

# On Integrations and Cross Ratios on Supermanifolds

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**Abstract.** (A) The conventional integration theory on supermanifolds had been constructed in order to have (an analog of) the Stokes formula in which a sub-supermanifold is of codimension  $1 = (1|0)$ . I review other integrations and formulate related open problems:

(1) On the  $1|1$ -dimensional superstring associated with the trivial bundle, in presence of a contact structure there is a special integration useful in describing super versions of elliptic functions. It is needed to construct a particular spinor representation of the Neveu-Schwarz superalgebra.

(2) Versions of the Stokes formula with “over-supermanifold” of codimension  $(0|-1)$  due to Shander and Palamodov should be developed further.

(3) Apply Shander’s integration with odd parameters over chains to inverse problems.

(4) Establish existence of conjectural integrations (apparently, not leading to any analog of the Stokes formula) related to various (super)traces on various Lie superalgebras and the corresponding (super)determinants.

(B) I offer analogs of the cross ratio for “classical superspaces”, including infinite-dimensional versions. Open problem: apply these invariants to the matrix-valued Riccati equations.

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This review unites several ostensibly unrelated topics and their “analytical continuations” and draws attention to open problems I find rather important: their solutions would lead us further to unpredictable results. I also wanted to give adequate references in certain cases under-represented in the literature and not as well-known as they deserve to be as, I hope, I demonstrated here why. The text contains definitions, construction, and problems (their solutions are future theorems).

For the basics on Linear Algebra in Superspaces and Supermanifold theory, see Bernstein’s lectures in the book [18] (or [9, 4] if you read Russian); for some useful facts, see [70, 66]. Of several sources listing the simple (and close to them) Lie superalgebras of finite dimension or  $\mathbb{Z}$ -graded of polynomial growth, I recommend most recent [24, 17, 13] with new results and insight. The list of these Lie superalgebras is rather long, in order not to reproduce it, I refer to [10].

## 1. Integration

### 1.1. Integrations with the Stokes formula

In recent notes [78], Witten recalls integration suggested in the mid-1970’s by J. Bernstein and me (see [6, 5]) and lists several applications. Here, I survey what is not covered in [78], and – a bit differently – what is.

In [6, 5], we offered an analog of integration theory on supermanifolds superizing the Stokes formula. We had at our disposal the following two ingredients:

(i1) the *differential forms*, i.e., polynomials in the differentials  $\hat{x}_i := dx_i$  of the coordinates on the given supermanifold; the *degree- $k$*  forms span the module  $\Omega^k$  over the algebra  $\mathcal{F}$  of functions.

(i2) *volume forms*, the latter constituted a rank 1 module  $\text{Vol}$  over  $\mathcal{F}$ ;

*under the change of coordinates  $x \mapsto y$  the generator  $\text{vol}(y(x))$  of  $\text{Vol}$  is multiplied by the Berezinian (superdeterminant) of the Jacobi matrix  $\frac{\partial y}{\partial x}$ .*

Each of the above notions (i1) and (i2) had to be carefully reconsidered in super setting because even the most innocent-looking notions and theorems displayed, in the super setting, inexplicable – at first glance – signs at unexpected places (e.g., the Foubini theorem), see [9].

We wanted to have some analog of the Stokes formula, so we needed two more ingredients:

(i3) Certain integrands of “degree” smaller than that of volume forms and a differential connecting integrands of degrees that differ by 1.

(i4) The notion of *supermanifold-with-boundary*, see [6]. (Observe that like manifold-with-boundary is not a manifold, supermanifold-with-boundary is not a supermanifold.)

Speaking of “something” to serve as integrand on a subsupermanifold we naturally wish to label these somethings by bidegree corresponding to the superdimension of the subsupermanifold.

To have an integration theory, one needs not only an object to be integrated (the *integrand*), but over what it is integrated (the *cycle*), and the *orientation*. The latter two notions turned out to be more involved than we originally thought; Shander clarified this in his development of integration theory, see [68] expounded in [9, Ch. 6]. In particular, Shander introduced the objects dual to cycles and also labelled by bidegree (as superdimension).

Actually, having defined *supermanifold-with-boundary*, we had enough to construct the integration theory with an analog of the Stokes formula: by setting  $\Sigma_{-i} := \text{Hom}_{\mathcal{F}}(\Omega^i, \text{Vol})$  we obtain a complex dual to the de Rham one with  $\Sigma_0 = \text{Vol}$  as forms of the highest degree. J. Bernstein and I called the elements of  $\Sigma_{-i} = \oplus \Sigma_{-i}$  *integrateBle* forms (the ones one can integrate) and explained how to integrate such forms in [6]. More precisely, recall that  $\Omega^i := T(E^i(V^*))$ , where  $V$  is the tangent space,  $E^i$  is the operator of raising to the  $i$ th exterior power, and  $T(W)$  designates the space of  $W$ -valued tensor fields on the  $(a|b)$ -dimensional superdomain  $\mathcal{U}$  for any  $\mathfrak{gl}(a|b)$ -module  $W$ . Then,  $\Sigma_{-i} := T(E^i(V) \otimes \text{str})$ , where  $\text{str}$  denotes the supertrace on  $\mathfrak{gl}(a|b)$  and the module it defines; clearly,  $\text{Vol} := T(\text{str})$ . Thus, locally,  $\Sigma_{-i}$  is the  $\text{Vol}$ -valued  $\mathcal{F}$ -module of polynomials in  $\partial_{\hat{x}_i}$ ; no generalized functions are needed to describe integrable forms, cf. [78].

(i5) The analog of exterior differential  $d : \Sigma_{-1} \longrightarrow \text{Vol}$ . To find such an operator and *investigate if it is unique*, we had to consider the super analog of Veblen’s problem solved, as on cue, by Rudakov shortly before, thus giving an invariant characterization of the exterior differential, spaces on which it is defined, and pointing a way to find its possible superanalogs.

### 1.1.1. Veblen's problem and Rudakov's solution of it in the unary case

The exterior differential  $d$  (instrumental in the Stokes formula) is, as is well known, an invariant operator; in [60] it is proven that  $d$  is the *only* invariant unary differential operator between spaces  $T(V)$  of tensor fields – sections of vector bundles (over the  $n$ -dimensional manifold) whose fibers are irreducible  $\mathfrak{gl}(n)$ -modules  $V$  with a vacuum vector. Remarkably, among all spaces  $T(V)$

$$\text{only the ones with } \dim V < \infty, \text{ namely the spaces } T(E^i(\text{id}^*)) = \Omega^i, \quad (1)$$

where  $\text{id}$  is the identity (tautological)  $\mathfrak{gl}(n)$ -module, have a non-trivial submodule:  $\text{Ker } d$ .

In order to overview the options on supermanifolds, it was necessary to describe all differential operators  $T(V) \rightarrow T(W)$  invariant with respect to arbitrary (invertible) changes of coordinates – obtain a super version of Rudakov's result.

On the  $n$ -dimensional manifold (locally: a domain)  $M$ , let  $W$  and  $V_i$ , where  $i = 1, \dots, k$ , be irreducible  $\mathfrak{gl}(n)$ -modules. The problem

*Describe all  $k$ -linear differential operators*

$$D: T(V_1) \otimes_{\mathcal{F}} T(V_2) \otimes_{\mathcal{F}} \cdots \otimes_{\mathcal{F}} T(V_k) \rightarrow T(W) \quad (2)$$

*invariant with respect to any changes of coordinates on  $M$ .*

was raised in 1928 by O. Veblen; for a review, see [25] and an update [14]. For unary ( $k = 1$  in (2)) operators on manifolds, it was solved by Rudakov [60] as a part of another problem, namely, the description of continuous in the discrete (or, dual to it,  $x$ -adic) topology irreducible vacuum vector modules over the simple Lie algebra  $\mathfrak{vect}(n|0)$  of formal or polynomial vector fields.

**Aside:** An interesting and curious fact: the solution of the binary ( $k = 2$  in (2)) version of Veblen's problem on manifolds (no “super”), i.e., classification of invariant bilinear differential operators, due to Grozman [21], brings several analogs of the Poisson bracket on Lie **super**algebras close to simple. (Up to dualization and permutation of arguments, there are no other invariant bilinear operators.)

## 1.2. Integrations unconventional: either without Stokes formula, or for another reason

For a graphic and short description of  $\mathcal{C}$ -points in the non-super setting, see [52].

### 1.2.1. Preliminaries: Automorphisms that preserve parity and their odd parameters

All parity preserving automorphisms over the ground field  $\mathbb{K}$  of the Grassmann algebra  $\Lambda^*(n)$  on  $n$  generators  $\xi_i$  are of the form (see [19])

$$\xi_j \mapsto \sum_r \varphi_j^r \xi_r + \sum_{s \geq 1} \sum_{j_1 < \cdots < j_{2s+1}} \varphi_j^{j_1 \cdots j_{2s+1}} \xi_{j_1} \cdots \xi_{j_{2s+1}}, \quad \text{where } \varphi_j^r, \varphi_j^{j_1 \cdots j_{2s+1}} \in \mathbb{K}, \quad (3)$$

with invertible matrix  $(\varphi_j^r)$ . Clearly, such automorphisms constitute the algebraic group (or a Lie group if  $\text{char } \mathbb{K} = 0$ ) whose Lie algebra consists of the even elements of the Lie superalgebra  $\mathfrak{vect}(0|n) := \mathfrak{der } \Lambda^*(n)$  of superderivations of  $\Lambda^*(n)$ .

Let  $\mathcal{U} := (U, C^\infty(\mathcal{U}) := C^\infty(U) \otimes \Lambda^*(n))$ , where  $U \subset \mathbb{R}^m$  is a domain, be an  $m|n$ -dimensional *superdomain* – the local chart of a supermanifold. To account

for the odd parameters (the boxed terms in (4)) we have to consider the functor  $\mathcal{C} \mapsto \text{Aut}_{\mathcal{C}}^{\bar{0}}(C^\infty(\mathcal{U}) \otimes \mathcal{C})$  from the category of supercommutative superalgebras to the category of groups, sending every  $\mathcal{C}$  to the group of the parity preserving (as indicated by the superscript  $\bar{0}$ )  $\mathcal{C}$ -linear automorphisms of the form (here the  $u_i$  are the coordinates in  $U$ , and  $u := (u_1, \dots, u_m)$ )

$$\left\{ \begin{array}{l} \varphi^*(u_i) = \varphi_i^0(u) + \sum_{r \geq 1} \sum_{i_1 < \dots < i_{2r}} \varphi_i^{i_1 \dots i_{2r}}(u) \xi_{i_1} \dots \xi_{i_{2r}} \\ \quad + \boxed{\sum_{r \geq 1} \sum_{i_1 < \dots < i_{2r+1}} \varphi_i^{i_1 \dots i_{2r+1}}(u) \xi_{i_1} \dots \xi_{i_{2r+1}}} \\ \varphi^*(\xi_j) = \sum_{r \geq 0} \sum_{j_1 < \dots < j_{2r+1}} \psi_j^{j_1 \dots j_{2r+1}}(u) \xi_{j_1} \dots \xi_{j_{2r+1}} \\ \quad + \boxed{\psi_i^0(u) + \sum_{r \geq 1} \sum_{j_1 < \dots < j_{2r}} \psi_j^{j_1 \dots j_{2r}}(u) \xi_{j_1} \dots \xi_{j_{2r}}} \end{array} \right. \quad (4)$$

where for all  $r$ , the elements

$$\varphi_i^0(u), \varphi_i^{i_1 \dots i_{2r}}(u), \psi_j^{j_1 \dots j_{2r+1}}(u) \in \mathcal{C}_{\bar{0}} \text{ and } \psi_i^0(u), \psi_j^{j_1 \dots j_{2r}}(u), \varphi_i^{i_1 \dots i_{2r+1}}(u) \in \mathcal{C}_{\bar{1}}$$

are the parameters of the infinite-dimensional supergroup of automorphisms of  $C^\infty(\mathcal{U})$  or, equivalently, of diffeomorphisms of the superdomain  $\mathcal{U}$ ; infinitesimally, these are parameters of the Lie superalgebra  $\mathfrak{vect}(m|n)$ . The automorphisms (4) are called  $\mathcal{C}$ -points of the supergroup of diffeomorphisms of  $\mathcal{U}$ ; the parity-preserving automorphisms correspond to the underlying group of this supergroup.

### 1.2.2. Automorphisms that do not preserve parity

In 1973 or so, I've astonished Berezin informing him that there are automorphisms not preserving parity of the Grassmann algebra  $\Lambda^\bullet(n) := \mathbb{C}[\xi]$ , where  $\xi = (\xi_1, \dots, \xi_n)$ , considered as an associative algebra, cf. [3], where all automorphisms are supposed to preserve parity. The first complete description of all automorphisms of  $\Lambda^\bullet(n)$  not preserving parity of the Grassmann algebra over a field of any characteristic is due to Djokovič (see [19]).

The group  $\text{Aut}(\Lambda^\bullet(n))$  of all automorphisms of  $\Lambda^\bullet(n)$  is just a bit larger than the group  $\text{Aut}^{\bar{0}}(\Lambda^\bullet(n))$  of parity preserving automorphisms: the general automorphism  $\varphi$  of  $\Lambda^\bullet(n)$  is of the form

$$\varphi(\xi_i) = \varphi_0(\xi_i)(1 + \psi(\xi)) \quad \text{for all } i \text{ and some } \psi(\xi) \in \Lambda^\bullet(n)_{\bar{1}}, \quad (5)$$

where  $\varphi_0$  is a parity preserving automorphism.

More generally, for the classification of non-preserving parity automorphisms of  $\mathbb{C}[x, \xi]$ , where  $x = (x_1, \dots, x_m)$ , see [45, 46] and for details, see [2]. The meaning of such general automorphisms is unclear at the moment, see [43].

### 1.2.3. Towards conjectural interpretations

By analogy with supersymmetries – which are wider than symmetries carried by groups or Lie algebras – these general automorphisms that do not preserve parity widen supersymmetries still further. More precisely, the even functions are invariantly defined (bosons do not depend on the coordinate systems), whereas to say

that a function is odd (corresponding to a fermion) is the same as to select a coordinate system modulo the subspace of even functions. *The notion of Fermion is not invariant.*

Recently, U. Iyer proved that the non-homogeneous with respect to parity subalgebras of Lie superalgebras, called *Volichenko algebras*, play the role of Lie algebras for the groups of  $\mathcal{C}$ -points of such general automorphisms of supercommutative superalgebras, see [28].

**Theorem 1.1.** *For  $n = 2k$ , the Berezin integral is invariant with respect to the general changes of coordinates (5).*

**Proof.** The integral is equal to the coefficient of the highest term of the integrand and Theorem follows since

$$\prod \varphi(\xi_i) = \prod \varphi_0(\xi_i). \quad \blacksquare$$

### 1.3. Miscellanea

#### 1.3.1. Non-trivial densities on supermanifolds with an odd non-degenerate differential 2-form (periplectic form)

For these, see H. Khudaverdian's papers [36, 37] and [51, Ch.4, §7].

#### 1.3.2. No invariant differential operator depends on higher jets of diffeomorphisms

Rudakov proved that there are no (non-scalar) invariant differential operators between spaces of objects depending on higher jets of the diffeomorphism (changes of coordinates); the proof of this fact constitutes the bulk of [60].

A. Schwarz (a.k.a. Shwarts and Šwarc) et al. attempted to integrate densities and objects depending on higher jets of the diffeomorphism, but all their non-trivial examples boil down to either pseudo-differential ([5]) or integrable forms (depending only on the 1-jets of diffeomorphisms). These *ad hoc* attempts suggest the following

#### 1.3.3. Problem

Prove that on supermanifolds, same as on manifolds, there are no invariant differential operators depending on higher jets of diffeomorphisms.

Observe that no *direct* superization of Rudakov's proof of the above problem on manifolds is visible; it seems, one needs totally new ideas.

#### 1.3.4. Analogs of the Berezin integral of tensors constituting irreducible infinite-dimensional fibers

Having obtained an analog of Rudakov's result for the general vectorial Lie superalgebra  $\mathbf{vect}(a|b)$ , see [7], we can be sure that

$$\begin{aligned} &\text{there is only one integration theory on supermanifolds} \\ &\text{provided the integration involves only tensors} \\ &\text{with irreducible finite-dimensional spaces.} \end{aligned} \quad (6)$$

The result of [7] does not preclude, however, unconventional integrations with infinite-dimensional fibers. In [25] the calculations from [7] are used to classify invariant differential operators between sections with infinite-dimensional fibers snubbed at in [7] for no reason except tradition: in the non-super case, the invariant operator exists only if fibers of tensor fields are finite-dimensional, see eq. (1).

Having stated that it is impossible to integrate differential forms on supermanifolds (functions polynomial in differentials of coordinates) in [6], Bernstein and I showed in [5] how to integrate *pseudo*-differential forms, i.e., functions *smooth* in differentials of coordinates:  $\hat{x} := dx$ . Observe that on supermanifolds of superdimension  $(a|b)$ , the space of pseudo-differential forms of a given degree  $\lambda \in \mathbb{K}$  is a module of finite rank over the algebra  $\mathcal{F}$  of functions only for  $b = 0$  or  $1$ .

In [9, Ch. 6], Shander singled out several interesting types of pseudo-differential forms. For example, one can consider *homogeneous* forms, i.e., such that

$$f(x, t\hat{x}) = t^a f(x, \hat{x}) \quad \text{for any } t \in \mathbb{R} \text{ and some } a \in \mathbb{R};$$

or *forms with compact support in  $\hat{x}$* , where  $x_i$  is any odd indeterminates; or *forms rapidly decreasing as  $\hat{x}_i \rightarrow \infty$*  for all odd  $x_i$ , etc. For example, on a  $1|n$ -dimensional supermanifold with even coordinates  $x$  and an odd coordinate  $\theta$ , a *semi-infinite pseudo-differential* form of degree  $a + b = \lambda \in \mathbb{C}$  is of the shape

$$f_{a,b}(x)\hat{x}_1^a\hat{x}_2^b + \sum_j f_{i,j}(x)\hat{x}_i^{a_i}\hat{x}_j^{a_j-1}\hat{x}_j + \sum_{j,k} f_{\lambda,j,k}(x)\hat{\theta}^{\lambda-1}\hat{x}_j\hat{x}_k + \dots$$

### 1.3.5. Problem

Compute the *semi-infinite cohomology of supermanifolds* related with semi-infinite pseudo-differential forms.

No examples of absolutely new invariants like these semi-infinite cohomology are calculated yet, except for examples due to A. Zorich (see [82], and a general discussion in [77]). For Shander's clarification of bigraded differential forms introduced in [77] and *how to integrate them with the help of odd parameters*, see [9, Ch. 6].

Observe that several authors published some computations of bigraded cohomology on supermanifolds, but these papers contain irreparable mistakes as explained, e.g., in [35].

### 1.4. A version of the Stokes formula with codimension $(0|-1)$ "over-supermanifold"

Around 1986, Shander wrote ca 80 pages elucidating 2+2 pages of [6, 5]. Shander's results were published ca 25 years later and so far only in Russian, see [9, Ch. 6]. There, Shander demonstrated the importance of odd parameters (see also [41]) and implicitly suggested a version of the Stokes formula with "oversupermanifold" of codimension  $(0|-1)$ .

In 1996, Palamodov published a note [58], where he gave an example of the Stokes formula in which the "boundary" of the  $0|0$ -dimensional point is a  $0|1$ -dimensional line.

#### 1.4.1. Problem

Develop integration via the ideas due to Shander and Palamodov.

### 1.5. Supertraces and superdeterminants

Since 1975, I wondered what if we stop insisting on having an analog of the Stokes formula? What remains of the paraphernalia needed for integration then?

The superdeterminant of the Jacobian, i.e., of the matrix  $\left(\frac{\partial y}{\partial x}\right)$  of the change of local coordinates on the supermanifold  $\mathcal{M}$  will definitely remain. Since it is easier to deal with Lie superalgebras than with supergroups, let me list analogs of trace for Lie superalgebras. Then, the Lie superalgebra  $\mathfrak{g}$  can be exponentiated to a Lie supergroup, and we can consider the super determinant defined (on the level of points over a supercommutative superalgebra  $\mathcal{C}$ ) via the formula

$$\det \exp X = e^{\text{tr}(X)} \quad \text{for any } X \in (\mathfrak{g} \otimes \mathcal{C})_{\bar{0}}. \quad (7)$$

The same procedure can be applied to any  $\mathbb{Z}$ -graded Lie superalgebra  $\mathfrak{g}$  of vector fields on whose component  $\mathfrak{g}_0$  a trace is defined. Let us consider examples.

(A) Let  $\mathcal{M}$  be a supermanifold with the fiber of the tangent bundle isomorphic to the space  $V$ ; let  $\mathfrak{g}$  be a Lie subsuperalgebra of  $\mathfrak{gl}(V)$  with its own trace  $\text{tr}$  (which by definition is *any linear function on  $\mathfrak{g}$  such that  $\text{tr}([x, y]) = 0$  for any  $x, y \in \mathfrak{g}$* ; in the super setting, this trace is often called *supertrace* for emphasis). Then, for the role of  $\text{Vol}$  over  $\mathcal{M}$ , we can take tensor fields of type  $\text{tr}$ , i.e., take sections of rank 1 module given by  $\text{tr}$  over the algebra  $\mathcal{F}$  of functions on  $\mathcal{M}$ ; the change of variables acts on  $\text{Vol}$  via the Lie derivative. The Lie superalgebra of infinitesimal transformations of this  $\text{Vol}$  is the *Cartan prolongation* (for its definitions, see [10, 25, 71]) of the pair  $(V, \mathfrak{g})$ , where  $V$  – the fiber of the tangent bundle on  $\mathcal{M}$  – is the tautological module over  $\mathfrak{g} \subset \mathfrak{gl}(V)$ .

The prime example of the Lie superalgebra with trace analogous to  $\mathfrak{gl}(V)$  is provided by the Poisson Lie superalgebra  $\mathfrak{g} = \mathfrak{po}(0|2n)$ . Indeed, there is a parametric family (deformation, physicists called it *quantization*) of Lie superalgebras  $\mathfrak{g}_t$  which at  $t = 0$  coincides with  $\mathfrak{po}(0|2n)$  and  $\mathfrak{g}_t \simeq \mathfrak{gl}(2^{n-1}|2^{n-1})$  for any  $t \neq 0$ , see [48]. Under quantization the trace given on  $\mathfrak{po}(0|2n)$  by the Berezin integral turns into the supertrace on  $\mathfrak{gl}(2^{n-1}|2^{n-1})$ .

(B) From various points of view  $\mathfrak{gl}(n)$  has at least two superanalogs: the “simple-minded” one,  $\mathfrak{gl}(n|m)$ , and the “queer” one,  $\mathfrak{q}(n)$  which corresponds to the case of the Schur lemma with irreducible representations that have no invariant subspaces, but have an invariant subspace. It is easy to show that over an algebraically closed field  $\mathbb{K}$  this subspace in the superspace  $V$  is invariant under an odd operator  $J$  such that  $J^2 = \lambda \text{id}$ , where  $\lambda \in \mathbb{K}^\times$  and  $\text{sdim } V = n|n$ . If the characteristic of  $\mathbb{K}$  is  $\neq 2$ , it is possible to selected a basis in  $V$  so that  $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$  we can realize the supermatrices of  $\mathfrak{q}(n) := \{X \in \mathfrak{gl}(n|n) \mid [X, J] = 0\}$  in the form  $(A, B) := \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ , where  $A, B \in \mathfrak{gl}(n)$ . On  $\mathfrak{q}(n)$ , the supertrace vanishes identically, but there is the *queer trace*  $\text{qtr} : (A, B) \mapsto \text{tr } B$ . On the group  $GQ(n; \mathcal{C})$  of invertible matrices of the form  $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ , where  $A \in \mathfrak{gl}(n; \mathcal{C}_{\bar{0}})$  and  $B \in \mathfrak{gl}(n; \mathcal{C}_{\bar{1}})$ , where  $\mathcal{C}$  is a supercommutative superalgebra, the *queer determinant* is defined by the formula

$$\text{qet}(A, B) := 1 + \tau \sum_{i \geq 1} \frac{\text{tr}((A^{-1}B)^{2i-1})}{2i-1} \in \mathcal{C}[\tau], \text{ where } p(\tau) = \bar{1}.$$

The queer trace is odd, and therefore *to describe the representation corresponding to it, and the queer determinant qet, we need an odd parameter  $\tau$* , see [9, Ch. 1]; compare with the first definitions in [8].

(C) Still other traces are related with the “quasiclassical limit” of str and qtr as  $t \rightarrow 0$ : the restriction of the above quantization is a parametric family  $\mathfrak{g}_t$  which at  $t = 0$  coincides with  $\mathfrak{po}(0|2n - 1)$  and  $\mathfrak{g}_t \simeq \mathfrak{q}(2^{n-1})$  for  $t \neq 0$ , see [48].

Under quantization the trace given on  $\mathfrak{po}(0|n)$  by the Berezin integral turns into qtr on  $\mathfrak{q}(2^{k-1})$  for  $n = 2k - 1$ , and into str on  $\mathfrak{gl}(2^{k-1}|2^{k-1})$  for  $n = 2k$ .

**1.6. Integration on 1|1-dimensional superstrings with contact structure**

Consider the form  $\alpha := dt + \tau d\tau$  on real or complex supermanifold  $\mathcal{M} = (M, \mathcal{O})$  of dimension 1|1 with coordinates  $t$  (even) and  $\tau$  (odd). Let  $\mathfrak{k}^L(1|1)$  be the Lie superalgebra of contact vector fields preserving the distribution singled out by  $\alpha$ ; the superscript  $L$  is for “Laurent” since  $\mathfrak{k}^L(1|1)$  can be considered as a subsuperalgebra in the Lie superalgebra of vector fields with Laurent coefficients (in  $t$ ). All tensor bundles on the spaces of whose sections the Lie superalgebra  $\mathfrak{k}^L(1|1)$  acts are glued of the simplest ones – linear (of rank  $1 = 1|0$  or  $0|1$ ) which are labelled by their “conformal weight” – the eigenvalue of the generator under the action of the vector field  $2t\partial_t + \tau\partial_\tau$ . The space of differential 1-forms does not have  $\mathfrak{k}^L(1|1)$ -invariant basis over the algebra of functions  $\mathcal{F}$ , only the subspace  $\mathcal{F}\alpha$  is invariant; the quotient modulo  $\mathcal{F}\alpha$  is isomorphic to the space of volume forms, i.e.,  $[d\tau] = \text{vol}(t, \tau)$ . Hence, we can integrate not only functions, but also 1-forms (with compact support):

$$\int_{\mathcal{M}} (f dt + g d\tau) = \int_{\mathcal{M}} (f(dt + \tau d\tau - \tau d\tau) + g d\tau) := \tag{8}$$

$$\int_{\mathcal{M}} (g - f\tau)[d\tau] = \int_{\mathcal{M}} (g - f\tau)\text{vol}(t, \tau) = \int_M (g_1 - f_0)dt, \tag{9}$$

for any functions  $f = f_0(t) + f_1(t)\tau$  and  $g = g_0(t) + g_1(t)\tau$ . This construction, although is based on Berezin’s integration, is another integration, existing in parallel with Berezin’s. It is used in Levin’s theory of elliptic functions on supermanifolds; for an exposition (in which the above-described mechanism is not explained since it was only a tool, not the object), see [53].

**1.6.1. Problem**

Construct explicitly (by giving formulas for the operators) the spinor representation (see [49]) of  $\mathfrak{k}^L(1|1)$  related with the super anti-symmetric non-degenerate bilinear form on  $(\mathcal{F} \otimes \mathcal{C})/\mathcal{C} \cdot 1$  (cf. [44]) given by

$$(f, g) := \int_{\mathcal{M}} f dg := \int_{\mathcal{M}} f(dt\partial_t g + d\tau\partial_\tau g) \tag{10}$$

$$= \int_{\mathcal{M}} (-1)^{p(f)}(dt + \tau d\tau - \tau d\tau)f\partial_t g + d\tau f\partial_\tau g \tag{11}$$

$$= \int_M (f_1\partial_\tau g - (-1)^{p(f)}\tau f\partial_t g)\text{vol}(t, \tau) = \int_M (f_1g_1 - (-1)^{p(g)}f_0\partial_t g_0)dt. \tag{12}$$

**1.7. TKK construction**

In the 1960s, a remarkable correspondence between Jordan algebras and certain  $\mathbb{Z}$ -graded Lie algebras became explicit, cf. [32]–[33], [38], and [73]. Kantor used this correspondence to list simple Jordan algebras (over  $\mathbb{C}$  and  $\mathbb{R}$ ) by the, so far, simplest known method. He clarified the mysterious relation of Jordan algebras with classical domains and actively studied certain generalizations of Jordan algebras associated

with  $\mathbb{Z}$ -graded simple Lie algebras  $\mathfrak{g} = \bigoplus_{-d \leq i \leq h} \mathfrak{g}_i$  of finite depth  $d$ . (Since all finite-dimensional simple Lie algebras are “symmetric”,  $\mathfrak{g}_i = (\mathfrak{g}_{-i})^*$ , their *height*  $h$  is equal to  $d$ .) The Lie superalgebras of supergravity and supertwistors also have symmetric  $\mathbb{Z}$ -gradings but of depth  $d > 1$ , see [51, 22]; so it is interesting to find generalizations of Jordan algebras (or, rather, any useful structures) related with such gradings.

Following Freudenthal and Springer, Kantor generalized products (17) to several arguments; this is natural for  $d > 1$ .

I suggest, contrariwise, to stick to formula (17), even for  $d > 1$ , with the minimal modification: fix  $j \in \mathfrak{g}_1$  and for  $\mathfrak{g}_- = \bigoplus_{i \leq 0} \mathfrak{g}_i$  set

$$x \circ y = [[j, x], y] \quad \text{for any } x, y \in \mathfrak{g}_-. \quad (13)$$

In this way, we obtain non-commutative generalizations of Jordan algebras. Superization is immediate: via the Sign Rule.

### 1.7.1. Problem

What are the relations additional to (18) (and commutativity or replacing it) these generalizations satisfy? What type of integrable systems are associated with them in the Sokolov-Svinolupov approach, cf. [26, 72]?

Tables in §3 (borrowed from [47] and completed with an exceptional example discovered by Shchepochkina) provide us with a list of simple Jordan superalgebras associated with the known simple  $\mathbb{Z}$ -graded Lie superalgebras of polynomial growth (SZGLSAPGs for short), cf. [16, 29, 30, 31, 48], namely, with  $\mathbb{Z}$ -gradings of depth 1 of SZGLSAPGs *including* finite-dimensional ones.

### 1.7.2. The Tits-Kantor-Köcher functor $\mathfrak{k}\mathfrak{a}\mathfrak{n}$

Let  $J$  be a Jordan algebra and  $j$  the tensor that determines the product in  $J$ , i.e.,  $j(x, y) := x \circ y$ . To  $J$ , Kantor assigned (see [29] with its epigraph “There are no Jordan algebras, there are only Lie algebras. I. Kantor”) a  $\mathbb{Z}$ -graded Lie algebra  $\mathfrak{k}\mathfrak{a}\mathfrak{n}(J) := \bigoplus_{|i| \leq 1} \mathfrak{k}\mathfrak{a}\mathfrak{n}(J)_i$ , a  $\mathbb{Z}$ -graded Lie subalgebra in the Lie algebra  $\mathfrak{vect}(J)$  of vector fields on  $J$  in the standard  $\mathbb{Z}$ -grading ( $\deg x_i = 1$ ) such that

$$\begin{aligned} \mathfrak{k}\mathfrak{a}\mathfrak{n}(J)_{-1} &:= \mathfrak{vect}(J)_{-1}, \\ \mathfrak{k}\mathfrak{a}\mathfrak{n}(J)_0 &:= \text{Span}(L_a, [L_a, L_b] \mid a, b \in J), \quad \text{where } L_a(x) := a \circ x, \\ \mathfrak{k}\mathfrak{a}\mathfrak{n}(J)_1 &:= \text{Span}(j, [L_a, j] \mid a \in J). \end{aligned}$$

There are versions of this construction due to Tits and Köcher which yield the same Lie algebra when the constructions are applied to a simple finite-dimensional Jordan algebra. But, when these constructions are applied to Jordan *superalgebras*, they can produce non-isomorphic Lie superalgebras, as demonstrated in [1]. However, Kantor’s idea was that Jordan algebras are determined by Lie algebras, so this result in [1] is opposite Kantor’s preferred direction of study.

### 1.7.3. Digression: on homological fields

I do not know what structure is related with an arbitrary odd element  $L \in \mathfrak{g}_1$  of the  $\mathbb{Z}$ -graded Lie superalgebra  $\mathfrak{g} = \bigoplus_{i \geq -1} \mathfrak{g}_i$ , but if  $L$  is *homological*, i.e.,  $[L, L] = 2L^2 = 0$ , then the formula

$$[x, y]' := [[L, x], y] \quad (14)$$

determines a Lie superalgebra structure on  $\Pi(\mathfrak{g}_{-1})$ . This structure had been first noticed, perhaps, by M. Gerstenhaber in the 1960s and rediscovered many times since then. It is interesting to describe in intrinsic terms the  $L$ 's which determine *simple* Lie (super)algebras.

*Non-degenerate* homological vector fields were first characterized, in connection with the problem of integration of differential equations on supermanifolds, by V. Shander, see [67], who gave their normal forms.

Shander did not consider fields with singularities; Vaintrob did: see, e.g., [75]. Vaintrob and Serganova showed that the study of singularities of homological fields, and their classification, turns out to be rather similar to the case of singularities of smooth functions, see [65]. Regrettably, the answer describing the elements  $L$  corresponding to the simple Lie (super)algebras is more trivial than expected, as I have established with Grozman:

Let, first,  $\mathfrak{g} = \mathfrak{gl}(n)$ ; denote the matrix units by  $\partial_i^j$ ; let  $(\partial_i^j)^* = x_j^i$  be the dual basis. Then,  $L = \sum \xi_j^i \xi_k^j \frac{\partial}{\partial \xi_k^i}$ , where  $\xi_j^i$  is the odd copy of  $x_j^i$ , is homological. Similarly, if the  $X_i$  form a basis of  $\mathfrak{g}$  and  $[X_i, X_j] = \sum c_{ij}^k X_k$ , then the operator  $L \in \mathbf{vect}(0 | \dim \mathfrak{g})$  of the form  $\sum c_{ij}^k X_i^* X_j^* X_k$ , where  $X_i^*$  is the dual of  $X_i$ , is homological.

Having observed that every simple finite-dimensional Lie algebra (over  $\mathbb{C}$ ) has a non-degenerate invariant symmetric bilinear form (NIS), we see that, for any such Lie algebra  $\mathfrak{g}$ , the vector field  $L$  is hamiltonian; its generating function is the sum of all elements of degree 1 and weight 0 with respect to the maximal torus of the Poisson Lie superalgebra  $\mathfrak{po}(0 | \dim \mathfrak{g})$ , this torus is the Cartan subalgebra of the 0th component isomorphic to  $\mathfrak{o}(n)$ .

Generalization to Lie superalgebras is straightforward. Still, observe that some simple Lie superalgebras have no NIS at all, some (e.g.,  $\mathfrak{q}(n)$ ) have an *odd* NIS, in which case  $L$  belongs to the Buttin superalgebra (with antibracket a.k.a. the Schouten bracket), not to the Poisson one. (For the classification of NISes on finite-dimensional Lie (super)algebras, and infinite-dimensional Lie (super)algebras of polynomial growth, see [12].)

### 1.8. SZGLSAPGs of depth 1 and height 1

All possible  $\mathbb{Z}$ -gradings of SZGLSAPGs  $\mathfrak{g}$  are listed in [29] for  $\dim \mathfrak{g} < \infty$  and in [40] for the other cases known in 1982. Our job is to pick those of them which are of depth 1, in particular, of the form  $\bigoplus_{|i| \leq 1} \mathfrak{g}_i$ , see Tables in §3.

Observe that Albert's notation for Jordan algebras were given in accordance with Cartan's notations for the corresponding Lie algebras. As follows from Serganova's classification of systems of simple roots of simple Lie superalgebras [63], Cartan's notations are highly inappropriate for Lie superalgebras (hence, for Jordan superalgebras); for more arguments, see [17].

## 2. Cross ratios

Here, I consider certain algebraic structures associated with several selected "classical superdomains". So far, nobody knows yet (as far as I know) what should be considered as such a superdomain, except for the case of finite-dimensional manifolds when all is clear. For supermanifolds, we just consider the most easy to handle

simple Lie supergroups and their Lie superalgebras whereas the elusive “right” definition requires, perhaps, “semisimple” or “almost simple” Lie superalgebras (for their definition, see [10]). So we take the road of least resistance:

The ground field is  $\mathbb{C}$ , the classical superdomains are considered as quotients of *simple* or close to them “quasi-reductive” Lie supergroups (see [64]) modulo *certain* maximal parabolic subsupergroups; for their list, see Tables. Consider the following questions:

- (Q1) What are the criteria for selecting *certain* sub(super)groups among other maximal ones in the ambient supergroup?
- (Q2) Why cosets modulo other parabolic subsupergroups are seldom considered in Differential Geometry? Observe that precisely these “other” cosets are the main topic of study in analytical mechanics of non-holonomic dynamical systems and in supergravity, see [51, Ch. 5], [11, 22].

In these questions, “super” is beside the point, so we can very well begin with manifolds. The classical domains are distinguished among symmetric spaces by the fact that the Lie algebra of the symmetry group of any classical domain (Hermitian symmetric space)  $M$  is a simple complex Lie algebra of the form

$$\mathfrak{g} = \bigoplus_{|i| \leq 1} \mathfrak{g}_i; \quad (16)$$

the tangent space to  $M$  at a fixed point can be identified with the  $\mathfrak{g}_{-1}$  and, on it, one can always define a Jordan algebra structure, by fixing any element  $j \in \mathfrak{g}_1$  and setting

$$x \circ y := [[j, x], y] \quad \text{for any } x, y \in \mathfrak{g}_{-1}. \quad (17)$$

The Jordan multiplication on the prime example – the space of  $n \times n$ -matrices is no less natural than the Lie multiplication, but the identity (18) it satisfies is less known than the Jacobi identity. (No wonder: it has no graphic interpretation, unlike Jacobi identity which is just the Leibniz rule for the adjoint action  $\text{ad}$  or the bracket.) Recall that a *Jordan algebra* is a commutative algebra  $J$  with product  $\circ$  satisfying, instead of associativity, the identity

$$(x^2 \circ y) \circ x = x^2 \circ (y \circ x). \quad (18)$$

In the very inspiring paper and book [54, 55], McCrimmon gave an account of some applications of Jordan algebras from antiquity to nowadays. All numerous literature known to me ([38, 54, 74] and references therein) strengthen my prejudice that *general* Jordan algebras are not very useful. In the contrary, *simple* Jordan algebras are related to several notions important in various problems. For simple Jordan algebras, their *generic norm* (see [34]) should be non-degenerate and this imposes additional constraints on the parabolic subalgebra. This answers the above questions (15), but tempts one to make use of the other coset spaces spoken about in (Q2), as well.

I wish to make similar use of simple Jordan *superalgebras*, especially infinite-dimensional ones, associated with infinite-dimensional superdomains listed in [47] and Tables §3.

In [34], Kantor generalized the cross ratio of four points from a function on  $\mathbb{P}^1$  to a function on certain quotients  $G/P$ , where  $G$  is a simple Lie group and  $P$  is its parabolic subgroup. The Lie algebra  $\mathfrak{g} := \text{Lie}(G)$  in these cases is of the form  $\mathfrak{g} = \bigoplus_{|i| \leq d} \mathfrak{g}_i$ .

Kantor's construction of cross-ratio naturally appears in a super setting [53]; this prompts me to try to decipher a part of [34]; see also [50, 61].

I will consider here the simplest case, the one where  $d = 1$ , and the corresponding Jordan (super)algebra is simple. In this case, one can generalize the cross ratio from  $\mathbb{P}^1 = \mathbf{Gr}_1^2$  to a collection of  $\mathrm{GL}(2m|2n)$ -invariants of four points on  $\mathbf{Gr}_{m|n}^{2m|2n}$ . Kantor defined the cross ratio as (here  $J = \mathrm{Mat}(n)$ )

$$(A, B, C, D) := (A - B)(C - B)^{-1}(C - D)(A - D)^{-1} \text{ for any } A, B, C, D \in J. \quad (19)$$

If  $mn = 0$ , then for any  $A, B, C, D \in (J \otimes \mathcal{C})_{\bar{0}}$  and any supercommutative superalgebra  $\mathcal{C}$ , I suggest to take for the *cross ratio* the collection of *all* coefficients of *all* powers of  $\lambda$ :

$$\det((A, B, C, D) - \lambda E), \text{ where } E \text{ is the unit matrix.} \quad (20)$$

By dimension considerations these are all the invariants of four points for the general, orthogonal and Lagrangian Grassmannians, but not for the quadrics.

If  $mn \neq 0$ , take the Berezinian instead of  $\det$  in eq. (20). The amount of *polynomially independent* invariants (coefficients of the powers of  $\lambda$ ) is infinite, cf. [66], but if we consider rational dependence, which is natural in super setting, the coefficients of the first  $n + m$  powers of  $\lambda$  generate the algebra of invariants and this collection is a natural candidate for the cross ratio.

On  $QGr_n^m$  which Manin called  $\Pi$ -symmetric Grassmannian  $\Pi\mathbf{Gr}_{n|n}^{m|m}$ , see [51, Ch. 5, §6, no. 4], we should take the queerdeterminant,  $\mathrm{qet}$ , instead of  $\det$  in eq. (20); the collection of invariants thus obtained is finite, see [69].

For (super)Grassmannians associates with loop (super)algebras and curved super-Grassmannians, we have to consider matrix-valued functions. In these two types of cases, eq. (20) returns a collection of functions, rather than numbers.

## 2.1. Related results

Four-point functions in  $N = 2$  superconformal field theories were studied in [20]. In [79], the matrix cross ratio is applied to Riccati equation. In [80], classical domains related with loop algebras are considered. See also the beautiful paper [56]. Together with [53] these works provide a wide range for applications of the new cross ratios introduced above.

## 2.2. Problems

- (1) What are the analogs of cross ratio related to quadrics, (twisted) loops, and in stringy cases?
- (2) What are the cross ratios on the classical superspaces associated with the exceptional superdomains, see the two bottom lines in Table 2, and the infinite-dimensional exception related with the subalgebra  $\mathfrak{kas}^L \subset \mathfrak{k}^L(1|6)$ , discovered by Shchepochkina [70]? ( $\mathfrak{kas}^L$  and its version  $\mathfrak{kas}$  with polynomial coefficients are often called ‘‘CK’’ superalgebras in the literature.)

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### 3. Tables ([47, 62])

In Table 1, let  $G$  be a simple Lie supergroup,  $P$  its parabolic subsupergroup corresponding to several omitted generators of a Borel subalgebra (description of these generators can be found in [23]). We say that the homogeneous superspace  $G/P$  is of *depth*  $d$  and *height*  $l$  if such are the depth and height of  $\mathfrak{g} = \text{Lie}(G)$  in the  $\mathbb{Z}$ -grading compatible with that of  $\text{Lie}(P)$ .

Note that all superspaces of Table 1 have an hermitian structure (hence, are of depth 1) except for  $PeGr$  (no hermitian structure),  $PeQ$  (no hermitian structure, height 2),  $CGr_{0,k}^{0,n}$  and  $SCGr_{0,k}^{0,n}$  (no hermitian structure, heights  $n-k$  and  $n-k-1$ , respectively).

Let  $\mathfrak{s}(\mathfrak{g})$  be the traceless part of  $\mathfrak{g}$  and  $\mathfrak{p}(\mathfrak{g}) = \mathfrak{g}/\text{center}$ ; let  $\mathfrak{g}_\varphi^{(m)}$  be the stationary subalgebra of the loop algebra with values in  $\mathfrak{g}$  singled out by the degree  $m$  automorphism  $\varphi$  of  $\mathfrak{g}$ , cf. [14]. For  $\mathfrak{g}_\varphi^{(m)}$  with the  $\mathbb{Z}$ -grading of type  $(d=1)$ , the last column of Table 1 contains  $(\mathfrak{g}_\varphi^{(m)})_0$ ; the maps  $-\text{st}: X \mapsto -X^{st}$ , the map

$$\Pi: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \quad \text{and} \quad \delta_x: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & xb \\ xc & d \end{pmatrix};$$

the automorphism  $A$  of  $\mathfrak{po}$  is defined on monomials  $f(\theta)$  as  $\text{id}$  if  $\frac{\partial f}{\partial \theta_1} = 0$  and  $-\text{id}$  otherwise;  $\text{irr}(\dots)$  is any of the two irreducible components;  $LGr$  and  $OGr$  stand for the Lagrangian and orthogonal Grassmannian, respectively; the dual domain is endowed with an asterisk as a left superscript.

The curved superquadric  $CQ_{m-2,0}$  has two series of infinite-dimensional “stringy” counterparts with  $\mathfrak{h}(0|m)$  replaced with the centerless  $N$ -extended Neveu-Schwarz algebra  $\mathfrak{k}^L(1|N)$  or Ramond algebra  $\mathfrak{k}^M(1|N)$ , and one exceptional analog associated with the exceptional stringy Lie superalgebra  $\mathfrak{kas}$  discovered by Shchepochkina, see [24, 70].

Table 2: An interpretation of the superGrassmannians given line-wise: the superGrassmannian of  $p|q$ -dimensional subsuperspaces in  $\mathbb{C}^{m|n}$  and same for  $n=m$ ,  $p=q$ ; superquadric of  $1|0$ -dimensional isotropic (relative a non-degenerate even form) lines in  $\mathbb{C}^{m|n}$ ; ortholagrangian superGrassmannian; queerGrassmannian; “odd” superquadric (relative a non-degenerate even form) of  $1|0$ -lines in  $\mathbb{C}^{n|n}$ ; odd-lagrangian superGrassmannian; curved superGrassmannian of  $0|1$ -dimensional subsupermanifolds in  $\mathbb{C}^{0|n}$ ; curved superquadric; the two finite-dimensional exceptions. The exception  $\mathfrak{kas}^L \subset \mathfrak{k}^L(1|6)$  is not included in Table 2 for the lack of clear description of the superdomain, cf. [81].

Table 1: Gradings of twisted loop (super)algebras corresponding to hermitian superdomains

$\mathfrak{g}_\varphi^{(m)}$	$\varphi$	grading elements from $\mathfrak{h}$	$(\mathfrak{g}_\varphi^{(m)})_0$
$\mathfrak{sl}(2m 2n)^{(2)}$	$(-st) \circ \text{Ad}_{\text{diag}(\Pi_{2m}, J_{2n})}$	$\text{diag}(1_m, -1_m, 1_n, -1_n)$	$\mathfrak{sl}(m n)^{(1)}$
$\mathfrak{sl}(2m)^{(2)}$	$(-t) \circ \text{Ad}_{\Pi_{2m}}$	$\text{diag}(1_m, -1_m)$	$\mathfrak{sl}(m)^{(1)}$
$\mathfrak{sl}(2n)^{(2)}$	$(t) \circ \text{Ad}_{J_{2n}}$	$\text{diag}(1_n, -1_n)$	$\mathfrak{sl}(n)^{(1)}$
$\mathfrak{sl}(n n)^{(2)}$	$\Pi$	$\text{diag}(1_p, 0_{n-p}, 1_p, 0_{n-p})$	$\mathfrak{sl}(\mathfrak{gl}(p p)_\Pi^{(2)} \oplus \mathfrak{gl}(n-p n-p)_\Pi^{(2)})$
$\mathfrak{sl}(n n)^{(2)}$	$\Pi \circ (-st)$	$\text{diag}(1_p, -1_{n-p}, -1_p, 1_{n-p})$	$\mathfrak{sl}(\mathfrak{gl}(p p)_{\Pi \circ (-st)}^{(2)} \oplus \mathfrak{gl}(n-p n-p)_{\Pi \circ (-st)}^{(2)})$
$\mathfrak{osp}(2m 2n)^{(2)}$	$\varphi_{m,n} \text{Ad}_{\text{diag}(1_{2m-1}, 1, 1_{2n})}$	$\text{diag}(J_2, 0_{2(m+n-1)})$	$(\mathfrak{cosp}(2m-2 2n))_{\varphi_{m-1,n}}^{(1)}$
$\mathfrak{o}(2m)^{(2)}$			$(\mathfrak{co}(2m-2))^{(1)}$
$\mathfrak{psq}(2n)^{(4)}$	$(-st) \circ \delta_{\sqrt{-1}}$	$\text{diag}(J_{2n}, J_{2n})$	$\mathfrak{psq}(n)_{\delta_{-1}}^{(2)}$
$(\mathfrak{h}'(2n))^{(2)}$	$A$		$(\mathfrak{h}'(0 2n-2) \oplus \Lambda(2n-2))_A^{(2)}$
$\mathfrak{psq}(n)^{(2)}$	$\delta_{-1}$	$\text{diag}(1_p, 0_{n-p}, 1_p, 0_{n-p})$	$\mathfrak{ps}(\mathfrak{q}(p)_{\delta_{-1}}^{(2)} \oplus \mathfrak{q}(n-p)_{\delta_{-1}}^{(2)})$

Table 2: Classical superspaces of depth 1

$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$	Underlying domain	Name of the superdomain
$\mathfrak{sl}(m n)$	$\mathfrak{sl}(\mathfrak{gl}(p q) \oplus \mathfrak{gl}(m-p n-q))$	$\text{id} \boxtimes \text{id}^*$	$Gr_p^m \times Gr_q^n$	$Gr_{p,q}^{m,n}$
$\mathfrak{psl}(m m)$	$\mathfrak{psl}(\mathfrak{gl}(p p) \oplus \mathfrak{gl}(m-p m-q))$	$\text{id} \boxtimes \text{id}^*$	$Gr_p^m \times Gr_p^m$	$Gr_{p,p}^{m,m}$
$\mathfrak{osp}(m 2n)$	$\mathfrak{cosp}(m-2 2n)$	$\text{id}$	$Q_{m-2}$	$Q_{m-2,n}$
$\mathfrak{osp}(2m 2n)$	$\mathfrak{gl}(m n)$	$\Lambda^2(\text{id})$	$*OGr_m \times LGr_n$	$OLGr_{m,n}$
$\mathfrak{sq}(n)$	$\mathfrak{sl}(\mathfrak{q}(p) \oplus \mathfrak{q}(n-p))$	$\text{irr}(\text{id} \boxtimes \text{id}^*)$	$Gr_p^n$	$QGr_p^n$
$\mathfrak{psq}(n)$	$\mathfrak{psl}(\mathfrak{q}(p) \oplus \mathfrak{q}(n-p))$			
$\mathfrak{pe}(n)$	$\mathfrak{cpe}(n-1)$	$\text{id}$	$\mathbb{C}P^{n-1}$	$PeQ_{n-1}$
$\mathfrak{spe}(n)$	$\mathfrak{cspe}(n-1)$	$\text{id}$		
$\mathfrak{pe}(n)$ $(\mathfrak{spe}(n))$	$\mathfrak{gl}(p n-p)$ $(\mathfrak{sl}(p n-p))$	$\Pi(S^2(\text{id}))$ or $\Pi(\Lambda^2(\text{id}))$	$Gr_p^n$	$PeGr_p^n$
$\mathfrak{vect}(0 n)$	$\mathfrak{vect}(0 n-k) \oplus \mathfrak{gl}(k; \Lambda(n-k))$	$\Lambda(k) \boxtimes \Pi(\text{id})$	curved	$CGr_{0,k}^{0,n}$
$\mathfrak{svect}(0 n)$	$\mathfrak{vect}(0 n-k) \oplus \mathfrak{sl}(k; \Lambda(n-k))$	$\Pi(Vol)$ if $k=1$	curved	$SCGr_{0,k}^{0,n}$
$\mathfrak{h}(0 m)$	$\mathfrak{h}(0 m-2) \oplus \Lambda(m-2) \cdot z$	$\Pi(\text{id})$	curved	$CQ_{m-2,0}$
$\mathfrak{h}'(0 m)$	$\mathfrak{h}'(0 m-2) \oplus \Lambda(m-2) \cdot z$	$\Pi(\text{id})$	curved	
$\mathfrak{k}^L(1 n+2)$	$\mathfrak{k}^L(1 n) \oplus \mathcal{F}(1 n)\alpha\beta$	$\Pi(\mathbb{C}[t^{-1}, t, \theta_1, \dots, \theta_n; \alpha, \beta])$	curved	$K^L\mathbb{Q}_{1 n}$
$\mathfrak{k}^M(1 n+2)$ for $n > 0$	$\mathfrak{k}^M(1 n) \oplus \mathcal{F}(1 n)\alpha\beta$	$\Pi(\mathbb{C}[t^{-1}, t, \theta_1, \dots, \sqrt{t}\theta_n; \alpha, \beta])$	curved	$K^M\mathbb{Q}_{1 n}$
$\mathfrak{osp}(4 2; \alpha)$	$\mathfrak{cosp}(2 2) \simeq \mathfrak{gl}(2 1)$	$\text{id}$	$\mathbb{C}P^1 \times \mathbb{C}P^1$	$E_\alpha$
$\mathfrak{ab}(3)$	$\mathfrak{cosp}(2 4)$	$L_{3\epsilon_1}$	$\mathbb{C}P^1 \times Q_5$	$AB(3)$

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