Journal of Lie Theory
Volume **33** (2023) 497–526

Combinatorial and Geometric Constructions Associated with the Kostant Cascade

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Communicated by D. A. Timashev

Dedicated to Alexander Grigorievich Elashvili on the occasion of his 80th birthday.

Abstract. Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}^+$ a fixed Borel subalgebra. Let Δ^+ be the set of positive roots associated with $\mathfrak{u}^+$ and $\mathcal{K} \subset \Delta^+$ the Kostant cascade. We elaborate on some constructions related to $\mathcal{K}$ and applications of $\mathcal{K}$. This includes the cascade element $x_{\mathcal{K}}$ in the Cartan subalgebra $\mathfrak{t}$ and properties of certain objects naturally associated with $\mathcal{K}$: an abelian ideal of $\mathfrak{b}$, a nilpotent G-orbit in $\mathfrak{g}$, and an involution of $\mathfrak{g}$.

Mathematics Subject Classification: 17B22, 17B20, 17B08, 14L30.

Key Words: Root system, cascade element, abelian ideal, Frobenius algebra, nilpotent orbit.

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1. Introduction

Let G be a simple algebraic group with $\mathrm{Lie}\,(G) = \mathfrak{g}$. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{u}^+ \oplus \mathfrak{t} \oplus \mathfrak{u}^-$. Then Δ is the root system of $(\mathfrak{g}, \mathfrak{t})$ and Δ^+ is the set of positive roots corresponding to $\mathfrak{u}^+$. The *Kostant cascade* is a set $\mathcal{K}$ of strongly orthogonal roots in Δ^+ that is constructed recursively starting with the highest root $\theta \in \Delta^+$, see Section 2. The construction of cascade goes back to B. Kostant, who used it for studying the center of the enveloping algebra of $\mathfrak{u}^+$. His construction is prominently

This research was funded by RFBR, project 20-01-00515.

ISSN 0949–5932 / \$2.50

used in some articles afterwards [9, 12, 18], but Kostant's own publications related to the cascade appear some 40 years later [16, 17]. The cascade is also crucial for computing the index of seaweed subalgebras of simple Lie algebras [13, 28].

Ever since I learned from A. Elashvili about the cascade at the end of 80s, I was fascinated by this structure. Over the years, I gathered a number of results related to the occurrences of $\mathcal{K}$ in various problems of Combinatorics, Invariant Theory, and Representation Theory. In [26], I give an application of $\mathcal{K}$ to the problem of classifying the nilradicals of parabolic subalgebras of $\mathfrak{g}$ that admit a commutative polarisation. (General results on commutative polarisations are due to Elashvili and Ooms, see [8].) Some other observations appear in this article.

Let $\Pi \subset \Delta^+$ be the set of simple roots. If $\gamma = \sum_{\alpha\in\Pi} a_\alpha \alpha$, then $[\gamma : \alpha] = a_\alpha$ and $\widetilde{ht}(\gamma) = \sum_{\alpha\in\Pi}[\gamma : \alpha]$ is the *height* of γ. The set of positive roots Δ^+ is a poset with respect to the root order "$\preccurlyeq$", and $\mathcal{K} = \{\beta_1, \dots, \beta_m\}$ inherits this structure so that $\theta = \beta_1$ is the unique maximal element of $\mathcal{K}$.

In Section 3, we define a rational element of $\mathfrak{t}$ associated with $\mathcal{K}$. Let $(\ ,\ )$ denote the restriction of the Killing form on $\mathfrak{g}$ to $\mathfrak{t}$. As usual, $\mathfrak{t}$ and $\mathfrak{t}^*$ are identified via $(\ ,\ )$ and $\mathfrak{t}^*_{\mathbb{Q}}$ is the $\mathbb{Q}$-linear span of Δ. The *cascade element* of $\mathfrak{t}_{\mathbb{Q}} \simeq \mathfrak{t}^*_{\mathbb{Q}}$ is

$$x_{\mathcal{K}} = \sum_{i=1}^{m} \frac{\beta_i}{(\beta_i,\beta_i)} = \frac{1}{2}\sum_{i=1}^{m} \beta_i^{\vee}. \tag{1}$$

The numbers $\gamma(x_{\mathcal{K}})$, $\gamma \in \Delta^+$, are the eigenvalues of $\operatorname{ad} x_{\mathcal{K}}$ on $\mathfrak{u}^+$, and we say that they form the spectrum of $x_{\mathcal{K}}$ on Δ^+. It follows from (1) that $\gamma(x_{\mathcal{K}}) \in \frac{1}{2}\mathbb{Z}$ and $\beta(x_{\mathcal{K}}) = 1$ for any $\beta \in \mathcal{K}$. We prove that $-1 \le \gamma(x_{\mathcal{K}}) \le 2$ for any $\gamma \in \Delta^+$ and if $\mathfrak{g}$ is not of type A_{2p}, then the eigenvalues are integral (Theorem 3.5). It is also shown that the spectrum of $x_{\mathcal{K}}$ on $\Delta^+ \setminus \mathcal{K}$ is symmetric relative to $1/2$, which means that if m_λ is the multiplicity of the eigenvalue λ, then $m_\lambda = m_{1-\lambda}$. If θ is a fundamental weight and $\alpha \in \Pi$ is the unique root such that $(\theta, \alpha) \ne 0$, then α is long and we prove that $\alpha(x_{\mathcal{K}}) = -1$. On the other hand, if θ is not fundamental, then $x_{\mathcal{K}}$ appears to be dominant. Let $\mathcal{Q}$ (resp. $\mathcal{Q}^\vee$) denote the *root* (resp. *coroot*) *lattice* in $\mathfrak{t}^*_{\mathbb{Q}}$. The corresponding dual lattices are

$$\text{the coweight lattice } \mathcal{P}^\vee := \mathcal{Q}^* \quad \& \quad \text{the weight lattice } \mathcal{P} := (\mathcal{Q}^\vee)^*.$$

Hence $x_{\mathcal{K}} \in \mathcal{P}^\vee$ unless $\mathfrak{g}$ is of type A_{2p}. Here $\mathcal{Q}^\vee \subset \mathcal{P}^\vee$ and we prove that $x_{\mathcal{K}} \in \mathcal{Q}^\vee$ if and only if every self-dual representation of $\mathfrak{g}$ is orthogonal (Section 4).

In [20, Section 3], A. Ooms describes an interesting feature of the Frobenius Lie algebras. Let $\mathfrak{q} = \operatorname{Lie}(Q)$ be a Frobenius algebra and $\xi \in \mathfrak{q}^*$ a regular linear form, i.e., $\mathfrak{q}^\xi = \{0\}$. Then the Kirillov form $\mathcal{B}_\xi$ is non-degenerate and it yields a linear isomorphism $\boldsymbol{i}_\xi : \mathfrak{q}^* \to \mathfrak{q}$. Letting $x_\xi = \boldsymbol{i}_\xi(\xi)$, Ooms proves that $(\operatorname{ad} x_\xi)^* = 1 - \operatorname{ad} x_\xi$, where $(\operatorname{ad} x_\xi)^*$ is the adjoint operator w.r.t. $\mathcal{B}_\xi$. This implies that the spectrum of $\operatorname{ad} x_\xi$ on $\mathfrak{q}$ is symmetric relative to $1/2$. We say that $x_\xi \in \mathfrak{q}$ is the *Ooms element* associated with $\xi \in \mathfrak{q}^*_{\mathsf{reg}}$. Let $\mathfrak{t}_{\mathcal{K}} \subset \mathfrak{t}$ be the $\mathbb{C}$-linear span of $\mathcal{K}$. Then $\mathfrak{b}_{\mathcal{K}} = \mathfrak{t}_{\mathcal{K}} \oplus \mathfrak{u}^+$ is a Frobenius Lie algebra, i.e., $\operatorname{ind} \mathfrak{b}_{\mathcal{K}} = 0$, see [26, Sect. 5]. In Section 5, we prove that

- if $\mathfrak{q}$ is an algebraic Lie algebra, then any Ooms element $x_\xi \in \mathfrak{q}$ is semisimple;
- $x_{\mathcal{K}}$ is an Ooms element for the Frobenius Lie algebra $\mathfrak{b}_{\mathcal{K}} = \mathfrak{t}_{\mathcal{K}} \oplus \mathfrak{u}^+$.

The latter provides a geometric explanation for the symmetry of the spectrum of $x_{\mathcal{K}}$ on $\Delta^+ \setminus \mathcal{K}$.

By [15], one associates an abelian ideal of $\mathfrak{b}$, $\mathfrak{a}_z$, to any $z \in \mathfrak{t}$ such that we have $\gamma(z) \in \{-1, 0, 1, 2\}$ for all $\gamma \in \Delta^+$. Namely, let $\Delta^+_z(i) = \{\gamma \in \Delta^+ \mid \gamma(z) = i\}$. There is a unique $w_z \in W$ such that the inversion set of w_z, $\mathcal{N}(w_z)$, equals $\Delta^+_z(1) \cup \Delta^+_z(2)$, and then $\Delta_{\langle z\rangle} := w_z\big(\Delta^+_z(-1) \cup -\Delta^+_z(2)\big)$ is the set of roots of $\mathfrak{a}_z$. We notice that $w_z(z)$ is anti-dominant and w_z is the element of minimal length having such property. Kostant's construction applies to $z = x_{\mathcal{K}}$ unless $\mathfrak{g}$ is of type A_{2p} and we obtain a complete description of $w_{\mathcal{K}} := w_{x_{\mathcal{K}}}$ and $\mathfrak{a}_{\mathcal{K}} := \mathfrak{a}_{x_{\mathcal{K}}}$ (Sections 6, 7).

In this setting, we prove that $w_{\mathcal{K}}(\theta) \in -\Pi$ and $-w_{\mathcal{K}}(x_{\mathcal{K}})$ is a fundamental coweight. That is, if $w_{\mathcal{K}}(\theta) = -\alpha_j$, then $-w_{\mathcal{K}}(x_{\mathcal{K}}) = 2\varpi_j/(\alpha_j, \alpha_j) =: \varpi_j^\vee$. Since $[\mathfrak{b}, \mathfrak{a}_{\mathcal{K}}] \subset \mathfrak{a}_{\mathcal{K}}$, the set of roots $\Delta_{\langle\mathcal{K}\rangle} = \Delta(\mathfrak{a}_{\mathcal{K}})$ is an upper ideal of the poset $(\Delta^+, \preccurlyeq)$. Therefore, $\Delta_{\langle\mathcal{K}\rangle}$ is fully determined by the set of *minimal* elements of $\Delta_{\langle\mathcal{K}\rangle}$ or the set of *maximal* elements of $\Delta^+ \setminus \Delta_{\langle\mathcal{K}\rangle}$.

Letting $\Delta^+_{\mathcal{K}}(i) = \Delta^+_{x_{\mathcal{K}}}(i)$ and $\Pi_{\mathcal{K}}(i) = \Pi \cap \Delta^+_{\mathcal{K}}(i)$, we prove that

1. $\min(\Delta_{\langle\mathcal{K}\rangle}) = w_{\mathcal{K}}(\Pi_{\mathcal{K}}(-1))$ and $\max(\Delta^+ \setminus \Delta_{\langle\mathcal{K}\rangle}) = -w_{\mathcal{K}}(\Pi_{\mathcal{K}}(1))$;
2. if $d_{\mathcal{K}} = 1 + \sum_{\alpha \in \Pi_{\mathcal{K}}(\geq 0)} [\theta : \alpha]$, then $\Delta_{\langle\mathcal{K}\rangle} = \{\gamma \mid \widetilde{ht}(\gamma) \geq d_{\mathcal{K}}\}$.

In order to verify (2), we use explicit formulae for $w_{\mathcal{K}}$. To get such formulae, we exploit a description of $w_{\mathcal{K}}^{-1}(\Pi)$ (Theorem 7.1). Then we check directly that $\widetilde{ht}(w_{\mathcal{K}}(\alpha)) = d_{\mathcal{K}}$ for any $\alpha \in \Pi_{\mathcal{K}}(-1)$ and that $\#\Pi_{\mathcal{K}}(-1) = \#\{\gamma \mid \widetilde{ht}(\gamma) = d_{\mathcal{K}}\}$.

In Section 8, we naturally associate an involution $\sigma_{\mathcal{K}} \in \mathsf{Aut}(\mathfrak{g})$ to $\mathcal{K}$ if $x_{\mathcal{K}} \in \mathcal{P}^\vee$, i.e., $\mathfrak{g}$ is not of type A_{2p}. It is proved $\sigma_{\mathcal{K}}$ is the unique, up to conjugation, inner involution such that the (-1)-eigenspace of $\sigma_{\mathcal{K}}$ contains a regular nilpotent element of $\mathfrak{g}$.

For any simple Lie algebra ($\mathfrak{sl}_{2p+1}$ included), we construct the nilpotent G-orbit associated with $\mathcal{K}$, see Section 9. Let $e_\gamma \in \mathfrak{g}_\gamma$ be a nonzero root vector ($\gamma \in \Delta$). Then $e_{\mathcal{K}} = \sum_{\beta \in \mathcal{K}} e_\beta \in \mathfrak{g}$ is nilpotent and the orbit $\mathcal{O}_{\mathcal{K}} = G{\cdot}e_{\mathcal{K}}$ does not depend on the choice of root vectors. Properties of $\mathcal{O}_{\mathcal{K}}$ essentially depend on whether θ is fundamental or not. We prove that if θ is fundamental then $(\mathrm{ad}\, e_{\mathcal{K}})^5 = 0$ and $(\mathrm{ad}\, e_{\mathcal{K}})^4 \neq 0$; whereas if θ is not fundamental then $(\mathrm{ad}\, e_{\mathcal{K}})^3 = 0$ (and, of course, $(\mathrm{ad}\, e_{\mathcal{K}})^2 \neq 0$). By [21], this means that $\mathcal{O}_{\mathcal{K}}$ is spherical if and only if θ is not fundamental. Here $[x_{\mathcal{K}}, e_{\mathcal{K}}] = e_{\mathcal{K}}$ and $x_{\mathcal{K}} \in \mathsf{Im}\,(\mathrm{ad}\, e_{\mathcal{K}})$. Therefore, $2x_{\mathcal{K}}$ is a *characteristic* of $e_{\mathcal{K}}$ and the weighted Dynkin diagram of $\mathcal{O}_{\mathcal{K}}$, $\mathcal{D}(\mathcal{O}_{\mathcal{K}})$ is determined by the dominant element in $W{\cdot}(2x_{\mathcal{K}})$. Actually, this dominant element is $-2w_{\mathcal{K}}(x_{\mathcal{K}})$, and if $\mathfrak{g} \neq \mathfrak{sl}_{2p+1}$, then $-2w_{\mathcal{K}}(x_{\mathcal{K}}) = 2\varpi_j^\vee$, cf. above. Hence, in these cases, $\mathcal{D}(\mathcal{O}_{\mathcal{K}})$ has the unique nonzero label on the node α_j and $\mathcal{O}_{\mathcal{K}}$ is even, cf. Tables 1, 2.

Notation. Throughout, G is a simple algebraic group with $\mathfrak{g} = \mathrm{Lie}\,(G)$. Then

– $\mathfrak{b}$ is a fixed Borel subalgebra of $\mathfrak{g}$ and $\mathfrak{u}^+ = \mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$;

– $\mathfrak{t}$ is a fixed Cartan subalgebra in $\mathfrak{b}$ and Δ is the root system of $(\mathfrak{g}, \mathfrak{t})$;

– $\Delta^\pm$ is the set of roots corresponding to $\mathfrak{u}^\pm$;

- $\Pi = \{\alpha_1, \dots, \alpha_n\}$ is the set of simple roots in Δ^+ and the corresponding fundamental weights are $\varpi_1, \dots, \varpi_n$;
- $\mathfrak{t}^*_{\mathbb{Q}}$ is the $\mathbb{Q}$-vector subspace of $\mathfrak{t}^*$ spanned by Δ, and $(\ , \ )$ is the positive-definite form on $\mathfrak{t}^*_{\mathbb{Q}}$ induced by the Killing form on $\mathfrak{g}$; as usual, $\gamma^\vee = 2\gamma/(\gamma,\gamma)$ for $\gamma \in \Delta$.
- For each $\gamma \in \Delta$, $\mathfrak{g}_\gamma$ is the root space in $\mathfrak{g}$ and $e_\gamma \in \mathfrak{g}_\gamma$ is a nonzero vector;
- If $\mathfrak{c} \subset \mathfrak{u}^+$ is a $\mathfrak{t}$-stable subspace, then $\Delta(\mathfrak{c}) \subset \Delta^+$ is the set of roots of $\mathfrak{c}$;
- θ is the highest root in Δ^+;
- $W \subset GL(\mathfrak{t})$ is the Weyl group.

Our main references for (semisimple) algebraic groups and Lie algebras are [10, 19]. In explicit examples related to simple Lie algebras, the Vinberg-Onishchik numbering of simple roots and fundamental weights is used, see e.g. [19, Table 1] or [10, Table 1].

2. Preliminaries on root systems and the Kostant cascade

We identify Π with the vertices of the Dynkin diagram of $\mathfrak{g}$. For any $\gamma \in \Delta^+$, let $[\gamma : \alpha]$ be the coefficient of $\alpha \in \Pi$ in the expression of γ via Π. The *support* of γ is $\mathsf{supp}(\gamma) = \{\alpha \in \Pi \mid [\gamma : \alpha] \ne 0\}$ and the *height of* γ is $\widetilde{ht}(\gamma) = \sum_{\alpha\in\Pi}[\gamma : \alpha]$. As is well known, $\mathsf{supp}(\gamma)$ is a connected subset of the Dynkin diagram. For instance, $\mathsf{supp}(\theta) = \Pi$ and $\mathsf{supp}(\alpha) = \{\alpha\}$. A root γ is *long*, if $(\gamma,\gamma) = (\theta,\theta)$. We write Δ_l (resp. Δ_s) for the set of long (resp. short) roots in Δ. In the simply-laced case, $\Delta_s = \varnothing$.

Let "$\preccurlyeq$" denote the *root order* in Δ^+, i.e., we write $\gamma \preccurlyeq \gamma'$ if $[\gamma : \alpha] \le [\gamma' : \alpha]$ for all $\alpha \in \Pi$. Then γ' covers γ if and only if $\gamma' - \gamma \in \Pi$, which implies that $(\Delta^+, \preccurlyeq)$ is a graded poset. Write $\gamma \prec \gamma'$ if $\gamma \preccurlyeq \gamma'$ and $\gamma \ne \gamma'$. An *upper ideal* of $(\Delta^+, \preccurlyeq)$ is a subset I such that if $\gamma \in I$ and $\gamma \prec \gamma'$, then $\gamma' \in I$. Therefore, I is an upper ideal if and only if $\mathfrak{c} = \bigoplus_{\gamma\in I}\mathfrak{g}_\gamma$ is a $\mathfrak{b}$-ideal of $\mathfrak{u}$ (i.e., $[\mathfrak{b},\mathfrak{c}] \subset \mathfrak{c}$).

For a dominant weight $\lambda \in \mathfrak{t}^*_{\mathbb{Q}}$, set $\Delta^\pm_\lambda = \{\gamma \in \Delta^\pm \mid (\lambda,\gamma) = 0\}$ and $\Delta_\lambda = \Delta^+_\lambda \cup \Delta^-_\lambda$. Then Δ_λ is the root system of a semisimple subalgebra $\mathfrak{g}_\lambda \subset \mathfrak{g}$ and $\Pi_\lambda = \Pi \cap \Delta^+_\lambda$ is the set of simple roots in Δ^+_λ. Set $\Delta^{>0}_\lambda = \{\gamma \in \Delta^+ \mid (\lambda,\gamma) > 0\}$. Then $\Delta^+ = \Delta^+_\lambda \sqcup \Delta^{>0}_\lambda$ and

- $\mathfrak{p}_\lambda = \mathfrak{g}_\lambda + \mathfrak{b}$ is a standard parabolic subalgebra of $\mathfrak{g}$;
- the set of roots for the nilradical $\mathfrak{n}_\lambda = \mathfrak{p}^{\mathsf{nil}}_\lambda$ is $\Delta^{>0}_\lambda$; it is also denoted by $\Delta(\mathfrak{n}_\lambda)$.

If $\lambda = \theta$, then the nilradical $\mathfrak{n}_\theta$ is a *Heisenberg Lie algebra*. In this case, $\mathcal{H}_\theta := \Delta(\mathfrak{n}_\theta)$ is said to be the *Heisenberg subset* (of Δ^+).

The construction of the Kostant cascade $\mathcal{K}$ in Δ^+ is recalled below, see also [12, Sect. 2], [18, Section 3a], and [16, 17]. Whenever we wish to stress that $\mathcal{K}$ is associated with $\mathfrak{g}$, we write $\mathcal{K}(\mathfrak{g})$ for it.

1. We begin with $(\mathfrak{g}\langle 1\rangle, \Delta\langle 1\rangle, \beta_1) = (\mathfrak{g}, \Delta, \theta)$ and consider the (possibly reducible) root system Δ_θ. The highest root $\theta = \beta_1$ is the unique element of the *first* (highest) level in $\mathcal{K}$. Let $\Delta_\theta = \bigsqcup_{j=2}^{d_2}\Delta\langle j\rangle$ be the decomposition into irreducible root systems and $\Pi\langle j\rangle = \Pi \cap \Delta\langle j\rangle$. Then $\Pi_\theta = \bigsqcup_{j=2}^{d_2}\Pi\langle j\rangle$ and $\{\Pi\langle j\rangle\}$ are the connected components of $\Pi_\theta \subset \Pi$.

2. Let $\mathfrak{g}\langle j\rangle$ be the simple subalgebra of $\mathfrak{g}$ with root system $\Delta\langle j\rangle$. Then $\mathfrak{g}_\theta = \bigoplus_{j=2}^{d_2}\mathfrak{g}\langle j\rangle$. Let β_j be the highest root in $\Delta\langle j\rangle^+ = \Delta\langle j\rangle\cap\Delta^+$. The roots $\beta_2,\dots,\beta_{d_2}$ are the *descendants* of β_1, and they form the *second* level of $\mathcal{K}$. Note that $\mathsf{supp}(\beta_j) = \Pi\langle j\rangle$, hence different descendants have disjoint supports.

3. Making the same step with each pair $(\Delta\langle j\rangle,\beta_j)$, $j=2,\dots,d_2$, we get a collection of smaller simple subalgebras inside each $\mathfrak{g}\langle j\rangle$ and smaller irreducible root systems inside $\Delta\langle j\rangle$. This provides the descendants for each β_j ($j=2,\dots,d_2$), i.e., the elements of the *third* level in $\mathcal{K}$. And so on...

4. This procedure eventually terminates and yields in this way a maximal set $\mathcal{K}=\{\beta_1,\beta_2,\dots,\beta_m\}$ of *strongly orthogonal* roots in Δ^+. (The latter means that $\beta_i\pm\beta_j\notin\Delta$ for all i,j). We say that $\mathcal{K}$ is the *Kostant cascade* in Δ^+.

Thus, each $\beta_i\in\mathcal{K}$ occurs as the highest root of a certain irreducible root system $\Delta\langle i\rangle$ inside Δ such that $\Pi\langle i\rangle = \Pi\cap\Delta\langle i\rangle^+$ is a basis for $\Delta\langle i\rangle$.

We think of $\mathcal{K}$ as poset such that $\beta_1=\theta$ is the unique maximal element and each β_i covers exactly its own descendants. If β_j is a descendant of β_i, then $\beta_j\prec\beta_i$ in $(\Delta^+,\preccurlyeq)$ and $\mathsf{supp}(\beta_j)\subsetneq\mathsf{supp}(\beta_i)$, while different descendants of β_i are not comparable in Δ^+. Therefore the poset structure of $\mathcal{K}$ is the restriction of the root order in Δ^+. The resulting poset $(\mathcal{K},\preccurlyeq)$ is called the *cascade poset*. The numbering of $\mathcal{K}$ is not canonical. We only require that it is a linear extension of $(\mathcal{K},\preccurlyeq)$, i.e., if β_j is a descendant of β_i, then $j>i$.

Using the decomposition $\Delta^+=\Delta^+_\theta\sqcup\mathcal{H}_\theta$ and induction on $\mathsf{rk}\,\mathfrak{g}$, one readily obtains the disjoint union determined by $\mathcal{K}$:

$$\Delta^+ = \bigsqcup_{i=1}^{m}\mathcal{H}_{\beta_i} = \bigsqcup_{\beta\in\mathcal{K}}\mathcal{H}_\beta, \tag{2}$$

where $\mathcal{H}_{\beta_i}$ is the Heisenberg subset in $\Delta\langle i\rangle^+$ and $\mathcal{H}_{\beta_1}=\mathcal{H}_\theta$. The geometric counterpart of this decomposition is the direct sum of vector spaces

$$\mathfrak{u}^+ = \bigoplus_{i=1}^{m}\mathfrak{h}_i,$$

where $\mathfrak{h}_i$ is the Heisenberg Lie algebra in $\mathfrak{g}\langle i\rangle$, with $\Delta(\mathfrak{h}_i)=\mathcal{H}_{\beta_i}$. In particular, $\mathfrak{h}_1=\mathfrak{n}_\theta$.

For any $\beta\in\mathcal{K}$, set $\Phi(\beta)=\Pi\cap\mathcal{H}_\beta$. Then $\Pi=\bigsqcup_{\beta\in\mathcal{K}}\Phi(\beta)$ and Φ is thought of as a map from $\mathcal{K}$ to 2^Π. Our definition of subsets $\Phi(\beta_i)$ yields the well-defined map $\Phi^{-1}:\Pi\to\mathcal{K}$, where $\Phi^{-1}(\alpha)=\beta_i$ if $\alpha\in\Phi(\beta_i)$. Note that $\alpha\in\Phi(\Phi^{-1}(\alpha))$. We also have $\#\Phi(\beta_i)\le 2$ and $\#\Phi(\beta_i)=2$ if and only if the root system $\Delta\langle i\rangle$ is of type $\mathbf{A}_n$ with $n\ge 2$.

The cascade poset $(\mathcal{K},\preccurlyeq)$ with the set $\Phi(\beta)$ attached to each β is called the *marked cascade poset* (MCP). In [26], we use $(\mathcal{K},\preccurlyeq,\Phi)$ for describing the nilradicals of parabolic subalgebras that admit a commutative polarisation. The Hasse diagrams of $(\mathcal{K},\preccurlyeq)$ are presented in Appendix A, where the Cartan label of the simple Lie algebra $\mathfrak{g}\langle j\rangle$ is attached to the node β_j. These diagrams (without Cartan labels) appear already in [12, Section 2].

Let us gather some properties of $(\mathcal{K},\preccurlyeq)$ that either are explained above or easily follow from the construction.

Lemma 2.1. *Let $(\mathcal{K}, \preccurlyeq, \Phi)$ be the* MCP *for a simple Lie algebra $\mathfrak{g}$.*

(1) *The partial order in $\mathcal{K}$ coincides with the restriction to $\mathcal{K}$ of the root order in Δ^+;*
(2) *$\beta_i, \beta_j \in \mathcal{K}$ are comparable if and only if $\mathsf{supp}(\beta_i) \cap \mathsf{supp}(\beta_j) \ne \varnothing$; and then one support is properly contained in the other;*
(3) *each β_j, $j \ge 2$, is covered by a unique element of $\mathcal{K}$;*
(4) *for any $\beta_j \in \mathcal{K}$, the interval $[\beta_j, \beta_1]_{\mathcal{K}} = \{\nu \in \mathcal{K} \mid \beta_j \preccurlyeq \nu \preccurlyeq \beta_1\} \subset \mathcal{K}$ is a chain.*
(5) *For $\alpha \in \Pi$, we have $\alpha \in \Phi(\beta_i)$ if and only if $(\alpha, \beta_i) > 0$.*

Clearly, $\#\mathcal{K} \le \mathsf{rk}\,\mathfrak{g}$ and the equality holds if and only if each β_i is a multiple of a fundamental weight for $\mathfrak{g}\langle i\rangle$. Recall that θ is a multiple of a fundamental weight of $\mathfrak{g}$ if and only if $\mathfrak{g}$ is not of type $\mathbf{A}_n$, $n \ge 2$.

It is well known that the following two conditions are equivalent: (1) $\mathrm{ind}\,\mathfrak{b} = 0$; (2) $\#\mathcal{K} = \mathsf{rk}\,\mathfrak{g}$, see e.g. [1, Prop. 4.2]. This happens exactly if $\mathfrak{g}$ is not of type $\mathbf{A}_n$ $(n \ge 2)$, $\mathbf{D}_{2n+1}$ $(n \ge 2)$, $\mathbf{E}_6$. Then Φ yields a bijection between $\mathcal{K}$ and Π.

For future reference, we record the following observation.

Lemma 2.2. *If $\mathfrak{g}$ is of type $\mathbf{B}_{2k+1}$ or $\mathbf{G}_2$, then $\mathcal{K}$ contains a unique short root, which is simple. In all other cases, all elements of $\mathcal{K}$ are long.*

Write $r_\gamma \in W$ for the reflection relative to $\gamma \in \Delta$.

Proposition 2.3. [16, Prop. 1.10] *The product $\omega_0 := r_{\beta_1} \cdot \ldots \cdot r_{\beta_m}$ does not depend on the order of factors and it is the longest element of W (i.e., $\omega_0(\Delta^+) = \Delta^-$). In particular, $\omega_0(\beta_i) = -\beta_i$ for each i.*

It follows from this that $\omega_0 = -1$ if and only if $m = \mathsf{rk}\,\mathfrak{g}$.

3. The cascade element of a Cartan subalgebra

In this section, we define a certain element of $\mathfrak{t}$ associated with the cascade $\mathcal{K}$ and consider its properties related to Δ. As usual, we identify $\mathfrak{t}$ and $\mathfrak{t}^*$ using the restriction of the Killing form to $\mathfrak{t}$.

Definition 3.1. The *cascade element* of $\mathfrak{t}$ is the unique element $x_{\mathcal{K}} \in \langle \beta_1, \ldots, \beta_m\rangle_{\mathbb{Q}}$ such that $\beta_i(x_{\mathcal{K}}) = 1$ for each i.

Since the roots $\{\beta_i\}$ are pairwise orthogonal, we have

$$x_{\mathcal{K}} = \sum_{i=1}^{m} \frac{\beta_i}{(\beta_i, \beta_i)} = \frac{1}{2}\sum_{i=1}^{m} \beta_i^\vee. \qquad (3)$$

Therefore, $\gamma(x_{\mathcal{K}}) \in \frac{1}{2}\mathbb{Z}$ for any $\gamma \in \Delta$, and it follows from Proposition 2.3 that $\omega_0(x_{\mathcal{K}}) = -x_{\mathcal{K}}$. If $\mathcal{K} \subset \Delta_l$, then one can also write $x_{\mathcal{K}} = \frac{1}{(\theta, \theta)} \sum_{i=1}^{m} \beta_i$.

• It is a typical pattern related to $\mathcal{K}$ and $x_{\mathcal{K}}$ that a certain property holds for series $\mathbf{A}_n$ and $\mathbf{C}_n$, but does not hold for the other simple types. The underlying reason is that

θ is a fundamental weight if and only if $\mathfrak{g}$ is not of type $\mathbf{A}_n$ or $\mathbf{C}_n$.

(Recall that $\theta = \varpi_1 + \varpi_n$ for A_n and $\theta = 2\varpi_1$ for C_n.) It is often possible to prove that a property does not hold if θ is fundamental, and then directly verify that that property does hold for $\mathfrak{sl}_{n+1}$ and $\mathfrak{sp}_{2n}$ (or vice versa).

- Yet another pattern is that one has to often exclude the series A_{2n} from consideration. The reason is that

the Coxeter number *of* $\mathfrak{g}$, $\mathsf{h} = \mathsf{h}(\mathfrak{g})$, *is odd if and only if* $\mathfrak{g}$ *is of type* A_{2n}.

(The same phenomenon occurs also in the context of the McKay correspondence.) Recall that $\mathsf{h} = 1 + \sum_{\alpha\in\Pi}[\theta : \alpha] = 1 + \widetilde{ht}(\theta)$.

To get interesting properties of $x_{\mathcal{K}}$, we need some preparations. Set $n_\alpha = [\theta : \alpha]$, i.e., $\theta = \sum_{\alpha\in\Pi} n_\alpha\alpha$. Suppose that θ is a fundamental weight, and let $\tilde\alpha$ be the unique simple root such that $(\theta, \tilde\alpha) \ne 0$. Then $(\theta, \tilde\alpha^\vee) = 1$ and $(\theta^\vee, \tilde\alpha) = 1$, hence $\tilde\alpha$ is long.

We have
$$(\theta, \theta) = (\theta, \sum_{\alpha\in\Pi} n_\alpha\alpha) = (\theta, n_{\tilde\alpha}\tilde\alpha) = \frac{1}{2} n_{\tilde\alpha}(\tilde\alpha, \tilde\alpha),$$
which means that $n_{\tilde\alpha} = 2$. Let $\tilde\Pi$ be the set of simple roots that are adjacent to $\tilde\alpha$ in the Dynkin diagram. Since $\tilde\alpha$ is long, one also has $\tilde\Pi = \{\nu \in \Pi \mid (\nu, \tilde\alpha^\vee) = -1\}$.

Then
$$1 = (\theta, \tilde\alpha^\vee) = n_{\tilde\alpha}(\tilde\alpha, \tilde\alpha^\vee) + \sum_{\nu\in\tilde\Pi} n_\nu(\nu, \tilde\alpha^\vee) = 4 - \sum_{\nu\in\tilde\Pi} n_\nu.$$

Hence $\sum_{\nu\in\tilde\Pi} n_\nu = 3$ and $\#(\tilde\Pi) \le 3$. Set $J = \{i \in [1, m] \mid (\tilde\alpha, \beta_i) < 0\}$. Then $1 \notin J$ and we proved in [25, Sect. 6] that

$$\tilde\alpha = \frac{1}{2}\left(\theta - \sum_{i\in J}\frac{(\tilde\alpha, \tilde\alpha)}{(\beta_i, \beta_i)}\beta_i\right) =: \frac{1}{2}(\theta - \sum_{i\in J} c_i\beta_i) \ \text{ and } \ \sum_{i\in J} c_i = 3, \tag{4}$$

see [25, Lemma 6.5]. Here $c_i \in \mathbb{N}$ and therefore $\#J \le 3$. Set $\tilde{\mathcal{K}} := \{\beta_i \mid i \in J\}$.

We say that $x \in \mathfrak{t}$ is *dominant*, if $\gamma(x) \ge 0$ for all $\gamma \in \Delta^+$.

Lemma 3.2. *If θ is fundamental, then $\tilde\alpha(x_{\mathcal{K}}) = -1$ and $(\theta - \tilde\alpha)(x_{\mathcal{K}}) = 2$. In particular, $x_{\mathcal{K}} \in \mathfrak{t}$ is not dominant.*

Proof. Take $\gamma = \tilde\alpha$. Using (3) and (4), we obtain

$$\tilde\alpha(x_{\mathcal{K}}) = \frac{1}{2}\left[\frac{(\theta, \theta)}{(\theta, \theta)} - \sum_{i\in J}\frac{c_i(\beta_i, \beta_i)}{(\beta_i, \beta_i)}\right] = (1 - 3)/2 = -1.$$

Then $(\theta - \tilde\alpha)(x_{\mathcal{K}}) = 1 + 1 = 2$. ∎

Conversely, if θ is *not* fundamental, then the example below shows that $x_{\mathcal{K}}$ *is* dominant.

Example 3.3. (1) For $\mathfrak{g} = \mathfrak{sl}_{n+1}$, one has $\beta_i = \varepsilon_i - \varepsilon_{n+2-i}$ with $i = 1, \dots, m$, where $m = [(n + 1)/2]$. Here $\sum_{j=1}^{n+1} \varepsilon_j = 0$ and $\varpi_j = \varepsilon_1 + \dots + \varepsilon_j$. Hence

$$(\theta, \theta){\cdot}x_{\mathcal{K}} = \sum_{i=1}^{m} \beta_i = \begin{cases} 2\varpi_p, & \text{if } n = 2p{-}1 \\ \varpi_p + \varpi_{p+1}, & \text{if } n = 2p. \end{cases}$$

In the matrix form, one has

$$x_{\mathcal{K}} = \begin{cases} \mathsf{diag}(1/2, \dots, 1/2, -1/2, \dots, -1/2), & \text{if } n = 2p{-}1 \\ \mathsf{diag}(1/2, \dots, 1/2, 0, -1/2, \dots, -1/2), & \text{if } n = 2p\ . \end{cases}$$

(2) For $\mathfrak{sp}_{2n}$, one has $\beta_i = 2\varepsilon_i$ with $i = 1, \dots, m = n$. Hence

$$(\theta, \theta){\cdot}x_{\mathcal{K}} = \textstyle\sum_{i=1}^{n} 2\varepsilon_i = 2\varpi_n .$$

Consider the multiset $\mathcal{M}_{\mathcal{K}}$ of values $\{\gamma(x_{\mathcal{K}}) \mid \gamma \in \Delta^+ \setminus \mathcal{K}\}$. That is, each value d is taken with multiplicity $m_d = \#\mathcal{R}_d$, where $\mathcal{R}_d = \{\gamma \in \Delta^+ \setminus \mathcal{K} \mid \gamma(x_{\mathcal{K}}) = d\}$.

Lemma 3.4. *For any d, there is a natural bijection between the sets $\mathcal{R}_d$ and $\mathcal{R}_{1-d}$. In particular, $m_d = m_{1-d}$, i.e., the multiset $\mathcal{M}_{\mathcal{K}}$ is symmetric w.r.t. $1/2$.*

Proof. For any $\gamma \in \Delta^+ \setminus \mathcal{K}$, there is a unique $j \in \{1, \dots, m\}$ such that $\gamma \in \Delta(\mathfrak{h}_j) \setminus \{\beta_j\}$, see (2). Then $\beta_j - \gamma \in \Delta(\mathfrak{h}_j)$ and $\gamma(x_{\mathcal{K}}) + (\beta_j - \gamma)(x_{\mathcal{K}}) = 1$. ∎

As we shall see in Section 5, there is a geometric reason for such a symmetry. It is related to the fact that a certain Lie algebra is Frobenius.

Theorem 3.5. *If $\mathfrak{g}$ is a simple Lie algebra, then $-1 \le \gamma(x_{\mathcal{K}}) \le 2$ for all $\gamma \in \Delta^+$. More precisely,*

(1) *If $\mathfrak{g}$ is of type $\mathbf{A}_{2p-1}$ or $\mathbf{C}_n$, then $\{\gamma(x_{\mathcal{K}}) \mid \gamma \in \Delta^+\} = \{0, 1\}$;*

(2) *If $\mathfrak{g}$ is of type $\mathbf{A}_{2p}$, then $\{\gamma(x_{\mathcal{K}}) \mid \gamma \in \Delta^+\} = \{0, \frac{1}{2}, 1\}$;*

(3) *For all other types, i.e., if θ is fundamental, we have*

$$\{\gamma(x_{\mathcal{K}}) \mid \gamma \in \Delta^+\} = \{-1, 0, 1, 2\}.$$

In particular, if $\mathfrak{g}$ is not of type $\mathbf{A}_{2p}$, then $\gamma(x_{\mathcal{K}}) \in \mathbb{Z}$ for any $\gamma \in \Delta^+$.

Proof. (1) Using explicit formulae for $\mathcal{K}$, all these assertions can be verified case-by-case. For instance, data of Example 3.3 provide a proof for (1) and (2). However, this does not explain the general constraints $-1 \le \gamma(x_{\mathcal{K}}) \le 2$. Below we provide a more conceptual argument, which also uncovers some additional properties of $x_{\mathcal{K}}$.

(2) If $m = \mathsf{rk}\,\mathfrak{g} = n$, then $\mathcal{K}$ is a basis for $\mathfrak{t}^*$. Hence γ can be written as $\gamma = \sum_{i=1}^{n} k_i\beta_i$, where $k_i = \frac{1}{2}(\gamma, \beta_i^\vee) \in \frac{1}{2}\mathbb{Z}$, and $\gamma(x_{\mathcal{K}}) = \sum_{i=1}^{n} k_i$. We are to prove that $-1 \le \sum_{i=1}^{n} k_i \le 2$.

If $\gamma = \beta_i$, then $\gamma(x_{\mathcal{K}}) = 1$. Therefore we assume below that $\gamma \notin \mathcal{K}$. If $\gamma \in \Delta(\mathfrak{h}_i)$ for some $i \ge 1$, then the whole argument can be performed for the simple Lie subalgebra $\mathfrak{g}\langle i\rangle \subset \mathfrak{g}$ and the cascade $\mathcal{K}(\mathfrak{g}\langle i\rangle) \subset \mathcal{K}$, which has the unique maximal element β_i. Since $\mathsf{rk}\,\mathfrak{g}\langle i\rangle < \mathsf{rk}\,\mathfrak{g}$ for $i \ge 2$, it suffices to prove the assertion for $i = 1$ and $\beta_1 = \theta$.

Assume that $\gamma \in \mathcal{H}_1 = \Delta(\mathfrak{h}_1)$ and $\gamma \ne \theta$. Then $(\gamma, \theta^\vee) = 1$, $\gamma = \frac{1}{2}\theta + \sum_{i=2}^{n} k_i\beta_i$, and

$$4(\gamma, \gamma) = (\theta, \theta) + \sum_{i \ge 2} (2k_i)^2 (\beta_i, \beta_i). \tag{5}$$

(i) For $\gamma \in \Delta_l^+$, it follows from (5) that $3 = \sum_{i\ge 2} 4k_i^2 {\cdot} \frac{(\beta_i, \beta_i)}{(\gamma, \gamma)}$. Since $\#(\mathcal{K} \cap \Delta_s^+) \le 1$ (Lemma 2.2), the only possibilities for the nonzero coefficients k_i are:

- $k_2, k_3, k_4 = \pm\frac{1}{2}$ with $\beta_2, \beta_3, \beta_4 \in \Delta_l^+$;

- $k_2 = \pm\frac{1}{2}$, $k_3 = \pm 1$ with $\beta_2 \in \Delta_l^+$, $\beta_3 \in \Delta_s^+$ and $(\beta_2,\beta_2)/(\beta_3,\beta_3) = 2$ [this happens only for B_{2p+1}];
- $k_2 = \pm\frac{3}{2}$ with $\beta_2 \in \Delta_s^+$ and $(\theta,\theta)/(\beta_2,\beta_2) = 3$ [this happens only for G_2].

In all these cases, we have $\gamma(x_{\mathcal{K}}) \in \{-1,0,1,2\}$, as required.

(ii) If $\gamma \in \Delta_s^+$ and $\dfrac{(\theta,\theta)}{(\gamma,\gamma)} = 2$, then (5) shows that $2 = \sum_{i\ge 2} 4k_i^2 \cdot \dfrac{(\beta_i,\beta_i)}{(\gamma,\gamma)} \ge \sum_{i\ge 2} 4k_i^2$. Since $\#(\mathcal{K} \cap \Delta_s^+) \le 1$, the only possibility here is:

- $k_2 = \pm\frac{1}{2}$ with $\beta_2 \in \Delta_l^+$ [this happens for B_n, C_n, and F_4].

Therefore $\gamma(x_{\mathcal{K}}) = \frac{1}{2} + k_2 \in \{0,1\}$.

(iii) If $\gamma \in \Delta_s^+$ and $\dfrac{(\theta,\theta)}{(\gamma,\gamma)} = 3$, then (5) shows that $1 = \sum_{i\ge 2} 4k_i^2 \cdot \dfrac{(\beta_i,\beta_i)}{(\gamma,\gamma)}$. Here the only possibility is $k_2 = \pm\frac{1}{2}$ and β_2 is short [this happens for G_2].

(3) If $m < \mathsf{rk}\,\mathfrak{g} = n$, then $\mathcal{K}$ is not a basis for $\mathfrak{t}^*$. Nevertheless, one can circumvent this obstacle as follows. Let $\omega_0 \in W$ be the longest element. Then $-\omega_0 \in GL(\mathfrak{t})$ takes Δ^+ to itself and $\beta_i \in \mathfrak{t}^{-\omega_0}$ for each i, see Prop. 2.3. Moreover, $\mathcal{K}$ is a basis for $\mathfrak{t}^{-\omega_0}$, see [25, Lemma 6.2]. Hence $\bar\gamma := \frac{1}{2}(\gamma - \omega_0(\gamma)) = \sum_{i=1}^m k_i\beta_i$, $k_i \in \frac{1}{2}\mathbb{Z}$, and

$$\sum_{i=1}^m k_i = \bar\gamma(x_{\mathcal{K}}) = \gamma(x_{\mathcal{K}}).$$

Therefore, the argument of part (2) applies to $\bar\gamma$ in place of γ. However, a new phenomenon may occur here. As above, we begin with $\gamma \in \Delta(\mathfrak{h}_1) \setminus \theta$. Then $\bar\gamma = \frac{1}{2}\theta + \sum_{i\ge 2} k_i\beta_i$. But in this case, $\bar\gamma$ is not necessarily a root and it may happen that $k_i = 0$ for $i \ge 2$. Then $\bar\gamma(x_{\mathcal{K}}) = 1/2$. (This does occur for $\mathfrak{g}$ of type A_{2p}: if $\gamma = \varepsilon_1 - \varepsilon_{p+1}$, then $\omega_0(\gamma) = \varepsilon_{2p+1} - \varepsilon_{p+1}$ and $\bar\gamma = \frac{1}{2}\theta$. Conversely, if $\bar\gamma = \frac{1}{2}\theta$, then $\widetilde{ht}(\theta) = 2\cdot\widetilde{ht}(\gamma)$. Hence the Coxeter number of $\mathfrak{g}$ is odd, and this happens only for A_{2p}.)

(4) Recall that $\omega_0 = -1$ if and only $m = \mathsf{rk}\,\mathfrak{g}$ and then $\bar\gamma = \gamma$. Therefore, part (2) can be thought of as a special case of a more general approach outlined in (3). ■

Remark 3.6. An analysis of possibilities for $\{k_i\}$ in the proof of Theorem 3.5 reveals the following features:

1. for any $\gamma \in \Delta^+$, $\bar\gamma$ is a linear combination of at most four different elements of $\mathcal{K}$;
2. if $\#\mathcal{K} = \mathsf{rk}\,\mathfrak{g}$ and $\mathcal{K} \subset \Delta_l^+$, then every $\gamma \in \Delta_l^+ \setminus \mathcal{K}$ is presented as a linear combination of exactly four different elements of $\mathcal{K}$;
3. if $\gamma(x_{\mathcal{K}}) = 2$ or $\gamma(x_{\mathcal{K}}) = -1$, then $\gamma \in \Delta_l^+$.

Example 3.7. Let $\mathfrak{g} = \mathfrak{so}_N$ be realised as the set of skew-symmetric $N \times N$-matrices w.r.t. the antidiagonal.

- For $N = 2n$, one has $\mathfrak{t} = \{\mathsf{diag}(x_1,\dots,x_n,-x_n,\dots,-x_1) \mid x_i \in \mathbb{C}\}$.
  - If $n = 2k$, then $\#\mathcal{K} = 2k$ and the entries of $x_{\mathcal{K}}$ are $x_{2i-1} = 1$ and $x_{2i} = 0$ for $i = 1,\dots,k$.
  - If $n = 2k+1$, then still $\#\mathcal{K} = 2k$, the entries x_j with $j \le 2k$ are the same as for $\mathfrak{so}_{4k}$, and $x_{2k+1} = 0$.

- For $N = 2n+1$, one has $\mathfrak{t} = \{\mathsf{diag}(x_1, \dots, x_n, 0, -x_n, \dots, -x_1) \mid x_i \in \mathbb{C}\}$ and $\#\mathcal{K} = n$. Here the entries of $x_{\mathcal{K}}$ are $x_{2i-1} = 1$ for $i \le [(n+1)/2]$ and $x_{2i} = 0$ for $i \le [n/2]$.

Using Examples 3.3 and 3.7, one readily computes the numbers $\{\alpha(x_{\mathcal{K}})\}$ with $\alpha \in \Pi$ for all classical Lie algebras. For the exceptional Lie algebras one can use explicit formulae for $\mathcal{K}$, see Appendix A. An alternative approach is to use the recursive construction of $\mathcal{K}$. One has $\Pi = \bigsqcup_{i=1}^{m} \Phi(\beta_i)$, and it suffices to describe the numbers $\alpha(x_{\mathcal{K}})$ for any simple Lie algebra and $\alpha \in \Phi(\beta_1) = \Phi(\theta)$.

1. If θ is fundamental, then $\Phi(\theta) = \{\tilde\alpha\} \subset \Pi_l$ and $\tilde\alpha(x_{\mathcal{K}}) = -1$ (Lemma 3.2).
2. If $\mathfrak{g} = \mathfrak{sp}_{2n}$, then $\theta = 2\varpi_1$ and $\Phi(\theta) = \{\alpha_1\} \subset \Pi_s$. Here $\theta = 2\alpha_1 + \beta_2$ and $\alpha_1(x_{\mathcal{K}}) = 0$:
3. For $\mathfrak{sl}_{n+1}$ $(n \ge 2)$, we have $\Phi(\theta) = \{\alpha_1, \alpha_n\}$. If $n \ge 3$, then $\alpha_1(x_{\mathcal{K}}) = \alpha_n(x_{\mathcal{K}}) = 0$; if $n = 2$, then $\theta = \alpha_1 + \alpha_2$ and $\alpha_1(x_{\mathcal{K}}) = \alpha_2(x_{\mathcal{K}}) = 1/2$.
4. For $\mathfrak{sl}_2$, one has $\theta = \alpha_1$ and $\alpha_1(x_{\mathcal{K}}) = 1$.

The resulting labelled diagrams are presented in Figures 1 and 2.

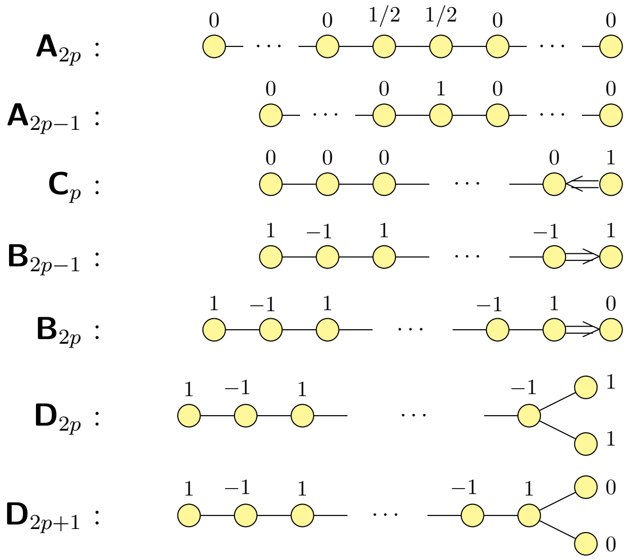


Figure 1: Numbers $\{\alpha(x_{\mathcal{K}}) \mid \alpha \in \Pi\}$ for the classical cases

Let us summarise the main features of the diagrams obtained.

Remark 3.8. (1) Fractional values occur only for $\mathfrak{g}$ of type $\mathbf{A}_{2p}$. For $\mathbf{A}_{2p-1}$ and $\mathbf{C}_p$, the dominant element $x_{\mathcal{K}}$ is a multiple of the fundamental weight ϖ_p. More precisely, $x_{\mathcal{K}} = \frac{2}{(\alpha_p, \alpha_p)}\varpi_p$ (cf. Example 3.3).

(2) All these diagrams have no marks '2' and all the marks $\{\alpha(x_{\mathcal{K}})\}$ are nonzero if and only if $\mathfrak{g}$ is of type $\mathbf{B}_{2p-1}, \mathbf{D}_{2p}, \mathbf{E}_7, \mathbf{E}_8, \mathbf{G}_2$. The subset $\{\alpha \in \Pi \mid \alpha(x_{\mathcal{K}}) = \pm 1\}$ is always connected and the marks '1' and '-1' alternate in this subset.

(3) If $\alpha(x_{\mathcal{K}}) = -1$ and $\alpha' \in \Pi$ is adjacent to α, then $\alpha'(x_{\mathcal{K}}) = 1$. Moreover, if $\alpha'(x_{\mathcal{K}}) = 1$, then $\alpha' \in \mathcal{K}$.

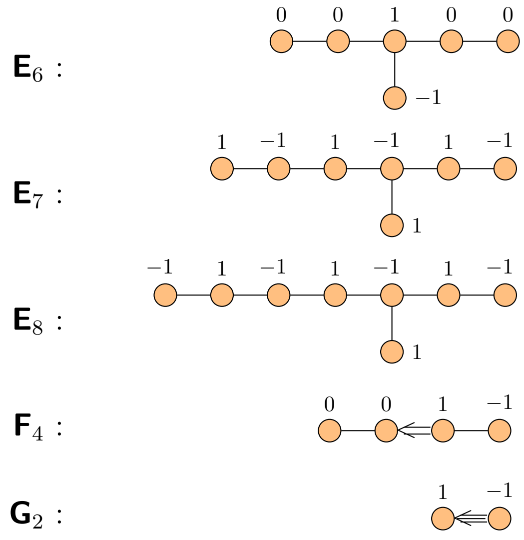
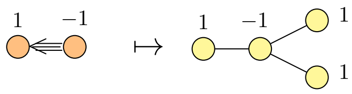


Figure 2: Numbers $\{\alpha(x_{\mathcal{K}}) \mid \alpha \in \Pi\}$ for the exceptional cases

(4) The numbers $\{\alpha(x_{\mathcal{K}})\}$ are compatible with the unfolding procedures

$$\mathbf{C}_p \mapsto \mathbf{A}_{2p-1}\,,\ \mathbf{B}_{p-1} \mapsto \mathbf{D}_p\,,\ \mathbf{F}_4 \mapsto \mathbf{E}_6\,, \text{ and } \mathbf{G}_2 \mapsto \mathbf{D}_4\,.$$

For instance,  .

4. The cascade element and self-dual representations of $\mathfrak{g}$

Consider the standard lattices in $\mathfrak{t}^*_{\mathbb{Q}}$ associated with Δ [19, Chap. 4, §2.8]:

- $\mathcal{Q} = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ – the root lattice;
- $\mathcal{Q}^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$ – the coroot lattice;
- $\mathcal{P} = \bigoplus_{i=1}^n \mathbb{Z}\varpi_i$ – the weight lattice;
- $\mathcal{P}^\vee = \bigoplus_{i=1}^n \mathbb{Z}\varpi_i^\vee$ – the coweight lattice, where $\varpi_i^\vee = 2\varpi_i/(\alpha_i,\alpha_i)$.

Then $\mathcal{P} \supset \mathcal{Q}$, $\mathcal{P}^\vee \supset \mathcal{Q}^\vee$, $\mathcal{P} = (\mathcal{Q}^\vee)^*$, and $\mathcal{P}^\vee = \mathcal{Q}^*$, where $\mathcal{L}^*$ stands for the dual lattice of $\mathcal{L}$. For instance, $\mathcal{P}^\vee = \mathcal{Q}^* = \{\nu \in \mathfrak{t}_{\mathbb{Q}} \mid (\nu,\gamma) \in \mathbb{Z}\ \ \forall \gamma \in \Delta\}$.

If $\mathfrak{g}$ is not of type $\mathbf{A}_{2p}$, then $x_{\mathcal{K}} \in \mathcal{P}^\vee$ (Theorem 3.5). However, then $x_{\mathcal{K}}$ does not always belong to $\mathcal{Q}^\vee$, and we characterise below the relevant cases.

If $\mathcal{M} \subset \mathfrak{t}_{\mathbb{Q}}$ is finite, then $|\mathcal{M}| := \sum_{m\in\mathcal{M}} m$. As usual, set $2\varrho = \sum_{\gamma\in\Delta^+}\gamma = |\Delta^+|$ and $2\varrho^\vee = \sum_{\gamma\in\Delta^+}\gamma^\vee = |(\Delta^\vee)^+|$. Then $\mathsf{h}(\mathfrak{g}) = (\varrho^\vee,\theta)+1$ and the *dual Coxeter number* of $\mathfrak{g}$ is $\mathsf{h}^* = \mathsf{h}^*(\mathfrak{g}) := (\varrho,\theta^\vee)+1$.

Lemma 4.1. *One has* $2\varrho = \sum_{i=1}^m \big(\mathsf{h}^*(\mathfrak{g}\langle j\rangle)-1\big)\beta_j$ *and* $2\varrho^\vee = \sum_{i=1}^m \big(\mathsf{h}(\mathfrak{g}\langle j\rangle)-1\big)\beta_j^\vee$.

Proof. Since we have $\Delta^+ = \bigsqcup_{i=1}^m \mathcal{H}_{\beta_i}$ and $\beta_1 = \theta$, it is sufficient to prove that $|\mathcal{H}_\theta^\vee| = \big(\mathsf{h}(\mathfrak{g})-1\big)\theta^\vee$ and $|\mathcal{H}_\theta| = \big(\mathsf{h}^*(\mathfrak{g})-1\big)\theta$. Since $\mathcal{H}_\theta \setminus \{\theta\}$ is the union of pairs $\{\gamma,\theta-\gamma\}$, where the roots γ and $\theta-\gamma$ have the same length, it is clear that $|\mathcal{H}_\theta^\vee| = a\theta^\vee$ for some $a \in \mathbb{N}$.

Then $$a = (\tfrac{1}{2}|\mathcal{H}_\theta^\vee|, \theta) = (\varrho^\vee, \theta) = \mathsf{h}(\mathfrak{g}) - 1.$$

The proof of the second relation is similar. ∎

Proposition 4.2. *The following conditions are equivalent:*

(1) $x_{\mathcal{K}} \in \mathcal{Q}^\vee$;

(1) *every self-dual representation of* $\mathfrak{g}$ *is orthogonal.*

Proof. By a classical result of Dynkin, if $\pi_\lambda : \mathfrak{g} \to \mathbb{V}_\lambda$ is an irreducible representation with highest weight λ, then π_λ is self-dual if and only if $\omega_0(\lambda) = -\lambda$. Then π_λ is orthogonal if and only if $(\varrho^\vee, \lambda) \in \mathbb{N}$ [5], cf. also [19, Exercises 4.2.12–13] or [10, Chap. 3, § 2.7]. Hence every self-dual representation of $\mathfrak{g}$ is orthogonal if and only if $\varrho^\vee \in \mathcal{Q}^\vee$.

Therefore, it suffices to prove that $\varrho^\vee - x_{\mathcal{K}} \in \mathcal{Q}^\vee$, if $\mathfrak{g}$ is not of type $\mathbf{A}_{2p}$. By Lemma 4.1, we have
$$\varrho^\vee - x_{\mathcal{K}} = \sum_{i=1}^{m} \frac{\mathsf{h}(\mathfrak{g}\langle j\rangle) - 2}{2} \cdot \beta_j^\vee .$$
It remains to observe that, for any simple $\mathfrak{g}$, the Coxeter numbers $\mathsf{h}(\mathfrak{g}\langle j\rangle)$, for $j = 1, \dots, m$, have the same parity, and if $\mathfrak{g}$ is not of type $\mathbf{A}_{2p}$, then all these numbers are even. ∎

Using Proposition 4.2 and [19, Table 3], one readily obtains that
$$x_{\mathcal{K}} \in \mathcal{Q}^\vee \iff \mathfrak{g} \in \{\mathbf{B}_{4p-1}, \mathbf{B}_{4p}, \mathbf{D}_{4p}, \mathbf{D}_{4p+1}, \mathbf{E}_6, \mathbf{E}_8, \mathbf{F}_4, \mathbf{G}_2\} .$$
Although the coefficients $[\varrho^\vee : \beta_i^\vee]$ are usually not integral, this does *not* necessarily mean that $\varrho^\vee \notin \mathcal{Q}^\vee$. For, the elements $\beta_1^\vee, \dots, \beta_m^\vee$ do not form a (part of a) basis for $\mathcal{Q}^\vee$.

5. The cascade element as the Ooms element of a Frobenius Lie algebra

Given an arbitrary Lie algebra $\mathfrak{q}$, one associates the *Kirillov form* $\mathcal{B}_\eta$ on $\mathfrak{q}$ to any $\eta \in \mathfrak{q}^*$. By definition, if $\langle\ ,\ \rangle : \mathfrak{q}^* \times \mathfrak{q} \to \mathbb{C}$ is the natural pairing and $x, y \in \mathfrak{q}$, then
$$\mathcal{B}_\eta(x, y) = \langle \eta, [x, y]\rangle = -\langle \mathrm{ad}^*(x)\eta, y\rangle.$$
Then $\mathcal{B}_\eta$ is skew-symmetric and $\mathsf{Ker}\,\mathcal{B}_\eta = \mathfrak{q}^\eta$, the stabiliser of η in $\mathfrak{q}$. The *index* of $\mathfrak{q}$ is $\mathrm{ind}\,\mathfrak{q} = \min_{\eta\in\mathfrak{g}^*} \dim \mathfrak{q}^\eta$, and $\mathfrak{q}^*_{\mathsf{reg}} = \{\xi \in \mathfrak{q}^* \mid \dim \mathfrak{q}^\xi = \mathrm{ind}\,\mathfrak{q}\}$ is the set of *regular elements* of $\mathfrak{q}^*$. Suppose that $\mathfrak{q}$ is *Frobenius*, i.e., there is $\xi \in \mathfrak{q}^*$ such that $\mathcal{B}_\xi$ is non-degenerate. Then $\mathfrak{q}^\xi = \{0\}$ and $\xi \in \mathfrak{q}^*_{\mathsf{reg}}$. The 2-form $\mathcal{B}_\xi$ yields a linear isomorphism between $\mathfrak{q}$ and $\mathfrak{q}^*$. Let $x_{\mathfrak{q},\xi} = x_\xi \in \mathfrak{q}$ correspond to ξ under that isomorphism. It is noticed by A. Ooms [20] that $\mathrm{ad}\, x_\xi$ enjoys rather interesting properties. Namely, using the non-degenerate form $\mathcal{B}_\xi$, one defines the adjoint operator $(\mathrm{ad}\, x_\xi)^* : \mathfrak{q} \to \mathfrak{q}$. By [20, Theorem 3.3], one has
$$(\mathrm{ad}\, x_\xi)^* = 1 - \mathrm{ad}\, x_\xi. \tag{6}$$
Therefore, if λ is an eigenvalue of $\mathrm{ad}\, x_\xi$ with multiplicity m_λ, then $1 - \lambda$ is also an eigenvalue and $m_\lambda = m_{1-\lambda}$. Hence $\mathrm{tr}_{\mathfrak{q}}(\mathrm{ad}\, x_\xi) = (\dim \mathfrak{q})/2$. We say that x_ξ is

the *Ooms element* associated with $\xi \in \mathfrak{q}^*_{\mathsf{reg}}$. Another way to define x_ξ is as follows. Since $\mathfrak{q}^\xi = \{0\}$, we have $\mathfrak{q}\cdot\xi = \mathfrak{q}^*$. Then $x_\xi \in \mathfrak{q}$ is the unique element such that $(\mathrm{ad}^* x_\xi)\cdot\xi = -\xi$.

If $\mathfrak{q} = \mathrm{Lie}\,(Q)$ is algebraic, then each element of $\mathfrak{q}$ has the Jordan decomposition [19, Ch. 3. § 3.7]. Furthermore, if $\mathfrak{q}$ is Frobenius and algebraic, then $\mathfrak{q}^*_{\mathsf{reg}}$ is the dense Q-orbit in $\mathfrak{q}^*_{\mathsf{reg}}$. Therefore, all Ooms elements in $\mathfrak{q}$ are Q-conjugate, and we can also write $x_\xi = x_{\mathfrak{q}}$ for an Ooms element in $\mathfrak{q}$.

Lemma 5.1. *If $\mathfrak{q}$ is Frobenius and algebraic, then any Ooms element x_ξ is semisimple.*

Proof. For the Jordan decomposition $x_\xi = (x_\xi)_s + (x_\xi)_n$, the defining relation $(\mathrm{ad}^* x_\xi)\cdot\xi = -\xi$ obviously implies that $\mathrm{ad}^*((x_\xi)_s)\cdot\xi = -\xi$. Then the uniqueness of the Ooms element associated with ξ shows that $x_\xi = (x_\xi)_s$ is semisimple. ■

Let $\mathsf{spec}(x_\xi)$ denote the multiset of eigenvalues of $\mathrm{ad}\, x_\xi$ in $\mathfrak{q}$. In other words, if λ is an eigenvalue and $\mathfrak{q}(\lambda)$ is the corresponding eigenspace, then $\lambda \in \mathsf{spec}(x_\xi)$ is taken with multiplicity $\dim \mathfrak{q}(\lambda)$. Then it follows from (6) that $\mathsf{spec}(x_\xi)$ is symmetric w.r.t. $1/2$.

In [26] we defined the *Frobenius envelope* of the nilradical $\mathfrak{p}^{\mathsf{nil}}$ of any standard parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. For $\mathfrak{u} = \mathfrak{b}^{\mathsf{nil}}$, this is done as follows. First set $\mathfrak{t}_{\mathcal{K}} = \langle \beta_1, \dots, \beta_m \rangle_{\mathbb{C}} = \langle \mathcal{K} \rangle_{\mathbb{C}} \subset \mathfrak{t}$. In more direct terms, $\mathfrak{t}_{\mathcal{K}} = \bigoplus_{i=1}^m [e_{\beta_i}, e_{-\beta_i}]$. Then $\mathfrak{t}_{\mathcal{K}}$ is an algebraic subalgebra of $\mathfrak{t}$, and the Frobenius envelope of $\mathfrak{u}$ is $\mathfrak{b}_{\mathcal{K}} = \mathfrak{t}_{\mathcal{K}} \oplus \mathfrak{u}$, which is an ideal of $\mathfrak{b}$. Note that $\mathfrak{b}_{\mathcal{K}} = \mathfrak{b}$ if and only if $\#\mathcal{K} = \mathsf{rk}\,\mathfrak{g}$. By [26, Proposition 5.1], we have $\mathrm{ind}\,\mathfrak{b}_{\mathcal{K}} = 0$, i.e., $\mathfrak{b}_{\mathcal{K}}$ is Frobenius. Here $\mathfrak{b}^*_{\mathcal{K}} \simeq \mathfrak{g}/\mathfrak{b}^{\perp}_{\mathcal{K}}$ can be identified with $\mathfrak{b}^-_{\mathcal{K}} = \mathfrak{t}_{\mathcal{K}} \oplus \mathfrak{u}^-$ as vector space and $\mathfrak{t}$-module. Furthermore, under this identification, we have

$$\xi_{\mathcal{K}} = \sum_{\beta \in \mathcal{K}} e_{-\beta} \in (\mathfrak{b}_{\mathcal{K}})^*_{\mathsf{reg}}.$$

Therefore, $(\mathrm{ad}^* x_{\mathcal{K}})\xi_{\mathcal{K}} = -\xi_{\mathcal{K}}$, i.e., the Ooms element $x_{\xi_{\mathcal{K}}}$ associated with $\xi_{\mathcal{K}}$ is nothing elde but the cascade element $x_{\mathcal{K}}$ from Section 3. Hence $\mathsf{spec}(x_{\mathcal{K}})$ is symmetric w.r.t. $1/2$. Note that since $\mathfrak{t}_{\mathcal{K}} \subset \mathfrak{b}_{\mathcal{K}}(0)$, $\bigoplus_{\beta \in \mathcal{K}} \mathfrak{g}_\beta \subset \mathfrak{b}_{\mathcal{K}}(1)$, and $\dim \mathfrak{t}_{\mathcal{K}} = \dim(\sum_{\beta \in \mathcal{K}} \mathfrak{g}_\beta) = \#\mathcal{K}$, the symmetry of the multiset $\mathsf{spec}(x_{\mathcal{K}})$ w.r.t. $1/2$ is equivalent to the symmetry established in Lemma 3.4.

In this situation, we have

$$\frac{1}{2}(\dim \mathfrak{u} + \#\mathcal{K}) = \frac{1}{2}\dim \mathfrak{b}_{\mathcal{K}} = \mathrm{tr}_{\mathfrak{b}_{\mathcal{K}}}(\mathrm{ad}\, x_{\mathcal{K}}) = \sum_{\gamma > 0} \gamma(x_{\mathcal{K}}) = 2\varrho(x_{\mathcal{K}}).$$

Since $\mathrm{ind}\,\mathfrak{u} = \#\mathcal{K} = m$ [12], the sum $\frac{1}{2}(\dim \mathfrak{u} + \#\mathcal{K})$ is the *magic number* associated with $\mathfrak{u}$. Comparing this with Lemma 4.1, we obtain

$$\frac{1}{2}(\dim \mathfrak{u} + m) = 2\varrho(x_{\mathcal{K}}) = \sum_{i=1}^{m} ((\mathsf{h}^*(\mathfrak{g}\langle j \rangle) - 1).$$

Remark 5.2. The case of $\mathfrak{g} = \mathfrak{sl}_{2n+1}$ in Theorem 3.5 shows that the eigenvalues of the Ooms element for the Frobenius algebra $\mathfrak{b}_{\mathcal{K}}$ are not always integral. Nevertheless, there are interesting classes of Frobenius algebras $\mathfrak{q}$ such that $\mathsf{spec}(x_{\mathfrak{q}}) \subset \mathbb{Z}$.

Using meander graphs of type A_n [4] or C_n [27], I can explicitly describe the Ooms element $x_{\mathfrak p}$ for any Frobenius *seaweed subalgebra* $\mathfrak p$ of $\mathfrak{sl}_{n+1}$ or $\mathfrak{sp}_{2n}$ and then prove that the eigenvalues of $\mathrm{ad}\, x_{\mathfrak p}$ belong to $\mathbb Z$. However, being symmetric with respect to $1/2$ and integral, the eigenvalues of such $x_{\mathfrak p}$ do not always confine to the interval $[-1,2]$.

6. The abelian ideal of $\mathfrak b$ associated with the cascade

If $\mathfrak g$ is not of type A_{2p}, then $x_{\mathcal K}\in\mathfrak t$ has the property $\{\gamma(x_{\mathcal K})\mid \gamma\in\Delta^+\}\subset\{-1,0,1,2\}$ (Theorem 3.5). By Kostant's extension of Peterson's theory [15, Section 3], every such element of $\mathfrak t$ determines an abelian ideal of $\mathfrak b$. In particular, one may associate an abelian ideal of $\mathfrak b$ to $x_{\mathcal K}$ (i.e., to $\mathcal K$) as long as $\mathfrak g$ is not of type A_{2p}. Our goal is to characterise this ideal. If $\mathfrak a$ is an abelian ideal of $\mathfrak b$, then $\mathfrak a$ is $\mathfrak t$-stable and $\mathfrak a\subset\mathfrak u^+$. Hence it suffices to determine the set of positive roots $\Delta(\mathfrak a)\subset\Delta^+$.

Recall that the *inversion set* of $w\in W$ is $\mathcal N(w)=\{\gamma\in\Delta^+\mid w(\gamma)\in\Delta^-\}$. Then $\mathcal N(w)$ and $\Delta^+\setminus\mathcal N(w)$ are *closed* (under root addition), i.e., if $\gamma',\gamma''\in\mathcal N(w)$ and $\gamma'+\gamma''\in\Delta^+$, then $\gamma'+\gamma''\in\mathcal N(w)$, and likewise for $\Delta^+\setminus\mathcal N(w)$. Conversely, if $S\subset\Delta^+$ and both S and $\Delta^+\setminus S$ are closed, then $S=\mathcal N(w)$ for a unique $w\in W$ [14, Prop. 5.10]. Below, $\gamma>0$ (resp. $\gamma<0$) is a shorthand for $\gamma\in\Delta^+$ (resp. $\gamma\in\Delta^-$).

Kostant's construction. Set $\mathfrak D_{\mathsf{ab}}=\{t\in\mathfrak t_{\mathbb Q}\mid -1\le\gamma(t)\le 2\ \ \forall\gamma\in\Delta^+\}$. Kostant associates the abelian ideal $\mathfrak a_z\lhd\mathfrak b$ to each $z\in\mathfrak D_{\mathsf{ab}}\cap\mathcal P^\vee$ [15, Theorem 3.2], i.e., if $\gamma(z)\in\{-1,0,1,2\}$ for any $\gamma\in\Delta^+$. Unlike the Peterson method, his construction exploits only W and does not invoke the affine Weyl group $\widehat W$ and "minuscule" elements in it. Set $\Delta_z^{\pm}(i)=\{\gamma\in\Delta^{\pm}\mid\gamma(z)=i\}$ and $\Delta_z(i)=\Delta_z^+(i)\cup\Delta_z^-(i)$. Note that $-\Delta_z^+(i)=\Delta_z^-(-i)$. We have

$$\Delta^+=\bigsqcup_{i=-1}^{2}\Delta_z^+(i) \tag{7}$$

and both subsets $\Delta_z^+(1)\sqcup\Delta_z^+(2)$ and $\Delta_z^+(-1)\sqcup\Delta_z^+(0)$ are closed. Therefore, there is a unique $w_z\in W$ such that $\mathcal N(w_z)=\Delta_z^+(1)\sqcup\Delta_z^+(2)$. By definition, then

$$\Delta(\mathfrak a_z)=w_z\big(\Delta_z^+(-1)\big)\sqcup w_z\big(-\Delta_z^+(2)\big)\subset\Delta^+.$$

Hence $\dim\mathfrak a_z=\#\big(\Delta_z^+(-1)\cup\Delta_z^+(2)\big)$. Note that $\Delta_z(0)$ is a root system in its own right, and $\Delta_z^+(0)$ is a set of positive roots in it. We say that the union in (7) is the *z-grading* of Δ^+, and if $\gamma\in\Delta_z^+(i)$, then i is the *z-degree* of γ.

Proposition 6.1. *Let $z\in\mathfrak D_{\mathsf{ab}}\cap\mathcal P^\vee$ be arbitrary.*

(i) *If $\gamma\in\Delta_z^+(0)$ is not a sum of two roots from $\Delta_z^+(0)$, then $w_z(\gamma)\in\Pi$;*

(ii) *if $\theta(z)=1$, i.e., $\theta\in\Delta_z^+(1)$, then $w_z(\theta)\in-\Pi$.*

Proof. Recall that $\mathcal N(w_z)=\Delta_z^+(1)\cup\Delta_z^+(2)$.

(i) Since $\gamma\notin\mathcal N(w_z)$, we have $w_z(\gamma)>0$. Assume that $w_z(\gamma)=\mu_1+\mu_2$, where $\mu_i>0$. Then $\gamma=w_z^{-1}(\mu_1)+w_z^{-1}(\mu_2)$. Letting $\gamma_i:=w_z^{-1}(\mu_i)$, we get the following possibilities.

(1) If $\gamma_1, \gamma_2 > 0$, then the z-grading of Δ^+ shows that there are further possibilities:

- $\gamma_1 \in \Delta_z^+(-1)$ and $\gamma_2 \in \Delta_z^+(1)$. But then $\mu_2 = w_z(\gamma_2) < 0$. A contradiction!
- $\gamma_1, \gamma_2 \in \Delta_z^+(0)$ – this contradicts the hypothesis on γ.

(2) Suppose that $\gamma_1 > 0$ and $\gamma_2 < 0$. Then $\gamma_1 \not\in \mathcal{N}(w_z)$ and $-\gamma_2 \in \mathcal{N}(w_z)$. Hence $\gamma_1 \in \Delta_z^+(\le 0)$ and $-\gamma_2 \in \Delta_z^+(\ge 1)$, i.e., $\gamma_2 \in \Delta_z^-(\le -1)$. It follows that $\gamma_1 + \gamma_2 = \gamma \in \Delta_z(\le -1)$. A contradiction!

Thus, cases (1) and (2) are impossible, and $w_z(\gamma)$ must be simple.

(ii) Since $\theta(z) = 1$, we have $w_z(\theta) < 0$. Assume that $w_z(\theta) = -\gamma_1 - \gamma_2$, where $\gamma_i > 0$. Then $\theta = w_z^{-1}(-\gamma_1) + w_z^{-1}(-\gamma_2)$ and $\mu_i := w_z^{-1}(-\gamma_i) > 0$ for $i = 1, 2$. Then $\mu_i \in \mathcal{N}(w_z)$ and hence $\mu_i(z) \ge 1$. Therefore $\theta(z) = \mu_1(z) + \mu_2(z) \ge 2$. A contradiction! ■

Remark 6.2. Note that $\gamma \in \Delta_z^+(0)$ is not a sum of two roots from $\Delta_z^+(0)$ if and only if γ belongs to the *base* (= set of simple roots) of $\Delta_z(0)$ that is contained in $\Delta_z^+(0)$.

Let $\mathcal{C} \subset \mathfrak{t}_{\mathbb{Q}}$ denote the *dominant Weyl chamber*, i.e., $\mathcal{C} = \{x \in \mathfrak{t}_{\mathbb{Q}} \mid \alpha(x) \ge 0\ \forall \alpha \in \Pi\}$.

Proposition 6.3. *We have $w_z(z) \in -\mathcal{C}$, i.e., it is anti-dominant. Moreover, w_z is the unique element of minimal length in W that takes z into $-\mathcal{C}$.*

Proof. For any $\lambda \in \mathfrak{t}_{\mathbb{Q}}^*$, let λ^+ be the *dominant* representative in $W{\cdot}\lambda$. By [11, Lemma 4.1], there is a unique element of minimal length $w_\lambda \in W$ such that $w_\lambda{\cdot}\lambda = \lambda^+$ and then $\mathcal{N}(w_\lambda) = \{\gamma \in \Delta^+ \mid (\gamma, \lambda) < 0\}$ (cf. also [24, Theorem 4.1]). Translating this into the assertion on the *anti-dominant* representative in $W{\cdot}z \subset \mathfrak{t}_{\mathbb{Q}}$, we see that the element of minimal length $\bar{w}$ that takes z into $-\mathcal{C}$ is defined the property that

$$\mathcal{N}(\bar{w}) = \{\gamma \in \Delta^+ \mid (\gamma, z) > 0\} = \Delta_z^+(1) \cup \Delta_z^+(2).$$

Therefore, $\bar{w} = w_z$. ■

We will use Kostant's construction with $z = x_{\mathcal{K}}$. Therefore, it is assumed below that $\mathsf{spec}(x_{\mathcal{K}}) \subset \mathbb{Z}$, which excludes the series $\mathbf{A}_{2p}$. For simplicity, write $\Delta_{\mathcal{K}}^+(i)$, $\mathfrak{a}_{\mathcal{K}}$, and $w_{\mathcal{K}}$ in place of $\Delta_{x_{\mathcal{K}}}^+(i)$, $\mathfrak{a}_{x_{\mathcal{K}}}$, and $w_{x_{\mathcal{K}}}$, respectively. By the symmetry of $\mathsf{spec}(x_{\mathcal{K}})$, one has $\dim \mathfrak{a}_{\mathcal{K}} = \#\Delta_{\mathcal{K}}^+(-1) + \#\Delta_{\mathcal{K}}^+(2) = 2\#\Delta_{\mathcal{K}}^+(2)$ and $\#\Delta_{\mathcal{K}}^+(0) + \#\mathcal{K} = \#\Delta_{\mathcal{K}}^+(1)$. Set $\Pi_{\mathcal{K}}(i) = \Pi \cap \Delta_{\mathcal{K}}^+(i)$.

Example 6.4. For $\mathfrak{sl}_{2n}$ and $\mathfrak{sp}_{2n}$, the element $x_{\mathcal{K}}$ is dominant and $\Delta_{\mathcal{K}}^+(2) = \varnothing$. Therefore, $\mathfrak{a}_{\mathcal{K}} = \{0\}$ in these cases. In the other cases, i.e., when θ is fundamental, $\mathfrak{a}_{\mathcal{K}}$ is a non-trivial abelian ideal of $\mathfrak{b}$. Anyway, for *all* simple Lie algebras except $\mathfrak{sl}_{2n+1}$, we obtain a non-trivial element $w_{\mathcal{K}} \in W$, which possesses some interesting properties, see below.

Proposition 6.5. *The rank of the root system $\Delta_{\mathcal{K}}(0)$ equals $\mathsf{rk}\,\mathfrak{g} - 1$ and the dominant weight $-w_{\mathcal{K}}(x_{\mathcal{K}})$ is a multiple of a fundamental weight. Namely, if $w_{\mathcal{K}}(\theta) = -\alpha_j \in -\Pi$ (cf. Proposition 6.1), then $-w_{\mathcal{K}}(x_{\mathcal{K}}) = \frac{2}{(\alpha_j,\alpha_j)}\varpi_j =: \varpi_j^\vee$.*

Proof. Clearly, $-w_{\mathcal{K}}(x_{\mathcal{K}})$ is a multiple of a fundamental weight if and only if the rank of the root system $\Delta_{\mathcal{K}}(0)=\{\gamma\in\Delta\mid\gamma(x_{\mathcal{K}})=0\}$ equals $\mathsf{rk}\,\mathfrak{g}-1$. Therefore, it suffices to point out $\mathsf{rk}\,\mathfrak{g}-1$ linearly independent roots in $\Delta^+_{\mathcal{K}}(0)$.

Since $\Pi_{\pm1}:=\Pi_{\mathcal{K}}(-1)\cup\Pi_{\mathcal{K}}(1)$ is connected and the roots from $\Pi_{\mathcal{K}}(-1)$ and $\Pi_{\mathcal{K}}(1)$ alternate in the Dynkin diagram (see Remark 3.8(2)), there are $(\#\Pi_{\pm1})-1$ edges therein and each edge gives rise to the root $\alpha_{i_1}+\alpha_{i_2}\in\Delta^+_{\mathcal{K}}(0)$. Together with the roots in $\Pi_{\mathcal{K}}(0)$, this yields exactly $\mathsf{rk}\,\mathfrak{g}-1$ linearly independent roots in $\Delta^+_{\mathcal{K}}(0)$. Actually, these roots form the base of $\Delta_{\mathcal{K}}(0)$ in $\Delta^+_{\mathcal{K}}(0)$.

If $w_{\mathcal{K}}(x_{\mathcal{K}})=-a_i\varpi_i$ and $w_{\mathcal{K}}(\theta)=-\alpha_j$, then

$$1=\theta(x_{\mathcal{K}})=w_{\mathcal{K}}(\theta)(w_{\mathcal{K}}(x_{\mathcal{K}}))=a_i(\alpha_j,\varpi_i).$$

Hence $i=j$ and $a_i=2/(\alpha_i,\alpha_i)$. ∎

Important characteristics of the abelian ideal $\mathfrak{a}_{\mathcal{K}}$ can be expressed via $w_{\mathcal{K}}\in W$. Since $\Delta(\mathfrak{a}_{\mathcal{K}})=:\Delta_{\langle\mathcal{K}\rangle}$ is an upper ideal of the poset $(\Delta^+,\preccurlyeq)$, it is completely determined by its subset $\min(\Delta_{\langle\mathcal{K}\rangle})$ of *minimal elements* w.r.t. the root order "$\preccurlyeq$". Similarly, the complement $\overline{\Delta_{\langle\mathcal{K}\rangle}}=\Delta^+\setminus\Delta_{\langle\mathcal{K}\rangle}$ is determined by the subset of its *maximal elements*, $\max(\overline{\Delta_{\langle\mathcal{K}\rangle}})$.

It follows from (7) and the definition of $\mathcal{N}(w_{\mathcal{K}})$ that

$$\Delta^+=w_{\mathcal{K}}\big(\Delta^+_{\mathcal{K}}(-1)\big)\sqcup w_{\mathcal{K}}\big(\Delta^+_{\mathcal{K}}(0)\big)\sqcup w_{\mathcal{K}}\big(-\Delta^+_{\mathcal{K}}(1)\big)\sqcup w_{\mathcal{K}}\big(-\Delta^+_{\mathcal{K}}(2)\big).$$

In this union, the first and last sets form $\Delta_{\langle\mathcal{K}\rangle}$, and two sets in the middle form $\overline{\Delta_{\langle\mathcal{K}\rangle}}$. Hence

$$w_{\mathcal{K}}^{-1}(\Delta_{\langle\mathcal{K}\rangle})=\Delta^+_{\mathcal{K}}(-1)\sqcup-\Delta^+_{\mathcal{K}}(2)=\Delta^+_{\mathcal{K}}(-1)\sqcup\Delta^-_{\mathcal{K}}(-2)\,; \tag{8}$$

$$w_{\mathcal{K}}^{-1}(\overline{\Delta_{\langle\mathcal{K}\rangle}})=\Delta^+_{\mathcal{K}}(0)\sqcup-\Delta^+_{\mathcal{K}}(1)=\Delta^+_{\mathcal{K}}(0)\sqcup\Delta^-_{\mathcal{K}}(-1)\,. \tag{9}$$

Theorem 6.6. *One has* $\min(\Delta_{\langle\mathcal{K}\rangle})=w_{\mathcal{K}}(\Pi_{\mathcal{K}}(-1))$. *In other words,*

$$\gamma\in\min(\Delta_{\langle\mathcal{K}\rangle})\Longleftrightarrow w_{\mathcal{K}}^{-1}(\gamma)\in\Pi_{\mathcal{K}}(-1).$$

Proof. We repeatedly use the following observation. For $\gamma\in\Delta_{\langle\mathcal{K}\rangle}$, it follows from (8) that if $w_{\mathcal{K}}^{-1}(\gamma)>0$, then $w_{\mathcal{K}}^{-1}(\gamma)\in\Delta^+_{\mathcal{K}}(-1)$; whereas if $w_{\mathcal{K}}^{-1}(\gamma)<0$, then $w_{\mathcal{K}}^{-1}(\gamma)\in-\Delta^+_{\mathcal{K}}(2)$.

(1) If $w_{\mathcal{K}}^{-1}(\gamma)=\alpha\in\Pi_{\mathcal{K}}(-1)$, then $\gamma\in\Delta_{\langle\mathcal{K}\rangle}$. Assume further that γ is not a minimal element of $\Delta_{\langle\mathcal{K}\rangle}$, i.e., $\gamma=\gamma'+\mu$ for some $\gamma'\in\Delta_{\langle\mathcal{K}\rangle}$ and $\mu>0$. Then $\alpha=w_{\mathcal{K}}^{-1}(\gamma')+w_{\mathcal{K}}^{-1}(\mu)$ and there are two possibilities:

(a) $w_{\mathcal{K}}^{-1}(\gamma')<0$ and $w_{\mathcal{K}}^{-1}(\mu)>0$;

(b) $w_{\mathcal{K}}^{-1}(\gamma')>0$ and $w_{\mathcal{K}}^{-1}(\mu)<0$.

For (a): By (8) and (9), one has $w_{\mathcal{K}}^{-1}(\gamma')\in\Delta^-_{\mathcal{K}}(-2)$ and $w_{\mathcal{K}}^{-1}(\mu)\in\Delta^+_{\mathcal{K}}(0)\cup\Delta^+_{\mathcal{K}}(-1)$. Then their sum belongs to $\Delta_{\mathcal{K}}(\le-2)=\Delta_{\mathcal{K}}(-2)$, and this cannot be $\alpha\in\Delta^+_{\mathcal{K}}(-1)$. Hence this case is impossible.

For (b): By (8) and (9), one has $w_{\mathcal{K}}^{-1}(\gamma')\in\Delta^+_{\mathcal{K}}(-1)$ and $w_{\mathcal{K}}^{-1}(\mu)\in\Delta^-_{\mathcal{K}}(-2)\cup\Delta^-_{\mathcal{K}}(-1)$. Then their sum again belongs to $\Delta_{\mathcal{K}}(-2)$, which is impossible, too.

Thus, $\gamma=w_{\mathcal{K}}(\alpha)$ must be a minimal element of $\Delta_{\langle\mathcal{K}\rangle}$.

(2) Suppose that $\gamma \in \Delta_{\langle\mathcal{K}\rangle}$ and $w_{\mathcal{K}}^{-1}(\gamma) \notin \Pi_{\mathcal{K}}(-1)$. By (8), there is a dichotomy:

(a) $w_{\mathcal{K}}^{-1}(\gamma) > 0$;

(b) $w_{\mathcal{K}}^{-1}(\gamma) < 0$.

For (a): By (8), one has $w_{\mathcal{K}}^{-1}(\gamma) \in \Delta_{\mathcal{K}}^{+}(-1)$. Since $w_{\mathcal{K}}^{-1}(\Delta_{\langle\mathcal{K}\rangle}) \cap \Pi = \Pi_{\mathcal{K}}(-1)$, we have $w_{\mathcal{K}}^{-1}(\gamma) = \mu_1 + \mu_2$ for some $\mu_1, \mu_2 > 0$. From the $x_{\mathcal{K}}$-grading of Δ^+, we deduce that $\mu_1 \in \Delta_{\mathcal{K}}^{+}(-1)$ and $\mu_2 \in \Delta_{\mathcal{K}}^{+}(0)$. Letting $\gamma_i = w_{\mathcal{K}}(\mu_i)$, we see that $\gamma_1 \in \Delta_{\langle\mathcal{K}\rangle}$ and $\gamma_2 > 0$. Hence $\gamma = \gamma_1 + \gamma_2$ is not a minimal element of $\Delta_{\langle\mathcal{K}\rangle}$.

For (b): Here $w_{\mathcal{K}}^{-1}(\gamma) \in -\Delta_{\mathcal{K}}^{+}(2)$. Since $\theta \in \Delta_{\mathcal{K}}^{+}(1)$, we have $w_{\mathcal{K}}^{-1}(\gamma) \neq -\theta$ and hence $w_{\mathcal{K}}^{-1}(\gamma) = \nu_1 - \nu_2$ for some $\nu_1, \nu_2 > 0$. Using the $x_{\mathcal{K}}$-grading of Δ^+, one again encounters two possibilities:

(i) $\nu_1 \in \Delta_{\mathcal{K}}^{+}(-1), \nu_2 \in \Delta_{\mathcal{K}}^{+}(1)$;

(ii) $\nu_1 \in \Delta_{\mathcal{K}}^{+}(0), \nu_2 \in \Delta_{\mathcal{K}}^{+}(2)$.

In both cases, $\gamma = w_{\mathcal{K}}(\nu_1) + w_{\mathcal{K}}(-\nu_2)$ and one of the summands lies in $\Delta_{\langle\mathcal{K}\rangle}$, while the other is positive. Hence $\gamma \notin \min(\Delta_{\langle\mathcal{K}\rangle})$. ■

Theorem 6.7. *One has* $\max(\overline{\Delta_{\langle\mathcal{K}\rangle}}) = -w_{\mathcal{K}}(\Pi_{\mathcal{K}}(1))$. *In other words,*

$$\gamma \in \max(\overline{\Delta_{\langle\mathcal{K}\rangle}}) \iff -w_{\mathcal{K}}^{-1}(\gamma) \in \Pi_{\mathcal{K}}(1).$$

Proof. To a large extent the proof is analogous to that of Theorem 6.6, and we skip similar arguments.

(1) If $-w_{\mathcal{K}}^{-1}(\gamma) \in \Pi_{\mathcal{K}}(1)$, then an argument similar to that in part (1) of Theorem 6.6 proves that $\gamma \in \max(\overline{\Delta_{\langle\mathcal{K}\rangle}})$.

(2) Conversely, suppose that $\gamma \in \overline{\Delta_{\langle\mathcal{K}\rangle}}$ and $-w_{\mathcal{K}}^{-1}(\gamma) \notin \Pi_{\mathcal{K}}(1)$. In view of (9), one has to handle two possibilities for $w_{\mathcal{K}}^{-1}(\gamma)$.

(a): If $w_{\mathcal{K}}^{-1}(\gamma) > 0$, then $w_{\mathcal{K}}^{-1}(\gamma) \in \Delta_{\mathcal{K}}^{+}(0)$. Arguing as in the proof of Theorem 6.6 (part (2)(a)), we show that $\gamma = \gamma_1 - \gamma_2$, where $\gamma_1 \in \overline{\Delta_{\langle\mathcal{K}\rangle}}$ and $\gamma_2 > 0$. Hence $\gamma \notin \max(\overline{\Delta_{\langle\mathcal{K}\rangle}})$.

(b): If $w_{\mathcal{K}}^{-1}(\gamma) < 0$, then $-w_{\mathcal{K}}^{-1}(\gamma) \in \Delta_{\mathcal{K}}^{+}(1) \setminus \Pi_{\mathcal{K}}(1)$. Hence $-w_{\mathcal{K}}^{-1}(\gamma) = \nu_1 + \nu_2$ for some $\nu_1, \nu_2 > 0$. There again are two possibilities:

(1) $\nu_1 \in \Delta_{\mathcal{K}}^{+}(0)$, $\nu_2 \in \Delta_{\mathcal{K}}^{+}(1)$ [$(0,1)$-decomposition of $-w_{\mathcal{K}}^{-1}(\gamma)$];

(2) $\nu_1 \in \Delta_{\mathcal{K}}^{+}(-1)$, $\nu_2 \in \Delta_{\mathcal{K}}^{+}(2)$ [$(-1,2)$-decomposition of $-w_{\mathcal{K}}^{-1}(\gamma)$].

In case (1), we get $\gamma = \gamma_2 - \gamma_1$, where $\gamma_2 := -w_{\mathcal{K}}(\nu_2) \in \overline{\Delta_{\langle\mathcal{K}\rangle}}$ and $\gamma_1 = w_{\mathcal{K}}(\nu_1) > 0$. Hence $\gamma \notin \max(\overline{\Delta_{\langle\mathcal{K}\rangle}})$.

In case (2), one similarly obtains the presentation of γ as difference of two elements of $\Delta_{\langle\mathcal{K}\rangle}$, which is useless for us. However, one can replace such a $(-1,2)$-decomposition of $-w_{\mathcal{K}}^{-1}(\gamma)$ with a $(0,1)$-decomposition, which is sufficient. Since $\nu_2 \in \Delta_{\mathcal{K}}^{+}(2)$, the root ν_2 is long (Remark 3.6(3)) and not simple (for, $\Pi_{\mathcal{K}}(2) = \varnothing$). Hence $(\nu_1, \nu_2) < 0$ and $\nu_2 = \nu_2' + \nu_2''$ with $\nu_1', \nu_2'' > 0$. W.l.o.g., we may assume that $(\nu_1, \nu_2') < 0$. One has three possibilities for the $x_{\mathcal{K}}$-degrees of (ν_1', ν_2''), i.e., $(1,1), (2,0), (0,2)$, and it is easily seen that $-w_{\mathcal{K}}^{-1}(\gamma) = (\nu_1 + \nu_2') + \nu_2''$ is either a $(0,1)$-decomposition, or still a $(-1,2)$-decomposition, with $\nu_2'' \in \Delta_{\mathcal{K}}^{+}(2)$. But in this last case we have $\widetilde{ht}(\nu_2'') < \widetilde{ht}(\nu_2)$, which provides the induction step. ■

Remark 6.8. Theorem 6.6 holds for *arbitrary* $z \in \mathfrak{D}_{\mathsf{ab}} \cap \mathcal{P}^\vee$ in place of $x_{\mathcal{K}}$, with certain amendments. That is, if $\theta(z) \le 1$, then the statement and the proof remain the same. If $\theta(z) = 2$, then $-w_z(\theta)$ has to be added to $\min \Delta(\mathfrak{a}_z)$. Certain complements of similar nature are also required in Theorem 6.7. I hope to elaborate on this topic in a subsequent publication.

7. An explicit description of $w_{\mathcal{K}} \in W$ and the ideals $\mathfrak{a}_{\mathcal{K}}$

The element $w_{\mathcal{K}} \in W$ and the abelian ideal $\mathfrak{a}_{\mathcal{K}}$ have many interesting properties, which can be verified case-by-case. To this end, we need explicit formulae for $w_{\mathcal{K}}$. Our main tool is the following

Theorem 7.1. *Given* $z \in \mathfrak{D}_{\mathsf{ab}} \cap \mathcal{P}^\vee$, *suppose that* $\mathsf{rk}\,\Delta_z(0) = \mathsf{rk}\,\Delta - 1$ *and* $\theta(z) = 1$. *Then*

$$w_z^{-1}(\Pi) = \{\text{the base of } \Delta_z(0) \ \text{ in } \ \Delta_z^+(0)\} \cup \{-\theta\}.$$

Proof. Set $p = \mathsf{rk}\,\Delta$, and let $\nu_1, \dots, \nu_{p-1} \in \Delta_z^+(0)$ be the base of $\Delta_z(0)$. By Proposition 6.1, we have $w_z(\nu_i) \in \Pi$ $(i = 1, \dots, p-1)$ and $w_z(\theta) \in -\Pi$. The assertion follows. ∎

By Proposition 6.5, Theorem 7.1 applies to $z = x_{\mathcal{K}}$. We demonstrate below how to use this technique for finding $w_{\mathcal{K}} \in GL(\mathfrak{t})$.

Example 7.2. (1) Let $\mathfrak{g}$ be of type $\mathbf{D}_{2n}$. Then $\Pi_{\mathcal{K}}(0) = \varnothing$ and the base of $\Delta_{\mathcal{K}}^+(0)$ corresponds to the edges of the Dynkin diagram, i.e., it consists of $\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3, \dots, \alpha_{2n-2} + \alpha_{2n-1}, \alpha_{2n-2} + \alpha_{2n}$, cf. the diagram for $\mathbf{D}_{2n}$ in Figure 1. Therefore, the root system $\Delta_{\mathcal{K}}(0)$ is of type $\mathbf{A}_{n-1} + \mathbf{D}_n$. More precisely,

$\{\alpha_1 + \alpha_2, \alpha_3 + \alpha_4, \dots, \alpha_{2n-3} + \alpha_{2n-2}\}$ is a base for $\Delta(\mathbf{A}_{n-1})$;

$\{\alpha_2 + \alpha_3, \alpha_4 + \alpha_5, \dots, \alpha_{2n-4} + \alpha_{2n-3}, \alpha_{2n-2} + \alpha_{2n-1}, \alpha_{2n-2} + \alpha_{2n}\}$ is a base for $\Delta(\mathbf{D}_n)$.

Since the Dynkin diagram of $\mathbf{A}_{n-1} + \mathbf{D}_n$ is obtained by removing the node α_n from the Dynkin diagram of $\mathbf{D}_{2n}$, we must have $w_{\mathcal{K}}^{-1}(\alpha_n) = -\theta$. Then an easy argument shows that

- $w_{\mathcal{K}}^{-1}(\alpha_1) = \alpha_{2n-3} + \alpha_{2n-2}, \dots, w_{\mathcal{K}}^{-1}(\alpha_{n-2}) = \alpha_3 + \alpha_4,\ w_{\mathcal{K}}^{-1}(\alpha_{n-1}) = \alpha_1 + \alpha_2$ – for the $\mathbf{A}_{n-1}$-part;
- $w_{\mathcal{K}}^{-1}(\alpha_{n+1}) = \alpha_2 + \alpha_3, \dots, w_{\mathcal{K}}^{-1}(\alpha_{2n-2}) = \alpha_{2n-4} + \alpha_{2n-3}$, $w_{\mathcal{K}}^{-1}(\{\alpha_{2n-1}, \alpha_{2n}\}) = \{\alpha_{2n-2} + \alpha_{2n-1},\ \alpha_{2n-2} + \alpha_{2n-1}\}$ – for the $\mathbf{D}_n$-part;

The only unclear point for the $\mathbf{D}_n$-part is how to distinguish $w_{\mathcal{K}}^{-1}(\alpha_{2n-1})$ and $w_{\mathcal{K}}^{-1}(\alpha_{2n})$. Using the expressions of simple roots of $\mathbf{D}_{2n}$ via $\{\varepsilon_i\}$, $i = 1, \dots, 2n$, we obtain two possibilities for $w_{\mathcal{K}}$ as a signed permutation on $\{\varepsilon_i\}$, where the only ambiguity concerns the sign of transformation $\varepsilon_{2n} \mapsto \pm\varepsilon_{2n}$. We then choose this sign so that the total number of minuses be even, see Example 7.3 below.

(2) Similar argument works for the other orthogonal series.

(3) For the exceptional Lie algebras, a certain ambiguity (due to the symmetry of the Dynkin diagram) occurs only for $\mathbf{E}_6$.

For the classical cases, our formulae for $w_{\mathcal{K}}$ use the explicit standard models of W as (signed) permutations on the set of $\{\varepsilon_i\}$. Write $\mathrm{ord}(w_{\mathcal{K}}(\mathfrak{g}))$ for the order of $w_{\mathcal{K}} = w_{\mathcal{K}}(\mathfrak{g})$.

Example 7.3. Here we provide formulae for $w_{\mathcal{K}}$ if $\mathfrak{g}$ is an orthogonal Lie algebra.

(1) If $\mathfrak{g}$ is of type $\mathbf{D}_{2n}$, then $w_{\mathcal{K}}$ is the following permutation:

$$\left(\begin{array}{ccccc|ccccc} \varepsilon_1 & \varepsilon_3 & \dots & \varepsilon_{2n-3} & \varepsilon_{2n-1} & \varepsilon_2 & \varepsilon_4 & \dots & \varepsilon_{2n-2} & \varepsilon_{2n} \\ \downarrow & \downarrow & \dots & \downarrow & \downarrow & \downarrow & \downarrow & \dots & \downarrow & \downarrow \\ -\varepsilon_n & -\varepsilon_{n-1} & \dots & -\varepsilon_2 & -\varepsilon_1 & \varepsilon_{n+1} & \varepsilon_{n+2} & \dots & \varepsilon_{2n-1} & (-1)^n\varepsilon_{2n} \end{array}\right)$$

The last sign is determined by the condition that the total number of minuses must be even for type $\mathbf{D}_N$.

(2) For $\mathfrak{g}$ of type $\mathbf{B}_{2n-1}$, one should merely omit the last column in the previous array.

(3) If $\mathfrak{g}$ is of type $\mathbf{D}_{2n+1}$, then the following adjustment works for $w_{\mathcal{K}}$:

$$\left(\begin{array}{ccccc|cccccc} \varepsilon_1 & \varepsilon_3 & \dots & \varepsilon_{2n-3} & \varepsilon_{2n-1} & \varepsilon_2 & \varepsilon_4 & \dots & \varepsilon_{2n-2} & \varepsilon_{2n} & \varepsilon_{2n+1} \\ \downarrow & \downarrow & \dots & \downarrow & \downarrow & \downarrow & \downarrow & \dots & \downarrow & \downarrow & \downarrow \\ -\varepsilon_n & -\varepsilon_{n-1} & \dots & -\varepsilon_2 & -\varepsilon_1 & \varepsilon_{n+1} & \varepsilon_{n+2} & \dots & \varepsilon_{2n-1} & \varepsilon_{2n} & (-1)^n\varepsilon_{2n+1} \end{array}\right)$$

(4) For $\mathfrak{g}$ of type $\mathbf{B}_{2n}$, one should merely omit the last column in the previous array.

- It follows that in all four cases $w_{\mathcal{K}}(\theta) = w_{\mathcal{K}}(\varepsilon_1+\varepsilon_2) = -\varepsilon_n+\varepsilon_{n+1} = -\alpha_n$, which agrees with Lemma 6.1(ii).

- For $\mathbf{D}_{2n+1}$, one has $\Pi_{\mathcal{K}}(0) = \{\alpha_{2n}, \alpha_{2n+1}\}$ (see Fig. 1) and $w_{\mathcal{K}}$ takes $\Pi_{\mathcal{K}}(0)$ to itself. Recall that here $\alpha_{2n} = \varepsilon_{2n} - \varepsilon_{2n+1}$ and $\alpha_{2n+1} = \varepsilon_{2n} + \varepsilon_{2n+1}$. The same happens for $\mathbf{B}_{2n}$, where $\Pi_{\mathcal{K}}(0) = \{\alpha_{2n} = \varepsilon_{2n}\}$, cf. Lemma 6.1(i).

Example 7.4. Here we provide formulae for $w_{\mathcal{K}}$ if $\mathfrak{g} = \mathfrak{sl}_{2n}$ or $\mathfrak{sp}_{2n}$.

(1) If $\mathfrak{g}$ is of type $\mathbf{A}_{2n-1}$, then

$$w_{\mathcal{K}}\colon \left(\begin{array}{ccccc|ccccc} \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_{n-1} & \varepsilon_n & \varepsilon_{n+1} & \varepsilon_{n+2} & \dots & \varepsilon_{2n-1} & \varepsilon_{2n} \\ \downarrow & \downarrow & \dots & \downarrow & \downarrow & \downarrow & \downarrow & \dots & \downarrow & \downarrow \\ \varepsilon_{n+1} & \varepsilon_{n+2} & \dots & \varepsilon_{2n-1} & \varepsilon_{2n} & \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_{n-1} & \varepsilon_n \end{array}\right)$$

It follows that $w_{\mathcal{K}}(\theta) = w_{\mathcal{K}}(\varepsilon_1 - \varepsilon_{2n}) = -\varepsilon_n + \varepsilon_{n+1} = -\alpha_n$, which agrees with Lemma 6.1(ii). Here $\Pi_{\mathcal{K}}(0) = \Pi \setminus \{\alpha_n\}$ (see Figure 1) and

$$w_{\mathcal{K}}(\alpha_i) = \begin{cases} \alpha_{i+n}, & i < n \\ \alpha_{i-n}, & i > n \end{cases}.$$

(2) If $\mathfrak{g}$ is of type $\mathbf{C}_n$, then

$$w_{\mathcal{K}}\colon \left(\begin{array}{ccccc} \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_{n-1} & \varepsilon_n \\ \downarrow & \downarrow & \dots & \downarrow & \downarrow \\ -\varepsilon_n & -\varepsilon_{n-1} & \dots & -\varepsilon_2 & -\varepsilon_1 \end{array}\right)$$

Here $w_{\mathcal{K}}(\theta) = -\alpha_n$, $\Pi_{\mathcal{K}}(0) = \Pi \setminus \{\alpha_n\}$, and $w_{\mathcal{K}}(\alpha_i) = \alpha_{n-i}$ for $i < n$.

- In both cases here, one has $w_{\mathcal{K}}^2 = 1$.

Example 7.5. (1) For $\mathfrak{g}$ of type $\mathbf{F}_4$, we write $(a_1a_2a_3a_4)$ for $\sum_{i=1}^4 a_i\alpha_i$. Then $w_{\mathcal{K}}(\alpha_i) = \alpha_i$ for $i = 1, 2$ and $w_{\mathcal{K}}(\alpha_3) = -(2421)$, $w_{\mathcal{K}}(\alpha_4) = (2431) = \theta - \alpha_4$. It follows that $w_{\mathcal{K}}(\theta) = -\alpha_4$.

(2) For $\mathfrak g$ of type E_6, we write $(a_1a_2a_3a_4a_5a_6)$ for $\gamma=\sum_{i=1}^6 a_i\alpha_i = \begin{smallmatrix}a_1a_2a_3a_4a_5\\ a_6\end{smallmatrix}$. Then $w_{\mathcal K}(\alpha_i)=\alpha_i$ for $i=1,2,4,5$ and $w_{\mathcal K}(\alpha_3)=-(122211)$, $w_{\mathcal K}(\alpha_6)=(123211)=\theta-\alpha_6$. It follows that $w_{\mathcal K}(\theta)=-\alpha_6$.

(3) For $\mathfrak g$ of type G_2, we have $\Pi_l=\{\alpha_2\}$, $w_{\mathcal K}=(r_{\alpha_2}r_{\alpha_1})^2$, and $w_{\mathcal K}(\theta)=-\alpha_2$.

- In all three cases, one has $w_{\mathcal K}^3=1$.

Example 7.6. (1) For E_7, we have $\Pi_{\mathcal K}(-1)=\{\alpha_2,\alpha_4,\alpha_6\}$ (see Figure 2) and

α_i	α_1	α_2	α_3	α_4	α_5	α_6	α_7
$w_{\mathcal K}(\alpha_i)$	$-\begin{smallmatrix}122210\\1\end{smallmatrix}$	$\begin{smallmatrix}122211\\1\end{smallmatrix}$	$-\begin{smallmatrix}112211\\1\end{smallmatrix}$	$\begin{smallmatrix}112221\\1\end{smallmatrix}$	$-\begin{smallmatrix}012221\\1\end{smallmatrix}$	$\begin{smallmatrix}012321\\1\end{smallmatrix}$	$-\begin{smallmatrix}111221\\1\end{smallmatrix}$

It follows that $w_{\mathcal K}(\theta)=-\alpha_7$. A direct calculation shows that $\mathrm{ord}(w_{\mathcal K})=18$.

(2) For E_8, we have $\Pi_{\mathcal K}(-1)=\{\alpha_1,\alpha_3,\alpha_5,\alpha_7\}$ (see Figure 2) and

α_i	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$w_{\mathcal K}(\alpha_i)$	$\begin{smallmatrix}0123431\\2\end{smallmatrix}$	$-\begin{smallmatrix}0123421\\2\end{smallmatrix}$	$\begin{smallmatrix}1123421\\2\end{smallmatrix}$	$-\begin{smallmatrix}1123321\\2\end{smallmatrix}$	$\begin{smallmatrix}1223321\\2\end{smallmatrix}$	$-\begin{smallmatrix}1223321\\1\end{smallmatrix}$	$\begin{smallmatrix}1233321\\1\end{smallmatrix}$	$-\begin{smallmatrix}1222321\\2\end{smallmatrix}$

It follows that $w_{\mathcal K}(\theta)=-\alpha_7$. A direct calculation shows that $\mathrm{ord}(w_{\mathcal K})=5$.

Remark 7.7. (1) We do not know a general formula for $\mathrm{ord}(w_{\mathcal K}(\mathfrak{so}_N))$. Using Example 7.3, it is not hard to prove that $\mathrm{ord}(w_{\mathcal K})$ takes the same value for $\mathsf{B}_{2n-1},\mathsf{D}_{2n},\mathsf{B}_{2n},\mathsf{D}_{2n+1}$. But explicit computations, up to $n=13$, show that the function $n\mapsto \mathrm{ord}(w_{\mathcal K}(\mathsf{D}_{2n}))$ behaves rather chaotically.

(2) If $\alpha\in\Pi_{\mathcal K}(0)$, then $w_{\mathcal K}(\alpha)\in\Pi_{\mathcal K}(0)$ as well. But this does not hold for arbitrary $z\in\mathfrak{D}_{\mathsf{ab}}$. In general, it can happen that $\alpha\in\Pi_z(0)$, but $w_z(\alpha)\in\Pi\setminus\Pi_z(0)$.

The main result of this section is an explicit uniform description of $\Delta(\mathfrak a_{\mathcal K})$.

Theorem 7.8. *Set* $d_{\mathcal K}=\widetilde{ht}(\theta)+1-\sum_{\alpha\in\Pi_{\mathcal K}(-1)}[\theta:\alpha]=1+\sum_{\alpha\in\Pi_{\mathcal K}(\ge 0)}[\theta:\alpha]$. *Then*

(i) $\widetilde{ht}(\gamma)=d_{\mathcal K}$ *if and only if* $w_{\mathcal K}^{-1}(\gamma)\in\Pi_{\mathcal K}(-1)$;

(ii) $\widetilde{ht}(\gamma)=d_{\mathcal K}-1$ *if and only if* $-w_{\mathcal K}^{-1}(\gamma)\in\Pi_{\mathcal K}(1)$;

(iii) $\Delta(\mathfrak a_{\mathcal K})=\{\gamma\in\Delta^+\mid \widetilde{ht}(\gamma)\ge d_{\mathcal K}\}$.

Proof. (1) For A_{2n-1} and C_n, $x_{\mathcal K}$ is dominant. Therefore $\Pi_{\mathcal K}(-1)=\varnothing$, $d_{\mathcal K}=\widetilde{ht}(\theta)+1=\mathsf{h}$ is the Coxeter number, and $\Delta(\mathfrak a_{\mathcal K})=\varnothing$, which agrees with (iii). Here θ is the only root of height $d_{\mathcal K}-1$, $\Pi_{\mathcal K}(1)=\{\alpha_n\}$, and $w_{\mathcal K}(\alpha_n)=-\theta$, see Example 7.4. This confirms (ii), whereas (i) is vacuous.

(2) In the other cases, $\Pi_{\mathcal K}(-1)\ne\varnothing$.

(i) Using the formulae from the Examples 7.3–7.6, one can directly verify that if $\alpha\in\Pi_{\mathcal K}(-1)$, then $\widetilde{ht}(w_{\mathcal K}(\alpha))=d_{\mathcal K}$. It also happens that, in all cases,

$$\#\{\gamma\mid \widetilde{ht}(\gamma)=d_{\mathcal K}\}=\#\Pi_{\mathcal K}(-1).$$

(ii) Again, this can be verified directly. Alternatively, this follows from (i), Theorem 6.6, and Theorem 6.7 (without further verifications).

(iii) This follows from (i) and Theorem 6.6. ∎

Note that $$\mathsf{h}-1=\widetilde{ht}(\theta)=\sum_{\alpha\in\Pi_{\mathcal{K}}(1)}[\theta:\alpha]+\sum_{\alpha\in\Pi_{\mathcal{K}}(0)}[\theta:\alpha]+\sum_{\alpha\in\Pi_{\mathcal{K}}(-1)}[\theta:\alpha]$$

and $$1=\theta(x_{\mathcal{K}})=\sum_{\alpha\in\Pi_{\mathcal{K}}(1)}[\theta:\alpha]-\sum_{\alpha\in\Pi_{\mathcal{K}}(-1)}[\theta:\alpha]\,.$$

Therefore, if $\Pi_{\mathcal{K}}(0)=\varnothing$, then $d_{\mathcal{K}}=(\mathsf{h}/2)+1$; whereas, if $\Pi_{\mathcal{K}}(0)\ne\varnothing$, then $d_{\mathcal{K}}>(\mathsf{h}/2)+1$.

Remark 7.9. A remarkable property of the abelian ideal $\mathfrak{a}_{\mathcal{K}}$ is that $\min\Delta(\mathfrak{a}_{\mathcal{K}})$ consists of *all* roots of a *fixed* height. This can be explained by the properties that $\Pi_{\mathcal{K}}(-1)\cup\Pi_{\mathcal{K}}(1)$ is connected in the Dynkin diagram and that $+1$ and -1 nodes alternate. For, if α_i and α_j are adjacent nodes, $\alpha_i\in\Pi_{\mathcal{K}}(-1)$, and $\alpha_j\in\Pi_{\mathcal{K}}(1)$, then $\alpha_i+\alpha_j$ is a simple root in $\Delta^+_{\mathcal{K}}(0)$. Hence

$$w_{\mathcal{K}}(\alpha_i)\in\min\Delta(\mathfrak{a}_{\mathcal{K}}),\ -w_{\mathcal{K}}(\alpha_j)\in\max(\Delta^+\setminus\Delta(\mathfrak{a}_{\mathcal{K}})),\text{ and } w_{\mathcal{K}}(\alpha_i+\alpha_j)\in\Pi.$$

Hence $\widetilde{ht}(w_{\mathcal{K}}(\alpha_i))=1+\widetilde{ht}(-w_{\mathcal{K}}(\alpha_j))$. In view of Theorems 6.6 and 6.7, together with the connectedness and the alternating property for $\Pi_{\mathcal{K}}(-1)\cup\Pi_{\mathcal{K}}(1)$, this relation propagates to any pair in $\min\Delta(\mathfrak{a}_{\mathcal{K}})\times\max(\Delta^+\setminus\Delta(\mathfrak{a}_{\mathcal{K}}))$.

But the exact value of the boundary height, which is $d_{\mathcal{K}}$ is our case, has no explanation.

8. The involution of $\mathfrak{g}$ associated with the cascade

In this section, we assume that $\mathsf{spec}(x_{\mathcal{K}})\in\mathbb{Z}$, i.e., $\mathfrak{g}\ne\mathfrak{sl}_{2n+1}$. Then partition (7) yields the partition of the whole root system $\Delta=\bigcup_{i=-2}^{2}\Delta_{\mathcal{K}}(i)$ such that $\Delta_{\mathcal{K}}(2)=\Delta^+_{\mathcal{K}}(2)$.

Set $\Delta_0=\Delta_{\mathcal{K}}(-2)\cup\Delta_{\mathcal{K}}(0)\cup\Delta_{\mathcal{K}}(2)$ and $\Delta_1=\Delta_{\mathcal{K}}(-1)\cup\Delta_{\mathcal{K}}(1)$. Consider the vector space decomposition $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$, where $\mathfrak{g}_0=\mathfrak{t}\oplus(\oplus_{\gamma\in\Delta_0}\mathfrak{g}_\gamma)$ and $\mathfrak{g}_1=\oplus_{\gamma\in\Delta_1}\mathfrak{g}_\gamma$. This is a $\mathbb{Z}_2$-grading and the corresponding involution of $\mathfrak{g}$, denoted $\sigma_{\mathcal{K}}$, is inner.

Lemma 8.1. *The involution $\sigma_{\mathcal{K}}$ has the property that*

$$\dim\mathfrak{g}_0=\dim\mathfrak{b}-\#\mathcal{K}\quad\textit{and}\quad\dim\mathfrak{g}_1=\dim\mathfrak{u}+\#\mathcal{K}.$$

Proof. This readily follows from the symmetry of $\mathsf{spec}(x_{\mathcal{K}})$ on the Frobenius envelope $\mathfrak{b}_{\mathcal{K}}$ and the equality $\dim\mathfrak{b}_{\mathcal{K}}=\dim\mathfrak{u}+\#\mathcal{K}$, see Section 5 or Lemma 3.4. For, one has

i	-2	-1	0	1	2
$\#\Delta^+_{\mathcal{K}}(i)$	0	a	$b-\#\mathcal{K}$	b	a
$\#\Delta^-_{\mathcal{K}}(i)$	a	b	$b-\#\mathcal{K}$	a	0

for some a,b. Then $\dim\mathfrak{u}=2a+2b-\#\mathcal{K}$, $\dim\mathfrak{g}_0=2a+2b+\mathsf{rk}\,\mathfrak{g}-2\#\mathcal{K}$ and $\dim\mathfrak{g}_1=2a+2b$. ■

Since $\operatorname{ind}\mathfrak{u}=\#\mathcal{K}$ and $\operatorname{ind}\mathfrak{b}+\operatorname{ind}\mathfrak{u}=\mathsf{rk}\,\mathfrak{g}$, the formulae of Lemma 8.1 mean that $\dim\mathfrak{g}_1=\dim\mathfrak{u}+\operatorname{ind}\mathfrak{u}=\dim\mathfrak{b}-\operatorname{ind}\mathfrak{b}$ and $\dim\mathfrak{g}_0=\dim\mathfrak{u}+\operatorname{ind}\mathfrak{b}=\dim\mathfrak{b}-\operatorname{ind}\mathfrak{u}$. Recall that $\mathfrak{t}_{\mathcal{K}}=\bigoplus_{i=1}^m[e_{\beta_i},e_{-\beta_i}]\subset\mathfrak{t}$.

Lemma 8.2. *The subalgebra $\mathfrak{t}_{\mathcal{K}}$ contains a regular semisimple element of $\mathfrak{g}$.*

Proof. By Eq. (2), if $\gamma \in \Delta^+$, then $\gamma \in \mathcal{H}_{\beta_j}$ for a unique $\beta_j \in \mathcal{K}$. That is, for any $\gamma \in \Delta^+$, there exists $\beta_j \in \mathcal{K}$ such that $(\gamma, \beta_j) > 0$. This implies that, for any $\gamma \in \Delta^+$, the set $\mathcal{H}_\gamma := \{\mu \in \langle \beta_1, \dots, \beta_m \rangle_{\mathbb{Q}} \mid (\gamma, \mu) = 0\}$ is a hyperplane in the $\mathbb{Q}$-vector space $\langle \beta_1, \dots, \beta_m \rangle_{\mathbb{Q}}$. Therefore, if

$$\nu \in \langle \beta_1, \dots, \beta_m \rangle_{\mathbb{Q}} \setminus (\cup_{\gamma \in \Delta^+} \mathcal{H}_\gamma),$$

then $(\gamma, \nu) \neq 0$ for *every* $\gamma \in \Delta^+$. Upon the identification of $\mathfrak{t}$ and $\mathfrak{t}^*$, this yields a required element of $\mathfrak{t}_{\mathcal{K}}$. ■

Theorem 8.3. *The involution $\sigma_{\mathcal{K}}$ has the property that $\mathfrak{g}_1$ contains a regular semisimple and a regular nilpotent element of $\mathfrak{g}$.*

Proof. For each $\beta_i \in \mathcal{K}$, consider the 3-dimensional subalgebra $\mathfrak{sl}_2(\beta_i)$ with basis $\{e_{\beta_i}, e_{-\beta_i}, [e_{\beta_i}, e_{-\beta_i}]\}$. Then $\mathfrak{sl}_2(\beta_i) \simeq \mathfrak{sl}_2$ and since the elements of $\mathcal{K}$ are strongly orthogonal, all these $\mathfrak{sl}_2$-subalgebras pairwise commute. Hence $\mathfrak{h} = \bigoplus_{j=1}^m \mathfrak{sl}_2(\beta_j)$ is a Lie algebra and $\mathfrak{t}_{\mathcal{K}}$ is a Cartan subalgebra of $\mathfrak{h}$. Recall that $\beta_j(x_{\mathcal{K}}) = 1$, i.e., $\beta_j \in \Delta^+_{\mathcal{K}}(1)$ for each j. By the very definition of $\sigma_{\mathcal{K}}$, this means that $\mathfrak{h} \cap \mathfrak{g}_0 = \mathfrak{t}_{\mathcal{K}}$ and $\mathfrak{h} \cap \mathfrak{g}_1 = (\bigoplus_{i=1}^m \mathfrak{g}_{\beta_i}) \oplus (\bigoplus_{i=1}^m \mathfrak{g}_{-\beta_i})$. Clearly, there is a Cartan subalgebra of $\mathfrak{h}$ that is contained in $\mathfrak{h} \cap \mathfrak{g}_1$ (because this is true for each $\mathfrak{sl}_2(\beta_i)$ separately). Combining this with Lemma 8.2, we see that $\mathfrak{g}_1$ contains a regular semisimple element of $\mathfrak{g}$.

Finally, for any involution σ, its (-1)-eigenspace $\mathfrak{g}_1$ contains a regular semisimple element if and only if it contains a regular nilpotent element. ■

Remark 8.4. (i) One can prove that if $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{h} = \bigoplus_{j=1}^m \mathfrak{sl}_2(\beta_j)$ that is contained in $\mathfrak{h} \cap \mathfrak{g}_1$, then $\mathfrak{c}$ is a maximal diagonalisable subalgebra of $\mathfrak{g}_1$. In other words, $\mathfrak{c} \subset \mathfrak{g}_1$ is a *Cartan subspace* associated with $\sigma_{\mathcal{K}}$. Therefore, the rank of the symmetric variety G/G_0 equals $\dim \mathfrak{c} = \#\mathcal{K}$.

(ii) The involution $\sigma_{\mathcal{K}}$ is the unique, up to G-conjugacy, *inner* involution such that $\mathfrak{g}_1$ contains a regular nilpotent element. Moreover, $\sigma_{\mathcal{K}}$ has the property that $\mathcal{O} \cap \mathfrak{g}_1 \neq \varnothing$ for *any nilpotent* G-orbit $\mathcal{O} \subset \mathfrak{g}$, see [2, Theorem 3].

(iii) If $\#\mathcal{K} = \mathsf{rk}\,\mathfrak{g}$, then $\dim \mathfrak{g}_1 = \dim \mathfrak{b}$, $\dim \mathfrak{g}_0 = \dim \mathfrak{u}$, and $\mathfrak{g}_1$ contains a Cartan subalgebra of $\mathfrak{g}$. In this case, $\sigma_{\mathcal{K}}$ is an *involution of maximal rank* and one has a stronger assertion that $\mathcal{O} \cap \mathfrak{g}_1 \neq \varnothing$ for *any* G-orbit $\mathcal{O}$ in $\mathfrak{g}$ ([2, Theorem 2]).

9. The nilpotent G-orbit associated with the cascade

Consider the nilpotent element associated with $\mathcal{K}$

$$e_{\mathcal{K}} := \sum_{\beta \in \mathcal{K}} e_\beta = \sum_{i=1}^m e_{\beta_i} \in \mathfrak{u}^+ \qquad (10)$$

Since the roots in $\mathcal{K}$ are linearly independent, the closure of $G{\cdot}e_{\mathcal{K}}$ contains the space $\bigoplus_{\beta \in \mathcal{K}} \mathfrak{g}_\beta$. Hence the nilpotent orbit $G{\cdot}e_{\mathcal{K}}$ does not depend on the choice of root vectors $e_\beta \in \mathfrak{g}_\beta$. The orbit $\mathcal{O}_{\mathcal{K}} := G{\cdot}e_{\mathcal{K}}$ is said to be the *cascade (nilpotent) orbit.* Our goal is to obtain some properties of this orbit.

Recall from Section 3 that if θ is fundamental, then $\tilde\alpha$ is the only (long!) simple root such that $(\theta, \tilde\alpha) > 0$ and $\tilde{\mathcal{K}} = \{\beta \in \mathcal{K} \mid (\beta, \tilde\alpha) < 0\}$. It then follows from (4) that $\#\tilde{\mathcal{K}} \le 3$ and $\#\tilde{\mathcal{K}} = 3$ if and only if the roots in $\tilde{\mathcal{K}}$ are long. Actually, one has $\#\tilde{\mathcal{K}} = 1$ for $\mathbf{G}_2$, $\#\tilde{\mathcal{K}} = 2$ for $\mathbf{B}_3$, and $\#\tilde{\mathcal{K}} = 3$ for the remaining cases with fundamental θ.

Theorem 9.1. (1) *Suppose that θ is fundamental, and let $\tilde\alpha$ be the unique simple root such that $(\theta, \tilde\alpha) \ne 0$. Then* (i) $(\mathrm{ad}\, e_{\mathcal{K}})^4(e_{-\theta+\tilde\alpha}) \ne 0$ *and* (ii) $(\mathrm{ad}\, e_{\mathcal{K}})^5 = 0$.
(2) *If θ is* not *fundamental, then* $(\mathrm{ad}\, e_{\mathcal{K}})^3 = 0$.

Proof. (1)(i) It follows from Eq. (10) that

$$(\mathrm{ad}\, e_{\mathcal{K}})^4 = \sum_{i_1,i_2,i_3,i_4} \mathrm{ad}(e_{\beta_{i_1}})\cdot \ldots \cdot \mathrm{ad}(e_{\beta_{i_4}}),$$

where the sum is taken over all possible quadruples of indices from $\{1, \ldots, m\}$. Set

$$\mathcal{A}_{i_1,i_2,i_3,i_4} = \mathrm{ad}(e_{\beta_{i_1}})\mathrm{ad}(e_{\beta_{i_2}})\mathrm{ad}(e_{\beta_{i_3}})\mathrm{ad}(e_{\beta_{i_4}}).$$

As the roots in $\mathcal{K}$ are strongly orthogonal, the ordering of factors in the operator $\mathcal{A}_{i_1,i_2,i_3,i_4}$ is irrelevant. Hence $\mathcal{A}_{i_1,i_2,i_3,i_4}$ depends only on the 4-multiset $\{i_1, i_2, i_3, i_4\}$. Furthermore, the *nonzero* operators corresponding to different 4-multisets are linearly independent. Therefore, to ensure that $(\mathrm{ad}\, e_{\mathcal{K}})^4 \ne 0$, it suffices to point out a 4-multiset $\mathcal{M}$ and a root vector e_γ such that $\mathcal{A}_{\mathcal{M}}(e_\gamma) \ne 0$. Of course, in place of 4-multisets of indices in $[1, m]$, one can deal with 4-multisets in $\mathcal{K}$.

Using (4) and $\tilde{\mathcal{K}} \subset \mathcal{K}$, one defines a natural 4-multiset $\tilde{\mathcal{M}}$ in $\mathcal{K}$. The first element of $\tilde{\mathcal{M}}$ is $\theta = \beta_1$ and then one takes each $\beta_i \in \tilde{\mathcal{K}}$ with multiplicity $k_i = (\tilde\alpha, \tilde\alpha)/(\beta_i, \beta_i)$. The resulting 4-multiset has the property that $(-\theta + \tilde\alpha) + (\theta + \sum_{i\in J} k_i\beta_i) = \theta - \tilde\alpha$ and $(-\theta + \tilde\alpha, \beta) < 0$ for any β in $\tilde{\mathcal{M}}$. This implies that

$$0 \ne \mathcal{A}_{\tilde{\mathcal{M}}}(e_{-\theta+\tilde\alpha}) \in \mathfrak{g}_{\theta-\tilde\alpha}.$$

(ii) Now we deal with 5-multisets of $\mathcal{K}$. Assume that $\tilde{\mathcal{M}} = \{\beta_{i_1}, \ldots, \beta_{i_5}\}$ and $\mathcal{A}_{\tilde{\mathcal{M}}} \ne 0$. Then there are $\gamma, \mu \in \Delta$ such that $0 \ne \mathcal{A}_{\tilde{\mathcal{M}}}(\mathfrak{g}_{-\mu}) \subset \mathfrak{g}_\gamma$. Hence $\gamma + \mu = \sum_{j=1}^5 \beta_{i_j}$ and $(\gamma + \mu)(x_{\mathcal{K}}) = 5$. But $\gamma(x_{\mathcal{K}}) \le 2$ for any $\gamma \in \Delta$ (Theorem 3.5(3)). This contradiction shows that $(\mathrm{ad}\, e_{\mathcal{K}})^5 = 0$.

(2) By Theorem 3.5(1)(2), we have $\gamma(x_{\mathcal{K}}) \le 1$ for any $\gamma \in \Delta$. Hence the equality $\gamma + \mu = \sum_{j=1}^3 \beta_{i_j}$ is impossible. This implies that $\mathcal{A}_{\tilde{\mathcal{M}}} = 0$ for any 3-multiset $\tilde{\mathcal{M}}$ of $\mathcal{K}$ and thereby $(\mathrm{ad}\, e_{\mathcal{K}})^3 = 0$. ∎

Let $\mathcal{N} \subset \mathfrak{g}$ be the set of nilpotent elements. Recall that the *height* of $e \in \mathcal{N}$, denoted $\mathsf{ht}(e)$ or $\mathsf{ht}(G{\cdot}e)$, is the maximal $l \in \mathbb{N}$ such that $(\mathrm{ad}\, e)^l \ne 0$ (see [22, Section 2]). By the Jacobson-Morozov theorem, $(\mathrm{ad}\, e)^2 \ne 0$ for any $e \in \mathcal{N}$, i.e., $\mathsf{ht}(G{\cdot}e) \ge 2$.

The *complexity* of a G-variety X, $c_G(X)$, is the minimal codimension of the B-orbits in X. If X is irreducible, then $c_G(X) = \mathrm{trdeg}\, \mathbb{C}(X)^B$, where $\mathbb{C}(X)^G$ is the field of B-invariant rational functions on X. The *rank* of an irreducible G-variety X is defined by the equality $c_G(X) + r_G(X) = \mathrm{trdeg}\, \mathbb{C}(X)^U$ [23]. If $c_G(X) = 0$, then X is said to be *spherical*.

Proposition 9.2.

(i) *If θ is fundamental, then the orbit $G{\cdot}e_{\mathcal{K}}$ is* not *spherical and* $\mathsf{ht}(G{\cdot}e_{\mathcal{K}})=4$.

(ii) *If θ is* not *fundamental, then the orbit $G{\cdot}e_{\mathcal{K}}$ is spherical and* $\mathsf{ht}(G{\cdot}e_{\mathcal{K}})=2$.

Proof. By [21, Theorem (0.3)], a nilpotent orbit $G{\cdot}e$ is spherical if and only if $(\mathrm{ad}\, e)^4=0$. Hence both assertions follow from Theorem 9.1. ■

9.1. A description of the cascade orbits

For the classical Lie algebras, we determine the partition corresponding to $\mathcal{O}_{\mathcal{K}}$. While for the exceptional Lie algebras, we point out a "minimal including regular subalgebra" in the sense of Dynkin [6].

I. In the classical cases, we use formulae for the height of nilpotent elements of $\mathfrak{sl}(\mathbb{V})$ or $\mathfrak{so}(\mathbb{V})$ or $\mathfrak{sp}(\mathbb{V})$ in terms of the corresponding partitions of $\dim\mathbb{V}$, see [22, Theorem 2.3].

- $\mathfrak{g}=\mathfrak{so}_{2N+1}$. Since $\mathsf{ht}(\mathcal{O}_{\mathcal{K}})=4$, the parts of $\boldsymbol{\lambda}(e_{\mathcal{K}})$ does not exceed 3, i.e., $\boldsymbol{\lambda}(e_{\mathcal{K}})=(3^a,2^b,1^c)$ with $a>0$ and $3a{+}2b{+}c=2N{+}1$. Then $\boldsymbol{\lambda}(e_{\mathcal{K}}^2)=(2^a,1,\dots,1)$. Hence $\mathsf{rk}\,(e_{\mathcal{K}})=2a+b$ and $\mathsf{rk}\,(e_{\mathcal{K}}^2)=a$. On the other hand, here $\#\mathcal{K}=N=\mathsf{rk}\,\mathfrak{g}$ and using the formulae for roots in $\mathcal{K}$ and thereby the explicit matrix form for $e_{\mathcal{K}}$, one readily computes that

$$\mathsf{rk}\,(e_{\mathcal{K}})=\begin{cases} N, & \text{if } N \text{ is even}\\ N{+}1, & \text{if } N \text{ is odd}\end{cases} \quad \text{and} \quad \mathsf{rk}\,(e_{\mathcal{K}}^2)=\tfrac{1}{2}\mathsf{rk}\,(e_{\mathcal{K}})=[(N{+}1)/2].$$

Therefore, if N is either $2j{-}1$ or $2j$, then $a=j$ and $b=0$.

Hence $\boldsymbol{\lambda}(e_{\mathcal{K}})=(3^j,1^{j-1})$ or $(3^j,1^{j+1})$, respectively.

- $\mathfrak{g}=\mathfrak{so}_{2N}$. Here $\mathsf{ht}(\mathcal{O}_{\mathcal{K}})=4$ and $\boldsymbol{\lambda}(e_{\mathcal{K}})=(3^a,2^b,1^c)$, with $a>0$ and $3a+2b+c=2N$. Then again $\mathsf{rk}\,(e_{\mathcal{K}})=2a+b$ and $\mathsf{rk}\,(e_{\mathcal{K}}^2)=a$. The explicit form of $\mathcal{K}$ and $e_{\mathcal{K}}$ shows that

$$\mathsf{rk}\,(e_{\mathcal{K}})=\begin{cases} N, & \text{if } N \text{ is even}\\ N{-}1, & \text{if } N \text{ is odd}\end{cases} \quad \text{and} \quad \mathsf{rk}\,(e_{\mathcal{K}}^2)=\tfrac{1}{2}\mathsf{rk}\,(e_{\mathcal{K}})=[N/2].$$

Therefore, $a=[N/2]$ and $b=0$ in both cases. Hence

$$\boldsymbol{\lambda}(e_{\mathcal{K}})=(3^j,1^j),\ \text{if } N=2j;\quad \boldsymbol{\lambda}(e_{\mathcal{K}})=(3^j,1^{j+2}),\ \text{if } N=2j+1.$$

- $\mathfrak{g}=\mathfrak{sl}_{N+1}$ or $\mathfrak{sp}_{2N}$. Then $\mathsf{ht}(\mathcal{O}_{\mathcal{K}})=2$, $\boldsymbol{\lambda}(e_{\mathcal{K}})=(2^a,1^b)$, and $a=\mathsf{rk}\,(e_{\mathcal{K}})=\#\mathcal{K}$. Therefore, $\boldsymbol{\lambda}(e_{\mathcal{K}})=(2^n)$ for $\mathfrak{sl}_{2n}$ and $\mathfrak{sp}_{2n}$, while $\boldsymbol{\lambda}(e_{\mathcal{K}})=(2^n,1)$ for $\mathfrak{sl}_{2n+1}$.

II. In the exceptional cases, one can use some old, but extremely helpful computations of E.B. Dynkin. Following Dynkin, we say that a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is *regular*, if it is normalised by a Cartan subalgebra. As in Section 8, consider $\mathfrak{h}:=\bigoplus_{i=1}^m\mathfrak{sl}_2(\beta_i)$. Then $\mathfrak{h}$ is normalised by $\mathfrak{t}$ and $e_{\mathcal{K}}\in\mathfrak{h}$ is a regular nilpotent element of $\mathfrak{h}$. Clearly, $\mathfrak{h}$ is a minimal regular semisimple subalgebra of $\mathfrak{g}$ meeting $\mathcal{O}_{\mathcal{K}}$.

For every nilpotent G-orbit $\mathcal{O}$ in an exceptional Lie algebra $\mathfrak{g}$, Dynkin computes all, up to conjugacy, minimal regular semisimple subalgebras of $\mathfrak{g}$ meeting $\mathcal{O}$, see Tables 16–20 in [6]. (This information is also reproduced, with a few corrections, in Tables 2–6 in [7].) Therefore, it remains only to pick the nilpotent orbit with a "minimal including regular subalgebra" of the required type, which also yields the corresponding *weighted Dynkin diagram* $\mathcal{D}(\mathcal{O}_{\mathcal{K}})$.

For the regular subalgebras $\mathfrak{sl}_2 \subset \mathfrak{g}$, Dynkin uses the Cartan label $\mathbf{A}_1$ (resp. $\widetilde{\mathbf{A}}_1$) if the corresponding root is long (resp. short). Therefore, we are looking for the "minimal including regular subalgebra" of type $m\mathbf{A}_1$, $m = \#\mathcal{K}$, if $\mathcal{K} \subset \Delta_l$; whereas for $\mathfrak{g}$ of type $\mathbf{G}_2$ we need the subalgebra of type $\mathbf{A}_1 + \widetilde{\mathbf{A}}_1$.

9.2. Another approach to $\mathcal{O}_\mathcal{K}$

By the very definition of $x_\mathcal{K} \in \mathfrak{t}$ and $e_\mathcal{K}$, we have $[x_\mathcal{K}, e_\mathcal{K}] = e_\mathcal{K}$. It is also clear that $x_\mathcal{K} \in \mathsf{Im}\,(\mathrm{ad}\, e_\mathcal{K})$. Therefore $h_\mathcal{K} = 2x_\mathcal{K}$ is a *characteristic* of $e_\mathcal{K}$ [10, Chap. 6, § 2.1]. Hence the weighted Dynkin diagram $\mathcal{D}(\mathcal{O}_\mathcal{K})$ is determined by the dominant representative in the Weyl group orbit $W{\cdot}h_\mathcal{K} \subset \mathfrak{t}$. Since the antidominant representative in $W{\cdot}x_\mathcal{K}$ is $w_\mathcal{K}(x_\mathcal{K})$ (Prop. 6.3) and $\omega_0(x_\mathcal{K}) = -x_\mathcal{K}$, the dominant representative is $-w_\mathcal{K}(x_\mathcal{K})$. Therefore, if $\mathfrak{g} \neq \mathfrak{sl}_{2n+1}$, then $W{\cdot}h_\mathcal{K} \cap \mathcal{C} = \{2\varpi_j^\vee\}$, where j is determined by the condition that $w_\mathcal{K}(\theta) = -\alpha_j$ (cf. Prop. 6.5). Thus, if $\mathfrak{g} \neq \mathfrak{sl}_{2n+1}$, then $\mathcal{O}_\mathcal{K}$ is even and $\mathcal{D}(\mathcal{O}_\mathcal{K})$ has the unique nonzero label "2" that corresponds to α_j.

Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ be the $\mathbb{Z}$-grading determined by $h_\mathcal{K} = 2x_\mathcal{K}$; that is, $h_\mathcal{K}$ has the eigenvalue i on $\mathfrak{g}(i)$. Then

$$\mathsf{ht}(e_\mathcal{K}) = \max\{i \mid \mathfrak{g}(i) \neq 0\} = 2\max\{\gamma(x_\mathcal{K}) \mid \gamma \in \Delta\}.$$

Using results of Section 3, we again see that $\mathsf{ht}(e_\mathcal{K}) \le 4$ and the equality occurs if and only if θ is fundamental. Note that

$$\dim \mathfrak{g}(2) = \#\big(\Delta^+_\mathcal{K}(1) \cup \Delta^-_\mathcal{K}(1)\big) = \#\big(\Delta^+_\mathcal{K}(1) \cup \Delta^+_\mathcal{K}(-1)\big) = \#\big(\Delta^+_\mathcal{K}(1) \cup \Delta^+_\mathcal{K}(2)\big),$$
$$\dim \mathfrak{g}(4) = \#\Delta^+_\mathcal{K}(2) = \#\Delta_\mathcal{K}(2) = \dim \mathsf{Im}\,(\mathrm{ad}\, e_\mathcal{K})^4,$$

and also $2\dim \mathfrak{g}(4) = \dim \mathfrak{a}_\mathcal{K}$.

In Tables 1 and 2, we point out $\dim \mathcal{O}_\mathcal{K}$ and $\mathcal{D}(\mathcal{O}_\mathcal{K})$. For classical cases, we provide the partition $\boldsymbol{\lambda}(e_\mathcal{K})$; while for the exceptional cases, the Dynkin-Bala-Carter notation for orbits is given, see e.g. [3, Ch. 8]. We also include dimensions of the spaces $\mathfrak{g}(2)$ and $\mathfrak{g}(4)$. In Table 1, the unique nonzero numerical mark "2" corresponds to the simple root α_j for all even cases. For $\mathbf{A}_{2j}$, two marks "1" correspond to the roots α_j and α_{j+1}. We also assume that $j \ge 2$ in the four orthogonal cases. (For, $\mathbf{B}_1 = \mathbf{A}_1$, $\mathbf{D}_2 = \mathbf{A}_1 + \mathbf{A}_1$, $\mathbf{B}_2 = \mathbf{C}_2$, and $\mathbf{D}_3 = \mathbf{A}_3$.)

Let us summarise the main properties of the cascade orbit in all simple $\mathfrak{g}$.

- The cascade orbit $\mathcal{O}_\mathcal{K}$ is even unless $\mathfrak{g}$ is of type $\mathbf{A}_{2j}$; this reflects the fact that $x_\mathcal{K} \in \mathcal{P}^\vee$ unless $\mathfrak{g}$ is of type $\mathbf{A}_{2j}$.
- If θ is fundamental, then $\mathsf{ht}(\mathcal{O}_\mathcal{K}) = 4$ and $\mathcal{O}_\mathcal{K}$ is *not* spherical.
- If θ is *not* fundamental, then $\mathsf{ht}(\mathcal{O}_\mathcal{K}) = 2$ and $\mathcal{O}_\mathcal{K}$ is spherical. Moreover, it appears that $\mathcal{O}_\mathcal{K}$ is the *maximal* spherical nilpotent orbit in these cases.
- Using the general formulae for the complexity and rank of nilpotent orbits in terms of $\mathbb{Z}$-gradings [21, Sect. (2.3)], one can prove that $c_G(\mathcal{O}_\mathcal{K}) = 2\dim \mathfrak{g}(4) = \dim \mathfrak{a}_\mathcal{K}$ and $r_G(\mathcal{O}_\mathcal{K}) = \dim \mathfrak{t}_\mathcal{K} = \#\mathcal{K}$ for all simple $\mathfrak{g}$.

Table 1: The cascade orbits for the classical Lie algebras

$\mathfrak{g}$	$\boldsymbol{\lambda}(e_{\mathcal{K}})$	$\dim \mathcal{O}_{\mathcal{K}}$	$\mathcal{D}(\mathcal{O}_{\mathcal{K}})$	$\dim \mathfrak{g}(2)$	$\dim \mathfrak{g}(4)$
$\mathbf{A}_{2j-1}$	(2^j)	$2j^2$	$0-\cdots-0-2-0-\cdots-0$	j^2	0
$\mathbf{A}_{2j}$	$(2^j,1)$	$2j^2+2j$	$0-\cdots-0-1-1-0-\cdots-0$	j^2	0
$\mathbf{C}_j$	(2^j)	j^2+j	$0-\cdots-0\Leftarrow 2$	$\binom{j+1}{2}$	0
$\mathbf{B}_{2j-1}$	$(3^j,1^{j-1})$	$5j^2-3j$	$0-\cdots-0-2-0-\cdots-0\Rightarrow 0$	$2j^2-j$	$\binom{j}{2}$
$\mathbf{D}_{2j}$	$(3^j,1^j)$	$5j^2-j$	$0-\cdots-0-2-0-\cdots-0<^{0}_{0}$	$2j^2$	$\binom{j}{2}$
$\mathbf{B}_{2j}$	$(3^j,1^{j+1})$	$5j^2+j$	$0-\cdots-0-2-0-\cdots-0\Rightarrow 0$	$2j^2+j$	$\binom{j}{2}$
$\mathbf{D}_{2j+1}$	$(3^j,1^{j+2})$	$5j^2+3j$	$0-\cdots-0-2-0-\cdots-0<^{0}_{0}$	$2j^2+2j$	$\binom{j}{2}$

Table 2: The cascade orbits for the exceptional Lie algebras

$\mathfrak{g}$	$\mathcal{O}_{\mathcal{K}}$	$\dim \mathcal{O}_{\mathcal{K}}$	$\mathcal{D}(\mathcal{O}_{\mathcal{K}})$	$\dim \mathfrak{g}(2)$	$\dim \mathfrak{g}(4)$
$\mathbf{E}_6$	A_2	42	$0-0-\underset{2}{\underset{\mid}{0}}-0-0$	20	1
$\mathbf{E}_7$	$\mathsf{A}_2+3\mathsf{A}_1$	84	$0-0-0-\underset{2}{\underset{\mid}{0}}-0-0$	35	7
$\mathbf{E}_8$	$2\mathsf{A}_2$	156	$0-0-0-0-\underset{0}{\underset{\mid}{0}}-0-2$	64	14
$\mathbf{F}_4$	A_2	30	$0-0\Leftarrow 0-2$	14	1
$\mathbf{G}_2$	$\mathsf{G}_2(a_1)$	10	$0 \Lleftarrow 2$	4	1

- If θ is fundamental, then ($\mathcal{O}_{\mathcal{K}}$ is even and) the node with mark "2", regarded as a node in the affine Dynkin diagram, determines the *Kac diagram* of the involution $\sigma_{\mathcal{K}}$. (See [10, Chap. 3, § 3.7] for the definition of the Kac diagram of a finite order inner automorphism of $\mathfrak{g}$.)
- If θ is not fundamental and $\mathcal{O}_{\mathcal{K}}$ is even, then the node with mark "2" and the extra node in the affine Dynkin diagram, together determine the Kac diagram of the involution $\sigma_{\mathcal{K}}$.

A. The elements of $\mathcal{K}$ and Hasse diagrams

Here we provide the lists of cascade elements and the Hasse diagrams of cascade posets $\mathcal{K}$ for all simple Lie algebras, see Figures 3–6. To each node β_j in the Hasse diagram, the Cartan label of the simple Lie algebra $\mathfrak{g}\langle j\rangle$ is attached. Recall that β_j is the highest root for $\mathfrak{g}\langle j\rangle$. If $\mathfrak{g}\langle j\rangle \simeq \mathfrak{sl}_2$ and β_j is *short*, then we use the Cartan label $\widetilde{\mathbf{A}}_1$. (This happens only for $\mathbf{B}_{2k+1}$ and $\mathbf{G}_2$.) It is also assumed that $\mathbf{A}_1=\mathbf{C}_1$. The main features are:

- $\Pi=\{\alpha_1,\dots,\alpha_{\mathsf{rk}\,\mathfrak{g}}\}$ and the numbering of Π follows [19, Table 1],
- $\beta_1=\theta$ is always the highest root,
- The numbering of the β_i's in the lists corresponds to that in the figures.

We use the standard ε-notation for the roots of classical Lie algebras, see [19, Table 1].

The cascade elements for the classical Lie algebras:

$\mathbf{A}_n, n \ge 2$**:** $\beta_i = \varepsilon_i - \varepsilon_{n+2-i} = \alpha_i + \cdots + \alpha_{n+1-i}$ $(i = 1, 2, \ldots, \left[\frac{n+1}{2}\right])$;

$\mathbf{C}_n, n \ge 1$**:** $\beta_i = 2\varepsilon_i = 2(\alpha_i + \cdots + \alpha_{n-1}) + \alpha_n$ $(i = 1, 2, ..., n-1)$ and $\beta_n = 2\varepsilon_n = \alpha_n$;

$\mathbf{B}_{2n}$, $\mathbf{D}_{2n}$, $\mathbf{D}_{2n+1}$ $(n \ge 2)$**:** $\beta_{2i-1} = \varepsilon_{2i-1} + \varepsilon_{2i}$, $\beta_{2i} = \varepsilon_{2i-1} - \varepsilon_{2i}$ $(i = 1, 2, \ldots, n)$;

$\mathbf{B}_{2n+1}, n \ge 1$**:** here $\beta_1, \ldots, \beta_{2n}$ are as above and $\beta_{2n+1} = \varepsilon_{2n+1}$.

For all orthogonal series, we have $\beta_{2i} = \alpha_{2i-1}$, $i = 1, \ldots, n$, while formulae for β_{2i-1} via Π slightly differ for different series. E.g. for $\mathbf{D}_{2n}$ one has $\beta_{2i-1} = \alpha_{2i-1} + 2(\alpha_{2i} + \cdots + \alpha_{2n-2}) + \alpha_{2n-1} + \alpha_{2n}$ $(i = 1, 2, \ldots, n-1)$ and $\beta_{2n-1} = \alpha_{2n}$.

The cascade elements for the exceptional Lie algebras:

$\mathbf{G}_2$**:** $\beta_1 = (32) = 3\alpha_1 + 2\alpha_2$, $\beta_2 = (10) = \alpha_1$;

$\mathbf{F}_4$**:** $\beta_1 = (2432) = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$, $\beta_2 = (2210)$, $\beta_3 = (0210)$, $\beta_4 = (0010) = \alpha_3$;

$\mathbf{E}_6$**:** $\beta_1 = \begin{smallmatrix}12321\\2\end{smallmatrix}$, $\beta_2 = \begin{smallmatrix}11111\\0\end{smallmatrix}$, $\beta_3 = \begin{smallmatrix}01110\\0\end{smallmatrix}$, $\beta_4 = \begin{smallmatrix}00100\\0\end{smallmatrix} = \alpha_3$;

$\mathbf{E}_7$**:** $\beta_1 = \begin{smallmatrix}123432\\2\end{smallmatrix}$, $\beta_2 = \begin{smallmatrix}122210\\1\end{smallmatrix}$, $\beta_3 = \begin{smallmatrix}100000\\0\end{smallmatrix} = \alpha_1$, $\beta_4 = \begin{smallmatrix}001210\\1\end{smallmatrix}$, $\beta_5 = \begin{smallmatrix}001000\\0\end{smallmatrix} = \alpha_3$,
$\beta_6 = \begin{smallmatrix}000010\\0\end{smallmatrix} = \alpha_5$, $\beta_7 = \begin{smallmatrix}000000\\1\end{smallmatrix} = \alpha_7$;

$\mathbf{E}_8$**:** $\beta_1 = \begin{smallmatrix}2345642\\3\end{smallmatrix}$, $\beta_2 = \begin{smallmatrix}0123432\\2\end{smallmatrix}$, $\beta_3 = \begin{smallmatrix}0122210\\1\end{smallmatrix}$, $\beta_4 = \begin{smallmatrix}0100000\\0\end{smallmatrix} = \alpha_2$, $\beta_5 = \begin{smallmatrix}0001210\\1\end{smallmatrix}$,
$\beta_6 = \begin{smallmatrix}0001000\\0\end{smallmatrix} = \alpha_4$, $\beta_7 = \begin{smallmatrix}0000010\\0\end{smallmatrix} = \alpha_6$, $\beta_8 = \begin{smallmatrix}0000000\\1\end{smallmatrix} = \alpha_8$.

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Figure 3: The cascade posets for $\mathbf{A}_p$ $(p \geq 2)$, $\mathbf{C}_n$ $(n \geq 1)$, $\mathbf{F}_4$, $\mathbf{G}_2$

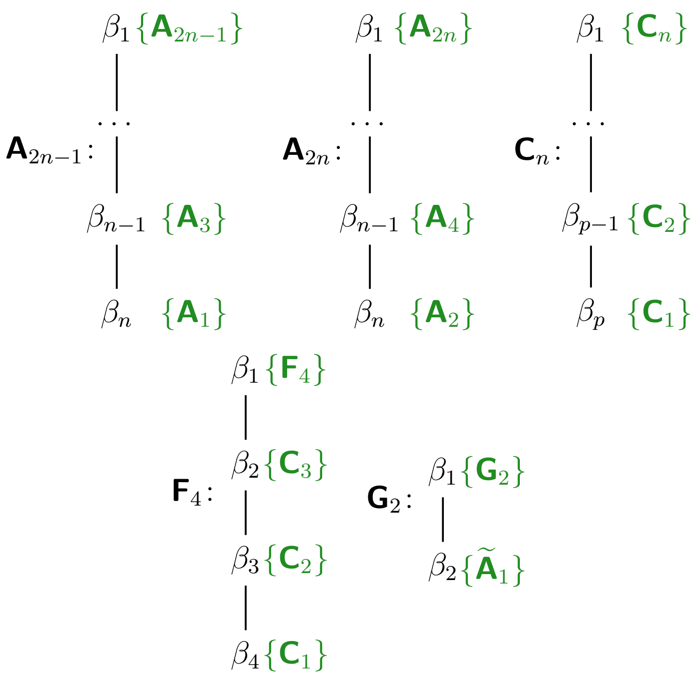

Figure 4: The cascade posets for series $\mathbf{B}_p$, $p \geq 3$

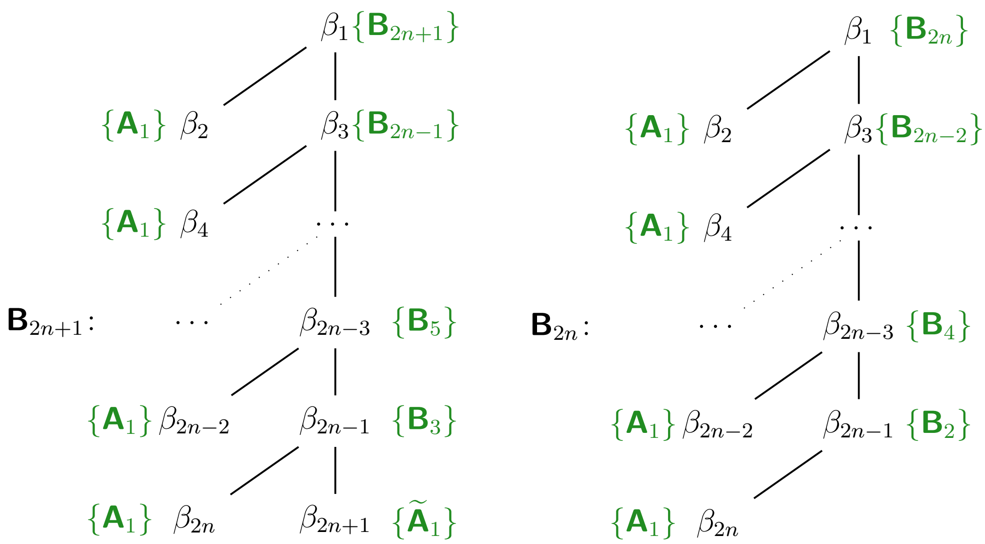

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Figure 5: The cascade posets for series $\mathbf{D}_p$, $p \geq 4$

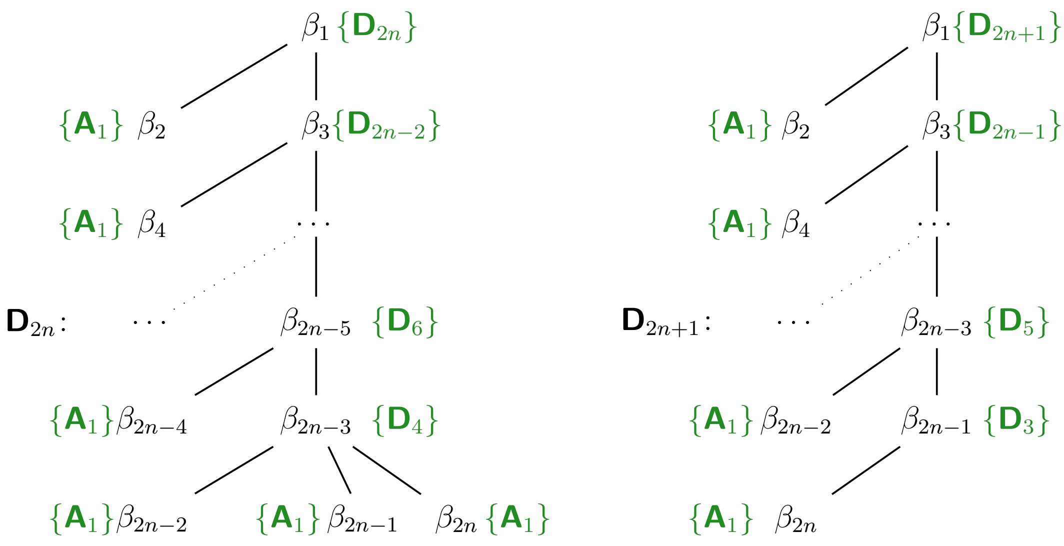
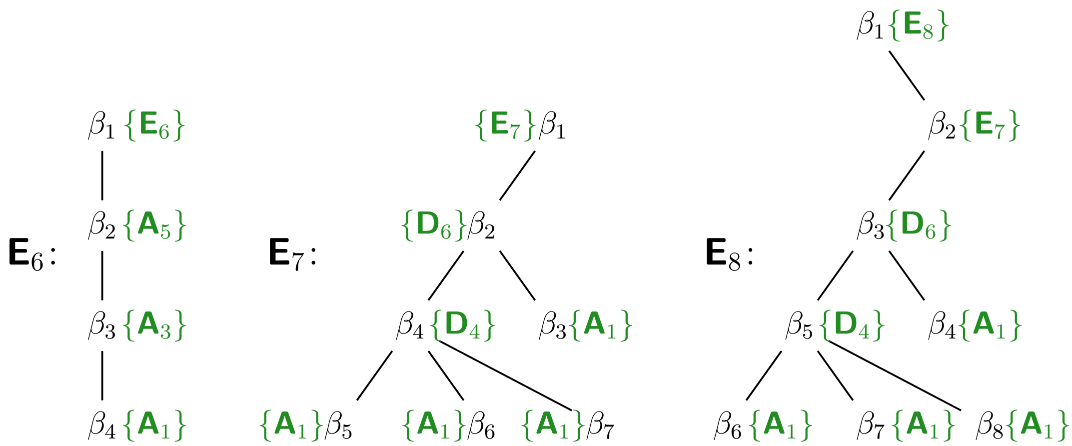


Figure 6: The cascade posets for $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$



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Received June 24, 2022
and in final form November 8, 2022