Journal of Lie Theory
Volume **33** (2023) 297–303

On Topologically Quasihamiltonian LC-Groups

Wolfgang Herfort

Communicated by K.-H. Neeb

Abstract. A topologically quasihamiltonian group G is defined by the property that any two closed subgroups X and Y give rise to a closed subgroup $\overline{XY} = \overline{YX}$. Y. N. Mukhin employed lattice theoretic arguments for proving that any such group with a connected component not a singleton set must be commutative. We reprove here this fact – using only standard arguments from topological group theory.

Mathematics Subject Classification: 22A05, 22A26.

Key Words: Quasihamiltonian locally compact groups, permutable subgroups.

1. Introduction

A group is *hamiltonian* provided every subgroup is normal. It was R. Dedekind, who classified in 1897 the finite groups with this property (see [1]). K. Iwasawa introduced in [7] the more general notion of *quasihamiltonian group*: A group G is *quasihamiltonian* provided any two subgroups X and Y satisfy $XY = YX$. Then clearly XY is a subgroup of G. There is a vast literature on the classification of such groups, see e.g. the book of R. Schmidt, [10]. Turning to topological groups, for locally compact groups, the full characterization of all such groups with $XY = YX$ a closed subgroup for any closed subgroups X and Y has been accomplished by K. H. Hofmann, F. G. Russo, and the present author in [4]. An important ingredient during the proof is the following weakening of this concept, introduced by F. Kümmich in [8], where he defined a locally compact group G to be *topologically quasihamiltonian* if, and only if, for every pair of closed subgroups X and Y, the equality

$$\overline{XY} = \overline{YX} \tag{1}$$

holds. If such a group G is *discrete* it is quasihamiltonian. In [8] Kümmich shows

Proposition 1.1. *When G is topologically quasihamiltonian so is every closed subgroup and any factor group of G.*

In the same paper he already showed that an almost connected topologically quasihamiltonian group must be abelian, more precisely:

ISSN 0949–5932 / $2.50

Theorem 1.2 (Satz in [8])**.** *A locally compact group G with a nontrivial identity component G_0 and compact factor group G/G_0 is topologically quasihamiltonian if, and only if, G is abelian.*

His original statement includes the addition that every compact totally disconnected topologically quasihamiltonian group is the projective limit of finite quasihamiltonian groups. Y. N. Mukhin, making use of Theorem 1.2, gave a complete description of all topologically quasihamiltonian groups in [9] (see also [3] for a broader discussion of these and related results). The issue of our paper is reproving

Theorem 1.3 (Consequence 2 in [9])**.** *Any locally compact topologically quasihamiltonian group G with nontrivial identity component G_0 is abelian.*

The motivation for presenting an alternative proof of the theorem is not only the lack of an English translation of Mukhin's article [9]. It was the search for a purely group theoretic approach based upon knowledge about discrete quasihamiltonian groups and Theorem 1.2, thereby avoiding lattice theoretic arguments. We note that in [11] W. Xi, M. Shlossberg, and D. Toller provided a quick proof of the above fact under the additional assumption G being *compactly covered*, i.e, every element being contained in a compact subgroup of G.

1.1. Some notation. We refer to [6] for standard notation concerning locally compact groups. However, during this paper, unless stated differently, direct sums, (semi)direct products, and, isomorphisms are meant to be both, algebraic and topological. For elements x and y in a group G we let $y^x = xyx^{-1}$ denote conjugation and $[x,y] = xyx^{-1}y^{-1}$ be their commutator. For a subgroup H of G and $x \in G$ define $H^x = \{h^x : h \in H\}$. For subsets A and B of G we denote by $[A,B]$ the subgroup of G topologically generated by $\{[a,b] : a \in A, b \in B\}$. The factor group $\mathbb{R}/\mathbb{Z}$ will be denoted by $\mathbb{T}$. A group G is *locally cyclic* if every finite subset is contained in a cyclic subgroup (see e.g. [2, p. 16]). For a topological group G we let G_0 denote the identity component.

We record an immediate consequence of [10, 2.4.11. Theorem]:

Lemma 1.4. *Let G be a discrete nonabelian quasihamiltonian group. Then the set $T(G)$ of torsion elements is a characteristic subgroup and $G/T(G)$ is abelian. Moreover, if $T(G) \neq \{1\}$, then $G/T(G)$ is locally cyclic.*

Lemma 1.5. *Let G be a topologically quasihamiltonian group. Then G contains an open subgroup $\widetilde{P}$ comprising the set P of all compact elements of G and the following statements hold:*

(i) *P is a subgroup of G.*

(ii) *$\widetilde{P}/G_0$ is the set of all compact elements of G/G_0 and $\widetilde{P}$ is an open characteristic abelian subgroup of G with torsion-free factor group $G/\widetilde{P}$. Furthermore, $P = \mathrm{comp}(\widetilde{P}) = \mathrm{comp}(G)$ is a closed subgroup of G.*

(iii) *Every closed subgroup of $\widetilde{P}$ is normal in G. Moreover, $\widetilde{P} \setminus \mathrm{comp}(\widetilde{P}) \subseteq Z(G)$.*

(iv) *$G/\widetilde{P}$ is locally cyclic.*

(v) *G is abelian if, and only if, $\mathrm{comp}(\widetilde{P}) \leq Z(G)$.*

Proof. (i) Let x and y belong to P. Then $\overline{\langle x\rangle}$ and $\overline{\langle y\rangle}$ are compact subgroups of G and their permutability implies that

$$\overline{\langle x\rangle}\,\overline{\langle y\rangle} = \overline{\langle y\rangle}\,\overline{\langle x\rangle}$$

is a compact subgroup of G. Since $xy \in \overline{\langle x\rangle}\,\overline{\langle y\rangle}$ conclude

$$\overline{\langle xy\rangle} \le \overline{\langle x\rangle}\,\overline{\langle y\rangle}$$

and thus $\overline{\langle xy\rangle}$ is compact. Hence $xy \in P$. Clearly, if x is a compact element, so is x^{-1}. Thus P is a subgroup of G.

(ii) As a consequence of [5, (7.7) Theorem] there is an open compact subgroup, say U, of the totally disconnected locally compact group G/G_0. Since, by Proposition 1.1, G/G_0 is topologically quasihamiltonian it follows from applying (i) to G/G_0 that $\mathrm{comp}(G/G_0)$ is a subgroup of G/G_0. Since $U \le \mathrm{comp}(G/G_0)$ it follows that $\mathrm{comp}(G/G_0)$ is an open and hence closed subgroup of G/G_0. We let $\widetilde{P}$ be the unique open subgroup of G with $\widetilde{P}/G_0 = \mathrm{comp}(G/G_0)$. Certainly $\widetilde{P}$ is characteristic in G since G_0 is characteristic in G and $\widetilde{P}/G_0$ is characteristic in G/G_0.

For showing that $\widetilde{P}$ is abelian pick $x, y \in \widetilde{P}$ and set $L := \overline{\langle x, y, G_0\rangle}$. Then L is almost connected and topologically quasihamiltonian by Proposition 1.1. Hence, by Theorem 1.2, L, and thus $\widetilde{P}$, is abelian.

Since $G/\widetilde{P}$ is torsion-free and discrete it follows that $P = \mathrm{comp}(G) \le \widetilde{P}$. Therefore $\mathrm{comp}(G) = \mathrm{comp}(\widetilde{P}) = P$. By [6, Corollary 7.55] $P = \mathrm{comp}(\widetilde{P})$ is a closed subgroup of $\widetilde{P}$ and hence of G.

For proving the torsion-freeness of $G/\widetilde{P}$ suppose there is $x \in G$ and $n \ge 1$ with $x^n \in \widetilde{P}$. Then $(xG_0/G_0)^n \in \mathrm{comp}(G/G_0)$ and therefore xG_0/G_0 is itself a compact element in G/G_0. It follows that $x \in \widetilde{P}$. Hence $G/\widetilde{P}$ is indeed torsion-free.

(iii) Since, by (ii), $\widetilde{P}$ is abelian it will suffice to show that $H^x = H$ holds for every monothetic subgroup $H = \overline{\langle y\rangle}$ of $\widetilde{P}$ and every $x \in G \setminus \widetilde{P}$. Note first that $\overline{\langle x\rangle} \cong \langle x\rangle \cong \mathbb{Z}$ is discrete.

Since G is topologically quasihamiltonian one must have

$$\overline{\langle x\rangle\overline{\langle y\rangle}} = \overline{\overline{\langle y\rangle}\langle x\rangle}. \qquad (*)$$

Since H is monothetic Weil's Lemma (see [6, 7.43]) implies that exactly one of the following is true

(A) $H = \langle y\rangle \cong \mathbb{Z}$ is discrete.

(B) $H = \overline{\langle y\rangle}$ is compact.

Let us show in both cases that $\langle x\rangle \cap \widetilde{P} = \{1\}$.

(A) Indeed, if $\langle x\rangle \cap \widetilde{P} \ne \{1\}$ then the algebraic isomorphism

$$\langle x\rangle/\langle x\rangle \cap \widetilde{P} \cong_{\mathrm{alg}} \overline{\langle x\rangle}\widetilde{P}/\widetilde{P}$$

and the finiteness of the group on the left hand side show that for some natural n one has $x^n \in \widetilde{P}$. Therefore, in the factor group G/G_0 the element $\xi := xG_0/G_0$ satisfies $\xi^n \in \mathrm{comp}(G/G_0)$ and thus $\xi \in \mathrm{comp}(G/G_0)$. Then, however, $x \in \widetilde{P}$ and since $\widetilde{P}$ is abelian, we certainly have that $H^x = H$.

(B) Since H is compact and $\langle x\rangle \cong \mathbb{Z}$ is discrete and thus does not contain compact elements we must have $H \cap \langle x\rangle = \{1\}$.

Equation $(*)$ implies that $xy \in \overline{\langle y\rangle\langle x\rangle}$ and thus there is a net $(y_\nu x^{k_\nu})$ with elements $y_\nu \in \langle y\rangle$ and integers k_ν, converging to xy. Since $\widetilde{P}$ is open, there is $\nu_0 \in \mathbb{N}$ such that

$$xyx^{-k_\nu}y_\nu^{-1} \in \widetilde{P} \text{ for } \nu \geq \nu_0,$$

and, since y and y_ν belong to $\widetilde{P}$, conclude for all $\nu \geq \nu_0$ that $x^{1-k_\nu} \in \langle x\rangle \cap \widetilde{P}$. Observing that $\langle x\rangle \cap \widetilde{P} = \{1\}$ one deduces that $k_\nu = 1$ for all $\nu \geq \nu_0$. Therefore the net $(y_\nu x)$ converges and thus the net (y_ν) converges to some element in H. Altogether $H^x \leq H$ follows. Replacing x by x^{-1} and observing that $\langle x\rangle = \langle x^{-1}\rangle$ one finds $H^{x^{-1}} \leq H$, i.e., $H \leq H^x$. Thus $H = H^x$.

For proving the extra statement pick $x \in G \setminus \widetilde{P}$ and $y \in \widetilde{P} \setminus \operatorname{comp}(\widetilde{P})$. Then $H := \langle y\rangle \cong \mathbb{Z}$ is a discrete monothetic subgroup of $\widetilde{P}$ and is therefore normal in G by what we just proved. Since any nontrivial automorphism of $\mathbb{Z}$ inverts the elements in $\mathbb{Z}$ either $y^x = y$ or else $y^x = y^{-1}$. In the first case we are done. Thus assume $y^x = y^{-1}$. Then $y^{x^2} = y$ and thus subgroup $\langle x^2\rangle$ is normal in the discrete subgroup $\langle x, y\rangle$ of G. By Proposition 1.1 $\langle x, y\rangle$ and also the factor group $L := \langle x, y\rangle/\langle x^2\rangle$ are quasihamiltonian. However, L is isomorphic to the infinite dihedral group which cannot be quasihamiltonian by Lemma 1.4, contradiction. Thus $[G \setminus \widetilde{P}, \widetilde{P} \setminus \operatorname{comp}(\widetilde{P})] = \{1\}$. Since (ii) implies that $\widetilde{P}$ is abelian we also have that $[\widetilde{P}, \widetilde{P} \setminus \operatorname{comp}(\widetilde{P})] = \{1\}$ so that $[G, \widetilde{P} \setminus \operatorname{comp}(\widetilde{P})] = \{1\}$ follows. Therefore $\widetilde{P} \setminus \operatorname{comp}(\widetilde{P}) \subseteq Z(G)$.

(iv) Let U be any open compact subgroup of G/G_0 and let $\widetilde{U}$ be the unique subgroup of G with $\widetilde{U}/G_0 = U$. Since $U \leq \operatorname{comp}(G/G_0)$ it follows that $\widetilde{U} \leq \widetilde{P}$. Therefore (iii) implies that $\widetilde{U} \trianglelefteq G$ and hence $U = \widetilde{U}/G_0 \triangleleft G/G_0$. Since $(\widetilde{P}/G_0)/U = \operatorname{tor}(G/U)$ and by Proposition 1.1 the discrete group G/U is topologically quasihamiltonian and therefore $(G/U)/\operatorname{tor}(G/U) \cong G/\widetilde{P}$ we conclude from Lemma 1.4 that $G/\widetilde{P}$ is locally cyclic.

(v) If G is abelian then certainly $\operatorname{comp}(\widetilde{P}) \leq Z(G)$.

For proving the converse, since $\widetilde{P}$ is abelian by (ii), we only need to prove that $x \notin \widetilde{P}$ commutes with any $y \in G$. Observe first that by (iii) $\widetilde{P} \setminus \operatorname{comp}(\widetilde{P}) \leq Z(G)$. As, by assumption, $\operatorname{comp}(\widetilde{P}) \leq Z(G)$, we may conclude from this that $\widetilde{P} \leq Z(G)$. Since, by (iv), $\overline{\langle x, y, \widetilde{P}\rangle}/\widetilde{P}$ is locally cyclic there are $t \in G \setminus \widetilde{P}$, u and v in $\widetilde{P}$, and integers k and l such that $x = t^k u$ and $y = t^l v$. Therefore, since $\widetilde{P} \leq Z(G)$,

$$xy = t^k u t^l v = t^{k+l} uv = t^l v t^k u = yx.$$

Hence G is abelian. ∎

Proof of Theorem 1.3. Let G be a locally compact topologically quasihamiltonian group and $G_0 \neq \{1\}$ and let $\widetilde{P}$ and $P = \operatorname{comp}(G)$ be as in Lemma 1.5.

Claim 1: We can assume that $\widetilde{P} = P = \operatorname{comp}(G)$ *is a proper subgroup of* G*. As a consequence,* G_0 *turns out to be compact.*

Since by Lemma 1.5(ii) the closed subgroup $\widetilde{P}$ is abelian we can assume $G \setminus \widetilde{P} \neq \emptyset$. Next let us assume that $\widetilde{P} \setminus P \neq \emptyset$ and show that G is abelian: indeed, by Lemma 1.5(ii) we have that $P = \mathrm{comp}(\widetilde{P})$ and therefore Lemma 1.5(iii) implies $\widetilde{P} \setminus P \leq Z(G)$. Pick $x \in \widetilde{P} \setminus P$. Then, for y arbitrary in P one has $xy \in \widetilde{P} \setminus P$ and hence x and xy belong to $Z(G)$. Thus $y \in Z(G)$ showing that $P = \mathrm{comp}(\widetilde{P}) \leq Z(G)$. Lemma 1.5(v) implies that G must be abelian.

The reduction to $P = \tilde{P}$ implies that $G_0 = P_0 = \tilde{P}_0$ is compact. It follows that $G_0 = \widetilde{P}_0$ is a locally compact abelian group consisting of compact elements only. Therefore, by the *Vector Splitting Theorem* (see [6, Theorem 7.57]), the connected component $G_0 = \widetilde{P}_0 = P_0$ is compact. Claim 1 holds.

By way of contradiction assume that G is not abelian. Then, as a consequence of Lemma 1.5(v) there exists $y \in \widetilde{P} \setminus Z(G)$. Since by Lemma 1.5(iii) the subgroup $\widetilde{P}$ is abelian there must exist $x \in G \setminus \widetilde{P}$ not commuting with y. Moreover one observes that by Lemma 1.5(iii) the compact subgroup $\overline{\langle y \rangle} G_0$ is normal in G and that $\langle x \rangle \cap \overline{\langle y \rangle} G_0 = \{1\}$. Hence we can make the following assumption:

$$G = \overline{\langle y \rangle} G_0 \rtimes \langle x \rangle \ \textit{and}\ z := [x, y] \neq 1. \qquad (\ddagger)$$

Claim 2: We have $P = \widetilde{P} = \overline{\langle y \rangle} G_0$.

By Claim 1 the component G_0 is compact and $P = \widetilde{P} = \mathrm{comp}(G)$. Therefore $y \in P$ and hence $\overline{\langle y \rangle} G_0$ is compact. Since the factor group $G/\overline{\langle y \rangle} G_0 \cong \mathbb{Z}$ is discrete it follows that $P = \overline{\langle y \rangle} G_0$.

Claim 3: We can assume G to be a Lie group, and for some $m \geq 0$, $P = \overline{\langle y \rangle} G_0 \cong \mathbb{Z}(m) \times \prod_{i=1}^{r} T_i$ for $T_i \cong \mathbb{T}$, where $1 \leq i \leq r$ and $r \geq 1$.

Recall from Claim 2 that $\widetilde{P} = P = \overline{\langle y \rangle} G_0$. Suppose that we know already that every topologically quasihamiltonian *Lie group* with nontrivial connected components must be abelian. The set $\mathcal{U}$ of open subsets U of G with $1 \in U$ but $z = [x, y] \notin U$ is a filter basis of open neighbourhoods of 1. By Claim 2 the set of compact elements $\widetilde{P} = P$ is open and thus we can assume $U \subseteq P$ for all $U \in \mathcal{U}$. Fix, for a moment, $U \in \mathcal{U}$. Since P is a locally compact abelian group one can find inside U a compact subgroup K of P with P/K a Lie group and $zK/K \neq 1$ (see e.g. [6, Corollory 7.54]). Since, as an immediate consequence of Lemma 1.5(iii), K turns out to be normal in G and as, by taking Equation ($\ddagger$) and Claim 2 into account, G/P can be seen to be discrete, it follows that G/K is a Lie group. As we assume Theorem 1.3 to hold for Lie groups, we may conclude that G/K is abelian. Hence $z = [x, y] \in K \leq U$, a contradiction.

Thus we can assume that G is a Lie group and so is $P = \mathrm{comp}(G)$. Therefore, by [6, Corollary 7.54], $P \cong \mathbb{T}^r \times D$ for some $r \geq 1$ (since $G_0 \neq \{1\}$) and D a discrete and hence finite subgroup of G. Since P/G_0 is cyclic it follows that $D \cong \mathbb{Z}(m)$ for some $m \geq 0$.

Claim 4: $G_0 \leq Z(G)$

By Claim 3 $P \cong \mathbb{Z}(m) \times \prod_{i=1}^{r} T_i$ with $T_i \cong \mathbb{T}$. We need to show that x centralizes every T_i. Fix $1 \leq i \leq r$. Then by Lemma 1.5(iii) $T_i^x = T_i$ and, as $\mathrm{Aut}(\mathbb{T}) \cong \mathbb{Z}(2)$ the element x acts by conjugation upon T_i either by inverting all elements or by centralizing them. In the latter case we are done. Suppose x inverts the elements in T_i. Then $T_i \cong \mathbb{T}$ contains an element t of order 4. In particular, $t^x = t^{-1} \neq t$. Consider the discrete subgroup $\Gamma := \langle t \rangle \rtimes \langle x \rangle$, which by Proposition 1.1 is quasihamiltonian. Since x^2 commutes with t we may pass to the factor group $\Gamma/\langle x^2 \rangle$ – which by Proposition 1.1 is quasihamiltonian. However, we arrive at a contradiction since $\Gamma/\langle x^2 \rangle$ is isomorphic to the dihedral group D_8, which is not quasihamiltonian. Thus $T_i \leq Z(G)$ and hence $G_0 \leq Z(G)$.

Finishing the proof:

Since $G_0 \leq Z(G)$ by Claim 4, in light of condition (‡), a final contradiction will arise if we show $z = [x, y] = 1$. The topological isomorphism $P \cong \mathbb{Z}(m) \times \prod_{i=1}^{r} T_i$ from Claim 3 and the fact that by Claim 4 $G_0 \leq Z(G)$, show that condition (‡), namely $1 \neq z = [x, u]$, must hold for some u with $\langle u \rangle \cong \mathbb{Z}(m)$. Replacing y by u, we therefore we can assume that $y^m = 1$. Since $\langle y \rangle^x = \langle y \rangle$ by Lemma 1.5(iii), and $\langle y \rangle$ is finite there must exist $k > 1$ with $y^{x^k} = y$. Factoring the discrete and hence closed normal subgroup $\langle x^k \rangle$ results in the compact Lie group $L := P \rtimes \langle x \rangle / \langle x^k \rangle \cong P \rtimes \mathbb{Z}(k)$. Since $L_0 \cong G_0 \neq \{1\}$ it follows from Theorem 1.2 that L is abelian and therefore x acts trivially on $\langle y \rangle$, i.e.. $z = [x, y] = 1$. Thus we arrive at the contradiction to assumption (‡). ■

Acknowledgements. I would like to express thanks to K. H. Hofmann for his detailed remarks on an early version of this note, M. Shlossberg (Beersheva) for an email discussion, and the referee for his quick and helpful report.
It is my great pleasure, on the occasion of his 90-th birthday, to thank K. H. Hofmann for his enormeous patience when reading my often quite incomplete and jumpy drafts, the numerous ideas he contributed, his detailed and inspiring email-discussions, responses accompanied with plain.tex-letters, containing lots of suggestions, leading to significant developments and improvements. I truly enjoy the privilege of this very fruitful collaboration with K. H. Hofmann and F. G. Russo.
My very best wishes and congratulations.

References

[1] R. Dedekind: *Üeber Gruppen, deren sämtliche Teiler Normalteiler sind*, Math. Ann. 48/4 (1897) 548–561.

[2] L. Fuchs: *Infinite Abelian Groups. Vol. I*, Pure and Applied Mathematics 36, Academic Press, New York (1970).

[3] W. Herfort, K. H. Hofmann, F. G. Russo: *Periodic Locally Compact Groups: A Study of a Class of Totally Disconnected Topological Groups*, Studies in Mathematics 71, DeGruyter, Berlin (2018).

[4] W. Herfort, K. H. Hofmann, F. G. Russo: *Locally compact groups with permutable closed subgroups*, Adv. Math. 390 (2021), art. no. 107894, 13 p.

[5] E. Hewitt, K. A. Ross: *Abstract Harmonic Analysis. I: Structure of Topological Groups. Integration Theory, Group Representations*, Grundlehren der Mathematischen Wissenschaften 115, Springer, Berlin (1963).

[6] K. H. Hofmann, S. A. Morris: *The Structure of Compact Groups*, 4th revised and augmented edition, De Gruyter Studies in Mathematics 25, De Gruyter, Berlin (2020).

[7] K. Iwasawa: *Über die endlichen Gruppen und die Verbände ihrer Untergruppen*, J. Fac. Sci. Imp. Univ. Tokyo, Sect. I. 4 (1941) 171–199.

[8] F. Kümmich: *Topologisch quasihamiltonsche Gruppen*, Arch. Math. (Basel) 29/4 (1977) 392–397.

[9] Y. N. Mukhin: *Topologically quasi-Hamiltonian groups (Russian)*, in: *Problems in Algebra, No. 4 (Russian)*, Universitetskoe, Minsk (1989) 83–89.

[10] R. Schmidt: *Subgroup Lattices of Groups*, Expositions in Mathematics 14, De Gruyter, Berlin (1994).

[11] W. Xi, M. Shlossberg, D. Toller: *Algebraic entropy on topologically quasihamiltonian groups*, Topology Appl. 272 (2020), art. no. 107093, 24 p.

Wolfgang Herfort, Department of Analysis and Scientific Computing, University of Technology, Wiedner Hauptstraße 8-10, A-1040 Vienna, Austria; w.herfort@tuwien.ac.at.

Received April 29, 2022
and in final form May 16, 2022