From Euler Elements and 3-Gradings to
Non-Compactly Causal Symmetric Spaces

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Abstract.   We discuss the interplay between causal structures of symmetric spaces and geometric aspects of Algebraic Quantum Field Theory (AQFT). The central focus is the set of Euler elements in a Lie algebra, i.e., elements whose adjoint action defines a 3-grading. In the first half of this article we survey the classification of reductive causal symmetric spaces from the perspective of Euler elements. This point of view is motivated by recent applications in AQFT. In the second half we obtain several results that prepare the exploration of the deeper connection between the structure of causal symmetric spaces and AQFT. In particular, we explore the technique of strongly orthogonal roots and corresponding systems of $\mathfrak{sl}_2$-subalgebras. Furthermore, we exhibit real Matsuki crowns in the adjoint orbits of Euler elements and we describe the group of connected components of the stabilizer group of Euler elements.

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1.   Introduction

From the group theoretic perspective, symmetric spaces are quotients $M = G/H$, where G is a Lie group, τ is an involutive automorphism of G and $H \subseteq G^\tau$ (the group of τ-fixed points) is an open subgroup. Symmetric spaces subsume quadrics on which pseudo-orthogonal groups act and Lie groups G on which the product group $G \times G$ acts by left and right translations. For an axiomatic approach to symmetric spaces we refer to O. Loos' monograph [25].

Causal symmetric spaces G/H carry a G-invariant field of pointed generating closed convex cones $C_m \subseteq T_m(M)$ in their tangent spaces. They subsume time-orientable Lorentzian symmetric spaces, but it is not required that the cones come from an invariant Lorentzian metric. They permit to study causality aspects of spacetimes in a highly symmetric environment. On some of these spaces the causal curves define a global order structure with compact intervals (they are called globally hyperbolic) and in this context one can also prove the existence of a global "time function" with group theoretic methods (see [35]). We refer to the monograph [17] for more details and a complete exposition of the classification of irreducible symmetric spaces.

Recent interest in causal symmetric spaces in relation with representation theory arose from their role as analogs of spacetime manifolds in the context of Algebraic Quantum Field Theory (AQFT) in the sense of Haag-Kastler, where one considers

nets of von Neumann algebras $\mathcal{M}(\mathcal{O})$ of operators on a fixed Hilbert space $\mathcal{H}$, associated to regions $\mathcal{O}$ in some spacetime manifold M ([11]). The hermitian elements of the algebra $\mathcal{M}(\mathcal{O})$ represent observables that can be measured in the "laboratory" $\mathcal{O}$. One typically requires the following properties:

(I)   Isotony: $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)$

(L)   Locality: $\mathcal{O}_1 \subseteq \mathcal{O}_2'$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)'$, where $\mathcal{O}'$ is the "causal complement" of $\mathcal{O}$, i.e., the maximal open subset that cannot be connected to $\mathcal{O}$ by causal curves.

(RS)   Reeh-Schlieder property: There exists a unit vector $\Omega \in \mathcal{H}$ that is cyclic for $\mathcal{M}(\mathcal{O})$ if $\mathcal{O} \neq \emptyset$.

(Cov)   Covariance: There is a Lie group G acting on M and a unitary representation $U: G \to \mathrm{U}(\mathcal{H})$ such that $U_g \mathcal{M}(\mathcal{O}) U_g^{-1} = \mathcal{M}(g\mathcal{O})$ for $g \in G$.

(BW)   Bisognano-Wichmann property: Ω is separating for some "wedge region" $W \subseteq M$ and there exists an element $h \in \mathfrak{g}$ with $\Delta^{-it/2\pi} = U(\exp th)$ for $t \in \mathbb{R}$, where Δ is the modular operator corresponding to $(\mathcal{M}(W), \Omega)$ in the sense of the Tomita-Takesaki Theorem [6, Thm. 2.5.14].

(Vac)   Invariance of vacuum: $U(g)\Omega = \Omega$ for every $g \in G$.

The (BW) property gives a geometrical meaning to the dynamics provided by the modular group Δ^{it} of von Neumann algebras associated to wedge regions with respect to the vacuum state. This also holds under the weaker hypothesis of the modular covariance property (MC)[1]. Under one of these assumptions, the relations among the modular groups of the wedge algebras can be used to reconstruct a positive energy representation of the Poincaré group, acting covariantly on the net of von Neumann algebras on Minkowski spacetime ([10, 2]). In particular, one can start with a finite configuration of von Neumann algebras with a cyclic and separating vector in some specific relative position to determine a large group of symmetries generated by their modular groups whose action on the family of von Neumann algebras is generating an AQFT on the spacetime manifold. For instance, in the chiral theories, such configurations are represented by half-sided modular inclusions ([45, 1]): An inclusion $\mathcal{A} \subset \mathcal{B} \subset \mathcal{B}(\mathcal{H})$ of von Neumann algebras with a common cyclic and separating vector $\Omega \in \mathcal{H}$, such that $\Delta_{\mathcal{B},\Omega}^{-it}\mathcal{A} \subset \mathcal{A}$ for all $t \geq 0$. On Minkowski spacetime these structures have been discussed by [46]. The homogeneous spacetimes occurring naturally in AQFT are causal symmetric spaces associated to their symmetry groups (Minkowski spacetime for the Poincaré group, de Sitter space for the Lorentz group and anti-de Sitter space for $\mathrm{SO}_{2,d}(\mathbb{R})$) and the localization in wedge regions is ruled by the acting group.

In our abstract context a natural question is, given a symmetry group G, to which extent such nets of von Neumann algebras exist on causal symmetric spaces. For representations $(U, \mathcal{H})$ of G for which the positive cone

$$C_U := \{x \in \mathfrak{g} \colon -i \cdot \partial U(x) \geq 0\} \tag{1}$$

spans $\mathfrak{g}$ such nets can be constructed. We refer to [39] for left invariant nets on reductive Lie groups, to [42] for left invariant nets on non-reductive Lie groups,

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[1] $\Delta_{\mathcal{M}(W),\Omega}^{-it}\mathcal{M}(\mathcal{O})\Delta_{\mathcal{M}(W),\Omega}^{it} = \mathcal{M}(\exp(2\pi th)\mathcal{O})$, $\mathcal{O} \subset M$.

and to [40] for invariant nets on compactly causal symmetric spaces. In all these constructions the non-triviality of the cone C_U is a crucial assumption, but this restricts the class of representations considerably. These papers construct so-called one-particle nets on symmetric spaces from which nets of von Neumann algebras can be obtained by second quantization functors. An abstract description of wedge spaces is introduced in [31]. Here the spectral condition $C_U \neq \{0\}$ is only needed to encode non-trivial inclusions among the wedge regions. For instance, $C_U = \{0\}$ for non-trivial representations of the Lorentz groups $\mathrm{SO}_{1,d}(\mathbb{R})_e$ and in de Sitter space there are no proper inclusions for wedge regions. On the other hand, [7] shows that covariant nets for the Lorentz group exist for de Sitter space. Moreover, according to [32] the potential generators $h \in \mathfrak{g}$ of the modular groups in (BW) are *Euler elements*, i.e., $\mathrm{ad}\,h$ defines a 3-grading

$$\mathfrak{g} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_0(h) \oplus \mathfrak{g}_{-1}(h), \quad \text{where} \quad \mathfrak{g}_\lambda(h) = \ker(\mathrm{ad}\,h - \lambda\mathbf{1}).$$

This leads to the question how the existence and the choice of the Euler element affects the geometry of symmetric spaces.

The goal of this article is twofold. First, we present an approach to reductive causal symmetric spaces and their classification from the perspective of Euler elements that should be accessible to a large readership beyond the Lie group community (Sections 2-4). Second, we intend to lay the foundation for the exploration of the deeper connection between the structure of causal symmetric spaces and AQFT (Sections 5-7). In particular, a better understanding of the locality condition (L) and "causal complements" is under development; see in particular [33] and [34] on wedge regions in causal symmetric spaces.

We recall some basic terminology concerning symmetric spaces and symmetric Lie algebras:

- A *symmetric Lie algebra* is a pair $(\mathfrak{g}, \tau)$, where $\mathfrak{g}$ is a finite-dimensional real Lie algebra and τ is an involutive automorphism of $\mathfrak{g}$. We write

  $$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \quad \text{with} \quad \mathfrak{h} = \mathfrak{g}^\tau = \ker(\tau - \mathbf{1}) \quad \text{and} \quad \mathfrak{q} = \mathfrak{g}^{-\tau} = \ker(\tau + \mathbf{1}).$$

- A *causal symmetric Lie algebra* is a triple $(\mathfrak{g}, \tau, C)$, where $(\mathfrak{g}, \tau)$ is a symmetric Lie algebra and $C \subseteq \mathfrak{q}$ is a pointed generating closed convex cone invariant under the group $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}) := \langle e^{\mathrm{ad}\,\mathfrak{h}} \rangle \subseteq \mathrm{Aut}(\mathfrak{g})$. We call $(\mathfrak{g}, \tau, C)$

  - *compactly causal (cc)* if C is *elliptic* in the sense that, for $x \in C^\circ$ (the interior of C), the operator $\mathrm{ad}\,x$ is semisimple with purely imaginary spectrum.

  - *non-compactly causal (ncc)* if C is *hyperbolic* in the sense that, for $x \in C^\circ$, the operator $\mathrm{ad}\,x$ is diagonalizable.

Let $(\mathfrak{g}, \tau, C)$ be an ncc symmetric Lie algebra, G a connected Lie group with Lie algebra $\mathfrak{g}$, τ_G an involution on G integrating the involution τ on G and $H \subseteq G^{\tau_G}$ be an open subgroup with $\mathrm{Ad}(H)C = C$. Then we obtain the structure of a *causal symmetric space* on $M = G/H$, specified by the G-invariant field of open convex cones $g.C^\circ \subseteq T_{gH}(M)$, $g \in G$.

If G is semisimple with finite center, then, for compactly causal spaces, there are closed causal curves, so that no global causal order exists on M, but non-compactly

causal spaces carry a global order which is *globally hyperbolic* in the sense that all order intervals are compact ([17, Thm. 5.3.5]). In general the semi-Riemannian metric on semisimple causal symmetric spaces is not Lorentzian and the cone C may not have a smooth boundary away from 0.

The two main examples of non-flat Lorentzian symmetric spaces are the de Sitter space and the anti de Sitter space. The *De Sitter space*

$$\mathrm{dS}^d := \{(x_0, x_1, \ldots, x_d) \in \mathbb{R}^{1,d} \colon x_0^2 - x_1^2 - \cdots - x_d^2 = -1\} \tag{2}$$

is a non-compactly causal irreducible symmetric space satisfying $G = \mathrm{SO}_{1,d}(\mathbb{R})_e$, $H = G_{\mathbf{e}_1} = \mathrm{SO}_{1,d-1}(\mathbb{R})_e$, and $C \subseteq T_{\mathbf{e}_1}(\mathrm{dS}^d) \cong \mathbf{e}_1^\perp$ given by

$$C = \{(x_0, 0, x_2, \ldots, x_d) \colon x_1 = 0, x_0 \geq 0, x_0^2 \geq x_2^2 + \cdots + x_d^2\}.$$

Likewise *anti-de Sitter space* is the compactly causal irreducible symmetric space

$$\mathrm{AdS}^d := \{(x_0, x_1, \ldots, x_d) \in \mathbb{R}^{2,d-1} \colon x_0^2 + x_1^2 - x_2^2 - \cdots - x_d^2 = 1\} \tag{3}$$

with $G = \mathrm{SO}_{2,d-1}(\mathbb{R})_e$, $H = G_{\mathbf{e}_1} \cong \mathrm{SO}_{1,d-1}(\mathbb{R})_e$, and $C \subseteq T_{\mathbf{e}_1}(\mathrm{AdS}^d) \cong \mathbf{e}_1^\perp$ given by

$$C = \{(x_0, 0, x_2, \ldots, x_d) \colon x_0 \geq 0, x_0^2 \geq x_2^2 + \cdots + x_d^2\}.$$

In Appendix C we recall the well-known result that any **irreducible** Lorentzian d-dimensional causal symmetric space is either locally isomorphic to anti-de Sitter space (if compactly causal) or to de Sitter space (if non-compactly causal). However, there are many **reducible** Lorentzian symmetric spaces (see Appendix C).

The contents of this paper is as follows: We start in Section 2 by introducing Euler elements and their classification, as presented in [31]. Since Euler elements correspond to 3-gradings of Lie algebras, their classification is also contained implicitly in the work of S. Kaneyuki (see the symmetric spaces of type $K_\varepsilon I$ in [19, Thm. 2.1], [20, p. 600] or [21]).

In Section 3 we explore the close relation between Euler elements, H-elements ([44]) and invariant cones. Recall that an element z of a reductive Lie algebra $\mathfrak{g}$ is called an *H-element* if $\ker(\mathrm{ad}\, z)$ is maximal compactly embedded in $\mathfrak{g}$ and $iz \in \mathfrak{g}_\mathbb{C}$ is an Euler element. In Section 3.1 we recall the classification of simple hermitian Lie algebras in terms of Euler elements of their complexification. Using maximal sets of strongly orthogonal roots, we explore in Section 3.2 how the symmetry of an Euler element is related to the projection of the root system to the subspace spanned by the strongly orthogonal roots: These projections are either of type C_r or BC_r, and the first case occurs if and only if h is symmetric, i.e., to $-h \in \mathrm{Inn}(\mathfrak{g})h$. This implies in particular that hermitian Lie algebras of tube type correspond to symmetric Euler elements in complex simple Lie algebras. This is closely related to their characterization in terms of the existence of a "morphism of hermitian Lie algebras" $\mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{g}$ whose range contains h ([44, Cor. III.1.6]). We recall in Section 3.3 how H-elements are related to Cartan involutions and in Section 3.4 we discuss the duality between H-elements and Euler elements. We conclude Section 3 with a review concerning invariant cones in hermitian Lie algebras and irreducible symmetric spaces (Section 3.5).

In Section 4 we present a classification of irreducible non-compactly causal symmetric spaces based on Euler elements. Here a key concept is that of a *causal Euler*

element that is explored in Section 4.1: If $(\mathfrak{g}, \tau)$ is a symmetric Lie algebra, an Euler element $h \in \mathfrak{q}$ is said to be causal if it is contained in the interior of an $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant pointed convex cone in $\mathfrak{q}$. Theorem 4.2 asserts that, for any pair (θ, h) of a Cartan involution θ and an Euler element satisfying $\theta(h) = -h$, the involution $\tau = \tau_h \theta$ with $\tau_h = e^{\pi i \, \mathrm{ad} \, h}$ makes h causal for $(\mathfrak{g}, \tau)$. In Section 4.3 we use this construction to classify irreducible non-compactly causal symmetric Lie algebras in terms of $\mathrm{Inn}(\mathfrak{g})$-orbits of Euler elements (Theorem 4.21). From the dual perspective, focusing on H-elements, this classification goes back to [43], which ties in naturally with [16, Prop. II.3], where it is shown that the maximal generating invariant cones in $\mathfrak{g}$ (and by duality the minimal ones) are parametrized by adjoint orbits of H-elements.

In general there are many locally isomorphic causal symmetric spaces G/H corresponding to the same triple $(\mathfrak{g}, \tau, C)$. The maximal one is the universal covering space of $\mathrm{Inn}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})^\tau$, but the "minimal model" is not as obvious. In Section 4.2 we show that the minimal symmetric space $\mathrm{Inn}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})^\tau$ for the irreducible symmetric Lie algebra $(\mathfrak{g}, \tau)$ is this minimal model for $(\mathfrak{g}, \tau, C)$ if and only if the corresponding causal Euler element is not symmetric; otherwise we have to pass to a two-fold covering. This can be understood in terms of wedge regions in non-compactly causal symmetric spaces as described in [41, Sect. 1]. The wedge region associated to an Euler element can be defined as the open subset of the causal space where the generator of the modular flow takes values in the open cones. This generalizes the fact that the flux of a uniformly accelerated observer in the Rindler wedge is indeed timelike. If h is a symmetric Euler element (h and $-h$ belong to the same orbit), then, on the *minimal symmetric space* $M = \mathrm{Inn}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})^\tau$ we cannot distinguish between C and $-C$. Hence the causality of the manifold M is lost. In order to preserve causality one has to consider the *minimal causal symmetric space* $\mathrm{Inn}(\mathfrak{g})/H_C$, where $H_C \subseteq \mathrm{Inn}(\mathfrak{g})^\tau$ is the subgroup of those elements g satisfying $gC = C$. In this case causally complementary wedge regions correspond to opposite Euler elements: if h determines the wedge region $W_h \subset M$, then $-h$ determine $W_h' \subset M$, where the prime refers to the locality condition (L). A prominent example is de Sitter spacetime which is discussed in Remark 4.20.

In Section 4.4 we present a structured table with the classification of the irreducible non-compactly causal symmetric Lie algebras from the perspective of causal Euler elements. Together with Theorem 4.21 it provides a complete classification of the local structure of non-compactly causal symmetric spaces given by non-compactly causal simple symmetric Lie algebras.

In Section 5 we introduce a key technical tool in the structure theory of causal symmetric spaces: maximal τ-invariant sets of strongly orthogonal long roots. In Section 5.1 we recall the construction of such sets from [43] and discuss their basic properties. In particular we connect Euler elements with strongly orthogonal roots and the corresponding $\mathfrak{sl}_2$-subalgebras (Proposition 5.2). An important application of this technique is Theorem 5.4 that characterizes irreducible non-compactly causal symmetric Lie algebras $(\mathfrak{g}, \tau, C)$ for which $\mathfrak{h} = \mathfrak{g}^\tau$ contains Euler elements as those for which the causal Euler element is symmetric. Quadruples $(\mathfrak{g}, \tau, C, h')$ with an Euler element $h' \in \mathfrak{h}$ are called *modular causal symmetric Lie algebras*. For these symmetric Lie algebras wedge regions in G/H corresponding to h' are studied in [40, 41]. In this context there are several equivalent ways to define wedge regions

(cf. [40, Thm. 7.1], [41, §§6,7]). In particular, a characterization by a KMS like condition implies that the Euler conjugation $\tau_{h'}$ implements a "wedge reflection". In a minimal irreducible causal symmetric space, the existence of a complementary wedge region, with respect to the causal structure, also corresponds to the property of the Euler element associated to the wedge being symmetric (cf. [31] and [33]).

In Section 6 we extend some of the results obtained in [41] for the special class of modular ncc spaces to general semisimple non-compactly causal symmetric spaces. Our main result is that, if the cone C is maximal and $(\mathfrak{g}, \tau)$ is semisimple without Riemannian ideals, then the connected component of h in the intersection of the adjoint orbit $\mathcal{O}_h = \mathrm{Inn}(\mathfrak{g})h$ with the real tube domain $\mathcal{T}_C = \mathfrak{h} + C^\circ \subseteq \mathfrak{g}$ is the Matsuki crown of the Riemannian symmetric space $\mathcal{O}_h^{\mathfrak{q}} := \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})h \cong e^{\mathrm{ad}\,\mathfrak{h}_{\mathfrak{p}}}h$, i.e.,

$$(\mathcal{T}_C \cap \mathcal{O}_h)_h = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})e^{\mathrm{ad}\,\Omega_{\mathfrak{q}_{\mathfrak{k}}}}h \subset \mathcal{O}_h \quad \text{for} \quad \Omega_{\mathfrak{q}_{\mathfrak{k}}} = \left\{ x \in \mathfrak{q}_{\mathfrak{k}} \colon \rho(\mathrm{ad}\,x) < \frac{\pi}{2} \right\}.$$

Here $\rho(\mathrm{ad}\,x)$ is the spectral radius of $\mathrm{ad}\,x$ (Theorem 6.6) and $\mathfrak{q}_{\mathfrak{k}} = \mathfrak{q} \cap \mathfrak{k}$ for a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $h \in \mathfrak{p}$. In [33] we actually show that the intersection of $\mathcal{T}_C$ with the set of Euler elements is connected.

Section 7 is devoted to an analysis of the group $\pi_0(G^h)$ of connected components of the centralizer G^h of an Euler element h in a simple real Lie algebra. By the polar decomposition $G^h = K^h \exp(\mathfrak{h}_{\mathfrak{p}})$, this group equals $\pi_0(K^h)$. As $K/K^h \cong \mathcal{O}_h^K := \mathrm{Ad}(K)h$ is a compact symmetric space, we discuss this problem in Section 7.1 in the context of compact symmetric spaces, where $\pi_0(K^h)$ appears as a quotient of $\pi_1(K/K^h)$ in the long exact homotopy sequence

$$\pi_1(K) \to \pi_1(K/K^h) \twoheadrightarrow \pi_0(K^h) \to \pi_0(K) = \mathbf{1}.$$

In Section 7.2 we explore this situation further, using that $\mathcal{O}_h^K$ actually is a symmetric R-space (cf. [26]). Here the strongly orthogonal roots come in handy and permit us to show that $\mathrm{Ad}(G)^h$ is connected if $(\mathfrak{g}, \tau)$ is either of complex type or non-split type (cf. Section 4.4), and if it is of split type or Cayley type, then it either is trivial or $\mathbb{Z}_2$ (see Theorem 7.8 for details). In particular $\mathrm{Ad}(G)^h$ has at most two connected components. In Section 7.3 we finally collect some consequences of this result such as the identity

$$\mathrm{Inn}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{g}^c) \cap \mathrm{Inn}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{g}) = \mathrm{Inn}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{k})^h e^{\mathrm{ad}\,\mathfrak{h}_{\mathfrak{p}}}$$

for the c-dual Lie algebra $\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q}$.

We conclude this paper with three short appendices containing some calculations in $\mathfrak{sl}_2(\mathbb{R})$ (Appendix A), some general facts on invariant cones and their extensions (Appendix B) and on Lorentzian symmetric spaces (Appendix C).

Notation:

- We write $e \in G$ for the identity element in the Lie group G and G_e for its identity component.
- For $x \in \mathfrak{g}$, we write $G^x := \{g \in G \colon \mathrm{Ad}(g)x = x\}$ for the stabilizer of x in the adjoint representation and $G_e^x = (G^x)_e$ for its identity component.
- For $h \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$, we write $\mathfrak{g}_\lambda(h) := \ker(\mathrm{ad}\,h - \lambda\mathbf{1})$ for the corresponding eigenspace in the adjoint representation.

- If $\mathfrak{g}$ is a Lie algebra, we write $\mathcal{E}(\mathfrak{g})$ for the set of *Euler elements* $h \in \mathfrak{g}$, i.e., $\operatorname{ad} h$ is non-zero and diagonalizable with $\operatorname{Spec}(\operatorname{ad} h) \subseteq \{-1, 0, 1\}$. We write $\mathcal{O}_h = \operatorname{Inn}(\mathfrak{g})h$ for the adjoint orbit of h and call it *symmetric* if $-h \in \mathcal{O}_h$.

- For a Lie subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$, we write $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{s}) = \langle e^{\operatorname{ad} \mathfrak{s}} \rangle \subseteq \operatorname{Aut}(\mathfrak{g})$ for the subgroup generated by $e^{\operatorname{ad} \mathfrak{s}}$. We call $\mathfrak{s}$ *compactly embedded* if the group $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{s})$ has compact closure.

- For a convex cone C in a vector space V, we write $C^\circ := \operatorname{int}_{C-C}(C)$ for the relative interior of C in its span.

- For a symmetric space $M = G/H$ we write $\operatorname{Exp} \colon T(M) \to M$ for the exponential function which, in the base point eH takes the form $\operatorname{Exp}_{eH}(x) = \exp xH$ if we identity $\mathfrak{q}$ with $T_{eH}(M)$.

2.   The classification of Euler elements

In this section we introduce Euler elements and their classification, as presented in [31]. Since Euler elements correspond to 3-gradings of Lie algebras, their classification is also contained implicitly in the work of S. Kaneyuki (cf. [20, p. 600] or [21]).

Definition 2.1.   We call an element h of the finite dimensional real Lie algebra $\mathfrak{g}$ an *Euler element* if $\operatorname{ad} h$ is non-zero and diagonalizable with $\operatorname{Spec}(\operatorname{ad} h) \subseteq \{-1, 0, 1\}$. Then the eigenspace decomposition of $\mathfrak{g}$ with respect to $\operatorname{ad} h$ defines a 3-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_0(h) \oplus \mathfrak{g}_{-1}(h), \quad \text{where} \quad \mathfrak{g}_\nu(h) = \ker(\operatorname{ad} h - \nu \operatorname{id}_{\mathfrak{g}})$$

Then $\tau_h(y_j) = (-1)^j y_j$ for $y_j \in \mathfrak{g}_j(h)$ defines an involutive automorphism of $\mathfrak{g}$ that can also be written as $e^{\pi i \operatorname{ad} h}$ such that

$$\mathfrak{g}^{\tau_h} = \mathfrak{g}_0(h) \quad \text{and} \quad \mathfrak{g}^{-\tau_h} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_{-1}(h).$$

We write $\mathcal{E}(\mathfrak{g})$ for the set of Euler elements in $\mathfrak{g}$. The orbit of an Euler element h under the group $\operatorname{Inn}(\mathfrak{g}) = \langle e^{\operatorname{ad} \mathfrak{g}} \rangle$ of *inner automorphisms* is denoted with $\mathcal{O}_h = \operatorname{Inn}(\mathfrak{g})h \subseteq \mathfrak{g}$. We say that h is *symmetric* if $-h \in \mathcal{O}_h$.

Definition 2.2.   Let θ be a Cartan involution of the semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{a} \subseteq \mathfrak{p}$ maximal abelian, so that we obtain the restricted root system $\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a})$. Then the Cartan-Killing form $\kappa(x, y) = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$ restricts to a scalar product on $\mathfrak{a}$. For $\alpha \in \Sigma$, we define the *coroot* $\alpha^\vee \in \mathfrak{a}$ as the unique element which is orthogonal to $\ker \alpha$ and satisfies

$$\alpha(\alpha^\vee) = 2.$$

Then the corresponding *reflection* is given on $\mathfrak{a}$ by

$$s_\alpha \colon \mathfrak{a} \to \mathfrak{a}, \quad s_\alpha(x) = x - \alpha(x)\alpha^\vee,$$

and on $\mathfrak{a}^*$ by $\qquad s_\alpha \colon \mathfrak{a}^* \to \mathfrak{a}^*, \quad s_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha.$

The Weyl group $\mathcal{W} := \mathcal{W}(\mathfrak{g}, \mathfrak{a})$ is the finite subgroup of $\operatorname{GL}(\mathfrak{a})$, generated by these reflections.

If $\mathfrak{g}$ is simple, we call a root *long* if its length is maximal ([18, §2.9]).

Remark 2.3. An element $h \in \mathfrak{a}$ is an Euler element if and only if it represents a so-called *minuscule weight* of the root system $\Sigma^\vee = \{\alpha^\vee : \alpha \in \Sigma\}$ dual to $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$. This follows from the remark at the end of §7, no. 3 in [5, Ch. 8], that also provides an interesting relation with the corresponding affine root system (see also Remark 4.15).

If $h \neq 0$ is hyperbolic, then there exists a maximal abelian hyperbolic subspace $\mathfrak{a}$ ($\mathfrak{a}$ consists of hyperbolic elements) containing h. Then h is an Euler element if and only if

$$\Sigma(\mathfrak{g}, \mathfrak{a})(h) \subseteq \{-1, 0, 1\}$$

where $\Sigma(\mathfrak{g}, \mathfrak{a})$ is the system of restricted roots of $\mathfrak{a}$ in $\mathfrak{g}$. Therefore Euler elements are easy to detect in terms of the structure of $\Sigma(\mathfrak{g}, \mathfrak{a})$. For a given set $\{\alpha_1, \ldots, \alpha_r\}$ of simple roots contained in the half-space $\{\lambda \in \mathfrak{a}^* : \lambda(h) \geq 0\}$ and with the highest root $\alpha = \sum_{j=1}^r n_j \alpha_j$, the different 3-gradings correspond to those indices j with $n_j = 1$.

The following theorem describes the conjugacy classes of Euler elements in simple real Lie algebras.

Theorem 2.4. ([31, Thm. 3.10]) *Suppose that $\mathfrak{g}$ is a non-compact simple real Lie algebra and that $\mathfrak{a} \subseteq \mathfrak{g}$ is maximal ad-diagonalizable with restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) \subseteq \mathfrak{a}^*$ of type X_n. We follow the conventions of the tables in [4] for the classification of irreducible root systems and the enumeration of the simple roots $\alpha_1, \ldots, \alpha_n$. For each $j \in \{1, \ldots, n\}$, we consider the uniquely determined element $h_j \in \mathfrak{a}$ satisfying $\alpha_k(h_j) = \delta_{jk}$. Then every Euler element in $\mathfrak{g}$ is conjugate under inner automorphism to exactly one h_j. For every irreducible root system, the Euler elements among the h_j are the following:*

$$A_n : h_1, \ldots, h_n, \quad B_n : h_1, \quad C_n : h_n, \quad D_n : h_1, h_{n-1}, h_n, \quad E_6 : h_1, h_6, \quad E_7 : h_7. \quad (4)$$

For the root systems BC_n, E_8, F_4 and G_2 no Euler element exists (they have no 3-grading). The symmetric Euler elements are

$$A_{2n-1} : h_n, \quad B_n : h_1, \quad C_n : h_n, \quad D_n : h_1, \quad D_{2n} : h_{2n-1}, h_{2n}, \quad E_7 : h_7. \quad (5)$$

3. Euler elements, H-elements and invariant cones

In this section we turn to the close relation between Euler elements, H-elements ([16, 44]) and invariant cones. In Section 3.1 we recall the classification of simple hermitian Lie algebras in terms of Euler elements. Using arguments involving maximal sets of strongly orthogonal roots, we explore in Section 3.2 how the symmetry of an Euler element is related to the projection of the root system to the subspace spanned by the strongly orthogonal roots: These projections are either of type C_r or BC_r, and the first case is equivalent to h being symmetric. This implies in particular that hermitian Lie algebras of tube type correspond to symmetric Euler elements in complex simple Lie algebras. We recall in Section 3.3 how H-elements are related to Cartan involutions and in Section 3.4 the duality between H-elements and Euler elements. We conclude this section with a review of invariant cones in hermitian Lie algebras and irreducible symmetric Lie algebras in Section 3.5.

3.1. Hermitian real forms of complex simple Lie algebras

Definition 3.1. Let $\mathfrak{g}$ be a reductive Lie algebra and let $z \in \mathfrak{g}$ be an element for which iz is an Euler element of $\mathfrak{g}_{\mathbb{C}}$. Then $\theta_z := e^{\pi \operatorname{ad} z}$ is an involution of $\mathfrak{g}$ whose fixed point set is $\ker(\operatorname{ad} z)$. We call z an *H-element* if $\ker(\operatorname{ad} z)$ is maximal compactly embedded, i.e., if θ_z restricts to a Cartan involution on the commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$.

We call a pair $(\mathfrak{g}, z)$, where $\mathfrak{g}$ is a reductive Lie algebra and $z \in \mathfrak{g}$ an H-element a *reductive Lie algebra of hermitian type* ([44]). A reductive Lie algebra containing an H-element is called *quasihermitian.*

Remark 3.2. If $\mathfrak{g}$ is a simple non-compact real Lie algebra, then an H-element exists if and only if $\mathfrak{g}$ is *hermitian,* i.e., for any Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the center $\mathfrak{z}(\mathfrak{k})$ of $\mathfrak{k}$ is non-zero. Then $\mathfrak{z}(\mathfrak{k}) = \mathbb{R}z$ for an H-element z. In particular iz is an Euler element in $\mathfrak{g}_{\mathbb{C}}$ and, for the conjugation σ with respect to the real form $\mathfrak{g}$, the involution $\theta = \sigma\tau_{iz}$ is a Cartan involution of $\mathfrak{g}_{\mathbb{C}}$ with $\theta(iz) = -iz$.

If, conversely, $\mathfrak{g}$ is a complex simple Lie algebra and $h \in \mathfrak{g}$ an Euler element, then there exists a Cartan involution θ of $\mathfrak{g}$ (automatically antilinear) with $\theta(h) = -h$, and then $\sigma := \theta\tau_h$ defines the real form $\mathfrak{h} := \mathfrak{g}^\sigma$ with $\mathfrak{h}_{\mathbb{C}} \cong \mathfrak{g}$. Then $\sigma(h) = -h$ implies that $z := ih \in \mathfrak{h}$ is an H-element, and thus $\mathfrak{h}$ is hermitian. By this construction, the Euler elements in the root systems of simple complex Lie algebras, listed in (4), specify the simple hermitian Lie algebras as the associated real forms:

$\mathfrak{h}$ (hermitian)	$\Sigma(\mathfrak{h}, \mathfrak{a}_{\mathfrak{h}})$	$\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$	$\Sigma(\mathfrak{g}, \mathfrak{a})$	Euler elt.
$\mathfrak{su}_{p,q}(\mathbb{C}), 1 \leq p \neq q$	$BC_p \ (p < q)$	$\mathfrak{sl}_{p+q}(\mathbb{C})$	A_{p+q-1}	h_p
	$C_p \ (p = q)$			
$\mathfrak{sp}_{2n}(\mathbb{R})$	C_n	$\mathfrak{sp}_{2n}(\mathbb{C})$	C_n	h_n
$\mathfrak{so}_{2,n}(\mathbb{R}), n > 2$	C_2	$\mathfrak{so}_{2+n}(\mathbb{C})$	$B_m \ (n{=}2m{-}1)$	h_1
			$D_m \ (n{=}2m{-}2)$	
$\mathfrak{so}^*(2n)$	$BC_m \ (n{=}2m{+}1)$	$\mathfrak{so}_{2n}(\mathbb{C})$	D_n	h_{n-1}, h_n
	$C_m \ (n = 2m)$			
$\mathfrak{e}_{6(-14)}$	BC_2	$\mathfrak{e}_6$	E_6	h_1, h_6
$\mathfrak{e}_{7(-25)}$	C_3	$\mathfrak{e}_7$	E_7	h_7

Table 1: Simple hermitian Lie algebras $\mathfrak{h}$ and corresponding Euler elements.

Note that $\mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{so}_{2,1}(\mathbb{R}) \cong \mathfrak{su}_{1,1}(\mathbb{C})$. More exceptional isomorphisms are discussed in [15, §17]. We recall that the hermitian real form $\mathfrak{so}^*(2n)$ of $\mathfrak{so}_{2n}(\mathbb{C})$ is given by

$$\mathfrak{so}^*(2n) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathfrak{gl}_{2n}(\mathbb{C}) : a^* = -a, b^* = b \right\}.$$

For hermitian Lie algebras $\mathfrak{h}$, the restricted root system $\Sigma = \Sigma(\mathfrak{h}, \mathfrak{a}_{\mathfrak{h}})$ is either of type C_r or BC_r (cf. [12, Lemmas 13-16], [30, Thm. 2], [36, Thm. XII.1.14] or Proposition 3.3 below). We say that $\mathfrak{g}$ is *of tube type* if the restricted root system is

of type C_r (cf. [23]). A comparison of Table 1 with the list (5) reveals that simple hermitian Lie algebras of tube type $\mathfrak{h}$ correspond to symmetric Euler elements in $\mathfrak{h}_\mathbb{C}$. Since the Euler elements h_{n-1} and h_n for the root system of type D_n are conjugate under a diagram automorphism, they correspond to isomorphic hermitian real forms. These Euler elements are symmetric if and only if n is even (Theorem 2.4).

$\mathfrak{h}$ (hermitian)	$\Sigma(\mathfrak{h}, \mathfrak{a}_\mathfrak{h})$	$\mathfrak{g} = \mathfrak{h}_\mathbb{C}$	$\Sigma(\mathfrak{g}, \mathfrak{a})$	symm. Euler element
$\mathfrak{su}_{n,n}(\mathbb{C})$	C_n	$\mathfrak{sl}_{2n}(\mathbb{C})$	A_{2n-1}	h_n
$\mathfrak{sp}_{2n}(\mathbb{R})$	C_n	$\mathfrak{sp}_{2n}(\mathbb{C})$	C_n	h_n
$\mathfrak{so}_{2,n}(\mathbb{R}), n > 2$	C_2	$\mathfrak{so}_{2+n}(\mathbb{C})$	$B_m \ (n{=}2m{-}1)$	h_1
			$D_m \ (n{=}2m{-}2)$	
$\mathfrak{so}^*(4n)$	C_n	$\mathfrak{so}_{4n}(\mathbb{C})$	D_{2n}	h_{2n-1}, h_{2n}
$\mathfrak{e}_{7(-25)}$	C_3	$\mathfrak{e}_7$	E_7	h_7

Table 2: Simple hermitian Lie algebras $\mathfrak{h}$ of tube type

3.2. Strongly orthogonal roots and symmetric Euler elements

In this subsection we consider maximal systems of long strongly orthogonal roots in Σ_1, where Σ is a 3-graded irreducible root system. As we shall see in Section 5.1, strongly orthogonal roots provide a powerful tool to reduce problems on groups and symmetric spaces to the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$.

Proposition 3.3. (Harish-Chandra) *Suppose that $\mathfrak{g}$ is simple with restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$. Let $h \in \mathfrak{a}$ be an Euler element and consider the corresponding 3-grading of the restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$, defined by*

$$\Sigma_j := \{\alpha \in \Sigma \colon \alpha(h) = j\}, \quad j = -1, 0, 1.$$

Let $\Gamma = \{\gamma_1, \ldots, \gamma_r\} \subseteq \Sigma_1$ be a maximal set of long strongly orthogonal roots (the sums and differences are no roots) and $\Sigma^+ \subseteq \Sigma$ a positive system containing Σ_1. The orthogonal projection pr_Γ from $\mathfrak{a}$ to the span of $\{\gamma_1, \ldots, \gamma_r\}$ is given by

$$\mathrm{pr}_\Gamma(\alpha) := \sum_{j=1}^r \frac{\alpha(\gamma_j^\vee)}{2} \gamma_j, \quad \alpha \in \mathfrak{a}^*.$$

We put $\mathcal{C}_0 := \Sigma_0 \cap \mathrm{pr}_\Gamma^{-1}(0)$ and consider the subsets

$$\mathcal{C}_j := \Sigma_0 \cap \mathrm{pr}_\Gamma^{-1}\left(\frac{\gamma_j}{2}\right), \quad \mathcal{P}_j := \Sigma_1 \cap \mathrm{pr}_\Gamma^{-1}\left(\frac{\gamma_j}{2}\right), \quad for \quad j = 1, \ldots, r,$$

and $\mathcal{C}_{jk} := \Sigma_0 \cap \mathrm{pr}_\Gamma^{-1}\left(\frac{\gamma_j - \gamma_k}{2}\right), \quad \mathcal{P}_{jk} := \Sigma_1 \cap \mathrm{pr}_\Gamma^{-1}\left(\frac{\gamma_j + \gamma_k}{2}\right) \quad for \ j < k.$

Then the following assertions hold:

$$\Sigma_1 = \Gamma \cup \bigcup_j \mathcal{P}_j \cup \bigcup_{j<k} \mathcal{P}_{jk} \quad and \quad \Sigma_0^+ \subseteq \mathcal{C}_0 \cup \bigcup_j \mathcal{C}_j \cup \bigcup_{j<k} \mathcal{C}_{jk} \tag{6}$$

where all the unions are disjoint. Further, $\mathrm{pr}_\Gamma(\Sigma) \setminus \{0\}$ either is a root system of type BC_r or of type C_r.

Proof.   Since every 3-graded root system arises in the complexification of a hermitian simple Lie algebra $\mathfrak{g}$ (cf. Theorem 2.4) (6) can be derived from [12, Lemmas 13–16] or [30, §2].

However, we have to show that our choice of Γ is equivalent to the construction of the set Γ' of strongly orthogonal roots in [12], which proceeds as follows: Let γ_1' be the maximal root (with respect to Σ^+), and choose γ_j' maximal in Σ_1 orthogonal to $\gamma_1', \ldots, \gamma_{j-1}'$.

The set Σ_1 is the weight set of the irreducible representation of the Lie subalgebra $\mathfrak{g}_{\mathbb{C},0}(h)$ on $\mathfrak{g}_{\mathbb{C},1}(h)$. The long roots in Σ_1 are the extremal weights of this representation. Therefore the Weyl group $\mathcal{W}_0$ of the subsystem Σ_0 acts transitively on this set. Harish-Chandra's results ([12, Lemmas 13-16]) show in particular that, for $r > 1$, the root system $(\gamma_1')^\perp$ is irreducible. Here the main point is that, for any $\alpha \in \mathcal{C}_0$, there exists a $\beta \in \Sigma_1$ with $\alpha + \beta \in \Sigma_1$. This implies the existence of $\beta' \in \Sigma_1 \setminus \alpha^\perp$.

Now we show that Harish-Chandra's set Γ' of strongly orthogonal roots is conjugate under $\mathcal{W}_0$ to Γ. As all long roots in Σ_1 are conjugate under the Weyl group $\mathcal{W}_0$, we may assume that $\gamma_1 = \gamma_1' \in \Gamma$. Now one proceeds inductively to see that in the irreducible root system $\gamma_1^\perp \cap \Sigma$, the roots γ_2 and γ_2' are conjugate under the Weyl group $\mathcal{W}_0$. We conclude that Γ is conjugate under the Weyl group $\mathcal{W}_0$ to Γ'.

The set $\mathrm{pr}_\Gamma(\Sigma) \setminus \{0\}$ is a root system of type BC_r if all sets $\mathcal{C}_j, \mathcal{P}_j, \mathcal{C}_{jk}, \mathcal{P}_{jk}$ are non-empty, and of type C_r if the sets $\mathcal{C}_j$ and $\mathcal{P}_j$ are empty ([30, §2]).   ∎

Example 3.4.   (a) For $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$, the restricted root system is of type A_2:

$$\Sigma = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3)\}.$$

For the 3-grading defined by the Euler element $h = \frac{1}{3}(2, -1, -1)$, we have

$$\Sigma_1 = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3\}, \quad \Sigma_0 = \{\pm(\varepsilon_2 - \varepsilon_3)\}, \quad \mathcal{W}_0 \cong S_2,$$

and Γ is $\{\varepsilon_1 - \varepsilon_2\}$ or $\{\varepsilon_1 - \varepsilon_3\}$. The projection $\mathrm{pr}_\Gamma(\Sigma) \setminus \{0\}$ is a root system of type BC_1.

(b) For $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{R})$, the restricted root system is o f type C_2:

$$\Sigma = \{\pm 2\varepsilon_1, \pm 2\varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_2\}.$$

For the 3-grading defined by $h := \frac{1}{2}(1, 1)$, we then have

$$\Sigma_1 = \{2\varepsilon_1, 2\varepsilon_2, \varepsilon_1 + \varepsilon_2\} \quad \text{and} \quad \Sigma_0 = \{\pm(\varepsilon_1 - \varepsilon_2)\}.$$

Here $\Gamma = \{2\varepsilon_1, 2\varepsilon_2\}$ is a maximal subset of long strongly orthogonal roots. The subset $\{\varepsilon_1 + \varepsilon_2\}$ is also maximal strongly orthogonal in Σ_1, but it consists of a short root.

Corollary 3.5.   *In the notation of* Proposition 3.3, *consider* $h_s := \frac{1}{2} \sum_{j=1}^r \gamma_j^\vee$. *Then the following statements are equivalent:*

(a)   *The Euler element h is symmetric.*

(b) $h = h_s$.

(c)   *The sets $\mathcal{P}_j$ and $\mathcal{C}_j$ are empty.*

(d) $\mathrm{pr}_\Gamma(\Sigma) \setminus \{0\}$ *is a root system of type C_r.*

Proof.   Let $s_\alpha(x) = x - \alpha(x)\alpha^\vee$ be the reflection corresponding to the root α. We consider the Euler element

$$h' := s_{\gamma_1} \cdots s_{\gamma_r}(h) = h - \sum_{j=1}^r \gamma_j(h)\gamma_j^\vee = h - \sum_{j=1}^r \gamma_j^\vee.$$

Since the Weyl group acts by automorphisms of the root system ([18, §1.2]), h' also is an Euler element. Next we observe that $\gamma_j(h') = -1$ for each j, $\alpha(h') = 1 - 1 = 0$ for $\alpha \in \mathcal{P}_j$ and $\alpha(h') = 1 - 2 = -1$ for $\alpha \in \mathcal{P}_{jk}$. Moreover, $\alpha(h') = 0$ for $\alpha \in \mathcal{C}_0$, $\alpha(h') = -1$ for $\alpha \in \mathcal{C}_j$ and $\alpha(h') = 0$ for $\alpha \in \mathcal{C}_{jk}$. This shows that $\alpha(h') \leq 0$ for all $\alpha \in \Sigma^+$, so that $h' \in \mathcal{W}.h$ is the unique element with this property. It follows that h is symmetric if and only if $h' = -h$.

(a) $\Leftrightarrow$ (b): The Euler element h is symmetric if and only if $h' = -h$, which is equivalent to $2h = \sum_{j=1}^r \gamma_j^\vee = 2h_s$, i.e., to $h = h_s$.

(b) $\Rightarrow$ (c): For $\alpha \in \mathcal{P}_j \cup \mathcal{C}_j$ we have $\alpha(h_s) = \frac{1}{2}$, which is impossible if h_s is an Euler element.

(c) $\Rightarrow$ (b): If all sets $\mathcal{P}_j$ and $\mathcal{C}_j$ are empty, then the discussion above shows that $\alpha(h) = -\alpha(h')$ for all roots α, hence that $h' = -h$ and thus $h = h_s$.

(c) $\Leftrightarrow$ (d) follows from Proposition 3.3.   ∎

The following proposition translates the information on symmetry of the Euler element to the Lie algebra context.

Proposition 3.6.   *Let $\mathfrak{g}$ be a simple real Lie algebra and $h \in \mathfrak{g}$ be an Euler element. Let $\mathfrak{a} \ni h$ be a maximal abelian hyperbolic subspace, $\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a})$ the corresponding set of restricted roots, $\Gamma = \{\gamma_1, \ldots, \gamma_r\} \subseteq \Sigma_1$ a maximal set of strongly orthogonal roots and $\mathfrak{g}(\gamma_j) \cong \mathfrak{sl}_2(\mathbb{R})$ corresponding $\mathfrak{sl}_2(\mathbb{R})$-subalgebras. Then the following stements are equivalent:*

(a)   *h is symmetric.*

(b)   *$h \in \sum_{j=1}^r \mathfrak{g}(\gamma_j)$.*

Proof.   (a) $\Rightarrow$ (b): From Corollary 3.5 we know that h is symmetric if and only if $h = h_s$, and h_s is contained in $\sum_{j=1}^r \mathfrak{g}(\gamma_j)$ because $\gamma_j^\vee \in \mathfrak{g}(\gamma_j)$.

(b) $\Rightarrow$ (a) follows from the fact that all Euler elements in $\mathfrak{sl}_2(\mathbb{R})$ are symmetric, and this is inherited by $\mathfrak{sl}_2(\mathbb{R})^r$.   ∎

3.3. Cartan involutions and H-elements

In this subsection we briefly discuss the relation between Cartan involutions and H-elements in quasihermitian semisimple Lie algebras.

Lemma 3.7.   *If $\mathfrak{g}$ is semisimple without compact ideals, then the following assertions hold:*

(a) *If σ is an involutive automorphism for which $\mathfrak{g}^\sigma$ is compactly embedded. Then σ is a Cartan involution.[2]*

(b) *If $C_\mathfrak{g} \subseteq \mathfrak{g}$ is a pointed generating invariant cone and $z \in C_\mathfrak{g}^\circ$ such that $h := iz$ is an Euler element in $\mathfrak{g}_\mathbb{C}$, then z is an H-element.*

---

[2] Every non-trivial involution on a compact simple Lie algebra shows that this conclusion is false if $\mathfrak{g}$ contains compact ideals.

Proof. (a) Let $\mathfrak{k} \supseteq \mathfrak{g}^\sigma$ be a maximal compactly embedded subalgebra and θ the corresponding Cartan involution with $\mathfrak{k} = \mathfrak{g}^\theta$. Then

$$\mathfrak{p} := \mathfrak{g}^{-\theta} = \mathfrak{k}^\perp \subseteq (\mathfrak{g}^\sigma)^\perp = \mathfrak{g}^{-\sigma},$$

where $\perp$ refers to the Cartan-Killing form. As $\mathfrak{g}$ contains no compact ideals, we get

$$\mathfrak{g} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$$

because the right hand side is an ideal containing all non-compact simple ideals. Since θ and σ coincide on $\mathfrak{p}$, they coincide, i.e., σ is a Cartan involution.

(b) First we observe that $\mathrm{Spec}(\mathrm{ad}\, z) = \{0, \pm i\}$ and that $\ker(\mathrm{ad}\, z)$ is compactly embedded because $z \in C_\mathfrak{g}^\circ$ ([36, Prop. V.5.11]). Hence $\sigma := e^{\pi\,\mathrm{ad}\, z} \in \mathrm{Aut}(\mathfrak{g})$ is an involution for which $\mathfrak{g}^\sigma = \ker(\mathrm{ad}\, z)$ is compactly embedded. Now (a) implies that θ_z is a Cartan involution, i.e., z is an H-element. ∎

Lemma 3.8. *Let $\mathfrak{g}$ be a simple Lie algebra and θ_1 and θ_2 two Cartan involutions. Then θ_1 and θ_2 commute if and only if $\theta_1 = \theta_2$.*

Proof. If $\theta_1 = \theta_2$ then they clearly commute. For the other direction write $\mathfrak{g}_j = \mathfrak{k}_j \oplus \mathfrak{p}_j$, $j = 1, 2$, for the Cartan decomposition corresponding the θ_j. Then $\mathfrak{k}_2$ is θ_1-stable and hence

$$\mathfrak{k}_2 = (\mathfrak{k}_2 \cap \mathfrak{k}_1) \oplus (\mathfrak{k}_2 \cap \mathfrak{p}_1).$$

As $\mathfrak{k}_2$ is compactly embedded, we have $\mathfrak{k}_2 \cap \mathfrak{p}_1 = \{0\}$, hence $\mathfrak{k}_2 \subseteq \mathfrak{k}_1$. We likewise obtain $\mathfrak{k}_1 \subseteq \mathfrak{k}_2$, hence equality. Then $\mathfrak{p}_1 = \mathfrak{p}_2$ is the orthogonal space with respect to the Cartan-Killing form and thus $\theta_1 = \theta_2$. ∎

Lemma 3.9. *If $\mathfrak{g}$ is a simple hermitian Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $z \in \mathfrak{z}(\mathfrak{k})$ is an H-element, then $\pm z$ are the only H-elements in $\mathfrak{k}$.*

Proof. Clearly, $\pm z$ are both H-elements in $\mathfrak{k}$, defining the same Cartan involution $\theta = \theta_z = e^{\pi\,\mathrm{ad}\, z}$. If $z' \in \mathfrak{k}$ is another H-element, then $[z, z'] = 0$, so that θ_z and $\theta_{z'}$ are two commuting Cartan involutions and hence $\theta_z = \theta_{z'}$ by Lemma 3.8, so that $z, z' \in \mathfrak{z}(\mathfrak{k})$. But $\dim \mathfrak{z}(\mathfrak{k}) = 1$ and hence $z' = \pm z$. ∎

3.4. The duality between Euler elements and H-elements

The following lemma relates Euler elements in $\mathfrak{p} = \mathfrak{g}^{-\theta}$ to symmetric Lie algebras $(\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}, \tau^c, z)$ of hermitian type.

Lemma 3.10. *Let*

$$\mathcal{A} := \{(\mathfrak{g}, \theta, h) \colon \mathfrak{g} \text{ semisimple}, h \in \mathcal{E}(\mathfrak{g}), \theta \text{ Cartan involution with } \theta(h) = -h\}$$

and

$$\mathcal{B} := \{(\mathfrak{g}, \tau, z) \colon \mathfrak{g} \text{ semisimple}, z \text{ H-element}, \tau \text{ involution with } \tau(z) = -z\}.$$

Then we have a bijection $\Phi \colon \mathcal{A} \to \mathcal{B}$, $\Phi(\mathfrak{g}, \theta, h) := (\mathfrak{g}^c, \tau^c, ih)$, where

$$\tau := \tau_h \theta, \quad \mathfrak{g}^c = \mathfrak{g}^\tau + i\mathfrak{g}^{-\tau}, \quad \tau^c(x + iy) = x - iy.$$

Proof. Let $(\mathfrak{g}, \theta, h) \in \mathcal{A}$. As θ commutes with τ_h, the product $\tau := \tau_h \theta$ is an involution on $\mathfrak{g}$. It satisfies

$$\mathfrak{h} := \mathfrak{g}^\tau = \mathfrak{k}^{\tau_h} \oplus \mathfrak{p}^{-\tau_h} \quad \text{and} \quad \mathfrak{q} := \mathfrak{g}^{-\tau} = \mathfrak{k}^{-\tau_h} \oplus \mathfrak{p}^{\tau_h},$$

so that
$$\mathfrak{g}^{\tau_h} = \ker(\mathrm{ad}\, h) = \mathfrak{h}_\mathfrak{k} \oplus \mathfrak{q}_\mathfrak{p} \quad \text{and} \quad h \in \mathfrak{q}_\mathfrak{p}. \tag{7}$$

We associate to $(\mathfrak{g}, \tau)$ the dual symmetric Lie algebra $(\mathfrak{g}^c, \tau^c)$. It contains $z := ih$ with centralizer $\ker(\operatorname{ad} z) = \mathfrak{h}_{\mathfrak{k}} + i\mathfrak{q}_{\mathfrak{p}} = \mathfrak{k}^c$ maximal compactly embedded in $\mathfrak{g}^c$ (where $\mathfrak{h} = \mathfrak{g}^\tau, \mathfrak{q} = \mathfrak{g}^{-\tau}$). Therefore $z \in i\mathfrak{q}_{\mathfrak{p}}$ is an H-element and the involution τ^c on $\mathfrak{g}^c$ satisfies $\tau^c(z) = -z$. This shows that $(\mathfrak{g}^c, \tau^c, ih) \in \mathcal{B}$.

If, conversely, $(\mathfrak{g}, \tau, z) \in \mathcal{B}$, then the dual symmetric Lie algebra $(\mathfrak{g}^c, \tau^c)$ contains the Euler element $h := -iz$ and $\theta := \tau^c \tau_h$ is a Cartan involution because its fixed point set

$$\mathfrak{k} := \mathfrak{h}^{\tau_h} + i\mathfrak{q}^{-\tau_h} = \mathfrak{h}_{\mathfrak{k}} + i\mathfrak{q}_{\mathfrak{p}}$$

is maximal compactly embedded in $\mathfrak{g}$. This shows that $(\mathfrak{g}^c, \theta, h) \in \mathcal{A}$. As $\tau^c = \theta\tau_h$, we have $\Phi(\mathfrak{g}^c, \theta, h) = (\mathfrak{g}, \tau, z)$. Therefore Φ is bijective. $\blacksquare$

Remark 3.11.   Since the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^c$ are real forms of the same complex Lie algebra $\mathfrak{g}_\mathbb{C}$, one can also represent the triples $(\mathfrak{g}, \theta, h)$ by triples (σ, θ, h), where σ is an antilinear involution of $\mathfrak{g}_\mathbb{C}$, θ is a linear involution of $\mathfrak{g}_\mathbb{C}$ commuting with σ such that $\sigma\theta$ is a Cartan involution of $\mathfrak{g}_\mathbb{C}$, and $h \in \mathcal{E}(\mathfrak{g}_\mathbb{C})$ satisfies $\sigma(h) = h$ and $\theta(h) = -h$.

Likewise elements of $\mathcal{B}$ can be represented by triples (σ, τ, z), where σ is an antilinear involution of $\mathfrak{g}_\mathbb{C}$, τ is a linear involution of $\mathfrak{g}_\mathbb{C}$ commuting with σ and $iz \in \mathcal{E}(\mathfrak{g}_\mathbb{C})$ satisfies $\sigma(z) = z$ and $\tau(z) = -z$. In these terms Φ maps (σ, θ, h) to $(\sigma\tau, \tau, ih) = (\sigma\tau_h\theta, \tau, ih)$.

3.5. Basic facts on invariant cones

3.5.1. Invariant cones in hermitian Lie algebras

Let $\mathfrak{g}$ be a simple hermitian Lie algebra and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. Then $\mathfrak{z}(\mathfrak{k}) = \mathbb{R}z$ is one-dimensional and generated by an H-element z satisfying $\theta = e^{\pi \operatorname{ad} z}$ (Remark 3.2). Now every closed convex $\operatorname{Inn}(\mathfrak{g})$-invariant cone $C_\mathfrak{g} \subseteq C$ satisfies

$$C_{\mathfrak{z}(\mathfrak{k})} := p_{\mathfrak{z}(\mathfrak{k})}(C_\mathfrak{g}) = C_\mathfrak{g} \cap \mathfrak{z}(\mathfrak{k}) \quad \text{and} \quad C^\circ_{\mathfrak{z}(\mathfrak{k})} = C^\circ_\mathfrak{g} \cap \mathfrak{z}(\mathfrak{k}) \tag{8}$$

where the projection $p_{\mathfrak{z}(\mathfrak{k})} \colon \mathfrak{g} \to \mathfrak{z}(\mathfrak{k})$ is the composition of the fixed point projection $\mathfrak{g} \to \mathfrak{k}$ for the compact group $\operatorname{Inn}_\mathfrak{g}(\mathfrak{z}(\mathfrak{k})) \cong \mathbb{T}$ and the fixed point projection for the compact group $\operatorname{Inn}_\mathfrak{g}(\mathfrak{k})$ (Proposition B.4). Therefore every non-trivial invariant cone $C_\mathfrak{g}$ either contains z or $-z$.

Conversely, it is easy to see with the Iwasawa decomposition that

$$C_\mathfrak{g}^{\min}(z) := \overline{\mathbb{R}_+ \operatorname{conv}(\operatorname{Inn}_\mathfrak{g}(z))}$$

is a pointed invariant cone ([14, Thm. 7.25]). For any invariant cone $C_\mathfrak{g} \subseteq \mathfrak{g}$ containing z we then have

$$C_\mathfrak{g}^{\min}(z) \subseteq C_\mathfrak{g}. \tag{9}$$

The Cartan-Killing form $\kappa(x, y) = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y)$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, so that

$$C_\mathfrak{g}^{\max}(z) := \{x \in \mathfrak{g} \colon (\forall y \in C_\mathfrak{g}^{\min}(z)) \; \kappa(x, y) \leq 0\}$$

is a pointed generating invariant cone containing z, hence also $C_\mathfrak{g}^{\min}(z)$. Dualizing (9) implies that every invariant cone $C_\mathfrak{g}$ containing z is contained in $C_\mathfrak{g}^{\max}(z)$:

$$C_\mathfrak{g}^{\min}(z) \subseteq C_\mathfrak{g} \subseteq C_\mathfrak{g}^{\max}(z). \tag{10}$$

If $\mathfrak{g}$ is a semisimple Lie algebra and $z \in \mathfrak{g}$ an H-element, then $\mathfrak{g} = \mathfrak{g}_k \oplus \mathfrak{g}_n$, where $\mathfrak{g}_k$ is the sum of all simple ideals commuting with z (these are compact) and $\mathfrak{g}_n$ is a sum of hermitian simple ideals. Applying the preceding discussion to all simple ideals in $\mathfrak{g}_n$, we obtain pointed generating invariant cones $C_{\mathfrak{g}_n}^{\min}(z) \subseteq C_{\mathfrak{g}_n}^{\max}(z)$ such that any pointed generating invariant cone $C_{\mathfrak{g}_n} \subseteq \mathfrak{g}_n$ containing z satisfies

$$C_{\mathfrak{g}_n}^{\min}(z) \subseteq C_{\mathfrak{g}_n} \subseteq C_{\mathfrak{g}_n}^{\max}(z).$$

We then put $C_{\mathfrak{g}}^{\min}(z) := C_{\mathfrak{g}_n}^{\min}(z)$ and $C_{\mathfrak{g}}^{\max}(z) := \mathfrak{g}_k \oplus C_{\mathfrak{g}_n}^{\max}(z).$ (11)

In this context it is still true that, (cf. [16, Prop. II.3]) for every pointed generating invariant cone $C_{\mathfrak{g}} \subseteq \mathfrak{g}$, there exists an H-element z with

$$C_{\mathfrak{g}}^{\min}(z) \subseteq C_{\mathfrak{g}} \subseteq C_{\mathfrak{g}}^{\max}(z).$$ (12)

3.5.2. Invariant cones in irreducible symmetric Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra with Cartan involution θ and $h \in \mathfrak{g}^{-\theta}$ an Euler element. For the involution $\tau = \tau_h \theta$ we then have for $\mathfrak{h} = \mathfrak{g}^\tau$ and $\mathfrak{q} = \mathfrak{g}^{-\tau}$

$$\mathfrak{h} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{h}_{\mathfrak{p}}, \quad \mathfrak{q} = \mathfrak{q}_{\mathfrak{k}} \oplus \mathfrak{q}_{\mathfrak{p}} \quad \text{with} \quad \mathfrak{g}^{\tau_h} = \ker(\operatorname{ad} h) = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{q}_{\mathfrak{p}}.$$

Here $\mathfrak{h}_{\mathfrak{k}} = \mathfrak{h} \cap \mathfrak{k}$ etc.. It follows that $z := ih \in \mathfrak{g}^c$ is an H-element and that its centralizer $\mathfrak{k}^c = \mathfrak{h}_{\mathfrak{k}} + i\mathfrak{q}_{\mathfrak{p}}$ is a maximal compact subalgebra of $\mathfrak{g}^c$ (Lemma 3.10). The closed convex $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant cone $C_{\mathfrak{q}}^{\min}(h) \subseteq \mathfrak{q}$ generated by h is pointed because it is contained in the pointed cone $-iC_{\mathfrak{g}^c}^{\min}(ih)$. By (8) h is contained in its interior (cf. also [17, Prop. 3.1.3]).

Lemma 3.12. (cf. [17, Prop. 3.1.11]) *If $(\mathfrak{g}, \tau, C)$ is irreducible ncc, then the following assertions hold:*

(a) *Every element in $C_{\mathfrak{q}}^\circ$ fixed by $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h}_{\mathfrak{k}})$ is contained in $\mathfrak{q}_{\mathfrak{p}}$.*

(b) *$\mathfrak{z}_{\mathfrak{q}_{\mathfrak{p}}}(\mathfrak{h}_{\mathfrak{k}}) = \mathbb{R}h$ for an Euler element h.*

(c) *$C^\circ \cap \mathfrak{q}_{\mathfrak{p}}$ contains an Euler element h.*

Proof.   The group $H := \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h})$ is the identity component of the group $\operatorname{Aut}(\mathfrak{g})^\tau$, hence closed. It has the polar decomposition $H = H_K e^{\operatorname{ad} \mathfrak{h}_{\mathfrak{p}}}$, where $H_K = \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h}_{\mathfrak{k}})$ ([15, Prop. 13.1.5]). As the simplicity of $\mathfrak{g}$ implies that $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}]$, the representation $\operatorname{Ad}_{\mathfrak{q}}$ of H on $\mathfrak{q}$ is faithful and its image in $\operatorname{GL}(\mathfrak{q})$ is closed with maximal compact subgroup $\operatorname{Ad}_{\mathfrak{q}}(H_K)$.

(a) Let $x \in C^\circ \cap \mathfrak{q}_{\mathfrak{p}}$ be fixed by the compact group H_K. As the stabilizer of x in $\operatorname{Ad}_{\mathfrak{q}}(H)$ is compact ([36, Prop. V.5.11]), $H^x = H_K$. Since x is hyperbolic, there exists an element $g \in H$ with $y := \operatorname{Ad}(g)x \in \mathfrak{q}_{\mathfrak{p}}$ ([24, Cor. II.9]). Then $\theta(y) = -y$ implies that the stabilizer H^y is θ-invariant. So $g_1 = h_1 e^{\operatorname{ad} z} \in H^y$ with $h_1 \in H_K$ and $z \in \mathfrak{h}_{\mathfrak{p}}$ implies $\theta(g_1)g_1 = e^{2\operatorname{ad} z} \in H^y$. The compactness of $H^y = gH^x g^{-1} = gH_K g^{-1}$ now entails $z = 0$. Hence $H^y \subseteq H_K$ and thus $H^y = H_K$ as H^y is maximally compact. We conclude that $gH_K g^{-1} = H_K$ which further implies $g \in H_K$, the stabilizer group of the base point in the Riemannian symmetric space H/H_K. This shows that $y = x \in \mathfrak{q}_{\mathfrak{p}}$.

(b) Let $x \in \mathfrak{q}_\mathfrak{p}$ be fixed by H_K, i.e., x commutes with $\mathfrak{h}_\mathfrak{k}$. Then $[x, \mathfrak{q}_\mathfrak{p}] \subseteq \mathfrak{h}_\mathfrak{k}$, and

$$\kappa([x, \mathfrak{q}_\mathfrak{p}], \mathfrak{h}_\mathfrak{k}) = \kappa(\mathfrak{q}_\mathfrak{p}, [x, \mathfrak{h}_\mathfrak{k}]) = \{0\}$$

implies $[x, \mathfrak{q}_\mathfrak{p}] = \{0\}$, so that x is central in the maximal compactly embedded subalgebra $\mathfrak{k}^c := \mathfrak{h}_\mathfrak{k} + i\mathfrak{q}_\mathfrak{p}$ of $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$. As $\mathfrak{g}$ is simple, either $\mathfrak{g}^c$ is simple as well (this happens if $(\mathfrak{g}, \tau)$ is not of complex type), or if $(\mathfrak{g}, \tau)$ is of complex type and then $(\mathfrak{g}^c, \tau) \cong (\mathfrak{h} \oplus \mathfrak{h}, \tau_{\mathrm{flip}})$. In the first case $\mathfrak{g}^c$ is simple hermitian, so that $\mathfrak{z}(\mathfrak{k}^c) = \mathbb{R}ix$ contains an H-element and thus x is a multiple of an Euler element. In the second case $\mathfrak{h}$ is also simple hermitian and $\mathfrak{k}^c = \mathfrak{h}_\mathfrak{k} \oplus \mathfrak{h}_\mathfrak{k}$ has 2-dimensional center with $\mathfrak{z}(\mathfrak{k}^c) \cap i\mathfrak{q}_\mathfrak{p} = \mathbb{R}(z, -z)$ for an H-element $z \in \mathfrak{h}_\mathfrak{k}$. Again, some positive multiple of x is an Euler element and $\mathfrak{z}_{\mathfrak{q}_\mathfrak{p}}(\mathfrak{h}_\mathfrak{k}) = \mathbb{R}x$.

(c) As the fixed point projection for the compact group H_K leaves the interior of $C_\mathfrak{q}$ invariant (Proposition B.4), $C_\mathfrak{q}^0$ contains an H_K-fixed point x. By (a) it is contained in $\mathfrak{q}_\mathfrak{p}$ and by (b) it is a multiple of some Euler element h. Hence $C_\mathfrak{q}^\circ$ contains h or $-h$. ∎

It follows from Lemma 3.12(c) that, for every $\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})$-invariant pointed generating closed convex cone $C_\mathfrak{q} \subseteq \mathfrak{q}$, the interior $C_\mathfrak{q}^\circ$ contains h or $-h$. Further, $h \in C_\mathfrak{q}^\circ$ implies

$$C_\mathfrak{q}^{\min}(h) \subseteq C_\mathfrak{q}.$$

Similar arguments as above, using the Cartan-Killing form on $\mathfrak{q}$, now lead to a maximal $\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})$-invariant cone $C_\mathfrak{q}^{\max}(h)$ with

$$C_\mathfrak{q}^{\min}(h) \subseteq C_\mathfrak{q} \subseteq C_\mathfrak{q}^{\max}(h) = C_\mathfrak{q}^{\min}(h)^\star = \{x \in \mathfrak{q} : (\forall y \in C_\mathfrak{q}^{\min}(h))\ \kappa(x, y) \geq 0\}.$$

If $\mathfrak{g}$ is only semisimple, we decompose it as $\mathfrak{g} = \mathfrak{g}_r \oplus \mathfrak{g}_s$, where $\mathfrak{g}_s$ is the sum of all simple ideals not commuting with h. We then obtain a pointed $\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})$-invariant cone $C_{\mathfrak{q}_s}^{\min}(h) \subseteq \mathfrak{q}_s := \mathfrak{q} \cap \mathfrak{g}_s$ whose dual cone $C_{\mathfrak{q}_s}^{\max}(h)$ with respect to the Cartan-Killing form satisfies

$$C_{\mathfrak{q}_s}^{\min}(h) \subseteq C_{\mathfrak{q}_s}^{\max}(h). \tag{13}$$

Both cones are adapted to the decomposition of $(\mathfrak{g}, \tau)$ into irreducible summands. Further, each pointed generating $\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})$-invariant cone $C_\mathfrak{q}$ containing h satisfies

$$C_\mathfrak{q}^{\min}(h) \subseteq C_\mathfrak{q} \subseteq C_\mathfrak{q}^{\max}(h). \tag{14}$$

Here the first inclusion is obvious, and the second one follows from the fact that h is also contained in the dual cone

$$C_\mathfrak{q}^\star = \{y \in \mathfrak{q} : (\forall x \in C_\mathfrak{q})\ \kappa(x, y) \geq 0\}.$$

This leads to $C_\mathfrak{q}^{\min}(h) \subseteq C_\mathfrak{q}^\star$, and thus to $C_\mathfrak{q} \subseteq C_\mathfrak{q}^{\min}(h)^\star = C_\mathfrak{q}^{\max}(h)$.

4. Euler elements and non-compactly causal symmetric spaces

In this section we present a classification of irreducible non-compactly causal symmetric spaces based on Euler elements. Here a key concept is that of a **causal** Euler element that is explored in Section 4.1: If $(\mathfrak{g}, \tau)$ is a symmetric Lie algebra and $h \in \mathfrak{q}$ an Euler element, then we call it causal if it is contained in the interior of an

$\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant pointed convex cone in $\mathfrak{q}$. Theorem 4.2 asserts that, for any pair (θ, h) of a Cartan involution θ and an Euler element h satisfying $\theta(h) = -h$, the involution $\tau = \tau_h \theta$ makes h causal for $(\mathfrak{g}, \tau)$. In Section 4.3 we use this construction to classify irreducible non-compactly causal symmetric Lie algebras in terms of $\mathrm{Inn}(\mathfrak{g})$-orbits of Euler elements (Theorem 4.21), which essentially goes back to [43].

In general there are many locally isomorphic causal symmetric spaces G/H corresponding to a causal symmetric Lie algebra $(\mathfrak{g}, \tau, C)$. It is therefore natural to ask for a "minimal model". In Section 4.2 we show that the minimal symmetric space $\mathrm{Inn}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})^\tau$ for the irreducible symmetric Lie algebra $(\mathfrak{g}, \tau)$ is this minimal model for $(\mathfrak{g}, \tau, C)$ if and only if the causal Euler element is not symmetric; otherwise we have to pass to a two-fold covering. In Section 4.4 we present a structured table with the classification of the irreducible non-compactly causal symmetric Lie algebras, based on causal Euler elements.

Throughout this section the Lie algebra $\mathfrak{g}$ is assumed to be *semisimple* if not stated otherwise.

4.1. Causal Euler elements

Definition 4.1. Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra and $h \in \mathcal{E}(\mathfrak{g}) \cap \mathfrak{q}$ an Euler element. We say that h is *causal* if there exists a pointed generating $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant cone $C \subseteq \mathfrak{q}$ with $h \in C^\circ$. We write $\mathcal{E}_c(\mathfrak{q}) \subseteq \mathcal{E}(\mathfrak{g}) \cap \mathfrak{q}$ for the *set of causal Euler elements in $\mathfrak{q}$*.

We start with a *reductive* Lie algebra $\mathfrak{g}$. Let $h \in \mathcal{E}(\mathfrak{g})$ be an Euler element. As h is hyperbolic, there exists a Cartan involution θ with $\theta(h) = -h$, where we assume that $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g}^{-\theta}$. To make h causal, one has to find an appropriate involution τ on $\mathfrak{g}$. It turns out that $\tau := \tau_h \theta$ is the natural choice making h a causal Euler element (cf. Lemma 3.10).

Theorem 4.2. (Euler elements vs. causal symmetric spaces) *Let $h \in \mathcal{E}(\mathfrak{g})$ be an Euler element in the reductive Lie algebra $\mathfrak{g}$ and θ a Cartan involution with $\theta(h) = -h$ and $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g}^{-\theta}$. Then the following assertions hold:*

(a) *$\tau := \tau_h \theta$ defines an involution on $\mathfrak{g}$.*

(b) *The symmetric Lie algebra $(\mathfrak{g}, \tau)$ is non-compactly causal and there exists a pointed generating $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant cone $C \subseteq \mathfrak{q}$ with $h \in C^\circ$.*

(c) *All ideals of $\mathfrak{g}$ contained in $\mathfrak{g}^\tau$ are compact.*

This theorem implies in particular that every Euler element is causal with respect to a suitably chosen involution τ on $\mathfrak{g}$. In [17] causal Euler elements in $\mathfrak{q}_{\mathfrak{p}}$ are called "cone generating elements" because they generate pointed generating invariant cones (see also [43] and Section 3.5.2).

Proof. Let $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ denote the composition of $\mathfrak{g}$ into the center and simple ideals. Recall that θ preservers all these ideals because all Cartan involutions are conjugate under $\mathrm{Inn}(\mathfrak{g})$ ([15, Cor. 13.2.13]) and all simple ideals possess Cartan involutions ([15, Thm. 13.2.10]). Now $h = h_z + h_1 + \cdots + h_n$, where either $h_j = 0$ or $h_j \in \mathfrak{g}_j$ is an Euler element. As θ and τ_h preserve $\mathfrak{g}_j$, the involution $\tau := \tau_h \theta$ defines on $\mathfrak{g}_j$ an involution τ_j which is Cartan if $h_j = 0$, and if this is not the case, then $h_j \in \mathfrak{q}_j := \mathfrak{g}_j^{-\tau_j}$. Note that $\mathrm{ad}\, h \neq 0$ implies that some h_j is non-zero. As all

ideals of $\mathfrak{g}$ contained in $\mathfrak{g}^\tau$ commute with h, they are contained in $\mathfrak{g}^\theta$, hence are compact Lie algebras.

From the discussion in Section 3.5.2 we know that h_j is contained in the interior of a pointed generating $\mathrm{Inn}_{\mathfrak{g}_j}(\mathfrak{h}_j)$-invariant cone $C_j := C_{\mathfrak{q}_j}^{\min}(h_j) \subseteq \mathfrak{q}_j$. As h_j is hyperbolic and contained in C_j°, the causal symmetric Lie algebra $(\mathfrak{g}_j, \tau_j, C_j)$ is ncc.

The Lie algebra
$$\mathfrak{h}_r := \sum_{h_j=0} \mathfrak{h}_j$$
is compactly embedded and

$$C_s := \sum_{h_j \neq 0} C_j = C_{\mathfrak{q}_s}^{\min}(h) \subseteq \mathfrak{q}_s := \sum_{h_j \neq 0} \mathfrak{q}_j$$

is pointed and generating.

Further $\mathfrak{h}_s := \displaystyle\sum_{h_j \neq 0} \mathfrak{h}_j$ satisfies $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_s)|_{\mathfrak{q}} \subseteq \mathrm{SL}(\mathfrak{q})$.

Hence Lemma B.2 implies the existence of a pointed generating $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant cone $C \subseteq \mathfrak{q} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{q}_r \oplus \mathfrak{q}_s$ with $C \cap \mathfrak{q}_s = C_s$ and $h \in C_s^\circ \subseteq C^\circ$. ∎

In the preceding proof, we have seen that it is convenient to decompose the Lie algebra $\mathfrak{g}$ as
$$\mathfrak{g} = \mathfrak{g}_k \oplus \mathfrak{g}_r \oplus \mathfrak{g}_s, \tag{15}$$
where $\mathfrak{g}_s$ is the sum of all simple ideals not commuting with h (the strictly ncc part), $\mathfrak{g}_r$ is the sum of all non-compact simple ideals commuting with h on which $\tau = \theta$ (the non-compact Riemannian part), and $\mathfrak{g}_k$ is the sum of all compact ideals (they commute with h). All these ideals are invariant under θ and $\tau = \tau_h \theta$, so that we obtain decompositions

$$\mathfrak{g}_s = \mathfrak{h}_s \oplus \mathfrak{q}_s, \quad \mathfrak{g}_r = \mathfrak{h}_r \oplus \mathfrak{q}_r \quad \text{and} \quad \mathfrak{g}_k = \mathfrak{h}_k, \tag{16}$$

where $\mathfrak{h}_r \oplus \mathfrak{h}_k$ is a compact ideal of $\mathfrak{h}$ and $\mathfrak{h}_r \oplus \mathfrak{q}_r$ is a Cartan decomposition of $\mathfrak{g}_r$.

Remark 4.3. Let $\mathfrak{q} = \mathfrak{q}_r \oplus \mathfrak{q}_s$ be the decomposition of $\mathfrak{q}$ into the Riemannian part $\mathfrak{q}_r$ and its orthogonal complement which is ncc. We write $p_s \colon \mathfrak{q} \to \mathfrak{q}_s$ for the projection with kernel $\mathfrak{q}_r$, which is the fixed point projection for the compact group $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_k \oplus \mathfrak{h}_r)$. Then every $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant closed convex cone $C \subseteq \mathfrak{q}$ satisfies

$$p_s(C) = C \cap \mathfrak{q}_s =: C_s \quad \text{and} \quad C_s^\circ = C^\circ \cap \mathfrak{q}_s \tag{17}$$

(Proposition B.4).

Corollary 4.4. *Let $(\mathfrak{g}, \tau)$ be a reductive symmetric Lie algebra and write it as $\mathfrak{g} = \mathfrak{g}_k \oplus \mathfrak{g}_r \oplus \mathfrak{g}_s$, where $\mathfrak{g}_k$ is the sum of all compact ideals of $\mathfrak{g}$ contained in $\mathfrak{h} = \mathfrak{g}^\tau$, $\mathfrak{g}_r$ is the sum of all Riemannian summands on which τ is a Cartan involution, and $\mathfrak{g}_s$ is the sum of all non-Riemannian irreducible summands. Then $\mathfrak{g}$ is non-compactly causal if and only if $\mathfrak{g}_s \neq \{0\}$ and $\mathfrak{g}_s$ is a sum of irreducible non-compactly causal symmetric Lie algebras.*

Proof. If $\mathfrak{g}_s$ is non-zero and non-compactly causal, then the proof of Theorem 4.2 implies that $(\mathfrak{g}, \tau)$ is non-compactly causal. If, conversely, this is the case, then Remark 4.3 shows that $\mathfrak{q}_s \neq \{0\}$ and that $(\mathfrak{g}_s, \tau)$ is non-compactly causal. ∎

Theorem 4.5. (Uniqueness of the causal Euler elements)   *Let $(\mathfrak{g}, \tau, C)$ be a semisimple ncc symmetric Lie algebra for which all ideals of $\mathfrak{g}$ contained in $\mathfrak{h}$ are compact, and θ a Cartan involution commuting with τ. Then the following assertions hold:*

(a) *$C_s^\circ \cap \mathfrak{q_p}$ contains a unique Euler element h and $\tau = \tau_h \theta$.*

(b) *$\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ acts transitively on $C_s^\circ \cap \mathcal{E}(\mathfrak{g})$.*

(c) *For every Euler element $h \in C_s^\circ$, the involution $\tau\tau_h$ is Cartan.*

Proof.   (a) We write $(\mathfrak{g}, \tau)$ as a direct sum of irreducible symmetric Lie algebras $(\mathfrak{g}_j, \tau_j)$, $j = 1, \ldots, n$ and a compact ideal $\mathfrak{g}_k$ of $\mathfrak{g}$ contained in $\mathfrak{h}$ (cf. [41, Prop. 2.14]). Let $x \in C$ be such that all components $x_j \in \mathfrak{q}_j = \mathfrak{q} \cap \mathfrak{g}_j$ are non-zero. Then

$$\mathcal{O}_x := \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})x = \mathcal{O}_{x_1} + \cdots + \mathcal{O}_{x_n} \subseteq C$$

implies that all the orbits $\mathcal{O}_{x_j}$ are contained in $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_j)$-invariant closed convex subsets not containing affine lines. If $\mathcal{O}_{x_j}$ is bounded, then $(\mathfrak{g}_j, \tau_j)$ is Riemannian, and if it is unbounded, $(\mathfrak{g}_j, \tau_j)$ is causal. Moreover, the fact that the projection of C° to $\mathfrak{q}_j$ consists of hyperbolic elements implies that $(\mathfrak{g}_j, \tau_j)$ is non-compactly causal. We enumerate the simple ideals in such a way that $(\mathfrak{g}_j, \tau_j)$ is Riemannian for $j \leq m$ and causal for $j > m$.

Proposition B.4 implies that the action of the compact group $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_\mathfrak{k})$ on C° has a fixed point $x = \sum_{j=m+1}^{n} x_j \in C^\circ$ with x_j contained in $\mathfrak{q}_{\mathfrak{p},j}$ (Lemma 3.12(a)). By Lemma 3.12, there exist $\lambda_j > 0$ for $j = m+1, \ldots, n$, such that $h_j := \lambda x_j$ is an Euler element in $\mathfrak{g}_j$. Then

$$h := \sum_{j=m+1}^{n} h_j \in \sum_{j=m+1}^{n} C_{\mathfrak{q}_j}^{\min}(h_j)^\circ \subseteq C_s^\circ$$

(see (14) and (17) for the last inclusion) is a causal Euler element in $\mathfrak{g}$. Further

$$\mathfrak{z}_{\mathfrak{g}}(h) = \sum_{j=1}^{m} \mathfrak{g}_j + \sum_{j=m+1}^{n} \mathfrak{z}_{\mathfrak{g}_j}(x_j) = \mathfrak{h}_\mathfrak{k} \oplus \mathfrak{q_p} \tag{18}$$

implies $\tau = \tau_h \theta$.

To verify the uniqueness of $h \in \mathfrak{q_p} \cap C_s^\circ$, let $h_1 \in \mathfrak{q_p} \cap C_s^\circ$ be an Euler element. We pick $\mathfrak{a} \subseteq \mathfrak{q_p}$ maximal abelian containing h_1 and observe that $h \in \mathfrak{a}$ follows from $[h, \mathfrak{q_p}] = \{0\}$. We choose a positive system $\Sigma^+ \subseteq \Sigma(\mathfrak{g}, \mathfrak{a})$ such that

$$C \cap \mathfrak{a} \subseteq C_{\mathfrak{a}}^{\max} := C_{\mathfrak{q}}^{\max} \cap \mathfrak{a} = \{x \in \mathfrak{a} \colon (\forall \alpha \in \Sigma_1)\ \alpha(x) \geq 0\}$$

([24, Thm. VI.6]). As all positive non-compact roots are positive on the interior of $C_{\mathfrak{a}}^{\max}$, we must have $\alpha(h_1) = 1$ for every $\alpha \in \Sigma_1$. Next we observe that Σ_1 spans $(\mathfrak{a} \cap \mathfrak{q}_s)^*$ ([24, §V]), so that we obtain $h = h_1$. Therefore h is the only Euler element in $\mathfrak{q_p} \cap C_s^\circ$.

(b) If $h_1 \in C_s^\circ$ is an Euler element, then it is in particular hyperbolic, hence conjugate under $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ to an element $h_2 \in \mathfrak{q_p}$ ([24, Cor. II.9]). Then $h_2 \in \mathfrak{q_p} \cap C_s^\circ$ is an Euler element, hence equal to the Euler element h from (a). This implies that $h_1 \in \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})h$, and thus $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ acts transitively on $C_s^\circ \cap \mathcal{E}(\mathfrak{g})$.

(c) The assertion holds for the unique Euler element $h \in C_s^\circ \cap \mathfrak{q_p}$ by (a). If $h_1 \in C_s^\circ$ is another Euler element, then (b) implies the existence of $\phi \in \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ with $h_1 = \phi(h)$. Then $\tau\tau_{h_1} = \tau\phi\tau_h\phi^{-1} = \phi\tau\tau_h\phi^{-1} = \phi\theta\phi^{-1}$ is a Cartan involution.   ∎

Assertion (b) in Theorem 4.5 has important consequences for the possible choices of an open subgroup $H \subseteq \mathrm{Inn}(\mathfrak{g})^\tau$ preserving the cone C.

Corollary 4.6. *If $(\mathfrak{g}, \tau, C)$ is a semisimple ncc symmetric Lie algebra for which $\mathfrak{h}$ contains no non-compact ideal of $\mathfrak{g}$, $h \in C_s^\circ$ is a causal Euler element, and $H \subseteq \mathrm{Inn}(\mathfrak{g})^\tau$ is an open subgroup preserving C, then the following assertions hold:*

(a) *$H = H_e H^h$, i.e., every connected component of H meets H^h, which is equivalent to $H.h$ being connected.*

(b) *H is closed and H^h is a maximal compact subgroup of H.*

(c) *$H^h = H^{\tau_h}$ and τ_h induces a Cartan involution on H.*

(d) *τ induces a Cartan involution on H^h for which $(H_e^h)^\tau = e^{\mathrm{ad}\, \mathfrak{h}_{\mathfrak{k}}}$ is connected.*

Proof. (a) From Theorem 4.5(b) and $H_e = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$, we derive immediately that $H.h \subseteq C_s^\circ \cap \mathcal{E}(\mathfrak{g}) = H_e.h$, so that $H = H_e H^h$.

(b) Since $\mathrm{Aut}(\mathfrak{g})^\tau$ is an algebraic group, it is closed and has only finitely many connected components. It contains H as an open subgroup, so that H is also closed. Further, H^h fixes the element h in the interior of C, so that H^h acts on $\mathfrak{q}$ as a relatively compact group. Next we use that $[\mathfrak{q}, \mathfrak{q}] + \mathfrak{q} \trianglelefteq \mathfrak{g}$ is an ideal of $\mathfrak{g}$ and since $\mathfrak{h}$ contains no non-compact ideal, it follows that $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{h}_k$, where $\mathfrak{h}_k \subseteq \mathfrak{h} \cap \mathfrak{k}$ is the sum of all compact ideals of $\mathfrak{g}$ contained in $\mathfrak{h}$. Therefore the closed subgroup H^h of $\mathrm{Aut}(\mathfrak{g})$ is compact. Its maximality in H now follows from the polar decomposition $H = H^h \exp(\mathfrak{h}_{\mathfrak{p}})$ and (a). Note that Riemannian components correspond to compact ideals of $\mathfrak{h}$, and the corresponding subgroups of $\mathrm{Inn}(\mathfrak{g})$ are compact, commute with τ and fix h.

(c) From $H^{\tau_h}.h \subseteq C_s^\circ \cap \mathfrak{q}_{\mathfrak{p}}$ and Theorem 4.5(a), it follows that H^{τ_h} fixes h, so that $H^{\tau_h} \subseteq H^h$. Conversely, every $h \in H^h \subseteq \mathrm{Aut}(\mathfrak{g})$ commutes with $\mathrm{ad}\, h$, hence also with τ_h, and thus $H^h = H^{\tau_h}$. In view of (b), τ_h induces a Cartan involution on H.

(d) On $H^h = H^{\tau_h}$ the involution $\tau = \tau_h \theta$ acts like θ. Restricting to the identity component H_e^h, we obtain (d) because the group of fixed points of a Cartan involution is connected. ∎

Corollary 4.7. (Characterization of causal Euler elements) *Let $(\mathfrak{g}, \tau)$ be a semisimple symmetric Lie algebra for which all ideals of $\mathfrak{g}$ contained in $\mathfrak{h}$ are compact and $h \in \mathcal{E}(\mathfrak{g})$ (cf. (16)). Consider the following assertions:* [3]

(a) *h is causal, i.e., there exists a pointed generating $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant closed convex cone $C \subseteq \mathfrak{q}$ with $h \in C^\circ$.*

(b) *$\tau \tau_h$ is a Cartan involution.*

(c) *ih is an H-element in $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$.*

Then (a) $\Leftarrow$ (b) $\Leftrightarrow$ (c) and if $h \in \mathfrak{q}_s$, then (a),(b),(c) are equivalent.

Proof. (a) $\Rightarrow$ (b): If $h \in \mathfrak{q}_s$, then this implication follows from Theorem 4.5(c). (b) $\Rightarrow$ (a) follows from Theorem 4.2 and the fact that $\tau \tau_h(h) = \tau(h) = -h$. (b) $\Rightarrow$ (c): Suppose that $\theta := \tau \tau_h$ is a Cartan involution of $\mathfrak{g}$. As $\theta(h) = -h$, Lemma 3.10 implies that ih is an H-element of $\mathfrak{g}^c$. (c) $\Rightarrow$ (b) follows immediately from Lemma 3.10 because $(\mathfrak{g}^c, \tau^c, ih) \in \mathcal{B}$. ∎

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[3] If h is not contained in $\mathfrak{q}_s$, then (b) does not follow from (a). Remark 4.8(b) provides an example for $h \in (C^\lambda)^\circ$ with $\lambda > 1$.

Remark 4.8. In Theorem 4.5(b) it is crucial to restrict to the cone $C_s = C \cap \mathfrak{q}_s$ in $\mathfrak{q}_s$.

(a) First we observe that, for a Riemannian symmetric Lie algebra $(\mathfrak{g}, \theta)$, where θ is a Cartan involution, we have $\mathfrak{p} = \mathfrak{q}_{\mathfrak{p}}$. If $h \in \mathfrak{p}$ is an Euler element then $\tau_h \theta$ is never a Cartan involution. The hyperbolic space $\mathbb{H}^d \cong \mathrm{SO}_{1,d}(\mathbb{R}) / \mathrm{SO}_d(\mathbb{R})$ is an example showing that this situation does in fact occur.

Moreover, $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{k})$ need not act transitively on $\mathcal{E}(\mathfrak{g}) \cap \mathfrak{p}$. If the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ ($\mathfrak{a} \subseteq \mathfrak{p}$ maximal abelian) is of type $A_n, n \geq 2$, $D_n, n \geq 4$ or E_6, then there exists more than one conjugacy class of Euler elements in $\mathfrak{p}$ (cf. Theorem 2.4).

(b) (cf. also Example 4.22 below) We consider the example

$$\mathfrak{g} = (\mathfrak{sl}_2(\mathbb{R}), \theta) \oplus (\mathfrak{sl}_2(\mathbb{R}), \tau_h), \quad \text{where} \quad h = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is ncc. For $G := \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ and $H := \mathrm{SO}_2(\mathbb{R}) \times \mathrm{SO}_{1,1}(\mathbb{R})$, the symmetric space

$$M = G/H \cong \mathbb{H}^2 \times \mathrm{dS}^2$$

is Lorentzian. A maximal abelian hyperbolic subspace is

$$\mathfrak{a} := \mathbb{R}h \oplus \mathbb{R}h \subseteq \mathfrak{q} \cong \mathbb{R}^4,$$

where $\qquad C_{\mathfrak{a}}^{\min} = \{0\} \oplus [0, \infty)h, \qquad C_{\mathfrak{a}}^{\max} = \mathbb{R}h \oplus [0, \infty)h,$

and the Riemannian Weyl group $\mathcal{W}_c = \{\mathbf{1}, \sigma\}$ (generated by the reflections corresponding to "compact" roots; see [24]) acts by $\sigma(x, y) = (-x, y)$. For any $\lambda > 0$, we obtain a Weyl group invariant cone $C_{\mathfrak{a}}^{\lambda}$ between the minimal and the maximal cone by

$$C_{\mathfrak{a}}^{\lambda} := \mathrm{cone}((\lambda, 1), (-\lambda, 1)).$$

The cone $C_{\mathfrak{a}}^{\lambda}$ contains a unique Euler element $(0, h)$ if and only if $\lambda < 1$. For $\lambda \geq 1$, it also contains the Euler elements $(\pm h, h)$. If $C^{\lambda} \subseteq \mathfrak{q}$ is the corresponding $\mathrm{Ad}(H)$-invariant cone with $C^{\lambda} \cap \mathfrak{a} = C_{\mathfrak{a}}^{\lambda}$, then the conclusion of Theorem 4.5(a)–(c) fail for $\lambda > 1$ and the cone C^{λ} if it is not first intersected with $\mathfrak{q}_s$.

Remark 4.9. Suppose that $h \in \mathcal{E}(\mathfrak{g})$ and θ is a Cartan involution with $\theta(h) = -h$ and $\tau = \tau_h \theta$. Then

$$\mathfrak{g}^{\tau_h} = \mathfrak{g}^{\tau\theta} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{q}_{\mathfrak{p}}.$$

Let G be a connected Lie group with semisimple Lie algebra $\mathfrak{g}$ on which τ_h^G, hence also τ^G, exist (cf. the discussion in [40, Rem. 2.12]). From the polar decomposition $G = K \exp(\mathfrak{p})$ we derive for the centralizer of h the decomposition

$$G^h = K^h \exp(\mathfrak{q}_{\mathfrak{p}}). \tag{19}$$

We also obtain $\qquad G^{\tau^G} = K^{\tau^G} \exp(\mathfrak{h}_{\mathfrak{p}}) = K^{\tau_h^G} \exp(\mathfrak{h}_{\mathfrak{p}}). \tag{20}$

In general $K^h \neq K^{\tau_h^G}$, as the simply connected covering group $G = \widetilde{\mathrm{SL}}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ shows ([40, Rem. 5.3]). If $Z(G) = \{e\}$, i.e., $G \cong \mathrm{Ad}(G) = \mathrm{Inn}(\mathfrak{g})$, then $G^h \subseteq G^{\tau_h^G}$ implies $K^h \subseteq K^{\tau_h^G} = K^{\tau^G}$.

Remark 4.10. For a semisimple symmetric Lie algebra $(\mathfrak{g}, \tau)$ for which all ideals of $\mathfrak{g}$ contained in $\mathfrak{h}$ are compact, Corollary 4.7 provides a map

$$\Gamma \colon \mathcal{E}_c(\mathfrak{q}_s) \to \mathrm{Cart}(\mathfrak{g})^{\tau}, \quad h \mapsto \tau\tau_h$$

to the set $\mathrm{Cart}(\mathfrak{g})^{\tau}$ of Cartan involutions on $\mathfrak{g}$ commuting with τ.

For a given Cartan involution θ commuting with τ, and $K := \mathrm{Inn}(\mathfrak{g})^\theta$, the transitivity of the action of $G := \mathrm{Inn}(\mathfrak{g})$ on $\mathrm{Cart}(\mathfrak{g})$ ([15, Cor. 13.2.13]) implies that

$$\mathrm{Cart}(\mathfrak{g}) \cong G/K \quad \text{and} \quad \mathrm{Cart}(\mathfrak{g})^\tau \cong \mathrm{Exp}_{eK}(\mathfrak{h}_\mathfrak{p}) = \mathrm{Inn}_\mathfrak{g}(\mathfrak{h}).\theta.$$

The equivariance of Γ with respect to $\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})$ thus shows that Γ maps any $\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})$-orbit in $\mathcal{E}_c(\mathfrak{q}_s)$ surjectively onto $\mathrm{Cart}(\mathfrak{g})^\tau$.

If $\Gamma(h) = \Gamma(h') = \theta$ for $h, h' \in \mathcal{E}_c(\mathfrak{q})$, then $h, h' \in \mathfrak{q}_\mathfrak{p}$. If $\mathfrak{g}$ is simple, then the following lemma implies that $h' \in \{\pm h\}$.

Lemma 4.11. *Let $(\mathfrak{g}, \tau)$ be a simple symmetric Lie algebra, $h \in \mathcal{E}_c(\mathfrak{q})$ and $\theta = \tau\tau_h$. Then the following assertions hold:*

(a) $\mathcal{E}_c(\mathfrak{q}) \cap \mathfrak{q}_\mathfrak{p} = \{\pm h\}$.

(b) $\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})$ *has two orbits in the set $\mathcal{E}_c(\mathfrak{q})$ of causal Euler elements in $\mathfrak{q}$. They are represented by $\pm h$ for any $h \in \mathcal{E}_c(\mathfrak{q})$.*

Proof.   (a) Let $\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q}$ be the dual symmetric Lie algebra. Then $\mathfrak{k}^c = \mathfrak{h}_\mathfrak{k} + i\mathfrak{q}_\mathfrak{p}$ is maximally compact in $\mathfrak{g}^c$ and $z := ih \in \mathfrak{z}(\mathfrak{k}^c)$ is an H-element with $\ker(\mathrm{ad}\, z) = \mathfrak{k}^c$.

We distinguish two cases according to $(\mathfrak{g}, \tau)$ being of complex type or not.

The complex case: If $\mathfrak{g}$ is complex and τ antilinear, then $\mathfrak{g} \cong \mathfrak{h}_\mathbb{C}$, where $\mathfrak{h}$ is simple hermitian. Then $\mathfrak{q} = i\mathfrak{h}$, and $\mathcal{E}_c(\mathfrak{q}) \cap \mathfrak{q}_\mathfrak{p} = \mathcal{E}_c(\mathfrak{q}) \cap i\mathfrak{h}_\mathfrak{k}$ consists of all elements of the form $h = iz$, where $z \in \mathfrak{k}_\mathfrak{q}^c = i\mathfrak{q}_\mathfrak{p}$ is an H-element. Hence the assertion follows from the fact that $\pm z$ are the only H-elements of $\mathfrak{g}^c$ contained in $\mathfrak{k}^c$ (Lemma 3.9).

The real case: If $\mathfrak{g}$ is not complex, then $\mathfrak{g}^c$ is a simple hermitian Lie algebra with $\mathfrak{k}^c = \mathfrak{h}_\mathfrak{k} + i\mathfrak{q}_\mathfrak{p}$ maximal compactly embedded. If $h, h' \in \mathfrak{q}_\mathfrak{p}$ are causal Euler elements, then $ih, ih' \in \mathfrak{k}^c$ are H-elements, and Lemma 3.9 implies that $h' \in \{\pm h\}$.

(b) Let $h \in \mathfrak{q}$ be a causal Euler element and consider the corresponding Cartan involution $\theta = \tau\tau_h$. Then $h \in \mathfrak{q}_\mathfrak{p}$ and any other causal Euler element $h_1 \in \mathfrak{q}$ is hyperbolic, hence conjugate under $\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})$ to an element of $\mathfrak{q}_\mathfrak{p}$ ([24, Cor. II.9]), and therefore by (a) to h or $-h$.   ∎

Remark 4.12.   For every irreducible ncc symmetric Lie algebra $(\mathfrak{g}, \tau, C)$ and $h \in \mathcal{E}(\mathfrak{g})$, the intersection $\mathcal{O}_h \cap C^\circ$ is either empty or a single orbit of $\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})$ (Theorem 4.5(b)). Therefore at most one $\mathrm{Inn}(\mathfrak{g})$-orbit in $\mathcal{E}(\mathfrak{g})$ intersects C°. In general $\mathrm{Inn}(\mathfrak{g})$ does not act transitively on $\mathcal{E}(\mathfrak{g})$. A typical example is $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ with $n - 1$ orbits of $\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})$ in $\mathcal{O}_h \cap \mathfrak{q}$ (cf. Example 4.22 below; see also [33, Rem. 4.15]).

Proposition 4.13.   (Classification of $\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})$-orbits in $\mathcal{O}_h \cap \mathfrak{q}$)   *Let $(\mathfrak{g}, \tau)$ be a reductive symmetric Lie algebra with $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{q}$. Further, let $h \in \mathfrak{q}_\mathfrak{p}$ be a causal Euler element with $\theta = \tau\tau_h$ and $\mathfrak{a} \subseteq \mathfrak{q}_\mathfrak{p}$ maximal abelian. Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ denote the corresponding set of restricted roots, $\Sigma_0 := \{\alpha \in \Sigma : \alpha(h) = 0\}$, and $\mathcal{W}_0 \subseteq \mathcal{W}$ the corresponding Weyl group. Then the map*

$$\mathcal{W}_0 \backslash \mathcal{W} / \mathcal{W}_0 \to (\mathcal{O}_h \cap \mathfrak{q})/\mathrm{Inn}_\mathfrak{g}(\mathfrak{h}), \quad \mathcal{W}_0 w \mathcal{W}_0 \mapsto \mathrm{Inn}_\mathfrak{g}(\mathfrak{h}).wh$$

is a bijection from the set of $\mathcal{W}_0$-double cosets in $\mathcal{W}$ to the set of orbits of the group $\mathrm{Inn}_\mathfrak{g}(\mathfrak{h})$ in $\mathcal{O}_h \cap \mathfrak{q}$.

Proof.   As $\mathfrak{z}_{\mathfrak{g}}(h) \cap \mathfrak{p} = \mathfrak{q}_{\mathfrak{p}}$, the subspace $\mathfrak{a}$ is also maximal abelian in $\mathfrak{p}$. Clearly, $\mathfrak{a}$ contains h. Then $\mathcal{O}_h \cap \mathfrak{a} = \mathcal{W}.h$ by [24, Thm. III.10] and elementary Coxeter theory implies that the stabilizer of h in $\mathcal{W}$ is $\mathcal{W}^h = \mathcal{W}_0$ ([36, Prop. V.2.7]). Every $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-orbit in $\mathcal{O}_h \cap \mathfrak{q}$ intersects $\mathfrak{a}$ ([24, Cor. II.9], [36, Prop. VII.2.10]) and for $x \in \mathfrak{a}$ we have

$$\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})x \cap \mathfrak{a} = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_{\mathfrak{k}})x \cap \mathfrak{a} = \mathcal{W}_0 x.$$

Therefore $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})\backslash(\mathcal{O}_h \cap \mathfrak{q}) \cong \mathcal{W}_0\backslash\mathcal{W}h \cong \mathcal{W}_0\backslash\mathcal{W}/\mathcal{W}_0.$ ∎

Remark 4.14.   Let $H := \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}) \supseteq H_K = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_{\mathfrak{k}})$ and $K := \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{k})$. We show that, if H_K fixes the causal Euler element h, then

$$H_K.x \cap \mathfrak{a} = \mathcal{W}_0 x \quad \text{for} \quad x \in \mathfrak{a}.$$

So let $x \in \mathfrak{a} \subseteq \mathfrak{q}_{\mathfrak{p}}$ and $g \in H_K \subseteq K^h$ with $x' := g.x \in \mathfrak{a}$. Then

$$\mathfrak{a}' := g.\mathfrak{a} \subseteq \mathfrak{q}_{\mathfrak{p}} \cap \mathfrak{z}_{\mathfrak{g}}(x')$$

is maximal abelian in $\mathfrak{q}_{\mathfrak{p}}$, hence in particular in $\mathfrak{q}_{\mathfrak{p}} \cap \mathfrak{z}_{\mathfrak{g}}(x')$, which also contains $\mathfrak{a}$. Therefore [24, Thm. III.3] implies the existence of a $g_1 \in \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{z}_{\mathfrak{g}}(x'))$ with $g_1.\mathfrak{a}' = \mathfrak{a}$. Then $g_2 := g_1 g \in H_K$ satisfies $x' = g_1.x' = g_2.x$ and $g_2.\mathfrak{a} = \mathfrak{a}$. Therefore $g_2 \in N_{H_K}(\mathfrak{a}) \subseteq N_K(\mathfrak{a})$ (where $N_B(\mathfrak{a}) = \{g \in B : \mathrm{Ad}(g)\mathfrak{a} = \mathfrak{a}\}$ is the normalizer of $\mathfrak{a}$ in B) acts on $\mathfrak{a}$ as a Weyl group element $w \in \mathcal{W}$ ([24, Def. III.9]). As H_K fixes h by assumption, we have $w \in \mathcal{W}^h = \mathcal{W}_0$ ([36, Prop. V.2.7]).

Remark 4.15.   (Supplement to Proposition 4.13) Let $\mathfrak{g}$ be a simple real Lie algebra. Euler elements in $\mathfrak{g}$ are classified by their representatives in a closed Weyl chamber $\mathfrak{a}_+ \subseteq \mathfrak{a}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan involution, and $\mathfrak{a} \subseteq \mathfrak{p}$ is maximal abelian. Theorem 2.4 provides a concrete list for all types of root systems $\Sigma(\mathfrak{g}, \mathfrak{a})$ for which Euler elements exist. For $r := \dim \mathfrak{a}$, the representatives h_j are labeled by a subset of $\{1, \ldots, r\}$. Any such Euler element defines an involution $\tau_j := \theta\tau_{h_j}$ for which $h_j \in \mathfrak{a} \subseteq \mathfrak{q}_j = \mathfrak{g}^{-\tau_j}$. The stabilizer group $\mathcal{W}^{h_j} \subseteq \mathcal{W} = \mathcal{W}(\mathfrak{g}, \mathfrak{a})$ is generated by the reflections fixing h_j ([36, Prop. V.2.7]). Now Proposition 4.13 applies to this situation and identifies the set of $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_j)$-orbits in $\mathcal{O}_{h_j} \cap \mathfrak{q}_j$ with the $\mathcal{W}^{h_j}$-double cosets in $\mathcal{W}$.

Example 4.16.   For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$, $n = p + q$, $p \leq q$, the Euler element

$$h_p := \frac{1}{p+q}\begin{pmatrix} q\mathbf{1}_p & 0 \\ 0 & -p\mathbf{1}_q \end{pmatrix},$$

and the Cartan involution $\theta(x) = -x^\top$, we obtain $\tau(x) = -\tau_h(x)^\top$, so that we may take

$$\mathfrak{a} = \Big\{ \mathrm{diag}(x_1, \ldots, x_n) : \sum_j x_j = 0 \Big\}.$$

Now $\mathcal{W} \cong S_n \supseteq \mathcal{W}_0 \cong S_p \times S_q,$

and the orbit space $\mathcal{W}/\mathcal{W}_0$ corresponds to the set of p-element subsets $F_p \subseteq \{1, \ldots, n\}$. The orbits of $\mathcal{W}_0$ on this set are represented by $F_p \cap \{1, \ldots, p\}$. Parametrized by the cardinality of this intersection, we have $p + 1$ orbits, and these correspond to $p + 1$ orbits of $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ in $\mathcal{O}_h \cap \mathfrak{q}$. They are obtained by permuting the entries of the diagonal matrix h_p.

4.2. On the causality of G/G^τ

In this section we show that the minimal irreducible symmetric space $\mathrm{Inn}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})^\tau$ is a causal symmetric space for the triple $(\mathfrak{g}, \tau, C)$ if and only if the corresponding causal Euler element is not symmetric; otherwise we have to pass to a two-fold covering to obtain a causal space.

Definition 4.17.   Suppose that $(\mathfrak{g}, \tau)$ is a semisimple symmetric Lie algebra.

(a) We call the corresponding symmetric space

$$\mathcal{M}(\mathfrak{g}, \tau) := \mathrm{Inn}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})^\tau$$

a *minimal symmetric space* associated to the symmetric Lie algebra $(\mathfrak{g}, \tau)$. All other connected symmetric spaces $M = G/H$ corresponding to $(\mathfrak{g}, \tau)$ are equivariant coverings of $\mathcal{M}(\mathfrak{g}, \tau)$ by the map

$$gH \mapsto \mathrm{Ad}(g)\tau\,\mathrm{Ad}(g)^{-1} \in \mathrm{Ad}(G).\tau \cong \mathcal{M}(\mathfrak{g}, \tau).$$

(b) If $(\mathfrak{g}, \tau, C)$ is a causal semisimple symmetric Lie algebra, $G = \mathrm{Inn}(\mathfrak{g})$, and

$$H_C := \{g \in G^\tau :\ \mathrm{Ad}(g)C = C\},$$

then we call G/H_C the *minimal causal symmetric space* associated to $(\mathfrak{g}, \tau, C)$.

Minimal symmetric spaces corresponding to causal symmetric Lie algebras $(\mathfrak{g}, \tau, C)$ are not always causal because G^τ does not always leave a pointed generating cone $C \subseteq \mathfrak{q}$ invariant, which is equivalent to $\mathrm{cone}(G^\tau.h)$ being pointed for a causal Euler element $h \in \mathcal{E}_c(\mathfrak{q})$. The following proposition and its corollary make this requirement easy to check.

Proposition 4.18.   *Suppose that $(\mathfrak{g}, \tau)$ is simple ncc and $h \in \mathcal{E}_c(\mathfrak{q})$ a causal Euler element, G a connected Lie group with Lie algebra $\mathfrak{g}$ to which τ integrates, and $H \subseteq G^\tau$ an open subgroup. Then two mutually exclusive cases occur:*

(a) $\mathrm{Ad}(H)h = \mathrm{Ad}(H_e)h$ *and G/H is causal.*

(b) $-h \in \mathrm{Ad}(H)h$ *and G/H is not causal.*

Proof.   Clearly, $\mathrm{Ad}(H)h \subseteq \mathcal{E}_c(\mathfrak{g}) \cap \mathfrak{q}$. In view of Lemma 4.11(b), we either have $\mathrm{Ad}(H)h = \mathrm{Ad}(H_e)h$ or $-h \in \mathrm{Ad}(H)h$. In the first case G/H is causal and in the second case it is not. ∎

The following theorem identifies the minimal causal symmetric space in terms of the symmetry property of the corresponding Euler element.

Theorem 4.19.   *Let $(\mathfrak{g}, \tau)$ be simple ncc with causal Euler element $h \in \mathcal{E}_c(\mathfrak{q})$ and $G = \mathrm{Inn}(\mathfrak{g})$. Then $\mathcal{M}(\mathfrak{g}, \tau) \cong G/G^\tau$ is causal if and only if h is not symmetric, i.e., $-h \notin \mathcal{O}_h$.*

Proof.   Let $H := \mathrm{Inn}(\mathfrak{g})^\tau$, so that $G/H \cong \mathcal{M}(\mathfrak{g}, \tau)$ and consider the pointed generating closed convex cone $C \subseteq \mathfrak{q}$ generated by $H_e.h$. If h is symmetric, then $-h = g.h$ for some $g \in G$. Proposition 4.13 implies the existence of a Weyl group

element $w \in \mathcal{W}$ with $w.h = -h$. As every $w \in \mathcal{W}$ is a restriction of $\mathrm{Ad}(k)$ to $\mathfrak{a}$ for some $k \in K = G^\theta$ (cf. [24, Def. III.9]), we may assume that $g \in G^\theta$. Further $g.h = -h$ implies that g commutes with τ_h, hence also with τ, i.e., $g \in G^\tau = H$. Therefore Proposition 4.18 implies that $\mathcal{M}(\mathfrak{g}, \tau)$ is not causal.

If, conversely, $\mathcal{M}(\mathfrak{g}, \tau)$ is not causal, then Proposition 4.18 implies immediately that $-h \in H.h \subseteq \mathcal{O}_h$, so that h is symmetric. ∎

Remark 4.20. (a) Suppose that $(\mathfrak{g}, \tau)$ is an irreducible ncc symmetric Lie algebra and h an Euler element with $\theta(h) = -h$ and $\tau = \tau_h \theta$. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ on which τ integrates to an involution τ^G and let $H \subseteq G^{\tau_G}$ be an open θ-invariant subgroup, so that

$$H = H_K \exp(\mathfrak{h}_\mathfrak{p}) \quad \text{for} \quad H_K = H^\theta = H \cap K,$$

and Lemma 4.11 implies that

$$H_K.h \subseteq \mathcal{E}_c(\mathfrak{q}) \cap \mathfrak{q}_\mathfrak{p} = \{\pm h\}.$$

Therefore the causality of G/H is equivalent to $H_K \subseteq G^h$ (cf. Proposition 4.18).

(b) If $G = \mathrm{Inn}(\mathfrak{g})$, then $K^h \subseteq K^{\tau_h} = K^\tau \subseteq G^\tau$, so that

$$H_{C^{\max}} := K^h e^{\mathrm{ad}\, \mathfrak{h}_\mathfrak{p}} \subseteq G^\tau$$

is the maximal open subgroup $H \subseteq G^\tau$ for which $G/H_{C^{\max}}$ is causal, i.e., $G/H_{C^{\max}}$ is a minimal causal symmetric space associated to $(\mathfrak{g}, \tau, C^{\max})$.

In view of Theorem 4.19, $H_{C^{\max}} = G^\tau$ if and only if h is not symmetric. We shall see in Theorem 5.4 below that, if $\mathfrak{h}$ contains Euler elements, then h is symmetric. Therefore $H_{C^{\max}}$ is an index 2-subgroup of $G^\tau = \mathrm{Inn}(\mathfrak{g})^\tau$.

(c) (Symmetric Euler element and de Sitter space) The symmetry group of de Sitter spacetime is $G = \mathrm{SO}_{1,d}(\mathbb{R})_e$. On its Lie algebra $\theta(x) = -x^\top$ is a Cartan involution. Let $h = h_{W_1} \in \mathfrak{g}$ be the Euler element such that

$$e^{th_{W_1}} p = (\cosh(t)p_0 + \sinh(t)p_1, \sinh(t)p_0 + \cosh(t)p_1, p_2, \ldots, p_d)$$

and the corresponding wedge reflection is implemented by the linear map

$$\tau_h = \mathrm{diag}(-1, -1, 1, 1, \ldots, 1) \in \mathrm{SO}_{1,d}(\mathbb{R}).$$

Then $$G^\tau \simeq G_{\mathbf{e}_1} \rtimes \{\mathbf{1}, r_{12}\} \simeq \mathrm{SO}_{1,d-1}(\mathbb{R})_e \rtimes \mathbb{Z}_2,$$

where $r_{12} = \mathrm{diag}(1, -1, -1, 1, \ldots, 1)$ is the point reflection in the x_1-x_2-plane. The orbit map of $\mathbf{e}_1 = (0, 1, 0, \ldots 0)$ induces a diffeomorphism $G/G_{\mathbf{e}_1} \to \mathrm{dS}^d$. Note that $G_{\mathbf{e}_1}$ is the identity component G_e^τ in G^τ. The involution r_{12} is not contained in G_e^τ. It normalizes $G_{\mathbf{e}_1}$, so that it acts by right multiplication on $G/G_{\mathbf{e}_1}$. We thus obtain a G-equivariant involution $\phi(gG_{\mathbf{e}_1}) := gr_{1,2}G_{\mathbf{e}_1}$ on $G/G_{\mathbf{e}_1}$. The corresponding involution ψ on $\mathrm{dS}^d \cong G/G_{\mathbf{e}_1}$ is G-equivariant and maps $\mathbf{e}_1$ to $r_{12}(\mathbf{e}_1) = -\mathbf{e}_1$, hence satisfies $\psi(x) = -x$ for all $x \in \mathrm{dS}^d$. As a consequence, the symmetric space G/G^τ is the projective de Sitter space $\mathrm{dS}^d/\{\pm\mathbf{1}\}$. This is not a causal space because multiplication by -1 reverses the causal structure on dS^d.

4.3. Classifying non-compactly causal structures by Euler elements

Let G be a connected Lie group with *simple* Lie algebra $\mathfrak{g}$. We now describe a bijection between the set $\mathcal{E}(\mathfrak{g})/G$ of G-orbits in $\mathcal{E}(\mathfrak{g})$ and the isomorphism classes of non-compactly causal symmetric Lie algebras $(\mathfrak{g}, \tau, C)$, where $C = C_{\mathfrak{q}}^{\min} \subseteq \mathfrak{q}$ is a minimal $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant pointed closed convex cone. We write

$$\mathcal{NCC}^{\min}(\mathfrak{g})$$

for the $\mathrm{Inn}(\mathfrak{g})$-orbits in the set of all pairs (τ, C) for which $(\mathfrak{g}, \tau, C)$ is an ncc symmetric Lie algebra and C is minimal.

Theorem 4.21. (Classification of ncc structures by Euler elements)
Let $\mathfrak{g}$ be simple. To $h \in \mathcal{E}(\mathfrak{g})$, we associate the ncc symmetric Lie algebra $(\mathfrak{g}, \tau, C^{\min}(h, \theta)) \in \mathcal{NCC}^{\min}(\mathfrak{g})$, where $\tau := \tau_h \theta$ for a Cartan involution θ satisfying $\theta(h) = -h$ and

$$C^{\min}(h, \theta) = \overline{\mathrm{cone}}(\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{g}^{\tau})h).$$

This assignment induces a bijection Ψ from the set $\mathcal{E}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ of adjoint orbits of Euler elements in $\mathfrak{g}$ onto $\mathcal{NCC}^{\min}(\mathfrak{g})$.

With the notation from Remark 4.15, we see in particular that, if $\mathfrak{a}_+ \subseteq \mathfrak{a}$ is a closed Weyl chamber (a fundamental domain for the $\mathcal{W}$-action), then

$$\mathcal{E}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g}) \cong (\mathcal{E}(\mathfrak{g}) \cap \mathfrak{a})/\mathcal{W} \cong \mathcal{E}(\mathfrak{g}) \cap \mathfrak{a}_+$$

is a finite set.

Proof. Theorem 4.2 implies that $(\mathfrak{g}, \tau, C^{\min}(h, \theta))$ is ncc with $h \in C^{\min}(h, \theta)^\circ$. If θ_1 is another Cartan involution with $\theta_1(h) = -h$, then $\theta_1\theta(h) = h$. Write $\theta_1 = e^{\mathrm{ad}\, x}\theta e^{-\mathrm{ad}\, x}$ with $\theta(x) = -x$ (cf. [15, Thm. 13.1.7]). Then

$$h = \theta_1\theta(h) = e^{\mathrm{ad}\, x}\theta e^{-\mathrm{ad}\, x}\theta(h) = e^{2\,\mathrm{ad}\, x}\theta\theta(h) = e^{2\,\mathrm{ad}\, x}h$$

implies $[x, h] = 0$ because $\mathrm{ad}\, x$ is diagonalizable. Therefore

$$\tau_1 := \tau_h\theta_1 = \tau_h e^{\mathrm{ad}\, x}\theta e^{-\mathrm{ad}\, x} = e^{\mathrm{ad}\, x}\tau e^{-\mathrm{ad}\, x},$$

and $$\qquad e^{\mathrm{ad}\, x} \colon (\mathfrak{g}, \tau, C^{\min}(h, \theta)) \to (\mathfrak{g}, \tau_1, C(h, \theta_1))$$

is an isomorphic of ncc symmetric Lie algebras fixing h. Therefore the isomorphism class of $(\mathfrak{g}, \tau, C^{\min}(h, \theta))$ does not depend on the choice of θ.

If $h_1 = g.h$, then $\theta_1 := g\theta g^{-1}$ is a Cartan involution of $\mathfrak{g}$ with $\theta_1(h_1) = -h_1$, so that

$$\tau_1 := \tau_{h_1}\theta_1 = g\tau g^{-1}$$

leads to an ncc symmetric Lie algebra $(\mathfrak{g}, \tau_1, C(h_1, \theta_1))$ with $C(h_1, \theta_1) = g.C^{\min}(h, \theta)$.

Clearly, $$\qquad g \colon (\mathfrak{g}, \tau, C^{\min}(h, \theta)) \to (\mathfrak{g}, \tau_1, C(h_1, \theta_1))$$

is an isomorphism of causal symmetric Lie algebras. This shows that the isomorphism class of $(\mathfrak{g}, \tau, C^{\min}(h, \theta))$ only depends on the orbit $\mathcal{O}_h \subseteq \mathcal{E}(\mathfrak{g})$.

If, conversely, $(\mathfrak{g}, \tau, C)$ is an ncc symmetric Lie algebra with C minimal, and θ a Cartan involution commuting with τ, then $C^\circ \cap \mathfrak{q}_{\mathfrak{p}}$ contains a causal Euler element h

by Theorem 4.5, and minimality implies $C = C^{\min}(h, \theta)$. Therefore Ψ is surjective. To see that it is also injective, we recall that any two Euler elements in C° are conjugate under $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$ (Theorem 4.5(b)), so that any two Euler elements defining isomorphic triples $(\mathfrak{g}, \tau, C)$ lie in the same $\mathrm{Inn}(\mathfrak{g})$-orbit. ∎

Example 4.22.   We consider the simple real Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$. For $1 \le p < n$ and $q := n - p$, we obtain an Euler element

$$h_p := \frac{1}{p+q} \begin{pmatrix} q\mathbf{1}_p & 0 \\ 0 & -p\mathbf{1}_q \end{pmatrix}$$

(cf. Example 4.16). Then

$$\tau_{h_p} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

and $\theta(x) = -x^\top$ is a Cartan involution with $\theta(h_p) = -h_p$. For $\tau = \tau_{h_p}\theta$ we then have

$$\tau(x) = -\tau_h(x^\top), \qquad \tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a^\top & c^\top \\ b^\top & -d^\top \end{pmatrix}.$$

Therefore $\mathfrak{h} = \mathfrak{so}_{p,q}(\mathbb{R})$ and

$$\mathfrak{q} = \left\{ \begin{pmatrix} a & b \\ -b^\top & d \end{pmatrix} : a^\top = a, d^\top = d, \mathrm{tr}(a) + \mathrm{tr}(d) = 0 \right\}.$$

A typical invariant cone in $\mathfrak{q}$ is

$$C = \left\{ \begin{pmatrix} a & b \\ -b^\top & d \end{pmatrix} \in \mathfrak{q} : \begin{pmatrix} a & b \\ b^\top & -d \end{pmatrix} \ge 0 \right\}.$$

It contains h_p in its interior.

Note that the subspace $\mathfrak{a}$ of diagonal matrices in $\mathfrak{q}$ is also maximal abelian in $\mathfrak{g}$, so that all G-orbits in $\mathcal{E}(\mathfrak{g})$ intersect $\mathfrak{a}$, hence also $\mathfrak{q}$ (cf. Example 4.16).

Remark 4.23.   (a) Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra of complex type, where $\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{h}$ is simple hermitian. If $\mathfrak{t} \subseteq \mathfrak{h}$ is a compactly embedded Cartan subalgebra, then $\mathfrak{a} := i\mathfrak{t} \subseteq \mathfrak{q} = i\mathfrak{h}$ is maximal hyperbolic abelian. As $\Sigma(\mathfrak{g}, \mathfrak{a})$ may have several 3-gradings, there are many Euler elements in $\mathfrak{a}$ which are not contained in an $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant cone in $i\mathfrak{h}$. This happens for $\mathfrak{h} = \mathfrak{su}_{p,q}(\mathbb{C})$, where we have $p+q-1$ $\mathrm{Inn}(\mathfrak{h})$-orbits of Euler elements in $i\mathfrak{h}$, represented by $h_1, \ldots, h_{p+q-1}$ (see Proposition 4.13 and Theorem 2.4). The only Euler elements contained in the interior of an invariant cone are those of the form $h = iz$, where $z \in \mathfrak{z}(\mathfrak{h}_{\mathfrak{k}})$ for a Cartan decomposition $\mathfrak{h} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{h}_{\mathfrak{p}}$. This is the case for $h = h_p$.

(b) The example in (a) shows that, for a simple symmetric Lie algebra $(\mathfrak{g}, \tau)$, there may be Euler elements in $\mathfrak{q}$ which are not contained in any hyperbolic $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})$-invariant cone $C \subseteq \mathfrak{q}$, i.e., not every Euler element in $\mathfrak{q}$ is causal for $(\mathfrak{g}, \tau)$. However, the picture changes if we are free to choose τ. Then Theorem 4.21 implies that every Euler element $h \in \mathcal{E}(\mathfrak{g})$ is causal for a suitable choice of τ.

4.4. The classification of irreducible ncc symmetric Lie algebras

We have seen above that irreducible non-compactly causal symmetric Lie algebras $(\mathfrak{g}, \tau)$ are classified by Euler element h in real simple Lie algebras $\mathfrak{g}$ and Cartan involutions θ on $\mathfrak{g}$ satisfying $\theta(h) = -h$ (Section 4.3). In this case $(\mathfrak{g}, \tau)$ with $\tau = \theta \tau_h$ is the corresponding causal symmetric Lie algebra. Along these lines, one obtains a complete classification of these structures, which is described in Table 3 below. It lists all irreducible non-compactly causal symmetric Lie algebras $(\mathfrak{g}, \tau)$ according to the subdivision into the following 4 types:

- Complex type: $\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$ and τ is complex conjugation with respect to $\mathfrak{h}$. In this case $\mathfrak{g}^c \cong \mathfrak{h}^{\oplus 2}$, so that $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}^c) = 2\,\mathrm{rk}_{\mathbb{R}}(\mathfrak{h})$.
- Cayley type (CT): $\tau = \tau_{h_1}$ for an Euler element $h_1 \in \mathfrak{h}$. Then we have $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}^c) = \mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) = \mathrm{rk}_{\mathbb{R}}(\mathfrak{h})$.
- Split type (ST): $\mathrm{rk}_{\mathbb{R}}(\mathfrak{h}) = \mathrm{rk}_{\mathbb{R}}(\mathfrak{g}^c)$ and $(\mathfrak{g}, \tau)$ is not of Cayley type.
- Non-split type (NST): $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}^c) = 2\,\mathrm{rk}_{\mathbb{R}}(\mathfrak{h})$ and $(\mathfrak{g}, \tau)$ is not of complex type.

Remark 4.24. (a) If $(\mathfrak{g}, \tau)$ is a simple symmetric Lie algebra, then either the dual symmetric Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$ is simple ($(\mathfrak{g}, \tau)$ is not of complex type) and if $(\mathfrak{g}, \tau)$ is of complex type, then $(\mathfrak{g}^c, \tau) \cong (\mathfrak{h} \oplus \mathfrak{h}, \tau_{\mathrm{flip}})$.

(b) For an irreducible ncc symmetric Lie algebra $(\mathfrak{g}, \tau)$, the Lie algebra $\mathfrak{g}$ is simple: Since $\tau = \theta \tau_h$ for a causal Euler element h, this follows from the fact that θ and τ_h, hence also τ, preserve all simple ideals.

(c) For an irreducible ncc symmetric Lie algebra $(\mathfrak{g}, \tau)$, either $\mathfrak{g}^c$ is simple hermitian or isomorphic to $(\mathfrak{h} \oplus \mathfrak{h}, \tau_{\mathrm{flip}})$ (if $\mathfrak{g}$ is complex).

In Table 3 we write $r = \mathrm{rk}_{\mathbb{R}}(\mathfrak{g}^c)$ and $s = \mathrm{rk}_{\mathbb{R}}(\mathfrak{h})$. Further $\mathfrak{a} \subseteq \mathfrak{p}$ is maximal abelian of dimension r. For root systems $\Sigma(\mathfrak{g}, \mathfrak{a})$ of type A_{n-1}, there are $n-1$ Euler elements $h_1, \ldots, h_{n-1}$, but for the other root systems there are less; see Theorem 2.4 for the concrete list. For $1 \leq j < n$ we write $j' := \min(j, n - j)$.

5. Strongly orthogonal roots

In this section we introduce a key technical tool in the structure theory of causal symmetric spaces: maximal τ-invariant sets of long strongly orthogonal roots. In Section 5.1 we recall the construction of such sets from [43] and discuss its basic properties. In particular we connect Euler elements with strongly orthogonal roots and the corresponding $\mathfrak{sl}_2$-subalgebras (Proposition 5.2). An important application of this technique is Theorem 5.4 that characterizes irreducible non-compactly causal symmetric Lie algebras $(\mathfrak{g}, \tau, C)$ for which $\mathfrak{h}$ contains Euler elements as those, for which the causal Euler element is symmetric.

5.1. A τ-invariant set of strongly orthogonal roots

Let $(\mathfrak{g}, \tau)$ be an irreducible non-compactly causal symmetric Lie algebra and recall that this implies that $\mathfrak{g}$ is simple ([41, Prop. 2.13], [17, Rem. 3.1.9]). We fix a causal Euler element $h \in \mathfrak{q}$ and the corresponding Cartan involution $\theta = \tau \tau_h$ (Theorem 4.5); then, by Lemma 3.12,

$$\mathfrak{g}^h = \ker(\mathrm{ad}\, h) = \mathfrak{h}_{\mathfrak{k}} + \mathfrak{q}_{\mathfrak{p}} \quad \text{and} \quad \mathfrak{z}(\mathfrak{g}^h) \cap \mathfrak{q}_{\mathfrak{p}} = \mathbb{R}h.$$

Table 3: Irreducible ncc symmetric Lie algebras with corresponding causal Euler elements $h \in \mathfrak{a}$

$\mathfrak{g}$	$\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$	r	$\mathfrak{h} = \mathfrak{g}^{\tau_h\theta}$	s	$\Sigma(\mathfrak{g},\mathfrak{a})$	h	$\mathfrak{g}_1(h)$
Complex type							
$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{su}_{j,n-j}(\mathbb{C})^{\oplus 2}$	$2j'$	$\mathfrak{su}_{j,n-j}(\mathbb{C})$	j'	A_{n-1}	h_j	$M_{j,n-j}(\mathbb{C})$
$\mathfrak{sp}_{2n}(\mathbb{C})$	$\mathfrak{sp}_{2n}(\mathbb{R})^{\oplus 2}$	$2n$	$\mathfrak{sp}_{2n}(\mathbb{R})$	n	C_n	h_n	$\mathrm{Sym}_n(\mathbb{C})$
$\mathfrak{so}_n(\mathbb{C}),\ n>4$	$\mathfrak{so}_{2,n-2}(\mathbb{R})^{\oplus 2}$	4	$\mathfrak{so}_{2,n-2}(\mathbb{R})$	2	$D_{[\frac{n}{2}]},\ B_{[\frac{n}{2}]}$	h_1	$\mathbb{C}^{n-2}$
$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{so}^*(2n)^{\oplus 2}$	$2[\frac{n}{2}]$	$\mathfrak{so}^*(2n)$	$[\frac{n}{2}]$	D_n	h_{n-1}, h_n	$\mathrm{Skew}_n(\mathbb{C})$
$\mathfrak{e}_6(\mathbb{C})$	$(\mathfrak{e}_{6(-14)})^{\oplus 2}$	4	$\mathfrak{e}_{6(-14)}$	2	E_6	h_1, h_6	$M_{1,2}(\mathbb{O})_{\mathbb{C}}$
$\mathfrak{e}_7(\mathbb{C})$	$(\mathfrak{e}_{7(-25)})^{\oplus 2}$	6	$\mathfrak{e}_{7(-25)}$	3	E_7	h_7	$\mathrm{Herm}_3(\mathbb{O})_{\mathbb{C}}$
Cayley type							
$\mathfrak{su}_{r,r}(\mathbb{C})$	$\mathfrak{su}_{r,r}(\mathbb{C})$	r	$\mathbb{R}\oplus\mathfrak{sl}_r(\mathbb{C})$	r	C_r	h_r	$\mathrm{Herm}_r(\mathbb{C})$
$\mathfrak{sp}_{2r}(\mathbb{R})$	$\mathfrak{sp}_{2r}(\mathbb{R})$	r	$\mathbb{R}\oplus\mathfrak{sl}_r(\mathbb{R})$	r	C_r	h_r	$\mathrm{Sym}_r(\mathbb{R})$
$\mathfrak{so}_{2,d}(\mathbb{R}),\ d>2$	$\mathfrak{so}_{2,d}(\mathbb{R})$	2	$\mathbb{R}\oplus\mathfrak{so}_{1,d-1}(\mathbb{R})$	2	C_2	h_2	$\mathbb{R}^{1,d-1}$
$\mathfrak{so}^*(4r)$	$\mathfrak{so}^*(4r)$	r	$\mathbb{R}\oplus\mathfrak{sl}_r(\mathbb{H})$	r	C_r	h_r	$\mathrm{Herm}_r(\mathbb{H})$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{7(-25)}$	3	$\mathbb{R}\oplus\mathfrak{e}_{6(-26)}$	3	C_3	h_3	$\mathrm{Herm}_3(\mathbb{O})$
Split type							
$\mathfrak{sl}_n(\mathbb{R})$	$\mathfrak{su}_{j,n-j}(\mathbb{C})$	j'	$\mathfrak{so}_{j,n-j}(\mathbb{R})$	j'	A_{n-1}	h_j	$M_{j,n-j}(\mathbb{R})$
$\mathfrak{so}_{n,n}(\mathbb{R})$	$\mathfrak{so}^*(2n)$	$[\frac{n}{2}]$	$\mathfrak{so}_n(\mathbb{C})$	$[\frac{n}{2}]$	D_n	h_{n-1}, h_n	$\mathrm{Skew}_n(\mathbb{R})$
$\mathfrak{so}_{p+1,q+1}(\mathbb{R}),\ p,q>1$	$\mathfrak{so}_{2,p+q}(\mathbb{R})$	2	$\mathfrak{so}_{1,p}(\mathbb{R})\oplus\mathfrak{so}_{1,q}(\mathbb{R})$	2	$B_{p+1}\ (p<q)$ $D_{p+1}\ (p=q)$	h_1	$\mathbb{R}^{p,q}$
$\mathfrak{e}_6(\mathbb{R})$	$\mathfrak{e}_{6(-14)}$	2	$\mathfrak{u}_{2,2}(\mathbb{H})$	2	E_6	h_1, h_6	$M_{1,2}(\mathbb{O}_{\mathrm{split}})$
$\mathfrak{e}_7(\mathbb{R})$	$\mathfrak{e}_{7(-25)}$	3	$\mathfrak{sl}_4(\mathbb{H})=\mathfrak{su}^*(8)$	3	E_7	h_7	$\mathrm{Herm}_3(\mathbb{O}_{\mathrm{split}})$
Non-split type							
$\mathfrak{sl}_n(\mathbb{H})$	$\mathfrak{su}_{2j,2n-2j}(\mathbb{C})$	$2j'$	$\mathfrak{u}_{j,n-j}(\mathbb{H})$	j'	A_{n-1}	h_j	$M_{j,n-j}(\mathbb{H})$
$\mathfrak{u}_{n,n}(\mathbb{H})$	$\mathfrak{sp}_{4n}(\mathbb{R})$	$2n$	$\mathfrak{sp}_{2n}(\mathbb{C})$	n	C_n	h_n	$\mathrm{Aherm}_n(\mathbb{H})$
$\mathfrak{so}_{1,d+1}(\mathbb{R})$	$\mathfrak{so}_{2,d}(\mathbb{R})$	2	$\mathfrak{so}_{1,d}(\mathbb{R})$	1	A_1	h_1	$\mathbb{R}^d$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{e}_{6(-14)}$	2	$\mathfrak{f}_{4(-20)}$	1	A_2	h_1, h_2	$M_{1,2}(\mathbb{O})$

In particular, any maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{q}_\mathfrak{p}$ contains h and is also maximal abelian in $\mathfrak{q}$ and $\mathfrak{p}$, so that

$$\dim \mathfrak{a} = \mathrm{rk}_\mathbb{R}(\mathfrak{g}) = \mathrm{rk}_\mathbb{R}(\mathfrak{g}^h). \tag{21}$$

Let $\mathfrak{c} \subseteq \mathfrak{g}$ be a Cartan subalgebra containing $\mathfrak{a}$. Then $\mathfrak{c}_\mathfrak{k} := \mathfrak{c} \cap \mathfrak{k} \subseteq \mathfrak{h}_\mathfrak{k}$ and $\mathfrak{c}$ is invariant under τ and θ, which coincide on $\mathfrak{c}$.

For the root decomposition of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{c}_\mathbb{C}$, we then have

$$\Sigma = \Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{c}_\mathbb{C}) \subseteq i\mathfrak{c}_\mathfrak{k}^* \oplus \mathfrak{a}^* \quad \text{with} \quad -\tau\alpha = \overline{\alpha} \quad \text{for} \quad \alpha \in \Sigma.$$

Further, h induces a 3-grading of the root system

$$\Sigma = \Sigma_{-1} \dot\cup \Sigma_0 \dot\cup \Sigma_1 \quad \text{with} \quad \Sigma_j = \{\alpha \in \Sigma \colon \alpha(h) = j\}.$$

The τ-invariance of $\mathfrak{c}$ implies that τ acts on Σ, and since $\tau(h) = -h$, we have

$$\tau\Sigma_0 = \Sigma_0 \quad \text{and} \quad \tau\Sigma_1 = \Sigma_{-1}.$$

Note that $\Sigma_0 = \Sigma(\mathfrak{g}^h, \mathfrak{c})$ is the root system of the subalgebra $\mathfrak{g}^h$.

According to [43, Thm. 3.4], there exists a maximal subset $\Gamma = \{\gamma_1, \ldots, \gamma_r\} \subseteq \Sigma_1$ of strongly orthogonal long roots, i.e., $\gamma_j \pm \gamma_k \notin \Sigma$ for $j \neq k$, such that $-\tau(\Gamma) = \Gamma$ (cf. Proposition 3.3). Then $\Gamma = \Gamma_0 \dot\cup \Gamma_1$, where $\Gamma_0 := \{\gamma \in \Gamma \colon -\tau\gamma = \gamma\}$ is the subset of real-valued roots in Γ, and $-\tau$ acts by complex conjugation without fixed points on Γ_1, so that this set has an even number of elements. For $r_0 := |\Gamma_0|$ and $r_1 := |\Gamma_1|/2$, we obviously have

$$r = r_0 + 2r_1 \quad \text{and put} \quad s := r_0 + r_1.$$

By [43, Lemma 4.3], $\qquad\qquad \mathrm{rk}_\mathbb{R}(\mathfrak{h}) = s. \tag{22}$

If $(\mathfrak{g}, \tau)$ if of Cayley type then $\mathfrak{h} = \mathfrak{g}^{h_1}$ for an Euler element $h_1 \in \mathfrak{h}$ (see Section 4.4), hence $\mathfrak{h}$ is of real rank s and $\mathfrak{g} \cong \mathfrak{g}^c$ is of real rank r. We therefore have $r = s$, which is equivalent to $r_1 = 0$.

The set Γ of strongly orthogonal roots specifies a subalgebra

$$\mathfrak{s}_\mathbb{C} := \sum_{\gamma \in \Gamma} \mathfrak{g}_\mathbb{C}^\gamma + \mathfrak{g}_\mathbb{C}^{-\gamma} + \mathbb{C}\gamma^\vee \cong \mathfrak{sl}_2(\mathbb{C})^r \tag{23}$$

of $\mathfrak{g}_\mathbb{C}$ invariant under the $\mathbb{C}$-linear extension of τ. The involution τ leaves all ideals in $\mathfrak{s}_\mathbb{C}$ corresponding to roots in Γ_0 invariant and induces flip involutions on the ideals corresponding to $(-\tau)$-orbits in Γ_1. We thus obtain for $\mathfrak{s} = \mathfrak{s}_\mathbb{C} \cap \mathfrak{g}$

$$\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})^{r_0} \oplus \mathfrak{sl}_2(\mathbb{C})^{r_1} \quad \text{and} \quad \mathfrak{s}^\tau \cong \mathfrak{so}_{1,1}(\mathbb{R})^{r_0} \oplus \mathfrak{su}_{1,1}(\mathbb{C})^{r_1}. \tag{24}$$

For any Euler element $h_\mathfrak{s} \in \mathfrak{s} \cap \mathfrak{q}_\mathfrak{p}$ whose kernel contains no non-zero ideal, the Lie subalgebra generated by $h_\mathfrak{s}$ and $\mathfrak{t}_\mathfrak{q}$, a maximal abelian subspace of $\mathfrak{q}_\mathfrak{k}$, is isomorphic to $\mathfrak{sl}_2(\mathbb{R})^r$. This is easily verified by considering the two cases $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{sl}_2(\mathbb{C})$ separately.

For the $\mathfrak{sl}_2(\mathbb{R})$-ideals of $\mathfrak{s}$ we have $\mathfrak{sl}_2(\mathbb{R})^c = \mathfrak{su}_{1,1}(\mathbb{C})$ and for the $\mathfrak{sl}_2(\mathbb{C})$ ideals, we have $\mathfrak{sl}_2(\mathbb{C})^c \cong \mathfrak{su}_{1,1}(\mathbb{C})^{\oplus 2}$, so that

$$\mathfrak{s}^c = \mathfrak{s}_\mathbb{C} \cap \mathfrak{g}^c \cong \mathfrak{su}_{1,1}(\mathbb{C})^r.$$

In particular $\mathfrak{s}^c \subseteq \mathfrak{g}^c$ is a subalgebra of full real rank $r = \mathrm{rk}_\mathbb{R}(\mathfrak{g}^c)$.

The Cayley transform $\kappa_h := e^{\frac{\pi i}{2} \operatorname{ad} h}$ induces a complex structure on $\mathfrak{h}_\mathfrak{p} + i\mathfrak{q}_\mathfrak{k}$ for which τ acts as an antilinear involution. Therefore κ_h maps $\mathfrak{h}_\mathfrak{p}$ bijectively to $i\mathfrak{q}_\mathfrak{k}$, and thus (22) implies that the maximal abelian subspaces of $\mathfrak{q}_\mathfrak{k}$ are also of dimension s. Note also that $\operatorname{ad} h$ defines a bijection $\mathfrak{h}_\mathfrak{p} \to \mathfrak{q}_\mathfrak{k}$, commuting with $\operatorname{Inn}_\mathfrak{g}(\mathfrak{h}_\mathfrak{k})$.

In view of (22), $\mathfrak{s} \cap \mathfrak{q}_\mathfrak{k}$ contains a maximal abelian subspace $\mathfrak{t}_\mathfrak{q}$ of $\mathfrak{q}_\mathfrak{k}$. With respect to the decomposition of $\mathfrak{s}$ in (24), we may choose

$$\mathfrak{t}_\mathfrak{q} = \mathfrak{so}_2(\mathbb{R})^{r_0+r_1} = \mathfrak{so}_2(\mathbb{R})^s.$$

Lemma 5.1. *For $x \in \mathfrak{t}_\mathfrak{q}$, we have $\rho(\operatorname{ad} x) = \rho(\operatorname{ad} x|_\mathfrak{s})$, where ρ denotes the spectral radius. With the basis*

$$z^j = \left(0, \ldots, 0, \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0, \cdots, 0\right), \quad j = 1, \ldots, s,$$

in $\mathfrak{so}_2(\mathbb{R})^s$ we have for $x = \sum_{j=1}^s x_j z^j$

$$\rho(\operatorname{ad} x) = \max\{|x_j| : j = 1, \ldots, s\}. \tag{25}$$

Proof. That $\rho(\operatorname{ad} x|_\mathfrak{s})$ equals $\max\{|x_j| : j = 1, \ldots, s\}$ follows by a matrix calculation in $\mathfrak{sl}_2(\mathbb{R})$. As complexification does not change the spectral radius, $\rho(\operatorname{ad} x|_\mathfrak{s}) = \rho(\operatorname{ad} x|_{\mathfrak{s}_\mathbb{C}})$. It therefore remains to observe that for semisimple elements

$$y = \sum_{j=1}^r y_j z^j \in \mathfrak{s}_\mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C})^r,$$

we have $\rho(\operatorname{ad} y) = \rho(\operatorname{ad} y|_{\mathfrak{s}_\mathbb{C}})$.

Clearly, $\rho(\operatorname{ad} y) \geq \rho(\operatorname{ad} y|_{\mathfrak{s}_\mathbb{C}})$. From (6) in Proposition 3.3 it follows that, for the simple ideals in $\mathfrak{s}_\mathbb{C}$, the Lie algebra $\mathfrak{g}_\mathbb{C}$ contains only simple modules of dimension 1, 2 and 3. This implies $\rho(\operatorname{ad} y) \leq \rho(\operatorname{ad} y|_{\mathfrak{s}_\mathbb{C}})$ for any semisimple element $y \in \mathfrak{s}_\mathbb{C}$. ∎

Proposition 5.2. *Let $(\mathfrak{g}, \tau, C)$ be a simple ncc symmetric Lie algebra. Pick a causal Euler element $h \in C^\circ$, a Cartan involution θ with $\theta(h) = -h$, so that $\tau = \tau_h\theta$, and $\mathfrak{t}_\mathfrak{q} \subseteq \mathfrak{q}_\mathfrak{k}$ maximal abelian. Then $\dim \mathfrak{t}_\mathfrak{q} = s = r_0 + r_1$, and the following assertions hold:*

(a) *The Lie algebra $\mathfrak{l}$ generated by h and $\mathfrak{t}_\mathfrak{q}$ is reductive.*

(b) *The commutator algebra $[\mathfrak{l}, \mathfrak{l}]$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})^s$.*

(c) *$\mathfrak{z}(\mathfrak{l}) = \mathbb{R}h_0$ for some hyperbolic element h_0 satisfying $\tau(h_0) = -h_0 = \theta(h_0)$ which is zero if and only if $\mathfrak{g}^c$ is of tube type or a sum of two ideals of tube type.*

(d) *The Lie algebra $\mathfrak{l}$ is τ-invariant with $\mathfrak{l}^\tau \cong \mathfrak{so}_{1,1}(\mathbb{R})^s$. It is also θ-invariant with $\mathfrak{l}^\theta = \mathfrak{t}_\mathfrak{q} = \mathfrak{so}_2(\mathbb{R})^s$.*

Note that (c) implies that $\mathfrak{l}$ is semisimple, i.e., $h \in [\mathfrak{l}, \mathfrak{l}]$, if and only if $\mathfrak{g}^c$ is of tube type.

Proof. The complex case: If $\mathfrak{g}$ is complex, then $\mathfrak{g} \cong \mathfrak{h}_\mathbb{C}$, where τ is complex conjugation, $\mathfrak{h}$ is simple hermitian and $\mathfrak{q} = i\mathfrak{h}$. For a Cartan decomposition $\mathfrak{h} = \mathfrak{h}_\mathfrak{k} \oplus \mathfrak{h}_\mathfrak{p}$, we then have $\mathfrak{k} = \mathfrak{h}_\mathfrak{k} + i\mathfrak{h}_\mathfrak{p}$ and $\mathfrak{p} = \mathfrak{h}_\mathfrak{p} + i\mathfrak{h}_\mathfrak{k}$. Now $\mathfrak{t}_\mathfrak{q} = i\mathfrak{a}$, where $\mathfrak{a} \subseteq \mathfrak{h}_\mathfrak{p}$ is maximal abelian and $s = \operatorname{rk}_\mathbb{R}(\mathfrak{h})$ and $r = \operatorname{rk}_\mathbb{R}(\mathfrak{g}^c) = \operatorname{rk}_\mathbb{R}(\mathfrak{h} \oplus \mathfrak{h}) = 2s$, so

that $r_0 = 0$. A maximal system $\{\gamma_1, \ldots, \gamma_s\}$ of long strongly orthogonal roots in $\Sigma(\mathfrak{h}, \mathfrak{a})$ now leads to a complex subalgebra

$$\mathfrak{s} := \sum_{j=1}^{s} \mathfrak{g}(\gamma_j) \cong \mathfrak{sl}_2(\mathbb{C})^s \subseteq \mathfrak{g}, \quad \text{where} \quad \mathfrak{g}(\alpha) := \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}],$$

which is invariant under $\operatorname{ad} h$. Next we recall that $\mathfrak{h}$ is of tube type if and only if the corresponding system of restricted roots is of type C_r. By Corollary 3.5 this is equivalent to h being symmetric and by Proposition 3.6, this is equivalent to $h \in \mathfrak{s}$. It is easy to see that h generates with $\mathfrak{t}_\mathfrak{q} \cong \mathfrak{so}_2(\mathbb{R})^s$ a subalgebra $\mathfrak{l} = \mathbb{R}h + \mathfrak{l}'$ with $\mathfrak{l}' \cong \mathfrak{sl}_2(\mathbb{R})^s$. The assertion thus follows with $r_0 = 0$ and $r_1 = s$. We write h_0 for the component of h in the center of $\mathfrak{l}$. Then $\tau(h_0) = -h_0$ follows from $\tau(h) = -h$. Likewise $\theta(h) = -h$ implies $\theta(h_0) = -h_0$, so that h_0 is hyperbolic.

The real case: If $\mathfrak{g}$ is not complex, then the c-dual Lie algebra $\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q}$ is simple hermitian. The subspace $i\mathfrak{t}_\mathfrak{q} \subseteq i\mathfrak{q}_\mathfrak{k} = \mathfrak{q}_\mathfrak{p}^c$ is maximal abelian. Since all maximal abelian subspaces in $\mathfrak{q}_\mathfrak{k}$ are conjugate under the group $\operatorname{Inn}_\mathfrak{g}(\mathfrak{h}_\mathfrak{k})$, we may w.l.o.g. assume that $i\mathfrak{t}_\mathfrak{q} \subseteq \mathfrak{s}_\mathbb{C}$, where $\mathfrak{s}_\mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C})^r$ is constructed from a $-\tau$-stable maximal set of strongly orthogonal roots as in (23) above. The action of τ and complex conjugation with respect to $\mathfrak{g}$ on $\mathfrak{s}_\mathbb{C}$ now show with (24) above that

$$\mathfrak{s} := \mathfrak{g} \cap \mathfrak{s}_\mathbb{C} \cong \mathfrak{sl}_2(\mathbb{R})^{r_0} \oplus \mathfrak{sl}_2(\mathbb{C})^{r_1} \quad \text{and} \quad \mathfrak{s}^\tau \cong \mathfrak{so}_{1,1}(\mathbb{R})^{r_0} \oplus \mathfrak{su}_{1,1}(\mathbb{C})^{r_1}.$$

As $\mathfrak{s}^{-\tau} \cap \mathfrak{k} \cong \mathfrak{so}_2(\mathbb{R})^{r_0+r_1}$ is contained in a maximal abelian subspace $\mathfrak{t}_\mathfrak{q} \subseteq \mathfrak{q}_\mathfrak{k}$, it follows that

$$\mathfrak{t}_\mathfrak{q} = \mathfrak{s}^{-\tau} \cap \mathfrak{k} = \mathfrak{q}_\mathfrak{k} \cap \mathfrak{s}.$$

As in the complex case, we see that $\mathfrak{g}^c$ is of tube type if and only if $h \in \mathfrak{s}_\mathbb{C}$ (Corollary 3.5, Proposition 3.6). Then h is an Euler element in $\mathfrak{s}$, contained in $\mathfrak{s}^{-\tau}$. Inspecting the configurations we obtain for the components in the simple ideals of $\mathfrak{s}$, it now follows easily that the Lie algebra $\mathfrak{l}$ generated by h and $\mathfrak{t}_\mathfrak{q}$ has the asserted form. Here the main point is to see that $\mathfrak{so}_2(\mathbb{R})$ generates with an Euler element in $i\mathfrak{sl}_2(\mathbb{R}) \subseteq \mathfrak{sl}_2(\mathbb{C})$, a 3-dimensional Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. ∎

5.2. Characterization of modular ncc symmetric Lie algebras

In this subsection we characterize non-compactly causal Lie algebras $(\mathfrak{g}, \tau, C)$ for which $\mathfrak{h}$ contains an Euler element by the symmetry of the corresponding causal Euler element. This condition plays an important role in [41], where we study the flows of Euler elements in $\mathfrak{h}$ on the symmetric space.

Definition 5.3. A *modular causal symmetric Lie algebra* is a quadruple $(\mathfrak{g}, \tau, C, h')$, where $(\mathfrak{g}, \tau, C)$ is a causal symmetric Lie algebra and $h' \in \mathfrak{h} = \mathfrak{g}^\tau$ is an Euler element. We sometimes call $(\mathfrak{g}, \tau, C)$ *modular* if such an Euler element h' exists.

Theorem 5.4. *For a simple ncc symmetric Lie algebra $(\mathfrak{g}, \tau, C)$ with causal Euler element $h \in \mathfrak{q}_\mathfrak{p}$ satisfying $\tau = \tau_h \theta$, the following statements are equivalent:*

(a) $\mathfrak{h} \cap \mathcal{E}(\mathfrak{g}) \neq \emptyset$.

(b) $\mathfrak{g}^c$ *is either simple of tube type or a direct sum of two such ideals.*

(c) h *is a symmetric Euler element, i.e.,* $-h \in \mathcal{O}_h = \operatorname{Inn}(\mathfrak{g})h$.

(d) *There exists a second Euler element h_1 such that $\tau_{h_1}(h) = -h$.*

(e) *There exists an $\mathfrak{sl}_2(\mathbb{R})$-subalgebra containing h that is invariant under τ and θ.*

(f) $\mathfrak{h} \cap \mathcal{O}_h \neq \emptyset$.

Proof. (a) $\Rightarrow$ (b): Suppose that $h_1 \in \mathfrak{h}$ is an Euler element. Then h_1 also is an Euler element in the dual Lie algebra $\mathfrak{g}^c$. Either $\mathfrak{g}^c$ is simple (if $\mathfrak{g}$ is not of complex type) or $(\mathfrak{g}^c, \tau^c) \cong (\mathfrak{h} \oplus \mathfrak{h}, \tau_{\mathrm{flip}})$ (if $\mathfrak{g}$ is of complex type). Now [31, Prop. 3.11(b)] implies that, in the first case, $\mathfrak{g}^c$ is of tube type, and, in the second case, the same argument shows that $\mathfrak{h}$ is of tube type (cf. also Corollary 3.5).

(b) $\Rightarrow$ (c): From Proposition 5.2(c) we infer that the Lie subalgebra $\mathfrak{l}$ generated by h and $\mathfrak{t}_{\mathfrak{q}}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})^s$. Therefore (c) follows from the fact that all Euler elements in $\mathfrak{sl}_2(\mathbb{R})$ are symmetric.

(c) $\Leftrightarrow$ (d): The equivalence of (c) and (d) follows from [31, Thm. 3.13(b)].

(d) $\Rightarrow$ (e): [31, Thm. 3.13] implies that we also have $\tau_h(h_1) = -h_1$, i.e., $h_1 \in \mathfrak{g}^{-\tau_h}$. As h_1 is hyperbolic, it is conjugate under $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{g}^{\tau_h}) = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{z}_{\mathfrak{g}}(h))$ to an element of $\mathfrak{g}^{-\tau_h} \cap \mathfrak{p}$ ([24, Cor. II.9]). We may therefore assume that $\theta(h_1) = -h_1$. Then $\tau = \tau_h\theta$ satisfies $\tau(h_1) = h_1$, i.e., $h_1 \in \mathfrak{h}$. [31, Thm. 3.13] further implies that $\mathfrak{b} := \mathrm{span}\{h, h_1, [h, h_1]\} \cong \mathfrak{sl}_2(\mathbb{R})$. This subalgebra is clearly invariant under θ and τ.

(e) $\Rightarrow$ (f): If $\mathfrak{b} \subseteq \mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, invariant under τ and θ and $h \in \mathfrak{b}$, then $\mathfrak{b}^\tau$ contains an Euler element h_1 of $\mathfrak{b}$, but then $h_1 \in \mathrm{Inn}(\mathfrak{b})h$ implies that $h_1 \in \mathcal{O}_h \cap \mathfrak{h}$.

(f) $\Rightarrow$ (a) is trivial. ∎

Corollary 5.5. *Let $(\mathfrak{g}, \tau) = (\mathfrak{h}_{\mathbb{C}}, \tau)$ be an irreducible ncc symmetric Lie algebra of complex type and $h \in \mathcal{E}_c(\mathfrak{q})$ a causal Euler element. Then the following are equivalent:*

(a) *The real form $\mathfrak{h}$ is of tube type.*

(b) *h is symmetric, i.e., $-h \in \mathcal{O}_h$.*

Proof. As $\mathfrak{g}^c \cong \mathfrak{h} \oplus \mathfrak{h}$, this follows from the equivalence of (b) and (c) in Theorem 5.4. ∎

Remark 5.6. Any pair of Euler elements (h, h_1) which are orthogonal in the sense that $\tau_h(h_1) = -h_1$ and $\tau_{h_1}(h) = -h$ (cf. [31, Thm. 3.13]) leads to an embedding of symmetric Lie algebras $(\mathfrak{l}, \tau_{\mathfrak{l}}) \hookrightarrow (\mathfrak{g}, \tau)$ with $h, h_1 \in \mathfrak{l} \cong \mathfrak{sl}_2(\mathbb{R})$. Here we may w.l.o.g. assume that $h_1 \in \mathfrak{h}_{\mathfrak{p}}$, so that $\mathfrak{l}_{\mathfrak{q}} = [h_1, \mathfrak{l}] = \mathbb{R}h + \mathbb{R}[h_1, h]$. The centralizer of h_1 in $L := \mathrm{SL}_2(\mathbb{R})$ is $L^{h_1} \cong \mathrm{SO}_{1,1}(\mathbb{R})$. The fact that $\mathfrak{l}$ contains an Euler element of $\mathfrak{g}$ implies that the adjoint action of L on $\mathfrak{g}$ descends to a homomorphism

$$\mathrm{Ad}(L) \cong \mathrm{PSL}_2(\mathbb{R}) \hookrightarrow \mathrm{Inn}(\mathfrak{g}),$$

which leads to an embedding

$$\mathrm{dS}^2 \cong L/L^{h_1} \hookrightarrow \mathrm{Inn}(\mathfrak{g})/\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{k})^h e^{\mathrm{ad}\,\mathfrak{h}_{\mathfrak{p}}}$$

and also to an embedding of *projective 2-dimensional de Sitter space*

$$\mathrm{PdS}^2 := \mathrm{dS}^2/\{\pm\mathbf{1}\} \hookrightarrow \mathrm{Inn}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})^\tau$$

(cf. Remark 4.20(c)).

6.   Real crown domains and real tubes

Let (G, τ^G) be a connected symmetric semisimple Lie group with non-compactly causal symmetric Lie algebra $(\mathfrak{g}, \tau, C)$, $H \subseteq G^\tau$ an open subgroup satisfying $\mathrm{Ad}(H)C = C$ and $M = G/H$ the associated non-compactly causal symmetric space. In this section we extend some of the results obtained in [41] for the special class of modular ncc spaces (Definition 5.3), where $\mathfrak{h}$ contains an Euler element, to general semisimple non-compactly causal symmetric Lie algebras.

Our main result is that, if the cone C is maximal and $(\mathfrak{g}, \tau)$ is semisimple without Riemannian ideals, then the connected component of h in the intersection of the adjoint orbit $\mathcal{O}_h$ with the real tube domain $\mathcal{T}_C = \mathfrak{h} + C^\circ \subseteq \mathfrak{g}$ is the Matsuki crown (see [29]) of the Riemannian symmetric space $\mathcal{O}_h^{\mathfrak{q}} = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})h = e^{\mathrm{ad}\, \mathfrak{h}_{\mathfrak{p}}}h$, i.e.,

$$\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})e^{\mathrm{ad}\, \Omega_{\mathfrak{q}_{\mathfrak{k}}}}h \quad \text{for} \quad \Omega_{\mathfrak{q}_{\mathfrak{k}}} = \left\{ x \in \mathfrak{q}_{\mathfrak{k}} : \rho(\mathrm{ad}\, x) < \frac{\pi}{2} \right\},$$

where $\rho(\mathrm{ad}\, x)$ is the spectral radius of $\mathrm{ad}\, x$ (Theorem 6.6) and $\mathfrak{q}_{\mathfrak{k}} = \mathfrak{q} \cap \mathfrak{k}$ for a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $h \in \mathfrak{p}$.

Definition 6.1. (a) Let $h \in \mathfrak{q}_{\mathfrak{p}} \cap C^\circ$ be a causal Euler element, Then $\mathfrak{z}_{\mathfrak{h}}(h) = \mathfrak{h}_{\mathfrak{k}}$ implies that
$$\mathcal{O}_h^{\mathfrak{q}} = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})h = e^{\mathrm{ad}\, \mathfrak{h}_{\mathfrak{p}}}h$$
is the non-compact Riemannian symmetric space associated to the symmetric Lie algebra $(\mathfrak{h}, \theta)$.

(b) For a convex cone $C \subseteq \mathfrak{q}$, we write $\mathcal{T}_C := \mathfrak{h} + C^\circ$ for the corresponding open *real tube domain* in $\mathfrak{g}$.

(c) We define the *Matsuki crown* of the Riemannian symmetric space $\mathcal{O}_h^{\mathfrak{q}}$ by

$$\mathcal{C}(\mathcal{O}_h^{\mathfrak{q}}) := \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h})e^{\mathrm{ad}\, \Omega_{\mathfrak{q}_{\mathfrak{k}}}}h = e^{\mathrm{ad}\, \mathfrak{h}_{\mathfrak{p}}}e^{\mathrm{ad}\, \Omega_{\mathfrak{q}_{\mathfrak{k}}}}h, \quad \text{where} \quad \Omega_{\mathfrak{q}_{\mathfrak{k}}} := \left\{ x \in \mathfrak{q}_{\mathfrak{k}} : \rho(\mathrm{ad}\, x) < \frac{\pi}{2} \right\}.$$

Here we have used the polar decomposition

$$\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}) = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_{\mathfrak{k}})e^{\mathrm{ad}\, \mathfrak{h}_{\mathfrak{p}}} = e^{\mathrm{ad}\, \mathfrak{h}_{\mathfrak{p}}}\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_{\mathfrak{k}}) \quad \text{and} \quad \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_{\mathfrak{k}})\Omega_{\mathfrak{q}_{\mathfrak{k}}} = \Omega_{\mathfrak{q}_{\mathfrak{k}}}$$

so see that $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_{\mathfrak{k}})e^{\mathrm{ad}\, \Omega_{\mathfrak{q}_{\mathfrak{k}}}}h = e^{\mathrm{ad}\, \Omega_{\mathfrak{q}_{\mathfrak{k}}}}h$.

Remark 6.2.   (The connection with Matsuki's domains) To see that the inverse image of our domain $\mathcal{C}(\mathcal{O}_h^{\mathfrak{q}})$ in G is one of the domains considered by Matsuki in [29, §1.2], let us recall his setting: On G we consider a Cartan involution θ with $\theta(h) = -h$ and the involution $\tau = \tau_h \theta$ and consider the connected groups

$$H := G_e^\tau \quad \text{and} \quad H' := G_e^{\tau\theta} = G_e^{\tau_h} = G_e^h.$$

Their Lie algebras are $\mathfrak{h} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{h}_{\mathfrak{p}}$ and $\mathfrak{h}' = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{q}_{\mathfrak{p}} = \ker(\mathrm{ad}\, h)$.

We choose a maximal abelian subspace $\mathfrak{t}_{\mathfrak{q}} \subseteq \mathfrak{q}_{\mathfrak{k}}$. As θ fixes $\mathfrak{t}_{\mathfrak{q}}$, it leaves the corresponding weight spaces $\mathfrak{g}_{\mathbb{C}}^\alpha$ invariant, so that $\mathfrak{g}_{\mathbb{C}}^\alpha = \mathfrak{k}_{\mathbb{C}}^\alpha \oplus \mathfrak{p}_{\mathbb{C}}^\alpha$, and Matsuki considers the subset

$$\Sigma(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathfrak{q}}) := \{\alpha \in \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathfrak{q}}) : \mathfrak{p}_{\mathbb{C}}^\alpha \neq \{0\}\}$$

and the domain

$$\mathfrak{t}_{\mathfrak{q}}^+ := \left\{ y \in \mathfrak{t}_{\mathfrak{q}} : (\forall \alpha \in \Sigma(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathfrak{q}})) \; |\alpha(y)| < \frac{\pi}{2} \right\} \supseteq \left\{ y \in \mathfrak{t}_{\mathfrak{q}} : \rho(\mathrm{ad}\, y) < \frac{\pi}{2} \right\} \qquad (26)$$

which in turn determines the domain

$$\mathcal{M}(H, H') := H \exp(\mathfrak{t}_{\mathfrak{q}}^+) H' \subseteq G.$$

The discussion of the following two examples, combined with Lemma 5.1 now implies that we have equality in (26).

(a) For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ and $h = \frac{1}{2} \operatorname{diag}(1, -1)$ we have $\theta(x) = -x^\top$,

$$\tau_h \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & -a \end{pmatrix} \quad \text{and} \quad \tau \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = - \begin{pmatrix} a & -b \\ -c & -a \end{pmatrix}^\top = \begin{pmatrix} -a & c \\ b & a \end{pmatrix}.$$

So we obtain $\mathfrak{k} = \mathfrak{so}_2(\mathbb{R}) = \mathfrak{q}_{\mathfrak{k}} \subseteq \mathfrak{q}$ and $\mathfrak{t}_{\mathfrak{q}} = \mathfrak{q}_{\mathfrak{k}} = \mathfrak{so}_2(\mathbb{R})$. We conclude that $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_{\mathfrak{q}}) = \Sigma(\mathfrak{p}_\mathbb{C}, \mathfrak{t}_{\mathfrak{q}})$.

(b) For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \supseteq \mathfrak{h} = \mathfrak{su}_{1,1}(\mathbb{C})$, we have $\theta(x) = -x^*$,

$$\mathfrak{h} = \left\{ \begin{pmatrix} ia & b \\ \bar{b} & -ia \end{pmatrix} : a \in \mathbb{R}, b \in \mathbb{C} \right\}, \quad \mathfrak{q} = i\,\mathfrak{su}_{1,1}(\mathbb{C})$$

and $\qquad \mathfrak{k} = \mathfrak{su}_2(\mathbb{C}), \quad \mathfrak{q}_{\mathfrak{k}} = \left\{ \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} : b \in \mathbb{C} \right\} \supseteq \mathfrak{t}_{\mathfrak{q}} = \mathfrak{so}_2(\mathbb{R}).$

In this case $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_{\mathfrak{q}}) = \{\pm\alpha\}$ with $\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 2i$, $\dim_\mathbb{C}(\mathfrak{g}_\mathbb{C}^\alpha) = 2$, and both roots occur in $\mathfrak{p}_\mathbb{C}$ for $\mathfrak{p} = i\,\mathfrak{su}_2(\mathbb{C})$.

Example 6.3.   We consider the hermitian simple Lie algebra $\mathfrak{g} = \mathfrak{su}_{p,q}(\mathbb{C})$. It is of tube type if and only if $p = q$ (cf. Section 3.1). We assume that $p \leq q$, so that $r := \operatorname{rk}_\mathbb{R}(\mathfrak{g}) = p$. Then

$$z := \frac{i}{p+q} \begin{pmatrix} q\mathbf{1}_p & 0 \\ 0 & -p\mathbf{1}_q \end{pmatrix}$$

is an H-element, the subspace of diagonal matrices $\mathfrak{t}$ is a compactly embedded Cartan subalgebra with

$$\Sigma(\mathfrak{g}, \mathfrak{t}) = \{\varepsilon_j - \varepsilon_k : j \neq k, j, k = 1, \dots, p+q\} \cong A_{p+q-1}.$$

We consider the adapted positive system

$$\Sigma^+ = \{\varepsilon_j - \varepsilon_k : 1 \leq j < k \leq p+q\}$$

with $\qquad \Sigma_1 = \{\varepsilon_j - \varepsilon_k : j \leq p < k\} = \{\alpha \in \Sigma : -i\alpha(z) > 0\}.$

A set of strongly orthogonal roots in Σ_1 is

$$\Gamma = \{\gamma_j : j = 1, \dots, p\}, \quad \gamma_j := \varepsilon_j - \varepsilon_{p+j}.$$

In $\mathfrak{g}(\gamma_j) \cong \mathfrak{su}_{1,1}(\mathbb{C})$, we have the H-element

$$z^j = \frac{i}{2}(E_{jj} - E_{j+p,j+p}).$$

We then obtain

$$z = z^0 + \sum_{j=1}^{p} z^j \quad \text{with} \quad z^0 = \frac{i}{p+q} \begin{pmatrix} \frac{q-p}{2} \cdot \mathbf{1}_{2p} & 0 \\ 0 & -p\mathbf{1}_{q-p} \end{pmatrix}.$$

Note that $z^0 = 0$ if and only if $p = q$, i.e., if $\mathfrak{g}$ is of tube type.

With the notation $\mathbf{x} = \operatorname{diag}(x_1, \ldots, x_{p+q})$, we then have

$$
\begin{aligned}
C_{\mathfrak{t}}^{\max} &= \{-i\mathbf{x} \in \mathfrak{t} \colon (\forall \alpha \in \Sigma_1)\ i\alpha(-ix) = \alpha(x) \geq 0\} \\
&= \{-i\mathbf{x} \colon x_j > x_k \ \text{ for } j \leq p, k > p\}
\end{aligned}
$$

and $\qquad C_{\mathfrak{t}}^{\min} = \operatorname{cone}\{-i\alpha^\vee \colon \alpha \in \Sigma_1\} = \operatorname{cone}\{-i(E_{jj} - E_{kk}) \colon j \leq p < k\}.$

Considering extreme points of a basis of this cone, we see that

$$
C_{\mathfrak{t}}^{\min} = \operatorname{cone}\left\{ -ix \colon x_j \geq 0,\ x_k \leq 0,\ j \leq p < k;\ \sum_j x_j = 0 \right\}.
$$

If $p < q$, then the entry in position $p+1$ of z^0 is negative, so that $z^0 \notin C_{\mathfrak{t}}^{\min}$.

The preceding example shows in particular that the following lemma does not hold for the minimal cone $C_{\mathfrak{q}}^{\min}$; see also Example 6.8 below.

Lemma 6.4. *Let $(\mathfrak{g}, \tau)$ be irreducible ncc, $h \in C_{\mathfrak{q}}^{\max}$ a causal Euler element and write h_0 for its central component in the subalgebra $\mathfrak{l}$ generated by h and $\mathfrak{t}_{\mathfrak{q}}$ (Proposition 5.2). Then $h_0 \in C_{\mathfrak{q}}^{\max}$.*

Proof. (a) We start with a discussion concerning hermitian Lie algebras, which corresponds to the "complex case". Let $\mathfrak{g}$ be simple hermitian and

$$
\mathfrak{l} \cong \mathbb{R}z + \mathfrak{sl}_2(\mathbb{R})^r \subseteq \mathfrak{g}
$$

the reductive subalgebra generated by an H-element $z \in \mathfrak{z}(\mathfrak{k})$ and the subalgebras $\mathfrak{g}(\gamma_j)$ associated to a set $\{\gamma_1, \ldots, \gamma_r\}$ of strongly orthogonal positive non-compact roots in $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, where $\mathfrak{t} \subseteq \mathfrak{k}$ is a Cartan subalgebra and

$$
\Delta_p^+ = \{\alpha \in \Delta \colon i\alpha(z) = 1\}.
$$

Then $\qquad z = z^0 + \sum_{j=1}^r z^j, \quad z^j = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R})$

and $\qquad \mathfrak{t}_{\mathfrak{l}} = \operatorname{span}\{z^j \colon j = 0, \ldots, r\} \subseteq \mathfrak{t} \cap \mathfrak{l}$

is a compactly embedded Cartan subalgebra of $\mathfrak{l}$. We then have

$$
z^j \in C_{\mathfrak{g}}^{\min} \quad \text{for} \quad j > 0,
$$

(see Section 3.5.1 for $C_{\mathfrak{g}}^{\min}$) but in general $z^0 \notin C_{\mathfrak{g}}^{\min}$ (Example 6.3). We claim that

$$
z^0 \in C_{\mathfrak{t}}^{\max} := C_{\mathfrak{g}}^{\max} \cap \mathfrak{t} = \{x \in \mathfrak{t} \colon (\forall \alpha \in \Delta_p^+)\ i\alpha(x) \geq 0\}.
$$

From (6) in Proposition 3.3 it follows that, for $\alpha \in \Delta_p^+$, we have

$$
i\alpha(z^j) \in \{\tfrac{1}{2}, 1\} \quad \text{and} \quad \sum_{j=1}^r i\alpha(z^j) \in \{\tfrac{1}{2}, 1\}.
$$

From $i\alpha(z) = 1$ it now follows that

$$
i\alpha(z^0) \in \left\{0, \tfrac{1}{2}\right\}, \quad \text{so that} \quad z^0 \in C_{\mathfrak{t}}^{\max}.
$$

(b) (The real situation) Now we prove the lemma. Let $(\mathfrak{g}, \tau)$ be simple ncc not of complex type; the complex type is covered by (a). Then $\mathfrak{g}^c$ is simple hermitian and

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^c \oplus i\mathfrak{g}^c.$$

We then have maximal invariant cones

$$C_{\mathfrak{q}}^{\max} \subseteq \mathfrak{q} \quad \text{and} \quad C_{i\mathfrak{g}^c}^{\max} \subseteq i\mathfrak{g}^c,$$

and [43, Lemma 7.10] (see also [24, Thm. VIII.1]) implies that

$$C_{\mathfrak{q}}^{\max} = \mathfrak{q} \cap C_{i\mathfrak{g}^c}^{\max}.$$

As the Euler element $h \in C_{\mathfrak{q}}^{\max,\circ} \cap \mathfrak{q}_{\mathfrak{p}}$ also is an Euler element in $i\mathfrak{k}^c \subseteq i\mathfrak{g}^c$. The central component $h_0 \in \mathfrak{z}(\mathfrak{l})$ coincides with the central component of h, considered as an Euler element in $i\mathfrak{g}^c$ (as in (a)). Therefore (a) implies that

$$h_0 \in \mathfrak{q} \cap C_{i\mathfrak{g}^c}^{\max} = C_{\mathfrak{q}}^{\max}. \qquad \blacksquare$$

We shall need the following lemma ([41, Lemma 4.8]).

Lemma 6.5. *Let (G, τ^G) be a connected symmetric reductive Lie group corresponding to the causal symmetric Lie algebra $(\mathfrak{g}, \tau, C_{\mathfrak{q}})$. If $D \subseteq \mathcal{T}_{C_{\mathfrak{q}}} = \mathfrak{h} + C_{\mathfrak{q}}^\circ \subseteq \mathfrak{q}$ is compact and $H \subseteq G^\tau$ an open subgroup with $\mathrm{Ad}(H)C_{\mathfrak{q}} = C_{\mathfrak{q}}$, then $\mathrm{Ad}(H)D$ is closed in $\mathfrak{g}$. Moreover, there exists a smooth $\mathrm{Ad}(H)$-invariant function $\psi \colon \mathcal{T}_{C_{\mathfrak{q}}} \to (0, \infty)$ such that $z_n \to z_0 \in \partial\mathcal{T}_{C_{\mathfrak{q}}}$ for $z_n \in \mathcal{T}_{C_{\mathfrak{q}}}$ implies $\psi(z_n) \to \infty$.*

The following theorem has a twin in [41, Thm. 4.10], where we consider only the modular case, where $\mathfrak{h}$ contains an Euler element, and this stronger assumption permits us to work with a general cone $C_{\mathfrak{q}}$. Here we work with general ncc spaces, where $\mathfrak{h}$ may not contain an Euler element, but in this case we have to assume that $C_{\mathfrak{q}}$ is maximal.

Theorem 6.6. (Crown Theorem for causal Euler elements) *Let $(\mathfrak{g}, \tau)$ be a semisimple ncc symmetric Lie algebra without τ-invariant Riemannian ideals and $h \in \mathfrak{q}$ a causal Euler element. Let*

$$H = \mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}) \subseteq G = \mathrm{Inn}(\mathfrak{g}) \quad \text{and} \quad C := C_{\mathfrak{q}}^{\max} \subseteq \mathfrak{q}$$

be the maximal pointed generating $\mathrm{Ad}(H)$-invariant cone containing h. Then the connected component of h in the open subset $\mathcal{O}_h \cap \mathcal{T}_C$ of $\mathcal{O}_h$ is the Matsuki crown $\mathcal{C}(\mathcal{O}_h^{\mathfrak{q}}) = \mathrm{Ad}(H)e^{\mathrm{ad}\,\Omega_{\mathfrak{q}_{\mathfrak{k}}}}h$.

In [33, §7] we show that $\mathcal{O}_h \cap \mathcal{T}_{C_{\mathfrak{q}}^{\max}}$ is actually connected.

Proof. Our assumptions on $(\mathfrak{g}, \tau)$ imply that all simple τ-invariant ideals are irreducible ncc and that C is the product of the maximal $\mathrm{Ad}(H)$-invariant cones in the irreducible factors. Therefore both sets whose equality is to be shown are adapted to the decomposition into irreducible factors and we may therefore assume that $(\mathfrak{g}, \tau)$ is irreducible and that $G = \mathrm{Inn}(\mathfrak{g})$ has trivial center (cf. [41, Prop. 2.14]). Then $K = G^\theta$ is compact and the polar decomposition of H implies that $H = H_K e^{\mathrm{ad}\,\mathfrak{h}_{\mathfrak{p}}}$ with $H_K = H^h$ (Corollary 4.6).

We will derive the theorem from the following three claims:

(a) $\mathcal{C}(\mathcal{O}_h^{\mathfrak{q}})$ is connected and open in $\mathcal{O}_h$.

(b) $\mathcal{C}(\mathcal{O}_h^{\mathfrak{q}}) \subseteq \mathcal{T}_C$.

(c) $\mathcal{C}(\mathcal{O}_h^{\mathfrak{q}})$ is relatively closed in $\mathcal{O}_h \cap \mathcal{T}_C$.

As $\mathcal{C}(\mathcal{O}_h^{\mathfrak{q}})$ is connected by definition (a), (b) and (c) imply that it is the connected component of h in the open subset $\mathcal{O}_h \cap \mathcal{T}_C$.

(a) To see that $\mathcal{C}(\mathcal{O}_h^{\mathfrak{q}})$ is connected, we use the polar decomposition $H = H_K \exp(\mathfrak{h}_{\mathfrak{p}})$. Then $\mathrm{Ad}(H_K)\Omega_{\mathfrak{q}_{\mathfrak{k}}} = \Omega_{\mathfrak{q}_{\mathfrak{k}}}$ and $\mathrm{Ad}(H_K)h = \{h\}$ (cf. Lemma 4.11). This implies that

$$\mathcal{C}(\mathcal{O}_h^{\mathfrak{q}}) = \mathrm{Ad}(H)e^{\mathrm{ad}\,\Omega_{\mathfrak{q}_{\mathfrak{k}}}}h = e^{\mathrm{ad}\,\mathfrak{h}_{\mathfrak{k}}}e^{\mathrm{ad}\,\Omega_{\mathfrak{q}_{\mathfrak{k}}}}\mathrm{Ad}(H_K)h = e^{\mathrm{ad}\,\mathfrak{h}_{\mathfrak{k}}}e^{\mathrm{ad}\,\Omega_{\mathfrak{q}_{\mathfrak{k}}}}h,$$

which is obviously connected.

Next we use [41, Lemma C.3(a)] to see that the exponential map

$$\mathrm{Exp}\colon \mathfrak{q}_{\mathfrak{k}} \to \mathcal{O}_h, \quad x \mapsto e^{\mathrm{ad}\,x}h$$

is regular in $x \in \Omega_{\mathfrak{q}_{\mathfrak{k}}}$ because $\rho(\mathrm{ad}\,x) < \pi/2 < \pi$. Now [41, Lemma C.3(b)] implies that the map

$$\Phi\colon H \times \Omega_{\mathfrak{q}_{\mathfrak{k}}} \to \mathcal{O}_h, \quad (g, x) \mapsto \mathrm{Ad}(g)e^{\mathrm{ad}\,x}h$$

is regular in (g, x) because $\mathrm{Spec}(\mathrm{ad}\,x) \subseteq (-\pi/2, \pi/2)i$ does not intersect $\left(\frac{\pi}{2} + \mathbb{Z}\pi\right)i$. This implies that the differential of Φ is surjective in each point of $H \times \Omega_{\mathfrak{q}_{\mathfrak{k}}}$; hence its image is open.

(b) We observe that both sides of (b) are $\mathrm{Ad}(H)$-invariant, and

$$\mathcal{C}(\mathcal{O}_h^{\mathfrak{q}}) = \mathrm{Ad}(H)e^{\mathrm{ad}\,\Omega_{\mathfrak{q}_{\mathfrak{k}}}}h = \mathrm{Ad}(H)e^{\mathrm{ad}\,\Omega_{\mathfrak{t}_{\mathfrak{q}}}}h$$

for a maximal abelian subspace $\mathfrak{t}_{\mathfrak{q}} \subseteq \mathfrak{q}_{\mathfrak{k}}$ and $\Omega_{\mathfrak{t}_{\mathfrak{q}}} := \Omega_{\mathfrak{q}_{\mathfrak{k}}} \cap \mathfrak{t}_{\mathfrak{q}}$. Here we use that H_K fixes h and $\mathrm{Ad}(H_K)\mathfrak{t}_{\mathfrak{q}} = \mathfrak{q}_{\mathfrak{k}}$. Therefore it suffices to show $e^{\mathrm{ad}\,x}h \in \mathcal{T}_C$ for

$$x \in \Omega_{\mathfrak{t}_{\mathfrak{q}}} = \left\{y \in \mathfrak{t}_{\mathfrak{q}}\colon \rho(\mathrm{ad}\,y) < \frac{\pi}{2}\right\}.$$

From Proposition 5.2 we infer that the causal Euler element h and $\mathfrak{t}_{\mathfrak{q}}$ generate a τ-invariant reductive subalgebra $\mathfrak{l}$ with

$$[\mathfrak{l}, \mathfrak{l}] \cong \mathfrak{sl}_2(\mathbb{R})^s \quad \text{and} \quad \mathfrak{l}^{\tau} \cong \mathfrak{so}_{1,1}(\mathbb{R})^s$$

in which h is a causal Euler element, contained in the interior of the pointed generating cone $C_{\mathfrak{l}} := C \cap \mathfrak{l} \subseteq \mathfrak{l}^{-\tau}$ which is invariant under $\mathrm{Inn}(\mathfrak{l}^{\tau})$. Let $s := r_0 + r_1$. For elements of $\mathfrak{sl}_2(\mathbb{R})$, we recall the notation from (38) and (39) above:

$$h^0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{27}$$

For a suitable isomorphism $[\mathfrak{l}, \mathfrak{l}] \to \mathfrak{sl}_2(\mathbb{R})^{r_0} \oplus \mathfrak{sl}_2(\mathbb{C})^{r_1}$, we obtain

$$h = h_0 + \sum_{j=1}^s h_j \quad \text{and} \quad \mathfrak{t}_{\mathfrak{q}} = \mathfrak{so}_2(\mathbb{R})^s,$$

where we write h_j for the Euler elements h^0 in the jth summand. For

$$x = \sum_{j=1}^s x_j \frac{e_j - f_j}{2} \in \mathfrak{t}_{\mathfrak{q}},$$

we then obtain with (41)

$$p_{\mathfrak{q}}(e^{\operatorname{ad} x}h) = \cosh(\operatorname{ad} x)(h) = h_0 + \sum_{j=1}^{s} \cos(x_j)h_j. \tag{28}$$

Since $\rho(\operatorname{ad} x) < \frac{\pi}{2}$ for $x \in \Omega_{\mathfrak{t}_{\mathfrak{q}}}$ and $ix_j \in \operatorname{Spec}(\operatorname{ad} x)$, we have $|x_j| < \frac{\pi}{2}$ for $j = 1, \ldots, s$, hence $\cos(x_j) \in (0, 1]$. As $h_0 \in C_{\mathfrak{l}}$ (Lemma 6.4) and $h_j \in \bar{C}_{\mathfrak{l}}$ for $j = 1, \ldots, s$, we have

$$p_{\mathfrak{q}}(e^{\operatorname{ad} x}h) \in C^{\circ}_{\mathfrak{a}_{\mathfrak{s}}} \quad \text{for} \quad C_{\mathfrak{a}_{\mathfrak{s}}} := C \cap \mathfrak{a}_{\mathfrak{s}}, \quad \mathfrak{a}_{\mathfrak{s}} := \operatorname{span}\{h_j : j = 0, \ldots, s\}.$$

Now $h \in C^{\circ} \cap \mathfrak{a}_{\mathfrak{s}}$ implies $C^{\circ}_{\mathfrak{a}_{\mathfrak{s}}} = C^{\circ} \cap \mathfrak{a}_s \subseteq C_{\mathfrak{l}}$ (Lemma B.1), so that

$$p_{\mathfrak{q}}(e^{\operatorname{ad} x}h) \in C^{\circ}.$$

This proves (b).

(c) We have to show that $\mathcal{C}(\mathcal{O}^{\mathfrak{q}}_h)$ is relatively closed in $\mathcal{O}_h \cap \mathcal{T}_C$. If $D \subseteq \Omega_{\mathfrak{t}_{\mathfrak{q}}}$ is compact, then $\operatorname{Ad}(H)e^{\operatorname{ad} D}h$ is closed in $\mathfrak{g}$ by Lemma 6.5, and by (a) it is contained in $\mathcal{T}_C$.

Now suppose that the sequence $\operatorname{Ad}(h_n)e^{\operatorname{ad} x_n}h$, $h_n \in H$, $x_n \in \Omega_{\mathfrak{t}_{\mathfrak{q}}}$, converges to some element in $\mathcal{T}_C$ which is not contained in $\mathcal{C}(\mathcal{O}^{\mathfrak{q}}_h)$. As we may assume that the bounded sequence $x_n \in \Omega_{\mathfrak{t}_{\mathfrak{q}}}$ converges in $\mathfrak{t}_{\mathfrak{q}}$, it converges by the preceding paragraph to a boundary point $y \in \partial\Omega_{\mathfrak{t}_{\mathfrak{q}}}$. Writing

$$y = \sum_{j=1}^{s} y_j \frac{e_j - f_j}{2},$$

we claim that there exists a j with $|y_j| = \frac{\pi}{2}$. As $\rho(\operatorname{ad} y|_{\mathfrak{l}}) = \max\{|y_j| : j = 1, \ldots, s\}$, this follows from $\rho(\operatorname{ad} y|_{\mathfrak{l}}) = \rho(\operatorname{ad} y)$ (Lemma 5.1). Now (28) and

$$C \cap \operatorname{span}\{h_0, h_1, \ldots, h_s\} \subseteq \mathbb{R}h_0 + \sum_{j=1}^{s} [0, \infty)h_j. \tag{29}$$

imply that

$$e^{\operatorname{ad} y}h \in \partial\mathcal{T}_{C^{\max}_{\mathfrak{q}}}.$$

For the H-invariant function ψ from Lemma 6.5, this leads to

$$\psi(\operatorname{Ad}(h_n)e^{\operatorname{ad} x_n}h) = \psi(e^{\operatorname{ad} x_n}h) \to \infty,$$

contradicting the convergences of the sequence $\operatorname{Ad}(h_n)e^{\operatorname{ad} x_n}h$ in $\mathcal{T}_{C^{\max}_{\mathfrak{q}}}$. ∎

Remark 6.7. In the special case where $(\mathfrak{g}, \tau)$ is Riemannian, $\mathfrak{q} = \mathfrak{q}_{\mathfrak{p}}$ and $\mathfrak{h} = \mathfrak{h}_{\mathfrak{k}}$, the real crown domain $\mathcal{C}(\mathcal{O}^{\mathfrak{q}}_h)$ reduces to a point. Hence there is no interesting analog of the preceding theorem in the Riemannian case, and we therefore assume that $(\mathfrak{g}, \tau)$ contains no Riemannian summands.

Example 6.8. We consider the Lie algebra $\mathfrak{g} = \mathfrak{su}_{2,1}(\mathbb{C})$ which is hermitian, but not of tube type. In particular $(\mathfrak{g}_{\mathbb{C}}, \tau)$, $\tau(z) = \bar{z}$, is ncc of complex type, but $\mathfrak{g}$ contains no Euler element. Concretely, we have

$$\mathfrak{su}_{2,1}(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ b^* & -\operatorname{tr} a \end{pmatrix} : a \in \mathfrak{u}_2(\mathbb{C}), b \in \mathbb{C}^2 \right\}.$$

For the Cartan involution $\theta(x) = -x^*$, we obtain

$$\mathfrak{k} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -\operatorname{tr} a \end{pmatrix} : a \in \mathfrak{u}_2(\mathbb{C}) \right\} \cong \mathfrak{u}_2(\mathbb{C})$$

with

$$\mathfrak{z}(\mathfrak{k}) = \mathbb{R}ih_c, \quad h_c = \frac{1}{3}\operatorname{diag}(1, 1, -2).$$

The subspace $\mathfrak{t} \subseteq \mathfrak{k}$ of diagonal matrices is a compactly embedded Cartan subalgebra of $\mathfrak{g}$. In $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ we have

$$\Delta_k = \{\pm(\varepsilon_1 - \varepsilon_2)\}, \quad \Delta_p^+ = \{\alpha \in \Delta_p \colon \alpha(h_c) = 1\} = \{\varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3\}$$

and $\Gamma := \{\varepsilon_1 - \varepsilon_3\}$ is a maximal system of strongly orthogonal roots. In $\mathfrak{p}$ the set Γ leads to the maximal abelian subspace

$$\mathfrak{a} = \mathbb{R}h \quad \text{with} \quad h := \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which generates with $\mathfrak{a}$ the Lie algebra

$$\mathfrak{s} = \mathfrak{t} + [\mathfrak{s}, \mathfrak{s}] \cong \mathfrak{gl}_2(\mathbb{R}) \quad \text{with} \quad [\mathfrak{s}, \mathfrak{s}] = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ \bar{b} & 0 & -a \end{pmatrix} \colon a \in i\mathbb{R}, b \in \mathbb{C} \right\} \cong \mathfrak{su}_{1,1}(\mathbb{C}).$$

In $\mathfrak{s}$ the element ih_c decomposes into $ih_c^0 + ih_c^1$, where ih_c^0 is central and $ih_c^1 \in [\mathfrak{s}, \mathfrak{s}]$. Concretely, we have

$$h_c^0 = \frac{1}{6}\operatorname{diag}(-1, 2, -1), \quad h_c^1 = \frac{1}{3}\operatorname{diag}(1, 0, -1).$$

In $i\mathfrak{t}$ we have the cone

$$C^{\max} := \{\operatorname{diag}(x_1, x_2, -x_1 - x_2) \colon 2x_1 + x_2 \geq 0, 2x_2 + x_1 \geq 0\}$$

$$\supseteq C^{\min} = \operatorname{cone}(\operatorname{diag}(1, 0, -1), \operatorname{diag}(0, 1, -1)).$$

We have $\qquad p_{i\mathfrak{g}}(e^{\frac{\pi i}{2}\operatorname{ad} h}h_c) = \cos\left(\frac{\pi}{2}\operatorname{ad} h\right)h_c = h_c^0 \in \partial C^{\max} \setminus C^{\min}$.

Therefore the real crown domain in $\operatorname{Ad}(G_{\mathbb{C}})h_c$ is **not** contained in the tube domain $\mathfrak{g} + iC_{\min,\mathfrak{g}} \subseteq \mathfrak{g}_{\mathbb{C}}$.

7. Components of the stabilizer group G^h

This section is devoted to an analysis of the group $\pi_0(G^h)$ of connected components of the centralizer G^h of an Euler element h in a simple real Lie algebra. With a polar decomposition $G^h = K^h \exp(\mathfrak{h}_{\mathfrak{p}})$, this group equals $\pi_0(K^h)$. As $K/K^h \cong \mathcal{O}_h^K := \operatorname{Ad}(K)h$ is a compact symmetric space, we discuss this problem in Section 7.1 in the context of compact symmetric spaces, where $\pi_0(K^h)$ appears as a quotient of $\pi_1(K/K^h)$ in the long exact homotopy sequence

$$\pi_1(K) \to \pi_1(K/K^h) \twoheadrightarrow \pi_0(K^h) \to \pi_0(K) = \mathbf{1}.$$

In Section 7.2 we explore this situation further, using that $\mathcal{O}_h^K$ actually is a symmetric R-space. Here the strongly orthogonal roots come in handy and permit us in Theorem 7.8 to show that G^h is connected if $(\mathfrak{g}, \tau)$ is either of complex type or non-split type (cf. Section 4.4), and if it is of split type or Cayley type, then it either is trivial or $\mathbb{Z}_2$. In Section 7.3 we finally collect some consequences of this result such as the identity

$$\operatorname{Inn}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{g}^c) \cap \operatorname{Inn}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{g}) = \operatorname{Inn}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{k})^h e^{\operatorname{ad} \mathfrak{h}_{\mathfrak{p}}}.$$

7.1. The fundamental group of a compact symmetric space

Let G be a connected symmetric Lie group with *compact* Lie algebra $\mathfrak{g}$, τ an involutive automorphism of G, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ the corresponding eigenspace decomposition, $H \subseteq G^\tau$ an open subgroup, $X = G/H$ the corresponding compact symmetric space and $q_X \colon G \to X$ the quotient map with the base point $e_X := q(e)$. We pick a maximal abelian subspace $\mathfrak{t}_\mathfrak{q} \subseteq \mathfrak{q}$ and enlarge it to a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$, so that $\mathfrak{t}$ is τ-invariant and $\mathfrak{t} = \mathfrak{t}_\mathfrak{h} \oplus \mathfrak{t}_\mathfrak{q}$.

We write $q_G \colon \widetilde{G} \to G$ for the simply connected covering. For a subgroup $B \subseteq G$, we write $B^* := q_G^{-1}(B)$ for its inverse image in $\widetilde{G}$. Then $X = G/H \cong \widetilde{G}/H^*$, which leads to an isomorphism $\widetilde{X} \cong \widetilde{G}/H_e^*$ and thus to $\pi_1(X) \cong \pi_0(H^*)$ ([15, Cor. 11.1.14]).

In G we have the tori $T_Q := \exp(\mathfrak{t}_\mathfrak{q}) \subseteq T := \exp(\mathfrak{t})$.

Now $T_X := \mathrm{Exp}(\mathfrak{t}_\mathfrak{q}) = q_X(T_Q) \subseteq X$ is a maximal flat torus whose fundamental group is the lattice
$$\Gamma_X := \{x \in \mathfrak{t}_\mathfrak{q} \colon \mathrm{Exp}(x) = e_X\}.$$

We likewise have $T_{\widetilde{X}} \subseteq \widetilde{X}$ and a discrete subgroup $\Gamma_{\widetilde{X}} \subseteq \Gamma_X$. On the group level we likewise define
$$\Gamma_G := \{x \in \mathfrak{t} \colon \exp_G x = e\} \cong \pi_1(T) \quad \text{and} \quad \Gamma_{\widetilde{G}} := \{x \in \mathfrak{t} \colon \exp_{\widetilde{G}} x = e\} \cong \pi_1(T^*).$$

In the following lemma note that $Z(G) = \{e\}$ if and only if $G = \mathrm{Inn}(\mathfrak{g})$.

Lemma 7.1. *The following assertions hold:*
  (a) *The inclusion $T \hookrightarrow G$ induces an isomorphism $\Gamma_G/\Gamma_{\widetilde{G}} \to \pi_1(G)$.*
  (b) *If $Z(G) = \{e\}$, then $\pi_1(G) \cong Z(\widetilde{G})$ and $\Gamma_G = \{x \in \mathfrak{t} \colon e^{\mathrm{ad}\,x} = \mathbf{1}\}$.*
  (c) *$G = \exp_G \mathfrak{h} \exp_G \mathfrak{q}$.*
  (d) *$H^* = H_e^* \exp(\{x \in \mathfrak{t}_\mathfrak{q} \colon e^{2\,\mathrm{ad}\,x} = \mathbf{1}\}) \subseteq H_e^*(H^* \cap T_Q)$.*
  (e) *$H = H_e(H \cap T_Q)$.*

Proof. (a) follows from [15, Cor. 14.2.10] (see also [13, Thm. VII.6.7]).

(b) If $Z(G) = \{e\}$, then G is semisimple and $Z(\widetilde{G}) = \ker q_G \cong \pi_1(G)$.

(c) First we assume that G is semisimple, hence compact. As a closed subgroup of G, the group H has only finitely many connected components and $\widehat{X} := G/H_e$ is still compact. The surjectivity of the Riemannian exponential function of $\widehat{X}$ ([13, Thm. I.10.3/4]) implies $\widehat{X} = \mathrm{Exp}(\mathfrak{q})$, and therefore $G = \exp(\mathfrak{q})H_e = \exp \mathfrak{q} \exp \mathfrak{h}$. Here we use that the exponential function of the compact group H_e is surjective. Now we turn to the general case. As the commutator group G' with the semisimple Lie algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is compact, the preceding argument yields
$$G' = \exp(\mathfrak{q} \cap \mathfrak{g}') \exp(\mathfrak{h} \cap \mathfrak{g}').$$

For the, possibly non-compact, group G, we now obtain
$$G = Z(G)_e G' = \exp(\mathfrak{h} \cap \mathfrak{z}(\mathfrak{g})) \exp(\mathfrak{q} \cap \mathfrak{z}(\mathfrak{g})) \exp(\mathfrak{q} \cap \mathfrak{g}') \exp(\mathfrak{h} \cap \mathfrak{g}') = \exp(\mathfrak{q}) \exp(\mathfrak{h}).$$

(d) (cf. [13, Thm. VII.9.1]) If $h \in H^*$, then $q_G(h) \in G^\theta$ implies that
$$z := h\theta(h)^{-1} \in \ker(q_G) \subseteq Z(\widetilde{G}).$$

In view of (c), there exists an $x \in \mathfrak{q}$ and $g \in H_e^* = \widetilde{G}^\tau$ with $h = \exp(x)g$, so that

$$z = h\theta(h)^{-1} = \exp(2x) \in Z(\widetilde{G}).$$

Now we pick $h_1 \in \widetilde{G}^\tau$ with $x_1 := \mathrm{Ad}(h_1)x \in \mathfrak{t}_\mathfrak{q}$ ([24, Thm. II.8]) and find

$$z = h_1 z h_1^{-1} = \exp(2\,\mathrm{Ad}(h_1)x) = \exp(2x_1).$$

Then $\mathbf{1} = \mathrm{Ad}(z) = e^{2\,\mathrm{ad}\,x_1}$, and (d) follows.

(e) follows immediately from (d) by applying q_G.   ∎

Proposition 7.2. *The homomorphism $\iota \colon \pi_1(T_X) \cong \Gamma_X \to \pi_1(X)$ induced by the inclusion $T_X \hookrightarrow X$ is surjective and induces an isomorphism*

$$\Gamma_X / \Gamma_{\widetilde{X}} \to \pi_1(X).$$

Proof. First we show that ι is surjective. We shall use the exact sequence

$$\pi_1(G) \to \pi_1(X) \overset{\delta}{\longrightarrow} \pi_0(H), \tag{30}$$

which is part of the long exact homotopy sequence of the principal H-bundle $G \to X$. Every element in $\pi_0(H)$ can be represented by an element $g = \exp x$ in $T_Q \cap H$ (Lemma 7.1(e)), and then $x \in \Gamma_X$. This shows that $\delta \circ \iota$ is surjective.

It remains to see that $\ker \delta \subseteq \operatorname{im} \iota$. So let $[\gamma] \in \ker \delta$. By (30), $[\gamma]$ comes from a loop in G, hence from a loop in T (Lemma 7.1(a)). We may therefore assume that $\gamma(t) = \exp(tx)H$ with $x \in \mathfrak{t}$ satisfying $\exp(x) = e$. Writing $x = x_\mathfrak{h} + x_\mathfrak{q}$ with $x_\mathfrak{h} \in \mathfrak{t}_\mathfrak{h}$ and $x_\mathfrak{q} \in \mathfrak{t}_\mathfrak{q}$, we then have

$$\gamma(t) = \exp(tx_\mathfrak{q})H = \mathrm{Exp}(tx_\mathfrak{q}) \quad \text{with} \quad \mathrm{Exp}(x_\mathfrak{q}) = \exp(x_\mathfrak{q})H = e_X.$$

This implies $x_\mathfrak{q} \in \Gamma_X$ and shows that ι is surjective.

That $\ker \iota$ is $\Gamma_{\widetilde{X}}$ follows immediately from the fact that, for $x \in \Gamma_X$, the curve $\gamma_x(t) := \mathrm{Exp}_X(tx)$ lifts to a loop in $\widetilde{X}$ if and only if $\mathrm{Exp}_{\widetilde{X}}(x) = e_{\widetilde{X}}$, which means that $x \in \Gamma_{\widetilde{X}}$.   ∎

Remark 7.3. [13, Thm. VII.8.5] also provides a description of the discrete subgroup $\Gamma_{\widetilde{X}} \subseteq \mathfrak{t}_\mathfrak{q}$ in terms of root data. We have

$$\Gamma_{\widetilde{X}} = \frac{1}{2}\Gamma_{T_Q^*} \quad \text{for} \quad \Gamma_{T_Q^*} := \{x \in \mathfrak{t}_\mathfrak{q} \colon \exp_{\widetilde{G}} x = e\}. \tag{31}$$

In fact, the simply connected covering space $\widetilde{X} \cong \widetilde{G}/H_e^*$ can be identified with the identity component $\widetilde{G}_e^{-\theta}$ of the set $\widetilde{G}^{-\theta} = \{g \in \widetilde{G} \colon \theta(g) = g^{-1}\}$ on which $\widetilde{G}$ acts transitively by $g.x := gx\theta(g)^{-1}$ and the stabilizer of e is the connected subgroup $\widetilde{G}^\theta$. This leads to the realization $T_{\widetilde{X}} = \mathrm{Exp}_{\widetilde{X}}(\mathfrak{t}_\mathfrak{q}) \subseteq \widetilde{G}$ with $\mathrm{Exp}_{\widetilde{X}}(x) = \exp(2x)$, and this proves (31). Moreover, by [13, Thm. VII.8.5], the subgroup $\Gamma_{\widetilde{X}} \subseteq \mathfrak{t}_\mathfrak{q}$ is generated by the elements of the form

$$\frac{2\pi i}{\langle \alpha, \alpha \rangle} A_\alpha, \quad \alpha \in \Sigma,$$

where $A_\alpha \in i\mathfrak{t}_\mathfrak{q}$ is the unique element satisfying $\kappa(A_\alpha, x) = \alpha(x)$ for $x \in \mathfrak{t}_\mathfrak{q}$ and the Cartan-Killing form κ.

Remark 7.4. The recent preprint [3] deals with a closely related problem: the determination of $\pi_0(G(\mathbb{R}))$ for a connected complex reductive group $G(\mathbb{C})$. Let $G := G(\mathbb{C})$ and write σ for the antiholomorphic involution with $G(\mathbb{R}) = G^\sigma$. We also write θ for the antiholomorphic involution on G for which $U := G^\theta$ is compact and assume that σ commutes with θ, so that θ induces a Cartan involution on $G(\mathbb{R})$. Then polar decomposition implies that

$$\pi_0(G(\mathbb{R})) = \pi_0(U^\sigma),$$

so that it is about the component group of U^σ, where U is a connected compact Lie group.

7.2. Application to symmetric R-spaces

In this subsection $\mathfrak{g}$ denotes a semisimple Lie algebra, $h \in \mathfrak{g}$ an Euler element and θ is a Cartan involution with $\theta(h) = -h$, i.e., $h \in \mathfrak{p}$.

Remark 7.5. (a) If τ_h is the involution defined by h, then $\mathfrak{p}$ decomposes into eigenspaces $\mathfrak{p}^{\tau_h}$ and $\mathfrak{p}^{-\tau_h}$ and the K-orbit $\mathcal{O}_h^K = \mathrm{Ad}(K)h \subseteq \mathfrak{p}$ is τ_h-invariant. Moreover, τ_h induces on this orbit an involution turning it into a so-called *extrinsic symmetric space*, for the definition see [9]. In [9] Ferus shows that all extrinsic symmetric submanifolds embed into such spaces.

(b) The compact symmetric spaces of the form $\mathcal{O}_h^K$ are precisely the symmetric R-spaces, when considered as homogeneous spaces of K ([26]). We refer to [33] for their interpretation as real flag manifolds G/P^- on which K acts transitively with point stabilizer $K \cap P^- = K^h$.

Let $G = \mathrm{Inn}(\mathfrak{g})$, $K = \mathrm{Inn}_\mathfrak{g}(\mathfrak{k})$, and consider the symmetric R-space

$$X := \mathrm{Ad}(K)h \subseteq \mathfrak{p}$$

defined by the Euler element in $h \in \mathfrak{p}$. We want to determine the group $\pi_0(G^h) \cong \pi_0(K^h)$ of connected components of G^h, resp., K^h.

We consider the torus $T_X := e^{\mathrm{ad}\, \mathfrak{t}_\mathfrak{q}}h \subseteq X$ with the fundamental group

$$\Gamma_X = \{x \in \mathfrak{t}_\mathfrak{q} : e^{\mathrm{ad}\, x}h = h\}.$$

We shall need the following piece of the long exact homotopy sequence of the K^h-principal bundle $K \to X$:

$$\pi_2(X) \hookrightarrow \pi_1(K^h) \to \pi_1(K) \xrightarrow{\pi_1(q)} \pi_1(X) \to \pi_0(K^h) \to \mathbf{1},$$

where $q\colon K \to X \cong K/K^h$ is the orbit map. Then the surjectivity of the homomorphism $\pi_1(X) \to \pi_0(K^h)$ implies that

$$\pi_0(K^h) \cong \mathrm{coker}(\pi_1(q)).$$

With Prop. 7.2 we see that $\pi_1(X) \cong \Gamma_X/\Gamma_{\widetilde{X}}$, so that we obtain a surjective map

$$\delta\colon \Gamma_X \to \pi_0(K^h), \quad x \mapsto [e^{\mathrm{ad}\, x}]. \tag{32}$$

We shall now determine the range of this map with the aid of Proposition 5.2, where we have seen that h and $\mathfrak{t}_\mathfrak{q}$ generate a reductive Lie algebra

$$\mathfrak{l} \cong \mathbb{R}h_0 \oplus \mathfrak{sl}_2(\mathbb{R})^s \supseteq \mathfrak{t}_\mathfrak{q} = \mathfrak{so}_2(\mathbb{R})^s, \qquad s = r_0 + r_1.$$

We write $L = \langle \exp \mathfrak{l} \rangle \subseteq G$ for the corresponding integral subgroup.

Lemma 7.6. *The following assertions hold:*

(a) *The torus $T_Q := \exp(\mathfrak{t_q})$ satisfies $T_Q^h = \exp(\Gamma_X) = Z(L')$, where $L' \subseteq L$ denotes the commutator subgroup.*

(b) *Write $L = Z(L)_e L_1 \cdots L_s$, where L_j is the integral subgroup corresponding to the j-th simple ideal $\mathfrak{l}_j$ in $\mathfrak{l}$. Then $Z(L) = Z(L)_e Z(L_1) \cdots Z(L_s)$, where the group $Z(L_j)$ is either trivial or contains two elements. In particular we have*

$$T_Q^h = Z(L_1) \cdots Z(L_s).$$

(c) *The subgroup $Z(L_1) \cdots Z(L_{r_0}) \subseteq T_Q^h$ maps surjectively onto $\pi_0(K^h)$.*

Proof.　　(a)　In $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{sl}_2(\mathbb{C})$ any element $x \in \mathfrak{so}_2(\mathbb{R})$ satisfying $e^{\operatorname{ad} x} h = h$ for an Euler element h also satisfies $e^{\operatorname{ad} x} = \mathbf{1}$. Therefore $T_Q^h \subseteq Z(L')$. The converse inclusion follows from $Z(L') \subseteq \exp(\mathfrak{t_q}) = T_Q$, which implies that $Z(L') \subseteq T_Q^h$.

(b)　As the center $Z(L_j)$ commutes for each j with all of L, we have $Z(L) = Z(L)_e Z(L_1) \cdots Z(L_s)$. The subgroup $Z(L_j)$ is the image of the center of a group of the type $\mathrm{SL}_2(\mathbb{R})$, hence contains at most two elements. Now $Z(L_1) \cdots Z(L_s) = Z(L')$, combined with (a), implies (b).

(c)　For $j > r_0$, the ideal $\mathfrak{l}_j$ is contained as $\mathfrak{sl}_2(\mathbb{R})$ in a complex τ-invariant Lie algebra $\mathfrak{s}_j \cong \mathfrak{sl}_2(\mathbb{C})$. If $S_j := \langle \exp \mathfrak{s}_j \rangle$, then $K_j := K \cap S_j \cong \mathrm{SU}_2(\mathbb{C})$ or $\mathrm{PSU}_2(\mathbb{C})$ and $\mathrm{Ad}(K_j)h$ is a 2-sphere, hence simply connected. Therefore K_j^h is connected and therefore $Z(L_j)$ does not contribute to $\pi_0(K^h)$. This implies (c). ∎

Remark 7.7.　　(a)　The fact that $h \in \mathfrak{l}$ is an Euler element of $\mathfrak{g}$ restricts the possibilities for irreducible $\mathfrak{l}$-submodules of $\mathfrak{g}$ significantly. Writing V_k^j for the k-dimensional simple complex module of the jth ideal $\mathfrak{l}_j$, we see that the only non-trivial complex simple $\mathfrak{l}$-submodules of $\mathfrak{g}_\mathbb{C}$ are the adjoint modules V_3^j, tensor products $V_2^{j_1} \otimes V_2^{j_2}$ for $j_1 \neq j_2$, and, if $h_0 \neq 0$, also 2-dimensional modules V_2^j on which h_0 acts by $\pm \frac{1}{2} \mathbf{1}$. On the other two types the central element h_0 acts trivially.

Let $z_j \in Z(L_j)$ be generators. Then $z_{j_1} z_{j_2}$ acts trivially on $V_3^{j_1}$ and $V_2^{j_1} \otimes V_2^{j_2}$, but non-trivially on $V_2^{j_1} \otimes V_2^{j_3}$ for $j_3 \neq j_1, j_2$.

(b)　Write $h \in \mathfrak{l}$ as $h = h_0 + h_1$ with $0 \neq h_0 \in \mathfrak{z}(\mathfrak{l})$ and $h_1 \in [\mathfrak{l}, \mathfrak{l}]$. Then the structure of the restricted root system implies that h_1 defines a 5-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{-1}(h_1) \oplus \mathfrak{g}_{-1/2}(h_1) \oplus \mathfrak{g}_0(h_1) \oplus \mathfrak{g}_{1/2}(h_1) \oplus \mathfrak{g}_1(h_1),$$

where h_0 commutes with $\mathfrak{g}_{\pm 1}(h_1)$. Therefore

$$\mathfrak{g}_{\mathrm{tt}} := \mathfrak{z}_{\mathfrak{g}}(h_0) = \mathfrak{g}_{-1}(h_1) \oplus (\mathfrak{g}_0(h_0) \cap \mathfrak{g}_0(h_1)) \oplus \mathfrak{g}_1(h_1) \subseteq \mathfrak{g}$$

is a 3-graded τ-invariant subalgebra in which h_1 is an Euler element. Let r_0', r_1' and $s' := r_0' + r_1'$ be the corresponding numbers. We claim that

$$r_0 = r_0', \quad r_1 = r_1' \quad \text{and} \quad s' = s. \tag{33}$$

In fact, as h_0 is hyperbolic $r' = \mathrm{rk}_\mathbb{R}(\mathfrak{g}_{\mathrm{tt}}^c) = \mathrm{rk}_\mathbb{R}(\mathfrak{g}^c) = r$. Further $\mathfrak{t_q} \subseteq \mathfrak{l} \subseteq \mathfrak{g}_{\mathrm{tt}}$ shows that $s' = \mathrm{rk}_\mathbb{R}(\mathfrak{h}_{\mathrm{tt}}) = s$. This implies that $r_1' = r' - s' = r - s = r_1$ and $r_0' = s' - r_1' = s - r_1 = r_0$.

Theorem 7.8. (The group $\pi_0(G^h)$) *Let* $G = \mathrm{Inn}(\mathfrak{g})$. *For the group* $\pi_0(K^h) \cong \pi_0(G^h)$, *the following assertions hold:*

(a) *If* $(\mathfrak{g}, \tau)$ *is of* complex type *or* non-split type, *then* G^h *is connected.*

(b) *If* $(\mathfrak{g}, \tau)$ *is of* Cayley type *or* split type, *then* G^h *is connected if* r *is odd. If* r *is even, then it is not connected and* $\pi_0(G^h) \cong \mathbb{Z}_2$ *in the following cases of Cayley type:*

- $r = 2$ *and* $\mathfrak{g} = \mathfrak{so}_{2,n}(\mathbb{R})$ *with* $n \geq 3$ *odd.*
- $r \geq 4$ *even and* $\mathfrak{g} = \mathfrak{sp}_{2r}(\mathbb{R})$,

and the following cases of split type:

- $\mathfrak{g} = \mathfrak{sl}_{2n}(\mathbb{R})$ *with* $h = \frac{1}{2}(\mathbf{1}_n, -\mathbf{1}_n)$ *and* n *even.*
- $\mathfrak{g} = \mathfrak{so}_{p,q}(\mathbb{R})$ *with* $p, q > 2$ *and* $p + q$ *odd.*
- $\mathfrak{g} = \mathfrak{so}_{2n,2n}(\mathbb{R})$.

Proof. (a) If $(\mathfrak{g}, \tau)$ is of *complex type*, then $\mathfrak{g} = \mathfrak{h}_\mathbb{C}$ and τ is antilinear. Here $\mathfrak{c} = \mathfrak{a}_\mathbb{C}$, where $i\mathfrak{a} \subseteq \mathfrak{h}_\mathfrak{k}$ is a compactly embedded Cartan subalgebra of $\mathfrak{h}$. For every root $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{c})$ we have $(-\tau)\alpha = \overline{\alpha}$. As the roots are complex linear on $\mathfrak{a}_\mathbb{C}$, no root is fixed by $-\alpha$, and thus $r_0 = 0$ and $s = r_1$. Hence K^h is connected by Lemma 7.6(c).

If $(\mathfrak{g}, \tau)$ is of *non-split type*, then $s = \mathrm{rk}_\mathbb{R}(\mathfrak{h})$ and $r = \mathrm{rk}_\mathbb{R}(\mathfrak{g}^c) = 2s$, so that $r_0 = 0$ and $r_1 = s$. Therefore Lemma 7.6(c) implies that K^h is connected.

(b) We first discuss *Cayley type* Lie algebras. Let

$$r := \mathrm{rk}_\mathbb{R}(\mathfrak{g}) \quad \text{and} \quad s = \mathrm{rk}_\mathbb{R}(\mathfrak{h}) = \mathrm{rk}_\mathbb{R}(\mathfrak{g}^h).$$

Then $s \leq r$ and there exists a subalgebra $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})^r$ containing h, so that we must have $s = r$, and therefore $r_1 = 0$.

If $r = 1$, then $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ and $K^h = Z(G)$ is trivial.

If $r > 1$, then the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ is of type C_r ($\mathfrak{g}$ is hermitian of tube type), which implies that the simple L-submodules of $\mathfrak{g}$ are the adjoint modules $\mathfrak{l}_j$ and the tensor products $V_{j_1,j_2} = V_2^{j_1} \otimes V_2^{j_2}$. Note that Cayley type algebras are hermitian of tube type, so that $h_0 = 0$ follows from Proposition 5.2(c). For any product $z := z_{j_1} \cdots z_{j_k} \in Z(L)$ with $j_1 < \ldots < j_k$ and $k < r$, we then find a tensor product V_{j_1,j_2} on which z acts non-trivially, and $z_1 \cdots z_r = \mathbf{1}$. This shows that $Z(L) \cong \mathbb{Z}_2^{r-1}$.

The restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ is of type BC_r with the subsystem $\Sigma(\mathfrak{g}^h, \mathfrak{a})$ of type A_{r-1}. The corresponding Weyl group is the symmetric group S_r which acts by permutations of the central involutions $z_1, \ldots, z_r$ of L. Since the Weyl group elements are induced by the normalizer of $\mathfrak{a}$ in the connected group $\mathrm{Inn}_\mathfrak{g}(\mathfrak{h}_\mathfrak{k})$, it follows that $z_1, \ldots, z_r$ all lie in the same connected component of K^h. As these elements are involutions, all even products $z_{j_1} \cdots z_{j_k}$ are contained in K_e^h. It follows that $|\pi_0(K^h)| \leq 2$ and, $z_1 \cdots z_r = \mathbf{1}$ further implies that $\pi_0(K^h)$ is trivial if r is odd. So the only cases that have to be inspected in detail arise for r even.

Case 1: $r = 2$. Then $\mathfrak{g} = \mathfrak{so}_{2,n}(\mathbb{R})$ for some $n \geq 3$. In this case $G \cong \mathrm{PSO}_{2,n}(\mathbb{R})_e$. Let $G^* := \mathrm{SO}_{2,n}(\mathbb{R})_e$ with maximal compact subgroup $K^* = \mathrm{SO}_2(\mathbb{R}) \times \mathrm{SO}_n(\mathbb{R})$.

Here $h \in \mathfrak{so}_{2,n}(\mathbb{R})$ is a diagonalizable rank-2 element whose ± 1-eigenspaces are isotropic, f.i., $\mathbf{e}_2 \pm \mathbf{e}_3$ (up to conjugacy). Hence its centralizer leaves the plane $F := \mathbb{R}\mathbf{e}_2 + \mathbb{R}\mathbf{e}_3$ invariant and we thus obtain a homomorphism $(K^*)^h \to \mathrm{O}_{1,1}(\mathbb{R})$. The centralizer of the Lorentz boost in $\mathfrak{so}_{1,1}(\mathbb{R})$ is $\mathrm{SO}_{1,1}(\mathbb{R})$ which is not connected and contains $-\mathrm{id}_{\mathbb{R}^2}$. As $\mathrm{diag}(-1,-1,-1,-1,1,\ldots) \in (K^*)^h$ maps to $-\mathrm{id}_{\mathbb{R}^2}$, it follows that $(K^*)^h$ has at least 2-connected components.

The invariance of F under $(K^*)^h$ also shows that it leaves $\mathbb{R}\mathbf{e}_2$ and $\mathbb{R}\mathbf{e}_3$ invariant, so that

$$(K^*)^h \subseteq \mathrm{S}(\mathrm{O}_1(\mathbb{R}) \times \mathrm{O}_1(\mathbb{R})) \times \mathrm{S}(\mathrm{O}_1(\mathbb{R}) \times \mathrm{O}_{n-1}(\mathbb{R}))$$

$$\cong \{\pm \mathbf{1}_{\mathbb{R}^2}\} \times \mathrm{S}(\mathrm{O}_1(\mathbb{R}) \times \mathrm{O}_{n-1}(\mathbb{R})),$$

where the group on the right has 4 connected components. As the group on the right maps surjectively onto $\mathrm{O}_{1,1}(\mathbb{R})$ and $(K^*)^h$ onto $\mathrm{SO}_{1,1}(\mathbb{R})$, we see that $(K^*)^h$ has exactly 2 components.

If n is odd, then $-\mathbf{1} \notin G^*$, so that $G \cong G^*$ and $K \cong K^*$. It follows that G^h has two connected components.

If n is even, then $-\mathbf{1} \in \mathrm{SO}_{2,n}(\mathbb{R})_e$ and $G \cong \mathrm{SO}_{2,n}(\mathbb{R})_e/\{\pm\mathbf{1}\}$. Then $-\mathbf{1} \notin (K^*)^h_e$ (consider the restriction to F) and the fact that $(K^*)^h$ has 2 connected components imply that K^h is connected.[4]

Case 2: $r \geq 3$. Let $E = \mathfrak{g}_1(h)$ be the euclidean Jordan algebra for which $\mathfrak{g}$ is the conformal Lie algebra.

- For $E = \mathrm{Sym}_r(\mathbb{R})$, we consider $G^* := \mathrm{Sp}_{2r}(\mathbb{R})$ with $(G^*)^h \cong \mathrm{GL}_r(\mathbb{R})$, which has 2 connected components and contains $Z(G^*) = \{\pm\mathbf{1}\}$. Therefore $G^h \cong (G^*)^h/\{\pm\mathbf{1}\}$ is connected if and only if $-\mathbf{1} \notin \mathrm{GL}_r(\mathbb{R})_e$, which is equivalent to $\det(-\mathbf{1}) = (-1)^r = -1$. This corresponds to r odd. If r is even, then G^h is not connected.

- For $E = \mathrm{Herm}_r(\mathbb{C})$ we consider $G^* := \mathrm{SU}_{r,r}(\mathbb{C})$ with

$$(G^*)^h \cong \{g \in \mathrm{GL}_r(\mathbb{C})\colon \det(g) \in \mathbb{R}\}.$$

This group has two connected components, corresponding to the sign of the real-valued determinant. It contains

$$Z(G^*) = \{\zeta \mathbf{1}_{2r}\colon \zeta \in \mathbb{C}^\times, \zeta^{2r} = 1\}$$

$$\cong \{\zeta \mathbf{1}_r\colon \zeta \in \mathbb{C}^\times, \det(\zeta \mathbf{1}_r) = \zeta^r \in \{\pm 1\}\} \cong C_{2r}.$$

If $\zeta \in \mathbb{C}^\times$ is an rth root of -1, then $\zeta \mathbf{1}_r \in Z(G^*) \cap (G^*)^h$ with $\det(\zeta \mathbf{1}_r) < 0$ shows that both connected components of $(G^*)^h$ intersect $Z(G^*)$.

Therefore $G^h = \mathrm{Ad}((G^*)^h) \cong (G^*)^h/Z(G^*)$ is connected.

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[4] The group $G = \mathrm{PSO}_{2,n}(\mathbb{R})_e$ acts by causal automorphisms on the causal compactification $\widehat{M} \cong (\mathbb{S}^1 \times \mathbb{S}^{n-1})/\{\pm\mathbf{1}\}$ of n-dimensional Minkowski space $M = \mathbb{R}^{1,n-1}$. If $\widehat{M}$ is orientable, then the connected group preserves the orientation. As the subgroup $G^h \subseteq \mathbb{R}_+^\times \mathrm{O}_{1,n-1}(\mathbb{R})$ fixes 0 and acts by linear maps, this implies that $G^h \subseteq \mathbb{R}_+^\times \mathrm{SO}_{1,n-1}(\mathbb{R})_e$, and since the latter group is connected, it follows that $G^h \cong \mathrm{SO}_{1,n-1}(\mathbb{R})_e$ is connected. The manifold $\widehat{M}$ is orientable if and only if the antipodal map is orientation preserving on $\mathbb{S}^{n-1}$, i.e., if n is even. If this is not the case, then $\widehat{M}$ is not orientable and $G^h \cong \mathbb{R}_+^\times \mathrm{O}_{1,n-1}(\mathbb{R})^\uparrow$ is not connected.

- For $E = \mathrm{Herm}_r(\mathbb{H})$ we consider

$$G^* := \mathrm{SO}^*(4r) := \{g \in \mathrm{SU}_{2r,2r}(\mathbb{C}) \colon g^\top A g = A\} \quad \text{for} \quad A = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

Then $\qquad K^* := G^* \cap \mathrm{U}_{4r}(\mathbb{C}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-\top} \end{pmatrix} : a \in \mathrm{U}_{2r}(\mathbb{C}) \right\} \cong \mathrm{U}_{2r}(\mathbb{C}).$

In the Lie algebra

$$\mathfrak{so}^*(4r) = \left\{ \begin{pmatrix} a & b \\ b^* & -a^\top \end{pmatrix} : a \in \mathfrak{u}_{2r}(\mathbb{C}), b \in \mathrm{Skew}_{2r}(\mathbb{C}) \right\}$$

an Euler element is given by

$$h = \frac{1}{2} \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \quad \text{for} \quad J := \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \in \mathrm{Skew}_{2r}(\mathbb{C}).$$

An element $k = \mathrm{diag}(a, a^{-\top})$, $a \in K^*$, commutes with h if and only if $J = aJa^\top = aJ\bar{a}^{-1}$, i.e., $aJ = J\bar{a}$. One readily checks that this is equivalent to a being of the form $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$, i.e., to $a \in \mathrm{U}_r(\mathbb{H}) \subseteq \mathrm{U}_{2r}(\mathbb{C})$. We conclude that $(K^*)^h \cong \mathrm{U}_r(\mathbb{H})$ is connected. This implies that $G^h = \mathrm{Ad}((G^*)^h)$ is connected.

- For $E = \mathrm{Herm}_3(\mathbb{O})$ and $\mathfrak{g} = \mathfrak{e}_{7(-25)}$, the rank $r = 3$ is odd, so that G^h is connected by the discussion above.

Now we turn to *split type* Lie algebras. Then

$$s = \mathrm{rk}_{\mathbb{R}}(\mathfrak{h}) = \mathrm{rk}_{\mathbb{R}}(\mathfrak{g}^c) = r, \quad \text{so that} \quad r_0 = r = s \quad \text{and} \quad r_1 = 0$$

(see Table 3). As in (c), we see that G^h is connected if r is odd and that it has at most 2 connected components.

- For $\mathfrak{g} = \mathfrak{sl}_{2n}(\mathbb{R})$ and $G^* = \mathrm{SL}_{2n}(\mathbb{R})$, we have

$$(G^*)^h = \mathrm{S}(\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R})),$$

which has two connected components. As $-\mathbf{1} \in (G^*)^h$ is contained in the identity component if and only if n is even, it follows that G^h is connected if and only if n is odd.

- For $\mathfrak{g} = \mathfrak{so}_{p,q}(\mathbb{R})$, we consider $G^* = \mathrm{SO}_{p,q}(\mathbb{R})_e$, $p, q > 2$, with maximal compact subgroup $K^* = \mathrm{SO}_p(\mathbb{R}) \times \mathrm{SO}_q(\mathbb{R})$. Here $h \in \mathfrak{so}_{p,q}(\mathbb{R})$ is a diagonalizable rank-2 element whose ± 1-eigenspaces are one-dimensional isotropic, f.i., $\mathbf{e}_p \pm \mathbf{e}_{p+1}$. Hence its centralizer $(G^*)^h$ leaves the plane $F := \mathbb{R}\mathbf{e}_p + \mathbb{R}\mathbf{e}_{p+1}$ invariant. This shows that

$$(K^*)^h \subseteq \mathrm{S}(\mathrm{O}_1(\mathbb{R}) \times \mathrm{O}_{p-1}(\mathbb{R})) \times \mathrm{S}(\mathrm{O}_1(\mathbb{R}) \times \mathrm{O}_{q-1}(\mathbb{R})) \cong \mathrm{O}_{p-1}(\mathbb{R}) \times \mathrm{O}_{q-1}(\mathbb{R}).$$

The fact that $(K^*)^h$ preserves the 1-dimensional subspaces generated by $\mathbf{e}_p \pm \mathbf{e}_{p+1}$ shows that $(K^*)^h$ has 2-connected components and that its restriction to the Minkowski plane F contains $-\mathbf{1}$.

If $p + q$ is odd, then $-\mathbf{1} \notin G^*$, so that $G \cong G^*$ and G^h has two connected components.

If $p + q$ is even, then $-\mathbf{1} \in G^*$ and $G \cong G^*/\{\pm\mathbf{1}\}$. Then $-\mathbf{1} \notin (K^*)_e^h$ (consider the restriction to F) implies that K^h is connected.

- We realize the split real form $\mathfrak{so}_{2n,2n}(\mathbb{R}) \subseteq \mathfrak{so}_{4n}(\mathbb{C})$ as

$$\mathfrak{so}_{2n,2n}(\mathbb{R}) = \left\{ X \in \mathfrak{gl}_{4n}(\mathbb{R}) \colon XI + IX^\top = 0 \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & -a^\top \end{pmatrix} \colon b, c \in \mathrm{Skew}_{2n}(\mathbb{R}), a \in \mathfrak{gl}_{2n}(\mathbb{R}) \right\}$$

for $I = \begin{pmatrix} 0 & \mathbf{1}_{2n} \\ \mathbf{1}_{2n} & 0 \end{pmatrix}$. This exhibits the Euler element

$$h := \frac{1}{2} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \quad \text{with} \quad \mathfrak{g}_1(h) \cong \mathrm{Skew}_{2n}(\mathbb{R})$$

and we obtain for $G^* := \mathrm{SO}_{2n,2n}(\mathbb{R})_e$ the subgroup

$$(G^*)^h = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-\top} \end{pmatrix} \colon a \in \mathrm{GL}_{2n}(\mathbb{R}) \right\} \cong \mathrm{GL}_{2n}(\mathbb{R})$$

with 2 connected components. Further, $Z(G^*) = \{\pm\mathbf{1}\} \subseteq (G^*)_e^h$, so that $G^h = \mathrm{Ad}((G^*)^h)$ also has 2 connected components.

- The split real form $\mathfrak{g} = \mathfrak{e}_7(\mathbb{R})$ of type E_7: Then $\mathfrak{g}^c$ is the hermitian real form of $\mathfrak{e}_7(\mathbb{C})$, hence of real rank $r = 3$. Therefore $r_0 = r = 3$ is odd, so that G^h is connected by the discussion above. ∎

7.3. The maximal compact subgroup of $H = G \cap G^c$

In this subsection we collect some consequences of our discussion of $\pi_0(G^h)$ in the preceding subsection. In particular, we show that, for the c-dual Lie algebra $\mathfrak{g}^c = \mathfrak{h} + i\mathfrak{q}$, we have
$$\mathrm{Inn}_{\mathfrak{g}_\mathbb{C}}(\mathfrak{g}^c) \cap \mathrm{Inn}_{\mathfrak{g}_\mathbb{C}}(\mathfrak{g}) = \mathrm{Inn}_{\mathfrak{g}_\mathbb{C}}(\mathfrak{k})^h e^{\mathrm{ad}\,\mathfrak{h}_\mathfrak{p}}.$$

Proposition 7.9. *Let $(\mathfrak{g}, \tau, C)$ be a semisimple non-compactly causal symmetric Lie algebra, where $\tau = \tau_h\theta$, $h \in \mathfrak{q}_s \cap C^\circ$ is a causal Euler element,*

$$G := \mathrm{Inn}_{\mathfrak{g}_\mathbb{C}}(\mathfrak{g}) \cong \mathrm{Inn}(\mathfrak{g}), \quad K := G^\theta = \mathrm{Inn}_\mathfrak{g}(\mathfrak{k}), \quad \text{and} \quad G^c := \mathrm{Inn}_{\mathfrak{g}_\mathbb{C}}(\mathfrak{g}^c).$$

Then $H := G \cap G^c$ is τ-invariant and satisfies

$$H = K^h \exp(\mathfrak{h}_\mathfrak{p}) \quad \text{and} \quad H \cap K = K^h.$$

Note that $\tau = \tau_h\theta$ and $G = \mathrm{Inn}(\mathfrak{g})$ imply $G^h \subseteq G^\tau$ because $K^h \subseteq K^{\tau_h} = K^\tau \subseteq G^\tau$ and $G^h = K^h \exp(\mathfrak{q}_\mathfrak{p})$. Therefore the preceding proposition shows in particular that

$$G \cap G^c \subseteq G^\tau. \tag{34}$$

Proof. First we write $\mathfrak{g}$ as a direct sum $\mathfrak{g}_k \oplus \mathfrak{g}_r \oplus \mathfrak{g}_s$ as in (15), where $\mathfrak{g}_s$ is the sum of all simple ideals not commuting with h (the strictly ncc part), $\mathfrak{g}_r$ is the sum of all non-compact simple ideals commuting with h on which $\tau = \theta$ (the non-compact Riemannian part), and $\mathfrak{g}_k$ is the sum of all compact ideals (they commute with h). As $h \in \mathfrak{q}_s$, the corresponding subgroups G_k and G_r are contained in G^h and $\mathfrak{h}_p \cap \mathfrak{g}_k = \{0\} = \mathfrak{h}_p \cap \mathfrak{g}_r$. Therefore we may assume that $\mathfrak{g} = \mathfrak{g}_s$ and, by decomposition into simple ideals, even that $(\mathfrak{g}, \tau)$ is irreducible non-compactly causal.

We start with the polar decomposition $G^h = K^h \exp(\mathfrak{p}^{-\tau_h})$ which implies that

$$H_K := H \cap K = G^c \cap K = G \cap K^c = K \cap K^c.$$

Using the strongly orthogonal roots and the subgroup $L \subseteq G$ (cf. Lemma 7.6), we find with Lemma 7.1(d), applied with the compact group K (G in the lemma) and the subgroup K^h (H in the lemma), that

$$K^h = K^h_e(K^h \cap Z(L)) \quad \text{with} \quad K^h \cap Z(L) \subseteq T_Q = \exp(\mathfrak{t}_\mathfrak{q}). \tag{35}$$

We claim that all generators $z_j \in Z(L_j)$ are contained in $G \cap G^c$. In fact, we have $\mathfrak{l}_j \cong \mathfrak{sl}_2(\mathbb{R})$, $\mathfrak{l}_j^c \cong \mathfrak{su}_{1,1}(\mathbb{C})$ and $\mathfrak{l}_{j,\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$. Clearly $-\mathbf{1} \in \mathrm{SL}_2(\mathbb{R}) \cap \mathrm{SU}_{1,1}(\mathbb{C})$ holds in $\mathrm{SL}_2(\mathbb{C})$, so that $z_j \in L_j \cap L_j^c$. This implies that $T_Q^h \subseteq Z(L) \subseteq L \cap L^c$ is contained in $H = G \cap G^c$, and hence by (35) that $K^h \subseteq H$. Since we also have $H_K \subseteq K^h$ (a consequence of $\mathrm{Ad}(H_K)$ fixing $h \in C$) (Lemma 4.11), we obtain $H_K = K^h$. ∎

Lemma 7.10. *If* $H' \subseteq G^\tau$ *is another open subgroup satisfying*

$$\mathrm{Ad}(H')C_\mathfrak{q}^{\max} = C_\mathfrak{q}^{\max}, \tag{36}$$

then $H' \subseteq H = K^h \exp(\mathfrak{h}_\mathfrak{p})$ *and we obtain an equivariant covering of ordered symmetric spaces*

$$M' := G/H' \to M = G/H, \quad gH' \mapsto gH. \tag{37}$$

Proof. In the polar decomposition $H' = H'_K \exp(\mathfrak{h}_\mathfrak{p})$ we have $H'_K \subseteq K^\tau = K^{\tau h}$. Since every element $g \in G^{\tau h}$ normalizes $\mathfrak{g}_0(h)$ and preserves $\mathfrak{g}_1(h) + \mathfrak{g}_{-1}(h)$, which are inequivalent representations of the Lie algebra $\mathfrak{g}_0(h)$, we either have $\mathrm{Ad}(g)\mathfrak{g}_{\pm 1}(h) = \mathfrak{g}_{\pm 1}(h)$ or $\mathfrak{g}_{\mp 1}(h)$. In the first case $\mathrm{Ad}(g)h = h$, and in the second $\mathrm{Ad}(g)h = -h \in -C_\mathfrak{q}^{\max}$. Therefore (36) implies $H'_K \subseteq K^h$, so that $H' \subseteq H = G_\mathcal{D}$. We therefore have an equivariant covering of homogeneous spaces as in (37). ∎

As an important application of the preceding discussion, we obtain the following result on the fundamental group of M, connecting it to connected components of the stabilizer of h, and **not** to connected components of H, as usual.

Proposition 7.11. *Let* $q_G \colon \widetilde{G} \to G$ *denote the simply connected covering group. Then* q_G *induces the universal covering*

$$q_M \colon \widetilde{M} := \widetilde{G}/\widetilde{G}^\tau \to M = G/H \quad \text{for} \quad H = K^h \exp(\mathfrak{h}_\mathfrak{p})$$

and $$\pi_1(M) \cong \pi_0(\widetilde{G}^h).$$

Proof. The subgroup $\widetilde{G}^\tau$ is connected because $\widetilde{G}$ is simply connected ([25, Thm. IV.3.4]). Thus $\widetilde{M}$ is the simply connected covering of $M = G/H \cong \widetilde{G}/q_G^{-1}(H)$. This further implies that
$$\pi_1(M) \cong \pi_0(q_G^{-1}(H))$$
([15, Cor. 11.1.14]). As $H = K^h \exp(\mathfrak{h}_\mathfrak{p})$, the inclusion $\ker(q_G) \subseteq \widetilde{K}^h$ leads to the polar decompositions

$$q_G^{-1}(H) = q_G^{-1}(H_K)\exp(\mathfrak{h}_\mathfrak{p}) = q_G^{-1}(K^h)\exp(\mathfrak{h}_\mathfrak{p}) = \widetilde{K}^h \exp(\mathfrak{h}_\mathfrak{p})$$

and $\widetilde{G}^h = \widetilde{K}^h \exp(\mathfrak{q}_\mathfrak{p})$, which imply that

$$\pi_0(\widetilde{G}^h) \cong \pi_0(\widetilde{K}^h) \cong \pi_0(q_G^{-1}(H)) \cong \pi_1(M). \qquad \blacksquare$$

A.   Some calculations in $\mathfrak{sl}_2(\mathbb{R})$

In this subsection we collect some formulas concerning the 3-dimensional Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ that we shall use below. For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, we fix the Cartan involution $\theta(x) = -x^\top$, so that

$$\mathfrak{k} = \mathfrak{so}_2(\mathbb{R}) \quad \text{and} \quad \mathfrak{p} = \{x \in \mathfrak{sl}_2(\mathbb{R}) \colon x^\top = x\}.$$

The basis elements

$$h^0 := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f^0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{38}$$

and

$$h^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(e^0 + f^0), \quad e^1 = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad f^1 = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \tag{39}$$

satisfy

$$[h^j, e^j] = e^j, \quad [h^j, f^j] = -f^j, \quad [e^j, f^j] = 2h^j \text{ and } \theta(e^j) = -f^j \text{ for } j = 1, 2.$$

For the involution

$$\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \quad \text{we have} \quad \mathfrak{h} = \mathfrak{g}^\tau = \mathbb{R}h^0 \quad \text{and} \quad \mathfrak{q} = \mathfrak{g}^{-\tau} = \mathbb{R}h^1 + \mathbb{R}(e^0 - f^0)$$

and

$$C = [0, \infty)e^0 + [0, \infty)f^0$$

is a hyperbolic $\operatorname{Inn}(\mathfrak{h})$-invariant cone in $\mathfrak{q}$, containing h^1 as a causal Euler element. The subspace $\mathfrak{t}_\mathfrak{q} := \mathbb{R}(e^0 - f^0) = \mathfrak{so}_2(\mathbb{R})$ of $\mathfrak{q}$ is maximal elliptic. For

$$x_0 := \frac{\pi}{4}(e^0 - f^0) = \frac{\pi}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{t}_\mathfrak{q}$$

and

$$g_0 := \exp(x_0) = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & \sin\left(\frac{\pi}{4}\right) \\ -\sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

we then have

$$\operatorname{Ad}(g_0)h^1 = h^0 \quad \text{and} \quad \operatorname{Ad}(g_0)h^0 = -h^1, \tag{40}$$

More generally, we have for $t \in \mathbb{R}$

$$e^{\frac{t}{2}\operatorname{ad}(e^0 - f^0)}h^0 = \cos(t)h^0 - \sin(t)h^1 \tag{41}$$

because

$$\left[\frac{e^0 - f^0}{2}, h^0\right] = -h^1, \quad \left[\frac{e^0 - f^0}{2}, h^1\right] = h^0.$$

B.   Some general facts on invariant cones

Lemma B.1.   *Let E be a finite dimensional real vector space, $C \subseteq E$ a closed convex cone and $E_1 \subseteq E$ a linear subspace. If the interior C° of C intersects E_1, then $C^\circ \cap E_1$ coincides with the relative interior C_1° of the cone $C_1 := C \cap E_1$ in E_1.*

Proof.   Clearly, $C^\circ \cap E_1 \subseteq C_1^\circ$. Pick $x_0 \in C^\circ \cap E_1$. If $x \in C_1 = C \cap E_1$ and $t \in [0, 1)$, then $x_t := x_0 + t(x - x_0) = (1 - t)x_0 + tx \in C^\circ$ shows that $C^\circ \cap E_1$ is dense in C_1. If, in addition, $x \in C_1^\circ$, then there exists an $s > 1$ with $x_s \in C_1$. Then the argument from above shows that

$$x = x_1 = x_0 + \frac{1}{s}(x_s - x_0) \in C^\circ. \qquad \blacksquare$$

Lemma B.2. (Cone Extension Lemma) *Let $E = E_1 \oplus E_2$ be a finite dimensional real vector space and $p_1 \colon E \to E_1$ the projection along E_2. Further, let $K \subseteq \mathrm{GL}(E_2) \subseteq \mathrm{GL}(E)$ be a compact subgroup and $H \subseteq \mathrm{SL}(E_1) \subseteq \mathrm{SL}(E)$ a subgroup. If $C_1 \subseteq E_1$ is pointed generating H-invariant closed convex cone and $h \in E$ with $p_1(h) \in C_1^\circ$, then there exists a $H \times K$-invariant pointed generating closed convex cone $C \subseteq E$ such that*

(a) $p_1(C) = C \cap E_1 = C_1$.

(b) $C^\circ \cap E_1 = C_1^\circ$.

(c) $h \in C^\circ$.

Proof.   As $H \subseteq \mathrm{SL}(E_1)$, the characteristic function

$$\phi \colon C_1^\circ \to (0, \infty), \quad \phi(x) = \int_{C_1^\star} e^{-\alpha(x)} \, d\alpha$$

is smooth and H-invariant with the property that $x_n \to x_0$ with $x_n \in C_1^\circ$ and $x_0 \in \partial C_1$ implies $\phi(x_n) \to \infty$ ([36, Thm. V.5.4]). Let

$$B := \{x \in C_1 \colon \phi(x) \leq 2\phi(p_1(h))\}.$$

This is a closed convex H-invariant subset with $C_1 = \mathrm{cone}(B) = \overline{\mathbb{R}_+ B}$. Further, let $D \subseteq E_2$ be a K-invariant compact 0-neighborhood such that $h - p_1(h) \in \mathrm{int}_{E_2}(D)$. Then $B + D$ is a $H \times K$-invariant closed convex subset not containing 0, and thus $C := \mathrm{cone}(B + D)$ is a $H \times K$-invariant pointed generating invariant cone which satisfies

$$C \cap E_1 \subseteq p_1(C) \subseteq C_1 \subseteq C \cap E_1.$$

Moreover, $h \in \mathrm{int}_{E_1}(B) + \mathrm{int}_{E_2}(D) \subseteq C^\circ$ and

$$C^\circ \cap E_1 \subseteq C_1^\circ = (0, \infty)B \subseteq C^\circ. \qquad \blacksquare$$

Example B.3.   The condition $H \subseteq \mathrm{SL}(E_1)$ is crucial, as the following example shows. We consider

$$E = \mathbb{R}^2, \quad K = \{\mathbf{1}\} \quad \text{and} \quad H = \{\mathrm{diag}(t, 1) \colon t > 0\}.$$

Then every open neighborhood $U \subseteq E$ of a point $(t, 0) \in E_1$ has the property that $\{0\} \times \mathbb{R}$ is contained in the closed convex cone generated by $H.U$. Therefore the cone $C = [0, \infty) \times \{0\}$ does not extend to a pointed generating H-invariant cone $\widehat{C}$ containing $(0, \infty) \times \{0\}$ in its interior. However, there are H-invariant pointed generating invariant cones $\widehat{C}$ with $\widehat{C} \cap E_1 = C$, but they contain C in their boundary.

Proposition B.4. ([37, Prop. 2.11]) *Let K be a compact group acting continuously on the finite-dimensional real vector space E by the representation $\pi \colon K \to \mathrm{GL}(E)$ and $p(v) := \int_K \pi(k)v \, d\mu_K(k)$ the corresponding fixed point projection, where μ_K is a normalized Haar measure on K. If $\Omega \subseteq E$ is an open or closed K-invariant convex subset, then*

$$p(\Omega) = \Omega \cap E^K.$$

The preceding proposition implies in particular for any convex invariant subset with non-empty interior the relation

$$p(\Omega^\circ) = \Omega^\circ \cap E^K \subseteq (\Omega \cap E^K)^\circ.$$

That we actually have equality follows from Lemma B.1.

C. Lorentzian symmetric spaces

Time-oriented Lorentzian symmetric spaces are in particular causal symmetric spaces. Not all such spaces are reductive, as the biinvariant Lorentzian structures on the 4-dimensional (solvable) oscillator group shows ([14]). If, however, $(\mathfrak{g}, \tau, \beta)$ is reductive and Lorentzian (β denoting the Lorentzian form on $\mathfrak{q}$), then we may assume that $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g}^{-\tau}$. Accordingly

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \bigoplus_{j=1}^n (\mathfrak{g}_j, \tau_j),$$

where each $(\mathfrak{g}_j, \tau_j)$ is irreducible and the corresponding direct sum decomposition

$$(\mathfrak{q}, \beta) = (\mathfrak{z}(\mathfrak{g}), \beta_{\mathfrak{z}}) \oplus \bigoplus_{j=1}^n (\mathfrak{q}_j, \beta_j)$$

is orthogonal. Therefore at most one summand contains timelike vectors and the other summands are space-like, hence correspond to Riemannian symmetric spaces. So two types occur:

(CL) The central type, where $\beta_{\mathfrak{z}}$ is Lorentzian and all forms β_j are negative definite.

(SL) The simple type, where some β_{j_0} is Lorentzian and all other summands are negative definite.

For the classification of Riemannian symmetric spaces, we refer to Helgason's monograph [13]. These determine the spaces of central type and to understand the other type, one needs to know the irreducible Lorentzian spaces, whose classification we recall below.

Theorem C.1. *If $(\mathfrak{g}, \tau)$ is an irreducible semisimple symmetric Lie algebra and the corresponding d-dimensional symmetric space M is Lorentzian, then it is locally isomorphic to de Sitter space dS^d or anti-de Sitter space AdS^d. Accordingly, $(\mathfrak{g}, \mathfrak{h})$ is isomorphic to*

$$(\mathfrak{so}_{1,d}(\mathbb{R}), \mathfrak{so}_{1,d-1}(\mathbb{R})) \quad or \quad (\mathfrak{so}_{2,d-1}(\mathbb{R}), \mathfrak{so}_{1,d-1}(\mathbb{R})).$$

Proof. Let $(\mathfrak{g}, \tau)$ be an irreducible semisimple symmetric Lie algebra. We are interested in a description of all Lorentzian symmetric spaces G/H. To see if $(\mathfrak{g}, \tau)$ is Lorentzian, we choose a Cartan involution θ of $\mathfrak{g}$ commuting with τ. We then have

$$\mathfrak{q} = \mathfrak{q}_{\mathfrak{k}} \oplus \mathfrak{q}_{\mathfrak{p}},$$

where the Cartan-Killing form of $\mathfrak{g}$ is negative definite on $\mathfrak{q}_{\mathfrak{k}}$ and positive definite on $\mathfrak{q}_{\mathfrak{p}}$. That a corresponding symmetric space G/H is Lorentzian is equivalent to

one of the two subspaces $\mathfrak{q}_\mathfrak{k}$ or $\mathfrak{q}_\mathfrak{p}$ to be one-dimensional, because this implies that a suitable multiple of the Cartan-Killing form on $\mathfrak{q}$ is a Lorentzian form. As $(\mathfrak{g}, \tau)$ is Lorentzian if and only if the dual pair $(\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}, \tau^c)$ is Lorentzian, and $(i\mathfrak{q})_\mathfrak{k} = i\mathfrak{q}_\mathfrak{p}$ and $(i\mathfrak{q})_\mathfrak{p} = i\mathfrak{q}_\mathfrak{k}$, we may assume that $\mathfrak{q}_\mathfrak{p}$ is one-dimensional. Then θ defines a dissecting involution on the associated 1-connected symmetric space G/H, where G is 1-connected and $H = G^\tau$. Then the classification of irreducible symmetric spaces with dissecting involutions in [38] implies that G/H is locally isomorphic to a quadric. In particular $\mathfrak{g} \cong \mathfrak{so}_{p,q}(\mathbb{R})$ and $\mathfrak{h} \cong \mathfrak{so}_{p-1,q}(\mathbb{R})$. The Lorentzian property now implies that $q = 1$ or $p = 2$, which corresponds to de Sitter space and Anti-de Sitter space. ∎

Remark C.2. In addition to irreducible Lorentzian symmetric spaces, there is also the class of indecomposable solvable symmetric spaces; classified by Cahen and Wallach in [8]. According to [8, Thm. 3], all indecomposable Lorentzian symmetric Lie algebras are either semisimple or solvable. A detailed exposition of the classification can be found in [22, §3.3]. The corresponding symmetric Lorentzian Lie algebras $(\mathfrak{g}, \tau, \beta)$, where $\beta: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is an invariant non-degenerate symmetric bilinear form, have the following structure. We start with a quadratic space $(\mathfrak{a}, \kappa_\mathfrak{a})$ of signature $(p + 2q, p)$ and orthogonal basis

$$\mathbf{e}_1, \ldots, \mathbf{e}_{2q}, \quad \mathbf{e}'_1, \ldots, \mathbf{e}'_{2p}$$

with $$\kappa_\mathfrak{a}(\mathbf{e}_j, \mathbf{e}_j) = 1, \quad \kappa_\mathfrak{a}(\mathbf{e}'_j, \mathbf{e}'_j) = \begin{cases} 1 & \text{for } j \leq p \\ -1 & \text{for } j > p. \end{cases}$$

We also have a skew-symmetric endomorphism $D \in \mathfrak{so}(\mathfrak{a}, \kappa_\mathfrak{a})$ acting by

$$D\mathbf{e}_j = \lambda_j \mathbf{e}_{j+q}, \quad D\mathbf{e}_{j+q} = -\lambda_j \mathbf{e}_j, \quad D\mathbf{e}'_j = \mu_j \mathbf{e}'_{j+p}, \quad D\mathbf{e}'_{j+p} = \mu_j \mathbf{e}'_j$$

with $$0 < \lambda_1 \leq \cdots \leq \lambda_q, \quad 0 < \mu_1 \leq \cdots \leq \mu_p.$$

Now $$\mathfrak{g} = \mathbb{R} \oplus \mathfrak{a} \oplus \mathbb{R}$$

with the Lie bracket

$$[(z, a, t), (z', a', t')] := (\kappa_\mathfrak{a}(Da, a'), tDa' - t'Da, 0)$$

and the invariant symmetric bilinear form

$$\beta((z, a, t), (z', a', t')) = zt' + z't + \kappa_\mathfrak{a}(a, a').$$

The involution τ is given by $\tau(z, a, t) = (-z, \tau_\mathfrak{a}(a), -t)$, and

$$\mathfrak{a}^{-\tau} = \text{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_q, \mathbf{e}'_1, \ldots, \mathbf{e}'_p\}, \quad \mathfrak{a}^\tau = \text{span}\{\mathbf{e}_{q+1}, \ldots, \mathbf{e}_{2q}, \mathbf{e}'_{p+1}, \ldots, \mathbf{e}'_{2p}\}.$$

For $p = 0$ these Lie algebras are called *oscillator algebras* and the form β on $\mathfrak{g}$ is Lorentzian because $\kappa_\mathfrak{a}$ is positive definite. In general the form β is positive definite on $\mathfrak{a}^{-\tau} = \mathfrak{a} \cap \mathfrak{q}$, so that the restriction of β to $\mathfrak{q} = \mathfrak{g}^{-\tau}$ is Lorentzian, but the form β on $\mathfrak{g}$ has signature $(p + 1 + 2q, p + 1)$.

In addition to Lorentzian symmetric spaces, also natural generalizations of conformal spacetime-geometries have been discussed from the physics perspective in [27], where the conformal compactifications of simple euclidean Jordan algebras are studied.

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