# Mapping Groups Associated with Real-Valued Function Spaces and Direct Limits of Sobolev-Lie Groups

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**Abstract.** Let $M$ be a compact smooth manifold of dimension $m$ (without boundary) and $G$ be a finite-dimensional Lie group, with Lie algebra $\mathfrak{g}$. Let $H^{>\frac{m}{2}}(M,G)$ be the group of all mappings $\gamma\colon M \to G$ which are $H^s$ for some $s > \frac{m}{2}$. We show that $H^{>\frac{m}{2}}(M,G)$ can be made a regular Lie group in Milnor's sense, modelled on the Silva space $H^{>\frac{m}{2}}(M,\mathfrak{g}) := \varinjlim_{s > \frac{m}{2}} H^s(M,\mathfrak{g})$, such that
$$H^{>\frac{m}{2}}(M,G) \;=\; \varinjlim_{s > \frac{m}{2}} H^s(M,G)$$
as a Lie group (where $H^s(M,G)$ is the Hilbert-Lie group of all $G$-valued $H^s$-mappings on $M$). We also explain how the (known) Lie group structure on $H^s(M,G)$ can be obtained as a special case of a general construction of Lie groups $\mathcal{F}(M,G)$ whenever function spaces $\mathcal{F}(U,\mathbb{R})$ on open subsets $U \subseteq \mathbb{R}^m$ are given, subject to simple axioms.

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## 1. Introduction and statement of results

Lie groups of mappings from a compact manifold $M$ to a finite-dimensional Lie group $G$ form an important class of infinite-dimensional Lie groups, as well as variants like gauge groups of principal $G$-bundles over $M$. See [27] for more context, as well as the references at the end of this introduction.

In this article, we describe a general construction principle for Lie groups of mappings when real-valued function spaces are given, satisfying suitable axioms. We then study ascending unions of the constructed mapping groups, in the special case of Sobolev-Lie groups.

For fixed $m \in \mathbb{N}$, consider a basis $\mathcal{U}$ of the topology of $\mathbb{R}^m$ satisfying suitable properties (a "good collection of open sets" in the sense of Definition 2.6). Suppose that, for each $U \in \mathcal{U}$, an integral complete locally convex space $\mathcal{F}(U,\mathbb{R})$ of bounded, continuous real-valued functions is given.[1] Then an integral complete locally convex space $\mathcal{F}(U,E)$ of $E$-valued maps can be defined in a natural way for each finite-dimensional real vector space $E$ (see 3.1). If four simple axioms (PF), (PB), (GL), and (MU) are satisfied, we say that the family $(\mathcal{F}(U,\mathbb{R}))_{U\in\mathcal{U}}$ is *suitable for Lie theory*

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[1] We say that a locally convex space $F$ is *integral complete* if the weak integral $\int_0^1 \gamma(t)\,dt$ exists in $F$ for each continuous map $\gamma\colon [0,1] \to F$. See [34] for a characterization.

(see Definition 3.3). For each $E$ as before, one can then define a locally convex space $\mathcal{F}(M, E)$ of $E$-valued functions for each compact $m$-dimensional smooth manifold $M$ without boundary, see 5.4. We can also define a set $\mathcal{F}(M, N)$ of $N$-valued functions on $M$, for each finite-dimensional smooth manifold $N$ (see 5.1). If $N$ is a Lie group, we obtain (with terminology as in 2.5):

**Proposition 1.1.** *For a given $m \in \mathbb{N}$, let $\mathcal{U}$ be a good collection of open subsets of $\mathbb{R}^m$ and $(\mathcal{F}(U, \mathbb{R}))_{U \in \mathcal{U}}$ be a family of integral complete locally convex spaces which is suitable for Lie theory. Let $M$ be a compact $m$-dimensional smooth manifold without boundary and $G$ be a finite-dimensional Lie group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, with Lie algebra $\mathfrak{g}$. Then $\mathcal{F}(M, G)$ can be made a $\mathbb{K}$-analytic BCH-Lie group whose Lie algebra can be identified with $\mathcal{F}(M, \mathfrak{g})$, such that*

$$\mathcal{F}(M, \exp_G) \colon \mathcal{F}(M, \mathfrak{g}) \to \mathcal{F}(M, G), \quad \gamma \mapsto \exp_G \circ \gamma$$

*is the exponential function of $\mathcal{F}(M, G)$.*

**Example 1.2.** For all $m \in \mathbb{N}$ and $s > \frac{m}{2}$, we can apply Proposition 1.1 to the Sobolev spaces $\mathcal{F}(U, \mathbb{R}) := H^s(U, \mathbb{R})$ on bounded open subsets $U \subseteq \mathbb{R}^m$ (see Section 7). We obtain Hilbert-Lie groups $H^s(M, G) := \mathcal{F}(M, G)$ with properties as described in the proposition.

Also the following two examples can be treated (see [17]).

**Example 1.3.** For all $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $\alpha \in ]0, 1]$, Proposition 1.1 can be applied to the Banach spaces $\mathcal{F}(U, \mathbb{R}) := C^{k,\alpha}(U, \mathbb{R})$ of $k$ times Hölder-differentiable functions on bounded open subsets $U \subseteq \mathbb{R}^m$ (see [17]). This yields Banach-Lie groups $C^{k,\alpha}(M, G) := \mathcal{F}(M, G)$ with properties as described in the proposition.

**Example 1.4.** For $m = 1$ and $p \in [1, \infty]$, Proposition 1.1 can be applied to the Banach spaces $\mathcal{F}(U, \mathbb{R}) := AC_{L^p}(U, \mathbb{R})$ of absolutely continuous functions with $L^p$-derivatives on bounded open intervals $U \neq \emptyset$ in $\mathbb{R}$ (see [17]). For $M := \mathbb{S}$ the unit circle, this yields Banach-Lie groups $AC_{L^p}(\mathbb{S}, G) := \mathcal{F}(\mathbb{S}, G)$ with properties as described in the proposition.

We then study direct limits of the Hilbert-Lie groups $H^s(M, G)$ as $s \searrow s_0$ for some $s_0 \geq m/2$. Using terminology as in 2.5 and 8.6, we obtain:

**Theorem 1.5.** *Let $m \in \mathbb{N}$, $s_0 \geq \frac{m}{2}$ be a real number, $M$ be a compact, $m$-dimensional smooth manifold without boundary and $G$ be a finite-dimensional Lie group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, with Lie algebra $\mathfrak{g}$. Then*

$$H^{>s_0}(M, G) := \bigcup_{s > s_0} H^s(M, G)$$

*can be made a $\mathbb{K}$-analytic BCH-Lie group over $\mathbb{K}$ whose Lie algebra can be identified with the locally convex direct limit*

$$H^{>s_0}(M, \mathfrak{g}) := \varinjlim_{s > s_0} H^s(M, \mathfrak{g}),$$

*such that the map $H^{>s_0}(M, \exp_G) \colon H^{>s_0}(M, \mathfrak{g}) \to H^{>s_0}(M, G)$, $\gamma \mapsto \exp_G \circ \gamma$ is the exponential function of $H^{>s_0}(M, G)$. The Lie group $H^{>s_0}(M, G)$ is $L^\infty_{\mathrm{rc}}$-regular and*

$C^0$-*regular. Each compact subset of* $H^{>s_0}(M,G)$ *is a compact subset of* $H^s(M,G)$ *for some* $s > s_0$. *Moreover*

$$H^{>s_0}(M,G) = \varinjlim_{s>s_0} H^s(M,G)$$

*holds in each of the categories of topological spaces, topological groups,* $C_{\mathbb{L}}^\infty$-*Lie groups for* $\mathbb{L} \in \{\mathbb{R}, \mathbb{K}\}$, *and* $C_{\mathbb{L}}^r$-*manifolds for* $r \in \mathbb{N}_0 \cup \{\infty\}$.

The Lie groups and manifolds we are referring to are Lie groups and manifolds modelled on locally convex spaces. The morphisms in the categories just mentioned are continuous maps, continuous group homomorphisms, group homomorphisms which are $C_{\mathbb{L}}^\infty$-maps, and $C_{\mathbb{L}}^r$-maps, respectively.

**General background of the studies.** Paradigmatic examples of mapping groups are Lie groups $C^k(M,G)$ of $C^k$-maps for $k \in \mathbb{N}_0 \cup \{\infty\}$, in particular for $k = \infty$ (see [11, 23, 25, 26, 31]). Lie groups $H^s(M,G)$ of Sobolev maps with real exponent $s > m/2$ have also been considered. See [21] for the case of loop groups (i.e., when $M$ is the unit circle $\mathbb{S} \subseteq \mathbb{C}$), using Fourier series for the definition. For a compact Lie group $G$ and a compact Riemannian manifold $(M,g)$, Lie groups $H^s(M,G)$ are constructed in [8, Theorem 1.2], referring to [9, Appendix A] for details where some proofs rely on integer exponents. A global approach using the Laplace operator of $(M,g)$ is used there to define Sobolev spaces. For real $s > m/2$ and a finite-dimensional Lie group $G$, Sobolev-Lie groups $H^s(M,G)$ also occur in [30, (7), p. 395].

Related studies of manifold structures on $C^k(M,N)$ for a finite-dimensional smooth manifold $N$ can be found, e.g., in [5, 20, 23]. Manifold structures on $H^s(M,N)$ for integers $s > \dim(M)/2$ are studied in [5, p. 781] and [22]; the possible generalization to real $s$ is broached in [22, Appendix B].

We mention that [29] pursues an axiomatic approach to global analysis, starting with the choice of a Banach space-valued section functor (see [29, §4]).[2] Starting with function spaces on open subsets of $\mathbb{R}^m$, as proposed in this article, constitutes a complementary, more elementary approach.

## 2.   Preliminaries

We write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A map between topological spaces shall be called a *topological embedding* if it is a homeomorphism onto its image. The word "vector space" refers to a real vector space, unless the contrary is stated. A subset $U$ of a $\mathbb{K}$-vector space $E$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is called *balanced* if $zx \in U$ for all $x \in U$ and $z \in \mathbb{K}$ with $|z| \leq 1$. All locally convex topological vector spaces are assumed Hausdorff. If $(E, \|\cdot\|)$ is a normed space, we let $B_r^E(x) := \{y \in E : \|y - x\| < r\}$ be the open ball of radius $r > 0$ around $x \in E$. We shall use $C^k$-maps between open subsets of locally convex spaces as introduced by Bastiani [1], and recall some concepts for the reader's convenience. For further information, see [10] and [18]

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[2] General Sobolev spaces occur in [29, §9], but an essential proof (of [29, Lemma 9.9]) presumes integer exponents.

(where also the corresponding manifolds and Lie groups are discussed), or also [20] (for Fréchet modelling spaces) and [26] (for sequentially complete spaces). If $U \subseteq \mathbb{R}^m$ is open and $E$ a finite-dimensional vector space, we let $C_c^\infty(U, E)$ be the vector space of all compactly supported smooth functions $\gamma \colon U \to E$.

**2.1.** Let $E$ and $F$ be locally convex spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $U \subseteq E$ be open. A mapping $f \colon U \to F$ is called $C_\mathbb{K}^0$ if it is continuous. We call $f$ a $C_\mathbb{K}^1$-map if $f$ is continuous, the directional derivative

$$df(x, y) := \lim_{z \to 0} \frac{1}{z}(f(x + zy) - f(x))$$

exists in $E$ for all $x \in U$ and $y \in E$ (where $z \in \mathbb{K} \setminus \{0\}$ with $x + zy \in U$), and $df \colon U \times E \to F$ is continuous. Recursively, for $k \in \mathbb{N}$ we say that $f$ is $C_\mathbb{K}^{k+1}$ if $f$ is $C_\mathbb{K}^1$ and $df$ is $C_\mathbb{K}^k$. If $f$ is $C_\mathbb{K}^k$ for all $k \in \mathbb{N}$, then $f$ is called $C_\mathbb{K}^\infty$.

The $C_\mathbb{R}^k$-maps are also referred to as $C^k$-maps. The $C_\mathbb{R}^\infty$-maps are also called *smooth*. The $C_\mathbb{C}^\infty$-maps are also called *complex analytic* (or $\mathbb{C}$-analytic); they are continuous and given locally by pointwise convergent series of continuous complex homogeneous polynomials (see [18, Corollary 2.1.9], cf. also Proposition 5.5 and Theorem 3.1 in [3]), but we shall not use this fact. For each $C_\mathbb{K}^1$-map $f \colon U \to F$ and each $x \in U$, the map $f'(x) \colon E \to F$, $y \mapsto df(x, y)$ is $\mathbb{K}$-linear.

**2.2.** Let $E$ and $F$ be real locally convex spaces and $U \subseteq E$ be an open subset. A function $f \colon U \to F$ is called *real analytic* (or $\mathbb{R}$-analytic) if $f$ admits a complex analytic extension $g \colon W \to F_\mathbb{C}$ to an open subset $W \subseteq E_\mathbb{C}$ (see [18, Definition 2.2.2], also [10] and [26]).

The following fact is useful.

**2.3.** Let $E$ and $F$ be complex locally convex spaces and $U \subseteq E$ be an open subset. A function $f \colon U \to F$ is complex analytic if and only if $f$ is $C_\mathbb{R}^\infty$ and $f'(x) \colon E \to F$ is $\mathbb{C}$-linear for each $x \in U$ (see [18, Corollary 2.1.9], also [10]).

**2.4.** $C_\mathbb{K}^r$-manifolds modeled on a locally convex topological $\mathbb{K}$-vector space $E$ for $r \in \mathbb{N}_0 \cup \{\infty\}$ and $\mathbb{K}$-analytic manifolds modelled on $E$ can be defined as expected, as well as tangent bundles and tangent maps; likewise $C_\mathbb{K}^\infty$-Lie groups modelled on $E$ and $\mathbb{K}$-analytic Lie groups (see [18, 27], also [2, 10]). In this article, Lie groups or manifolds may always have infinite dimension (unless the contrary is stated).

By definition, a $\mathbb{K}$-*analytic diffeomorphism* is an invertible $\mathbb{K}$-analytic map between $\mathbb{K}$-analytic manifolds whose inverse is $\mathbb{K}$-analytic. If $V$ is an open subset of a locally convex space $E$, we identify its tangent bundle with $V \times E$ as usual. If $M$ is a $C^1$-manifold and $f \colon M \to V$ a $C^1$-map, we write $df$ for the second component of the tangent map $Tf \colon TM \to TV = V \times E$.

See [18], [26], and [27] for basic concepts concerning infinite-dimensional Lie groups (like the Lie algebra $\mathfrak{g} := L(G) := T_eG$, the Lie algebra homomorphism $L(f) := T_e(f)$ associated with a smooth group homomorphism $f$ and the notion of an exponential map $\exp_G \colon \mathfrak{g} \to G$). See [18] for the next concept (cf. also [11, 26]).

**2.5.** A $\mathbb{K}$-analytic Lie group $G$ is called a *BCH-Lie group* if it has an exponential function $\exp_G$ which restricts to a $\mathbb{K}$-analytic diffeomorphism from an open zero-neighbourhood in the Lie algebra $\mathfrak{g}$ of $G$ onto an open identity-neighbourhood in $G$.

**Definition 2.6.** Let $m \in \mathbb{N}$. A set $\mathcal{U}$ of open subsets of $\mathbb{R}^m$ will be called a *good collection of open subsets* if the following conditions are satisfied:

(a) $\mathcal{U}$ is a basis for the topology of $\mathbb{R}^m$.

(b) If $U \in \mathcal{U}$ and $K \subseteq U$ is a compact non-empty subset, then there exists $V \in \mathcal{U}$ with compact closure $\overline{V}$ in $\mathbb{R}^m$ such that $K \subseteq V$ and $\overline{V} \subseteq U$.

(c) If $U \subseteq \mathbb{R}^m$ is an open set and $W \in \mathcal{U}$ is a relatively compact subset of $U$, then there exists $V \in \mathcal{U}$ such that $V$ is a relatively compact subset of $U$ and $\overline{W} \subseteq V$.

(d) If $\phi \colon U \to V$ is a $C^\infty$-diffeomorphism between open subsets $U$ and $V$ of $\mathbb{R}^m$ and $W \in \mathcal{U}$ is a relatively compact subset of $U$, then $\phi(W) \in \mathcal{U}$.

**Example 2.7.** The following are good collections of open subsets of $\mathbb{R}^m$:

(a) The set of all open subsets of $\mathbb{R}^m$, and the set of all open bounded subsets;

(b) If $m = 1$, the set of all relatively compact, open intervals $I \neq \emptyset$ in $\mathbb{R}$.

The simple verification is left to the reader.

**Remark 2.8.** In an appendix to the preprint version of this article, we show that also bounded open subsets $U \subseteq \mathbb{R}^m$ with $C^\infty$-boundary form a good collection of open sets (see arXiv:2210.01246). We shall not use this fact here.

## 3. Axioms for function spaces

Fix $m \in \mathbb{N}$. If $U \subseteq \mathbb{R}^m$ is an open subset, we let $BC(U, \mathbb{R})$ be the vector space of all bounded continuous functions $f \colon U \to \mathbb{R}$ and make it a Banach space using the supremum norm $\|\cdot\|_\infty$. Let $\mathcal{U}$ be a good collection of open subsets of $\mathbb{R}^m$. For $U \in \mathcal{U}$, let a vector subspace $\mathcal{F}(U, \mathbb{R})$ of $BC(U, \mathbb{R})$ be given; assume that $\mathcal{F}(U, \mathbb{R})$ is equipped with an integral complete locally convex vector topology making the inclusion $\mathcal{F}(U, \mathbb{R}) \to BC(U, \mathbb{R})$ continuous.

Given $U \in \mathcal{U}$, we can then associate an integral complete locally convex space $\mathcal{F}(U, E)$ to each finite-dimensional real vector space $E$:

**3.1.** If $b_1, \ldots, b_n$ is a basis for $E$, where $n := \dim(E)$, we define

$$\mathcal{F}(U, E) := \sum_{k=1}^{n} \mathcal{F}(U, \mathbb{R}) b_k$$

and give it the locally vector topology making the map

$$\mathcal{F}(U, \mathbb{R})^n \to \mathcal{F}(U, E), \quad (f_1, \ldots, f_n) \mapsto \sum_{k=1}^{m} f_k b_k \tag{1}$$

an isomorphism of topological vector spaces.

Note that $\mathcal{F}(U, E)$ and its topology are independent of the choice of basis.

**3.2.** If we have $E = E_1 \oplus E_2$ with vector subspaces $E_1$ and $E_2$, we can choose a basis $b_1, \ldots, b_k$ for $E_1$ and a basis $b_{k+1}, \ldots, b_n$ for $E_2$. We easily deduce that $\mathcal{F}(U, E) = \mathcal{F}(U, E_1) \oplus \mathcal{F}(U, E_2)$ as a topological vector space. For all finite-dimensional vector spaces $F_1$ and $F_2$, we therefore have

$$\mathcal{F}(U, F_1 \times F_2) \cong \mathcal{F}(U, F_1) \times \mathcal{F}(U, F_2).$$

If $W$ is an open subset of $E$, we let $\mathcal{F}(U, W)$ be the set of all $\gamma \in \mathcal{F}(U, E)$ such that $\gamma(U) + Q \subseteq W$ for some 0-neighbourhood $Q \subseteq E$.

**Definition 3.3.** We say that $(\mathcal{F}(U, \mathbb{R}))_{U \in \mathcal{U}}$ as before is a *family of locally convex spaces suitable for Lie theory* if the following axioms are satisfied for all finite-dimensional real vector spaces $E$ and $F$:

**Pushforward Axiom (PF):** For all $U, V \in \mathcal{U}$ such that $V$ is relatively compact in $U$ and for each smooth map $f : U \times E \to F$, we have
  (1) $f_*(\gamma) := f \circ (\mathrm{id}_V, \gamma|_V) \in \mathcal{F}(V, F)$ for all $\gamma \in \mathcal{F}(U, E)$ and
  (2) the map $f_* : \mathcal{F}(U, E) \to \mathcal{F}(V, F), \quad \gamma \mapsto f_*(\gamma)$ is continuous.

**Pullback Axiom (PB):** Let $U$ be an open subset of $\mathbb{R}^m$ and $V, W \in \mathcal{U}$ such that $W$ has a compact closure contained in $U$. Let $\Theta : U \to V$ be a $C^\infty$-diffeomorphism. Then $\gamma \circ \Theta|_W \in \mathcal{F}(W, E)$ for all $\gamma \in \mathcal{F}(V, E)$ and $\mathcal{F}(\Theta|_W, E) : \mathcal{F}(V, E) \to \mathcal{F}(W, E), \gamma \mapsto \gamma \circ \Theta|_W$ is a continuous map.

**Globalization Axiom (GL):** If $U, V \in \mathcal{U}$ with $V \subseteq U$ and $\gamma \in \mathcal{F}(V, E)$ has compact support, then the map $\widetilde{\gamma} : U \to E$ defined by $\widetilde{\gamma}(x) = \gamma(x)$ if $x \in V$ and $\widetilde{\gamma}(x) = 0$ if $x \in U \setminus \mathrm{supp}(\gamma)$ is in $\mathcal{F}(U, E)$ and for each compact subset $K$ of $V$ the map
$$e_{U,V,K}^E : \mathcal{F}_K(V, E) \to \mathcal{F}(U, E), \quad \gamma \mapsto \widetilde{\gamma}$$
is continuous, where $\mathcal{F}_K(V, E) := \{\gamma \in \mathcal{F}(V, E) : \mathrm{supp}(\gamma) \subseteq K\}$ is endowed with the topology induced by $\mathcal{F}(V, E)$.

**Multiplication Axiom (MU):** If $U \in \mathcal{U}$ and $h \in C_c^\infty(U, \mathbb{R})$, then
  (1) $h\gamma \in \mathcal{F}(U, E)$ for all $\gamma \in \mathcal{F}(U, E)$ and
  (2) the map $m_h^E : \mathcal{F}(U, E) \to \mathcal{F}(U, E), \gamma \mapsto h\gamma$ is continuous.

**Remark 3.4.** As the map in (1) is an isomorphism of topological vector spaces, we see that Axioms (PB), (GL), and (MU) hold in general whenever they hold for $E := \mathbb{R}$. Likewise, Axiom (PF) holds in general whenever it holds for $F := \mathbb{R}$.

**Remark 3.5.** Concerning Axiom (MU), observe that if $\mathcal{F}(U, E)$ is a Fréchet space and $h\gamma \in \mathcal{F}(U, E)$ for each $\gamma \in \mathcal{F}(U, E)$, then $m_h^E$ is continuous.

[The multiplication operator $M_h : BC(U, E) \to BC(U, E), \gamma \mapsto h\gamma$ being continuous, its graph $\mathrm{graph}(M_h)$ is closed in $BC(U, E) \times BC(U, E)$. As the inclusion map $\iota : \mathcal{F}(U, E) \to BC(U, E)$ is continuous, we deduce that $(\iota \times \iota)^{-1}(\mathrm{graph}(M_h)) = \mathrm{graph}(m_h^E)$ is closed in $\mathcal{F}(U, E) \times \mathcal{F}(U, E)$. The continuity of $m_h^E$ now follows from the Closed Graph Theorem.]

Likewise, the continuity of the linear mapping $\mathcal{F}(V, E) \to \mathcal{F}(W, E)$ in (PB) is automatic if $\mathcal{F}(V, E)$ and $\mathcal{F}(W, E)$ are Fréchet spaces, using that the linear mapping $BC(V, E) \to BC(W, E)$, $\gamma \mapsto \gamma \circ \Theta|_W$ is continuous with operator norm $\leq 1$ (if we endow $E$ with a norm defining its topology and spaces of bounded continuous functions to $E$ with the supremum norm).

Likewise, $e_{U,V,K}^E$ is continuous in (GL) if $\mathcal{F}_K(V, E)$ and $\mathcal{F}(U, E)$ are Fréchet spaces. In fact, endowing $BC_K(V, E) := \{\gamma \in BC(V, E) \colon \mathrm{supp}(\gamma) \subseteq K\}$ with the supremum norm, the map $BC_K(V, E) \to BC(U, E)$, $\gamma \mapsto \widetilde{\gamma}$ which extends functions by 0 is a linear isometry.

## 4. Basic consequences of the axioms

Let $m \in \mathbb{N}$, $\mathcal{U}$ be a good collection of subsets of $\mathbb{R}^m$ and $(\mathcal{F}(U, \mathbb{R}))_{U \in \mathcal{U}}$ be a family of locally convex spaces which is suitable for Lie theory. We record consequences of the four axioms.

**Lemma 4.1.** *Let $E$ be a finite-dimensional real vector space and $U, W \in \mathcal{U}$ such that $W$ is relatively compact in $U$. Then $\gamma|_W \in \mathcal{F}(W, E)$ holds for each $\gamma \in \mathcal{F}(U, E)$ and the restriction map*

$$r_{W,U}^E \colon \mathcal{F}(U, E) \to \mathcal{F}(W, E), \quad \gamma \mapsto \gamma|_W$$

*is continuous.*

**Proof.** We can take $V := U$ and $\Theta := \mathrm{id}_U$ in Axiom (PB). ∎

Lemma 4.1 and Axiom (GL) imply:

**Lemma 4.2.** *Let $E$ be a finite-dimensional real vector space and $U, W \in \mathcal{U}$ such that $W$ is relatively compact in $U$. Let $K \subseteq W$ be compact. Then*

$$r_{W,U,K}^E \colon \mathcal{F}_K(U, E) \to \mathcal{F}_K(W, E), \quad \gamma \mapsto \gamma|_W$$

*is an isomorphism of topological vector spaces.* ∎

Also the following maps are useful.

**Lemma 4.3.** *Let $E$ and $F$ be finite-dimensional vector spaces. Let $U, V \in \mathcal{U}$ such that $V$ is relatively compact in $U$, and $\Phi \colon E \to F$ be a smooth map. Then $\Phi \circ \gamma|_V \in \mathcal{F}(V, F)$ holds for each $\gamma \in \mathcal{F}(U, E)$ and the mapping $\mathcal{F}(U, E) \to \mathcal{F}(V, F)$, $\gamma \mapsto \Phi \circ \gamma|_V$ is continuous.*

**Proof.** Axiom (PF) applies to $f \colon U \times E \to F$, $(x, y) \mapsto \Phi(y)$. ∎

**Definition 4.4.** Let $E$ be a finite-dimensional real vector space and $U$ be an open subset of $\mathbb{R}^m$. We let $\mathcal{F}_{\mathrm{loc}(U,E)}$ be the set of all functions $\gamma \colon U \to E$ with the following property: For each $V \in \mathcal{U}$ which is a relatively compact subset of $U$, the restriction $\gamma|_V$ is in $\mathcal{F}(V, E)$.

Note that each $\gamma \in \mathcal{F}_{\mathrm{loc}}(U, E)$ is continuous, and that $\mathcal{F}_{\mathrm{loc}}(U, E)$ is a vector subspace of $E^U$. If $U \in \mathcal{U}$, then $\mathcal{F}(U, E) \subseteq \mathcal{F}_{\mathrm{loc}(U,E)}$, by Lemma 4.1. We give $\mathcal{F}_{\mathrm{loc}(U,E)}$ the initial topology with respect to the restrictions maps

$$\rho_{V,U}^E \colon \mathcal{F}_{\mathrm{loc}(U,E)} \to \mathcal{F}(V, E), \quad \gamma \mapsto \gamma|_V$$

for all $V \in \mathcal{U}$ which are relatively compact in $U$.

As the restriction maps are linear and separate points, $\mathcal{F}_{\mathrm{loc}(U,E)}$ is a Hausdorff locally convex space.

**Lemma 4.5.** *Let $E$ be a finite-dimensional vector space. If $U$ and $V$ are open subsets of $\mathbb{R}^m$ such that $V \subseteq U$, then $\gamma|_V \in \mathcal{F}_{\mathrm{loc}}(V,E)$ for each $\gamma \in \mathcal{F}_{\mathrm{loc}}(U,E)$ and the restriction map*

$$\delta_{V,U}^E \colon \mathcal{F}_{\mathrm{loc}}(U,E) \to \mathcal{F}_{\mathrm{loc}}(V,E), \quad \gamma \mapsto \gamma|_V$$

*is continuous and linear.*

**Proof.** Each $W \in \mathcal{U}$ which is relatively compact in $V$ is also relatively compact in $U$, whence $(\gamma|_V)|_W = \gamma|_W \in \mathcal{F}(W,E)$ by definition of $\mathcal{F}_{\mathrm{loc}}(U,E)$. Hence $\gamma|_V \in \mathcal{F}_{\mathrm{loc}}(V,E)$. Since $\rho_{W,V}^E \circ \delta_{V,U}^E = r_{W,V}^E \circ \rho_{V,U}^E$ is continuous for each $W$, the map $\delta_{V,U}^E$ is continuous. ∎

**Lemma 4.6.** *Let $E$ and $F$ be finite-dimensional vector spaces and $U$ be an open subset of $\mathbb{R}^m$. For each $\Phi \in C^\infty(E,F)$, the assignment $\gamma \mapsto \Phi \circ \gamma$ defines a continuous map $\mathcal{F}_{\mathrm{loc}(U,\Phi)} \colon \mathcal{F}_{\mathrm{loc}(U,E)} \to \mathcal{F}_{\mathrm{loc}(U,F)}$.*

**Proof.** Let $\gamma \in \mathcal{F}_{\mathrm{loc}(U,E)}$. If $V \in \mathcal{U}$ is relatively compact in $U$, we have $\overline{V} \subseteq W$ for some $W \in \mathcal{U}$ which is relatively compact in $U$, by Definition 2.6 (c). Since $\gamma|_W = \rho_{W,U}^E(\gamma) \in \mathcal{F}(W,E)$, we have $\Phi \circ \gamma|_V = \Phi \circ (\gamma|_W)|_V \in \mathcal{F}(V,F)$, by Lemma 4.3. Thus $\Phi \circ \gamma \in \mathcal{F}_{\mathrm{loc}}(U,F)$. For $V$ and $W$ as before, the map

$$h_V \colon \mathcal{F}(W,E) \to \mathcal{F}(V,F), \quad \eta \mapsto \Phi \circ \eta|_V$$

is continuous, by Lemma 4.3. Thus $\rho_{V,U}^F \circ \mathcal{F}_{\mathrm{loc}}(U,\Phi) = h_V \circ \rho_{W,U}^E$ is continuous. The topology on $\mathcal{F}_{\mathrm{loc}}(U,F)$ being initial with respect to the maps $\rho_{V,U}^F$, we deduce that $\mathcal{F}_{\mathrm{loc}}(U,\Phi)$ is continuous. ∎

We also need the following variant.

**Lemma 4.7.** *Let $E$ and $F$ be finite-dimensional vector spaces and $U$ be an open subset of $\mathbb{R}^m$. Let $\Psi \colon Q \to F$ be a smooth function on an open subset $Q \subseteq E$. Then $\Psi \circ \gamma \in \mathcal{F}_{\mathrm{loc}}(U,F)$ for each $\gamma \in \mathcal{F}_{\mathrm{loc}}(U,E)$ such that $\gamma(U) \subseteq Q$.*

**Proof.** Let $\gamma \in \mathcal{F}_{\mathrm{loc}}(U,E)$ with $\gamma(U) \subseteq Q$. For each $W \in \mathcal{U}$ which is relatively compact in $U$, there exists $V \in \mathcal{U}$ such that $\overline{W} \subseteq V$ and $V$ is relatively compact in $U$, by Definition 2.6 (c). Then $\gamma|_V \in \mathcal{F}(V,E)$. The image $K := \gamma(\overline{V})$ is a compact subset of $Q$. There exists $\xi \in C_c^\infty(Q,\mathbb{R})$ such that $\xi|_K = 1$. We define $\Phi(y) := \xi(y)\Psi(y)$ for $y \in Q$ and $\Phi(y) := 0$ for $y \in E \setminus K$. Then $\Phi \in C^\infty(E,F)$ and $\Phi(y) = \Psi(y)$ for each $y \in \gamma(V)$, whence

$$(\Psi \circ \gamma)|_W = \Psi \circ (\gamma|_W) = \Phi \circ (\gamma|_W) = (\Phi \circ (\gamma|_V))|_W \in \mathcal{F}(W,F)$$

by Lemma 4.3. Thus $\Psi \circ \gamma \in \mathcal{F}(U,F)$. ∎

**Lemma 4.8.** *Let $E$ be a finite-dimensional vector space, $U$ and $V$ be open subsets of $\mathbb{R}^m$ and $\Theta \colon U \to V$ be a $C^\infty$-diffeomorphism. Then $\gamma \circ \Theta \in \mathcal{F}_{\mathrm{loc}(U,E)}$ holds for all $\gamma \in \mathcal{F}_{\mathrm{loc}(V,E)}$. Moreover, the linear map $\mathcal{F}_{\mathrm{loc}}(\Theta,E) \colon \mathcal{F}_{\mathrm{loc}}(V,E) \to \mathcal{F}_{\mathrm{loc}}(U,E)$, $\gamma \mapsto \gamma \circ \Theta$ is continuous.*

**Proof.** Let $W \in \mathcal{U}$ be relatively compact in $U$. By Definition 2.6 (c), there exists $P \in \mathcal{U}$ such that $\overline{P}$ is compact and $\overline{W} \subseteq P \subseteq \overline{P} \subseteq U$. Then $Q := \Theta(P)$ is a relatively compact subset of $V$ and $Q \in \mathcal{U}$ by Definition 2.6 (d).

Hence $\gamma|_Q \in \mathcal{F}(Q, E)$. By Axiom (PB), we have

$$(\gamma \circ \Theta)|_W = (\gamma|_Q \circ \Theta|_P^Q)|_W \in \mathcal{F}(W, E).$$

Thus, $\gamma \circ \Theta \in \mathcal{F}_{\mathrm{loc}(U,E)}$. By Axiom (PB), the mapping

$$\rho_{W,U}^E \circ \mathcal{F}_{\mathrm{loc}}(\Theta, E) = \mathcal{F}((\Theta|_P^Q)|_W, E) \circ \rho_{Q,V}^E$$

is continuous for each $W$ (choosing $P$ and $Q$ for $W$ as before). Hence $\mathcal{F}_{\mathrm{loc}}(\Theta, E)$ is continuous. ∎

**Lemma 4.9.** *Let $E$ be a finite-dimensional vector space and assume $U \in \mathcal{U}$ and $h \in C_c^\infty(U, \mathbb{R})$. Then $h\gamma \in \mathcal{F}(U, E)$ holds for all $\gamma \in \mathcal{F}_{\mathrm{loc}}(U, E)$ and the mapping $\mu_h^E \colon \mathcal{F}_{\mathrm{loc}}(U, E) \to \mathcal{F}(U, E),\ \gamma \mapsto h\gamma$ is continuous.*

**Proof.** By Definition 2.6 (b), there exists $V \in \mathcal{U}$ with $K := \mathrm{supp}(h) \subseteq V$ such that $V$ is relatively compact in $U$. Let $\gamma \in \mathcal{F}_{\mathrm{loc}}(U, E)$. Then $\gamma|_V \in \mathcal{F}(V, E)$. By Axiom (MU), $(h\gamma)|_V = h|_V\gamma|_V \in \mathcal{F}(V, E)$, entailing that $(h\gamma)|_V \in \mathcal{F}_K(V, E)$. Consequently, $h\gamma = (h|_V\gamma|_V)^{\sim} \in \mathcal{F}(U, E)$ and $h\gamma = (e_{U,V,K}^E \circ m_{h|_V}^E \circ \rho_{V,U}^E)(\gamma) \in \mathcal{F}(U, E)$ depends continuously on $\gamma \in \mathcal{F}_{\mathrm{loc}}(U, E)$ by continuity of $\rho_{V,U}^E$, Axiom (MU), and Axiom (GL). ∎

**Lemma 4.10.** *Let $E$ be a finite-dimensional vector space, $U_1, \ldots, U_n$ be open subsets of $\mathbb{R}^m$ and $\gamma_j \in \mathcal{F}_{\mathrm{loc}}(U_j, E)$ for $j \in \{1, \ldots, n\}$ such that*

$$\gamma_j|_{U_j \cap U_k} = \gamma_k|_{U_j \cap U_k} \quad \text{for all } j, k \in \{1, \ldots, n\}.$$

*If $V \in \mathcal{U}$ is relatively compact in $U_1 \cup \ldots \cup U_n$, then $\gamma \in \mathcal{F}(V, E)$ holds for the map $\gamma \colon V \to E$ defined piecewise via $\gamma(x) := \gamma_j(x)$ for $x \in V \cap U_j$.*

**Proof.** By Definition 2.6 (c), we find $W \in \mathcal{U}$ which is relatively compact in $U_1 \cup \cdots \cup U_n$ and contains $\overline{V}$. Since $\mathcal{U}$ is a basis for the topology of $\mathbb{R}^m$, using the compactness of $\overline{V}$ we find $W_1, \ldots, W_\ell \in \mathcal{U}$ with $\overline{V} \subseteq \bigcup_{i=1}^\ell W_i$ such that, for each $i \in \{1, \ldots, \ell\}$, the set $W_i$ is relatively compact in $U_{j(i)} \cap W$ for some $j(i) \in \{1, \ldots, n\}$. Let $h_1, \ldots, h_\ell, h_0$ be a $C^\infty$-partition of unity on $\mathbb{R}^m$ subordinate to $W_1, \ldots, W_\ell, \mathbb{R}^m \setminus \overline{V}$. For each $i \in \{1, \ldots, \ell\}$, the support $L_i := \mathrm{supp}(h_i) \subseteq W_i$ of $h_i$ in $\mathbb{R}^m$ is compact, as $W_i$ is relatively compact in $\mathbb{R}^m$. Now $\gamma$ can be written

$$\gamma = \sum_{i=1}^\ell ((h_i|_{W_i}\gamma_{j(i)}|_{W_i})^{\sim})|_V,$$

where the tilde indicates the extension by 0 to an element of $\mathcal{F}(W, E)$. Thus $\gamma = \sum_{i=1}^\ell (r_{V,W}^E \circ e_{W,W_i,L_i}^E \circ m_{h_i|_{W_i}}^E \circ \rho_{W_i,U_{j(i)}}^E)(\gamma_{j(i)}) \in \mathcal{F}(V, E)$. ∎

**Remark 4.11.** Let $\mathcal{E}$ be the vector subspace of $\prod_{j=1}^n \mathcal{F}_{\mathrm{loc}}(U_j, E)$ given by the $n$-tuples $(\gamma_1, \ldots, \gamma_n)$ such that $\gamma_j|_{U_j \cap U_k} = \gamma_k|_{U_j \cap U_k}$ for all $j, k \in \{1, \ldots, n\}$. Endow $\mathcal{E}$ with the topology induced by $\prod_{k=1}^n \mathcal{F}_{\mathrm{loc}}(U_k, E)$. The final formula of the preceding proof shows that the linear map

$$\mathrm{glue} \colon \mathcal{E} \to \mathcal{F}(V, E), \quad (\gamma_1, \ldots, \gamma_n) \mapsto \gamma$$

(with $\gamma$ as in Lemma 4.10) is continuous.

## 5. Associated function spaces on manifolds

Let $m$, $\mathcal{U}$ and $(\mathcal{F}(U, \mathbb{R}))_{U \in \mathcal{U}}$ be as in the preceding section.

**Definition 5.1.** Let $M$ be a compact smooth manifold of dimension $m$ and $N$ a smooth manifold of dimension $n$, both without boundary. We let $\mathcal{F}(M, N)$ be the set of all functions $\gamma \colon M \to N$ with the following property: For each $x \in M$, there exists a chart $\phi \colon M \supseteq U_\phi \to V_\phi \subseteq \mathbb{R}^m$ of $M$ with $V_\phi \in \mathcal{U}$ and a chart $\psi \colon N \supseteq U_\psi \to V_\psi \subseteq \mathbb{R}^n$ of $N$ such that $x \in U_\phi$, $\gamma(U_\phi) \subseteq U_\psi$, and $\psi \circ \gamma \circ \phi^{-1} \in \mathcal{F}(V_\phi, \mathbb{R}^n)$.

Since $\psi \circ \gamma \circ \phi^{-1}$ is continuous, $\gamma|_{U_\phi}$ is continuous in the preceding situation. Hence each $\gamma \in \mathcal{F}(M, N)$ is continuous.

**Lemma 5.2.** *Let $M$ be an $m$-dimensional compact smooth manifold, $N$ be a smooth manifold of dimension $n$ and $\gamma \colon M \to N$ be a continuous map. Then $\gamma \in \mathcal{F}(M, N)$ if and only if $\psi \circ \gamma \circ \phi^{-1} \in \mathcal{F}_{\mathrm{loc}}(V_\phi, \mathbb{R}^n)$ for each chart $\phi \colon U_\phi \to V_\phi$ of $M$ and each chart $\psi \colon U_\psi \to V_\psi$ of $N$ such that $\gamma(U_\phi) \subseteq U_\psi$.*

**Proof.** If $\gamma \in \mathcal{F}(M, N)$, let $\phi \colon U_\phi \to V_\phi$ be a chart of $M$ and $\psi \colon U_\psi \to V_\psi$ be a chart of $N$ such that $\gamma(U_\phi) \subseteq U_\psi$. Let $W \in \mathcal{U}$ be relatively compact in $V_\phi$. By definition, for each point $x \in M$ there exist a chart $\phi_x \colon U_x \to V_x \subseteq \mathbb{R}^m$ of $M$ with $V_x \in \mathcal{U}$ and a chart $\psi_x \colon A_x \to B_x \subseteq \mathbb{R}^n$ of $N$ such that $x \in U_x$, $\gamma(U_x) \subseteq A_x$ and $\psi_x \circ \gamma \circ \phi_x^{-1} \in \mathcal{F}(V_x, \mathbb{R}^n)$. Since $M = \bigcup_{x \in M} U_x$, there exists a finite subcover $U_{x_1}, \ldots, U_{x_r}$ of $M$. We have

$$g_i := \psi_{x_i} \circ \gamma \circ \phi_{x_i}^{-1} \in \mathcal{F}(V_{x_i}, \mathbb{R}^n) \subseteq \mathcal{F}_{\mathrm{loc}}(V_{x_i}, \mathbb{R}^n)$$

for all $i \in \{1, \ldots, r\}$. Then $g_i|_{\phi_{x_i}(U_\phi \cap U_{x_i})} \in \mathcal{F}_{\mathrm{loc}}(\phi_{x_i}(U_\phi \cap U_{x_i}), \mathbb{R}^n)$, since $\phi_{x_i}(U_\phi \cap U_{x_i})$ is an open subset of $V_{x_i}$ (see Lemma 4.5). For each $i \in \{1, \ldots, r\}$, consider the $C^\infty$-diffeomorphisms

$$\Theta_i := \phi_{x_i} \circ \phi^{-1} \colon \phi(U_\phi \cap U_{x_i}) \to \phi_{x_i}(U_\phi \cap U_{x_i})$$

and $\Phi_i := \psi \circ \psi_{x_i}^{-1} \colon \psi_{x_i}(U_\psi \cap A_{x_i}) \to \psi(U_\psi \cap A_{x_i})$. By Lemma 4.8, we have

$$g_i \circ \Theta_i \in \mathcal{F}_{\mathrm{loc}}(\phi(U_\phi \cap U_{x_i}), \mathbb{R}^n),$$

whence $\Phi_i \circ g_i \circ \Theta_i \in \mathcal{F}_{\mathrm{loc}}(\phi(U_\phi \cap U_{x_i}), \mathbb{R}^n)$, by Lemma 4.7. Note that

$$(\psi \circ f \circ \phi^{-1})|_{\phi(U_\phi \cap U_{x_i})} = \Phi_i \circ g_i \circ \Theta_i;$$

in fact, for all $y \in \phi(U_\phi \cap U_{x_i})$, we have

$$(\Phi_i \circ g_i \circ \Theta_i)(y) = ((\psi \circ \psi_{x_i}^{-1}) \circ (\psi_{x_i} \circ \gamma \circ \phi_{x_i}^{-1}) \circ (\phi_{x_i} \circ \phi^{-1}))(y) = (\psi \circ \gamma \circ \phi^{-1})(y).$$

Since $\overline{W} \subseteq V_\phi = \bigcup_{i=1}^r \phi(U_\phi \cap U_{x_i})$, we are in the situation of Lemma 4.10 and deduce that $\psi \circ \gamma \circ \phi^{-1}|_W \in \mathcal{F}(W, \mathbb{R}^n)$. Thus $\psi \circ \gamma \circ \phi^{-1} \in \mathcal{F}_{\mathrm{loc}(V_\phi, \mathbb{R}^n)}$.

Conversely, let $\gamma \colon M \to N$ be a function such that, for each $x \in M$, there are a chart $\phi \colon U_\phi \to V_\phi \subseteq \mathbb{R}^m$ of $M$ and a chart $\psi \colon U_\psi \to V_\psi \subseteq \mathbb{R}^n$ of $N$ such that $x \in U_\phi$ holds, $\gamma(U_\phi) \subseteq U_\psi$ and $\psi \circ \gamma \circ \phi^{-1} \in \mathcal{F}_{\mathrm{loc}}(V_\phi, \mathbb{R}^n)$. Since $\mathcal{U}$ is a basis for the topology on $\mathbb{R}^m$, there exists $W \in \mathcal{U}$ such that $\phi(x) \in W$ and $W$ is relatively compact in $V_\phi$. Then $(\psi \circ \gamma \circ \phi^{-1})|_W \in \mathcal{F}(W, \mathbb{R}^n)$. Thus $\eta := \phi|_W \colon \phi^{-1}(W) \to W$ is a chart of $M$ such that $x \in \phi^{-1}(W)$, $\gamma(\phi^{-1}(W)) \subseteq U_\psi$ and $\psi \circ \gamma \circ \eta^{-1} = (\psi \circ \gamma \circ \phi^{-1})|_W \in \mathcal{F}(W, \mathbb{R}^n)$. Hence $\gamma \in \mathcal{F}(M, N)$. ∎

**Lemma 5.3.** *Let $\Phi\colon N_1 \to N_2$ be a smooth map between finite-dimensional smooth manifolds, and $M$ be a compact $m$-dimensional smooth manifold. Then $\Phi \circ \gamma \in \mathcal{F}(M, N_2)$ for each $\gamma \in \mathcal{F}(M, N_1)$.*

**Proof.** For each $x \in M$, there exists a chart $\psi_2\colon U_2 \to V_2 \subseteq \mathbb{R}^{n_2}$ of $N_2$ with $\Phi(\gamma(x)) \in U_2$ and a chart $\psi_1\colon U_1 \to V_1 \subseteq \mathbb{R}^{n_1}$ on $N_1$ such that $\gamma(x) \in U_1$ and $\Phi(U_1) \subseteq U_2$. Moreover, there exists a chart $\phi\colon U \to V \subseteq \mathbb{R}^m$ for $M$ with $x \in U$ such that $\gamma(U) \subseteq U_1$. By Definition 2.6 (a), there exists $W \in \mathcal{U}$ such that $\phi(x) \in W$ and $W$ is relatively compact in $V$. Now $\psi_1 \circ \gamma \circ \phi^{-1} \in \mathcal{F}_{\mathrm{loc}}(V, \mathbb{R}^{n_1})$, by Lemma 5.2. Since $\psi_2 \circ \Phi \circ \psi_1^{-1}$ is smooth, Lemma 4.7 shows that $\psi_2 \circ \Phi \circ \gamma \circ \phi^{-1} = (\psi_2 \circ \Phi \circ \psi_1^{-1}) \circ (\psi_1 \circ \gamma \circ \phi^{-1}) \in \mathcal{F}_{\mathrm{loc}}(V, \mathbb{R}^{n_2})$. Hence $\psi_2 \circ (\Phi \circ \gamma) \circ \phi^{-1}|_W \in \mathcal{F}(W, \mathbb{R}^{n_2})$. Thus $\Phi \circ \gamma \in \mathcal{F}(M, N_2)$. ∎

As a special case of Definition 5.1, taking $N := E$ we defined $\mathcal{F}(M, E)$ whenever $M$ is an $m$-dimensional compact smooth manifold and $E$ a finite-dimensional real vector space.

**5.4.** We give $\mathcal{F}(M, E)$ the initial topology $\mathcal{O}$ with respect to the mappings

$$\mathcal{F}(M, E) \to \mathcal{F}_{\mathrm{loc}}(V_\phi, E), \quad \gamma \mapsto \gamma \circ \phi^{-1},$$

for $\phi\colon U_\phi \to V_\phi \subseteq \mathbb{R}^m$ in the maximal $C^\infty$-atlas $\mathcal{A}$ of $M$. As the latter maps are linear and separate points on $\mathcal{F}(M, E)$, the topology $\mathcal{O}$ makes $\mathcal{F}(M, E)$ a Hausdorff locally convex space. By transitivity of initial topologies (see [18, Lemma A.2.7]), the topology $\mathcal{O}$ is also initial with respect to the maps

$$\mathcal{F}(\phi^{-1}|_W, E)\colon \mathcal{F}(M, E) \to \mathcal{F}(W, E), \quad \gamma \mapsto \gamma \circ \phi^{-1}|_W,$$

for $\phi\colon U_\phi \to V_\phi \subseteq \mathbb{R}^m$ in $\mathcal{A}$ and $W \in \mathcal{U}$ such that $W$ is relatively compact in $V_\phi$.

Finitely many pairs $(\phi, W)$ suffice to define the topology $\mathcal{O}$.

**Proposition 5.5.** *Let $\phi_j\colon U_j \to V_j \subseteq \mathbb{R}^m$ be charts for $M$ for $j \in \{1, \ldots, k\}$ and $W_j \in \mathcal{U}$ such that $W_j$ is relatively compact in $V_j$ and $M = \bigcup_{j=1}^k \phi_j^{-1}(W_j)$. Then*

$$\Theta\colon (\mathcal{F}(M, E), \mathcal{O}) \to \prod_{j=1}^k \mathcal{F}(W_j, E), \quad \gamma \mapsto (\gamma \circ \phi_j^{-1}|_{W_j})_{j=1}^k$$

*is a linear map and a topological embedding with closed image. The image $\mathrm{im}(\Theta)$ is the set $S$ of all $(\gamma_j)_{j=1}^k \in \prod_{j=1}^k \mathcal{F}(W_j, E)$ such that $\gamma_i(\phi_i(x)) = \gamma_j(\phi_j(x))$ for all $i, j \in \{1, \ldots, k\}$ and $x \in \phi_i^{-1}(W_i) \cap \phi_j^{-1}(W_j)$.*

**Proof.** Let $\mathcal{T}$ be the initial topology on $\mathcal{F}(M, E)$ with respect to the maps $\mathcal{F}(\phi_j^{-1}|_{W_j}, E)\colon \mathcal{F}(M, E) \to \mathcal{F}(W_j, E)$, $\gamma \mapsto \gamma \circ \phi_j^{-1}|_{W_j}$ for $j \in \{1, \ldots, k\}$. Then $\mathcal{T} \subseteq \mathcal{O}$. To see that $\mathcal{O} \subseteq \mathcal{T}$ holds, we have to show that $\mathcal{T}$ makes $\mathcal{F}(\phi^{-1}|_W, E)$ continuous for each chart $\phi\colon U \to V \subseteq \mathbb{R}^m$ of $M$ and each $W \in \mathcal{U}$ which is relatively compact in $V$. Abbreviate $Q_j := \phi_j^{-1}(W_j)$ for $j \in \{1, \ldots, k\}$. Let $\mathcal{E} \subseteq \prod_{j=1}^k \mathcal{F}_{\mathrm{loc}}(\phi(U \cap Q_j), E)$ be the set of those $(\gamma_j)_{j=1}^k$ with $\gamma_i(x) = \gamma_j(x)$ for all $x \in \phi(U \cap Q_i \cap Q_j)$. Then $W \subseteq \bigcup_{j=1}^n \phi(U \cap Q_j)$. Let

$$\mathrm{glue}\colon \mathcal{E} \to \mathcal{F}(W, E)$$

be the continuous linear glueing map, as in Remark 4.11. For $j \in \{1, \ldots, k\}$, consider the $C^\infty$-diffeomorphism $\Theta_j := \phi_j \circ \phi^{-1}\colon \phi(U \cap Q_j) \to \phi_j(U \cap Q_j)$. The

inclusion map $\lambda_j \colon \mathcal{F}(W_j, E) \to \mathcal{F}_{\mathrm{loc}}(W_j, E)$ is continuous linear and so is the map $\mathcal{F}_{\mathrm{loc}(\Theta_j, E)} \colon \mathcal{F}_{\mathrm{loc}}(\phi_j(U \cap Q_j), E) \to \mathcal{F}_{\mathrm{loc}}(\phi(U \cap Q_j), E)$. Abbreviate

$$h_j := \mathcal{F}_{\mathrm{loc}}(\Theta_j, E) \circ \delta^E_{\phi_j(U \cap Q_j), W_j} \circ \lambda_j \circ \mathcal{F}(\phi_j^{-1}|_{W_j}, E).$$

Then $h_j(\gamma) = (\gamma \circ \phi^{-1})|_{\phi(U \cap Q_j)}$ for all $\gamma \in \mathcal{F}(M, E)$ and $j \in \{1, \dots, k\}$, entailing that $(h_j(\gamma))_{j=1}^k \in \mathcal{E}$ and $\gamma \circ \phi^{-1}|_W = \mathrm{glue}((h_j(\gamma))_{j=1}^k)$. Thus the mapping $\mathcal{F}(\phi^{-1}|_W, E) = \mathrm{glue} \circ (h_1, \dots, h_k)$ is continuous on $(\mathcal{F}(M, E), \mathcal{T})$.

By the preceding, $\mathcal{O} = \mathcal{T}$, whence the linear map $\Theta$ is a topological embedding. Since $\mathcal{F}(W_j, E) \subseteq BC(W_j, E)$ and the inclusion map is continuous, the point evaluation $\mathcal{F}(W_j, E) \to E$, $\gamma \mapsto \gamma(x)$ is continuous for each $x \in W_j$. As it is also linear, we see that $S$ is a closed vector subspace of $\prod_{j=1}^k \mathcal{F}(W_j, E)$. We easily see that $\mathrm{im}(\Theta) \subseteq S$. If $(\gamma_j)_{j=1}^k \in S$, then $\gamma \colon M \to E$, $\gamma(x) := \gamma_j(\phi_j(x))$ if $x \in \phi_j^{-1}(W_j) =: Q_j$ with $j \in \{1, \dots, k\}$ is well defined and continuous, as $\gamma|_{Q_j} = \gamma_j \circ \phi_j|_{Q_j}$ is continuous. Since $\eta_j := \phi_j|_{Q_j} \colon Q_j \to W_j$ is a chart of $M$ with range in $\mathcal{U}$ such that $\gamma \circ \eta_j = \gamma_j \in \mathcal{F}(W_j, E)$, going back to Definition 5.1 we see that $\gamma \in \mathcal{F}(M, E)$. By construction, $\Theta(\gamma) = (\gamma_j)_{j=1}^k$. Thus $S \subseteq \mathrm{im}(\Theta)$ and thus $S = \mathrm{im}(\Theta)$. ∎

We record an immediate consequence:

**Corollary 5.6.** *The locally convex space $\mathcal{F}(M, E)$ is integral complete. For $(\phi_1, W_1), \dots, (\phi_k, W_k)$ as in Proposition 5.5, we have:*
  (a) *If $\mathcal{F}(W_j, E)$ is a Banach space for all $j \in \{1, \dots, k\}$, then $\mathcal{F}(M, E)$ is a Banach space.*
  (b) *If $\langle ., . \rangle_j$ is a scalar product on $\mathcal{F}(W_j, E)$ such that the associated norm defines its topology and makes it a Hilbert space, then*

$$\mathcal{F}(M, E) \times \mathcal{F}(M, E) \to \mathbb{R}, \quad (\gamma, \eta) \mapsto \langle \gamma, \eta \rangle := \sum_{j=1}^k \langle \gamma \circ \phi_j^{-1}|_{W_j}, \eta \circ \phi_j^{-1}|_{W_j} \rangle$$

  *is a scalar product making $\mathcal{F}(M, E)$ a Hilbert space whose associated norm defines the given topology $\mathcal{O}$ on $\mathcal{F}(M, E)$.* ∎

**Lemma 5.7.** *The inclusion map $\lambda_M^E \colon \mathcal{F}(M, E) \to C(M, E)$, $\gamma \mapsto \gamma$ is continuous. For each $x \in M$, the point evaluation*

$$\mathrm{ev}_x \colon \mathcal{F}(M, E) \to E, \quad \gamma \mapsto \gamma(x)$$

*is continuous and linear. Let $\mathcal{F}(M, U) := \{\gamma \in \mathcal{F}(M, E) \colon \gamma(M) \subseteq U\}$ for an open subset $U \subseteq E$. Then $\mathcal{F}(M, U)$ is an open subset of $\mathcal{F}(M, E)$.*

**Proof.** For a norm $\|\cdot\|$ on $E$, the topology on $C(M, E)$ can be obtained by the corresponding supremum norm $\|\cdot\|_\infty$. Using $(\phi_1, W_1), \dots, (\phi_k, W_k)$ as in Proposition 5.5 and $Q_j := \phi_j^{-1}(W_j)$, we have

$$\|\gamma\|_\infty = \max_{j=1,\dots,k} \|\gamma|_{Q_j}\|_\infty = \max_{j=1,\dots,k} \|\gamma \circ \phi_j^{-1}|_{W_j}\|_\infty = \max_{j=1,\dots,k} \|\mathcal{F}(\phi_j^{-1}|_{W_j}, E)(\gamma)\|_\infty.$$

As $\mathcal{F}(\phi_j^{-1}|_{W_j}, E) \colon \mathcal{F}(M, E) \to \mathcal{F}(W_j, E)$ is continuous linear and the supremum norm on $\mathcal{F}(W_j, E)$ is continuous, we see that $\|\lambda_M^E(\gamma)\|_\infty = \|\gamma\|_\infty$ is continuous in $\gamma \in \mathcal{F}(M, E)$, entailing that the linear map $\lambda_M^E$ is continuous. Since $C(M, U)$ is open in $C(M, E)$, we deduce that $\mathcal{F}(M, U) = (\lambda_M^E)^{-1}(C(M, U))$ is open in $\mathcal{F}(M, E)$. Finally, use that point evaluations on $C(M, E)$ are continuous. ∎

**Lemma 5.8.** *If $M$ is a compact smooth manifold of dimension $m$, $E$ and $F$ are finite-dimensional vector spaces, $U \subseteq E$ is open and $f \colon M \times U \to F$ a smooth mapping, then $f_*(\gamma) := f \circ (\mathrm{id}_M, \gamma) \in \mathcal{F}(M, F)$ for each $\gamma \in \mathcal{F}(M, U) \subseteq \mathcal{F}(M, E)$ and the map $f_* \colon \mathcal{F}(M, U) \to \mathcal{F}(M, F)$ is continuous.*

**Proof.** Let $\gamma \in \mathcal{F}(M, U)$. Then $\gamma(M)$ is a compact subset of $U$, and we find an open, relatively compact subset $V \subseteq U$ such that $\gamma(M) \subseteq V$. There exists a smooth function $\xi \colon U \to \mathbb{R}$ with compact support $K := \mathrm{supp}(\xi) \subseteq U$ such that $\xi|_V = 1$. We define a function $g \colon M \times E \to F$ piecewise via $g(x, y) := \xi(y) f(x, y)$ if $(x, y) \in M \times U$ and $g(x, y) := 0$ if $(x, y) \in M \times (E \setminus K)$. Then $g$ is smooth. Let $k \in \mathbb{N}$, charts $\phi_j \colon U_j \to V_j \subseteq \mathbb{R}^m$ for $M$ and sets $W_j \in \mathcal{U}$ for $j \in \{1, \ldots, k\}$ be as in Proposition 5.5. By Definition 2.6 (c), for each $j \in \{1, \ldots, k\}$ there exists $Y_j \in \mathcal{U}$ such that $Y_j$ is relatively compact in $V_j$ and $\overline{W}_j \subseteq Y_j$. The map

$$g_j \colon Y_j \times E \to F, \quad (x, y) \mapsto g(\phi_j^{-1}(x), y)$$

is smooth, for each $j \in \{1, \ldots, k\}$. By Axiom (PF), $(g_j)_*(\eta) := g_j \circ (\mathrm{id}_{W_j}, \eta|_{W_j}) \in \mathcal{F}(W_j, F)$ for each $\eta \in \mathcal{F}(Y_j, E)$ and the map

$$(g_j)_* \colon \mathcal{F}(Y_j, E) \to \mathcal{F}(W_j, F)$$

is continuous. For each $\eta \in \mathcal{F}(M, E)$, we have

$$g_*(\eta) \circ \phi_j^{-1}|_{W_j} = g \circ (\mathrm{id}_M, \eta) \circ \phi_j^{-1}|_{W_j} = (g_j)_*(\eta \circ \phi_j^{-1}|_{Y_j}) \in \mathcal{F}(W_j, F),$$

whence $g_*(\eta) \in \mathcal{F}(M, F)$. Since $\mathcal{F}(\phi_j^{-1}|_{W_j}, F) \circ g_* = (g_j)_* \circ \mathcal{F}(\phi_j^{-1}|_{Y_j}, E)$ is continuous for $j \in \{1, \ldots, k\}$, Proposition 5.5 shows that $g_*$ is continuous. For each $\eta$ in the open neighbourhood $P := \mathcal{F}(M, V)$ of $\gamma$ in $\mathcal{F}(M, U)$, we have $f_*(\eta) = g_*(\eta) \in \mathcal{F}(M, F)$. Notably, $f_*|_P = g_*|_P$ is continuous, whence $f_*$ is continuous at $\gamma$. ∎

**Lemma 5.9.** *If $E$ and $F$ are finite-dimensional vector spaces and $\Phi \colon U \to F$ is a smooth function on an open subset $U \subseteq E$, then $\Phi \circ \gamma \in \mathcal{F}(M, F)$ for each $\gamma \in \mathcal{F}(M, U) \subseteq \mathcal{F}(M, E)$ and the map $\mathcal{F}(M, \Phi) \colon \mathcal{F}(M, U) \to \mathcal{F}(M, F)$, $\gamma \mapsto \Phi \circ \gamma$ is continuous.*

**Proof.** Lemma 5.8 applies as $\Phi \circ \gamma = f_*(\gamma)$ for the $C^\infty$-map $f \colon M \times U \to F$, $(x, y) \mapsto \Phi(y)$. ∎

Mappings to products correspond to pairs of mappings.

**Lemma 5.10.** *If $E_1$ and $E_2$ are finite-dimensional vector spaces, we consider the projections $\mathrm{pr}_j \colon E_1 \times E_2 \to E_j$, $(x_1, x_2) \mapsto x_j$ for $j \in \{1, 2\}$. Then*

$$\Xi := (\mathcal{F}(M, \mathrm{pr}_1), \mathcal{F}(M, \mathrm{pr}_2)) \colon \mathcal{F}(M, E_1 \times E_2) \to \mathcal{F}(M, E_1) \times \mathcal{F}(M, E_2)$$

*is an isomorphism of topological vector spaces.*

**Proof.** The mappings $\lambda_1 \colon E_1 \to E_1 \times E_2$, $x_1 \mapsto (x_1, 0)$ and $\lambda_2 \colon E_2 \to E_1 \times E_2$, $x_2 \mapsto (0, x_2)$ are continuous and linear, as well as $\mathrm{pr}_1$, $\mathrm{pr}_2$ and the projections

$$\pi_j \colon \mathcal{F}(M, E_1) \times \mathcal{F}(M, E_2) \to \mathcal{F}(M, E_j), \quad (\gamma_1, \gamma_2) \mapsto \gamma_j$$

for $j \in \{1, 2\}$.

By Lemma 5.9, $\Xi$ and the linear map

$$\Theta := \mathcal{F}(M, \lambda_1) \circ \pi_1 + \mathcal{F}(M, \lambda_2) \circ \pi_2 \colon \mathcal{F}(M, E_1) \times \mathcal{F}(M, E_2) \to \mathcal{F}(M, E_1 \times E_2)$$

are continuous. We readily check that $\Xi \circ \Theta$ and $\Theta \circ \Xi$ are the identity maps, whence $\Xi$ is an isomorphism of topological vector spaces with $\Xi^{-1} = \Theta$. ∎

**Remark 5.11.** Likewise, using Lemma 4.6 instead of Lemma 5.9 we see that $\mathcal{F}_{\mathrm{loc}}(U, E_1 \times E_2) \cong \mathcal{F}_{\mathrm{loc}}(U, E_1) \times \mathcal{F}_{\mathrm{loc}}(U, E_2)$ for all open subsets $U \subseteq \mathbb{R}^m$ and finite-dimensional vector spaces $E_1$ and $E_2$.

**Lemma 5.12.** *Let $N_1$ and $N_2$ be finite-dimensional $C^\infty$-manifolds and $M$ a compact $m$-dimensional $C^\infty$-manifold. Then $\mathcal{F}(M, N_1 \times N_2) = \mathcal{F}(M, N_1) \times \mathcal{F}(M, N_2)$, identifying functions to $N_1 \times N_2$ with the pair of components.*

**Proof.** For $j \in \{1, 2\}$, the map $\pi_j \colon N_1 \times N_2 \to N_j$, $(x_1, x_2) \mapsto x_j$ is smooth. Hence, if $\gamma \in \mathcal{F}(M, N_1 \times N_2)$, then $\pi_j \circ \gamma \in \mathcal{F}(M, N_j)$, by Lemma 5.3. Conversely, let $\gamma_j \in \mathcal{F}(M, N_j)$ for $j \in \{1, 2\}$. Then the map $\gamma := (\gamma_1, \gamma_2) \colon M \to N_1 \times N_2$ is continuous. For $x \in M$, there are charts $\psi_j \colon U_j \to V_j \subseteq \mathbb{R}^{n_j}$ of $N_j$ for $j \in \{1, 2\}$ such that $\gamma_j(x) \in U_j$. There exists a chart $\phi \colon U \to V \subseteq \mathbb{R}^m$ of $M$ such that $x \in U$ and $\gamma(U) \subseteq U_1 \times U_2$. There exists $W \in \mathcal{U}$ such that $\phi(x) \in W$ and $W$ is relatively compact in $V$. Then $h := (\psi_1 \times \psi_2) \circ \gamma \circ \phi^{-1} \in \mathcal{F}_{\mathrm{loc}}(V, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ (using Remark 5.11), as its components $\psi_j \circ \gamma_j \circ \phi^{-1}$ are in $\mathcal{F}_{\mathrm{loc}}(V, \mathbb{R}^{n_j})$, by Lemma 5.2. Thus $h|_W \in \mathcal{F}(W, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. As a consequence, $\gamma \in \mathcal{F}(M, N_1 \times N_2)$. ∎

We need a more technical variant of the above maps.

**Lemma 5.13.** *Let $E$ and $F$ be finite-dimensional vector spaces, $N$ be a finite-dimensional smooth manifold, $M$ be a compact $m$-dimensional smooth manifold, $g \colon N \times E \to F$ be a $C^\infty$-map and $\gamma \in \mathcal{F}(M, N)$. Define $f \colon M \times E \to F$ via $f(x, y) := g(\gamma(x), y)$. Then $f_*(\eta) := f \circ (\mathrm{id}_M, \eta) = g \circ (\gamma, \eta) \in \mathcal{F}(M, F)$ for each $\eta \in \mathcal{F}(M, E)$ and the map $f_* \colon \mathcal{F}(M, E) \to \mathcal{F}(M, F)$ is continuous.*

**Proof.** Since $M$ is compact, we find $k \in \mathbb{N}$ and charts $\phi_j \colon U_j \to V_j \subseteq \mathbb{R}^m$ of $M$ for $j \in \{1, \ldots, k\}$, and relatively compact subsets $W_j \subseteq V_j$ such that $W_j \in \mathcal{U}$ and $M = \bigcup_{j=1}^k Q_j$ with $Q_j := \phi_j^{-1}(W_j)$, such that $\gamma(U_j) \subseteq P_j$ for some chart $\psi_j \colon A_j \to B_j \subseteq \mathbb{R}^n$ of $N$ and a relatively compact, open subset $P_j \subseteq A_j$. Let $\xi_j \colon B_j \to \mathbb{R}$ be a smooth map with compact support $K_j := \mathrm{supp}(\xi_j)$ such that $\xi_j|_{\psi_j(P_j)} = 1$. Then $g_j(x, y) := \xi_j(x) g(\psi_j^{-1}(x), y)$ for $(x, y) \in B_j \times E$, $g_j(x, y) := 0$ for $(x, y) \in (\mathbb{R}^n \setminus K_j) \times E$ defines a smooth map $g_j \colon \mathbb{R}^n \times E \to F$. By Lemma 4.6, the map

$$\mathcal{F}_{\mathrm{loc}}(V_j, g_j) \colon \mathcal{F}_{\mathrm{loc}}(V_j, \mathbb{R}^n \times E) \to \mathcal{F}_{\mathrm{loc}}(V_j, F), \quad \theta \mapsto g_j \circ \theta$$

is continuous. Identifying $\mathcal{F}_{\mathrm{loc}}(V_j, \mathbb{R}^n \times E)$ with $\mathcal{F}_{\mathrm{loc}}(V_j, \mathbb{R}^n) \times \mathcal{F}_{\mathrm{loc}}(V_j, E)$, we deduce from Lemma 5.2 that

$$\begin{aligned}
(g \circ (\gamma, \eta)) \circ \phi_j^{-1}|_{W_j} &= (g_j \circ (\psi_j \circ \gamma \circ \phi_j^{-1}, \eta \circ \phi_j^{-1}))|_{W_j} \\
&= (\rho_{W_j, V_j}^F \circ \mathcal{F}_{\mathrm{loc}}(V_j, g_j))(\psi_j \circ \gamma \circ \phi_j^{-1}, \eta \circ \phi_j^{-1})
\end{aligned}$$

is in $\mathcal{F}(W_j, F)$. Hence $g \circ (\gamma, \eta) \in \mathcal{F}(M, F)$.

Now $h_j \colon \mathcal{F}(M, E) \to \mathcal{F}_{\mathrm{loc}}(V_j, E)$, $\eta \mapsto \eta \circ \phi_j^{-1}$ is a continuous map by definition of the topology on $\mathcal{F}(M, E)$. Using Lemma 4.6, we deduce that the mapping $\mathcal{F}(M, E) \to \mathcal{F}(W_j, F)$,

$$\eta \mapsto f_*(\eta) \circ \phi_j^{-1}|_{W_j} = (\rho_{W_j, V_j}^F \circ \mathcal{F}_{\mathrm{loc}}(V_j, g_j))(\psi_j \circ \gamma \circ \phi_j^{-1}, h_j(\eta))$$

is continuous. Hence $f_*$ is continuous, by Proposition 5.5.  ∎

**Proposition 5.14.**    *If $M$ is an $m$-dimensional compact $C^\infty$-manifold, $E$ and $F$ are finite-dimensional vector spaces, $U$ is open in $E$ and $f \colon M \times U \to F$ a $C^\infty$-map, then also the map $f_* \colon \mathcal{F}(M, U) \to \mathcal{F}(M, F)$, $\gamma \mapsto f \circ (\mathrm{id}_M, \gamma)$ is smooth.*

**Proof.**    By Lemma 5.8, $f_*$ is continuous. We show by induction on $k \in \mathbb{N}$ that $f_*$ is $C^k$ for all $E$, $F$, $U$ and $f$ as in 5.14. To see that $f_*$ is $C^1$, let $\gamma \in \mathcal{F}(M, U)$ and $\eta \in \mathcal{F}(M, E)$. We claim that the directional derivative $d(f_*)(\gamma, \eta)$ exists and

$$d(f_*)(\gamma, \eta)(x) = d_2 f(x, \gamma(x), \eta(x))$$

holds for all elements $x \in M$; here $d_2 f \colon M \times U \times E \to F$ is the smooth mapping $(x, y, z) \mapsto d(f_x)(y, z)$ with $f_x := f(x, \cdot) \colon U \to F$. If this is true, then

$$d(f_*)(\gamma, \eta) = (d_2 f)_*(\gamma, \eta)$$

if we identify $\mathcal{F}(M, E) \times \mathcal{F}(M, E)$ with $\mathcal{F}(M, E \times E)$ by means of the isomorphism of topological vector spaces (and hence $C^\infty$-diffeomorphism) described in Lemma 5.10. The map $(d_2 f)_* \colon \mathcal{F}(M, U \times E) \to \mathcal{F}(M, F)$ being continuous, $f$ is $C^1$. If $k \geq 2$, then $(d_2 f)_*$ is $C^{k-1}$ by induction and thus $f$ is $C^k$.

Proof of the claim. As $\gamma(M)$ and $\eta(M)$ are compact in $U$ and $E$, respectively, there is $\varepsilon > 0$ with $\gamma(M) + {]{-}\varepsilon, \varepsilon[}\, \eta(M) \subseteq U$. As $(d_2 f)_*$ is continuous, also

$$g \colon [0, 1] \times \, ]{-}\varepsilon, \varepsilon[ \, \to \mathcal{F}(M, F), \quad (s, t) \mapsto d_2 f \circ (\mathrm{id}_M, \gamma + st\eta, \eta)$$

is continuous. Since $\mathcal{F}(M, F)$ is integral complete, the weak integral

$$h(t) := \int_0^1 g(s, t)\, ds$$

exists in $\mathcal{F}(M, F)$ for all $t \in \, ]{-}\varepsilon, \varepsilon[$. By continuity of parameter-dependent integrals (see [18, Lemma 1.1.11]), $h \colon \, ]{-}\varepsilon, \varepsilon[ \, \to \mathcal{F}(M, F)$ is continuous. For $t \in \, ]{-}\varepsilon, \varepsilon[ \, \backslash \{0\}$, consider the difference quotient

$$\Delta(t) := \frac{1}{t}(f_*(\gamma + t\eta) - f_*(\gamma)).$$

For $x \in M$, let $\mathrm{ev}_x \colon \mathcal{F}(M, F) \to F$ be the continuous linear point evaluation at $x$. Since weak integrals and continuous linear maps commute [18, Exercise 1.1.3 (a)], using the Mean Value Theorem [18, Proposition 1.2.6] we see that

$$
\begin{aligned}
\mathrm{ev}_x(\Delta(t)) \;=\; & \Delta(t)(x) = \frac{1}{t}\big(f(x, \gamma(x) + t\eta(x)) - f(x, \gamma(x))\big) \\
=\; & \int_0^1 d_2 f(x, \gamma(x) + st\eta(x), \eta(x))\, ds \\
=\; & \mathrm{ev}_x\left(\int_0^1 d_2 f \circ (\mathrm{id}_M, \gamma + st\eta, \eta)\, ds\right) = \mathrm{ev}_x(h(t)).
\end{aligned}
$$

As the point evaluations separate points, we deduce that $\Delta(t) = h(t)$, which converges to $h(0) = \int_0^1 d_2 f \circ (\mathrm{id}_M, \gamma, \eta)\, ds = d_2 f \circ (\mathrm{id}_M, \gamma, \eta)$ as $t \to 0$. ∎

Setting $f(x, y) := \Phi(y)$, we deduce:

**Corollary 5.15.** *If $M$ is an $m$-dimensional compact $C^\infty$-manifold, $E$ and $F$ are finite-dimensional vector spaces, $U$ is an open subset of $E$ and $\Phi \colon U \to F$ is smooth, then also $\mathcal{F}(M, \Phi) \colon \mathcal{F}(M, U) \to \mathcal{F}(M, F)$, $\gamma \mapsto \Phi \circ \gamma$ is a smooth map.* ∎

**5.16.** If $E$ is a finite-dimensional complex vector space with $\mathbb{C}$-basis $b_1, \ldots, b_n$, then $b_1, \ldots, b_n, ib_1, \ldots, ib_n$ is an $\mathbb{R}$-basis for $E$ and $E = F \oplus iF$ as a real vector space, using the real span $F$ of $b_1, \ldots, b_n$. For each $U \in \mathcal{U}$, we then have

$$\mathcal{F}(U, E) = \mathcal{F}(U, F) \oplus i\mathcal{F}(U, F)$$

as a real vector space and we easily check that the operation

$$\mathbb{C} \times \mathcal{F}(U, E) \to \mathcal{F}(U, E), \quad (t + is)(\gamma + i\eta) := (t\gamma - s\eta) + i(s\gamma + t\eta)$$

makes $\mathcal{F}(U, E)$ a complex locally convex space. As in the real case, the complex topological vector space structure is independent of the basis.

**5.17.** If $E$ is a finite-dimensional complex vector space, then the mappings $\mathcal{F}(\phi^{-1}|_W, E) \to \mathcal{F}(W, E)$ are complex linear in the situation of 5.4. Hence 5.4 provides a complex locally convex vector space structure on $\mathcal{F}(M, E)$.

**Corollary 5.18.** *If $E$ and $F$ are $\mathbb{K}$-vector spaces for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ in the situation of Corollary 5.15 and the map $\Phi$ is $\mathbb{K}$-analytic, then also the mapping $\mathcal{F}(M, \Phi) \colon \mathcal{F}(M, U) \to \mathcal{F}(M, F)$, $\gamma \mapsto \Phi \circ \gamma$ is $\mathbb{K}$-analytic.*

**Proof.** If $\mathbb{K} = \mathbb{C}$, define $f \colon M \times U \to F$ via $f(x, y) := \Phi(y)$. We know that $\mathcal{F}(M, \Phi) = f_*$ is smooth over $\mathbb{R}$ with directional derivatives $d(f_*)(\gamma, \eta) = (df_2)_*(\gamma, \eta) = (d\Phi) \circ (\gamma, \eta)$. As the latter are complex linear in $\eta$ for fixed $\gamma$, the map $f_*$ is complex analytic by 2.3.

If $\mathbb{K} = \mathbb{R}$, then $\Phi$ has a $\mathbb{C}$-analytic extension $g \colon V \to F_\mathbb{C}$ for some open subset $V \subseteq E_\mathbb{C}$ with $U \subseteq V$. Since $\mathcal{F}(M, g)$ is a $\mathbb{C}$-analytic extension for $\mathcal{F}(M, \Phi)$ which is defined on an open subset in $\mathcal{F}(M, E_\mathbb{C}) = \mathcal{F}(M, E)_\mathbb{C}$ and takes values in $\mathcal{F}(M, F_\mathbb{C}) = \mathcal{F}(M, F)_\mathbb{C}$, the map $\mathcal{F}(M, \Phi)$ is $\mathbb{R}$-analytic. ∎

## 6. The Lie groups $\mathcal{F}(M, G)$

To prove Proposition 1.1, let $m$, $\mathcal{U}$, and $(\mathcal{F}(U, \mathbb{R}))_{U \in \mathcal{U}}$ be as in Section 4. Let $M$ be a compact $C^\infty$-manifold of dimension $m$ and $G$ be a finite-dimensional Lie group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, with Lie algebra $\mathfrak{g}$. Let $\mu_G \colon G \times G \to G$ be the group multiplication and $\eta_G \colon G \to G$, $g \mapsto g^{-1}$ be the inversion map. Then $\mathcal{F}(M, G)$ is a subgroup of the group $G^M$ of all mappings $M \to G$. In fact,

$$\gamma_1 \gamma_2 := \mu_G \circ (\gamma_1, \gamma_2) = \mathcal{F}(M, \mu_G)(\gamma_1, \gamma_2) \in \mathcal{F}(M, G)$$

for all $\gamma_1, \gamma_2 \in \mathcal{F}(M, G)$, by Lemmas 5.3 and 5.12. Likewise, $\gamma^{-1} := \eta_G \circ \gamma = \mathcal{F}(M, \eta_G)(\gamma) \in \mathcal{F}(M, G)$ for each $\gamma \in \mathcal{F}(M, G)$, by Lemma 5.3. We now give $\mathcal{F}(M, G)$ a $\mathbb{K}$-analytic manifold structure as described in Proposition 1.1.

There exists a balanced open 0-neighbourhood $Q \subseteq \mathfrak{g}$ such that $P := \exp_G(Q)$ is open in $G$ and $\phi := \exp_G|_Q^P \colon Q \to P$ is a $\mathbb{K}$-analytic diffeomorphism. There exists a balanced open 0-neighbourhood $V \subseteq Q$ such that $U := \exp_G(V)$ satisfies $UU \subseteq P$. Since $V = -V$, we have $U = U^{-1}$. Lemma 5.3 implies that $\phi \circ \gamma \in \mathcal{F}(M, P)$ for each $\gamma \in \mathcal{F}(M, Q)$ and that

$$\Theta := \mathcal{F}(M, \phi) \colon \mathcal{F}(M, Q) \to \mathcal{F}(M, P), \quad \gamma \mapsto \phi \circ \gamma$$

is a bijection (with inverse $\eta \mapsto \phi^{-1} \circ \eta$). We give $\mathcal{F}(M, P)$ the $\mathbb{K}$-analytic manifold structure modelled on $\mathcal{F}(M, \mathfrak{g})$ which turns $\mathcal{F}(M, \phi)$ into a $\mathbb{K}$-analytic diffeomorphism. Then $\mathcal{F}(M, U)$ is open in $\mathcal{F}(M, P)$, as $\mathcal{F}(M, V)$ is open in $\mathcal{F}(M, Q)$. Since $f \colon U \times U \to P$, $(x, y) \mapsto xy^{-1}$ is $\mathbb{K}$-analytic, also

$$g \colon V \times V \to Q, \quad (x, y) \mapsto \phi^{-1}(\phi(x)\phi(y)^{-1})$$

is $\mathbb{K}$-analytic, whence

$$\mathcal{F}(M, g) \colon \mathcal{F}(M, V \times V) = \mathcal{F}(M, V) \times \mathcal{F}(M, V) \to \mathcal{F}(M, Q)$$

is $\mathbb{K}$-analytic, by Corollary 5.18. As

$$\mathcal{F}(M, f) = \Theta \circ \mathcal{F}(M, g) \circ (\Theta^{-1} \times \Theta^{-1})|_{\mathcal{F}(M,U) \times \mathcal{F}(M,U)},$$

also the map $\mathcal{F}(M, f)$ is $\mathbb{K}$-analytic, which takes $(\gamma, \eta) \in \mathcal{F}(M, U) \times \mathcal{F}(M, U)$ to $f \circ (\gamma, \eta) = \gamma\eta^{-1} \in \mathcal{F}(M, P)$. We now use the smoothness of the adjoint action $\mathrm{Ad} \colon G \times \mathfrak{g} \to \mathfrak{g}$, $(g, y) \mapsto \mathrm{Ad}_g(y)$. Given $\gamma \in \mathcal{F}(M, G)$, consider the inner automorphism $I_\gamma \colon \mathcal{F}(M, G) \to \mathcal{F}(M, G)$, $\eta \mapsto \gamma\eta\gamma^{-1}$. We deduce with Lemma 5.13 that $\mathrm{Ad} \circ (\gamma, \eta) \in \mathcal{F}(M, \mathfrak{g})$ for all $\eta \in \mathcal{F}(M, \mathfrak{g})$ and that the linear map

$$\beta \colon \mathcal{F}(M, \mathfrak{g}) \to \mathcal{F}(M, \mathfrak{g}), \quad \eta \mapsto \mathrm{Ad} \circ (\gamma, \eta)$$

is continuous (and hence $\mathbb{K}$-analytic). Thus $W := \beta^{-1}(\mathcal{F}(M, V)) \cap \mathcal{F}(M, V)$ is an open 0-neighbourhood in $\mathcal{F}(M, V)$ such that $\beta(W) \subseteq \mathcal{F}(M, V)$. Also, $\Theta(W)$ is open in $\mathcal{F}(M, P)$. As $\gamma(x)\exp_G(\eta(x))\gamma(x)^{-1} = \exp_G(\mathrm{Ad}_{\gamma(x)}(\eta(x)))$ for all $\eta \in W$ and $x \in M$, we have

$$I_\gamma \circ \Theta|_W = \Theta \circ \beta|_W,$$

whence $I_\gamma(\Theta(W)) \subseteq \mathcal{F}(M, P)$ and $I_\gamma|_{\Theta(W)} \colon \Theta(W) \to \mathcal{F}(M, P)$ is $\mathbb{K}$-analytic. By the familiar local description of Lie group structures, there is a uniquely determined $\mathbb{K}$-analytic manifold structure on $\mathcal{F}(M, G)$ which is modelled on $\mathcal{F}(M, \mathfrak{g})$, turns $\mathcal{F}(M, G)$ into a $\mathbb{K}$-analytic Lie group, and such that $\mathcal{F}(M, U)$ is open in $\mathcal{F}(M, G)$ and the latter induces the given $\mathbb{K}$-analytic manifold structure thereon (see Proposition 18 in [4, Chapter III, §1, no. 9], whose hypothesis that the modelling space be Banach is not needed in the proof).

By construction, $\Phi := \mathcal{F}(M, \phi|_V) \colon \mathcal{F}(M, V) \to \mathcal{F}(M, U)$ is a $\mathbb{K}$-analytic diffeomorphism onto an open identity neighbourhood in the Lie group $\mathcal{G} := \mathcal{F}(M, G)$. Identifying $T_0\mathcal{F}(M, V) = \{0\} \times \mathcal{F}(M, \mathfrak{g})$ with $\mathcal{F}(M, G)$ via $(0, v) \mapsto v$, we obtain an isomorphism

$$\alpha := T_0\Phi \colon \mathcal{F}(M, \mathfrak{g}) \to T_e\mathcal{G}$$

of topological vector spaces. Let $[.,.]_{\mathfrak{g}}$ be the Lie bracket on $\mathfrak{g}$ and $[.,.]$ be the Lie bracket on $\mathcal{F}(M, \mathfrak{g})$ making $\alpha$ an isomorphism of Lie algebras to $L(\mathcal{G})$. Then $[.,.]$ is the pointwise Lie bracket, i.e.,

$$[\gamma, \eta](x) = [\gamma(x), \eta(x)]_{\mathfrak{g}} \quad \text{for all } \gamma, \eta \in \mathcal{F}(M, \mathfrak{g}) \text{ and } x \in M.$$

To see this, consider the point evaluation $\mathrm{ev}_x \colon \mathcal{G} \to G$, $\gamma \mapsto \gamma(x)$ at $x \in M$ and the point evaluation $\varepsilon_x \colon \mathcal{F}(M, \mathfrak{g}) \to \mathfrak{g}$, $\gamma \mapsto \gamma(x)$. Since $\mathrm{ev}_x \circ \Phi = \exp_G \circ \varepsilon_x$, the homomorphism $\varepsilon_x$ is $\mathbb{K}$-analytic on some identity neighbourhood and hence $\mathbb{K}$-analytic. We also deduce that $T_e(\mathrm{ev}_x) \circ T_0\Phi = T_0 \exp_G \circ T_0\varepsilon_x$ and thus

$$L(\mathrm{ev}_x) \circ \alpha = \varepsilon_x \tag{2}$$

with $L(\mathrm{ev}_x) := T_e(\mathrm{ev}_x)$. Hence

$$[\gamma, \eta](x) = \varepsilon_x([\gamma, \eta]) = L(\mathrm{ev}_x)([\alpha(\gamma), \alpha(\eta)])$$
$$= [L(\mathrm{ev}_x)(\alpha(\gamma)), L(\mathrm{ev}_x)(\alpha(\eta))]_{\mathfrak{g}} = [\gamma(x), \eta(x)]_{\mathfrak{g}}$$

for all $\gamma, \eta \in \mathcal{F}(M, \mathfrak{g})$. Note that $(L(\mathrm{ev}_x)(\alpha(\gamma)))_{x \in M} = \gamma$ by (2), whence the inverse map $\alpha^{-1} \colon L(\mathcal{G}) \to \mathcal{F}(M, \mathfrak{g})$ is given by $\alpha^{-1}(v) = (L(\mathrm{ev}_x)(v))_{x \in M}$. We claim:

$$\mathcal{F}(M, \exp_G) \circ \alpha^{-1} \colon L(\mathcal{G}) \to G, \quad v \mapsto \exp_G \circ \alpha^{-1}(v)$$

*is the exponential function* $\exp_{\mathcal{G}}$ *of* $\mathcal{G}$. Since $\mathcal{F}(M, \exp_G)$ is a local $\mathbb{K}$-analytic diffeomorphism at $0$ (as it coincides with $\Phi$ on some $0$-neighbourhood), then also $\exp_{\mathcal{G}}$ will be a local $\mathbb{K}$-analytic diffeomorphism at $0$, and thus $\mathcal{G}$ is a BCH-Lie group. To prove the claim and complete the proof of Proposition 1.1, let $v \in L(\mathcal{G})$ and abbreviate $\gamma := \alpha^{-1}(v)$. Then

$$c \colon \mathbb{R} \to \mathcal{G}, \quad t \mapsto \exp_G \circ (t\gamma)$$

is a homomorphism of groups and smooth as it coincides with the smooth map $t \mapsto \Phi(t\gamma)$ on some $0$-neighbourhood. By the preceding, $\dot{c}(0) = T_0\Phi(\gamma) = \alpha(\gamma) = v$, whence $\exp_{\mathcal{G}}(v) = c(1) = \exp_G \circ \alpha^{-1}(v)$.

## 7. Sobolev spaces are suitable for Lie theory

We now show that the theory discussed in Sections 3 to 6 applies to Sobolev spaces with real exponents $s > m/2$. We start with notation and basic facts.

**7.1.** For $m \in \mathbb{N}$ and $s \in [0, \infty[$, let $H^s(\mathbb{R}^m, \mathbb{R})$ be the real Hilbert space of equivalence classes $[\gamma]$ modulo functions vanishing almost everywhere of $\mathcal{L}^2$-functions $\gamma \colon \mathbb{R}^m \to \mathbb{R}$ (with respect to Lebesgue-Borel measure $\lambda_m$) such that $y \mapsto (1 + \|y\|_2^2)^{s/2} \widehat{\gamma}(y)$ is an $\mathcal{L}^2$-function as well, where $\widehat{\gamma}$ is the Fourier transform (see [22, Appendix B]; cf. [7, Chapter 6.A] as well as Section 1.3.1 and Exercise 1.2.5 in [19], with $p = 2$). Here $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^m$. The scalar product on $H^s(\mathbb{R}^m, \mathbb{R})$ is given by

$$\langle [\gamma], [\eta] \rangle_{H^s} := \int_{\mathbb{R}^m} (1 + \|y\|_2^2)^{s/2} \widehat{\gamma}(y) \overline{\widehat{\eta}(y)} \, d\lambda_m(y)$$

for $[\gamma], [\eta] \in H^s(\mathbb{R}^m, \mathbb{R})$.

We let $\|\cdot\|_{H^s}$ be the corresponding norm, taking $[\gamma]$ to $\sqrt{\langle[\gamma],[\gamma]\rangle_{H^s}}$. For $s > m/2$, each $[\gamma] \in H^s(\mathbb{R}^m, \mathbb{R})$ has a unique bounded, continuous representative $\gamma$ and we identify the equivalence class with this representative. Moreover, the inclusion map

$$H^s(\mathbb{R}^m, \mathbb{R}) \to BC(\mathbb{R}^m, \mathbb{R})$$

is continuous (see Lemma 6.5 in [7] and the subsequent Remark 1 in [7]). By definition, $H^s(\mathbb{R}^m, \mathbb{R}) \subseteq H^t(\mathbb{R}^m, \mathbb{R})$ for $s \geq t \geq 0$ and

$$\|\gamma\|_{H^t} \leq \|\gamma\|_{H^s} \quad \text{for all } \gamma \in H^s(\mathbb{R}^m, \mathbb{R}). \tag{3}$$

**7.2.**    If $U \subseteq \mathbb{R}^m$ is a bounded open set and $s > m/2$, we define

$$H^s(U, \mathbb{R}) := \{\gamma|_U \colon \gamma \in H^s(\mathbb{R}^m, \mathbb{R})\}$$

and give this space the quotient norm with respect to the linear surjection

$$q_U^s \colon H^s(\mathbb{R}^m, \mathbb{R}) \to H^s(U, \mathbb{R}), \quad \gamma \mapsto \gamma|_U$$

whose kernel is closed as the restriction map $BC(\mathbb{R}^m, \mathbb{R}) \to BC(U, \mathbb{R})$ is continuous linear (with operator norm $\leq 1$). Note that each function $\gamma \in H^s(U, \mathbb{R})$ is bounded, being the restriction of a bounded function. The restriction of $q_U^s$ to the orthogonal complement of $\ker(q_U^s)$ in the Hilbert space $H^s(\mathbb{R}^m, \mathbb{R})$ is a surjective linear isometry

$$(\ker(q_U^s))^\perp \to H^s(U, \mathbb{R}),$$

whose inverse provides an isometric linear map $\mathcal{E}_U^s \colon H^s(U, \mathbb{R}) \to H^s(\mathbb{R}^m, \mathbb{R})$ which is an extension operator: $\mathcal{E}_U^s(\gamma)|_U = \gamma$ for all $\gamma \in H^s(U, \mathbb{R})$.

**7.3.**    For a finite-dimensional vector space $E$, define $H^s(\mathbb{R}^m, E)$ and $H^s(U, E)$ as in 3.1. For $E \cong \mathbb{R}^n$, the restriction map $q_U^{s,E} \colon H^s(\mathbb{R}^m, E) \to H^s(U, E)$, $\gamma \mapsto \gamma|_U$ then corresponds to $(q_U^s)^n$, whence it is a quotient map. For an open subset $V \subseteq E$, let $H^s(U, V)$ be the set of all $\gamma \in H^s(U, E)$ such that $\gamma(U) + Q \subseteq V$ for some $0$-neighbourhood $Q \subseteq E$. Then $H^s(U, V)$ is open in $H^s(U, E)$, using continuity of the restriction map $H^s(U, E) \to BC(U, E)$.

We shall use the following fact:

**Lemma 7.4.**    *Let $m, n \in \mathbb{N}$, $s \in {]m/2, \infty[}$ and $f \colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be a bounded smooth map with bounded partial derivatives such that $f(., 0) \in L^2(\mathbb{R}^m, \mathbb{R})$. Then $f_*(\gamma) := f \circ (\mathrm{id}_{\mathbb{R}^m}, \gamma) \in H^s(\mathbb{R}^m, \mathbb{R})$ for all $\gamma \in H^s(\mathbb{R}^m, \mathbb{R}^n)$ and the mapping $f_* \colon H^s(\mathbb{R}^m, \mathbb{R}^n) \to H^s(\mathbb{R}^m, \mathbb{R})$ is continuous.*

**Proof.**    It is known that $H^s(\mathbb{R}^m, \mathbb{R})$ coincides with the Triebel-Lizorkin space $F_{2,2}^s$ (see (vii) in the proposition stated on [33, p. 14]). Thus, the assertion follows from Theorem 1 on p. 387 and Theorem 2 on p. 389 in [33].    ∎

**Proposition 7.5.**    *For $m \in \mathbb{N}$ and $s > m/2$, the Sobolev spaces $H^s(U, \mathbb{R})$ on bounded open subsets $U \subseteq \mathbb{R}^m$ form a family of function spaces which is suitable for Lie theory.*

**Proof.** Axiom (PF). Let $U \subseteq \mathbb{R}^m$ be a bounded open subset and $V \subseteq U$ be a relatively compact open subset. Let $E$ be a finite-dimensional vector space and $f: U \times E \to \mathbb{R}$ be a smooth map. Given $\gamma \in H^s(U, E)$, we have $\overline{\gamma(U)} \subseteq W$ for a relatively compact open set $W \subseteq E$. Let $\xi: U \to \mathbb{R}$ and $\chi: E \to \mathbb{R}$ be compactly supported smooth functions such that $\xi|_V = 1$ and $\chi|_W = 1$. We get a function $g \in C_c^\infty(\mathbb{R}^m \times E, \mathbb{R})$ via $g(x, y) := \xi(x)\chi(y)f(x, y)$ for $(x, y) \in U \times E$, $g(x, y) := 0$ for $(x, y) \in (\mathbb{R}^m \setminus \mathrm{supp}(\xi)) \times E$. Lemma 7.4 shows that $g_*(\eta) := g \circ (\mathrm{id}_{\mathbb{R}^m}, \eta) \in H^s(\mathbb{R}^m, \mathbb{R})$ for each $\eta \in H^s(\mathbb{R}^m, E)$ and $g_*: H^s(\mathbb{R}^m, E) \to H^s(\mathbb{R}^m, \mathbb{R})$ is continuous. Now $h(\eta|_U) := g \circ (\mathrm{id}_V, \eta|_V) = g_*(\eta)|_V \in H^s(V, \mathbb{R})$ is well defined. Since $h \circ q_U^{s,E} = q_U^s \circ g_*$ is continuous, $h: H^s(U, E) \to H^s(V, \mathbb{R})$ is continuous. For each $\eta$ in the open $\gamma$-neighbourhood $H^s(U, W)$, we have $f_*(\eta) := f \circ (\mathrm{id}_V, \eta|_V) = h(\eta)$. Notably, $f_*(\gamma) \in H^s(V, \mathbb{R})$ and $f_*: H^s(U, E) \to H^s(V, \mathbb{R})$ is continuous at $\gamma$.

Axiom (PB). Given an open subset $U \subseteq \mathbb{R}^m$, let $H_{\mathrm{loc}}^s(U, \mathbb{R})$ be the set of all functions $\gamma: U \to \mathbb{R}$ such that $\gamma|_Q \in H^s(Q, \mathbb{R})$ for each relatively compact open subset $Q \subseteq U$. Let $V$ and $W$ be bounded open subsets of $\mathbb{R}^m$ such that $\overline{W} \subseteq U$. Let $\Theta: U \to V$ be a $C^\infty$-diffeomorphism. If $\gamma \in H^s(V, \mathbb{R})$, then $\gamma \in H_{\mathrm{loc}}^s(V, \mathbb{R})$ in particular, whence $\gamma \circ \Theta \in H_{\mathrm{loc}}^s(U, \mathbb{R})$ by [7, Corollary 6.25] and hence

$$H^s(\Theta|_W, \mathbb{R})(\gamma) := \gamma \circ \Theta|_W \in H^s(W, \mathbb{R}).$$

By Remark 3.5, the map $H^s(\Theta|_W, \mathbb{R}): H^s(V, \mathbb{R}) \to H^s(W, \mathbb{R})$ is continuous.

Axiom (GL). Let $U$ and $V$ be bounded open subsets in $\mathbb{R}^m$ such that $\overline{V} \subseteq U$, and $K \subseteq V$ be compact. If $\gamma \in H^s(V, \mathbb{R})$ with $\mathrm{supp}(\gamma) \subseteq K$, there exists $\eta \in H^s(\mathbb{R}^m, \mathbb{R})$ with $\eta|_U = \gamma$. Define $\widetilde{\gamma}(x) := \gamma(x)$ for $x \in V$, $\widetilde{\gamma}(x) := 0$ for $x \in U \setminus \mathrm{supp}(\gamma)$. Let $\xi \in C_c^\infty(\mathbb{R}^m, \mathbb{R})$ such that $\xi|_K = 1$ and $\mathrm{supp}(\xi) \subseteq V$. Then $\xi\eta \in H^s(\mathbb{R}^m, \mathbb{R})$ by [7, Proposition 6.12] and $\widetilde{\gamma} = (\xi\eta)|_U \in H^s(U, \mathbb{R})$. By Remark 3.5, the map $H_K^s(V, \mathbb{R}) \to H^s(U, \mathbb{R})$, $\gamma \mapsto \widetilde{\gamma}$ is continuous.

Axiom (MU). If $U \subseteq \mathbb{R}^m$ is a bounded open subset and $h \in C_c^\infty(U, \mathbb{R})$, let $\widetilde{h} \in C_c^\infty(\mathbb{R}^m, \mathbb{R})$ be the extension of $h$ by $0$. If $\gamma \in H^s(U, \mathbb{R})$, let $\eta \in H^s(\mathbb{R}^m, \mathbb{R})$ with $\eta|_U = \gamma$. By [7, Proposition 6.12], $\widetilde{h}\gamma \in H^s(\mathbb{R}^m, \mathbb{R})$, whence $m_h(\gamma) := h\gamma = (\widetilde{h}\eta)|_U \in H^s(U, \mathbb{R})$. By Remark 3.5, $m_h$ is continuous. $\blacksquare$

## 8. The Lie groups $H^{>s_0}(M, G)$

We begin with preparations for the proof of Theorem 1.5.

**Lemma 8.1.** *Let $m \in \mathbb{N}$, $\mathcal{U}$ be a good collection of open subsets of $\mathbb{R}^m$ and $(\mathcal{F}_j(U, \mathbb{R}))_{U \in \mathcal{U}}$ be a family of Banach spaces which is suitable for Lie theory, for $j \in \{1, 2\}$. Assume that $\mathcal{F}_1(U, \mathbb{R}) \subseteq \mathcal{F}_2(U, \mathbb{R})$ for each $U \in \mathcal{U}$, and the inclusion map $\mathcal{F}_1(U, \mathbb{R}) \to \mathcal{F}_2(U, \mathbb{R})$ is continuous. Then we have:*

(a) *$\mathcal{F}_1(M, N) \subseteq \mathcal{F}_2(M, N)$ holds for each compact $m$-dimensional $C^\infty$-manifold $M$ and each finite-dimensional $C^\infty$-manifold $N$. Moreover, the inclusion map $\mathcal{F}_1(M, E) \to \mathcal{F}_2(M, E)$ is continuous for each finite-dimensional vector space $E$.*

(b) *If the mappings $\kappa_{V,U}: \mathcal{F}_1(U, \mathbb{R}) \to \mathcal{F}_2(V, \mathbb{R})$, $\gamma \mapsto \gamma|_V$ are compact operators for all $U, V \in \mathcal{U}$ such that $V$ is relatively compact in $U$, then the inclusion mappings $\kappa_M^E: \mathcal{F}_1(M, E) \to \mathcal{F}_2(M, E)$ are compact operators for all finite-dimensional vector spaces $E$ and all $M$ as in (a).*

**Proof.** (a) The hypothesis implies that $(\mathcal{F}_1)_{\mathrm{loc}}(U, E) \subseteq (\mathcal{F}_2)_{\mathrm{loc}}(U, E)$ for each open subset $U \subseteq \mathbb{R}^m$ and finite-dimensional vector space $E$, with continuous linear inclusion map. The first assertion follows (using Lemma 5.2) and also the second.

(b) If $E \cong \mathbb{R}^n$, then $\kappa^E_{V,U} \colon \mathcal{F}_1(U, E) \to \mathcal{F}_2(V, E)$, $\gamma \mapsto \gamma|_V$ corresponds to $(\kappa_{V,U})^n$ for all $U, V \in \mathcal{U}$ with $V$ relatively compact in $U$, whence $\kappa^E_{V,U}$ is a compact operator. There exist $k \in \mathbb{N}$ and charts $\phi_i \colon U_i \to V_i \subseteq \mathbb{R}^m$ of $M$ for $i \in \{1, \ldots, k\}$ and sets $W_{i,2} \in \mathcal{U}$ which are relatively compact in $V_i$ such that $M = \bigcup_{i=1}^n \phi_i^{-1}(W_{i,2})$. By Definition 2.6 (b), we find $W_{i,1} \in \mathcal{U}$ such that $W_{i,1}$ is relatively compact in $V_i$ and $\overline{W_{i,2}} \subseteq W_{i,1}$. By Proposition 5.5, the map

$$\Theta_j \colon \mathcal{F}_j(M, E) \to \prod_{i=1}^k \mathcal{F}_j(W_{i,j}, E), \quad \gamma \mapsto (\gamma \circ \phi_i^{-1}|_{W_{i,j}})_{i=1}^k$$

is a linear topological embedding with closed image for $j \in \{1, 2\}$. Now

$$h := \prod_{i=1}^k \kappa^E_{W_{i,2}, W_{i,1}} \colon \prod_{i=1}^k \mathcal{F}_1(W_{i,1}, E) \to \prod_{i=1}^k \mathcal{F}_2(W_{i,2}, E)$$

is a compact operator. Since $\Theta_2 \circ \kappa^E_M = h \circ \Theta_1$ is a compact operator, so is $\kappa^E_M$. ∎

Following [7, p. 200], given $s > m/2$ and a bounded open subset $U \subseteq \mathbb{R}^m$, we let $\mathring{H}^s(U, \mathbb{R})$ be the closure of $\{\gamma \in C^\infty_c(\mathbb{R}^m, \mathbb{R}) \colon \mathrm{supp}(\gamma) \subseteq U\}$ in $H^s(\mathbb{R}^m, \mathbb{R})$.

**Lemma 8.2.** *For each bounded, open subset $U \subseteq \mathbb{R}^m$ and relatively compact, open subset $V \subseteq U$, the continuous linear map $q^s_{V,U} \colon \mathring{H}^s(U, \mathbb{R}) \to H^s(V, \mathbb{R})$, $\eta \mapsto \eta|_V$ is surjective and hence a quotient map.*

**Proof.** Being a restriction of $q^s_V$, the map $q^s_{V,U}$ is continuous and linear. To see surjectivity, let $\gamma \in H^s(V, \mathbb{R})$. There is $\eta \in H^s(\mathbb{R}^m, \mathbb{R})$ such that $\eta|_V = \gamma$. Now the Schwartz space $\mathcal{S}(\mathbb{R}^m, \mathbb{R})$ of rapidly decreasing smooth functions is dense in $H^s(\mathbb{R}^m, \mathbb{R})$ (cf. [7, p. 192]) and the inclusion mapping $\mathcal{S}(\mathbb{R}^m, \mathbb{R}) \to H^s(\mathbb{R}^m, \mathbb{R})$ is continuous, whence $C^\infty_c(\mathbb{R}^m, \mathbb{R})$ is dense in $H^s(\mathbb{R}^m, \mathbb{R})$ (being dense in $\mathcal{S}(\mathbb{R}^m, \mathbb{R})$ by [32, Theorem 7.10 (a)]). Thus, we find a sequence $(\eta_n)_{n \in \mathbb{N}}$ in $C^\infty_c(\mathbb{R}^m, \mathbb{R})$ such that $\eta_n \to \eta$ in $H^s(\mathbb{R}^m, \mathbb{R})$ as $n \to \infty$. Pick $\xi \in C^\infty_c(\mathbb{R}^m, \mathbb{R})$ such that $\mathrm{supp}(\xi) \subseteq U$ and $\xi|_V = 1$. Then $\xi \eta_n \to \xi \eta$ in $H^s(\mathbb{R}^m, \mathbb{R})$ (see [7, Proposition 6.12]), entailing that $\xi \eta \in \mathring{H}^s(U, \mathbb{R})$. Moreover, $(\xi \eta)|_V = \eta|_V = \gamma$. By the Open Mapping Theorem, the continuous linear surjection $q^s_{V,U}$ is an open map and hence a quotient map. ∎

We recall a known fact.

**Lemma 8.3.** *Let $m \in \mathbb{N}$ and $s > t > m/2$. For all bounded open subsets $U$ in $\mathbb{R}^m$, we have $H^s(U, \mathbb{R}) \subseteq H^t(U, \mathbb{R})$. The inclusion map $\theta_U$ is a compact operator.*

**Proof.** Let $W$ be a bounded open subset of $\mathbb{R}^m$ with $\overline{U} \subseteq W$. By Rellich's Theorem [7, Theorem 6.14], the inclusion map $h \colon \mathring{H}^s(W, \mathbb{R}) \to \mathring{H}^t(W, \mathbb{R})$ is a compact operator. Then $\theta_U \circ q^s_{U,W} = q^t_{U,W} \circ h$ is a compact operator and hence continuous, whence $\theta_U$ is continuous. If $B \subseteq \mathring{H}^s(W, \mathbb{R})$ is a bounded open $0$-neighbourhood, then $q^s_{U,W}(B)$ is a bounded $0$-neighbourhood in $H^s(U, \mathbb{R})$. Since $\theta_U(q^s_{U,W}(B))$ is relatively compact, $\theta_U$ is a compact operator. ∎

**8.4.** A locally convex space $E$ is called a *Silva space* (or (DFS)-space) if $E$ is the locally convex direct limit of some Banach spaces $E_1 \subseteq E_2 \subseteq \cdots$, such that all inclusion maps $E_j \to E_{j+1}$ are compact operators (see, e.g., [6] or [18, Appendix B.13]). Every Silva space is complete. It is *compact regular* in the sense that each compact subset $K \subseteq E$ is a compact subset of some $E_j$. The locally convex topology $\mathcal{O}$ on $E$ then also makes $E$ the direct limit of the $E_n$ as a topological space (see the cited works). Thus, a subset $U \subseteq E$ is open if and only if $U \cap E_j$ is open in $E_j$ for each $j \in \mathbb{N}$.

**Lemma 8.5.** *Let $m \in \mathbb{N}$, $\mathcal{U}$ be a good collection of open subsets of $\mathbb{R}^m$ and $(\mathcal{F}_j(U, \mathbb{R}))_{U \in \mathcal{U}}$ be a family of Banach spaces which is suitable for Lie theory, for each $j \in \mathbb{N}$. For all $j \in \mathbb{N}$ and $U \in \mathcal{U}$, assume that $\mathcal{F}_j(U, \mathbb{R}) \subseteq \mathcal{F}_{j+1}(U, \mathbb{R})$ with continuous inclusion map. For all $j \in \mathbb{N}$ and $U, V \in \mathcal{U}$ with $V$ relatively compact in $U$, assume that the map $\mathcal{F}_j(U, \mathbb{R}) \to \mathcal{F}_{j+1}(V, \mathbb{R})$, $\gamma \mapsto \gamma|_V$ is a compact operator. Then the following holds:*

(a) *For each finite-dimensional vector space $E$, the locally convex direct limit topology makes $\mathcal{F}(M, E) := \bigcup_{j \in \mathbb{N}} \mathcal{F}_j(M, E)$ a Silva space.*

(b) *If $E$ and $F$ are finite-dimensional vector spaces, $U \subseteq E$ open and $\Phi \colon U \to F$ a $C^\infty$-map, then $\mathcal{F}(M, U) := \{\gamma \in \mathcal{F}(M, E) \colon \gamma(M) \subseteq U\}$ is an open subset of $\mathcal{F}(M, E)$ and the map*

$$\mathcal{F}(M, \Phi) \colon \mathcal{F}(M, U) \to \mathcal{F}(M, F), \quad \gamma \mapsto \Phi \circ \gamma$$

*is smooth. If $E$ and $F$ are $\mathbb{K}$-vector spaces for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\Phi$ is $\mathbb{K}$-analytic, then also $\mathcal{F}(M, \Phi)$ is $\mathbb{K}$-analytic.*

**Proof.** (a) By Lemma 8.1, $\mathcal{F}_j(M, E) \subseteq \mathcal{F}_{j+1}(M, E)$ and the inclusion map is a compact operator.

(b) $\mathcal{F}(M, U)$ is open in the Silva space $\mathcal{F}(M, E)$ as $\mathcal{F}(M, U) \cap \mathcal{F}_j(M, E) = \mathcal{F}_j(M, U)$ is open in $\mathcal{F}_j(M, E)$ for each $j \in \mathbb{N}$. The inclusion map $\Lambda_j^F \colon \mathcal{F}_j(M, F) \to \mathcal{F}(M, F)$ is continuous and linear. Since $\mathcal{F}(M, \Phi)|_{\mathcal{F}_j(M, U)} = \Lambda_j^F \circ \mathcal{F}_j(M, \Phi)$ is smooth for each $j \in \mathbb{N}$ by Corollary 5.15, also $\mathcal{F}(M, \Phi)$ is smooth (see [13, Lemma 9.7]). The complex analytic case follows in the same way, using Corollary 5.18. If $\Phi$ is real analytic, pick a complex analytic extension $\Psi \colon V \to F_{\mathbb{C}}$ of $\Phi$, defined on an open subset $V \subseteq E_{\mathbb{C}}$. Then $\mathcal{F}(M, \Psi)$ is a complex analytic extension for $\mathcal{F}(M, \Phi)$. ∎

Before we can prove Theorem 1.5, we recall further terminology.

**8.6.** Let $G$ be a Lie group with neutral element $e$ and Lie algebra $\mathfrak{g} := T_e G$. Let $G \times TG \to TG$, $(g, v) \mapsto g.v$ be the left action of $G$ on its tangent bundle given by $g.v := TL_g(v)$, where $L_g \colon G \to G$, $x \mapsto gx$. Given $k \in \mathbb{N}_0 \cup \{\infty\}$, endow $C^k([0, 1], \mathfrak{g})$ with the topology of uniform convergence of $C^k$-functions $\gamma \colon [0, 1] \to \mathfrak{g}$ and their derivatives up to $k$th order. The Lie group $G$ is called $C^k$-*regular* if, for each $\gamma \in C^k([0, 1], \mathfrak{g})$, the initial value problem

$$\dot{\eta}(t) = \eta(t).\gamma(t), \quad \eta(0) = e \tag{4}$$

has a (necessarily unique) solution $\eta \colon [0, 1] \to G$ and the associated evolution map $C^k([0, 1], \mathfrak{g}) \to G$, $\gamma \mapsto \eta(1)$ is smooth (see [15]). Every $C^k$-regular Lie group is $C^\infty$-regular, a concept going back to [26] (for sequentially complete $\mathfrak{g}$).

We shall also encounter $L_{\mathrm{rc}}^{\infty}$-regularity of Lie groups modelled on sequentially complete locally convex spaces, a more specialized property introduced in [16] (see 1.13 and Definition 5.16 in loc. cit.) We shall not repeat the concept here but recall that $L_{\mathrm{rc}}^{\infty}$-regularity implies $C^0$-regularity (cf. [16, Corollary 5.21]).

**Proof of Theorem 1.5.** *The modelling space.* We pick $s_1 > s_2 > \cdots$ in $]s_0, \infty[$ such that $s_j \to s_0$ as $j \to \infty$. For each $j \in \mathbb{N}$, we have $H^{s_j}(M, \mathfrak{g}) \subseteq H^{s_{j+1}}(M, \mathfrak{g})$ and the inclusion map is a compact operator, as a consequence of Lemmas 8.1 and 8.3. Thus, the locally convex direct limit topology makes

$$H^{>s_0}(M, \mathfrak{g}) = \bigcup_{j \in \mathbb{N}} H^{s_j}(M, \mathfrak{g}) = \varinjlim H^{s_j}(M, \mathfrak{g})$$

a Silva space. Note that $]s_0, \infty[$ is a directed set for the opposite of the usual order. As $(s_j)_{j \in \mathbb{N}}$ is a cofinal subsequence of the latter set, we have

$$\varinjlim_{s > s_0} H^s(M, \mathfrak{g}) = \varinjlim H^{s_j}(M, \mathfrak{g})$$

in a standard way. The same argument allows $]s_0, \infty[$ to be replaced with the set $\{s_j : j \in \mathbb{N}\}$ in the direct limit properties described in Theorem 1.5.

*The group.* By Lemma 8.1 (a), $H^{s_j}(M, G)$ is a subgroup of $H^{s_{j+1}}(M, G)$ for each $j \in \mathbb{N}$. We give $H^{>s_0}(M, G) = \bigcup_{j \in \mathbb{N}} H^{s_j}(M, G)$ the unique group structure making $H^{s_j}(M, G)$ a subgroup for each $j \in \mathbb{N}$.

*The map $H^{>s_0}(M, \exp_G)$.* For each $\gamma \in H^{>s_0}(M, \mathfrak{g})$, we have $\gamma \in H^{s_j}(M, \mathfrak{g})$ for some $j \in \mathbb{N}$ and hence

$$H^{>s_0}(M, \exp_G)(\gamma) := \exp_G \circ \gamma = H^{s_j}(M, \exp_G)(\gamma) \in H^{s_j}(M, G) \subseteq H^{>s_0}(M, G),$$

using Lemma 5.3.

*The adjoint action on $H^{>s_0}(M, \mathfrak{g})$.* If $\gamma \in H^{>s_0}(M, G)$, then $\gamma \in H^{s_j}(M, G)$ for some $j \in \mathbb{N}$. For all $i \geq j$, we then have $\gamma \in H^{s_i}(M, G)$ and the proof of Proposition 1.1 shows that

$$\beta_i \colon H^{s_i}(M, \mathfrak{g}) \to H^{s_i}(M, \mathfrak{g}), \quad \eta \mapsto \mathrm{Ad} \circ (\gamma, \eta)$$

is a continuous linear map (where $\mathrm{Ad} \colon G \times \mathfrak{g} \to \mathfrak{g}$ is the adjoint action). Then also the linear map

$$\beta \colon H^{>s_0}(M, \mathfrak{g}) \to H^{>s_0}(M, \mathfrak{g}), \quad \eta \mapsto \mathrm{Ad} \circ (\gamma, \eta) \tag{5}$$

is continuous, as $\beta = \varinjlim_{i \geq j} \beta_i$.

*The Lie group structure.* We already saw that $H^{>s_0}(M, G)$ is a group under pointwise operations and that $\exp_G \circ \gamma \in H^{>s_0}(M, G)$ for all $\gamma \in H^{>s_0}(M, \mathfrak{g})$. To construct the Lie group structure on $H^{>s_0}(M, G)$, replace $\mathcal{F}$ with $H^{>s_0}$ in the remaining steps of the proof of Proposition 1.1 and make the following changes: We use Lemma 8.5 in place of Corollary 5.18; we use the continuity of $\beta$ in (5) just established instead of Lemma 5.13.

As a result $\mathcal{G} := H^{>s_0}(M, G)$ is a $\mathbb{K}$-analytic BCH-Lie group modelled on $H^{>s_0}(M, \mathfrak{g})$. For each $x \in M$, the point evaluation $\mathrm{ev}_x \colon \mathcal{G} \to G$ is a $\mathbb{K}$-analytic group homomorphism and

$$\alpha^{-1} \colon L(\mathcal{G}) \to H^{>s_0}(M, \mathfrak{g}), \quad v \mapsto (L(\mathrm{ev}_x)(v))_{x \in M}$$

is an isomorphism of topological Lie algebras if we give $H^{>s_0}(M, \mathfrak{g})$ the pointwise Lie bracket. Moreover, $H^{>s_0}(M, \exp_G) \circ \alpha^{-1}$ is the exponential function of $\mathcal{G}$.

*Existence of a direct limit chart.* With $P$, $Q$, $U$, $V$, $\phi$ as in the preceding adaptation of the proof of Proposition 1.1, the map

$$\Phi := H^{>s_0}(M, \phi|_V) \colon H^{>s_0}(M, V) \to H^{>s_0}(M, U)$$

is a $\mathbb{K}$-analytic diffeomorphism and $\Phi^{-1}$ is a chart for $H^{>s_0}(M, G)$ whose restriction to $H^{>s_0}(M, U) \cap H^{s_j}(M, G) = H^{s_j}(M, U)$ is the chart

$$H^{s_j}(M, \phi^{-1}|_U) \colon H^{s_j}(M, U) \to H^{s_j}(M, V), \quad \gamma \mapsto \phi^{-1}|_U \circ \gamma$$

of the Lie group $H^{s_j}(M, G)$ around $e$. Thus $\Phi^{-1}$ is a strict direct limit chart for $H^{>s_0}(M, G) = \bigcup_{j \in \mathbb{N}} H^{s_j}(M, G)$ around $e$ as in [13, Definition 2.1].

*Regularity.* Since $H^{>s_0}(M, \mathfrak{g}) = \varinjlim H^{s_j}(M, \mathfrak{g})$ is a Silva space and thus compact regular, the Lie group $H^{>s_0}(M, G) = \bigcup_{j \in \mathbb{N}} H^{s_j}(M, G)$ is $L^\infty_{\mathrm{rc}}$-regular by [16, Proposition 8.10] and hence $C^0$-regular.

*Direct limit properties.* Since $\mathcal{G} := H^{>s_0}(M, G) = \bigcup_{j \in \mathbb{N}} H^{s_j}(M, G)$ has a direct limit chart and $L(\mathcal{G}) \cong H^{>s_0}(M, \mathfrak{g}) = \varinjlim H^{s_j}(M, \mathfrak{g})$ is a Silva space, [13, Proposition 9.8 (i)] shows that $\mathcal{G} = \varinjlim H^{s_j}(M, G)$ as a topological group, $C^\infty_{\mathbb{L}}$-Lie group for $\mathbb{L} \in \{\mathbb{R}, \mathbb{K}\}$, and as a $C^r_{\mathbb{L}}$-manifold for all $r \in \mathbb{N}_0 \cup \{\infty\}$.

*Compact subsets.* Since $\mathcal{G}$ has a direct limit chart and

$$L(\mathcal{G}) \colon H^{>s_0}(M, \mathfrak{g}) = \bigcup_{j \in \mathbb{N}} H^{s_j}(M, \mathfrak{g})$$

is compact regular, each compact subset $K$ of $\mathcal{G}$ is a compact subset of $H^{s_j}(M, G)$ for some $j \in \mathbb{N}$, by [14, Lemma 6.1]. ∎

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