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Geodesic Bicombings and a Metric Crandall-Liggett Theory

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Abstract. We develop an abstract and general Crandall-Liggett theory in the setting of metric geometry that generalizes the well-known one originally developed for solving certain classes of differential equations on Banach spaces. The metric spaces considered are complete metric spaces equipped with a conical geodesic bicombing, a distinguished collection of metric geodesics that satisfy a weak global non-positive curvature condition. The cone of invertible positive linear operators on a Hilbert space, or more generally the cone of positive invertible elements on a unital C^*-algebra, equipped with the Thompson metric is a motivating example for the type of metric space we consider. Some examples of application of our results arose in that setting, but generalize to spaces with geodesic bicombings.

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1. Introduction

In this paper we seek to make some substantial connections among three significant topics. The first topic began with the development of two-variable operator means, a foundational paper being that of Ando and Kubo [14] in 1980. The geometric operator mean, with particular emphasis on the matrix geometric mean, received considerable attention. The multivariable matrix geometric mean was introduced independently by Moakher [19] and Bhatia and Holbrook [4] in 2005 and 2006 and the theory has advanced rapidly since. Lim and the author [17] extended many of the basic results to the cone of positive operators on a Hilbert space and more recently the theory has been extended to the positive cone of a unital C^*-algebra [15], [18].

The open cone of positive invertible elements on a C^*-algebra with identity has an interesting geometric structure, primarily that of a symmetric cone, that has received considerable attention, see, for example, [16]. It also has a natural metric structure arising from the Thompson metric, for which the metric topology agrees with the relative topology from the C^*-algebra norm. There are multiple metric geodesics between points with respect to this metric, but there is a canonical choice for these, namely the maps $t \mapsto x\#_t y := x^{1/2}(x^{-1/2}yx^{-1/2})^t x^{1/2}$, which runs from x to y as t runs from 0 to 1. Members $x\#_t y$ of this geodesic are called weighted geometric means (of two variables). The mapping $\sigma(x, y, t) = x\#_t y$ is an important example

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of what was called a "conical geodesic bicombing" by Thurston. Bicombings have re-emerged in recent years and several papers treating various aspects of them exist in the recent literature, e.g., [3],[2], [8], [9],[10]. Bicombings are basically metric spaces with a constant speed geodesic defined on $[0,1]$ singled out continuously between each pair of points, and conical bicombings satisfy a weak non-negative curvature condition. Bicombings are the setting in which we work in this paper and form the second source on which our work is based.

The third and main background source is the machinery associated with the famous 1971 paper of Crandall and Liggett [7]. The Crandall-Liggett generation theorem of that paper describes how to solve a broad class of differential equations on Banach spaces by exponentiating operators. Their solution is given in terms of a semigroup of operators $\{S(t)\}$ indexed by the non-negative reals $\mathbb{R}^+$. What we realized was that their basic techniques can be generalized to conical bicombings. We identify three basic properties (J1), (J2), and (J3) that are needed, and show that when these are satisfied on a complete metric space with a conical bicombing, then a semigroup action of transformations $\{S(t) : t \geq 0\}$ is generated. Our development was inspired by Shapiro's approach [21] to the Crandall-Liggett theory, and generalizes his work to the setting of bicombings. We briefly review the Banach space application of the Crandall-Liggett approach in Section 5.

That this machinery can supply some powerful tools has recently been demonstrated by Lim and Palfia in [18] in the context of studying the Karcher geometric mean on the positive cone of a unital C^*-algebra. Using Crandall-Liggett tools they have been able to solve some difficult questions that had been open. Our work in Sections 6 and 7 is closely related to parts of their work, but done in a more general setting.

The semigroup flow results given here are reminiscent of the gradient flow theory that has been developed for finite-dimensional CAT-(0) spaces (or Hadamard spaces) [1]. However, in the general theory of bicombings one does not have the minimizers present to define the resolvent functions J_λ associated with that theory, so one must look for other approaches. The Crandall-Liggett machinery offers the possibility of finding alternative routes to desired ends in the setting of metric spaces equipped with conical bicombings.

2. Geodesic Bicombings

Let (X,d) be a metric space. A *standardized geodesic*, or *geodesic* for short, is a map $\alpha\colon [0,1] \to X$ such that $d(\alpha(s),\alpha(t)) = |s-t|d(\alpha(0),\alpha(1))$ for all $s,t \in [0,1]$. Suppose now that (X,d) denotes a metric space equipped with a continuous map $\sigma\colon X \times X \times [0,1] \to X$ such that $\sigma_{x,y}\colon [0,1] \to X$ defined by $\sigma_{xy}(t) = \sigma(x,y,t)$ is a geodesic with $\sigma_{xy}(0) = x$ and $\sigma_{xy}(1) = y$ for all $x,y \in X$. Thus σ identifies a distinguished geodesic between x and y for each $(x,y) \in X \times X$ and reduces to $\sigma_{x,x}(t) = x$ for the degenerate case $y = x$. Such maps σ were called *geodesic bicombings* by Epstein and Thurston [11, p.83], and they have received renewed attention in recent years. We shorten the terminology simply to bicombing and also typically denote $\sigma_{xy}(t)$ by $x\#_t y$. With respect to this notation we now have $\sigma(x,y,t) = \sigma_{x,y}(t) = x\#_t y$. The fact that $\sigma_{x,y}$ is a geodesic between x and y readily yields that $d(x, x\#_t y) = td(x,y)$, $d(x\#_t y, y) = (1-t)d(x,y)$ for all $x,y \in X$, $0 \leq t \leq 1$.

A bicombing $\sigma\colon X\times X\times[0,1]\to X$ is:

- *reversible* if $x\#_t y = y\#_{1-t}x$ for all $x,y\in X$ and $t\in[0,1]$;
- *consistent* if $\sigma_{xy}((1-\lambda)t+\lambda s)=\sigma_{pq}(\lambda)$ whenever $x,y\in X$, $0\le s\le t\le 1$, $p:=\sigma_{xy}(s)$, $q:=\sigma_{xy}(t)$, and $\lambda\in[0,1]$;
- *conical* if $d(x\#_t y, x'\#_t y')\le(1-t)d(x,x')+td(y,y')$ for all $x,x',y,y'\in X$, $t\in[0,1]$; for the case $x=y$ this reduces to $d(x,x'\#_t y')=d(x\#_t x, x'\#_t y')\le(1-t)d(x,x')+td(x,y')$.
- *convex* if $t\mapsto d(x\#_t y, x'\#_t y')$ is a convex function on the interval $[0,1]$ for all $x,y,x',y'\in X$.

Remark 2.1. The following are some elementary and well-known properties about bicombings.

(1) A bicombing σ on (X,d) is consistent if and only if for all points $p,q\in X$, whenever $p'=\sigma_{pq}(s)$ and $q'=\sigma_{pq}(t)$ for $0\le s\le t\le 1$, then $\sigma_{p'q'}([0,1])\subseteq\sigma_{pq}([0,1])$.

(2) A bicombing σ is conical if and only if we have $d(x\#_t y, x\#_t z)\le td(y,z)$ and $d(x\#_t z, y\#_t z)\le(1-t)d(x,y)$ for all $x,y,z\in X$, $t\in[0,1]$.

(3) A conical bicombing on a complete metric space admits a reversible conical bicombing.

(4) For bicombings, consistent conical $\Rightarrow$ convex $\Rightarrow$ conical.

The following elementary lemma is useful.

Lemma 2.2. *Let (X,d,σ) be a consistent bicombing. Then for all $x,y\in X$ and $s,t\in[0,1]$ we have*

$$x\#_s(x\#_t y)=x\#_{st}\,y,\quad (x\#_s y)\#_t y = x\#_r y,\ \textit{where } r=s+t-st.$$

Proof. As remarked previously $d(x,x\#_t y)=td(x,y)$. Hence

$$d(x,x\#_s(x\#_t y))=sd(x,x\#_t y)=std(x,y)=d(x,x\#_{st}y).$$

Using the consistency of the bicombing, we see also that $x\#_s(x\#_t y)$ and $x\#_{st}y$ both lie on the geodesic from x to y and hence they must be equal. The other case follows similarly. ■

A motivating example of a bicombing for this work is the cone $\mathbb{P}$ of positive invertible elements of a unital C^*-algebra.

Example 2.3. Let $\mathbb{A}$ be a unital C^*-algebra and let $\mathbb{P}$ be the cone of positive invertible elements. A natural and useful metric for $\mathbb{P}$ is the *Thompson metric.* One simple definition of it is $d(x,y)=\|\log(x^{-1/2}yx^{-1/2})\|$. The topology induced by the Thompson metric is the same as the relative topology arising from the norm topology on $\mathbb{A}$. There is a natural bicombing on $\mathbb{P}$ that is conical, reversible, consistent and convex. This bicombing $\sigma\colon\mathbb{P}\times\mathbb{P}\times[0,1]\to\mathbb{P}$ is given by

$$\sigma(x,y,t)=x\#_t y:=x^{1/2}(x^{-1/2}yx^{-1/2})^t x^{1/2}.$$

For more details on the preceding example see [17] and [18], and the references there.

Other examples of metric spaces with a bicombing include:

- Banach spaces,
- CAT(0) spaces (also called Hadamard or NPC spaces),
- The space of integrable measures $\mathbf{P}^1(X)$ equipped with the Wasserstein metric $\mathfrak{w}$ for X a metric space.

We remark that there are midpoint structures equivalent to bicombings. In a metric space (X,d) a point m is a midpoint for the pair (a,b) if

$$d(a,m) = (1/2)d(a,b) = d(m,b).$$

A *continuous midpoint map* on X is a continuous map $m\colon X\times X\to X$ such that $m(a,b)$ is a midpoint of (a,b) for every $(a,b)\in X\times X$, and in this case the triple (X,d,m) is called a *continuous midpoint space.* For a bicombing (X,d,σ) the map $\mu(x,y)=\sigma(x,y,1/2)$ is a continuous midpoint map, so (X,d,μ) is a continuous midpoint structure. Horvath in [13] has given for complete metric spaces an inverse construction that determines a bicombing from a continuous midpoint structure, and therewith determines a one-to-one correspondence between bicombings and continuous midpoint structures.

3. A combinatorial inequality

We assume in this section a double indexed family

$$\{\gamma(i,j)\in\mathbb{R}^+ : 0\le i\le n,\ 0\le j\le m\}$$

of non-negative real numbers with $\gamma(0,0)=0$ satisfying the following inequality for all $i,j\ge 1$,

$$\gamma(i,j)\le p\gamma(i-1,j-1)+q\gamma(i-1,j), \tag{1}$$

where p and q are fixed and $0\le p,q$ and $p+q=1$. Our goal in this section is to present an efficient combinatorial inequality derived from inequality (1) bounding $\gamma(n,m)$, where $0<m\le n$, by a sum of terms of the form $a_i\gamma(i,0)$ and and $b_j\gamma(0,j)$ where $i<n$ and $j\le m$.

To illustrate an approach, we begin with a derivation reminiscent of a binomial expansion. Using inequality (1) we observe

$$\begin{aligned}\gamma(n,m) &\le p\gamma(n-1,m-1)+q\gamma(n-1,m)\\ &\le p^2\gamma(n-2,m-2)+2pq\gamma(n-2,m-1)+q^2\gamma(n-2,m)\\ &\le p^3\gamma(n-3,m-3)+3p^2q\gamma(n-3,m-2)+3pq^2\gamma(n-3,m-1)\\ &\quad +q^3\gamma(n-3,m).\end{aligned}$$

Here we have repeatedly applied inequality (1) to the terms on the right-hand side of the inequality to obtain the next inequality. The preceding process can be continued until all the terms are of the form $(i,0)$ or $(0,j)$ preceded by some appropriate coefficient. The case $m=n$ is the easiest.

Lemma 3.1. *If $m=n>0$, then*

$$\gamma(n,n)\le\sum_{k=0}^{n}\binom{n}{k}p^kq^{n-k}\gamma(0,n-k) \tag{2}$$

Proof. If one continues the preceding expansion for the case $m=n$ to the n-th row, then that row is precisely the right-hand side of the inequality of the lemma. ∎

The case $0 < m < n$ is more difficult and in this case one obtains terminating terms both of the type $(0,k)$ and $(j,0)$. Nonetheless these terms can be appropriately combined to be presented in the following form.

Lemma 3.2. *For $0 < m < n$ equation* (1) *implies the following inequality:*

$$\gamma(n,m) \leq \sum_{k=0}^{m-1} \binom{n}{k} p^k q^{n-k} \gamma(0, m-k) + \sum_{k=m}^{n} \binom{k-1}{m-1} p^m q^{k-m} \gamma(n-k, 0). \qquad (3)$$

A version of this lemma appears as a key lemma, Lemma 1.3, in Crandall's and Liggett's fundamental paper [7]. The main inductive step of their proof is omitted with the disclaimer "The (somewhat awkward) induction is left to the reader." Joel Shapiro [21] has later given a direct "diagrammatic" proof expanding our illustration above to its completion and showing that the summing of the $\gamma(i,j)$ terms with one entry 0 yields the preceding formula.

4. The Crandall-Liggett machinery

Let $\{J_\lambda \colon X \to X : \lambda > 0\}$ be a family of maps on a metric space (X,d) equipped with a conical bicombing. We require that properties (J1), (J2), and (J3) be satisfied.

(J1) (contraction) $d(J_\lambda(x), J_\lambda(y)) \leq d(x,y)$.

(J2) (resolvent condition) For $0 < \tau < \lambda$, $J_\lambda(x) = J_\tau(J_\lambda(x) \#_{\frac{\tau}{\lambda}} x)$.

(J3) $d(J_\lambda(x), x) \leq K_{x,T}\lambda$ for all $x \in X$ and $0 < \lambda \leq T$ and some constant $K_{x,T}$ depending only on x and $T > 0$.

Remark 4.1. In specific situations where the Crandall-Liggett machinery is employed the constant $K_{x,T}$ typically takes a more concrete form, and frequently one that is independent of T.

Lemma 4.2. *For $0 < \tau < \lambda$, $0 < m,n$, and $x \in X$*

$$d(J_\tau^n(x), J_\lambda^m(x)) \leq \frac{\tau}{\lambda} d(J_\tau^{n-1}(x), J_\lambda^{m-1}(x)) + \frac{\lambda-\tau}{\lambda} d(J_\tau^{n-1}(x), J_\lambda^m(x)). \qquad (4)$$

Proof. We note by the resolvent condition (J2) that

$$d(J_\tau^n(x), J_\lambda^m(x)) = d(J_\tau^n(x), J_\tau(J_\lambda^m(x) \#_{\tau/\lambda} J_\lambda^{m-1}(x))),$$

and from the assumptions that J_τ is contractive and d is conical we obtain

$$\begin{aligned} d(J_\tau^n(x), J_\tau(J_\lambda^m(x) \#_{\tau/\lambda} J_\lambda^{m-1}(x))) &\leq d(J_\tau^{n-1}(x), J_\lambda^m(x) \#_{\tau/\lambda} J_\lambda^{m-1}(x)) \\ \leq \frac{\lambda-\tau}{\lambda} d(J_\tau^{n-1}(x), J_\lambda^m(y)) + \frac{\tau}{\lambda} d(J_\tau^{n-1}(x), J_\lambda^{m-1}(y)) \\ = \frac{\tau}{\lambda} d(J_\tau^{n-1}(x), J_\lambda^{m-1}(x)) + \frac{\lambda-\tau}{\lambda} d(J_\tau^{n-1}(x), J_\lambda^m(x)). \end{aligned}$$

■

Remark 4.3. If we choose $0 < m < n$ and $t > 0$ and set $p = \frac{m}{n}$, $q = 1-p = \frac{n-m}{n}$, then equation (4) can be written alternatively as

$$d(J_{t/n}^n(x), J_{t/m}^m(x)) \leq p d(J_{t/n}^{n-1}(x), J_{t/m}^{m-1}(x)) + q d(J_{t/n}^{n-1}(x), J_{t/m}^m(x)). \qquad (5)$$

Lemma 4.4. *For $x \in D$, $d(J_\lambda^n(x), x) \le n\lambda K_{x,T}$ if (J1), (J2), (J3) hold.*

Proof. From the triangle inequality and properties (J1) and (J3)

$$d(J_\lambda^n(x), x) \le \sum_{i=1}^{n} d(J_\lambda^i(x), J_\lambda^{i-1}(x)) \le \sum_{i=1}^{n} d(J_\lambda(x), x) \le n\lambda K_{x,T}. \qquad \blacksquare$$

Lemma 4.5. *For integers $0 < m < n$ and $t > 0$, set $p = \frac{m}{n}$ and $q = 1 - p = \frac{n-m}{n}$. Then for $x \in X$*

$$\begin{aligned} d(J_{t/n}^n(x), J_{t/m}^m(x)) \;\le\; & \sum_{k=0}^{m-1} \binom{n}{k} p^k q^{n-k} d(x, J_{t/m}^{m-k}(x)) \\ & + \sum_{k=m}^{n} \binom{k-1}{m-1} p^m q^{k-m} d(J_{t/n}^{n-k}(x), x). \end{aligned} \tag{6}$$

Proof. This follows directly from Lemma 3.2 where we take $\gamma_{i,j} = d(J_{t/n}^i(x), J_{t/m}^j(x))$ with $J_{t/n}^0(x) = x = J_{t/m}^0(x)$. We note by Remark 4.3 that inequality (1) holds. $\blacksquare$

In what follows we remain in the setting of the previous lemma with fixed $0 < m < n$ and $0 < t \le T_{x,T}$, and with $p = \frac{m}{n}$ and $q = 1 - p = \frac{n-m}{n}$.

We next give our version in the general setting of metric spaces of a key result of the Crandall-Liggett generation theorems. The proof is a mild adaptation of the probabilistic proof of Shapiro [21].

Theorem 4.6. *Let (X, d, σ) be a complete metric space equipped with a conical bicombing σ and let $\{J_\lambda \colon X \to X : \lambda > 0\}$ be a family of maps satisfying properties (J1)-(J3). Then for any t, $0 < t \le T$, integers $0 < m < n$, and $x \in X$,*

$$d(J_{t/n}^n(x), J_{t/m}^m(x)) \le 2tK_{x,T}\sqrt{\frac{1}{m} - \frac{1}{n}}. \tag{7}$$

Hence the pointwise limit $S(t) = \lim_n J_{t/n}^n$ exists for each $t > 0$.

Proof. By Lemma 4.4 we have $d(x, J_{t/m}^{m-k}(x)) \le (m-k)(t/m)K_{x,T}$ and similarly $d(J_{t/n}^{n-k}(x), x) \le (n-k)(t/n)K_{x,T}$. Applying these inequalities to (6) yields

$$d(J_{t/n}^n(x), J_{t/m}^m(x)) \le \left[\sum_{k=0}^{m-1} \frac{m-k}{m} \binom{n}{k} p^k q^{n-k} + \sum_{k=m}^{n} \frac{n-k}{n} \binom{k-1}{m-1} p^m q^{k-m}\right] tK_{x,T}. \tag{8}$$

Let X denote a binomial random variable that takes on the value k for $0 \le k \le n$ with probability $\binom{n}{k} p^k q^{n-k}$. It is well-known that X has mean $E(X) = np = m$ and variance $Var(X) = npq$. Hence

$$\begin{aligned} \left[\sum_{k=0}^{m-1} \frac{m-k}{m} \binom{n}{k} p^k q^{n-k}\right] tK_{x,T} \;&\le\; \left[\sum_{k=0}^{n} |m-k| \binom{n}{k} p^k q^{n-k}\right] (t/m)K_{x,T} \\ &=\; E\left(|E(X) - X|\right)(t/m)K_{x,T}. \end{aligned}$$

By Jensen's inequality $(E(|E(X)-X|))^2 \le E(|E(X)-X|^2) = \mathrm{Var}(X) = npq$, so

$$\frac{t}{m}K_{x,T}E(|E(X)-X|) \le \frac{t}{m}K_{x,T}(npq)^{1/2} = \frac{t}{m}K_{x,T}\left(m\frac{n-m}{n}\right)^{1/2} = tK_{x,T}\sqrt{\frac{1}{m}-\frac{1}{n}}.$$

To bound the second summation of inequality (8) at the beginning of the proof, we consider a negative binomial random Y that takes on the value k for $m \le k$ with probability $\binom{k-1}{m-1}p^m q^{k-m}$. Such a random variable has expectation $E(Y)=m/p=n$ and variance $\mathrm{Var}(Y) = mq/p^2 = \frac{n(n-m)}{m}$.

$$\begin{aligned}\left[\sum_{k=m}^{n}\frac{n-k}{n}\binom{k-1}{m-1}p^m q^{k-m}\right]tK_{x,T} &\le \left[\sum_{k=m}^{\infty}|(n-k)|\binom{k-1}{m-1}p^m q^{k-m}\right]\frac{t}{n}K_{x,T}\\ &= E(|E(Y)-Y|)\frac{t}{n}K_{x,T}\end{aligned}$$

By Jensen's inequality $(E(|E(Y)-Y|))^2 \le E(|E(Y)-Y|^2)) = \mathrm{Var}(Y) = \frac{n(n-m)}{m}$, so

$$\frac{t}{n}K_{x,T}E(|E(Y)-Y|) \le \frac{t}{n}K_{x,T}\left(\frac{n(n-m)}{m}\right)^{1/2} = tK_{x,T}\sqrt{\frac{1}{m}-\frac{1}{n}}.$$

Combining together the preceding, we obtain

$$d(J^n_{t/n}(x), J^m_{t/m}(x)) \le 2t\sqrt{\frac{1}{m}-\frac{1}{n}}K_{x,T}.$$

Thus the sequence $\{J^n_{t/n}(x)\}$ is Cauchy, so its limit $S(t)(x) = \lim_n J^n_{t/n}(x)$ exists. ■

Since the composition of contractions is again a contraction and the pointwise limit of contactions is a contraction, we have the following corollary.

Corollary 4.7. *Under the hypotheses of the preceding theorem each of the maps $S(t)\colon X \to X$, $t > 0$ is contractive.*

In the following lemmas we maintain the setting of Theorem 4.6.

Lemma 4.8. *For each positive integer m and $x \in X$*

$$\lim_{n\to\infty}(J^n_{t/n})^m(x) = (S(t))^m(x) = S(mt)(x).$$

Proof. The equalities hold trivially for $m = 1$. Assume the first equality holds for $m = k \ge 1$. Then for $x \in X$

$$\begin{aligned}&d((J^n_{t/n})^{k+1}(x), (S(t))^{k+1}(x))\\ &\le d(J^n_{t/n}(J^n_{t/n})^k(x), J^n_{t/n}(S(t))^k(x)) + d(J^n_{t/n}(S(t)^k(x)), S(t)(S(t)^k(x))\\ &\le d((J^n_{t/n})^k(x), S(t)^k(x)) + d(J^n_{t/n}(S(t)^k(x)), S(t)(S(t)^k(x))).\end{aligned}$$

The first term after the last inequality goes to 0 by the inductive hypothesis, and the second by the definition of $S(t)$. We conclude that $\lim_n(J^n_{t/n}(t))^m = (S(t))^m$. For the second equality we observe that for every m

$$(S(t))^m(x) = \lim_{n\to\infty}(J^n_{t/n})^m(x) = \lim_{n\to\infty}J^{mn}_{mt/mn}(x) = S(mt)(x).$$ ■

Proposition 4.9. *Under the hypotheses of Theorem* 4.6*, the function* $t \mapsto S(t)(x)$ *is Lipschitz continuous in* t *with Lipschitz constant* $K_{x,T}$ *on* $(0,T]$ *for each* $T>0$*, hence extends to a Lipschitz function on* $[0,T]$*, and hence defines a function continuous on* $[0,\infty)$*.*

Proof. Pick $0<\tau<\lambda\le T$. We set $m=n$, $\mu=\tau/n$, $\lambda=t/n$, $p=\tau/\lambda$, and $q=1-p=(\lambda-\tau)/\lambda$. We fix n and set $\gamma(i,j)=d(J^i_{\tau/n}(x),J^j_{\lambda/n}(x))$ for $i,j\ge 0$. It follows from Lemma 4.2 that inequality (1) of the preceding section is satisfied (see also Remark 4.3). Thus by Lemma 3.1

$$d(J^n_{\tau/n}(x),J^n_{\lambda/n}(x))\le\sum_{k=0}^{n}\binom{n}{k}p^kq^{n-k}d(x,J^{n-k}_{\lambda/n}(x)).$$

By Lemma 4.4 $d(x,J^{n-k}_{\lambda/n}(x))\le(n-k)(\lambda/n)K_{x,T}$. We thus obtain

$$d(J^n_{\tau/n}(x),J^n_{\lambda/n}(x))\le\left[\sum_{k=0}^{n}(n-k)\binom{n}{k}p^kq^{n-k}\right]\frac{\lambda}{n}K_{x,T}.$$

We rewrite the expression in closed brackets as

$$\sum_{k=0}^{n}(n-k)\binom{n}{k}p^kq^{n-k}=n\sum_{k=0}^{n}\binom{n}{k}p^kq^{n-k}-\sum_{k=0}^{n}k\binom{n}{k}p^kq^{n-k}=n-\sum_{k=0}^{n}k\binom{n}{k}p^kq^{n-k}.$$

Let X be a binomial random variable taking value k with probability $\binom{n}{k}p^kq^{n-k}$. Then $E(X)=\sum_{k=0}^{n}k\binom{n}{k}p^kq^{n-k}$, which since X is binomial must be $np=n\tau/\lambda$. Thus from the preceding lines

$$d(J^n_{\tau/n}(x),J^n_{\lambda/n}(x))\le\left[n-n\frac{\tau}{\lambda}\right]\frac{\lambda}{n}K_{x,T}=(\lambda-\tau)K_{x,T}.$$

Taking the limit as $n\to\infty$ yields $d(S(\tau)(x),S(\lambda)(x))\le(\lambda-\tau)K_{x,T}$. Since X is a complete metric space, the map extends to a Lipschitz-continuous map on $[0,T]$. ■

From Lemma 4.8 and Proposition 4.9 we can deduce the semigroup property of S: $S(s+t)=S(s)S(t)$.

Corollary 4.10. *For* $s,t>0$*,* $S(s+t)=S(s)S(t)$*.*

Proof. Let j,k,m,n be positive integers. Then by Lemma 4.8

$$\begin{aligned}S\left(\frac{j}{k}+\frac{m}{n}\right)&=S\left(\frac{jn+km}{kn}\right)=S\left(\frac{1}{kn}\right)^{jn+km}\\&=S\left(\frac{1}{kn}\right)^{jn}S\left(\frac{1}{kn}\right)^{km}=S\left(\frac{jn}{kn}\right)S\left(\frac{km}{kn}\right)=S\left(\frac{j}{k}\right)S\left(\frac{m}{n}\right).\end{aligned}$$

Thus $S(s+t)(x)=S(s)S(t)(x)$ for all s,t rational. It follows from Proposition 4.9 that both sides are continuous functions of t for fixed rational s, and thus by density of the rationals the corollary holds for rational s and all $t>0$. A similar argument in the variable s completes the proof. ■

We collect together the preceding results.

Theorem 4.11. *Let* (X, d, σ) *be a complete metric space equipped with a conical bicombing* σ *and let* $\{J_\lambda \colon X \to X : \lambda > 0\}$ *be a family of maps satisfying properties* (J1)–(J3)*. Defining* $S(\lambda)(x) = \lim_n J^n_{\lambda/n}(x)$ *for each* $x \in X$ *and* $\lambda > 0$ *and* $S(0)(x) = x$ *gives rise to a semigroup of operators* $\{S(\lambda) : \lambda \in \mathbb{R}^+\}$ (*where* $\mathbb{R}^+ = [0, \infty)$) *on* X *satisfying the following properties:*

(1) *For fixed* $x \in X$*,* $0 < T$*, the map* $\lambda \mapsto S(\lambda)(x)$ *is Lipschitz continuous on* $[0, T]$ *with Lipschitz constant* $K_{x,T}$*.*

(2) *The semigroup action* $\pi \colon \mathbb{R}^+ \times X \to X$ *given by* $\pi(\lambda, x) = S(\lambda)(x)$ *is continuous.*

Proof. That $\{S(\lambda) : \lambda > 0\}$ operates on X as a semigroup follows from Corollary 4.10. That adding S_0 to $\{S_\lambda\}_{\lambda>0}$ still operates as a semigroup is immediate from S_0 being the identity.

For $0 < \lambda \le T$, $d(J^n_{\lambda/n}(x), x) \le n(\lambda/n)K_{x.T} = \lambda K_{x,T}$ from Lemma 4.4. Taking the limit as $n \to \infty$ yields $d(S(\lambda)(x), x) \le \lambda K_{x,T}$. It follows that $\lim_{\lambda \to 0^+} S(\lambda)(x) = x = S_0(x)$. Then from Proposition 4.9 and the last part of its proof we deduce item (1).

Let $0 \le \lambda < T$ and $x \in X$. Then we have for $0 < \tau \le T$ and $y \in X$,

$$d(S(\tau)y, S(\lambda)x) \le d(S(\tau)y, S(\tau)x) + d(S(\tau)x, S(\lambda)x) \le d(y, x) + K_{T,x}|\tau - \lambda|,$$

where the last inequality comes from Corollary 4.7 (and the fact the identity is also contractive) and (1). It follows that (2) holds at all points (λ, x). ∎

Definition 4.12. The collection $\{S(t) \colon X \to X : t \in \mathbb{R}^+\}$ with the operation $S(s)S(t) = S(s+t)$ is called the *semigroup generated by* $\{J_\lambda : \lambda > 0\}$ and the corresponding action $\pi \colon \mathbb{R}^+ \times X \to X$ defined by $\pi(\lambda, x) = S(\lambda)(x)$ is the *generated semigroup action.*

Proposition 4.13. *Assume* (J1) *of the axioms for* $\{J_\lambda\}_{\lambda>0}$ *is replaced by the stronger axiom:* (J1*) *there exists* $r > 0$ *such that for all* $\lambda > 0$ *and* $x, y \in X$

$$d(J_\lambda(x), J_\lambda(y)) \le \frac{1}{1 + r\lambda} d(x, y) \tag{J1*}$$

Then for all $t > 0$ *and* $x, y \in X$*,*

$$d(S(t)x, S(t)y) \le e^{-rt} d(x, y). \tag{9}$$

Proof. Repeated application of (J1*) yields

$$d(J^n_{t/n}(x), J^n_{t/n}(y)) \le \left(\frac{1}{1 + r(t/n)}\right)^n d(x, y) = \left(1 + \frac{rt}{n}\right)^{-n} d(x, y).$$

Taking the limit as $n \to \infty$ yields the desired result. ∎

5. The Banach space setting

The original goal of the Crandall-Liggett generation theorem [7] was to solve on a Banach space the initial value problem

$$\frac{d}{dt} x(t) + A(x(t)) = 0, \qquad x(0) = x_0 \tag{10}$$

for the case that A is accretive.

The preceding sections present an alternative derivation rooted in later work of Shapiro [21] of significant parts of the original work of Crandall and Liggett [7] in a general metric setting. Here we quickly recall their work, as it pertains to the preceding. Let $(X, \|\cdot\|)$ be a Banach space. It is straightforward to verify that $\sigma\colon X \times X \times [0,1] \to X$ defined by $\sigma(x,y,t) = x\#_t y = (1-t)x + ty$ is a bicombing on X that is conical.

We introduce a mild variant of our main theorem, Theorem 4.11, of the preceding section.

Theorem 5.1. *Let (X,d,σ) be a complete metric space equipped with a conical bicombing σ and let $\{J_\lambda\colon X \to X : \lambda > 0\}$ be a family of maps satisfying properties* (J1), (J2)*, and the following variant of* (J3):

(J3*) *For all $x \in D \subseteq X$, $d(J_\lambda(x), x) \le K_{x,T}\lambda$ for all $0 < \lambda \le T$ and some constant $K_{x,T}$ depending only on x and $T > 0$.*

Defining $S(\lambda)(x) = \lim_n J^n_{\lambda/n}(x)$ for each $x \in D$ and $\lambda > 0$ and $S(0)(x) = x$ gives rise to a semigroup of operators $\{S(\lambda) : \lambda \in \mathbb{R}^+\}$ on D satisfying the following properties:

(1) *For fixed $x \in D$, $0 < T$, the map $\lambda \mapsto S(\lambda)(x)$ is Lipschitz continuous on $[0,T]$ with Lipschitz constant $K_{x,T}$.*

(2) *The semigroup action $\pi\colon \mathbb{R}^+ \times D \to X$ given by $\pi(\lambda,x) = S(\lambda)(x)$ is continuous.*

(3) *The semigroup action of $\{S(\lambda)\}_{\lambda\ge 0}$ extends to $\overline{D}$.*

The same proof as given in the previous section carries through in this setting when applied to the points of D. Since each $S(\lambda)$ is a contraction by Lemma 4.7, it extends to a contraction on $\overline{D}$. The fact each $S(\lambda)$ is a contraction on $\overline{D}$ makes the proof that the extended $\{S(\lambda) : \lambda \ge 0\}$ acts continuously as a semigroup on $\overline{D}$ straightforward.

Let I be the identity map (and relation). A multifunction on a Banach space X is a subset $A \subseteq X \times X$. Set $A^{-1} := \{(y,x) \in X \times X : (x,y) \in A\}$. Note that we can add, scalar multiply, and compose relations in a manner analogous to that used for operators on X. The domain of and range of A are given by

$$\mathrm{Dom}(A) = \{x \in X : \exists y \in X,\ (x,y) \in A\}, \quad \mathrm{Ran}(A) = \{y \in X : \exists x \in X,\ (x,y) \in A\}.$$

Note that $\mathrm{Ran}(A)$ is the domain of A^{-1}.

We say the multifunction A is $\mathfrak{m}$-*accretive* if

(i) $\mathrm{Ran}(A) = X$,

(ii) the relation $(I + \lambda A)^{-1}$ is actually an operator (function) from X into X for each $\lambda > 0$, and

(iii) this function is contractive with respect to the norm of X.

We set $J_\lambda = (I + \lambda A)^{-1}$ for each $\lambda > 0$.

We show that $\{J_\lambda : \lambda > 0\}$ satisfies the defining properties (J1), (J2), (J3) introduced in the previous section. Since the relation A is accretive, the contractive condition (J1) is satisfied by definition. We show that the resolvent condition is satisfied. Let $0 < \tau < \lambda$ and let $x \in X$. If $y = J_\lambda(x) = (I + \lambda A)^{-1}(x)$, then $x \in (I + \lambda A)(y) = y + \lambda A(y)$. We then have

$$\begin{aligned} x \in y + \lambda A(y) &\Leftrightarrow \frac{\tau}{\lambda}x \in \frac{\tau}{\lambda}y + \tau A(y) \\ &\Leftrightarrow \frac{\tau}{\lambda}x + \frac{\lambda-\tau}{\lambda}y \in y + \tau A(y) = (I+\tau A)(y) \\ &\Leftrightarrow J_\tau\left(\frac{\tau}{\lambda}x + \frac{\lambda-\tau}{\lambda}J_\lambda(x)\right) = y = J_\lambda(x). \end{aligned}$$

Thus the resolvent condition (J2) is satisfied.

We turn to (J3). Suppose that $x \in X = \mathrm{Dom}(A)$, and let $y \in A(x)$. Then

$$J_\lambda(x+\lambda y) \in (I+\lambda A)^{-1}[(I+\lambda A)(x)] = x,$$

and thus $\quad d(x, J_\lambda(x)) = \|x - J_\lambda(x)\| = \|J_\lambda(x+\lambda y) - J_\lambda(x)\| \le \lambda\|y\|,$

where the last inequality follows from (J1). Thus we may take $K_{x,T} = \|y\|$, where $y \in A(x)$, in (J3). Hence in the setting of Banach spaces Theorem 5.1 translates to the following version of the Crandall-Liggett theorem.

Theorem 5.2. *Let $(X, \|\cdot\|)$ be a Banach space and let A be an $\mathfrak{m}$-accretive multifunction on X. Let $\{J_\lambda\colon X \to X : \lambda > 0\}$ be defined by $J_\lambda(x) = (I+\lambda A)^{-1}$. Then the family of maps satisfies properties* (J1), (J2)*, and*

(J3*) *For all $x \in \mathrm{Dom}(A)$, $d(J_\lambda(x), x) \le \lambda\|y\|$ for all $y \in A(x)$.*

Defining $S(\lambda)(x) = \lim_n J^n_{\lambda/n}(x)$ for each $x \in D$ and $\lambda > 0$ and $S(0)(x) = x$ gives rise to a semigroup of operators $\{S(\lambda) : \lambda \in \mathbb{R}^+\}$ on D satisfying the following properties:

(1) *For fixed $x \in D$, $0 < T$, the map $\lambda \mapsto S(\lambda)(x)$ is Lipschitz continuous on $[0,T]$ with Lipschitz constant $K_{x,T}$.*

(2) *The semigroup action $\pi\colon \mathbb{R}^+ \times D \to X$ given by $\pi(\lambda, x) = S(\lambda)(x)$ is continuous.*

(3) *The semigroup action of $\{S(\lambda)\}_{\lambda\ge 0}$ extends to $\overline{D}$.*

As shown in [7] or [21] the function $x(t) = S(t)(x_0)$ is then a solution of Equation (10) for $x_0 \in \overline{D}$.

6. Semigroup actions from contractive maps

The study of semigroups acting on topological spaces is a key component of topological dynamics, and there is a particular emphasis on continuous actions of the additive semigroup $\mathbb{R}^+$ of non-negative reals on topological spaces. The general version of the Crandall-Liggett machinery presented here allows for a variety of new constructions of such actions. In this section we give a construction of an action arising from a contractive map on a conical bicombing that is consistent.

Let (X, d, σ) be a complete metric space endowed with consistent conical bicombing σ. Let $F\colon X \to X$ be a contractive map.

Lemma 6.1. *For $\lambda > 0$ and and $x \in X$ the map $G_{\lambda,x}(y) = F(y)\#_{\frac{1}{1+\lambda}}x$ is a strict contraction with Lipschitz constant $\lambda/(1+\lambda)$, and hence has a unique fixed point, denoted $J_\lambda(x)$.*

Proof. By the conical property we have

$$d(G_{\lambda,x}(y), G_{\lambda,x}(z)) = d\left(F(y)\#_{\frac{1}{1+\lambda}}x, F(z)\#_{\frac{1}{1+\lambda}}x\right) \le \frac{\lambda}{\lambda+1}d(F(y),F(z)).$$

Hence by the Banach fixed point theorem there is a unique fixed point. ■

Remark 6.2. From the preceding we see that $J_\lambda(x)$ is characterized as the unique member of X satisfying

$$J_\lambda(x) = F(J_\lambda(x))\#_{\frac{1}{1+\lambda}}x. \tag{11}$$

Lemma 6.3. *For contractive F, J_λ in equation (11) is also contractive.*

Proof. Using the conical property and the non-expansiveness of F, we obtain

$$\begin{aligned} d(J_\lambda(x), J_\lambda(y) &= d(F(J_\lambda(x))\#_{\frac{1}{1+\lambda}}x, F(J_\lambda(y))\#_{\frac{1}{1+\lambda}}y) \\ &\le \frac{\lambda}{1+\lambda}d(F(J_\lambda(x)), F(J_\lambda(y))) + \frac{1}{1+\lambda}d(x,y) \\ &\le \frac{\lambda}{1+\lambda}d(J_\lambda(x), J_\lambda(y)) + \frac{1}{1+\lambda}d(x,y). \end{aligned}$$

Combining like terms in the extremes and multiplying through by $(1+\lambda)$ we obtain $d(J_\lambda(x), J_\lambda(y)) \le d(x,y)$. ■

Lemma 6.4. *For contractive F and $x \in X$ we have $d(x, J_\lambda(x)) \le \lambda d(x, F(x))$.*

Proof. Since $J_\lambda(x) = F(J_\lambda(x))\#_{\frac{1}{1+\lambda}}x$, from the geodesic property we obtain the second line of the following:

$$\begin{aligned} d(x, J_\lambda(x)) = d(x, F(J_\lambda(x))\#_{\frac{1}{1+\lambda}}x) &= \frac{\lambda}{1+\lambda}d(x, F(J_\lambda(x))) \\ &\le \frac{\lambda}{1+\lambda}\left[d(x,F(x)) + d(F(x), F(J_\lambda(x)))\right] \\ &\le \frac{\lambda}{1+\lambda}d(x,F(x)) + \frac{\lambda}{1+\lambda}d(x, J_\lambda(x)) \end{aligned}$$

Subtracting the second term on the last line from both sides of the inequality and multiplying through by $1+\lambda$ yields the desired inequality. ■

Lemma 6.5. *Suppose that the bicombing on X is consistent. Then X satisfies the resolvent condition: For $0 < \tau < \lambda$,*

$$J_\lambda(x) = J_\tau(J_\lambda(x)\#_{\frac{\tau}{\lambda}}x).$$

Proof. Let $z = J_\lambda(x)\#_{\tau/\lambda}x$. Then the equation $y = F(y)\#_{\frac{1}{1+\tau}}z$, has a unique solution in y, which we have denoted $J_\tau(z) = J_\tau(J_\lambda(x)\#_{\frac{\tau}{\lambda}}x)$. We show that $J_\lambda(x)$ is also a solution, and then the two must be equal since the solution is unique. Substituting $J_\lambda(x)$ for y in the right-hand side of $y = F(y)\#_{\frac{1}{1+\tau}}z$, we obtain

$$\begin{aligned} &F(J_\lambda(x))\#_{\frac{1}{1+\tau}}z = F(J_\lambda(x))\#_{\frac{1}{1+\tau}}\left(J_\lambda(x)\#_{\frac{\tau}{\lambda}}x\right) \\ &= F(J_\lambda(x))\#_{\frac{1}{1+\tau}}\left(\left(F(J_\lambda(x))\#_{\frac{1}{1+\lambda}}x\right)\#_{\frac{\tau}{\lambda}}x\right) \\ &= F(J_\lambda(x))\#_{\frac{1}{1+\tau}}\left(F(J_\lambda(x))\#_{\frac{1+\tau}{1+\lambda}}x\right) = F(J_\lambda(x))\#_{\frac{1}{1+\lambda}}x = J_\lambda(x), \end{aligned}$$

where we have used Lemma 2.2 in the last two lines, which we can do since all points we consider lie on the geodesic with endpoints x and $F(J_\lambda(x))$. ■

From the preceding three lemmas and Theorem 4.11 we obtain the following result.

Theorem 6.6. *Let (X, d, σ) be a complete metric space equipped with a consistent conical bicombing σ. Let $F\colon X \to X$ be a contractive map. Then the map $G_{\lambda,x}\colon X \to X$ defined by $G_{\lambda,x}(y) = F(y)\#_{1/(\lambda+1)}x$ for $x \in X$ and $\lambda > 0$ has a unique fixed point, denoted $J_\lambda(x)$. The family of maps $\{J_\lambda\colon X \to X : \lambda > 0\}$ satisfies properties (J1)-(J3). Defining $S(\lambda)(x) = \lim_n J^n_{\lambda/n}(x)$ for each $x \in X$ and $\lambda > 0$ and $S(0)(x) = x$ gives rise to a semigroup of operators $\{S(\lambda) : \lambda \in \mathbb{R}^+\}$ on X satisfying the following properties:*

(1) For fixed $x \in X$, the map $\lambda \mapsto S(\lambda)(x)$ is Lipschitz continuous on $\mathbb{R}^+$ with Lipschitz constant $d(x, F(x))$.

(2) The semigroup action $\pi\colon \mathbb{R}^+ \times X \to X$ given by $\pi(\lambda, x) = S(\lambda)(x)$ is continuous.

Remark 6.7. The construction of this section was inspired by and generalizes that given in [18, Section 5]. In that work the authors used the flow as a tool in their analysis of the operator Karcher mean. It is an interesting and open problem to clarify how the flow derived in in Theorem 6.6 is connected to the contractive map F hypothesized in the theorem.

7. Barycentric metric spaces

For a metric space X, let $\mathcal{B}(X)$ be the algebra of Borel sets, the smallest σ-algebra containing the open sets. Let $\mathcal{P}(X)$ be the set of all probability measures on $(X, \mathcal{B}(X))$ with support that is separable and has measure 1 and $\mathcal{P}_0(X)$ the set of all $\mu \in \mathcal{P}(X)$ of the form $\mu = \frac{1}{n}\sum_{j=1}^n \delta_{x_j}$ with $n \in \mathbb{N}$, where δ_x is the point measure of mass 1 at x.

Remark 7.1. It is known, apparently not widely so, that in any metric space X the support of a Borel probability measure, the points for which each neighborhood has positive measure, is separable. Additionally the support has measure 1 if the metric space is separable, but for general metric spaces one needs to require that this be true.

Let $\mathcal{P}^1(X) \subseteq \mathcal{P}(X)$ be the set of probability measures μ with *finite first moment*: for some (and hence all) $x \in X$,

$$\int_X d(x, y)d\mu(y) < \infty.$$

For metric spaces X and Y, a continuous $f\colon X \to Y$ induces a *push-forward* map $f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ defined by $f_*(\mu)(B) = \mu(f^{-1}(B))$ for $\mu \in \mathcal{P}(X)$ and $B \in \mathcal{B}(Y)$. Note that $\operatorname{supp}(f_*(\mu)) = f(\operatorname{supp}(\mu))^-$, the closure of the image of the support of μ; in particular $f_*(\mu)$ has separable support.

We say that $\omega \in \mathcal{P}(X \times X)$ is a *coupling* for $\mu, \nu \in \mathcal{P}(X)$ and that μ, ν are *marginals* for ω if for all $B \in \mathcal{B}(X)$ we have $\omega(B \times X) = \mu(B)$ and $\omega(X \times B) = \nu(B)$.

Equivalently μ and ν are the push-forwards of ω under the projection maps π_1 and π_2 resp. We note that one such coupling is the product measure $\mu \times \nu$, and that for any coupling ω it must be the case that $\mathrm{supp}(\omega) \subseteq \mathrm{supp}(\mu) \times \mathrm{supp}(\nu)$. We denote the set of all couplings by $\Pi(\mu, \nu)$.

The Wasserstein distance $\mathfrak{w}$ (alternatively Kantorovich-Rubinstein distance) on $\mathcal{P}^1(X)$ is defined by

$$\mathfrak{w}(\mu_1, \mu_2) := \inf_{\pi \in \Pi(\mu_1, \mu_2)} \int_{X \times X} d(x, y) d\pi(x, y).$$

It is known that $\mathfrak{w}$ is a complete metric on $\mathcal{P}^1(X)$ whenever X is a complete metric space and $\mathcal{P}_0(X)$, the set of all uniform finitely supported probability measures, is $\mathfrak{w}$-dense in $\mathcal{P}^1(X)$ [5, 22]. In computer science circles the metric is known as the "earth-movers" metric. For finitely supported probabilities, it is known if you break up the mass of one probability measure and move the masses to the support of another in such a way to obtain the second measure, multiply each mass moved by the distance it is moved, and sum over all movements, then the Wasserstein distance is the infimum of such sums.

We henceforth assume that the metric space X is complete. A *contracting barycentric map* is a 1-Lipschitz map $\beta\colon (\mathcal{P}^1(X), \mathfrak{w}) \to (X, d)$ such that $\beta(\delta_x) = x$ for all $x \in X$; see [22, Remark 6.4]. A metric space is said to be a *barycentric metric space* if it admits a contracting barycentric map. Descombes [8] has shown the following important equivalence.

Theorem 7.2. *Let (X, d) denote a complete metric space. Then the following statements are equivalent:*

1. *X admits a reversible conical bicombing.*
2. *X is a barycentric metric space.*

The construction of a contracting barycentric map on a space admitting a reversible conical bicombing has its origins in work of Es-Sahib and Heinich [12], which was extended by Navas [20] to Busemann spaces. The work was further extended (and the proof shortened) by Descombes in [8, Section 2] to complete metric spaces with a reversible conical bicombing.

Remark 7.3. It is mentioned in the preceding Descombes reference that for barycenters of finitely supported measures (which may be treated as weighted means on finite subsets), if one is restricted to a convex subset isometric to a convex subspace of a normed linear space, the barycenter agrees (up to the isometry in question) with the usual convex combination. In particular if $\beta\colon \mathcal{P}^1(X) \to X$ is the contracting barycentric map, $\beta((1-t)\delta_x + t\delta_y) = x\#_t y$ for $0 \le t \le 1$.

Definition 7.4. Let (X, d) be a complete metric space equipped with a contracting barycentric map $\beta\colon \mathcal{P}^1(X) \to X$. Fix some $\mu \in \mathcal{P}^1(X)$. For $\lambda > 0$ we define $J_\lambda\colon X \to X$ by

$$J_\lambda(x) = \beta\left(\frac{\lambda}{1+\lambda}\mu + \frac{1}{1+\lambda}\delta_x\right). \tag{12}$$

Proposition 7.5. *The family* $\{J_\lambda : 0 < \lambda\}$ *satisfies conditions* (J1) *and* (J3):

(J1) $d(J_\lambda(x), J_\lambda(y)) \le \frac{1}{1+\lambda} d(x,y)$ *for* $x, y \in X$, $0 < \lambda$;

(J3) $d(J_\lambda(x), x) \le \frac{\lambda}{1+\lambda} \int_X d(x,a)\, d\mu(a)$ *for* $x \in X$, $0 < \lambda$.

Proof. (J1) Let $x, y \in X$. We can transport the mass $\frac{\lambda}{1+\lambda}\mu + \frac{1}{1+\lambda}\delta_x$ to the mass $\frac{\lambda}{1+\lambda}\mu + \frac{1}{1+\lambda}\delta_y$ by transporting the mass $\frac{1}{1+\lambda}$ from x to y. Thus we have the Wasserstein distance between the two masses bounded above by (actually equal to) $\frac{1}{1+\lambda} d(x,y)$. Condition (J1) now follows since β is a contractive mapping.

(J3) The following intuitive argument can be formalized in a straightforward manner. The distance between $\frac{\lambda}{1+\lambda}\mu + \frac{1}{1+\lambda}\delta_x$ and $\delta_x = \frac{\lambda}{1+\lambda}\delta_x + \frac{1}{1+\lambda}\delta_x$ is bounded by $\frac{\lambda}{1+\lambda}\mathfrak{w}(\mu, \delta_x)$ since we are moving a smaller amount $\lambda/(1+\lambda)$ of the mass μ to x. Since β is contractive,

$$d(J_\lambda(x), x) = d\left(\beta\left(\frac{\lambda}{1+\lambda}\mu + \frac{1}{1+\lambda}\delta_x\right), \beta(\delta_x)\right) \le \frac{\lambda}{1+\lambda}\mathfrak{w}(\mu, \delta_x).$$

Finally we note that since $\pi = \mu \times \delta_x$ is a coupling for the pair μ, δ_x,

$$\mathfrak{w}(\mu, \delta_x) \le \int_X d(u,v) d\pi(u,v) = \int_X d(u,v)(d\mu \times d\delta_x)(u,v) = \int_X d(u,x) d\mu(u),$$

which completes the proof. ■

We do not have proof that (J2), the resolvent condition, holds in general. Indeed we suspect that in the preceding setting it holds only in special cases. But whenever it does hold, then the general Crandall-Liggett machinery of Section 4 applies. In the special setting of positive cones of C^*-algebras (Example 2.3) Palfia and Lim in [18] have established the resolvent condition for the Karcher barycentric map. The constructions of this and the preceding section first appeared there, and were basic tools to establish important properties of the Karcher operator mean, among them a law of large numbers.

Problem. Characterize those barycentric maps for which the resolvent condition (J2) is valid. In particular does (J2) hold for the barycentric map mentioned before Remark 7.3?

References

[1] M. Bačák: *Convex Analysis and Optimization in Hadamard Spaces*, De Gruyter Series in Nonlinear Analysis and Applications 22, De Gruyter, Berlin (2014).

[2] G. Basso: *Fixed point theorems for metric spaces with a conical geodesic bicombing*, Ergodic Theory Dyn. Systems 38 (2018) 164–1657.

[3] G. Basso: *Extending and improving conical bicombings*, arXiv: 2005.13941 (2020).

[4] R. Bhatia, J. Holbrook: *Riemannian geometry and matrix geometric means*, Linear Algebra Appl. 413 (2006) 594–618.

[5] F. Bolley: *Separability and completeness for the Wasserstein distance*, in: *Séminaire de Probabilités XLI*, Lecture Notes in Mathematics 1934, Springer, Berlin (2008) 371–377.

[6] K. Craig: *The exponential formula for the Wasserstein metric*, ESAIM Control Optim. Calc. Var. 22 (2016) 169–187.

[7] M. G. Crandall, T. M. Liggett: *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math. 93 (1971) 265–295.

[8] D. Descombes: *Asymptotic rank of spaces with bicombings*, Math. Zeitschrift 284 (2016) 947–960.

[9] D. Descombes, M. Dominic: *Maël injective subsets of* $\ell_\infty(I)$, Adv. Math. 317 (2017) 91–107.

[10] D. Descombes, U. Lang: *Convex geodesic bicombings and hyperbolicity*, Geom. Dedicata 77 (2015) 367–384.

[11] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, W. P. Thurston: *Word Processing in Groups*, Jones and Bartlett Publishers, Boston (1992).

[12] A. Es-Sahib, H. Heinich: *Barycentre canonique pour un espace métrique à courbure négative*, in: Séminaire de Probabilités XXXIII, Lecture Notes in Mathematics 1709, Springer, Berlin (1999) 355–370.

[13] C. Horvath: *A note on metric spaces with continuous midpoints*, Annals of the Academy of Romanian Scientists, Series on Mathematics and its Applications 1 (2009) 255–288.

[14] F. Kubo, T. Ando: *Means of positive linear operators*, Math. Ann. 246 (1980) 205–224.

[15] J. Lawson: *Existence and uniqueness of the Karcher mean on unital* C^**-algebras*, J. Math. Analysis Appl. 483/2 (2020), art. no. 123625, 16 p.

[16] J. Lawson, Y. Lim: *Metric convexity of symmetric cones*, Osaka J. Math. 44 (2007) 795–816.

[17] J. Lawson, Y. Lim: *Karcher means and Karcher equations of positive operators*, Trans. Amer. Math. Soc. Ser. B 1 (2014) 1–22.

[18] Y. Lim, M. Palfia: *Existence, uniqueness and an ODE approach to the* L_1*-Karcher mean*, Advances Math. 376 (2021), art. no. 107435, 45 p.

[19] M. Moakher: *A differential geometric approach to the geometric mean of symmetric positive-definite matrices*, SIAM J. Matrix Analysis Appl. 26 (2005) 735–747.

[20] A. Navas,: *An* L_1 *ergodic theorem with values in a non-positively curved space via a canonical barycenter map*, Ergodic Theory Dyn. Systems 33 (2013) 609–623.

[21] J. Shapiro: *The Crandall-Liggett generation theorem*, internet manuscript, joelshapiro.org/Pubvit/Downloads/rulla_crandlig.pdf (2015).

[22] K.-T. Sturm: *Probability measures on metric spaces of nonpositive curvature*, in: *Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces*, P. Auscher et al. (eds.), Contemporary Mathematics 338, American Mathematical Society, Providence (2003) 357–390.

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