this implies that $\mu = 0$, proving that $u \in M_\lambda$.

The remainder of the proof is a formal consequence of the existence of the non-singular pairing. The functions $(t^r)^\sim$ corresponding to the coordinate polynomials t^r on N^- are weight vectors for the action of \mathfrak{h} for the weight $r - \lambda$. Hence \mathcal{P}^\sim is a weight module with the multiplicities defined by Lemma 3.8. We shall prove that every non-zero ℓ -invariant subspace W of \mathcal{P}^\sim contains the vector 1^\sim defined by the constant function 1 on N^- . Now W is a sum of weight spaces and if it does not contain 1^\sim , then W is contained in the sum of all weight spaces corresponding to the weights $\lambda - r$ where $r = (r_\alpha)$ with some $r_\alpha > 0$. Now

$$\langle Hm_1, m_2 \rangle = -\langle m_1, Hm_2 \rangle$$

for all $H \in \mathfrak{h}$, $m_1 \in \mathcal{P}^\sim$, $m_2 \in V_\lambda$. This shows that the weight space of \mathcal{P}^\sim for the weight θ is orthogonal to the weight space of V_λ for the weight ϕ unless $\theta = -\phi$. Let v be a non-zero vector of highest weight λ in V_λ . Since $W \subset \mathcal{P}$ is contained in the span of weights other than $-\lambda$, we have $\langle W, v \rangle = 0$. Hence, for all $g \in \mathcal{U}(\mathfrak{g})$, $w \in W$ we have $\langle \ell(g)w, v \rangle = 0$. So $\langle w, g^T v \rangle = 0$ for all $g \in \mathcal{U}(\mathfrak{g})$. But v is cyclic for V_λ and so we have $\langle w, V_\lambda \rangle = 0$ for all $w \in W$. This means that $W = 0$, contradicting the hypothesis that $W \neq 0$. Thus every non-zero submodule of \mathcal{P}^\sim contains the submodule \mathcal{I}^\sim generated by 1^\sim . This submodule is then the unique irreducible submodule of \mathcal{P}^\sim . The weights of \mathcal{I}^\sim are of the form $-\lambda + d$, where d is a positive integral linear combination of the simple roots, and 1^\sim has weight $-\lambda$. It is then clear that 1^\sim is the lowest weight of \mathcal{I}^\sim . This fact, together with its irreducibility, characterizes it uniquely. Theorem 3.12 is thus fully proved. ■

Remark 3.13. Notice that, as usual, given a highest weight representation of \mathfrak{g} in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, the corresponding irreducible representation will occur as a quotient (see [31], Ch. V, Sec. 3, Prop. 5.15). However here, since the perfect duality with the representation in $L(W)$ (W open in G), we have that the irreducible corresponding representation \mathcal{I} in $L(W)$ will occur as a *submodule*. Our Thm. 2.11 will also specify a criterion to establish irreducibility in this setting.

3.5. Representations of the real group

We now come to the real group G_0 . Let W be a G_0 stable neighbourhood of the identity in G , that is $g \cdot w \in W$ for all $w \in W$, $g \in G_0$. Then we can define a G_0 action on $L(W)$ as follows (see [21], p. 17):

$$(g \cdot f)(w) = f(g^{-1}w), \quad \forall g \in G_0, \quad \forall w \in W.$$

Theorem 3.14. *Let $S = G_0B^+$ and $F^1 = L(S)$. Then:*

- (1) $F^1 \neq 0$ if and only if F^1 contains an element ψ which is an analytic continuation of 1^\sim to S , i.e., which coincides with 1^\sim on $S \cap \Gamma$. In this case $\psi \in (F^1)^\infty$.
- (2) If $F^{11} = \text{Cl}(\mathcal{U}(\mathfrak{g})\psi)$ (Fréchet closure), then F^{11} is a Fréchet module for G_0 , is K_0 -finite, and its K_0 -finite part, which is also its A_0 -finite part, is the irreducible lowest weight module of lowest weight $-\lambda$.
- (3) In particular $\lambda(H_\alpha)$ is an integer ≥ 0 for all compact positive roots.

Proof. (1) F^1 is a Fréchet module for G_0 . We use Lemma 3.10 with $\Gamma^1 = S$ and $\Gamma^2 = (S \cap \Gamma)^0$ where 0 refers to the connected component containing the unit element of G . The restriction map $F^1 \rightarrow F^2$ is a continuous injection which is A_0 -equivariant, so that $F^1(\tau) \subset F^2(\tau) = F(\tau)$ for all characters τ of A_0 . Thus F^1 is A_0 -finite, hence also K_0 -finite, with $(F^1)^0 = \sum_{\tau} F^1(\tau)$ as the space of its K_0 -finite vectors (cf. Lemma 3.7). Suppose first that $F^1 \neq 0$. The weights corresponding to the characters τ are $-\lambda + r^*$ (earlier notation) with $(t_r)^\sim$ as the corresponding eigenfunction, the weight $-\lambda$ corresponding to 1^\sim . Suppose now that the weights in $(F^1)^0$ do not include $-\lambda$. Then the corresponding eigenfunctions are the $(t_r)^\sim$ where $r = (r_\alpha) = 0$, and so vanish at 1. Thus all elements of $(F^1)^0$ vanish at 1, hence all elements of F^1 vanish at 1 as $(F^1)^0$ is dense in F^1 . By left translation by elements of G_0 all elements of F^1 vanish everywhere on S , i.e., $F^1 = 0$. This contradicts the assumption that $L(S) \neq 0$. Hence $F^1(\tau_0) \neq 0$ for the trivial character τ_0 of A_0 .

Since

$$F^1(\tau_0) \subset F^2(\tau_0) = \mathbb{C}1^\sim,$$

we see that $F^1(\tau_0)$ contains an element ψ which restricts to 1^\sim on $(S \cap \Gamma)^0$. The converse is obvious; if ψ extends 1^\sim , then $F^1(\tau_0) \neq 0$, hence $F^1 \neq 0$.

(2) The algebra $\mathcal{U}(\mathfrak{g})$ acts on $(F^1)^0$ and the restriction map commutes with the actions of $\mathcal{U}(\mathfrak{g})$. Let $J = \mathcal{U}(\mathfrak{g})\psi$. Then J injects onto a non-zero submodule of \mathcal{I}^\sim , hence J maps onto \mathcal{I}^\sim isomorphically. Thus J is the lowest weight module for the weight $-\lambda$. Now ψ is both K_0 -finite and ζ -finite, where ζ is the center of $\mathcal{U}(\mathfrak{g})$, the latter because $\mathcal{I}^\sim \cong J$ has an infinitesimal character (as all lowest (or highest) weight modules have). Hence Theorems 11 and 12 of [42], p. 312, apply to tell us that $\text{Cl}(J)$ is invariant under G , is a Fréchet module whose K_0 -finite part is precisely J .

(3) The last statement is clear because F^{11} is a Harish-Chandra module and ψ is killed certainly by the compact negative roots, showing that $-\lambda(-H_\alpha) = \lambda(H_\alpha) \geq 0$ for all compact positive roots. This finishes the proof of the theorem. ■

The shortcoming of Theorem 3.14 is that it does not tell us when $L(S) \neq 0$. For instance, suppose that the real form G_0 is actually the maximal compact subgroup U_0 , i.e., $G_0 = U_0$. Then $S = U_0 B^+ = G$, so that $L = L(S)$ is the space of global sections of the holomorphic bundle. Then the result above is essentially the Borel-Weil-Bott theorem in degree 0. Indeed, the space of global sections, say H , is finite dimensional and carries an action of $\mathcal{U}(\mathfrak{g})$ from the left. If λ is a dominant integral linear function, then for the corresponding representation with highest weight λ the matrix element $a_{11}(g)$ in a weight basis (v_i) , $0 \leq i \leq N$ with v_0 as the highest weight vector is a global section of the sheaf so that $L \neq 0$. Then we know that the module J generated by a_{11} is irreducible and has lowest weight $-\lambda$. We claim that $J = L$; otherwise we can find a complementary module R such that $L = J \oplus R$, contradicting the fact that R must contain a_{11} .

3.6. Analytic continuation of 1^\sim when the positive system of roots is admissible

We shall now assume that the center \mathfrak{c}_0 of \mathfrak{k}_0 has dimension 1, and that P is an admissible system of positive roots. From the theory on the Lie algebra this means the following. We have

$$\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-;$$

\mathfrak{p}^\pm are stable under \mathfrak{k} and are irreducible, or what comes to the same thing, stable and irreducible under K_0 (see Sec. 2). Since \mathfrak{h}_0 is a CSA of \mathfrak{g}_0 also, it follows that \mathfrak{p}^\pm are sums of root spaces. The positive system of roots is $P = P_k \cup P_n$ where P_k is a positive system of compact roots, and P_n is the set of roots whose root spaces are contained in \mathfrak{p}^+ . Thus

$$\mathfrak{p}^\pm = \sum_{\beta \in P_n} \mathfrak{g}_\beta^\pm.$$

We know that the simple system for P is of the form $\{\alpha_1, \dots, \alpha_r, \beta\}$ where $\{\alpha_1, \dots, \alpha_r\}$ form the simple system for the compact roots, and β is non-compact. Moreover the positive non-compact roots are of the form

$$\gamma = m_1\alpha_1 + \dots + m_r\alpha_r + \beta$$

where the m_i are integers ≥ 0 . In particular $[\mathfrak{p}^\pm, \mathfrak{p}^\pm] = 0$ (same sign in both). Obviously \mathfrak{p}^\pm are ideals in \mathfrak{n}^\pm . Let P^\pm be the complex analytic subgroups defined by \mathfrak{p}^\pm .

Let $\mathfrak{n}_k^\pm = \sum_{\beta \in P_k} \mathfrak{g}_\beta^\pm$. Then $\mathfrak{n}_k^\pm = \mathfrak{n}^\pm \cap \mathfrak{k}$.

The subgroups of G defined by \mathfrak{n}^\pm are closed and $N^\pm = N_k^\pm P^\pm$. We can then set up the map

$$f : P^- \times K \times P^+ \longrightarrow G \quad f(p^-, k, p^+) = p^- k p^+.$$

Lemma 3.15. *The set $\Omega = P^- K P^+$ is open in G and the map f is a complex analytic diffeomorphism onto Ω .*

Proof. This is standard and is valid under much greater generality. First we compute the differential of f . We have, after a standard calculation,

$$df_{(p_1, k, p_2)}(X_1, Z, X_2) = (X_1 + Z^k + X_2^{kp_2})^{(kp_2)^{-1}}$$

from which it is clear df is bijective everywhere. We must show that f is one-one. This reduces to showing that $P^- \cap K P^+ = \{1\}$. Let $p_1 \in K P^+$. Since $p_1 = k p^+$ where $k \in K$, $p^+ \in P^+$, and $\text{Ad}(K)$ leaves \mathfrak{p}^+ invariant, it follows that $\text{Ad}(p_1)$ leaves \mathfrak{p}^+ invariant. Now P^- is nilpotent, even abelian, and so we can write $p_1 = \exp X_1$ where $0 \neq X_1 \in \mathfrak{p}^-$. Then $e^{\text{ad}(X_1)} = \text{Ad}(p_1)$ leaves \mathfrak{p}^+ invariant. Because $\text{ad}(X_1)$ is nilpotent, it is a polynomial in $e^{\text{ad}(X_1)}$. Hence $\text{ad}(X_1)$ leaves \mathfrak{p}^+ invariant. Using a suitable lexicographic ordering of the roots we can write $X_1 = c_1 X_{-\beta_1} + X_{-\beta_2} + \dots$ where $\beta_1 < \beta_2 < \dots$ are non-compact roots and $c_1 \neq 0$. Clearly

$$[X_1, X_{\beta_1}] = -c_1 H_{\beta_1} + Y \quad \text{where } Y \in \mathfrak{n}^-.$$

Hence $\text{Ad}(p_1)(X_{\beta_1}) = e^{\text{ad}(X_1)}(X_{\beta_1}) = X_{\beta_1} - c_1 H_{\beta_1} + Y'$ where $Y' \in \mathfrak{n}^-$.

But this must be in \mathfrak{p}^+ . Thus $H_{\beta_1} \in \mathfrak{n}^- + \mathfrak{n}^+$ which is impossible. ■

Ω is the big cell corresponding to the parabolic subgroups P^\pm , known also as the *Harish-Chandra open cell*. In particular, K , P^\pm are closed in Ω , hence locally closed in G , hence closed in G . We write θ for the automorphism of \mathfrak{g} whose fixed point set is \mathfrak{k} , namely the Cartan involution.

Let $\mathfrak{u} = \mathfrak{k}_0 + i\mathfrak{p}_0$. Then \mathfrak{u} is a compact form of \mathfrak{g} . One knows that

$$\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{h}_0 \oplus \mathfrak{n}^+$$

is an Iwasawa decomposition of \mathfrak{g} (viewed as a real Lie algebra), see [31], Ch. VI. Let U and A^+ be the real analytic subgroups of G defined by \mathfrak{u} and $i\mathfrak{h}_0$ respectively. Then U is a maximal compact subgroup of G and simply connected, A^+ is a vector space, and $G = UA^+N^+$ is the Iwasawa decomposition of G (viewed as a real Lie group). We write θ^\sim and η for the conjugations of \mathfrak{g} with respect to \mathfrak{u} and \mathfrak{g}_0 respectively. Since G is simply connected it follows that θ , θ^\sim , and η all lift to G ; we denote them by the same symbol. The actions of these on the Lie algebra are given by the following table:

	\mathfrak{k}_0	$i\mathfrak{k}_0$	\mathfrak{p}_0	$i\mathfrak{p}_0$
θ	id	id	-id	-id
θ^\sim	id	-id	-id	id
η	id	-id	id	-id

It follows from this that $\theta^\sim = \eta\theta$ both on \mathfrak{g} and G . Finally, we have

$$\gamma(\mathfrak{n}^+) = \mathfrak{n}^-, \quad \gamma(N^+) = N^- \quad (\gamma = \eta, \theta^\sim).$$

This follows from Lemma 3.3 and Lemma 2.21. We are thus able to formulate the crucial lemma.

Lemma 3.16. *We have $G_0B^+ \subset P^-KP^+$.*

Proof. Let us denote:

$$\mathfrak{n}_k^+ = \text{Lie}(N_k^+), \quad \mathfrak{n}_n^+ = \text{Lie}(N_n^+), \quad (\mathfrak{n}_k^+ = \sum_{\alpha>0 \text{ compact}} \mathfrak{g}_\alpha, \mathfrak{n}_n^+ = \sum_{\alpha>0 \text{ noncompact}} \mathfrak{g}_\alpha)$$

Since KP^+ contains $AN_k^+P^+ = B^+$, we have $KP^+B^+ = KP^+$ and so it is enough to prove that $G_0 \subset P^-KP^+$. Now $G_0 = K_0 \exp \mathfrak{p}_0$, while the left side is invariant by left multiplication by K_0 ; indeed,

$$K_0P^-KP^+ \subset KP^-KP^+ = P^-KKP^+ = P^-KP^+.$$

Hence it suffices to prove that $\exp \mathfrak{p}_0 \subset P^-KP^+$. Let $q = \exp X$ where $X \in \mathfrak{p}_0$. Write $p = \exp(X/2)$ so that $q = p^2$. Then by the Iwasawa decomposition for G we have $p = uan^+$ where $u \in U, a \in A^+, n^+ \in N^+$. We apply θ^\sim to this relation. We have $\theta^\sim(u) = u$ while $\theta^\sim = -\text{id}$ on $i\mathfrak{k}_0$, so that $\theta^\sim(a) = a^{-1}$. Further we have observed that θ^\sim takes N^+ to N^- . On the other hand,

$$p = \eta(p) = \theta^\sim(\theta(p)) = \theta^\sim(p^{-1})$$

so that $\theta^\sim(p) = p^{-1}$. Hence we have

$$p^{-1} = ua^{-1}n_1, \quad \text{for } n_1 \in N^-, \text{ giving } p = n_1^{-1}au^{-1}.$$

Multiplying this by $p = uan^+$ we get

$$\exp X = q = p^2 = n_1^{-1}a^2n^+$$

showing that $q \in N^-AN^+$. But

$$N^-AN^+ = P^-N_n^-AN_k^+P^+ \subset P^-KP^+$$

proving what we want. ■

To get our result on analytic continuation of 1^\sim we need some preparation. Let λ be an integral linear function on \mathfrak{h} which is dominant for compact positive roots, namely $\lambda(H_\alpha) \geq 0$ for all compact positive roots α . We wish to examine when there is an irreducible holomorphic representation σ_λ of K with a highest weight χ_λ , i.e., σ_λ admits a non-zero vector v such that

$$\sigma_\lambda(h)v = \chi_\lambda(h)v \quad (h \in A), \quad \sigma_\lambda(X_\alpha)v = 0 \quad (\alpha > 0 \quad \text{and compact}).$$

(Here we use the same symbol for a representation on the group and the differentiated representation on the Lie algebra). We are interested in the case when \mathfrak{k}_0 has a non-zero center \mathfrak{c}_0 . Let \mathfrak{k}'_0 be the derived algebra $[\mathfrak{k}_0, \mathfrak{k}_0]$; then \mathfrak{k}'_0 is semisimple and $\mathfrak{k}_0 = \mathfrak{k}'_0 \oplus \mathfrak{c}_0$. As usual, we drop the suffix 0 when we complexify. Thus \mathfrak{k} is the center of \mathfrak{k} , $\mathfrak{k}' := [\mathfrak{k}, \mathfrak{k}]$ is semisimple, $\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{c}$, and $\mathfrak{h}' := \mathfrak{h} \cap \mathfrak{k}'$ is a CSA of \mathfrak{k}' . Also, $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{k}' \oplus \mathfrak{c}$. Since K is the fixed point set of θ , we know that K is an algebraic group and so is C , the connected component of the center of K . We have $K = K'C$; moreover $Z = K' \cap C$ is finite since it is contained in the center of K' . Note that $C \subset A$. We then have the following lemma.

Lemma 3.17. *Let χ_λ be a holomorphic character of A such that λ is dominant for compact positive roots. For the existence of an holomorphic finite dimensional representation of K with highest weight λ it is necessary and sufficient that the representation of \mathfrak{k}' whose highest weight is the restriction of λ to \mathfrak{h}' lifts to a holomorphic representation of K' .*

Proof. The condition is obviously necessary. We now prove its sufficiency. Let K_1 be the universal covering group of K' . Clearly the representation of \mathfrak{k}' with highest weight λ lifts to a holomorphic representation τ' of K_1 . We thus have a representation $\tau := \tau' \otimes \chi_\lambda$ of $K_1 \times C$. The map $k, c \mapsto kc$ from $K_1 \times C$ to K is onto K and we shall show that τ is trivial on the kernel of this map. Indeed, if (k, c) is in the kernel, we have $kc = 1$ and so $k = c^{-1} \in \mathbb{Z}$. The representation τ' is a scalar on the center of K' and so $\tau'(t) = a(t)1$ for $t \in \mathbb{Z}$. But $\mathbb{Z} \subset A \cap K'$ since $A \cap K'$ is a maximal torus of K' and $A \cap K'$ acts as the character χ_λ on the highest weight vector. Hence $a(t) = \chi_\lambda(t)$ for $t \in \mathbb{Z}$. But then

$$\tau((k, c)) = \tau'(k)\chi_\lambda(c) = \chi_\lambda(c)^{-1}\chi_\lambda(c) = 1. \quad \blacksquare$$

Let σ_λ be the representation of K thus defined. It is obvious that it is the required one. We say that λ is of K -type if the condition of the lemma is satisfied. In the notation of the lemma, λ is of K -type if and only if the representation τ_1 is trivial on the kernel of the covering map $K_1 \rightarrow K'$.

Theorem 3.18. *Suppose that $\dim(\mathfrak{c}_0) = 1$. Let λ be integral and $\lambda(H_\alpha)$ be an integer ≥ 0 for all compact positive roots. Assume that λ is of K -type. Then 1^\sim extends analytically to an element $\psi \in L(S)$. In particular $L(S) \neq 0$ and $(F_\lambda)^{11}$ carries the representation whose K_0 -finite part is the irreducible lowest weight representation of lowest weight $-\lambda$. Finally and conversely, if λ is not of K -type, then $L(S) = 0$.*

Proof. We have a representation $\sigma = \sigma_\lambda$ of K with highest weight vector v of weight χ_λ . Since P^+ is normal in KP^+ and $KP^+/P^+ \cong K$, we can view σ as a representation of $Q^+ = KP^+$ trivial on P^+ .

We define the holomorphic function f on K by

$$R\sigma(k)v = f(k)v$$

where R is the unique projection on $\mathbb{C}v$ modulo the sum of the remaining weight spaces. Extend f to a holomorphic function g on $\Omega = P^-KP^+$ by

$$g(p^-kp^+) = f(k) \quad (k \in K, p^\pm \in P^\pm).$$

We now claim that the restriction of g to $S = G_0B^+$ is the analytic continuation of 1^\sim . Since Ω contains the big cell $\Gamma = N^-AN^+$ it is enough to show that the restriction of g to Γ is just 1^\sim . Fix $n^\pm \in N^\pm$, $h \in A$. As

$$N^\pm = P^\pm N_k^\pm = N_k^\pm P^\pm$$

we can write $n^- = p^-n_k^-$ and $n^+ = n_k^+p^+$ ($n_k^\pm \in N_k^\pm, p^\pm \in P^\pm$) and so

$$g(n^-hn^+) = g(p^-n_k^-hn_k^+p^+) = f(n_k^-hn_k^+) = \chi_\lambda(h)f(n_k^-).$$

It is thus enough to verify that $f(n_k^-) = 1$. But it is clear that $\sigma(nk)v \cong v$ modulo the sum of weight spaces of weight less than λ . Hence $f(n_k^-) = 1$ as we wanted to check. This proves the theorem. ■

Corollary 3.19. *Let K'_0 be the analytic subgroup of K_0 defined by $\mathfrak{k}'_0 := [\mathfrak{k}_0, \mathfrak{k}_0]$. If K'_0 is simply connected, Theorem 3.18 is valid for all integral λ which are dominant with respect to the compact positive roots.*

Proof. Since K'_0 is simply connected, and is a compact form of K' , it follows that K' is simply connected. So, for any integral λ dominant with respect to the positive compact roots, the corresponding irreducible representation lifts to K' . Lemma 3.17 now shows that this representation extends to K . To prove the last statement, assume that $L(S) \neq 0$. It is clear that the K'_0 -module generated by ψ has the lowest weight representation of \mathfrak{k}'_0 as its infinitesimal representation (same argument as for G_0). Call this π_λ . So $\pi_{-\lambda}$ lifts to a representation of K'_0 . Now K'_0 is semisimple and is the compact form of K' , and so $\pi_{-\lambda}$ extends to a holomorphic representation of K' . This is contragredient to the representation with highest weight λ which then lifts to K . By Lemma 3.17 we conclude that λ is of K -type. ■

Example 3.20. If $G = \mathrm{SU}(n, 1)$, K'_0 is $\mathrm{SU}(n)$ and so the above Corollary applies. If $G = \mathrm{SO}(2, 2k)$, then K'_0 is $\mathrm{SO}(2k)$ and the condition of λ being of K -type is non-trivial.

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