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Cohomologies and Deformations of Pre-Lie-Morphism Triples

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Abstract. A (pre-)Lie-morphism triple consists of two (pre-)Lie algebras and a (pre-)Lie algebra homomorphism between them. We give cohomologies of pre-Lie-morphism triples and show that the cohomology of a pre-Lie-morphism triple can be deduced from a cohomology of the Lie-morphism triple. As applications of cohomologies of pre-Lie-morphism triples, we study linear deformations and formal deformations of pre-Lie-morphism triples.

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Key Words: Pre-Lie morphism triple, cohomology, deformation.

1. Introduction

Pre-Lie algebras are a class of nonassociative algebras coming from the study of convex homogeneous cones, affine manifolds and affine structures on Lie groups, and cohomologies of associative algebras. They also appeared in many fields of mathematics and mathematical physics, such as symplectic structures on Lie groups and Lie algebras, integrable systems, Poisson brackets and infinite dimensional Lie algebras, vertex algebras, quantum field theory, operads and numerical analysis. See the survey [4, 20] and the references therein for more details.

The cohomology theory is a classical approach associating invariants to a mathematical structure. Cohomology controls deformations and extension problems of the corresponding algebraic structures. Cohomology theories of various kinds of algebras have been developed and studied in [7, 15, 17, 18]. See the review paper [16] for more details. The cohomologies of Lie algebra homomorphisms were introduced by Nijenhuis and Richardson in [22]. Then the cohomology of two Lie algebras with a Lie algebra homomorphism between them were studied by Y. Frégier in [13], which can control simultaneous deformations of these two Lie algebras with the Lie algebra homomorphism. We call two Lie algebras with a Lie algebra homomorphism between them a Lie-morphism triple. See [9] for more details on the cohomology of a Lie-morphism triple with coefficients in a representation and its applications.

In this paper, following the idea given by Y. Frégier in [13], we study the cohomologies of pre-Lie-morphism triples. Here a pre-Lie-morphism triple consists of two pre-Lie algebras and a pre-Lie algebra homomorphism between them. Pre-Lie-morphism triples arise from Rota-Baxter operators, Nijenhuis operators, symplectic

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Lie algebra homomorphisms, $\mathfrak{s}$-matrices, Cayley maps and B-series. Based on the close relationship between the cohomologies of pre-Lie algebras and the cohomologies of Lie algebras, we find that the cohomology of a pre-Lie-morphism triple can be deduced from a new cohomology of the Lie-morphism triple. As applications of the cohomology of pre-Lie-morphism triples, we study linear deformations and formal deformations of pre-Lie-morphism triples. We show that equivalent linear deformations and infinitesimal formal deformations of pre-Lie-morphism triples are in the same cohomology class and an order n deformation of a pre-Lie-morphism triple is extendable if and only if its obstruction class in the second cohomology group is trivial. We also introduce the notion of a Nijenhuis pair on a pre-Lie-morphism triple and show that it can generate a trivial linear deformation. We show that compatible $\mathcal{O}$-operators on pre-Lie algebras can give rise to Nijenhuis pairs naturally. See [10, 23] for more details about Nijenhuis operators on pre-Lie algebras and Lie algebras.

The paper is organized as follows. In Section 2, we recall representations, cohomologies and homomorphisms of pre-Lie algebras. Then some examples of pre-Lie algebra homomorphisms are given. In Section 3, we give the cohomology of a pre-Lie-morphism triple, and study the relationship between the cohomologies of pre-Lie-morphism triples and the cohomologies of the sub-adjacent Lie-morphism triples. In Section 4, we use the cohomology of pre-Lie-morphism triples to study linear deformations and formal deformations of them. In particular, we introduce the notion of a Nijenhuis pair on the pre-Lie-morphism triple, which generates trivial linear deformations. We also construct Nijenhuis pairs from compatible $\mathcal{O}$-operators on pre-Lie algebras.

In this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0 and all the vector spaces are taken over $\mathbb{K}$. Unless otherwise noted, all vector spaces we consider will be finite dimensional.

2. Preliminaries

In this section, we recall representations, cohomologies and homomorphisms of pre-Lie algebras and give some examples.

Definition 2.1. A *pre-Lie algebra* is a pair $(\mathfrak{g}, \cdot_\mathfrak{g})$, where $\mathfrak{g}$ is a vector space and $\cdot_\mathfrak{g} : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ is a bilinear multiplication satisfying that for all $x, y, z \in \mathfrak{g}$, the associator $(x, y, z) = (x \cdot_\mathfrak{g} y) \cdot_\mathfrak{g} z - x \cdot_\mathfrak{g} (y \cdot_\mathfrak{g} z)$ is symmetric in x, y, i.e.

$$(x, y, z) = (y, x, z), \text{ or equivalently, } (x \cdot_\mathfrak{g} y) \cdot_\mathfrak{g} z - x \cdot_\mathfrak{g} (y \cdot_\mathfrak{g} z) = (y \cdot_\mathfrak{g} x) \cdot_\mathfrak{g} z - y \cdot_\mathfrak{g} (x \cdot_\mathfrak{g} z).$$

Let $(\mathfrak{g}, \cdot_\mathfrak{g})$ be a pre-Lie algebra. The commutator $[x, y]_\mathfrak{g} = x \cdot_\mathfrak{g} y - y \cdot_\mathfrak{g} x$ defines a Lie algebra structure on $\mathfrak{g}$, which is called the *sub-adjacent Lie algebra* of $(\mathfrak{g}, \cdot_\mathfrak{g})$ and denoted by $\mathfrak{g}^c$. Furthermore, $L : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ with $x \to L_x$, where $L_x y = x \cdot_\mathfrak{g} y$, for all $x, y \in \mathfrak{g}$, gives a representation of the Lie algebra $\mathfrak{g}^c$ on $\mathfrak{g}$. See [4] for more details.

Definition 2.2. Let $(\mathfrak{g}, \cdot_\mathfrak{g})$ be a pre-Lie algebra and V a vector space. A *representation* of $\mathfrak{g}$ on V consists of a pair (ρ, μ), where $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation of the Lie algebra $\mathfrak{g}^c$ on V and $\mu : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a linear map satisfying

$$\rho(x)\mu(y)u - \mu(y)\rho(x)u = \mu(x \cdot_\mathfrak{g} y)u - \mu(y)\mu(x)u, \quad \forall\, x, y \in \mathfrak{g},\ u \in V. \tag{1}$$

Usually, we denote a representation by $(V;\rho,\mu)$. It is obvious that $(\mathbb{K};\rho=0,\mu=0)$ is a representation, which is called the *trivial representation*. Let $R:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ be a linear map with $x\longrightarrow R_x$, where the linear map $R_x:\mathfrak{g}\longrightarrow\mathfrak{g}$ is defined by $R_x(y)=y\cdot_{\mathfrak{g}}x$, for all $x,y\in\mathfrak{g}$. Then $(\mathfrak{g};\rho=L,\mu=R)$ is also a representation, which is called the *regular representation*. Define two linear maps $L^*,R^*:\mathfrak{g}\longrightarrow\mathfrak{gl}(\mathfrak{g}^*)$ with $x\longrightarrow L^*_x$ and $x\longrightarrow R^*_x$ respectively (for all $x\in\mathfrak{g}$) by

$$\langle L^*_x(\xi),y\rangle=-\langle\xi,x\cdot y\rangle,\quad\langle R^*_x(\xi),y\rangle=-\langle\xi,y\cdot x\rangle,\quad\forall x,y\in\mathfrak{g},\xi\in\mathfrak{g}^*.\tag{2}$$

Then $(\mathfrak{g}^*;\rho=\mathrm{ad}^*=L^*-R^*,\mu=-R^*)$ is a representation of $(\mathfrak{g},\cdot_{\mathfrak{g}})$, which is called the *coregular representation*. In fact, it is the dual representation of the regular representation $(\mathfrak{g};L,R)$.

The cohomology complex for a pre-Lie algebra $(\mathfrak{g},\cdot_{\mathfrak{g}})$ with coefficients in a representation $(V;\rho,\mu)$ is given as follows ([4]). The cohomology of the pre-Lie algebra $\mathfrak{g}$ with coefficients in V is the cohomology of the cochain complex $(\otimes_{n\geq1}C^n_{\mathrm{preLie}}(\mathfrak{g},V),\mathrm{d})$, where $C^n_{\mathrm{preLie}}(\mathfrak{g},V)=\mathrm{Hom}(\wedge^{n-1}\mathfrak{g}\otimes\mathfrak{g},V)$ and the coboundary operator

$$\mathrm{d}:C^n_{\mathrm{preLie}}(\mathfrak{g},V)\longrightarrow C^{n+1}_{\mathrm{preLie}}(\mathfrak{g},V)$$

is given by

$$\begin{aligned}\mathrm{d}f(x_1,\cdots,x_{n+1})=&\sum_{i=1}^{n}(-1)^{i+1}\rho(x_i)f(x_1,\cdots,\hat{x_i},\cdots,x_{n+1})\\&+\sum_{i=1}^{n}(-1)^{i+1}\mu(x_{n+1})f(x_1,\cdots,\hat{x_i},\cdots,x_n,x_i)\\&-\sum_{i=1}^{n}(-1)^{i+1}f(x_1,\cdots,\hat{x_i},\cdots,x_n,x_i\cdot_{\mathfrak{g}}x_{n+1})\\&+\sum_{1\leq i<j\leq n}(-1)^{i+j}f([x_i,x_j]_{\mathfrak{g}},x_1,\cdots,\hat{x_i},\cdots,\hat{x_j},\cdots,x_{n+1}),\end{aligned}\tag{3}$$

for all $x_i\in\mathfrak{g},\ i=1,\cdots,n+1$. We denote the corresponding n-th cohomology group by $H^k_{\mathrm{preLie}}(\mathfrak{g},V)$.

Definition 2.3. Let $(\mathfrak{g},\cdot_{\mathfrak{g}})$ and $(\mathfrak{h},\cdot_{\mathfrak{h}})$ be two pre-Lie algebras. A *homomorphism* between $\mathfrak{g}$ and $\mathfrak{h}$ is a linear map $\phi:\mathfrak{g}\to\mathfrak{h}$ satisfying

$$\phi(x\cdot_{\mathfrak{g}}y)=\phi(x)\cdot_{\mathfrak{h}}\phi(y),\quad\forall\ x,y\in\mathfrak{g}.\tag{4}$$

Moreover, we call two pre-Lie algebras $(\mathfrak{g},\cdot_{\mathfrak{g}})$ and $(\mathfrak{h},\cdot_{\mathfrak{h}})$ with a pre-Lie algebra homomorphism ϕ a *pre-Lie-morphism triple*. We denote it by $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$, or simply $(\mathfrak{g},\mathfrak{h},\phi)$. ■

It is obvious that if ϕ is a homomorphism from a pre-Lie algebra $\mathfrak{g}$ to a pre-Lie algebra $\mathfrak{h}$, then ϕ is a homomorphism from the sub-adjacent Lie algebra $\mathfrak{g}^c$ to the sub-adjacent Lie algebra $\mathfrak{h}^c$. We call the Lie-morphism triple $(\mathfrak{g}^c,\mathfrak{h}^c,\phi)$ a *sub-adjacent Lie-morphism triple* of the pre-Lie-morphism triple $(\mathfrak{g},\mathfrak{h},\phi)$.

Let D be a derivation on a commutative associative algebra $(A,\cdot_A)$. By defining a product

$$x*_Dy=x\cdot_AD(y),\quad\forall\ x,y\in A,\tag{5}$$

we have a pre-Lie algebra $(A,*_D)$ which is given in [14].

Example 2.4. Let D_1 be a derivation on a commutative associative algebra $(A, \cdot_A)$ and D_2 a derivation on a commutative associative algebra $(B, \cdot_B)$. Assume that $f : A \to B$ is an associative algebra homomorphism satisfying $f \circ D_1 = D_2 \circ f$. Then f is a pre-Lie algebra homomorphism from the pre-Lie algebra $(A, *_{D_1})$ to the pre-Lie algebra $(B, *_{D_2})$. ■

Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ be a Lie algebra. Recall that $\omega \in \wedge^2\mathfrak{g}^*$ is a 2-cocycle on $\mathfrak{g}$ if

$$\omega([x,y]_\mathfrak{g}, z) + \omega([z,x]_\mathfrak{g}, y) + \omega([y,z]_\mathfrak{g}, x) = 0, \quad \forall\, x,y,z \in \mathfrak{g}. \tag{6}$$

A *symplectic Lie algebra*, denoted by $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \omega)$, is a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ together with a nondegenerate 2-cocycle $\omega \in \wedge^2\mathfrak{g}^*$ on $\mathfrak{g}$.

Lemma 2.5. ([8]) *Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \omega)$ be a symplectic Lie algebra. Then there exists a pre-Lie algebra structure "$\cdot_\mathfrak{g}$" on $\mathfrak{g}$ given by*

$$\omega(x \cdot_\mathfrak{g} y, z) = -\omega(y, [x,z]_\mathfrak{g}), \quad \forall\, x,y,z \in \mathfrak{g}, \tag{7}$$

such that the sub-adjacent Lie algebra is exactly $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ itself.

Example 2.6. Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \omega_1)$ and $(\mathfrak{h}, [\cdot,\cdot]_\mathfrak{h}, \omega_2)$ be two symplectic Lie algebras. Assume that $\phi : \mathfrak{g} \to \mathfrak{h}$ is a symplectic Lie algebra homomorphism, i.e.

$$\phi([x,y]_\mathfrak{g}) = [\phi(x), \phi(y)]_\mathfrak{h}, \quad \omega_2(\phi(x), \phi(y)) = \omega_1(x,y), \quad \forall\, x,y \in \mathfrak{g}.$$

Then ϕ is a pre-Lie algebra homomorphism from the pre-Lie algebra $(\mathfrak{g}, \cdot_\mathfrak{g})$ to the pre-Lie algebra $(\mathfrak{h}, \cdot_\mathfrak{h})$. ■

Definition 2.7. ([1]) Let $(\mathfrak{g}, \cdot_\mathfrak{g})$ be a pre-Lie algebra and $\mathcal{R} : \mathfrak{g} \longrightarrow \mathfrak{g}$ a linear operator. If $\mathcal{R}$ satisfies

$$\mathcal{R}(x) \cdot_\mathfrak{g} \mathcal{R}(y) = \mathcal{R}(\mathcal{R}(x) \cdot_\mathfrak{g} y + x \cdot_\mathfrak{g} \mathcal{R}(y) + \lambda x \cdot_\mathfrak{g} y), \quad \forall x,y \in \mathfrak{g}, \tag{8}$$

then $\mathcal{R}$ is called a *Rota-Baxter operator of weight λ* on $\mathfrak{g}$. ■

Example 2.8. Let $\mathcal{R}$ be a Rota-Baxter operator of weight λ on a pre-Lie algebra $(\mathfrak{g}, \cdot_\mathfrak{g})$. Define a new operation $\cdot^\mathcal{R} : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$x \cdot^\mathcal{R} y = \mathcal{R}(x) \cdot_\mathfrak{g} y + x \cdot_\mathfrak{g} \mathcal{R}(y) + \lambda x \cdot_\mathfrak{g} y, \ \forall x,y \in \mathfrak{g}.$$

Then $(\mathfrak{g}, \cdot^\mathcal{R})$ is a pre-Lie algebra and $\mathcal{R}$ is a pre-Lie algebra homomorphism from the pre-Lie algebra $(\mathfrak{g}, \cdot^\mathcal{R})$ to the pre-Lie algebra $(\mathfrak{g}, \cdot_\mathfrak{g})$. ■

Definition 2.9. ([23]) Let $(\mathfrak{g}, \cdot_\mathfrak{g})$ be a pre-Lie algebra. Then a linear operator $N : \mathfrak{g} \longrightarrow \mathfrak{g}$ is called a *Nijenhuis operator* if

$$N(x) \cdot_\mathfrak{g} N(y) = N\big(N(x) \cdot_\mathfrak{g} y + x \cdot_\mathfrak{g} N(y) - N(x \cdot_\mathfrak{g} y)\big), \ \forall x,y \in \mathfrak{g}. \tag{9}$$

Example 2.10. Let $(\mathfrak{g}, \cdot_\mathfrak{g})$ be a pre-Lie algebra and N a Nijenhuis operator on $\mathfrak{g}$. Define a new operation $\cdot_N : \otimes^2\mathfrak{g} \to \mathfrak{g}$ by

$$x \cdot_N y = N(x) \cdot_\mathfrak{g} y + x \cdot_\mathfrak{g} N(y) - N(x \cdot_\mathfrak{g} y), \ \forall x,y \in \mathfrak{g}. \tag{10}$$

Then $(\mathfrak{g}, \cdot_N)$ is a pre-Lie algebra and N is a pre-Lie algebra homomorphism from $(\mathfrak{g}, \cdot_N)$ to $(\mathfrak{g}, \cdot_\mathfrak{g})$. ■

Definition 2.11. ([3]) Let $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ be a pre-Lie algebra and $(V; \rho, \mu)$ a representation. A linear map $T : V \longrightarrow \mathfrak{g}$ is called an $\mathcal{O}$*-operator* on a pre-Lie algebra $\mathfrak{g}$ associated to a representation $(V; \rho, \mu)$ if it satisfies

$$T(u) \cdot_{\mathfrak{g}} T(v) = T\big(\rho(T(u))(v) + \mu(T(v))(u)\big), \quad \forall u, v \in V. \quad \blacksquare \tag{11}$$

Note that a Rota-Baxter operator of weight zero on a pre-Lie algebra $\mathfrak{g}$ is exactly an $\mathcal{O}$-operator associated to the regular representation $(\mathfrak{g}; L, R)$.

Example 2.12. Let T be an $\mathcal{O}$-operator on a pre-Lie algebra $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ associated to a representation $(V; \rho, \mu)$. Define a new operation $\cdot^T : \otimes^2\mathfrak{g} \to \mathfrak{g}$ by

$$x \cdot^T y = \rho(T(u))(v) + \mu(T(v))(u), \ \forall x, y \in \mathfrak{g}. \tag{12}$$

Then $(\mathfrak{g}, \cdot^T)$ is a pre-Lie algebra and T is a pre-Lie algebra homomorphism from $(\mathfrak{g}, \cdot^T)$ to $(\mathfrak{g}, \cdot_{\mathfrak{g}})$. $\blacksquare$

Let $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ be a pre-Lie algebra and $r \in \mathrm{Sym}^2(\mathfrak{g})$. We introduce $[\![r, r]\!] \in \wedge^2\mathfrak{g} \otimes \mathfrak{g}$ as follows for all $\xi, \eta, \zeta \in \mathfrak{g}^*$:

$$[\![r, r]\!](\xi, \eta, \zeta) = -\langle \xi, r^\sharp(\eta) \cdot_{\mathfrak{g}} r^\sharp(\zeta)\rangle + \langle \eta, r^\sharp(\xi) \cdot_{\mathfrak{g}} r^\sharp(\zeta)\rangle + \langle \zeta, [r^\sharp(\xi), r^\sharp(\eta)]^c\rangle, \tag{13}$$

where $r^\sharp : \mathfrak{g}^* \to \mathfrak{g}$ is given by $\langle r^\sharp(\xi), \eta\rangle = r(\xi, \eta)$, for all $\xi, \eta \in \mathfrak{g}^*$.

Definition 2.13. ([2]) Let $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ be a pre-Lie algebra. If $r \in \mathrm{Sym}^2(\mathfrak{g})$ and satisfies $[\![r, r]\!] = 0$, then r is called an $\mathfrak{s}$*-matrix* in $(\mathfrak{g}, \cdot_{\mathfrak{g}})$. $\blacksquare$

Remark 2.14. An $\mathfrak{s}$-matrix in a pre-Lie algebra $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ is a solution of the S-equation in $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ which is an analogue of the classical Yang-Baxter equation in a Lie algebra. It plays an important role in the theory of pre-Lie bialgebras, see [2] for more details. $\blacksquare$

It was shown in [2] that $r \in \mathrm{Sym}^2(\mathfrak{g})$ is an $\mathfrak{s}$-matrix in the pre-Lie algebra $\mathfrak{g}$ if and only if $r^\sharp : \mathfrak{g}^* \to \mathfrak{g}$ is an $\mathcal{O}$-operator associated to the coregular representation $(\mathfrak{g}^*; \mathrm{ad}^*, -R^*)$.

Example 2.15. Let $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ be a pre-Lie algebra and r an $\mathfrak{s}$-matrix. Then $(\mathfrak{g}^*, \cdot_r)$ is a pre-Lie algebra, where the multiplication $\cdot^r : \otimes^2\mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ is given by

$$\xi \cdot^r \eta = \mathrm{ad}^*_{r^\sharp(\xi)}\eta - R^*_{r^\sharp(\eta)}\xi, \quad \forall\ \xi, \eta \in \mathfrak{g}^*. \tag{14}$$

Furthermore, $r^\sharp : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ is a pre-Lie algebra homomorphism from $(\mathfrak{g}^*, \cdot^r)$ to $(\mathfrak{g}, \cdot_{\mathfrak{g}})$. $\blacksquare$

3. Cohomologies of pre-Lie-morphism triples

In this section, we give a cohomology of the pre-Lie-morphism triple, and show that the cohomology for a pre-Lie-morphism triple can be deduced from a new cohomology of the sub-adjacent Lie-morphism triple.

3.1. Cohomologies of pre-Lie-morphism triples

Let ϕ be a homomorphism between pre-Lie algebras $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ and $(\mathfrak{h}, \cdot_{\mathfrak{h}})$. Define $\rho_\phi : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ and $\mu_\phi : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ by

$$\rho_\phi(x)u = \phi(x) \cdot_{\mathfrak{h}} u, \quad \mu_\phi(x)u = u \cdot_{\mathfrak{h}} \phi(x), \quad \forall\ x \in \mathfrak{g}, u \in \mathfrak{h}. \tag{15}$$

Furthermore, we have

Proposition 3.1. *The pair* (ρ_ϕ, μ_ϕ) *is a representation of the pre-Lie algebra* $\mathfrak{g}$ *on* $\mathfrak{h}$.

Proof. Since ϕ is a pre-Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$, ϕ is a Lie algebra homomorphism from sub-adjacent Lie algebra $\mathfrak{g}^c$ to $\mathfrak{h}^c$. Then we have

$$\begin{aligned}\rho_\phi([x,y]_\mathfrak{g})u &= \phi([x,y]_\mathfrak{g})\cdot_\mathfrak{h} u\\ &= (\phi(x)\cdot_\mathfrak{h}\phi(y))\cdot_\mathfrak{h} u-(\phi(y)\cdot_\mathfrak{h}\phi(x))\cdot_\mathfrak{h} u\\ &= \phi(x)\cdot_\mathfrak{h}(\phi(y)\cdot_\mathfrak{h} u)-\phi(y)\cdot_\mathfrak{h}(\phi(x)\cdot_\mathfrak{h} u)\\ &= \rho_\phi(x)\rho_\phi(y)u-\rho_\phi(y)\rho_\phi(x)u.\end{aligned}$$

This means $\rho_\phi([x,y]_\mathfrak{g})=\rho_\phi(x)\rho_\phi(y)-\rho_\phi(y)\rho_\phi(x)$.
Furthermore, since $\mathfrak{h}$ is a pre-Lie algebra, we have

$$\begin{aligned}&\rho_\phi(x)\mu_\phi(y)u-\mu_\phi(y)\rho_\phi(x)u-\mu_\phi(x\cdot_\mathfrak{g} y)u+\mu_\phi(y)\mu_\phi(x)u\\ &=\phi(x)\cdot_\mathfrak{h}(u\cdot_\mathfrak{h}\phi(y))-(\phi(x)\cdot_\mathfrak{h} u)\cdot_\mathfrak{h}\phi(y)-u\cdot_\mathfrak{h}(\phi(x)\cdot_\mathfrak{h}\phi(y))+(u\cdot_\mathfrak{h}\phi(x))\cdot_\mathfrak{h}\phi(y)\\ &=0,\end{aligned}$$

which implies $\rho_\phi(x)\mu_\phi(y)z-\mu_\phi(y)\rho_\phi(x)z=\mu_\phi(x\cdot_\mathfrak{g} y)z-\mu_\phi(y)\mu_\phi(x)z$.
Thus (ρ_ϕ, μ_ϕ) is a representation of the pre-Lie algebra $\mathfrak{g}$ on $\mathfrak{h}$ ■

Now we give the cohomology complex for pre-Lie-morphism triples.
Let $\phi:(\mathfrak{g},\cdot_\mathfrak{g})\to(\mathfrak{h},\cdot_\mathfrak{h})$ be a pre-Lie algebra homomorphism.
Define the set of k-cochains by

$$\mathcal{C}^k_{\mathrm{preLie}}(\phi,\phi)=C^{k+1}_{\mathrm{preLie}}(\mathfrak{g},\mathfrak{g})\oplus C^{k+1}_{\mathrm{preLie}}(\mathfrak{h},\mathfrak{h})\oplus C^{k}_{\mathrm{preLie}}(\mathfrak{g},\mathfrak{h}), k\geq 0,$$

in which we set $C^0_{\mathrm{preLie}}(\mathfrak{g},\mathfrak{h})=0$.
The coboundary operator $\delta:\mathcal{C}^k_{\mathrm{preLie}}(\phi,\phi)\to\mathcal{C}^{k+1}_{\mathrm{preLie}}(\phi,\phi)$ is defined by

$$\delta_{\mathrm{preLie}}(f_1,f_2,f_3)=(\mathrm{d}_\mathfrak{g} f_1,\mathrm{d}_\mathfrak{h} f_2,\mathrm{d}_\phi f_3+(-1)^k(\phi\circ f_1-\phi^*\circ f_2)),$$

where $\mathrm{d}_\mathfrak{g}$ and $\mathrm{d}_\mathfrak{h}$ are the coboundary operators of the pre-Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ associated to their regular representations respectively, d_ϕ is the coboundary operator of the pre-Lie algebra $\mathfrak{g}$ associated to the representation $(\mathfrak{h};\rho_\phi,\mu_\phi)$, and $f_1\in C^{k+1}_{\mathrm{preLie}}(\mathfrak{g},\mathfrak{g})$, $f_2\in C^{k+1}_{\mathrm{preLie}}(\mathfrak{h},\mathfrak{h})$, $f_3\in C^{k}_{\mathrm{preLie}}(\mathfrak{g},\mathfrak{h})$. Here $\phi\circ f_1\in C^{k+1}_{\mathrm{preLie}}(\mathfrak{g},\mathfrak{h})$ and $\phi^*\circ f_2\in C^{k+1}_{\mathrm{preLie}}(\mathfrak{g},\mathfrak{h})$ are defined by

$$\begin{aligned}(\phi\circ f_1)(x_1,\cdots,x_{k+1}) &= \phi\big(f_1(x_1,\cdots,x_{k+1})\big),\\ (\phi^*\circ f_2)(x_1,\cdots,x_{k+1}) &= f_2(\phi(x_1),\cdots,\phi(x_{k+1})),\end{aligned}$$

where $x_1,\cdots,x_{k+1}\in\mathfrak{g}$.

Proposition 3.2. *We have* $\delta_{\mathrm{preLie}}\circ\delta_{\mathrm{preLie}}=0$.
Hence $(\mathcal{C}^*_{\mathrm{preLie}}(\phi,\phi)=\otimes_{n\geq 0}C^n_{\mathrm{preLie}}(\phi,\phi),\delta_{\mathrm{preLie}})$ *is a cochain complex.*

Proof. For $f_1 \in C^{k+1}_{\mathrm{preLie}}(\mathfrak{g},\mathfrak{g})$, we have

$$\begin{aligned}
\mathrm{d}_\phi(\phi\circ f_1)(x_1,\cdots,x_{k+2}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\phi(x_i)\cdot_{\mathfrak{h}}\phi(f_1(x_1,\cdots,\hat{x_i},\cdots,x_{k+2}))\\
&+\sum_{i=1}^{k+1}(-1)^{i+1}\phi(f_1(x_1,\cdots,\hat{x_i},\cdots,x_{k+1},x_i))\cdot_{\mathfrak{h}}\phi(x_{k+2})\\
&-\sum_{i=1}^{k+1}(-1)^{i+1}\phi(f_1(x_1,\cdots,\hat{x_i},\cdots,x_{k+1},x_i\cdot_{\mathfrak{g}}x_{k+2}))\\
&+\sum_{1\le i<j\le k+1}(-1)^{i+j}\phi(f_1([x_i,x_j]_{\mathfrak{g}},x_1,\cdots,\hat{x_i},\cdots,\hat{x_j},\cdots,x_{k+2}))\\
&=\phi(\sum_{i=1}^{k+1}(-1)^{i+1}x_i\cdot_{\mathfrak{g}}f_1(x_1,\cdots,\hat{x_i},\cdots,x_{k+2}))\\
&+\phi(\sum_{i=1}^{k+1}(-1)^{i+1}f_1(x_1,\cdots,\hat{x_i},\cdots,x_{k+1},x_i))\cdot_{\mathfrak{g}}x_{k+2})\\
&-\phi(\sum_{i=1}^{k+1}(-1)^{i+1}f_1(x_1,\cdots,\hat{x_i},\cdots,x_{k+1},x_i\cdot_{\mathfrak{g}}x_{k+2}))\\
&+\phi(\sum_{1\le i<j\le k+1}(-1)^{i+j}f_1([x_i,x_j]_{\mathfrak{g}},x_1,\cdots,\hat{x_i},\cdots,\hat{x_j},\cdots,x_{k+2}))\\
&=\phi(\mathrm{d}_{\mathfrak{g}}(f_1))(x_1,\cdots,x_{k+2}).
\end{aligned}$$

Then we see
$$\mathrm{d}_\phi(\phi(f_1))=\phi(\mathrm{d}_{\mathfrak{g}}(f_1)). \tag{16}$$

For $f_2\in C^{k+1}_{\mathrm{preLie}}(\mathfrak{h},\mathfrak{h})$, we have

$$\begin{aligned}
\mathrm{d}_\phi(\phi^*f_2)(x_1,\cdots,x_{k+2}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\phi(x_i)\cdot_{\mathfrak{h}}f_2(\phi(x_1,\cdots,\hat{x_i},\cdots,x_{k+2}))\\
&+\sum_{i=1}^{k+1}(-1)^{i+1}f_2(\phi(x_1,\cdots,\hat{x_i},\cdots,x_{k+1},x_i))\cdot_{\mathfrak{h}}\phi(x_{k+2})\\
&-\sum_{i=1}^{k+1}(-1)^{i+1}f_2(\phi(x_1,\cdots,\hat{x_i},\cdots,x_{k+1},x_i\cdot_{\mathfrak{g}}x_{k+2}))\\
&+\sum_{1\le i<j\le k+1}(-1)^{i+j}f_2(\phi([x_i,x_j]_{\mathfrak{g}},x_1,\cdots,\hat{x_i},\cdots,\hat{x_j},\cdots,x_{k+2}))\\
&=\sum_{i=1}^{k+1}(-1)^{i+1}\phi(x_i)\cdot_{\mathfrak{h}}f_2(\phi(x_1,\cdots,\hat{x_i},\cdots,x_{k+2}))\\
&+\sum_{i=1}^{k+1}(-1)^{i+1}f_2(\phi(x_1,\cdots,\hat{x_i},\cdots,x_{k+1},x_i))\cdot_{\mathfrak{h}}\phi(x_{k+2})\\
&-\sum_{i=1}^{k+1}(-1)^{i+1}f_2(\phi(x_1),\cdots,\hat{x_i},\cdots,\phi(x_{k+1}),\phi(x_i)\cdot_{\mathfrak{h}}\phi(x_{k+2}))\\
&+\sum_{1\le i<j\le k+1}(-1)^{i+j}f_2([\phi(x_i),\phi(x_j)]_{\mathfrak{h}},\phi(x_1),\cdots,\hat{x_i},\cdots,\hat{x_j},\cdots,\phi(x_{k+2}))\\
&=\phi^*\mathrm{d}_{\mathfrak{h}}(f_2)(x_1,\cdots,x_{k+2}).
\end{aligned}$$

Thus
$$\mathrm{d}_\phi(\phi^* f_2) = \phi^* \mathrm{d}_\mathfrak{h}(f_2). \tag{17}$$

Furthermore, by (16), (17) and the fact that $\mathrm{d}_\mathfrak{g} \circ \mathrm{d}_\mathfrak{g} = 0$, $\mathrm{d}_\mathfrak{h} \circ \mathrm{d}_\mathfrak{h} = 0$ and $\mathrm{d}_\phi \circ \mathrm{d}_\phi = 0$, we have

$$\begin{aligned}
&\delta_{\mathrm{preLie}} \circ \delta_{\mathrm{preLie}}(f_1, f_2, f_3) = \delta_{\mathrm{preLie}}(\mathrm{d}_\mathfrak{g} f_1, \mathrm{d}_\mathfrak{h} f_2, \mathrm{d}_\phi f_3 + (-1)^k(\phi \circ f_1 - \phi^* \circ f_2)) \\
&= (\mathrm{d}_\mathfrak{g}\mathrm{d}_\mathfrak{g} f_1, \mathrm{d}_\mathfrak{h}\mathrm{d}_\mathfrak{h} f_2, \mathrm{d}_\phi(\mathrm{d}_\phi f_3 + (-1)^k(\phi \circ f_1 - \phi^* \circ f_2)) + (-1)^{k+1}(\phi \circ \mathrm{d}_\mathfrak{g} f_1 - \phi^* \circ \mathrm{d}_\mathfrak{h} f_2)) \\
&= \big(0, 0, (-1)^k\big(\mathrm{d}_\phi(\phi(f_1)) - \phi(\mathrm{d}_\mathfrak{g}(f_1) + \phi^* \mathrm{d}_\mathfrak{h}(f_2) - \mathrm{d}_\phi(\phi^* f_2))\big)\big) = (0,0,0),
\end{aligned}$$

which implies $\delta_{\mathrm{preLie}} \circ \delta_{\mathrm{preLie}} = 0$. ■

Definition 3.3. The cohomology of the cochain complex $(\mathcal{C}^*_{\mathrm{preLie}}(\phi, \phi), \delta_{\mathrm{preLie}})$ is called a *cohomology of the pre-Lie-morphism triple* $(\mathfrak{g}, \mathfrak{h}, \phi)$. The corresponding k-th cohomology group, denoted by $H^k_{\mathrm{preLie}}(\phi, \phi)$, is called the *$k$-th cohomology group for the pre-Lie-morphism triple* $(\mathfrak{g}, \mathfrak{h}, \phi)$.

3.2. From cohomologies of pre-Lie-morphism triples to cohomologies of the sub-adjacent Lie-morphism triples

In this subsection, we show that the cohomology of a pre-Lie-morphism triple can be deduced from a new cohomology of the sub-adjacent Lie-morphism triple.

The Chevalley-Eilenberg cohomology theory for a Lie algebra $(\mathfrak{g}, [-,-]_\mathfrak{g})$ associated to a representation $(V;\rho)$ is given as follows. Denote by $C^n_{\mathrm{CE}}(\mathfrak{g}, V) := \mathrm{Hom}(\wedge^n \mathfrak{g}, V)$, the space of n-cochains. The corresponding Chevalley-Eilenberg coboundary operator $\partial_\rho : C^n_{\mathrm{CE}}(\mathfrak{g}, V) \longrightarrow C^{n+1}_{\mathrm{CE}}(\mathfrak{g}, V)$ is given by

$$\begin{aligned}
\partial_\rho f(x_1, \ldots, x_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \rho(x_i) f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) \\
&\quad + \sum_{1 \le i < j \le n+1} (-1)^{i+j} f([x_i, x_j]_\mathfrak{g}, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1})
\end{aligned} \tag{18}$$

for all $f \in C^n_{\mathrm{CE}}(\mathfrak{g}, V)$ and $x_1, x_2, \cdots, x_{n+1} \in \mathfrak{g}$. We denote the corresponding n-th cohomology group by $H^n_{\mathrm{CE}}(\mathfrak{g}, V)$.

Let $(\mathfrak{g}, \cdot_\mathfrak{g})$ be a pre-Lie algebra and $(V; \rho, \mu)$ a representation. Define the homomorphism $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathrm{Hom}(\mathfrak{g}, V))$ by

$$\varrho(x)(f)(y) = \rho(x) f(y) - \mu(y) f(x) - f(x \cdot_\mathfrak{g} y), \quad \forall\, f \in \mathrm{Hom}(\mathfrak{g}, V), x, y \in \mathfrak{g}. \tag{19}$$

Then it was shown in [11] that ϱ is a representation of the sub-adjacent Lie algebra $\mathfrak{g}^c$ on the vector space $\mathrm{Hom}(\mathfrak{g}, V)$. Furthermore, we define

$$\Psi : C^k_{\mathrm{CE}}(\mathfrak{g}, \mathrm{Hom}(\mathfrak{g}, V)) \to C^{k+1}_{\mathrm{preLie}}(\mathfrak{g}, V)$$

by
$$\Psi(f)(x_1, \cdots, x_k, y) = f(x_1, \cdots, x_k)(y), \quad \forall\, x_1, \cdots, x_k, y \in \mathfrak{g}. \tag{20}$$

Theorem 3.4. (see [11, 21]) *The mapping* Ψ *is a cochain map from the complex* $(C^*_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},V)),\partial_\varrho)$ *to the complex* $(C^*_{\mathrm{preLie}}(\mathfrak{g},V),\mathrm{d})$*, where* ∂_ϱ *is the coboundary operator of the sub-adjacent Lie algebra* $\mathfrak{g}^c$ *associated to representation* ϱ *and* d *is the coboundary operator of the pre-Lie algebra* $\mathfrak{g}$ *associated to representation* (ρ,μ)*. Moreover,* Ψ *induces an isomorphism between the corresponding cohomology groups:*

$$H^n_{\mathrm{preLie}}(\mathfrak{g},V)\cong H^{n-1}_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},V)),\quad n=1,2\cdots. \tag{21}$$

Let ϕ be a homomorphism between pre-Lie algebras $(\mathfrak{g},\cdot_\mathfrak{g})$ and $(\mathfrak{h},\cdot_\mathfrak{h})$. Define $\varrho_\mathfrak{g}:\mathfrak{g}\to\mathfrak{gl}(\mathrm{Hom}(\mathfrak{g},\mathfrak{g}))$, $\varrho_\mathfrak{h}:\mathfrak{h}\to\mathfrak{gl}(\mathrm{Hom}(\mathfrak{h},\mathfrak{h}))$ and $\varrho_\phi:\mathfrak{g}\to\mathfrak{gl}(\mathrm{Hom}(\mathfrak{g},\mathfrak{h}))$ by

$$\varrho_\mathfrak{g}(x)(f_1)(y) = x\cdot_\mathfrak{g} f_1(y)-f_1(x)\cdot_\mathfrak{g} y-f_1(x\cdot_\mathfrak{g} y), \tag{22}$$

$$\varrho_\mathfrak{h}(u)(f_2)(v) = u\cdot_\mathfrak{h} f_2(v)-f_2(u)\cdot_\mathfrak{h} v-f_2(u\cdot_\mathfrak{h} v), \tag{23}$$

$$\varrho_\phi(x)(f_3)(y) = \rho_\phi(x)f_3(y)-\mu_\phi(y)f_3(x)-f_3(x\cdot_\mathfrak{g} y), \tag{24}$$

where $f_1\in\mathrm{Hom}(\mathfrak{g},\mathfrak{g})$, $f_2\in\mathrm{Hom}(\mathfrak{h},\mathfrak{h})$, $f_3\in\mathrm{Hom}(\mathfrak{g},\mathfrak{h})$ and (ρ_ϕ,μ_ϕ) is given by (15). Then $\varrho_\mathfrak{g}$, $\varrho_\mathfrak{h}$ and ϱ_ϕ are representations of the corresponding sub-adjacent Lie algebras.

In the following, we generalize Theorem 3.4 to the case of pre-Lie-morphism triples. Define the set of k-cochains by

$$\mathcal{C}^k_{\mathrm{CE}}(\phi,\phi)=C^{k+1}_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},\mathfrak{g}))\oplus C^{k+1}_{\mathrm{CE}}(\mathfrak{h},\mathrm{Hom}(\mathfrak{h},\mathfrak{h}))\oplus C^{k}_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},\mathfrak{h})), k\geq 0,$$

in which we set $C^0_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},\mathfrak{h}))=0$.

The coboundary operator $\delta_{\mathrm{CE}}:\mathcal{C}^k_{\mathrm{CE}}(\phi,\phi)\to\mathcal{C}^{k+1}_{\mathrm{CE}}(\phi,\phi)$ is defined by

$$\delta_{\mathrm{CE}}(f_1,f_2,f_3)=(\partial_\mathfrak{g}f_1,\partial_\mathfrak{h}f_2,\partial_\phi f_3+(-1)^k(\phi\circ f_1-\phi^*\circ f_2)),$$

for all $f_1\in C^{k+1}_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},\mathfrak{g})), f_2\in C^{k+1}_{\mathrm{CE}}(\mathfrak{h},\mathrm{Hom}(\mathfrak{h},\mathfrak{h})), f_3\in C^{k}_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},\mathfrak{h}))$, where $\partial_\mathfrak{g}$, $\partial_\mathfrak{h}$ and ∂_ϕ are Chevalley-Eilenberg coboundary operators of the sub-adjacent Lie algebras $\mathfrak{g}^c$, $\mathfrak{h}^c$ and $\mathfrak{g}^c$ associated to representations $\varrho_\mathfrak{g}$, $\varrho_\mathfrak{h}$ and ϱ_ϕ, respectively. Here $\phi\circ f_1\in C^{k+1}_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},\mathfrak{h}))$ and $\phi^*\circ f_2\in C^{k+1}_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},\mathfrak{h}))$ are defined by

$$\begin{aligned}(\phi\circ f_1)(x_1,\cdots,x_{k+1})(y) &= \phi\big(f_1(x_1,\cdots,x_{k+1})(y)\big),\\ (\phi^*\circ f_2)(x_1,\cdots,x_{k+1})(y) &= f_2(\phi(x_1),\cdots,\phi(x_{k+1}))(\phi(y)),\end{aligned}$$

where $x_1,\cdots,x_{k+1},y\in\mathfrak{g}$. We denote the k-th cohomology group corresponding to $\partial_\mathfrak{g}$, $\partial_\mathfrak{h}$ and ∂_ϕ by $H^k_{\mathrm{CE}}(\mathfrak{g},\mathfrak{g})$, $H^k_{\mathrm{CE}}(\mathfrak{h},\mathfrak{h})$ and $H^k_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},\mathfrak{h}))$, respectively.

Proposition 3.5. *With the above notations, we have* $\delta_{\mathrm{CE}}\circ\delta_{\mathrm{CE}}=0$*. Hence* $(\mathcal{C}^*_{\mathrm{CE}}(\phi,\phi)=\otimes_{n\geq 0}\mathcal{C}^n_{\mathrm{CE}}(\phi,\phi),\delta_{\mathrm{CE}})$ *is a cochain complex. We denote the corresponding* k*-th cohomology group by* $H^k_{\mathrm{CE}}(\phi,\phi)$*.*

Proof. For $f_1\in C^k_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},\mathfrak{g}))$, we have

$$\begin{aligned}&\partial_\phi(\phi\circ f_1)(x_1,\cdots,x_{k+1})(y)\\ &=\sum_{i=1}^{k+1}(-1)^{i+1}\phi(x_i)\cdot_\mathfrak{h}\phi(f_1(x_1,\cdots,\hat{x_i},\cdots,x_{k+1})(y))\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{k+1}(-1)^{i+1}\phi(f_1(x_1,\cdots,\hat{x_i},\cdots,x_{k+1})(x_i))\cdot_{\mathfrak{h}}\phi(y)\\
&-\sum_{i=1}^{k+1}(-1)^{i+1}\phi(f_1(x_1,\cdots,\hat{x_i},\cdots,x_{k+1})(x_i\cdot_{\mathfrak{g}}y))\\
&+\sum_{1\le i<j\le k+1}(-1)^{i+j}\phi(f_1([x_i,x_j]_{\mathfrak{g}},x_1,\cdots,\hat{x_i},\cdots,\hat{x_j},\cdots,x_{k+1})(y))\\
=&\ \phi(\sum_{i=1}^{k+1}(-1)^{i+1}x_i\cdot_{\mathfrak{g}}f_1(x_1,\cdots,\hat{x_i},\cdots,x_{k+1})(y))\\
&-\phi(\sum_{i=1}^{k+1}(-1)^{i+1}f_1(x_1,\cdots,\hat{x_i},\cdots,x_{k+1})(x_i)\cdot_{\mathfrak{g}}y)\\
&-\phi(f_1([x_i,x_j]_{\mathfrak{g}},x_1,\cdots,\hat{x_i},\cdots,\hat{x_j},\cdots,x_{k+1})(x_i\cdot_{\mathfrak{g}}y))\\
&+\phi(\sum_{1\le i<j\le k+1}(-1)^{i+j}f_1([x_i,x_j]_{\mathfrak{g}},x_1,\cdots,\hat{x_i},\cdots,\hat{x_j},\cdots,x_{k+1})(y))\\
=&\ \phi(\partial_{\mathfrak{g}}(f_1)(x_1,\cdots,x_{k+1})(y))=(\phi\circ\partial_{\mathfrak{g}}(f_1))(x_1,\cdots,x_{k+1})(y).
\end{aligned}
$$

Thus
$$\partial_\phi(\phi\circ f_1)=\phi\circ\partial_{\mathfrak{g}}(f_1). \tag{25}$$

Similarly, for $f_2\in C^{k+1}_{\mathrm{CE}}(\mathfrak{h},\mathrm{Hom}(\mathfrak{h},\mathfrak{h}))$, we have

$$\partial_\phi(\phi^*\circ f_2)=\phi^*\circ\partial_{\mathfrak{h}}(f_2). \tag{26}$$

Furthermore, by (25), (26) and the fact that $\partial_{\mathfrak{g}}\circ\partial_{\mathfrak{g}}=0$, $\partial_{\mathfrak{h}}\circ\partial_{\mathfrak{h}}=0$ and $\partial_\phi\circ\partial_\phi=0$, we have

$$
\begin{aligned}
&\delta_{\mathrm{CE}}\circ\delta_{\mathrm{CE}}(f_1,f_2,f_3)=\delta_{\mathrm{CE}}(\partial_{\mathfrak{g}}f_1,\partial_{\mathfrak{h}}f_2,\partial_\phi f_3+(-1)^k(\phi\circ f_1-\phi^*\circ f_2))\\
&\ =(\partial_{\mathfrak{g}}\circ\partial_{\mathfrak{g}}f_1,\partial_{\mathfrak{h}}\circ\partial_{\mathfrak{h}}f_2,\partial_\phi\circ(\partial_\phi f_3\\
&\qquad\quad+(-1)^k(\phi\circ f_1-\phi^*\circ f_2))+(-1)^{k+1}(\phi\circ\partial_{\mathfrak{g}}f_1-\phi^*\circ\partial_{\mathfrak{h}}f_2))\\
&\ =\big(0,0,(-1)^k\big(\partial_\phi(\phi\circ f_1)-\phi\circ\partial_{\mathfrak{g}}(f_1)+\phi^*\circ\partial_{\mathfrak{h}}(f_2)-\partial_\phi(\phi^*\circ f_2)\big)\big)=(0,0,0),
\end{aligned}
$$

which implies that $\delta_{\mathrm{CE}}\circ\delta_{\mathrm{CE}}=0$. ■

Remark 3.6. In [9], the author gives a cohomology of a Lie-morphism triple with coefficients in a representation. Our cohomology for the sub-adjacent Lie-morphism triple in this paper is not fit in the mentioned above cohomology theory. ■

Proposition 3.7. *If $H^{k+1}_{\mathrm{CE}}(\mathfrak{g},\mathfrak{g})$, $H^{k+1}_{\mathrm{CE}}(\mathfrak{h},\mathfrak{h})$ and $H^{k}_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},\mathfrak{h}))$ are trivial, then $H^k_{\mathrm{CE}}(\phi,\phi)$ is also trivial.*

Proof. Let $f=(f_1,f_2,f_3)$ be a k-cocycle. Then we have

$$\partial_{\mathfrak{g}}f_1=0,\quad \partial_{\mathfrak{h}}f_2=0,\quad \partial_\phi f_3=(-1)^{k+1}(\phi\circ f_1-\phi^*\circ f_2).$$

Since $H^{k+1}_{\mathrm{CE}}(\mathfrak{g},\mathfrak{g})$ and $H^{k+1}_{\mathrm{CE}}(\mathfrak{h},\mathfrak{h})$ are trivial, there exist in consequence $g_1\in H^k_{\mathrm{CE}}(\mathfrak{g},\mathfrak{g})$ and $g_2\in H^k_{\mathrm{CE}}(\mathfrak{h},\mathfrak{h})$ such that

$$f_1=\partial_{\mathfrak{g}}g_1,\quad f_2=\partial_{\mathfrak{h}}g_2.$$

By the fact that $\phi\circ(\partial_{\mathfrak{g}} f_1)=\partial_\phi(\phi\circ f_1)$ and $\phi^*(\partial_{\mathfrak{h}}\circ f_2)=\partial_\phi(\phi^*\circ f_2)$, we have

$$\begin{aligned}\partial_\phi f_3&=(-1)^{k+1}(\phi\circ f_1-\phi^*\circ f_2)=(-1)^{k+1}(\phi\circ(\partial_{\mathfrak{g}} g_1)-\partial_\phi(\phi\circ g_1))\\&=(-1)^{k+1}(\partial_\phi(\phi\circ g_1)-\partial_\phi(\phi^*\circ g_2)),\end{aligned}$$

which gives $\partial_\phi(f_3-(-1)^{k+1}(\phi\circ g_1-\phi^*\circ g_2))=0$. Since $H^k_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},\mathfrak{h}))$ is trivial, there exists $g_3\in C^{k-1}_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},\mathfrak{h}))$ such that

$$\partial_\phi g_3=f_3-(-1)^{k+1}(\phi\circ g_1-\phi^*\circ g_2).$$

This implies $(f_1,f_2,f_3)=\delta_{\mathrm{CE}}(g_1,g_2,g_3)$. Thus $H^k_{\mathrm{CE}}(\phi,\phi)$ is trivial. ■

Furthermore, we define $\Phi: C^k_{\mathrm{CE}}(\phi,\phi)\to C^{k+1}_{\mathrm{preLie}}(\phi,\phi)$ by

$$\Phi(f_1,f_2,f_3)=(\Phi_1(f_1),\Phi_2(f_2),\Phi_3(f_3)),\tag{27}$$

where $\Phi_1: C^{k+1}_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},\mathfrak{g}))\to C^{k+2}_{\mathrm{preLie}}(\mathfrak{g},\mathfrak{g})$, $\Phi_2: C^{k+1}_{\mathrm{CE}}(\mathfrak{h},\mathrm{Hom}(\mathfrak{h},\mathfrak{h}))\to C^{k+2}_{\mathrm{preLie}}(\mathfrak{h},\mathfrak{h})$ and $\Phi_3: C^{k}_{\mathrm{CE}}(\mathfrak{g},\mathrm{Hom}(\mathfrak{g},\mathfrak{h}))\to C^{k+1}_{\mathrm{preLie}}(\mathfrak{g},\mathfrak{h})$ are given by

$$\begin{aligned}\Phi_1(f_1)(x_1,\cdots,x_{k+1},y)&=f_1(x_1,\cdots,x_{k+1})(y),\\ \Phi_2(f_2)(u_1,\cdots,u_{k+1},v)&=f_2(u_1,\cdots,u_{k+1})(v),\\ \Phi_3(f_3)(x_1,\cdots,x_k,y)&=f_3(x_1,\cdots,x_k)(y)\end{aligned}$$

for all $x_1,\cdots,x_{k+1},y\in\mathfrak{g}$, $u_1,\cdots,u_{k+1},v\in\mathfrak{h}$.

Theorem 3.8. *The map Φ is an isomorphic cochain map from the complex $(C^*_{\mathrm{CE}}(\phi,\phi),\delta_{\mathrm{CE}})$ to the complex $(C^*_{\mathrm{preLie}}(\phi,\phi),\delta_{\mathrm{preLie}})$.*
Moreover, Φ induces an isomorphism between the corresponding cohomology groups:

$$H^k_{\mathrm{preLie}}(\phi,\phi)\cong H^{k-1}_{\mathrm{CE}}(\phi,\phi),\quad k=2,3,\cdots.\tag{28}$$

Proof. It is obvious that Φ is an isomorphism. In the following, we will show that $\delta_{\mathrm{CE}}\circ\Phi=\Phi\circ\delta_{\mathrm{preLie}}$. By direct calculation, we have

$$\begin{aligned}&\delta_{\mathrm{CE}}\circ\Phi(f_1,f_2,f_3)\\&=(\partial_{\mathfrak{g}}\Phi_1(f_1),\partial_{\mathfrak{h}}\Phi_2(f_2),\partial_\phi\Phi_3(f_3)+(-1)^k(\phi\circ\Phi_1(f_1)-\phi^*\circ\Phi_2(f_2)));\\&\Phi\circ\delta_{\mathrm{preLie}}(f_1,f_2,f_3)\\&=(\Phi_1(\mathrm{d}_{\mathfrak{g}} f_1),\Phi_2(\mathrm{d}_{\mathfrak{h}} f_2),\Phi_3(\mathrm{d}_\phi f_3)+(-1)^k\Phi_3(\phi\circ f_1-\phi^*\circ f_2)).\end{aligned}$$

By Theorem 3.4, we have

$$\partial_{\mathfrak{g}}\Phi_1(f_1)=\Phi_1(\mathrm{d}_{\mathfrak{g}} f_1),\quad \partial_{\mathfrak{h}}\Phi_2(f_2)=\Phi_2(\mathrm{d}_{\mathfrak{h}} f_2),\quad \partial_\phi\Phi_3(f_3)=\Phi_3(\mathrm{d}_\phi f_3).$$

Thus $\delta_{\mathrm{CE}}\circ\Phi=\Phi\circ\delta_{\mathrm{preLie}}$ holds if and only if

$$\phi\circ\Phi_1(f_1)-\phi^*\circ\Phi_2(f_2)=\Phi_3(\phi\circ f_1-\phi^*\circ f_2).$$

By direct calculation, we have

$$\begin{aligned}&\phi\circ\Phi_1(f_1)(x_1,\cdots,x_{k+1},y)=\phi(f_1(x_1,\cdots,x_{k+1})(y)),\\&=(\phi\circ f_1)(x_1,\cdots,x_{k+1})(y)=\Phi_3(\phi\circ f_1)(x_1,\cdots,x_{k+1},y),\end{aligned}$$

which implies $\phi\circ\Phi_1(f_1)=\Phi_3(\phi\circ f_1)$.
Similarly, we also have $\phi^*\circ\Phi_2(f_2)=\Phi_3(\phi^*\circ f_2)$.

Thus $\delta_{\mathrm{CE}} \circ \Phi = \Phi \circ \delta_{\mathrm{preLie}}$ holds, i.e. the map Φ is an isomorphic cochain map from the cochain complex $(C^*_{\mathrm{CE}}(\phi,\phi), \delta_{\mathrm{CE}})$ to the cochain complex $(C^*_{\mathrm{preLie}}(\phi,\phi), \delta_{\mathrm{preLie}})$. The second conclusion follows directly. ■

At last, by applying the close relation between the cohomology of a pre-Lie-morphism triple and that of the sub-adjacent Lie-morphism triple, we give the cohomologies of free pre-Lie algebra homomorphisms. See [6] for more details on free pre-Lie algebras.

Proposition 3.9. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two free pre-Lie algebras, and $\phi : \mathfrak{g} \to \mathfrak{h}$ a pre-Lie algebra homomorphism. Then we have*

$$H^k_{\mathrm{preLie}}(\phi,\phi) = 0, \quad k \geq 3$$

Proof. It was shown in [5, 12] that the sub-adjacent Lie algebra of a free pre-Lie algebra is a free Lie algebra. Then $\mathfrak{g}^c$ and $\mathfrak{h}^c$ are free Lie algebras, and $\phi : \mathfrak{g}^c \to \mathfrak{h}^c$ is a Lie algebra homomorphism. It is well-known that for $k \geq 2$, the k-th cohomology group of a free Lie algebra with coefficients in any representation is trivial. Thus $H^k_{\mathrm{CE}}(\mathfrak{g},\mathfrak{g})$, $H^k_{\mathrm{CE}}(\mathfrak{h},\mathfrak{h})$ and $H^k_{\mathrm{CE}}(\mathfrak{g}, \mathrm{Hom}(\mathfrak{g},\mathfrak{h}))$ for $k \geq 2$ are trivial. Also by Proposition 3.7, $H^k_{\mathrm{CE}}(\phi,\phi) = 0$ for $k \geq 2$. Furthermore, by Theorem 3.8, we have $H^k_{\mathrm{preLie}}(\phi,\phi) = 0$ for $k \geq 3$. ■

4. Deformations of pre-Lie-morphism triples

In this section, we use cohomologies of pre-Lie-morphism triples to study linear deformations and formal deformations of them. We show that equivalent linear deformations and infinitesmial deformations are in the same first cohomology group, and an order n deformation of a pre-Lie-morphism triple is extendable if and only if its obstruction class in the second cohomology group is trivial.

4.1. Linear deformations of pre-Lie-morphism triples

Let $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ and $(\mathfrak{h}, \cdot_{\mathfrak{h}})$ be two pre-Lie algebras, and ϕ a pre-Lie algebra homomorphism from $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ to $(\mathfrak{h}, \cdot_{\mathfrak{h}})$. Also let $\omega \in \mathrm{Hom}(\otimes^2\mathfrak{g}, \mathfrak{g})$, $\varpi \in \mathrm{Hom}(\otimes^2\mathfrak{h}, \mathfrak{h})$ and $\theta \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$. Consider a t-parameterized family of multiplications $\cdot_t : \mathfrak{g}\otimes\mathfrak{g} \longrightarrow \mathfrak{g}$, $*_t : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ and linear maps $\phi_t : \mathfrak{g} \longrightarrow \mathfrak{h}$ given by

$$\begin{aligned} x \cdot_t y &= x \cdot_{\mathfrak{g}} y + t\omega(x, y), \\ u *_t v &= u \cdot_{\mathfrak{h}} v + t\varpi(u, v), \\ \phi_t(x) &= \phi(x) + t\theta(x), \end{aligned}$$

where $x, y \in \mathfrak{g}$ and $u, v \in \mathfrak{h}$. If $\mathfrak{g}_t = (\mathfrak{g}, \cdot_t)$ and $\mathfrak{h}_t = (\mathfrak{h}, *_t)$ are pre-Lie algebras, and

$$\phi_t : \mathfrak{g}_t \to \mathfrak{h}_t \text{ module } t^3$$

are pre-Lie algebra homomorphisms for all t, we say that (ω, ϖ, θ) generates a *linear deformation* of $(\mathfrak{g}, \mathfrak{h}, \phi)$. We denote the linear deformation of $(\mathfrak{g}, \mathfrak{h}, \phi)$ by $(\mathfrak{g}_t, \mathfrak{h}_t, \phi_t)$. It is straightforward to check that (ω, ϖ, θ) generates a linear deformation of $(\mathfrak{g}, \mathfrak{h}, \phi)$ if and only if

$$\begin{aligned} 0 = {} & x \cdot_{\mathfrak{g}} \omega(y, z) - y \cdot_{\mathfrak{g}} \omega(x, z) + \omega(y, x) \cdot_{\mathfrak{g}} z - \omega(x, y) \cdot_{\mathfrak{g}} z \\ & - \omega(y, x \cdot_{\mathfrak{g}} z) + \omega(x, y \cdot_{\mathfrak{g}} z) + \omega([x, y]_{\mathfrak{g}}, z), \end{aligned} \tag{29}$$

$$0=\omega(\omega(x,y),z)-\omega(x,\omega(y,z))-\omega(\omega(y,x),z)+\omega(y,\omega(x,z)), \tag{30}$$

$$\begin{aligned}0=&\,u\cdot_{\mathfrak{h}}\varpi(v,w)-v\cdot_{\mathfrak{h}}\varpi(u,w)+\varpi(v,u)\cdot_{\mathfrak{h}}w-\varpi(u,v)\cdot_{\mathfrak{h}}w\\&-\varpi(v,u\cdot_{\mathfrak{h}}w)+\varpi(u,v\cdot_{\mathfrak{h}}w)+\varpi([u,v]_{\mathfrak{h}},w),\end{aligned} \tag{31}$$

$$0=\varpi(\varpi(u,v),w)-\varpi(u,\varpi(v,w))-\varpi(\varpi(v,u),w)+\varpi(v,\varpi(u,w)), \tag{32}$$

$$0=\phi(x)\cdot_{\mathfrak{h}}\theta(y)+\theta(x)\cdot_{\mathfrak{h}}\phi(y)-\theta(x\cdot_{\mathfrak{g}}y)-\phi(\omega(x,y))+\varpi(\phi(x),\phi(y)), \tag{33}$$

$$0=\theta(\omega(x,y))-\varpi(\phi(x),\theta(y))-\varpi(\theta(x),\phi(y))-\theta(x)\cdot_{\mathfrak{h}}\theta(y). \tag{34}$$

Note that (29), (31) and (33) give $\delta_{\mathrm{preLie}}(\omega,\varpi,\theta)=0$, (30) means that $(\mathfrak{g},\omega)$ is a pre-Lie algebra, and (32) implies that $(\mathfrak{h},\varpi)$ is a pre-Lie algebra.

Definition 4.1. Two linear deformations $(\mathfrak{g}_t,\mathfrak{h}_t,\phi_t)$ and $(\mathfrak{g}'_t,\mathfrak{h}'_t,\phi'_t)$ of a pre-Lie-morphism triple $(\mathfrak{g},\mathfrak{h},\phi)$, generated by (ω,ϖ,θ) and $(\omega',\varpi',\theta')$ respectively, are said to be *equivalent* if there exist a family of pre-Lie algebra isomorphisms $\mathrm{Id}_{\mathfrak{g}}+tN:\mathfrak{g}_t\to\mathfrak{g}'_t$ and $\mathrm{Id}_{\mathfrak{h}}+tS:\mathfrak{h}_t\to\mathfrak{h}'_t$ with $N\in\mathfrak{gl}(\mathfrak{g})$ and $S\in\mathfrak{gl}(\mathfrak{h})$, such that

$$\phi'_t\circ(\mathrm{Id}_{\mathfrak{g}}+tN)=(\mathrm{Id}_{\mathfrak{h}}+tS)\circ\phi_t.$$

A lineal deformation $(\mathfrak{g}_t,\mathfrak{h}_t,\phi_t)$ of a pre-Lie-morphism triple $(\mathfrak{g},\mathfrak{h},\phi)$ is said to be *trivial* if it is equivalent to $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$. ■

By direct calculation, $(\mathfrak{g}_t,\mathfrak{h}_t,\phi_t)$ and $(\mathfrak{g}'_t,\mathfrak{h}'_t,\phi'_t)$ of a pre-Lie-morphism triple $(\mathfrak{g},\mathfrak{h},\phi)$ are equivalent if and only if

$$\begin{aligned}\omega(x,y)-\omega'(x,y)&=x\cdot_{\mathfrak{g}}N(y)+N(x)\cdot_{\mathfrak{g}}y-N(x\cdot_{\mathfrak{g}}y), &(35)\\ N\omega(x,y)&=\omega'(x,N(y))+\omega'(N(x),y)+N(x)\cdot_{\mathfrak{g}}N(y), &(36)\\ \omega'(N(x),N(y))&=0, &(37)\\ \varpi(u,v)-\varpi'(u,v)&=u\cdot_{\mathfrak{h}}S(v)+S(u)\cdot_{\mathfrak{h}}v-S(u\cdot_{\mathfrak{h}}v), &(38)\\ S\varpi(u,v)&=\varpi'(u,S(v))+\varpi'(S(u),v)+S(u)\cdot_{\mathfrak{h}}S(v), &(39)\\ \varpi'(S(u,S(v))&=0, &(40)\\ \theta-\theta'&=\phi\circ N-S\circ\phi, &(41)\\ \theta'\circ N&=S\circ\theta. &(42)\end{aligned}$$

Note that (35), (38) and (41) imply $(\omega,\varpi,\theta)-(\omega',\varpi',\theta')=\delta_{\mathrm{preLie}}(N,S,0)$. We summarize the above discussion intothe following theorem:

Theorem 4.2. *Let $(\mathfrak{g}_t,\mathfrak{h}_t,\phi_t)$ be a linear deformation of a pre-Lie-morphism triple $(\mathfrak{g},\mathfrak{h},\phi)$ generated by (ω,ϖ,θ). Then (ω,ϖ,θ) is closed, i.e. $\delta_{\mathrm{preLie}}(\omega,\varpi,\theta)=0$. Furthermore, if two linear deformations $(\mathfrak{g}_t,\mathfrak{h}_t,\phi_t)$ and $(\mathfrak{g}'_t,\mathfrak{h}'_t,\phi'_t)$ of the pre-Lie-morphism triple $(\mathfrak{g},\mathfrak{h},\phi)$, generated by (ω,ϖ,θ) and $(\omega',\varpi',\theta')$ respectively, are equivalent, then (ω,ϖ,θ) and $(\omega',\varpi',\theta')$ are in the same cohomology class in $H^1_{\mathrm{preLie}}(\phi,\phi)$.*

Now we consider trivial linear deformations of a pre-Lie-morphism triple $(\mathfrak{g},\mathfrak{h},\phi)$. Then (35)–(42) reduce to

$$\begin{aligned}\omega(x,y)&=x\cdot_{\mathfrak{g}}N(y)+N(x)\cdot_{\mathfrak{g}}y-N(x\cdot_{\mathfrak{g}}y), &(43)\\ N\omega(x,y)&=N(x)\cdot_{\mathfrak{g}}N(y), &(44)\\ \varpi(u,v)&=u\cdot_{\mathfrak{h}}S(v)+S(u)\cdot_{\mathfrak{h}}v-S(u\cdot_{\mathfrak{h}}v), &(45)\end{aligned}$$

$$\begin{aligned} S\varpi(u,v) &= S(u)\cdot_{\mathfrak{h}} S(v), &(46)\\ \theta &= \phi\circ N - S\circ\phi, &(47)\\ S\circ\theta &= 0. &(48)\end{aligned}$$

It follows from (43) and (44) that N satisfies

$$N(x)\cdot_{\mathfrak{g}} N(y) - N(x\cdot_{\mathfrak{g}} N(y) + N(x)\cdot_{\mathfrak{g}} y - N(x\cdot_{\mathfrak{g}} y)) = 0. \tag{49}$$

This means that N is a Nijenhuis operator on the pre-Lie algebra $(\mathfrak{g},\cdot_{\mathfrak{g}})$. Similarly, by the identities (45) and (46), we have

$$S(u)\cdot_{\mathfrak{h}} S(v) - S(u\cdot_{\mathfrak{h}} S(v) + S(u)\cdot_{\mathfrak{h}} v - S(u\cdot_{\mathfrak{h}} v)) = 0. \tag{50}$$

That is to say, S is a Nijenhuis operator on the pre-Lie algebra $(\mathfrak{h},\cdot_{\mathfrak{h}})$. Applying the identities (47) and (48), we see that N and S satisfy

$$S\circ\phi\circ N = S^2\circ\phi. \tag{51}$$

Definition 4.3. Let $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$ be a pre-Lie-morphism triple. A pair (N,S), where $N\in\mathfrak{gl}(\mathfrak{g})$ and $S\in\mathfrak{gl}(\mathfrak{h})$, is called a *Nijenhuis pair* on the pre-Lie-morphism triple $(\mathfrak{g},\mathfrak{h},\phi)$ if N is a Nijenhuis operator on the pre-Lie algebra $(\mathfrak{g},\cdot_{\mathfrak{g}})$, S is a Nijenhuis operator on the pre-Lie algebra $(\mathfrak{h},\cdot_{\mathfrak{h}})$ and the identity (51) holds. ■

We have seen that a trivial linear deformation of a pre-Lie-morphism triple $(\mathfrak{g},\mathfrak{h},\phi)$ could give rise to a Nijenhuis pair. In fact, the converse is also true.

Theorem 4.4. *Let $(\mathfrak{g},\mathfrak{h},\phi)$ be a pre-Lie-morphism triple, and (N,S) a Nijenhuis pair on $(\mathfrak{g},\mathfrak{h},\phi)$. Then a linear deformation of $(\mathfrak{g},\mathfrak{h},\phi)$ can be obtained by defining*

$$\begin{aligned} \omega(x,y) &= \mathrm{d}_{\mathfrak{g}}N(x,y), &(52)\\ \varpi(u,v) &= \mathrm{d}_{\mathfrak{h}}S(u,v), &(53)\\ \theta(x) &= \phi(N(x)) - S(\phi(x)), \quad \forall\, x,y\in\mathfrak{g}, u,v\in\mathfrak{h}. &(54)\end{aligned}$$

Furthermore, this linear deformation is trivial.

Proof. By (52)–(54), we have $(\omega,\varpi,\theta) = \delta_{\mathrm{preLie}}(N,S,0)$. Then (ω,ϖ,θ) is closed. This implies that (29), (31) and (33) hold. Since N is a Nijenhuis operator on $\mathfrak{g}$ and S is a Nijenhuis operator on $\mathfrak{h}$, we have (30) and (32) , respectively. Also by (51), we have

$$\begin{aligned}&\theta(\omega(x,y)) - \varpi(\phi(x),\theta(y)) - \varpi(\theta(x),\phi(y)) - \theta(x)\cdot_{\mathfrak{h}}\theta(y)\\ &= S(\phi(x))\cdot_{\mathfrak{h}} S(\phi(y)) - S(S(\phi(x))\cdot_{\mathfrak{h}}\phi(y) + \phi(x)\cdot_{\mathfrak{h}} S(\phi(y)) - S(\phi(x)\cdot_{\mathfrak{h}}\phi(y)))\\ &= 0.\end{aligned}$$

That is to say, (34) holds.
At last, we see that

$$\begin{aligned}(\mathrm{Id}_{\mathfrak{g}}+tN)(x\cdot_t y) &= (\mathrm{Id}_{\mathfrak{g}}+tN)(x)\cdot_{\mathfrak{g}}(\mathrm{Id}_{\mathfrak{g}}+tN)(y),\\ (\mathrm{Id}_{\mathfrak{h}}+tS)(u*_t v) &= (\mathrm{Id}_{\mathfrak{h}}+tS)(u)\cdot_{\mathfrak{h}}(\mathrm{Id}_{\mathfrak{h}}+\mathrm{tS})(\mathrm{v}),\\ \phi((\mathrm{Id}_{\mathfrak{g}}+tN)(x)) &= (\mathrm{Id}_{\mathfrak{h}}+tS)((\phi+t\theta)(x)), \quad \forall\, x,y\in\mathfrak{g}, u,v\in\mathfrak{h}.\end{aligned}$$

Thus the linear deformation is trivial. ■

Let $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ be a pre-Lie algebra and $(V;\rho,\mu)$ a representation and let furthermore $T_1, T_2 : V \longrightarrow \mathfrak{g}$ be two $\mathcal{O}$-operators on the pre-Lie algebra $\mathfrak{g}$ associated to the representation $(V;\rho,\mu)$. Recall that T_1 and T_2 are *compatible* if for any k_1, k_2, $k_1T_1 + k_2T_2$ is still an $\mathcal{O}$-operator.

A pair of compatible $\mathcal{O}$-operators with some conditions can give rise to a Nijenhuis operator as follows.

Proposition 4.5. ([23])*Let $T_1, T_2 : V \longrightarrow \mathfrak{g}$ be two $\mathcal{O}$-operators on a pre-Lie algebra $\mathfrak{g}$ associated to a representation $(V;\rho,\mu)$. Suppose that T_2 is invertible. If T_1 and T_2 are compatible, then $N = T_1 \circ T_2^{-1}$ is a Nijenhuis operator on the pre-Lie algebra $(\mathfrak{g}, \cdot_{\mathfrak{g}})$.*

Given two compatible $\mathcal{O}$-operators T_1 and T_2 on the pre-Lie algebra $\mathfrak{g}$ associated to the representation $(V;\rho,\mu)$, by Example 2.12, $((V, \cdot^{T_1}), (\mathfrak{g}, \cdot_{\mathfrak{g}}), T_1)$ and $((V, \cdot^{T_2}), (\mathfrak{g}, \cdot_{\mathfrak{g}}), T_2)$ are pre-Lie-morphism triples.

Lemma 4.6. *Let $T_1, T_2 : V \longrightarrow \mathfrak{g}$ be two $\mathcal{O}$-operators on the pre-Lie algebra $\mathfrak{g}$ associated to the representation $(V;\rho,\mu)$. Suppose that T_2 is invertible. If T_1 and T_2 are compatible, then*

(i) *$N \circ T_1 = T_1 \circ S$ and S is a Nijenhuis operator on the pre-Lie algebra $(V, \cdot^{T_1})$, where $N = T_1 \circ T_2^{-1}$ and $S = T_2^{-1} \circ T_1$;*

(ii) *$N \circ T_2 = T_2 \circ S$ and S is a Nijenhuis operator on the pre-Lie algebra $(V, \cdot^{T_2})$, where $N = T_1 \circ T_2^{-1}$ and $S = T_2^{-1} \circ T_1$.*

Proof. This follows from Theorem 4.4 and Proposition 4.21 in [19]. ■

Proposition 4.7. *Let $T_1, T_2 : V \longrightarrow \mathfrak{g}$ be two $\mathcal{O}$-operators on the pre-Lie algebra $\mathfrak{g}$ associated to the representation $(V;\rho,\mu)$. Suppose that T_2 is invertible. If T_1 and T_2 are compatible, then (N, S) is a Nijenhuis pair on $((V, \cdot_{T_1}), (\mathfrak{g}, \cdot_{\mathfrak{g}}), T_1)$, and is also a Nijenhuis pair on $((V, \cdot_{T_2}), (\mathfrak{g}, \cdot_{\mathfrak{g}}), T_2)$, where $N = T_1 \circ T_2^{-1}$ and $S = T_2^{-1} \circ T_1$. Furthermore, $((V, \cdot_S^{T_1}), (\mathfrak{g}, \cdot_N), T_1)$ and $((V, \cdot_S^{T_2}), (\mathfrak{g}, \cdot_N), T_2)$ are pre-Lie-morphism triples, where the multiplication $\cdot_S^{T_i} : \otimes^2 V \longrightarrow V$ for $i = 1, 2$ is given by*

$$u \cdot_S^{T_i} v \;=\; S(u) \cdot^{T_i} v + u \cdot^{T_i} S(v) - S(u \cdot^{T_i} v), \quad \forall\, u, v \in V. \tag{55}$$

Proof. By Lemma 4.6, (N, S) is a Nijenhuis pair on $((V, \cdot_{T_1}), (\mathfrak{g}, \cdot_{\mathfrak{g}}), T_1)$ or $((V, \cdot_{T_2}), (\mathfrak{g}, \cdot_{\mathfrak{g}}), T_2)$. By the fact that T_1 is an $\mathcal{O}$-operators on the pre-Lie algebra $\mathfrak{g}$, we have $T_1(u \cdot^{T_1} v) = T_1(u) \cdot_{\mathfrak{g}} T_1(v)$ for all $u, v \in \mathfrak{g}$. Furthermore, by $N \circ T_1 = T_1 \circ S$, we have

$$\begin{aligned} T_1(u \cdot_S^{T_1} v) &= T_1(S(u) \cdot^{T_1} v + u \cdot^{T_1} S(v) - S(u \cdot^{T_1} v)) \\ &= N(T_1(u)) \cdot_{\mathfrak{g}} T_1(v) + T_1(u) \cdot^{T_1} N(T_1(v)) - N(T_1(u) \cdot_{\mathfrak{g}} T_1(v)) \\ &= T_1(u) \cdot_N T_1(v). \end{aligned}$$

Thus $((V, \cdot_S^{T_1}), (\mathfrak{g}, \cdot_N), T_1)$ is a pre-Lie-morphism triple. Similarly, we obtain a pre-Lie-morphism triple $((V, \cdot_S^{T_2}), (\mathfrak{g}, \cdot_N), T_2)$. ■

By Example 2.15 and Proposition 4.7, we have

Corollary 4.8. *Let $r_1, r_2 \in \mathrm{Sym}^2(\mathfrak{g})$ be two $\mathfrak{s}$-matrices on the pre-Lie algebra $\mathfrak{g}$. Assume that r_2 is invertible. If r_1 and r_2 are compatible in the sense that any linear combination of r_1 and r_2 is still an $\mathfrak{s}$-matrix, then $(N = r_1^\sharp \circ (r_2^\sharp)^{-1}, S = (r_2^\sharp)^{-1} \circ r_1^\sharp)$ is a Nijenhuis pair on $((\mathfrak{g}^*, \cdot^{r_1}), (\mathfrak{g}, \cdot_{\mathfrak{g}}), r_1^\sharp)$, and is also a Nijenhuis pair on $((\mathfrak{g}^*, \cdot^{r_2}), (\mathfrak{g}, \cdot_{\mathfrak{g}}), r_2^\sharp)$, where the operation $\cdot^{r_i}$ for $i = 1, 2$ is given by (14).*

4.2. Formal deformations of pre-Lie-morphism triples

Let $\mathbb{K}[[t]]$ be a ring of power series in one variable t. For any vector space V, we let $V[[t]]$ denote the vector space of formal power series in t with coefficients in V.

Definition 4.9. Let $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ be a pre-Lie algebra. A *formal deformation* of $\mathfrak{g}$ is a sequence of bilinear maps $\mu_k : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ for $k \geqslant 0$ with μ_0 being the pre-Lie algebra product $\cdot_{\mathfrak{g}}$ on $\mathfrak{g}$, such that the $\mathbb{K}[[t]]$-bilinear product $\cdot_t$ on $\mathfrak{g}[[t]]$ determined by

$$x \cdot_t y = \sum_{n=0}^{+\infty} t^n \mu_n(x, y), \quad \forall\, x, y \in \mathfrak{g} \tag{56}$$

is a pre-Lie algebra product. ■

The rule of the pre-Lie algebra product $\cdot_t$ on $\mathfrak{g}[[t]]$ is, for all $k \geqslant 0$, equivalent to

$$\sum_{i+j=k} \big(\mu_i(\mu_j(x, y), z) - \mu_i(x, \mu_j(y, z))\big) = \sum_{i+j=k} \big(\mu_i(\mu_j(y, x), z) - \mu_i(y, \mu_j(x, z))\big). \tag{57}$$

Let ϕ be a homomorphism between pre-Lie algebras $(\mathfrak{g}, \cdot_{\mathfrak{g}})$ and $(\mathfrak{h}, \cdot_{\mathfrak{h}})$. We consider a power series

$$\phi_t = \sum_{i=0}^{+\infty} \Phi_i t^i, \quad \Phi_i \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{h}), \tag{58}$$

with $\Phi_0 = \phi$. Here ϕ_t is a $\mathbb{K}[[t]]$-module map from $\mathfrak{g}[[t]]$ to $\mathfrak{h}[[t]]$ given by extending ϕ.

Definition 4.10. Let $((\mathfrak{g}, \cdot_{\mathfrak{g}}), (\mathfrak{h}, \cdot_{\mathfrak{h}}), \phi)$ be a pre-Lie-morphism triple. A *formal deformation* of $((\mathfrak{g}, \cdot_{\mathfrak{g}}), (\mathfrak{h}, \cdot_{\mathfrak{h}}), \phi)$ is a triple $((\mathfrak{g}[[t]], \cdot_t), (\mathfrak{h}[[t]], *_t), \phi_t)$, where $(\mathfrak{g}[[t]], \cdot_t)$ is a formal deformation of $(\mathfrak{g}, \cdot_{\mathfrak{g}})$, $(\mathfrak{h}[[t]], *_t)$ is a formal deformation of $(\mathfrak{h}, \cdot_{\mathfrak{h}})$ and ϕ_t satisfies

$$\phi_t(x \cdot_t y) = \phi_t(x) *_t \phi_t(y), \quad \forall\, x, y \in \mathfrak{g}, \tag{59}$$

where $\cdot_t$ is given (56), $*_t$ is given by

$$u *_t v = \sum_{n=0}^{+\infty} t^n \nu_n(u, v), \quad \forall\, u, v \in \mathfrak{h} \tag{60}$$

and ϕ_t is given by (58). ■

The rule of the pre-Lie algebra product $*_t$ on $\mathfrak{h}[[t]]$ is, for all $k \geqslant 0$, equivalent to

$$\sum_{i+j=k} \big(\nu_i(\nu_j(u, v), w) - \nu_i(u, \nu_j(v, w))\big) = \sum_{i+j=k} \big(\nu_i(\nu_j(v, u), w) - \nu_i(v, \nu_j(u, w))\big), \tag{61}$$

and (59) is equivalent to

$$\sum_{i+j=s} \Phi_i(\mu_j(x, y)) = \sum_{i+j+k=s} \nu_k(\Phi_i(x), \Phi_j(y)), \quad \forall\, s \geqslant 0. \tag{62}$$

Proposition 4.11. *Let* $((\mathfrak{g}[[t]],\cdot_t),(\mathfrak{h}[[t]],*_t),\phi_t)$ *be a formal deformation of a pre-Lie-morphism triple* $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$. *Then* (μ_1,ν_1,Φ_1) *is a* 1*-cocycle for the pre-Lie-morphism triple* $(\mathfrak{g},\mathfrak{h},\phi)$.

Proof. For $k=1$ in (57), we have

$$x\cdot_{\mathfrak{g}}\mu_1(y,z)-y\cdot_{\mathfrak{g}}\mu_1(x,z)+\mu_1(y,x)\cdot_{\mathfrak{g}}z-\mu_1(x,y)\cdot_{\mathfrak{g}}z$$
$$-\mu_1(y,x\cdot_{\mathfrak{g}}z)+\mu_1(x,y\cdot_{\mathfrak{g}}z)-\mu_1([x,y]_{\mathfrak{g}},z)=0.$$

For $k=1$ in (61), we have

$$u\cdot_{\mathfrak{h}}\nu_1(v,w)-v\cdot_{\mathfrak{h}}\nu_1(u,w)+\nu_1(v,u)\cdot_{\mathfrak{h}}w-\nu_1(u,v)\cdot_{\mathfrak{h}}w$$
$$-\nu_1(v,u\cdot_{\mathfrak{h}}w)+\nu_1(u,v\cdot_{\mathfrak{h}}w)-\nu_1([u,v]_{\mathfrak{h}},w)=0.$$

For $s=1$ in (62), we have

$$\phi(x)\cdot_{\mathfrak{h}}\Phi_1(y)+\Phi_1(x)\cdot_{\mathfrak{h}}\phi(y)-\Phi_1(x\cdot_{\mathfrak{g}}y)-\phi(\mu_1(x,y))+\nu_1(\phi(x),\phi(y))=0.$$

The above equations imply $\delta_{\mathrm{preLie}}(\mu_1,\nu_1,\Phi_1)=0$. ■

The 1-cocycle (μ_1,ν_1,Φ_1) given in Proposition 4.11 is called the *infinitesimal* of the formal deformation $((\mathfrak{g}[[t]],\cdot_t),(\mathfrak{h}[[t]],*_t),\phi_t)$ of $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$.

Definition 4.12. Let $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$ be a pre-Lie-morphism triple.
Two formal deformations $((\mathfrak{g}[[t]],\cdot_t),(\mathfrak{h}[[t]],*_t),\phi_t)$ and $((\mathfrak{g}[[t]],\cdot'_t),(\mathfrak{h}[[t]],*'_t),\phi'_t)$ of $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$ are said to be *equivalent* if there exist $\Psi_t:\mathfrak{g}[[t]]\to\mathfrak{g}[[t]]$ and $\Theta_t:\mathfrak{h}[[t]]\to\mathfrak{h}[[t]]$ given by

$$\Psi_t=\sum_{i=0}^{+\infty}t^i\Psi_i,\quad \Theta_t=\sum_{i=0}^{+\infty}t^i\Theta_i, \tag{63}$$

with $\Psi_0=\mathrm{Id}_{\mathfrak{g}}$, $\Psi_i\in\mathrm{End}(\mathfrak{g})$, $\Theta_0=\mathrm{Id}_{\mathfrak{h}}$ and $\Theta_i\in\mathrm{End}(\mathfrak{h})$, such that for $x,y\in\mathfrak{g}$, $u,v\in\mathfrak{h}$, the following conditions hold:

$$\Psi_t(x\cdot_t y) = \Psi_t(x)\cdot'_t\Psi_t(y), \tag{64}$$
$$\Theta_t(u*_t v) = \Theta_t(u)*'_t\Theta_t(v), \tag{65}$$
$$\Theta_t\circ\phi_t = \phi'_t\circ\Psi_t. \tag{66}$$

A formal deformation $((\mathfrak{g}[[t]],\cdot_t),(\mathfrak{h}[[t]],*_t),\phi_t)$ of $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$ is said to be *trivial* if the formal deformation $((\mathfrak{g}[[t]],\cdot_t),(\mathfrak{h}[[t]],*_t),\phi_t)$ is equivalent to the pre-Lie-morphism triple $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$.

Proposition 4.13. *If* $((\mathfrak{g}[[t]],\cdot_t),(\mathfrak{h}[[t]],*_t),\phi_t)$ *and* $((\mathfrak{g}[[t]],\cdot'_t),(\mathfrak{h}[[t]],*'_t),\phi'_t)$ *are equivalent formal deformations of a pre-Lie-morphism triple* $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$, *then their infinitesimals are in the same cohomology class.*

Proof. By (64), we have

$$\mu_1(x,y)-\mu'_1(x,y)=\Psi_1(x)\cdot_{\mathfrak{g}}y+x\cdot_{\mathfrak{g}}\Psi_1(y)+\Psi_1(x\cdot_{\mathfrak{g}}y),\quad \forall\, x,y\in\mathfrak{g}.$$

By (65), we have

$$\nu_1(u,v)-\nu_1'(u,v)=\Theta_1(u)\cdot_{\mathfrak{h}} v+u\cdot_{\mathfrak{h}}\Theta_1(v)+\Theta_1(u\cdot_{\mathfrak{h}} v),\quad \forall\, u,v\in\mathfrak{h}.$$

By (66), we have $$\Phi_1-\Phi_1'=\phi\circ\Psi_1-\Theta_1\circ\phi.$$

Thus we have $$(\mu_1,\nu_1,\Phi_1)-(\mu_1',\nu_1',\Phi_1')=\delta_{\mathrm{preLie}}(\Psi_1,\Theta_1,0).$$ ■

It is routine to check that

Proposition 4.14. *Let $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$ be a pre-Lie-morphism triple such that $H^1_{\mathrm{preLie}}(\phi,\phi)=0$. Then all formal deformations of $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$ are trivial.*

Let $(\mathfrak{g},\cdot_{\mathfrak{g}})$ be a pre-Lie algebra. Recall that an *order n-deformation* of $\mathfrak{g}$ is a sequence of bilinear maps $\mu_i:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ for $0\le i\le n$ with μ_0 being the pre-Lie algebra product $\cdot_{\mathfrak{g}}$ on $\mathfrak{g}$, such that the $\mathbb{K}[[t]]/(t^{n+1})$-bilinear product $\cdot_t$ on $\mathfrak{g}[[t]]/(t^{n+1})$ determined by

$$x\cdot_t y=\sum_{k=0}^{n}t^k\mu_k(x,y),\quad \forall\, x,y\in\mathfrak{g}$$

is a pre-Lie algebra product.

We denote this order n-deformation of $\mathfrak{g}$ by $\{\mu_1,\cdots,\mu_n\}$. Furthermore, if there exists a 1-cochain $\mu_{n+1}:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ such that $\{\mu_1,\cdots,\mu_n,\mu_{n+1}\}$ is an order $(n+1)$-deformation of $\mathfrak{g}$, the order n-deformation $\{\mu_1,\cdots,\mu_n\}$ of $\mathfrak{g}$ is called to be *extendable*.

Definition 4.15. Let $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$ be a pre-Lie-morphism triple. An *order n-deformation* of $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$ is a triple $((\mathfrak{g}[[t]],\cdot_t),(\mathfrak{h}[[t]],*_t),\phi_t)$, where $(\mathfrak{g}[[t]],\cdot_t)$ is an order n-deformation of $(\mathfrak{g},\cdot_{\mathfrak{g}})$, $(\mathfrak{h}[[t]],*_t)$ is an order n-deformation of $(\mathfrak{h},\cdot_{\mathfrak{h}})$ and ϕ_t satisfies

$$\phi_t(x\cdot_t y)=\phi_t(x)*_t\phi_t(y),\quad \forall\, x,y\in\mathfrak{g}, \tag{67}$$

where

$$x\cdot_t y=\sum_{i=0}^{n}t^i\mu_i(x,y),\quad u*_t v=\sum_{i=0}^{n}t^i\nu_i(u,v),\quad \phi_t=\sum_{i=0}^{n}\Phi_i t^i$$

for $x,y\in\mathfrak{g}$, $u,v\in\mathfrak{h}$. Furthermore, if there exists a 1-cochain

$$(\mu_{n+1},\nu_{n+1},\Phi_{n+1})\in\mathcal{C}^1_{\mathrm{preLie}}(\phi,\phi)$$

such that $((\mathfrak{g}[[t]],\tilde{\cdot}_t),(\mathfrak{h}[[t]],\tilde{*}_t),\tilde{\phi}_t)$ is an order $(n+1)$-deformation of $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$, where

$$x\tilde{\cdot}_t y=\sum_{i=0}^{n+1}t^i\mu_i(x,y),\quad u\tilde{*}_t v=\sum_{i=0}^{n+1}t^i\nu_i(u,v),\quad \tilde{\phi}_t=\sum_{i=0}^{n+1}\Phi_i t^i,$$

then the order n-deformation $((\mathfrak{g}[[t]],\cdot_t),(\mathfrak{h}[[t]],*_t),\phi_t)$ of $((\mathfrak{g},\cdot_{\mathfrak{g}}),(\mathfrak{h},\cdot_{\mathfrak{h}}),\phi)$ is called to be *extendable*. ■

Theorem 4.16. *Let* $((\mathfrak{g}[[t]], \cdot_t), (\mathfrak{h}[[t]], *_t), \phi_t)$ *be an order* n*-deformation of a pre-Lie-morphism triple* $((\mathfrak{g}, \cdot_{\mathfrak{g}}), (\mathfrak{h}, \cdot_{\mathfrak{h}}), \phi)$. *Define* $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathcal{C}^2_{\mathrm{preLie}}(\phi, \phi)$ *by*

$$\begin{aligned}
&\Omega_1(x, y, z)\\
&= \sum_{i+j=n+1, i,j\geq 1} \big(\mu_i(\mu_j(x, y), z) - \mu_i(x, \mu_j(y, z)) - \mu_i(\mu_j(y, x), z) + \mu_i(y, \mu_j(x, z))\big),\\
&\Omega_2(u, v, w)\\
&= \sum_{i+j=n+1, i,j\geq 1} \big(\nu_i(\nu_j(u, v), w) - \nu_i(u, \nu_j(v, w)) - \nu_i(\nu_j(v, u), w) + \nu_i(v, \nu_j(u, w))\big),\\
&\Omega_3(x, y)\\
&= \sum_{i+j=n+1, i,j\geq 1} \big(\Phi_i(\mu_j(x, y)) - \Phi_i(x) \cdot_{\mathfrak{h}} \Phi_j(y) - \nu_i(\phi(x), \Phi_j(y)) - \nu_i(\Phi_j(x), \phi(y))\big)\\
&\quad - \sum_{i+j+k=n+1, i,j,k\geq 1} \nu_i(\Phi_j(x), \Phi_k(y)),
\end{aligned}$$

where $x, y, z \in \mathfrak{g}$ *and* $u, v, w \in \mathfrak{h}$. *Then the* 2*-cochain* Ω *is closed, that is,* $\delta_{\mathrm{preLie}}\Omega = 0$.

Moreover, the order n*-deformation* $((\mathfrak{g}[[t]], \cdot_t), (\mathfrak{h}[[t]], *_t), \phi_t)$ *of* $((\mathfrak{g}, \cdot_{\mathfrak{g}}), (\mathfrak{h}, \cdot_{\mathfrak{h}}), \phi)$ *is extendable if and only if* $[\Omega] = 0$ *in* $H^2_{\mathrm{preLie}}(\phi, \phi)$.

Proof. The proof of this theorem is a tedious but straightforward calculation and similar to the dialgebra case in [24]. ∎

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