Journal of Lie Theory
Volume **33** (2023) 609–639

The Unbroken Spectra of Frobenius Seaweed Algebras

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Communicated by A. Fialowski

Abstract. We show that if $\mathfrak{g}$ is a Frobenius seaweed, then the spectrum of the adjoint of a principal element consists of an unbroken set of integers whose multiplicities have a symmetric distribution. Our methods are combinatorial.

Mathematics Subject Classification: 17B20, 05E15.

Key Words: Frobenius Lie algebra, seaweed, biparabolic, principal element, Dynkin diagram, spectrum, regular functional, Weyl group.

1. Introduction

Notation: All Lie algebras will be finite dimensional over $\mathbb{C}$, and the Lie multiplication will be denoted by $[-,-]$.

The *index* of a Lie algebra is an important algebraic invariant and, for *seaweed algebras*, is bounded by the algebra's rank: $\text{ind}\,\mathfrak{g} \leq \text{rk}\,\mathfrak{g}$, (see [8]). More formally, the index of a Lie algebra $\mathfrak{g}$ is given by

$$\text{ind}\,\mathfrak{g} = \min_{F\in\mathfrak{g}^*} \dim(\ker(B_F)),$$

where F is a linear form on $\mathfrak{g}$, and B_F is the associated skew-symmetric bilinear *Kirillov form*, defined by $B_F(x,y) = F([x,y])$ for $x, y \in \mathfrak{g}$. On a given $\mathfrak{g}$, index-realizing functionals are called *regular* and exist in profusion, being dense in both the Zariski and Euclidean topologies of $\mathfrak{g}^*$.

Of particular interest are Lie algebras which have index zero. Such algebras are called *Frobenius* and have been studied extensively from the point of view of invariant theory [20] and are of special interest in deformation and quantum group theory stemming from their connection with the classical Yang-Baxter equation (see [10] and [11]). A regular functional F on a Frobenius Lie algebra $\mathfrak{g}$ is called a *Frobenius functional*; equivalently, $B_F(-,-)$ is non-degenerate. Suppose $B_F(-,-)$ is non-degenerate and let $[F]$ be the matrix of $B_F(-,-)$ relative to some basis $\{x_1,\dots,x_n\}$ of $\mathfrak{g}$. In [1], Belavin and Drinfel'd showed that

$$\sum_{i,j}[F]^{-1}_{ij}x_i \wedge x_j$$

is the infinitesimal of a *Universal Deformation Formula* (UDF) based on $\mathfrak{g}$.

ISSN 0949–5932 / \$2.50

A UDF based on $\mathfrak{g}$ can be used to deform the universal enveloping algebra of $\mathfrak{g}$ and also the function space of any Lie group which contains $\mathfrak{g}$ in its Lie algebra of derivations. Despite the existence proof of Belavin and Drinfel'd, explicit UDF's are sparse. One does have the well-known exponential and quasi-exponential formula. These are based, respectively, on the abelian [4] and non-abelian [5] Lie algebras of dimension two (see also [13]).

A Frobenius functional can be algorithmically produced as a by-product of the Kostant Cascade (see [14] and [19]). If F is a Frobenius functional on $\mathfrak{g}$, then the natural map $\mathfrak{g} \to \mathfrak{g}^*$ defined by $x \mapsto F([x, -])$ is an isomorphism. The image of F under the inverse of this map is called a *principal element* of $\mathfrak{g}$ and will be denoted $\widehat{F}$. It is the unique element of $\mathfrak{g}$ such that

$$F \circ \operatorname{ad} \widehat{F} = F([\widehat{F}, -]) = F.$$

As a consequence of Proposition 3.1 in [21], Ooms established that the spectrum of the adjoint of a principal element of a Frobenius Lie algebra is independent of the principal element chosen to compute it (see also [11], Theorem 3). Generally, the eigenvalues of $\operatorname{ad}\widehat{F}$ can take on virtually any value (see [9] for examples). But, in their formal study of principal elements [12], Gerstenhaber and Giaquinto showed that if $\mathfrak{s}$ is a Frobenius seaweed subalgebra of $A_{n-1} = \mathfrak{sl}(n)$, then the spectrum of the adjoint of a principal element of $\mathfrak{s}$ consists entirely of integers. More generally, a *seaweed algebra* is defined as a subalgebra of a simple Lie algebra $\mathfrak{g}$ which is formed by two weakly opposite parabolic subalgebras of $\mathfrak{g}$, i.e., two parabolic subalgebras $\mathfrak{p}$ and $\mathfrak{p}'$ such that $\mathfrak{p} + \mathfrak{p}' = \mathfrak{g}$. In this case, $\mathfrak{s} = \mathfrak{p} \cap \mathfrak{p}'$ is called a *seaweed algebra* (or simply, *seaweed*). The *type* of $\mathfrak{s}$ is taken from its parent simple algebra. So, the integrality result of [12] applies to Frobenius type-A seaweeds.[1]

Subsequently in [6], the last three of the current authors showed that the spectrum of a type-A seaweed must actually consist of an *unbroken* sequence of integers centered at one half.[2] Moreover, the dimensions of the associated eigenspaces are shown to have a symmetric distribution.

The goal of this paper is to establish the following theorem, which asserts that the above-described unbroken symmetric spectral phenomena for type-A seaweeds is exhibited in all seaweed types.

Theorem 1.1. *If $\mathfrak{g}$ is a Frobenius seaweed and $\widehat{F}$ is a principal element of $\mathfrak{g}$, then the spectrum of $\operatorname{ad}\widehat{F}$ consists of an unbroken set of integers centered at one-half. Moreover, the dimensions of the associated eigenspaces form a symmetric distribution.*

Remark 1.2. In the prequel to this article [6], the type-A unbroken symmetric spectrum result is established using a combinatorial argument based on the graph-theoretic meander construction of Dergachev and Kirillov [8]. The combinatorial

[1] Joseph used different methods to strongly extend this integrality result to all seaweed subalgebras of semisimple Lie algebras [16].

[2] The paper of Coll et al. appears as a follow-up to the *Lett. in Math. Physics* article by Gerstenhaber and Giaquinto [12], where they claim that the eigenvalues of the adjoint representation of a semisimple principal element of a Frobenius seaweed subalgebra of $\mathfrak{sl}(n)$ consists of an unbroken sequence of integers. However, M. Dufflo, in a private communication to those authors, indicated that their proof contained an error.

arguments heavily leverage the results of [6], but the inductions here are predicated on the basis-independent "orbit meander" construction of Joseph [16].

Remark 1.3. To establish Theorem 1.1, we first combinatorially establish that the spectrum is symmetric about one-half. We hasten to add that Ooms had previously (1980) established the symmetry result for all Frobenius Lie algebras using a more algebraic approach [21], cf. [12].

Remark 1.4. In [9] Diatta and Manga show that any Frobenius Lie algebra can be embedded into $\mathfrak{sl}(n)$ for some n. They suggest that it would be interesting if one could find an obstruction to embedding the algebra as a seaweed. The unbroken spectrum provides such an obstruction.

2. Seaweeds

Let $\mathfrak{g}$ be a simple Lie algebra equipped with a triangular decomposition

$$\mathfrak{g} = \mathfrak{u}_+ \oplus \mathfrak{h} \oplus \mathfrak{u}_-, \tag{1}$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let Δ be its root system where Δ_+ are the *positive roots* on $\mathfrak{u}_+$ and Δ_- are the *negative roots* roots on $\mathfrak{u}_-$, and let Π denote the set of *simple roots* of $\mathfrak{g}$. Given $\beta \in \Delta$, let $\mathfrak{g}_\beta$ denote its corresponding root space, and let x_β denote the element of weight β in a Chevalley basis of $\mathfrak{g}$. Given a subset $\pi_1 \subseteq \Pi$, let $\mathfrak{p}_{\pi_1}$ denote the parabolic subalgebra of $\mathfrak{g}$ generated by all $\mathfrak{g}_\beta$ such that $-\beta \in \Pi$ or $\beta \in \pi_1$. Such a parabolic subalgebra is called *standard* with respect to the Borel subalgebra $\mathfrak{u}_- \oplus \mathfrak{h}$, and it is known that every parabolic subalgebra is conjugate to exactly one standard parabolic subalgebra.

As noted in the Introduction, the formation of a seaweed subalgebra of $\mathfrak{g}$ requires two weakly opposite parabolic subalgebras of $\mathfrak{g}$. For this reason, seaweeds are elsewhere called *biparabolic* (see [16]). Given a subset $\pi_2 \subseteq \Pi$, let $\mathfrak{p}^-_{\pi_2}$ denote the parabolic subalgebra of $\mathfrak{g}$ generated by all $\mathfrak{g}_\beta$ such that $\beta \in \Pi$ or $-\beta \in \pi_2$. Given two subsets $\pi_1, \pi_2 \subseteq \Pi$, we have $\mathfrak{p}_{\pi_1} + \mathfrak{p}_{\pi_2} = \mathfrak{g}$. We now define the seaweed

$$\mathfrak{p}(\pi_1 \mid \pi_2) = \mathfrak{p}_{\pi_1} \cap \mathfrak{p}^-_{\pi_2},$$

which is said to be *standard* with respect to the triangular decomposition in (1). Any seaweed is conjugate to a standard one, so it suffices to work with standard seaweeds only. Note that an arbitrary seaweed may be conjugate to more than one standard seaweed (see [22]).

We will often assume that $\pi_1 \cup \pi_2 = \Pi$, for if not then $\mathfrak{p}(\pi_1 \mid \pi_2)$ can be expressed as a direct sum of seaweeds. Additionally, we use superscripts and subscripts to specify the type and rank of the containing simple Lie algebra $\mathfrak{g}$. For example $\mathfrak{p}^{\mathrm{C}}_n(\pi_1 \mid \pi_2)$ is a seaweed subalgebra of $C_n = \mathfrak{sp}(2n)$, the symplectic Lie algebra of rank n.

It will be convenient to visualize the simple roots of a seaweed by constructing a graph, which we call a "split Dynkin diagram". Suppose $\mathfrak{g}$ has rank n, and let $\Pi = \{\alpha_n, \ldots, \alpha_1\}$, where α_1 is the exceptional root for types B, C, and D. Draw two horizontal lines of n vertices, say $v^+_n, \ldots, v^+_1$ on top and $v^-_n, \ldots, v^-_1$ on the bottom. Color v^+_i black if $\alpha_i \in \pi_1$, color v^-_i black if $\alpha_i \in \pi_2$, and color all other vertices white. Furthermore, if $\alpha_i, \alpha_j \in \pi_1$ are not orthogonal, connect v^+_i and v^+_j

with an edge in the standard way used in Dynkin diagrams. Do the same for bottom vertices according to the roots in π_2. A *maximally connected component* of a split Dynkin diagram is defined in the obvious way, and such a component is of type B, C, or D if it contains the exceptional root α_1; otherwise the component is of type A. See Examples 2.1 - 2.4 for what will, res become our type-A, type-C, and type-D running examples, respectively.

Example 2.1. Define the seaweed $\mathfrak{p}_9^{\mathrm{A}}(\Upsilon_1 \mid \Upsilon_2)$ by the following sets:

$$\Upsilon_1 = \{\alpha_9, \alpha_7, \alpha_6, \alpha_4, \alpha_3, \alpha_2, \alpha_1\} \quad \text{and} \quad \Upsilon_2 = \{\alpha_9, \alpha_8, \alpha_7, \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1\}.$$

See Figure 1 for the split Dynkin diagram of $\mathfrak{p}_9^{\mathrm{A}}(\Upsilon_1 \mid \Upsilon_2)$.

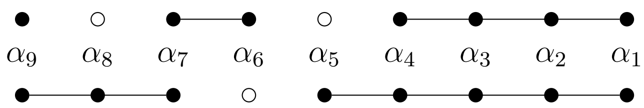

Figure 1: The split Dynkin diagram of $\mathfrak{p}_9^{\mathrm{A}}(\Upsilon_1 \mid \Upsilon_2)$

Example 2.2. Define the seaweed $\mathfrak{p}_8^{\mathrm{B}}(\Pi_1 \mid \Pi_2)$ by the following sets:

$$\Pi_1 = \{\alpha_8, \alpha_7, \alpha_6, \alpha_3, \alpha_2, \alpha_1\} \text{ and } \Pi_2 = \{\alpha_8, \alpha_7, \alpha_5, \alpha_4, \alpha_3, \alpha_2\}.$$

See Figure 2 for the split Dynkin diagram of $\mathfrak{p}_8^{\mathrm{B}}(\Pi_1 \mid \Pi_2)$.

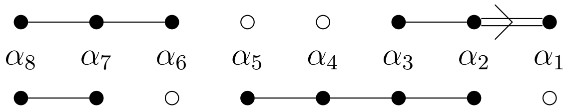

Figure 2: The split Dynkin diagram of $\mathfrak{p}_8^{\mathrm{B}}(\Pi_1 \mid \Pi_2)$

Example 2.3. Define the seaweed $\mathfrak{p}_8^{\mathrm{C}}(\Pi_1 \mid \Pi_2)$ by the following sets:

$$\Pi_1 = \{\alpha_8, \alpha_7, \alpha_6, \alpha_3, \alpha_2, \alpha_1\} \text{ and } \Pi_2 = \{\alpha_8, \alpha_7, \alpha_5, \alpha_4, \alpha_3, \alpha_2\}.$$

See Figure 3 for the split Dynkin diagram of $\mathfrak{p}_8^{\mathrm{C}}(\Pi_1 \mid \Pi_2)$.

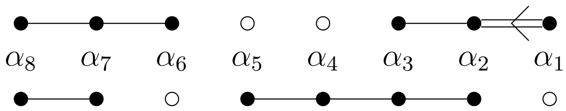

Figure 3: The split Dynkin diagram of $\mathfrak{p}_8^{\mathrm{C}}(\Pi_1 \mid \Pi_2)$

Example 2.4. Define the seaweed $\mathfrak{p}_{14}^{\mathrm{D}}(\Psi_1 \mid \Psi_2)$ by the following sets:

$$\Psi_1 = \{\alpha_{14}, \alpha_{13}, \alpha_{12}, \alpha_{11}, \alpha_{10}, \alpha_9, \alpha_8, \alpha_7, \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1\},$$

and
$$\Psi_2 = \{\alpha_{14}, \alpha_{13}, \alpha_{12}, \alpha_{11}, \alpha_9, \alpha_8, \alpha_7, \alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2\}.$$

See Figure 4 for the split Dynkin diagram of $\mathfrak{p}_{14}^{\mathrm{D}}(\Psi_1 \mid \Psi_2)$.

Given a seaweed $\mathfrak{p}(\pi_1 \mid \pi_2)$ of a simple Lie algebra $\mathfrak{g}$, let W denote the Weyl group of its root system Δ, generated by the reflections s_α such that $\alpha \in \Pi$.

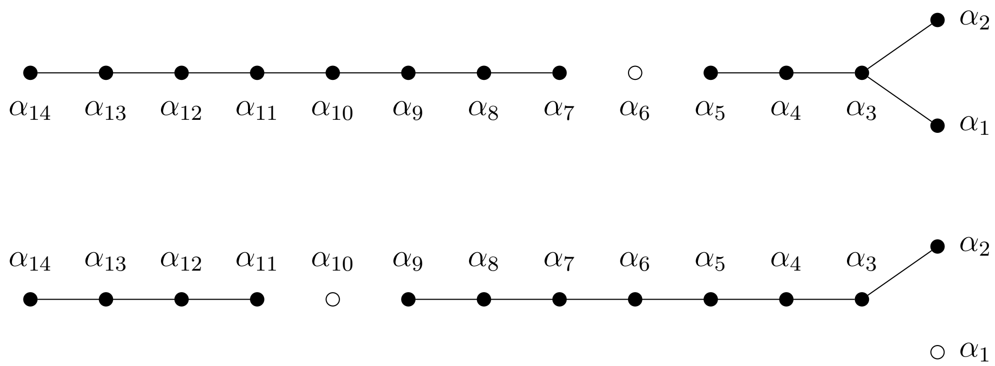

Figure 4: The split Dynkin diagram for $\mathfrak{p}_{14}^{\mathrm{D}}(\Psi_1 \mid \Psi_2)$

For $j = 1, 2$ define W_{π_j} to be the subgroup of W generated by s_α such that $\alpha \in \pi_j$. Let w_j denote the unique longest (in the usual Coxeter sense) element of W_{π_j}, and define an action

$$i_j\alpha = \begin{cases} -w_j\alpha, & \text{for all } \alpha \in \pi_j; \\ \alpha, & \text{for all } \alpha \in \Pi \setminus \pi_j. \end{cases}$$

Note that i_j is an involution. For components of types B, C, and D_k with k even, the longest element $w_j = -id$. However, if k is odd,

$$w_j\alpha_i = \begin{cases} \alpha_2, & \text{if } i = 1; \\ \alpha_1, & \text{if } i = 2; \\ \alpha_i, & \text{if } i \geq 3. \end{cases}$$

To visualize the action of i_j, we append dashed edges to the split Dynkin diagram of a seaweed. Specifically, we draw a dashed edge from v_i^+ to v_j^+ if $i_1\alpha_i = \alpha_j$, and we draw a dashed edge from v_i^- to v_j^- if $i_2\alpha_i = \alpha_j$. For simplicity, we will omit drawing the looped edge in the case that $i_j\alpha_i = \alpha_i$. We call the resulting graph the *orbit meander* of the associated seaweed. See Figures 5–8.

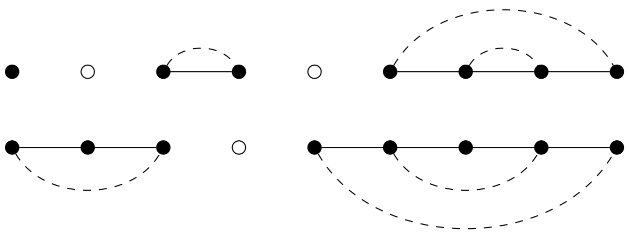

Figure 5: The orbit meander of $\mathfrak{p}_9^{\mathrm{A}}(\Upsilon_1 \mid \Upsilon_2)$

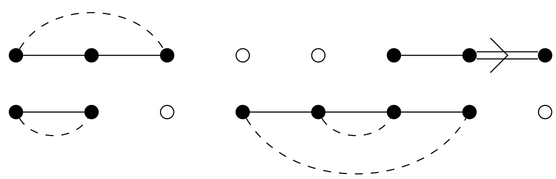

Figure 6: The orbit meander of $\mathfrak{p}_8^{\mathrm{B}}(\Pi_1 \mid \Pi_2)$

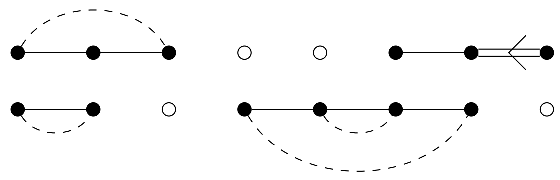

Figure 7: The orbit meander of $\mathfrak{p}_8^{\mathrm{C}}(\Pi_1 \mid \Pi_2)$

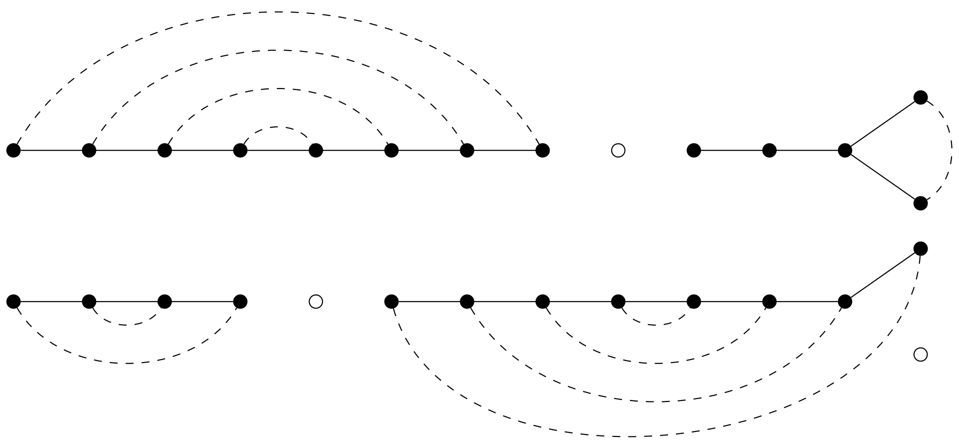

Figure 8: The orbit meander of $\mathfrak{p}_{14}^{\mathrm{D}}(\Psi_1 \mid \Psi_2)$

It turns out that the index of $\mathfrak{p}(\pi_1 \mid \pi_2)$ is governed by the orbits of the cyclic group $< i_1 i_2 >$ acting on Π.

Theorem 2.5. (Joseph [18], Lemma 4.2) *Given subsets $\pi_1, \pi_2 \subseteq \Pi$ such that $\pi_1 \cup \pi_2 = \Pi$, let $\pi_\cup = \Pi \setminus (\pi_1 \cap \pi_2)$. The seaweed $\mathfrak{p}(\pi_1 \mid \pi_2)$ is Frobenius if and only if every $< i_1 i_2 >$ orbit contains exactly one element from $\pi_\cup$.*

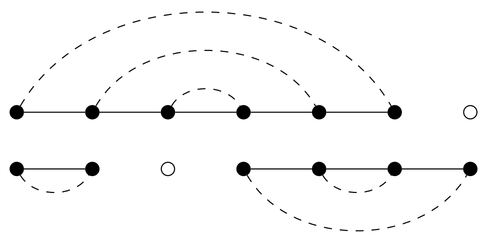

Figure 9: The orbit meander of $\mathfrak{p}_7^{\mathrm{A}}(\{\alpha_7, \alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2\} \mid \{\alpha_7, \alpha_6, \alpha_4, \alpha_3, \alpha_2, \alpha_1\})$

Example 2.6. We show that the seaweed in each of our running examples is Frobenius. The type-A seaweed $\mathfrak{p}_9^{\mathrm{A}}(\Upsilon_1 \mid \Upsilon_2)$ is Frobenius as its $< i_1 i_2 >$ orbits are $\{\alpha_9, \alpha_6, \alpha_7\}$, $\{\alpha_8\}$, $\{\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1\}$, and each orbit contains exactly one element from $\pi_\cup = \{\alpha_8, \alpha_6, \alpha_5\}$. The type-B and type-C seaweeds are Frobenius as each seaweed's $< i_1 i_2 >$ orbits are $\{\alpha_7, \alpha_6, \alpha_8\}$, $\{\alpha_5, \alpha_2\}$, $\{\alpha_4, \alpha_3\}$, $\{\alpha_1\}$, and each orbit contains exactly one element from $\pi_\cup = \{\alpha_6, \alpha_5, \alpha_4, \alpha_1\}$. Similarly, the type-D seaweed $\mathfrak{p}_{14}^{\mathrm{D}}(\Psi_1 \mid \Psi_2)$ is Frobenius as its $< i_1 i_2 >$ orbits are $\{\alpha_6, \alpha_5\}$, $\{\alpha_{14}, \alpha_{11}, \alpha_{10}, \alpha_7, \alpha_4\}$, and $\{\alpha_{13}, \alpha_{12}, \alpha_9, \alpha_8, \alpha_3, \alpha_2, \alpha_1\}$, each of which contains exactly one element from $\pi_\cup = \{\alpha_{10}, \alpha_6, \alpha_1\}$.

Example 2.7. For an example of a seaweed that is not Frobenius, consider the orbit meander of $\mathfrak{p}_7^{\mathrm{A}}(\{\alpha_7, \alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2\} \mid \{\alpha_7, \alpha_6, \alpha_4, \alpha_3, \alpha_2, \alpha_1\})$ shown in Figure 9 below. The $< i_1 i_2 >$ orbits $\{\alpha_7, \alpha_3\}$ and $\{\alpha_6, \alpha_2\}$ contain no elements from $\pi_\cup = \{\alpha_5, \alpha_1\}$, and the orbit $\{\alpha_5, \alpha_4, \alpha_1\}$ contains two elements from $\pi_\cup$.

3. Principal elements

Given a Frobenius seaweed with Frobenius functional F and corresponding principal element $\widehat{F}$, the eigenvalues of $\operatorname{ad} \widehat{F}$ are independent of which Frobenius functional is chosen [21]. We call these eigenvalues the *spectrum* of the seaweed. In this section, we describe an algorithm for computing them.

Let $\mathfrak{p}(\pi_1 \mid \pi_2)$ be Frobenius and F_{π_1,π_2} be an associated Frobenius functional with principal element $\widehat{F}_{\pi_1,\pi_2}$. (We will simply write F and $\widehat{F}$ when the seaweed is understood.) Let σ be a maximally connected component of either π_1 or π_2, and for convenience let $\sigma = \{\alpha_k, \alpha_{k-1}, \ldots, \alpha_1\}$ where α_1 is the exceptional root if σ is of type B, C, or D.

Each eigenvalue of $\operatorname{ad} \widehat{F}$ can be expressed as a linear combination of elements $\alpha_i(\widehat{F})$ where α_i is a simple root. We call such numbers *simple eigenvalues*. In many cases, the simple eigenvalues are determined.

Lemma 3.1. (Joseph [15], Section 5) *In Table* 1 *below, the given value is* $\alpha_i(\widehat{F})$ *if* σ *is a maximally connected component of* π_1*, and it is* $-\alpha_i(\widehat{F})$ *if* σ *is a maximally connected component of* π_2*. In either case it is assumed that* $\alpha_i \in \sigma$*.*

Type	$\pm\alpha_i(\widehat{F})$	$\pm\alpha_i(\widehat{F})$
$A_k : k \geq 1$	1, if $i_j\alpha_i = \alpha_i$	
$B_{2k-1} : k \geq 2$	$(-1)^{i-1}$, if $1 \leq i \leq 2k-1$	
$B_{2k} : k \geq 2$	$(-1)^i$, if $2 \leq i \leq 2k$	0, if $i = 1$
$C_k : k \geq 2$	0, if $2 \leq i \leq k$	1, if $i = 1$
$D_{2k} : k \geq 2$	$(-1)^i$, if $3 \leq i \leq 2k$	1, if $i = 1, 2$
$D_{2k+1} : k \geq 2$	$(-1)^{i-1}$, if $3 \leq i \leq 2k+1$	

Table 1: Values of $\pm\alpha_i(\widehat{F})$

For the cases not covered by Table 1, the following lemma (which includes a corrected typo from [15]) can be applied.

Lemma 3.2. (Joseph [15], Section 5) *For each equation below,* σ *is assumed to be a maximally connected component of* π_1*. If* σ *is a maximally connected component of* π_2*, then replace* $\alpha_i \mapsto -\alpha_i$ *and* $i_1 \mapsto i_2$*. Then*

$$\alpha_i(\widehat{F}) + i_1\alpha_i(\widehat{F}) = \begin{cases} 1, & \text{if } \sigma \text{ is of type } A_k \text{ and } (\alpha_i, i_1\alpha_i) < 0; \\ 0, & \text{if } \sigma \text{ is of type } A_k \text{ and } (\alpha_i, i_1\alpha_i) = 0; \\ 0, & \text{if } \sigma \text{ is of type } D_{2k+1} \text{ and } i = 1. \end{cases}$$

Here, (α, β) denotes the standard inner product on the Euclidean representation of the simple roots. Using Table 1 and applying Lemma 3.2, we compute the simple eigenvalues for each running example. See Figures 10–13, where each simple eigenvalue is noted above, below, or next to the appropriate vertex in the orbit meander for this seaweed.

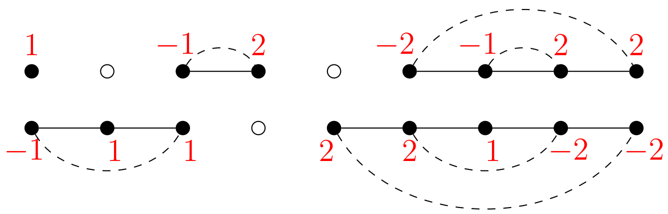

Figure 10: The simple eigenvalues of $\mathfrak{p}_9^{\mathrm{A}}(\Upsilon_1 \mid \Upsilon_2)$

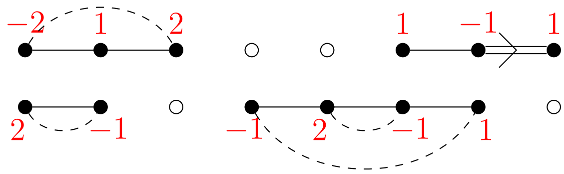

Figure 11: The simple eigenvalues of $\mathfrak{p}_8^{\mathrm{B}}(\Pi_1 \mid \Pi_2)$

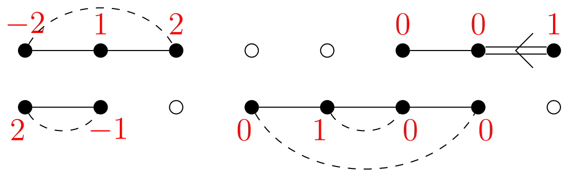

Figure 12: The simple eigenvalues of $\mathfrak{p}_8^{\mathrm{C}}(\Pi_1 \mid \Pi_2)$

Remark 3.3. Modulo the arrow which emanates from the exceptional root, observe that the components and Weyl action for $\mathfrak{p}_8^{\mathrm{B}}(\Pi_1 \mid \Pi_2)$ and $\mathfrak{p}_8^{\mathrm{C}}(\Pi_1 \mid \Pi_2)$ are identical. Even so, the simple eigenvalues of the type-B and type-C components differ. Curiously, this affects the simple eigenvalues associated to the simple eigenvalues of same the type-A components of $\mathfrak{p}_8^{\mathrm{B}}(\Pi_1 \mid \Pi_2)$ and $\mathfrak{p}_8^{\mathrm{C}}(\Pi_1 \mid \Pi_2)$ leading to different spectra for these two seaweeds.

Remark 3.4. We will use the Greek letter σ when referring to the simple roots of a maximally connected component of an orbit meander. However, there are times when it will be more convenient to consider the set of eigenvalues associated to a set of consecutive vertices in an orbit meander. If A is a set of consecutive vertices in an orbit meander, let $\mathcal{E}(A)$ be the eigenvalues associated to A. That is, if $A = \{v_k, v_{k-1}, ..., v_j\}$, let $\sigma = \{\alpha_k, \alpha_{k-1}, ..., \alpha_j\}$. We make the following notational convention:

$$\mathcal{E}(A) := \mathcal{E}(\sigma).$$

We have the following corollary of Lemma 3.2 that will be used to prove symmetry and the unbroken property.

Theorem 3.5. *If σ is a maximally connected component of type A_k, then* $\sum_{i=1}^{k} \alpha_i(\widehat{F}) = 1$.

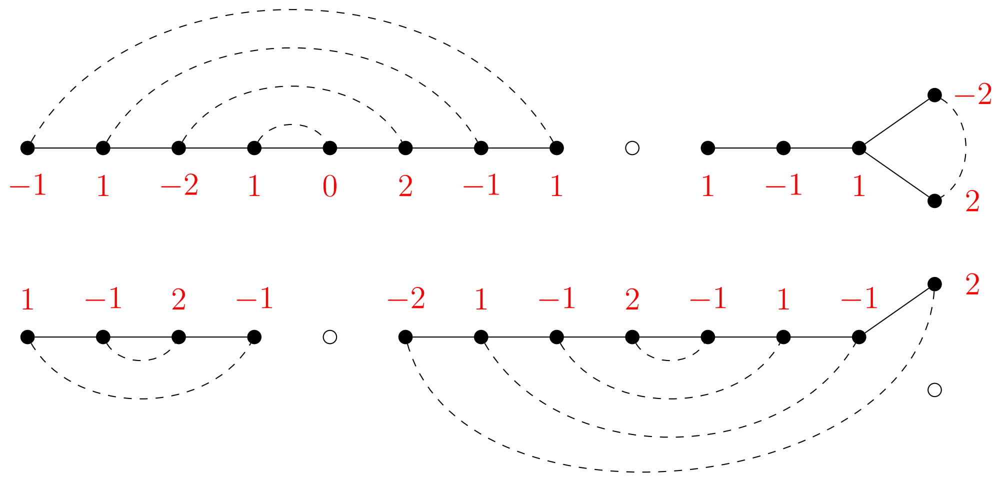

Figure 13: The simple eigenvalues of $\mathfrak{p}^{\mathrm{D}}_{14}(\Psi_1 \mid \Psi_2)$

Proof. If k is odd, then $\alpha_i(\widehat{F}) = -\alpha_{k+1-i}(\widehat{F})$ for $i < \frac{k+1}{2}$, and $\alpha_{\frac{k+1}{2}}(\widehat{F}) = 1$. If k is even, then $\alpha_i(\widehat{F}) = -\alpha_{k+1-i}(\widehat{F})$ for $i < \frac{k}{2}$, and $\alpha_{\frac{k}{2}}(\widehat{F}) + \alpha_{\frac{k}{2}+1}(\widehat{F}) = 1$. ■

It will be convenient to use the following notation. Let σ be a maximally connected component of π_1, and let $\sigma = \{\alpha_k, \alpha_{k-1}, ..., \alpha_1\}$ be of type A_k. The positive roots of σ are of the form $\alpha_j + \alpha_{j-1} + \cdots + \alpha_i$, where $k \geq j \geq i \geq 1$. If α is a positive root with $j + i \neq k + 1$, call $\overline{\alpha}$ its *symmetric (positive) root*, where

$$\overline{\alpha} = \begin{cases} \alpha_{i-1} + \alpha_{i-2} + \cdots + \alpha_{k+1-j}, & \text{if } k \geq j > i \geq \left\lceil \frac{k}{2} \right\rceil \geq 1; \\ \alpha_{i-1} + \alpha_{i-2} + \cdots + \alpha_{k+1-j}, & \text{if } \left| j - \left\lceil \frac{k}{2} \right\rceil \right| > \left| i - \left\lceil \frac{k}{2} \right\rceil \right|; \\ \alpha_{j+1} + \alpha_{j+2} + \cdots + \alpha_{k+1-i}, & \text{if } \left| j - \left\lceil \frac{k}{2} \right\rceil \right| < \left| i - \left\lceil \frac{k}{2} \right\rceil \right|; \\ \alpha_{j+1} + \alpha_{j+2} + \cdots + \alpha_{k+1-i}, & \text{if } k \geq \left\lceil \frac{k}{2} \right\rceil \geq j > i \geq 1. \end{cases}$$

As a corollary of Theorem 3.5, the symmetric roots α and $\overline{\alpha}$ satisfy the following relation:

$$\alpha(\widehat{F}) + \overline{\alpha}(\widehat{F}) = 1 \tag{2}$$

Symmetric roots satisfy the relation (2), and since $\alpha(\widehat{F}) = 1$ when $j + i = k + 1$, we say the positive root α has no symmetric root. We call $\overline{\alpha}(\widehat{F})$ the *symmetric eigenvalue* of $\alpha(\widehat{F})$ or that $\overline{\alpha}(\widehat{F})$ is an eigenvalue *symmetric to* $\alpha(\widehat{F})$.

Example 3.6. Referring to, for example, Figure 12, consider the type-A component on the bottom $\sigma = \{\alpha_5, \alpha_4, \alpha_3, \alpha_2\}$. The root symmetric to $\alpha = \alpha_5 + \alpha_4 + \alpha_3$ is $\overline{\alpha} = \alpha_2$, and the root symmetric to $\alpha = \alpha_5 + \alpha_4$ is $\overline{\alpha} = \alpha_3 + \alpha_2$. The root $\alpha = \alpha_5 + \alpha_4 + \alpha_3 + \alpha_2$ has no symmetric root.

4. Proof of Theorem 1.1

The proof will consist of two parts.

Part I: Symmetry

We will partition the multiset of eigenvalues according to the maximally connected components σ of π_1 and π_2. We will prove that each member of this partition is symmetric and unbroken.

If σ is a Type-A maximally connected component of π_1, then we say the multiset of eigenvalues from σ is

$$\mathcal{E}(\sigma) = \{\beta(\widehat{F}) \mid \beta \in \mathbb{N}\sigma \cap \Delta_+\} \cup \{0^{\lceil |\sigma|/2 \rceil}\}, \tag{3}$$

if σ is of Type B, C or D_k with k even, then

$$\mathcal{E}(\sigma) = \{\beta(\widehat{F}) \mid \beta \in \mathbb{N}\sigma \cap \Delta_+\} \cup \{0^{|\sigma|}\}, \tag{4}$$

and if σ is of Type D_k with k odd, then

$$\mathcal{E}(\sigma) = \{\beta(\widehat{F}) \mid \beta \in \mathbb{N}\sigma \cap \Delta_+\} \cup \{0^{|\sigma|-1}\}, \tag{5}$$

where the exponent denotes the multiplicity, and $\mathbb{N}\sigma = \{\sum c_i\alpha_i \mid c_i > 0 \text{ and } \alpha_i \in \sigma\}$. We proceed similarly if σ is a maximally connected component of π_2 except that Δ_+ is replaced by Δ_- in equations (3)–(5). We demonstrate these computations for each running example.

Example 4.1. In the type-A running example $\mathfrak{p}_9^{\mathrm{A}}(\Upsilon_1 \mid \Upsilon_2)$, on the top there are three type-A components: $\sigma_1 = \{\alpha_9\}$, $\sigma_2 = \{\alpha_7, \alpha_6\}$, and $\sigma_3 = \{\alpha_4, \alpha_3, \alpha_2, \alpha_1\}$. There are two type-A components on the bottom, namely: $\sigma_4 = \{\alpha_9, \alpha_8, \alpha_7\}$ and $\sigma_5 = \{\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1\}$. To compute $\mathcal{E}(\sigma_i)$, note, for example, that the positive roots for the computation of $\mathcal{E}(\sigma_3)$ are elements of the set

$$B_{\sigma_3} = \{\alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_4+\alpha_3, \alpha_3+\alpha_2, \alpha_2+\alpha_1, \alpha_4+\alpha_3+\alpha_2, \alpha_3+\alpha_2+\alpha_1, \alpha_4+\alpha_3+\alpha_2+\alpha_1\}.$$

Applying each of $\beta \in B_{\sigma_3}$ to $\widehat{F}$ yields the multiset

$$\{-2, -1, 2, 2, -3, 1, 4, -1, 3, 1\} = \{-3, -2, -1, -1, 1, 1, 2, 2, 3, 4\}.$$

Since $|\sigma_3| = 4$, we have by equation (3) that

$$\mathcal{E}(\sigma_3) = \{-3, -2, -1, -1, 1, 1, 2, 2, 3, 4\} \cup \{0, 0\} = \{-3, -2, -1^2, 0^2, 1^2, 2^2, 3, 4\}.$$

Similar computations yield

$$\mathcal{E}(\sigma_1) = \{0, 1\}, \quad \mathcal{E}(\sigma_2) = \{-1, 0, 1, 2\}, \quad \mathcal{E}(\sigma_4) = \{-1, 0^3, 1^3, 2\},$$
$$\mathcal{E}(\sigma_5) = \{-4, -3, -2^2, -1^2, 0^3, 1^3, 2^2, 3^2, 4, 5\}.$$

Example 4.2. In the type-B running example $\mathfrak{p}_8^{\mathrm{B}}(\Pi_1 \mid \Pi_2)$, on the top there is a single type-A component $\sigma_1 = \{\alpha_8, \alpha_7, \alpha_6\}$ and a single type-B component $\sigma_2 = \{\alpha_3, \alpha_2, \alpha_1\}$. There are two type-A components on the bottom: $\sigma_3 = \{\alpha_8, \alpha_7\}$ and $\sigma_4 = \{\alpha_5, \alpha_4, \alpha_3, \alpha_2\}$. For the type-A components, computations similar to those in Example 4.1 yield

$$\mathcal{E}(\sigma_1) = \{-2, -1, 0^2, 1^2, 2, 3\}, \quad \mathcal{E}(\sigma_3) = \{-1, 0, 1, 2\}, \text{ and } \mathcal{E}(\sigma_4) = \{-1^2, 0^4, 1^4, 2^2\}.$$

For the type-B component, the positive roots for the computation of $\mathcal{E}(\sigma_2)$ are elements of the set

$$B_{\sigma_2} = \{\alpha_3, \alpha_2, \alpha_1, \alpha_3+\alpha_2, \alpha_2+\alpha_1, \alpha_3+\alpha_2+\alpha_1, \alpha_2+2\alpha_1, \alpha_3+\alpha_2+2\alpha_1, \alpha_3+2\alpha_2+2\alpha_1\}.$$

Applying each of $\beta \in B_{\sigma_2}$ to $\widehat{F}$ yields the multiset

$$\{1, -1, 1, 0, 0, 1, 1, 2, 1\} = \{-1, 0, 0, 1, 1, 1, 1, 1, 2\}.$$

Since $|\sigma_2| = 3$, we have by equation (4) that

$$\mathcal{E}(\sigma_2) = \{-1, 0, 0, 1, 1, 1, 1, 1, 2\} \cup \{0, 0, 0\} = \{-1, 0^5, 1^5, 2\}.$$

Example 4.3. In the type-C running example $\mathfrak{p}_8^{\rm C}(\Pi_1 \mid \Pi_2)$, on the top there is a single type-A component $\sigma_1 = \{\alpha_8, \alpha_7, \alpha_6\}$ and a single type-C component $\sigma_2 = \{\alpha_3, \alpha_2, \alpha_1\}$. There are two type-A components on the bottom: $\sigma_3 = \{\alpha_8, \alpha_7\}$ and $\sigma_4 = \{\alpha_5, \alpha_4, \alpha_3, \alpha_2\}$. For the type-A components, computations similar to those in Example 4.1 yield

$$\mathcal{E}(\sigma_1) = \{-2, -1, 0^2, 1^2, 2, 3\}, \quad \mathcal{E}(\sigma_3) = \{-1, 0, 1, 2\}, \text{ and } \mathcal{E}(\sigma_4) = \{0^6, 1^6\}.$$

For the type-C component, the positive roots for the computation of $\mathcal{E}(\sigma_2)$ are elements of the set

$$B_{\sigma_2} = \{\alpha_3, \alpha_2, \alpha_1, \alpha_3+\alpha_2, \alpha_2+\alpha_1, \alpha_3+\alpha_2+\alpha_1, 2\alpha_2+\alpha_1, \alpha_3+2\alpha_2+\alpha_1, 2\alpha_3+2\alpha_2+\alpha_1\}.$$

Applying each of $\beta \in B_{\sigma_2}$ to $\widehat{F}$ yields the multiset

$$\{0,0,1,0,1,1,1,1,1\} = \{0,0,0,1,1,1,1,1,1\}.$$

Since $|\sigma_2| = 3$, we have by equation (4) that

$$\mathcal{E}(\sigma_2) = \{0,0,1,0,1,1,1,1,1\} \cup \{0,0,0\} = \{0^6, 1^6\}.$$

Example 4.4. Finally, in the type-D running example, $\mathfrak{p}_{14}^{\rm D}(\Psi_1 \mid \Psi_2)$, on the top there again is single type-A component $\sigma_1 = \{\alpha_{14}, \alpha_{13}, \alpha_{12}, \alpha_{11}, \alpha_{10}, \alpha_9, \alpha_8, \alpha_7\}$ and there is a single type-D component $\sigma_2 = \{\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1\}$. There are also two type-A components on the bottom: $\sigma_3 = \{\alpha_{14}, \alpha_{13}, \alpha_{12}, \alpha_{11}\}$ and $\sigma_4 = \{\alpha_9, \alpha_8, \alpha_7, \alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2\}$. For the type-A components, computations similar to those in Example 4.1, yield the multisets:

$$\mathcal{E}(\sigma_1) = \{-2^2, -1^7, 0^{11}, 1^{11}, 2^7, 3^2\}, \quad \mathcal{E}(\sigma_3) = \{-1^2, 0^4, 1^4, 2^2\}, \text{ and}$$
$$\mathcal{E}(\sigma_4) = \{-2^2, -1^7, 0^{11}, 1^{11}, 2^7, 3^2\}.$$

For the type-D component, the positive roots for the computation of $\mathcal{E}(\sigma_2)$ are elements of the set

$$B_{\sigma_2} = \left\{ \begin{array}{c} \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_5+\alpha_4, \alpha_4+\alpha_3, \alpha_3+\alpha_2, \alpha_3+\alpha_1, \\ \alpha_5+\alpha_4+\alpha_3, \alpha_4+\alpha_3+\alpha_2, \alpha_4+\alpha_3+\alpha_1, \alpha_3+\alpha_2+\alpha_1, \\ \alpha_5+\alpha_4+\alpha_3+\alpha_2, \alpha_5+\alpha_4+\alpha_3+\alpha_1, \alpha_4+\alpha_3+\alpha_2+\alpha_1, \alpha_4+2\alpha_3+\alpha_2+\alpha_1, \\ \alpha_5+\alpha_4+\alpha_3+\alpha_2+\alpha_1, \alpha_5+\alpha_4+2\alpha_3+\alpha_2+\alpha_1, \alpha_5+2\alpha_4+2\alpha_3+\alpha_2+\alpha_1 \end{array} \right\}.$$

Applying each of $\beta \in B_{\sigma_2}$ to $\widehat{F}$ yields the multiset

$$\{-2^2, -1^3, 0^3, 1^7, 2^3, 3^2\}.$$

Since, $|\sigma_2| = 5$, we have by equation (5) that

$$\mathcal{E}(\sigma_2) = \{-2^2, -1^3, 0^3, 1^7, 2^3, 3^2\} \cup \{0^4\} = \{-2^2, -1^3, 0^7, 1^7, 2^3, 3^2\}.$$

Lemma 4.5. *If $\mathfrak{p}(\pi_1 \mid \pi_2)$ is a Frobenius seaweed, then the multisets of eigenvalues contributed by each maximally connected component form a multiset partition of the spectrum of the seaweed.*

We apply Lemma 4.5 to obtain the spectrum for each running example. We take the union of the sets $\mathcal{E}(\sigma_i)$ found in Examples 4.1–4.4.

This data is consolidated in Tables 2–5.

Eigenvalue	-4	-3	-2	-1	0	1	2	3	4	5
Multiplicity	1	2	3	6	10	10	6	3	2	1

Table 2: Spectrum of $\mathfrak{p}_9^{\mathrm{A}}(\Upsilon_1 \mid \Upsilon_2)$ with multiplicities

Eigenvalue	-2	-1	0	1	2	3
Multiplicity	1	5	12	12	5	1

Table 3: Spectrum of $\mathfrak{p}_8^{\mathrm{B}}(\Pi_1 \mid \Pi_2)$ with multiplicities

Eigenvalue	-2	-1	0	1	2	3
Multiplicity	1	2	15	15	2	1

Table 4: Spectrum of $\mathfrak{p}_8^{\mathrm{C}}(\Pi_1 \mid \Pi_2)$ with multiplicities

Eigenvalue	-2	-1	0	1	2	3
Multiplicity	6	19	33	33	19	6

Table 5: Spectrum of $\mathfrak{p}_{14}^{\mathrm{D}}(\Psi_1 \mid \Psi_2)$ with multiplicities

The following example illustrates a Frobenius type-D seaweed containing an even-sized type-D component.

Example 4.6. Define the seaweed $\mathfrak{p}_{11}^{\mathrm{D}}(\Phi_1 \mid \Phi_2)$ by the following sets:

$$\Phi_1 = \{\alpha_{11}, \alpha_{10}, \alpha_9, \alpha_8, \alpha_7, \alpha_6, \alpha_4, \alpha_3, \alpha_2, \alpha_1\},$$

and

$$\Phi_2 = \{\alpha_{10}, \alpha_9, \alpha_7, \alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2\}.$$

See Figure 14 for the simple eigenvalues of $\mathfrak{p}_{11}^{\mathrm{D}}(\Phi_1 \mid \Phi_2)$.

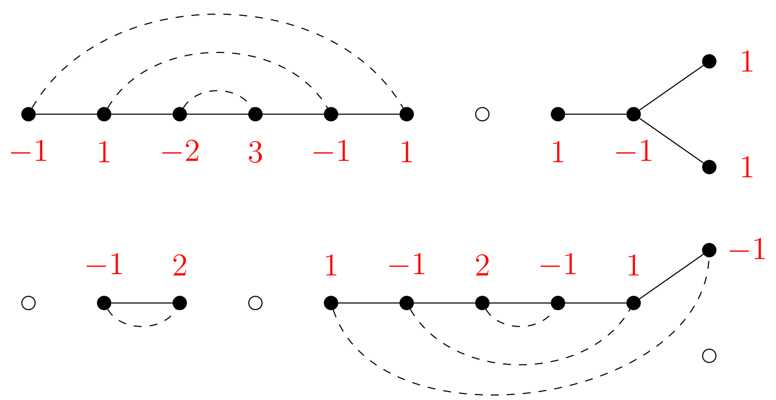

Figure 14: The simple eigenvalues of $\mathfrak{p}_{11}^{\mathrm{D}}(\Phi_1 \mid \Phi_2)$

The eigenvalues for $\mathfrak{p}_{11}^{\mathrm{D}}(\Phi_1 \mid \Phi_2)$ can be found using computations similar to those in Example 4.4 and are given in Table 6.

Eigenvalue	-2	-1	0	1	2	3
Multiplicity	2	9	23	23	9	2

Table 6: Spectrum of $\mathfrak{p}_{11}^{\mathrm{D}}(\Phi_1 \mid \Phi_2)$ with multiplicities

Notice that in Tables 2 - 6, $\mathcal{E}(\sigma_i)$ is symmetric about one half for each i. Furthermore, the multiplicities form a symmetric distribution. As the following theorem shows, these observations are not coincidences.

Theorem 4.7. *Let $\mathfrak{p}(\pi_1 \mid \pi_2)$ be a Frobenius seaweed. For each maximally connected component σ of π_1 or π_2, let r_i be the multiplicity of the eigenvalue i in $\mathcal{E}(\sigma)$. The sequence (i) is symmetric about one-half. Moreover, $r_{-i} = r_{i+1}$ for each eigenvalue i.*

Proof. We prove this in the case that σ is a maximally connected component of π_1. The case that σ is a maximally connected component of π_2 is similar and is omitted.

Case 1: Type A

Suppose σ is of type A_k. Then $\mathcal{E}(\sigma)$ is comprised of elements of the form

$$\alpha_j(\widehat{F}) + \alpha_{j-1}(\widehat{F}) + \cdots + \alpha_i(\widehat{F}),$$

where $k \geq j \geq i \geq 1$ along with $\lceil \frac{k}{2} \rceil$ zeros per Equation (3).

Each element $\alpha(\widehat{F})$ in $\mathcal{E}(\sigma)$ with $j + i \neq k + 1$ has a symmetric eigenvalue $\overline{\alpha}$ with $\alpha(\widehat{F}) + \overline{\alpha}(\widehat{F}) = 1$. Moreover, there are $\lceil \frac{k}{2} \rceil$ positive roots $\alpha_j + \alpha_{j-1} ... + \alpha_i$ with $k \geq j \geq i \geq 1$ and $j + i = k + 1$. These satisfy

$$\alpha_j(\widehat{F}) + \alpha_{j-1}(\widehat{F}) + \cdots + \alpha_i(\widehat{F}) = 1$$

and are in bijective correspondence with the zeros from Equation (3). Therefore, the multiset of eigenvalues from σ is symmetric about one-half.

Case 2: Type B

Suppose σ is of type B, with odd cardinality. For convenience, reorder the indices of the simple roots so that $\sigma = \{\alpha_{2k-1}, \alpha_{2k-2}, \ldots, \alpha_1\}$ as in Lemma 3.1. Then $\mathcal{E}(\sigma)$ is comprised of elements of the form

$$\alpha_j(\widehat{F}) + \alpha_{j-1}(\widehat{F}) + \cdots + \alpha_i(\widehat{F}),$$

where $2k - 1 \geq j \geq i \geq 1$, or

$$\alpha_j(\widehat{F}) + \cdots + \alpha_{i+1}(\widehat{F}) + 2\alpha_i(\widehat{F}) + \cdots + 2\alpha_1(\widehat{F}),$$

where $2k–1 \geq j > i \geq 2$, along with $2k–1$ zeros per Equation (4). We need only consider eigenvalues $-1, 0, 1,$ and 2, and these eigenvalues have multiplicities given by

$$r_{-1} = \sum_{i=1}^{k-1} i = \binom{k}{2}, \quad r_0 = \left(\sum_{i=1}^{k-1} 2i\right) + (2k-1) + \left(\sum_{i=1}^{k-2} i\right) = \frac{k(3k-1)}{2},$$

$$r_1 = \left(\sum_{i=1}^{k} i\right) + \left(\sum_{i=1}^{k-1} 2i\right) = \frac{k(3k-1)}{2}, \quad r_2 = \sum_{i=1}^{k-1} i = \binom{k}{2}.$$

If σ is of type B with even cardinality, we reorder the indices of the simple roots so that $\sigma = \{\alpha_{2k}, \ldots, \alpha_1\}$ as in Lemma 3.1. Then $\mathcal{E}(\sigma)$ is comprised of elements of the form

$$\alpha_j(\widehat{F}) + \alpha_{j-1}(\widehat{F}) + \cdots + \alpha_i(\widehat{F}),$$

where $2k \geq j \geq i \geq 1$, or

$$\alpha_j(\widehat{F}) + \cdots + \alpha_{i+1}(\widehat{F}) + 2\alpha_i(\widehat{F}) + \cdots + 2\alpha_1(\widehat{F}),$$

where $2k \geq j > i \geq 2$, along with $2k$ zeros per Equation (4). Then

$$r_{-1} = \sum_{i=1}^{k-1} i = \binom{k}{2}, \quad r_0 = \left(\sum_{i=1}^{k} 2(i-1)+1\right) + (2k) + \left(\sum_{i=1}^{k-1} i\right) = 3\binom{k+1}{2},$$

$$r_1 = \left(\sum_{i=1}^{k} i+1\right) + (k) + \left(\sum_{i=1}^{k-1} 2i\right) = 3\binom{k+1}{2}, \quad r_2 = \sum_{i=1}^{k-1} i = \binom{k}{2}.$$

Case 3: Type C

Suppose σ is of type C. We again reorder the indices of the simple roots so $\sigma = \{\alpha_k, \alpha_{k-1}, \ldots, \alpha_1\}$ as in Lemma 3.1. Then $\mathcal{E}(\sigma)$ is comprised of elements of the form

$$\alpha_j(\widehat{F}) + \alpha_{j-1}(\widehat{F}) + \cdots + \alpha_i(\widehat{F}),$$

where $k \geq j \geq i \geq 1$, or

$$\alpha_j(\widehat{F}) + \cdots + \alpha_{i+1}(\widehat{F}) + 2\alpha_i(\widehat{F}) + \cdots + 2\alpha_2(\widehat{F}) + \alpha_1(\widehat{F}),$$

where $k \geq j \geq i \geq 2$, along with k zeros per Equation (4).

Counting the multiplicity of the eigenvalue 0, we see that there are $\binom{k-1}{2}$ eigenvalues of the form $\alpha_j(\widehat{F}) + \cdots + \alpha_i(\widehat{F})$ where $k \geq j > i \geq 2$, $k-1$ eigenvalues of the form $\alpha_i(\widehat{F})$ where $k \geq i \geq 2$, and k zeros per Equation (4) included in $\mathcal{E}(\sigma)$. Thus $r_0 = \binom{k-1}{2} + (k-1) + k = \binom{k+1}{2}$.

Counting the multiplicity of the eigenvalue 1, we see that there are k eigenvalues of the form $\alpha_j(\widehat{F}) + \cdots + \alpha_1(\widehat{F})$ where $k \geq j \geq 1$, $k-1$ eigenvalues of the form $2\alpha_i(\widehat{F}) + \ldots 2\alpha_2(\widehat{F}) + \alpha_1(\widehat{F})$ where $k \geq i \geq 2$, and $\binom{k-1}{2}$ eigenvalues of the form $\alpha_j(\widehat{F}) + \cdots + \alpha_{i+1}(\widehat{F}) + 2\alpha_i(\widehat{F}) + \cdots + 2\alpha_2(\widehat{F}) + \alpha_1(\widehat{F})$ where $k \geq j > i \geq 2$. It follows that $r_1 = \binom{k+1}{2}$ as well.

Case 4: Type D

Suppose σ is of type D, with odd cardinality. For convenience, reorder the indices of the simple roots so that $\sigma = \{\alpha_{2k+1}, \alpha_{2k}, \ldots, \alpha_1\}$ as in Lemma 3.1. Then $\mathcal{E}(\sigma)$ is comprised of elements of the form

$$\alpha_j(\widehat{F}) + \alpha_{j-1}(\widehat{F}) + \cdots + \alpha_i(\widehat{F}),$$

where $2k+1 \geq j \geq i \geq 1$ except the case with $j = 2$ and $i = 1$, or

$$\alpha_j(\widehat{F}) + \alpha_{j-1}(\widehat{F}) + \cdots + \alpha_{i+2}(\widehat{F}) + \alpha_i(\widehat{F}),$$

where $2k+1 \geq j \geq i = 1$, or

$$\alpha_j(\widehat{F}) + \cdots + \alpha_{i+1}(\widehat{F}) + 2\alpha_i(\widehat{F}) + \cdots + 2\alpha_3(\widehat{F}) + \alpha_2(\widehat{F}) + \alpha_1(\widehat{F}),$$

where $2k+1 \geq j > i \geq 2$, along with $2k$ zeros per Equation (5).

If $\alpha_2(\widehat{F}) = -2$, then

$$r_{-2} = \sum_{i=1}^{k} 1 = k, \qquad r_{-1} = \sum_{i=1}^{k} 1 + \sum_{i=1}^{k-1} i = \frac{k(k+1)}{2},$$

$$r_0 = \left(\sum_{i=1}^{k-1} 2i\right) + 2k + \left(\sum_{i=1}^{k-1} i\right) = \frac{k(3k+1)}{2},$$

$$r_1 = \left(\sum_{i=0}^{k-1} 2i+1\right) + \left(\sum_{i=1}^{k} i\right) = \frac{k(3k+1)}{2},$$

$$r_2 = \sum_{i=1}^{k} 1 + \sum_{i=1}^{k-1} i = \frac{k(k+1)}{2}, \qquad r_3 = \sum_{i=1}^{k} 1 = k.$$

If $\alpha_2(\widehat{F}) = -1$, then

$$r_{-1} = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}, \quad r_0 = \left(\sum_{i=0}^{k-1} 2i+1\right) + 2k + \sum_{i=1}^{k-1} i = \frac{3k(k+1)}{2},$$

$$r_1 = \sum_{i=1}^{k} i + \left(\sum_{i=1}^{k} 2i\right) = \frac{3k(k+1)}{2}, \quad r_2 = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

Suppose σ is of type D, with even cardinality. For convenience, reorder the indices of the simple roots so that $\sigma = \{\alpha_{2k}, \ldots, \alpha_1\}$ as in Lemma 3.1. Then $\mathcal{E}(\sigma)$ is comprised of elements of the form

$$\alpha_j(\widehat{F}) + \alpha_{j-1}(\widehat{F}) + \cdots + \alpha_i(\widehat{F}),$$

where $2k \geq j \geq i \geq 1$ except the case with $j = 2$ and $i = 1$, or

$$\alpha_j(\widehat{F}) + \alpha_{j-1}(\widehat{F}) + \cdots + \alpha_{i+2}(\widehat{F}) + \alpha_i(\widehat{F}),$$

where $2k \geq j \geq i = 1$, or

$$\alpha_j(\widehat{F}) + \cdots + \alpha_{i+1}(\widehat{F}) + 2\alpha_i(\widehat{F}) + \cdots + 2\alpha_3(\widehat{F}) + \alpha_2(\widehat{F}) + \alpha_1(\widehat{F}),$$

where $2k \geq j > i \geq 2$, along with $2k$ zeros per Equation (4). We then have

$$r_{-1} = \sum_{i=1}^{k-1} i = \frac{k(k-1)}{2}, \quad r_0 = \left(\sum_{i=1}^{k-1} 2i\right) + 2k + \left(\sum_{i=1}^{k-1} i\right) = \frac{k(3k+1)}{2},$$

$$r_1 = \left(\sum_{i=1}^{k} i\right) + \left(\sum_{i=0}^{k-1} 2i+1\right) = \frac{k(3k+1)}{2}, \quad r_2 = \sum_{i=1}^{k-1} i = \frac{k(k-1)}{2}. \qquad \blacksquare$$

We have the following corollary that will be used to prove the unbroken property.

Corollary 4.8. *Let $\mathfrak{p}(\pi_1 \mid \pi_2)$ be a Frobenius seaweed, and let σ be a maximally connected component of type A, B, C, or D. If $\mathcal{E}(\sigma)$ is an unbroken multiset, then $\mathcal{E}(\sigma) \cup [-\mathcal{E}(\sigma)]$ is an unbroken multiset. Moreover, if x is symmetric to any eigenvalue in $\mathcal{E}(\sigma)$, then $\mathcal{E}(\sigma) \cup \{x\}$ is an unbroken multiset.*

Part II: Unbroken

A key concept for showing the spectrum is unbroken is a "U-turn" in an orbit of the orbit meander. By U-turn, we mean an application of Lemma 3.2 in the case that $(\alpha_i, i_1\alpha_i) < 0$. Since this case applies only to components of type A, if $(\alpha_i, i_1\alpha_i) < 0$, then α_i and $i_1\alpha_i$ must be adjacent. It follows that the type-A component must have an even number of vertices. To see this, note that in the orbit meander, α_i and $i_1\alpha_i$ are connected by a dashed edge. A component of type A with an odd number of vertices does not have any adjacent vertices connected by a dashed edge since the middle vertex is isolated. For example, the orbit meander in Figure 12 has two U-turns: one in the orbit $\{\alpha_7, \alpha_6, \alpha_8\}$ and one in the orbit $\{\alpha_4, \alpha_3\}$. We will find it convenient to break U-turns into two types of U-turns: *right U-turns* and *left U-turns*.

Now, arrange all orbits so that the first entry is a fixed point in $\pi_1 \cap \pi_2$. If a U-turn involves a dashed edge from v_i^- to v_{i-1}^-, or a dashed edge from v_i^+ to v_{i+1}^+, we call this a right U-turn. Similarly, if a U-turn involves a dashed edge from v_i^- to v_{i+1}^-, or a dashed edge from v_i^+ to v_{i-1}^+, we call this a left U-turn. See Figure 15; the right U-turn is represented by a red dashed arc, and the left U-turns are blue.

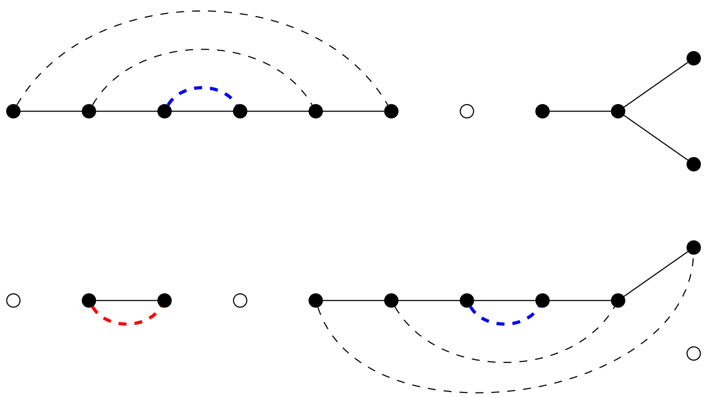

Figure 15: The orbit meander of $\mathfrak{p}_{11}^{\mathrm{D}}(\Phi_1 \mid \Phi_2)$ with U-turns highlighted

The orbit meander for $\mathfrak{p}_{11}^{\mathrm{D}}(\Phi_1 \mid \Phi_2)$ has four orbits:

$$\mathcal{O}_1 = \{\alpha_2, \alpha_7, \alpha_{10}, \alpha_9, \alpha_8\}, \quad \mathcal{O}_2 = \{\alpha_3, \alpha_6, \alpha_{11}\}, \quad \mathcal{O}_3 = \{\alpha_4, \alpha_5\}, \quad \text{and} \quad \mathcal{O}_4 = \{\alpha_1\}.$$

Note that $\mathcal{O}_2$ and $\mathcal{O}_4$ contain no U-turns, $\mathcal{O}_1$ has two U-turns (a right and a left) and $\mathcal{O}_3$ has a single left U-turn. As it turns out, an orbit cannot have more than two U-turns. Moreover, if an orbit has two U-turns, one must be a right U-turn and one must be a left U-turn. We record this in the following Lemma.

Lemma 4.9 (U-turn Lemma). *An orbit in an orbit meander contains at most two U-turns. Moreover, if an orbit contains two U-turns, one must be a right U-turn and one must be a left U-turn.*

Proof. The orbit meander for a Frobenius type-A seaweed has exactly two maximally connected components of even size, so an orbit can have at most two U-turns. For a seaweed of type B, C, or D, it is possible to have more than two maximally connected components of even size. Indeed, start from a fixed point contained in

$\pi_1 \cap \pi_2$. Without loss of generality, we proceed with the proof in the case that the first U-turn is a right U-turn.

The orbit may make a second U-turn. However, it cannot make a second right U-turn, as the orbit would self-intersect and never terminate at an element of $\pi_{\cup}$. Therefore, a second U-turn must be to the left. But now the orbit has previously traced edges on both the left and right. Additional U-turns are not allowed. ■

Simple eigenvalues can increase in absolute value by one only through a U-turn. This observation together with the U-turn Lemma gives us the following critical result.

Theorem 4.10. *Let $\mathfrak{g}$ be a Frobenius seaweed of type A or type D. If $\widehat{F}$ is a principal element of $\mathfrak{g}$ and $\alpha_i(\widehat{F})$ is a simple eigenvalue, then*

$$\alpha_i(\widehat{F}) \in \{-2, -1, 0, 1, 2, 3\}.$$

In particular, the simple eigenvalues are bounded in absolute value by three.

Remark 4.11. Theorem 4.10 combined with the results of [6] gives us the following complete table of possible simple eigenvalues for each classical case.

Component type	Possible values of $\alpha_i(\widehat{F})$
A	$-2, -1, 0, 1, 2, 3$
B	$-1, 0, 1$
C	$0, 1$
D	$-2, -1, 0, 1, 2, 3$

Table 7: Values of $\alpha_i(\widehat{F})$

To ease discourse, we note the following bases for the subsequent induction before detailing the winding-up moves. These bases are established in [2] and follow from a general combinatorial formula for the index of a type-D seaweed.

Theorem 4.12. *The following seaweeds are the bases for induction.*

(i) D_{2k} *with* $k \geq 2$ *is* $\mathfrak{p}^D_{2k}(\{\alpha_{2k}, \alpha_{2k-1}, ..., \alpha_1\} \mid \emptyset)$.

(ii) D_{2k+1} *with* $k \geq 1$ $\mathfrak{p}^D_{2k+1}(\{\alpha_{2k+1}, \alpha_{2k}, ..., \alpha_1\} \mid \{\alpha_2\})$, *and*

(iii) D_{2k+1} *with* $k \geq 1$ $\mathfrak{p}^D_{2k+1}(\{\alpha_{2k+1}, \alpha_{2k}, ..., \alpha_1\} \mid \{\alpha_3, \alpha_2\})$.

To prove the unbroken property, we will find it convenient to express these seaweeds in terms of sequences of flags defining them. Let $C_{\leq n}$ denote the set of sequences of positive integers whose sum is less than or equal to n, and call each integer in the string a *part*. Let $\mathcal{P}(X)$ denote the power set of a set X. Then given $\underline{a} = (a_1, a_2, \ldots, a_m) \in C_{\leq n}$, define a bijection $\varphi : C_{\leq n} \to \mathcal{P}(\Pi)$ by

$$\varphi(\underline{a}) = \{\alpha_{n+1-a_1}, \alpha_{n+1-(a_1+a_2)}, \ldots, \alpha_{n+1-(a_1+a_2+\cdots+a_m)}\}.$$

Then define $\mathfrak{p}_n(\underline{a} \mid \underline{b}) = \mathfrak{p}\,(\Pi \setminus \varphi(\underline{a}) \mid \Pi \setminus \varphi(\underline{b}))$, and let $\mathcal{M}_n(\underline{a} \mid \underline{b})$ denote the orbit meander of $\mathfrak{p}_n(\underline{a} \mid \underline{b})$.

Example 4.13. For example,

$$\mathfrak{p}_8^{\mathrm{C}}(\{\alpha_6, \alpha_5, \alpha_4, \alpha_3\} \mid \{\alpha_8, \alpha_7, \alpha_6, \alpha_4, \alpha_3, \alpha_2, \alpha_1\}) = \mathfrak{p}_8^{\mathrm{C}}((1, 1, 5, 1) \mid (4)).$$

We now write the bases for induction from Theorem 4.12 using sequences of integers.

Theorem 4.14. *The following seaweeds are the bases for the induction.*

(i)′ $\mathfrak{p}_q^{\mathrm{D}}\,(1^q \mid \emptyset)$ *with* q *even,*

(ii)′ $\mathfrak{p}_q^{\mathrm{D}}\,(1^{q-2}, 2 \mid \emptyset)$ *with* q *odd,*

(iii)′ $\mathfrak{p}_q^{\mathrm{D}}\,(1^{q-3}, 3 \mid \emptyset)$ *with* q *odd.*

Example 4.15. The following Figure 16 illustrates the inductive bases of Theorem 4.14.

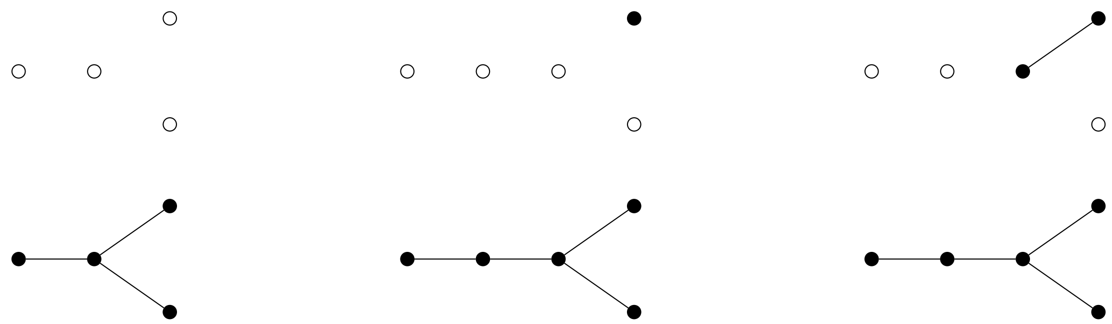

Figure 16: Type-D inductive bases with $q = 4$ and $q = 5$

If $a_1 + \cdots + a_m = n$, then each part a_i corresponds to a maximally connected component σ of cardinality $|\sigma| = a_i - 1$, all of which are of type A. If $a_1 + \cdots + a_m = r < n$, then each part a_i corresponds to a type-A maximally connected component σ of cardinality $|\sigma| = a_i - 1$, and there is one additional maximally connected component of cardinality $n - r$, which is of type B, C, or D if $n - r > 1$.

The following "Winding-up" lemma can be used to develop any Frobenius orbit meander of any size or configuration. It can be regarded as the inverse graph-theoretic rendering of Panyushev's well-known reduction [22].

Lemma 4.16. (Coll et al. [**3**], Lemma 4.2) *Let* $\mathcal{M}_n(\underline{c} \mid \underline{d})$ *be any type-B or type-C Frobenius orbit meander, and without loss of generality, assume that* $\sum c_i = q + \sum d_i = n$*. Then* $\mathcal{M}_n(\underline{c} \mid \underline{d})$ *is the result of a sequence of the following moves starting from* $\mathcal{M}_q(1^q \mid \emptyset)$*. Starting with an orbit meander* $\mathcal{M} = \mathcal{M}_n\,(a_1, a_2, \ldots, a_m \mid b_1, b_2, \ldots, b_t)$*, create an orbit meander* $\mathcal{M}'$ *by one of of the following:*

(1) **Block Creation:** $\mathcal{M}' = \mathcal{M}_{n+a_1}(2a_1, a_2, \ldots, a_m \mid a_1, b_1, b_2, \ldots, b_t)$,

(2) **Rotation Expansion:** $\mathcal{M}' = \mathcal{M}_{n+a_1-b_1}(2a_1 - b_1, a_2, a_3, ..., a_m \mid a_1, b_2, b_3, ..., b_t)$, *provided that* $a_1 > b_1$,

(3) **Pure Expansion:** $\mathcal{M}' = \mathcal{M}_{n+a_2}(a_1 + 2a_2, a_3, a_4, \ldots, a_m \mid a_2, b_1, b_2, \ldots, b_t)$,

(4) **Flip-Up:** $\mathcal{M}' = \mathcal{M}_n(b_1, b_2, \ldots, b_t \mid a_1, a_2, \ldots, a_m)$.

Similarly, any type-D Frobenius orbit meander is the result of a sequence of the same moves but starting from one of the bases given in Theorem 4.14.

Remark 4.17. While we previously found it convenient to order the vertices of an orbit meander from right to left, to ease notation in the following proof, we will find it convenient to relabel the vertices going from left to right. That is, if an orbit meander is defined by vertices $\{v_n^+, v_{n-1}^+, ..., v_1^+\}$ and $\{v_n^-, v_{n-1}^-, ..., v_1^-\}$, redefine the orbit meander by

$$\{v_n^+, v_{n-1}^+, ..., v_1^+\} \mapsto \{w_1^+, w_2^+, ..., w_n^+\}$$
$$\{v_n^-, v_{n-1}^-, ..., v_1^-\} \mapsto \{w_1^-, w_2^-, ..., w_n^-\}$$

This is done so that the induction is done on the block whose leftmost vertex is labeled w_1^+ rather than v_n^+.

Theorem 4.18. *If $\mathcal{M}(\pi_1 \mid \pi_2) = \mathcal{M}(\underline{a} \mid \underline{b})$ is any Frobenius orbit meander, then $\mathcal{E}(\sigma)$ is unbroken for every maximally connected component σ.*

Proof. The proof is by induction on the number of Winding-up moves from Lemma 4.16 and that $\mathcal{E}(\sigma)$ is unbroken for every maximally connected component σ. Since $\mathcal{E}(\sigma)$ always contains 0, Theorem 4.18 implies the unbroken property of Theorem 1.1.

The base of the induction in type-A is $\mathcal{M}_1(1 \mid \emptyset)$, which has unbroken spectrum $\{0, 1\}$. The base of the induction for either type B or type C is an orbit meander $\mathcal{M}_q(1^q \mid \emptyset)$ where q is a positive integer. There is one maximally connected component σ, of type B or type C, and by Theorem 4.7, $\mathcal{E}(\sigma)$ is unbroken either way.

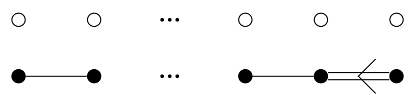

Figure 17: The Frobenius orbit meander $\mathcal{M}_q^{\mathrm{C}}(1^q \mid \emptyset)$

The base of the induction for type D is one of the orbit meanders from Example 4.15. The first two cases follow immediately from Theorem 4.7. For the last, observe the orbit meander consists of a type-A component and a type-D component. The eigenvalues contributed by the type-A component are $\{-1, 0, 1, 2\}$. The eigenvalues contributed by the type-D component are unbroken and contain 0 by Theorem 4.7. Their union is therefore unbroken.

Let $\mathcal{M} = \mathcal{M}(\underline{a} \mid \underline{b})$ be an orbit meander of classical type. Suppose $\mathcal{E}(\sigma)$ is unbroken for each maximally connected component σ in $\mathcal{M}$. Let $\mathcal{M}'$ be the orbit meander resulting from applying one of the Winding-up moves from Lemma 4.16 to $\mathcal{M}$. For the inductive step, we need not consider the Flip-up move since this merely replaces $\mathcal{M}$ with an inverted isomorphic copy and consequently has no effect on the eigenvalue calculations. So, there are three cases we need to consider: block creation, rotation expansion, and pure expansion. We show $\mathcal{E}(\sigma)$ is unbroken for each maximally connected component σ in $\mathcal{M}'$ that is not in $\mathcal{M}$. The following sets of vertices will assist in the computation of the eigenvalues. Note that the ordering of the sets in columns three and four of the following table are set up so that the vertices in the sets are in the order of the inducted-upon orbit meander.

Move	Base $\mathcal{M}$	New $\mathcal{M}'$	New σ's
Block Creation	$A = \{w_1^+, w_2^+, ..., w_{a_1-1}^+\}$	$B' = \{w_1^+, w_2^+, ..., w_{a_1-1}^+\}$ $C' = \{w_{a_1}^+\}$ $A' = \{w_{a_1+1}^+, w_{a_1+2}^+, ..., w_{2a_1-1}^+\}$ $D' = \{w_1^-, w_2^-, ..., w_{a_1-1}^-\}$	$\sigma_1 = B' \cup C' \cup A'$ $\sigma_2 = D'$
Rotation Expansion	$A = \{w_1^+, w_2^+, ..., w_{b_1-1}^+\}$ $B = \{w_{b_1}^+, w_{b_1+1}^+, ..., w_{a_1-1}^+\}$ $C = \{w_1^-, w_2^-, ..., w_{b_1-1}^-\}$	$A' = \{w_1^+, w_2^+, ..., w_{a_1-b_1}^+\}$ $C' = \{w_{a_1-b_1+1}^+, w_{a_1-b_1+2}^+, ..., w_{a_1-1}^+\}$ $B' = \{w_{a_1}^+, w_{a_1+1}^+, ..., w_{2a_1-b_1-1}^+\}$ $D' = \{w_1^-, w_2^-, ..., w_{a_1-1}^-\}$	$\sigma_1 = A' \cup C' \cup B'$ $\sigma_2 = D'$
Pure Expansion	$A = \{w_1^+, w_2^+, ..., w_{a_1-1}^+\}$ $B = \{w_{a_1+1}^+, w_{a_1+2}^+, ..., w_{a_1+a_2-1}^+\}$	$B' = \{w_1^+, w_2^+, ..., w_{a_2-1}^+\}$ $E' = \{w_{a_2}^+\}$ $A' = \{w_{a_2+1}^+, w_{a_2+2}^+, ..., w_{a_1+a_2-1}\}$ $F' = \{w_{a_1+a_2}^+\}$ $C' = \{w_{a_1+a_2+1}^+, w_{a_1+a_2+2}^+, ..., w_{a_1+2a_2-1}^+\}$ $D' = \{w_1^-, w_2^-, ..., w_{a_2-1}^-\}$	$\sigma_1 = B' \cup E' \cup A' \cup F' \cup C'$ $\sigma_2 = D'$

Table 8: Sets in the Induction Proof

Gray vertices in the orbit meanders associated to each Winding up move are vertices not impacted by the induction. Furthermore, for brevity, the orbit meanders do not extend beyond the relevant top blocks.

Case 1: Block Creation:

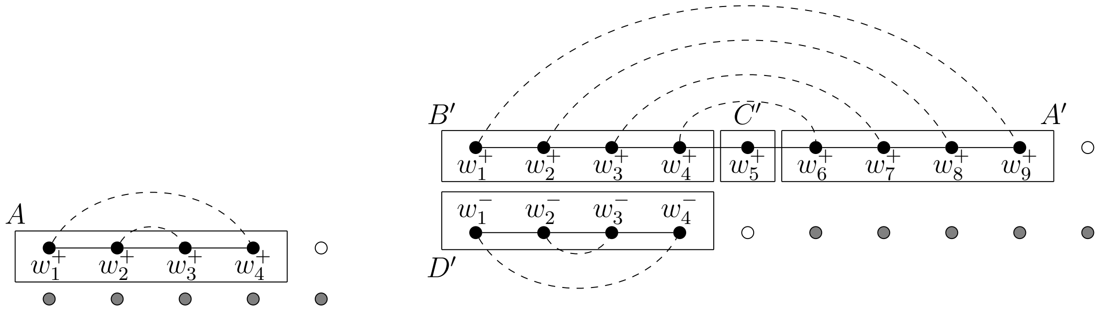

Figure 18: Block Creation applied to $\mathcal{M}$ with $a_1 = 5$ (left) to obtain $\mathcal{M}'$ (right)

We have
$$\mathcal{E}(A') = \mathcal{E}(A), \tag{6}$$
$$\mathcal{E}(B') = -\mathcal{E}(A), \tag{7}$$
$$\mathcal{E}(D') = \mathcal{E}(A). \tag{8}$$

Note that $\mathcal{E}(A)$ is unbroken by induction. So, by Equation (8), $\mathcal{E}(\sigma_2)$ is unbroken. Any number in $\mathcal{E}(A' \cup C')$ is in either $\mathcal{E}(A')$ or $\mathcal{E}(A') + 1$, so by equation (6), $\mathcal{E}(A' \cup C')$ is unbroken. By symmetry and equation (7), $\mathcal{E}(B' \cup C')$ is unbroken. Hence $\mathcal{E}(B' \cup C') \cup \mathcal{E}(A' \cup C')$ is unbroken. Any remaining eigenvalue in $\mathcal{E}(\sigma_1)$ is either one or symmetric to an eigenvalue in $\mathcal{E}(B' \cup C') \cup \mathcal{E}(A' \cup C')$, so $\mathcal{E}(\sigma_1)$ is unbroken.

Case 2: Rotation Expansion:

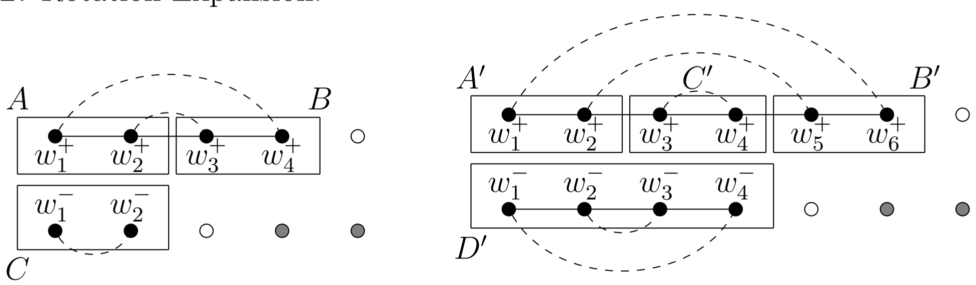

Figure 19: Rotation Expansion applied to $\mathcal{M}$ with $a_1 = 5$ and $b_1 = 3$ (left) to obtain $\mathcal{M}'$ (right)

We have
$$\mathcal{E}(D') = \mathcal{E}(A \cup B), \tag{9}$$
$$\mathcal{E}(C') = -\mathcal{E}(C). \tag{10}$$

By equation (9), $\mathcal{E}(\sigma_2)$ is unbroken. Without loss of generality,
$$\mathcal{E}(A' \cup C') \quad = \quad -\mathcal{E}(A \cup B), \tag{11}$$
so $\mathcal{E}(A' \cup C') \subseteq \mathcal{E}(\sigma_1)$ is unbroken. All other eigenvalues in $\mathcal{E}(\sigma_1)$ are symmetric to eigenvalues in $\mathcal{E}(A' \cup C')$, and consequently, $\mathcal{E}(\sigma_1)$ is unbroken.

Case 3: Pure Expansion:

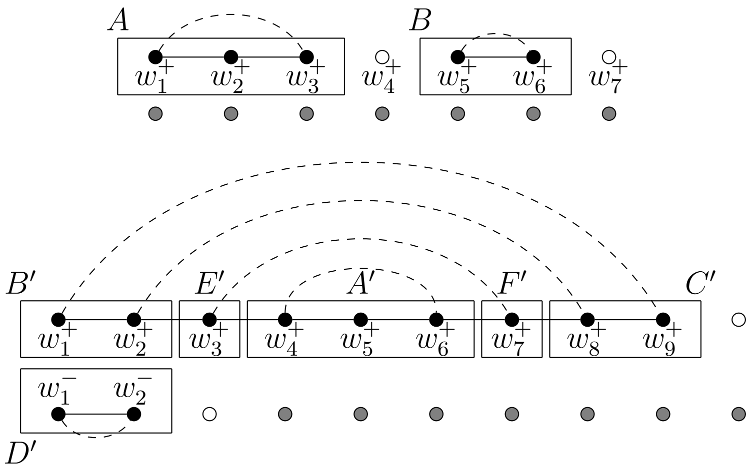

Figure 20: Pure Expansion applied to $\mathcal{M}$ with $a_1 = 4$ and $a_2 = 3$ (top) to obtain $\mathcal{M}'$ (bottom)

We have

$$\mathcal{E}(B') = -\mathcal{E}(B), \tag{12}$$

$$\mathcal{E}(A') = \mathcal{E}(A), \tag{13}$$

$$\mathcal{E}(C') = \mathcal{E}(B), \tag{14}$$

$$\mathcal{E}(D') = \mathcal{E}(B). \tag{15}$$

By equation (15), $\mathcal{E}(\sigma_2)$ is unbroken. Let $\gamma = \alpha_{a_2}(\widehat{F})$ for $\alpha_{a_2} \in \pi_1$. By Theorem 4.10, $\gamma = 1, 2,$ or 3. Since

$$\alpha_1(\widehat{F}) + \alpha_2(\widehat{F}) + ... + \alpha_{a_2-1}(\widehat{F}) = -1,$$

and $\gamma = 1, 2,$ or 3, $\mathcal{E}(B' \cup E')$ contains $0, 1,$ or 2. In particular,

$$\alpha_1(\widehat{F}) + \alpha_2(\widehat{F}) + ... + \alpha_{a_2-1}(\widehat{F}) + \alpha_{a_2}(\widehat{F}) = \gamma - 1.$$

It is clear that if a_2 is even, then the eigenvalues in $\mathcal{E}(B' \cup E') \setminus \mathcal{E}(B')$ are

$$\{\gamma, \gamma + \alpha_{a_2-1}(\widehat{F}), \gamma + \alpha_{a_2-2}(\widehat{F}), ..., \gamma + \alpha_{\frac{a_2}{2}+1}(\widehat{F}), \gamma - 1,$$
$$\gamma - 1 + \alpha_{a_2-1}(\widehat{F}), \gamma - 1 + \alpha_{a_2-2}(\widehat{F}), ..., \gamma - 1 + \alpha_{\frac{a_2}{2}+1}(\widehat{F})\}.$$

By Theorem 4.10, $|\alpha_i(\widehat{F})| = 0, 1, 2,$ or 3 for all i.

If $|\alpha_i(\widehat{F})| \neq 3$ for some $i \in \{a_2 - 1, a_2 - 2, ..., \frac{a_2}{2} + 1\}$, then $\mathcal{E}(B' \cup E') \setminus \mathcal{E}(B')$ is unbroken. Moreover, since

$$\alpha_{a_2+1}(\widehat{F}) + \alpha_{a_2+2}(\widehat{F}) + ... + \alpha_{a_1+a_2-1}(\widehat{F}) = 1,$$

the multiset

$$\mathcal{E}(B' \cup E' \cup A') \setminus [\mathcal{E}(B' \cup E') \ \cup \ \mathcal{E}(A')]$$

is unbroken; but since $\{0, 1\} \subseteq \mathcal{E}(A')$, the multiset $\mathcal{E}(B' \cup E' \cup A')$ is unbroken and contains 0. Similarly, the multiset $\mathcal{E}(A' \cup F' \cup C')$ is unbroken and contains 0. Therefore, $\mathcal{E}(B' \cup E' \cup A') \cup \mathcal{E}(A' \cup F' \cup C')$ is unbroken and contains 0. Any remaining eigenvalue in $\mathcal{E}(\sigma_1)$ is symmetric to an eigenvalue in $\mathcal{E}(B' \cup E' \cup A') \cup \mathcal{E}(A' \cup F' \cup C')$. Therefore, $\mathcal{E}(\sigma_1)$ is unbroken.

If $|\alpha_i(\widehat{F})| = 3$ for all $i \in \{a_2-1, a_2-2, ..., \frac{a_2}{2}+1\}$, then $\mathcal{E}(B' \cup E') \setminus \mathcal{E}(B')$ is not necessarily unbroken. Moreover, any numbers preventing $\mathcal{E}(B' \cup E') \setminus \mathcal{E}(B')$ from being unbroken must be congruent (mod 3). Let x be any such number. Observe that

$$\mathcal{E}(B' \cup E') \setminus \mathcal{E}(B') = -[\mathcal{E}(C' \cup F') \setminus \mathcal{E}(C')].$$

Then x is symmetric to some number in $\mathcal{E}(C' \cup F') \setminus \mathcal{E}(C')$ and exists somewhere in $\mathcal{E}(\sigma_1)$. The rest of the argument follows similarly to the previous argument: that is, when $|\alpha_i(\widehat{F})| \neq 3$ for each $i \in \{a_2-1, a_2-2, ..., \frac{a_2}{2}+1\}$.

If a_2 is odd, then the eigenvalues in $\mathcal{E}(B' \cup E') \setminus \mathcal{E}(B')$ are the same as in the previous case in addition to $\gamma + \alpha_{\lfloor \frac{a_2}{2} \rfloor}(\widehat{F})$. But since

$$\alpha_{a_2+1}(\widehat{F}) + \alpha_{a_2+2}(\widehat{F}) + ... + \alpha_{a_1+a_2-1}(\widehat{F}) = 1,$$

the multiset

$$\mathcal{E}(B' \cup E' \cup A') - [\mathcal{E}(B' \cup E') \cup \mathcal{E}(A')]$$

is unbroken. That $\mathcal{E}(\sigma_1)$ is unbroken follows from the argument where a_2 is even. ■

Remark 4.19. The proof of Theorem 4.18, in particular, the proof for Case 3, would have held even if the simple eigenvalues from Theorem 4.10 were bounded in absolute value by 4. The type-A proof by Coll et al. in [6] requires the bound to be 3.

5. Exceptional cases

We now consider seaweeds of the exceptional Lie algebras: E_6, E_7, E_8, F_4, and G_2. We continue to use the Bourbaki ordering of the simple roots. We let $\mathfrak{p}_n^{X}$ denote a seaweed of exceptional type, where $X \in \{E, F, G\}$ and n is the rank. In this notation, $\mathfrak{p}_2^{G}$ is a seaweed of type G_2. In the case where the pairs of included simple roots are explicit, we use the notation $\mathfrak{p}_n^{X}(\pi_1 \mid \pi_2)$.

For components of all exceptional types except E_6, the longest element of W_{π_j} is given by $w_j = -id$. For a component of type E_6,

$$w_j\alpha_i = \begin{cases} \alpha_6, & \text{if } i = 1; \\ \alpha_5, & \text{if } i = 3; \\ \alpha_i, & \text{if } i = 2, 4. \end{cases}$$

We visualize these actions and construct an exceptional orbit meander as in the classical cases. See Example 5.1.

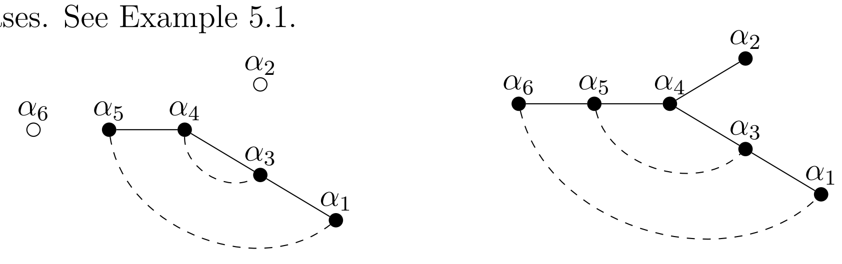

Figure 21: The seaweed $\mathfrak{p}_6^{E}(\Theta_1 \mid \Theta_2)$

Example 5.1. Define the seaweed $\mathfrak{p}_6^{E}(\Theta_1 \mid \Theta_2)$ by

$$\Theta_1 = \{\alpha_5, \alpha_4, \alpha_3, \alpha_1\}, \qquad \Theta_2 = \{\alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1\}.$$

See Figure 21 above for the orbit meander of $\mathfrak{p}_6^{E}(\Theta_1 \mid \Theta_2)$. Note that this seaweed is Frobenius by Theorem 2.5.

Again, as in Section 3, we have the exceptional analogues of Lemmas 3.1 and 3.2.

Lemma 5.2. (Joseph [15], Section 5) *In Table 9 below, the given value is* $\alpha_i(\widehat{F})$ *if* σ *is a maximally connected component of* π_1*, and it is* $-\alpha_i(\widehat{F})$ *if* σ *is a maximally connected component of* π_2*. In either case, it is assumed that* $\alpha_i \in \sigma$*.*

Type	$\pm\alpha_i(\widehat{F})$	$\pm\alpha_i(\widehat{F})$
E_6	-1, if $i = 2$	1, if $i = 4$
E_7	-1, if $i = 1, 4, 6$	1, if $i = 2, 3, 5, 7$
E_8	-1, if $i = 1, 4, 6, 8$	1, if $i = 2, 3, 5, 7$
F_4	$(-1)^i$, if $i = 1, 2$	0, if $i = 3, 4$
G_2	-1, if $i = 1$	1, if $i = 2$

Table 9: Values of $\pm\alpha_i(\widehat{F})$

Table 9 can be used to find all simple eigenvalues for comopnents not of type E_6. For components of type E_6, the following lemma can be applied.

Lemma 5.3. (Joseph [15], Section 5) *For the equation below,* σ *is assumed to be a component of* π_1 *of type* E_6*. If* σ *is a maximally connected component of* π_2*, then replace* $\alpha_i \mapsto -\alpha_i$ *and* $i_1 \mapsto i_2$*. Then*

$$\alpha_i(\widehat{F}) + i_1\alpha_i(\widehat{F}) = 0 \text{ if } i = 1, 3.$$

To establish Theorem 1.1 for exceptional seaweeds, we first examine seaweeds with components of type E_6 in Section 6. Because the Weyl action is non-trivial on an E_6 component, this case requires more analysis. The other exceptional Lie algebras are treated in Section 7.

6. The exceptional Lie algebra $\mathbf{E_6}$

We begin with an example computation. We use the seaweed in Example 5.1. We first compute the simple eigenvalues, then the eigenvalues, and finally the spectrum.

Example 6.1. Figure 22 shows the simple eigenvalues for $\mathfrak{p}_6^{\mathrm{E}}(\Theta_1 \mid \Theta_2)$.

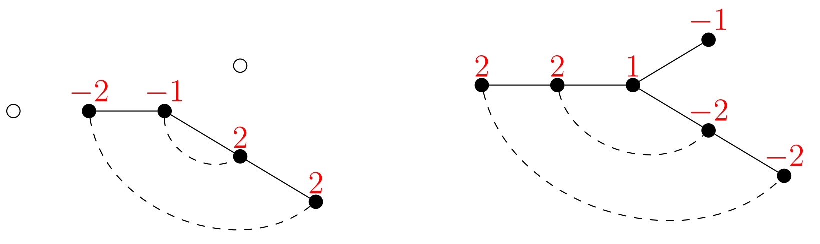

Figure 22: The simple eigenvalues of $\mathfrak{p}_6^{\mathrm{E}}(\Theta_1 \mid \Theta_2)$

We partition the multiset of eigenvalues according to the maximally connected components σ of π_1 and π_2. Let σ be a maximally connected component of π_1. If σ is of Type E_6, then $\mathcal{E}(\sigma) = \{\beta(\widehat{F}) \mid \beta \in \mathbb{N}\sigma \cap \Delta_+\} \cup \{0^4\}$.

Positive Root	Eigenvalue
$2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + \alpha_5 + \alpha_6$	1
$2\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + \alpha_5 + \alpha_6$	2
$2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$	1
$2\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$	3
$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$	-1
$2\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_6$	5
$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$	-3
$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$	1
$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_6$	-1
$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$	3
$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	0
$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4$	1
$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$	1
$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6$	-2
$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$	2
$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5$	-4
$\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5$	-1
$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	0
$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_6$	4
$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_6$	3
$\alpha_1 + \alpha_4 + \alpha_5$	-3
$\alpha_1 + \alpha_3 + \alpha_4$	-2
$\alpha_1 + \alpha_2 + \alpha_4$	2
$\alpha_2 + \alpha_3 + \alpha_4$	1
$\alpha_3 + \alpha_4 + \alpha_6$	5
$\alpha_1 + \alpha_5$	-4
$\alpha_1 + \alpha_4$	0
$\alpha_3 + \alpha_4$	-1
$\alpha_2 + \alpha_4$	3
$\alpha_3 + \alpha_6$	4
α_1	-2
α_2	-1
α_3	-2
α_4	1
α_5	2
α_6	2

Table 10: Eigenvalues associated with the component $\sigma_2 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$

To compute the eigenvalues of $\mathfrak{p}_6^{\mathrm{E}}(\Theta_1 \mid \Theta_2)$, note that this seaweed has a single type-A component $\sigma_1 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5\}$ on the top and a single type-E_6 component on the bottom

$$\sigma_2 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}.$$

We compute $\mathcal{E}(\sigma_1)$ as before to yield

$$\mathcal{E}(\sigma_1) = \{-3^1, -2^1, -1^2, 0^2, 1^2, 2^2, 3^1, 4^1\}.$$

We now compute $\mathcal{E}(\sigma_2)$. The positive roots for the computation of $\mathcal{E}(\sigma_2)$ form a thirty-six element set. See Table 10, where the left column lists the positive roots, and the right column lists the associated eigenvalue. Consequently,

$$\begin{aligned}\mathcal{E}(\sigma_2) &= \{-4^2, -3^2, -2^4, -1^5, 0^3, 1^7, 2^5, 3^4, 4^2, 5^2\} \cup \{0^4\}\\ &= \{-4^2, -3^2, -2^4, -1^5, 0^7, 1^7, 2^5, 3^4, 4^2, 5^2\}.\end{aligned}$$

The union of the sets $\mathcal{E}(\sigma_1)$ and $\mathcal{E}(\sigma_2)$ gives the spectrum. See Table 11.

Eigenvalue	-4	-3	-2	-1	0	1	2	3	4	5
Multiplicity	2	3	5	7	9	9	7	5	3	2

Table 11: Spectrum of $\mathfrak{p}_6^{\mathrm{E}}(\Theta_1 \mid \Theta_2)$ with multiplicities

Using Theorem 2.5, it is straightforward to show, by exhaustion, that there are, up to isomorphism, seventy-four Frobenius seaweed subalgebras of E_6. [3] Of these, fourteen contain a component of type E_6. (See Appendix A.) The simple eigenvalues associated with the E_6 component take on one of nine possible configurations, which we note in Remark 6.2 below.

Remark 6.2. We list the simple eigenvalues according to the order of simple roots. Observe that the first configuration occurs in the bottom of the orbit meander of $\mathfrak{p}_6^{\mathrm{E}}(\Theta_1 \mid \Theta_2)$.

Configuration	Simple eigenvalues
1	-2 -1 -2 1 2 2
2	-2 -1 1 1 -1 2
3	-1 -1 2 1 -2 1
4	1 -1 -2 1 2 -1
5	1 -1 -1 1 1 -1
6	2 -1 -1 1 1 -2
7	-1 -1 -1 1 1 1
8	-1 -1 1 1 -1 1
9	1 -1 1 1 -1 -1

Table 12: Configurations and simple eigenvalues for an E_6 component

[3] There are, up to isomorphism, two Frobenius seaweeds in G_2, eight in F_4, seventy-four in E_6, one hundred fourty-three in E_7, and three hundred one in E_8.

For each of the configurations in Table 12, computations similar to those used in Example 6.1 can be used to show that the spectrum of any E_6 component consists of an unbroken sequence of integers centered at one-half, and the associated eigenspace multiplicities form a symmetric distribution.

We next consider the remaining exceptional Lie algebras.

7. The exceptional Lie algebras E_7, E_8, F_4, and G_2

Because Table 9 gives all simple eigenvalues in each of these cases, we need only consider the eigenvalues associated to a component of each type. We use Lemma 5.2 to find the simple eigenvalues for each component.

Case 1: A component of type E_7

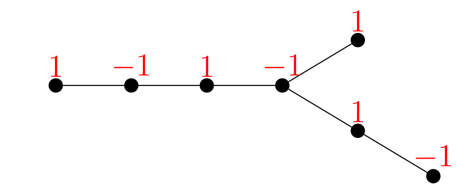

Figure 23: The simple eigenvalues of a component σ of type E_7

The eigenvalues of σ are given by $\mathcal{E}(\sigma) = \{\beta(\widehat{F}) \mid \beta \in \mathbb{N}\sigma \cap \Delta_+\} \cup \{0^7\}$.

The spectrum of σ and the multiplicities of associated eigenspaces are listed below (see Table 13).

Eigenvalue	-1	0	1	2
Multiplicity	7	28	28	7

Table 13: Spectrum of σ with multiplicities

Case 2: A component of type E_8

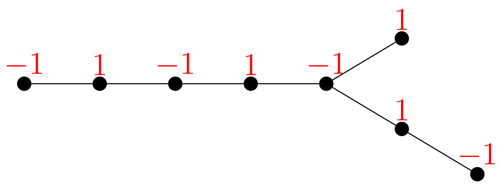

Figure 24: The simple eigenvalues of a component σ of type E_8

The eigenvalues of σ are given by $\mathcal{E}(\sigma) = \{\beta(\widehat{F}) \mid \beta \in \mathbb{N}\sigma \cap \Delta_+\} \cup \{0^8\}$.

The spectrum of σ and the multiplicities of associated eigenspaces are listed below (see Table 14).

Eigenvalue	-1	0	1	2
Multiplicity	14	50	50	14

Table 14: Spectrum of σ with multiplicities

Case 3: A component of type F_4

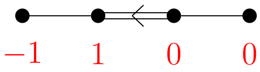

Figure 25: Simple eigenvalues for a component σ of type F_4

The eigenvalues of σ are given by $\mathcal{E}(\sigma) = \{\beta(\widehat{F}) \mid \beta \in \mathbb{N}\sigma \cap \Delta_+\} \cup \{0^4\}$.
The spectrum of σ and the multiplicities of associated eigenspaces are listed below (see Table 15).

Eigenvalue	-1	0	1	2
Multiplicity	1	13	13	1

Table 15: The spectrum of σ with multiplicities

Case 4: A component of type G_2

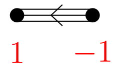

Figure 26: Simple eigenvalues for a component σ of type G_2

The eigenvalues of σ are given by $\mathcal{E}(\sigma) = \{\beta(\widehat{F}) \mid \beta \in \mathbb{N}\sigma \cap \Delta_+\} \cup \{0^2\}$.
The spectrum of σ and the multiplicities of associated eigenspaces are listed below (see Table 16).

Eigenvalue	-1	0	1	2
Multiplicity	1	3	3	1

Table 16: Spectrum with multiplicities

Since the spectrum of any component of exceptional type consists of an unbroken sequence of integers centered at one-half, and the associated eigenspace multiplicities form a symmetric distribution, we conclude Theorem 1.1 holds for the exceptional Lie algebras.

8. Epilogue

The unbroken spectrum property of a Frobenius seaweed subalgebra $\mathfrak{g}$ of a simple Lie algebra is a computable algebraic invariant of $\mathfrak{g}$ but it is not characteristic – as the following example illustrates.

Example 8.1. Consider the poset $\mathcal{P} = \{1, 2, 3, 4\}$ with $1, 2 \preceq 3 \preceq 4$ and no relations other than those following from these. Letting $\mathbb{C}$ be the ground field, one may construct an associative matrix algebra $A(\mathcal{P}, \mathbb{C})$ which is the span over $\mathbb{C}$ of $e_{i,j}$, $i \preceq j$ with multiplication given by $e_{i,j}e_{l,k} = e_{i,k}$ if $j = l$ and 0 otherwise. The underlying vector space of $A(\mathcal{P}, \mathbb{C})$ becomes a Lie algebra $\mathfrak{g}(\mathcal{P}, \mathbb{C})$ under commutator multiplication and, if one considers only the elements of trace zero, may be regarded as a Lie subalgebra of $A_4 = \mathfrak{sl}(4)$: in fact, a Frobenius

Lie subalgebra with Frobenius functional $F = e^*_{1,4} + e^*_{2,4} + e^*_{2,3}$, principal element $\widehat{F} = \text{diag}\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$, and unbroken spectrum $\{0^4, 1^4\}$. The algebra $\mathfrak{g}(\mathcal{P}, \mathbb{C})$ has rank 3 and dimension 8. However, the only Frobenius seaweed subalgebra of $\mathfrak{sl}(4)$ with the same rank and dimension is $\mathfrak{p}_4^{\text{A}}(\{\alpha_3, \alpha_1\} \mid \{\alpha_3, \alpha_2\})$, but the latter has spectrum given by the multiset $\{-1, 0^3, 1^3, 2\}$. See [7].

9. Appendix A - Frobenius seaweeds in $\mathbf{E_6}$

We provide in the following a list of Frobenius seaweed subalgebras in E_6. To ease notation, we will denote, for example, a seaweed

$$\mathfrak{p}_6^{\text{E}}(\{\alpha_5, \alpha_4, \alpha_3, \alpha_1\} \mid \{\alpha_6, \alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1\}) \text{ by } \{5,4,3,1\}, \{6,5,4,3,2,1\}.$$

Note that the seaweeds listed in 1–14 contain a component of type E_6.

1	$\{6,5,4,3,2,1\}, \{6,5,4,3\}$
2	$\{6,5,4,3,2,1\}, \{6,5,4,2\}$
3	$\{6,5,4,3,2,1\}, \{5,4,3,1\}$
4	$\{6,5,4,3,2,1\}, \{4,3,2,1\}$
5	$\{6,5,4,3,2,1\}, \{6,5,4\}$
6	$\{6,5,4,3,2,1\}, \{6,5,3\}$
7	$\{6,5,4,3,2,1\}, \{6,5,1\}$
8	$\{6,5,4,3,2,1\}, \{6,4,3\}$
9	$\{6,5,4,3,2,1\}, \{6,3,1\}$
10	$\{6,5,4,3,2,1\}, \{5,4,1\}$
11	$\{6,5,4,3,2,1\}, \{5,3,1\}$
12	$\{6,5,4,3,2,1\}, \{4,3,1\}$
13	$\{6,5,4,3,2,1\}, \{6,3\}$
14	$\{6,5,4,3,2,1\}, \{5,1\}$
15	$\{5,4,3,2,1\}, \{6,5,4\}$
16	$\{6,4,3,2,1\}, \{6,5,3,2,1\}$
17	$\{6,5,3,2,1\}, \{6,4,3,2\}$
18	$\{6,5,4,2,1\}, \{6,5,3,2,1\}$
19	$\{6,5,4,2,1\}, \{6,4,3,2\}$
20	$\{6,5,4,3,1\}, \{6,5,4,2,1\}$
21	$\{6,5,4,3,1\}, \{6,4,3,2,1\}$
22	$\{6,5,4,3,1\}, \{6,5,2,1\}$
23	$\{6,5,4,3,1\}, \{5,2,1\}$
24	$\{6,5,4,3,2\}, \{4,3,2,1\}$
25	$\{6,5,4,3,2\}, \{4,3,1\}$
26	$\{6,5,4,3,2\}, \{4,2,1\}$
27	$\{6,5,4,3,2\}, \{3,2,1\}$
28	$\{6,5,4,3,2\}, \{2,1\}$
29	$\{5,3,2,1\}, \{6,5,4,3,1\}$

30	$\{5,3,2,1\}, \{6,5,4,2,1\}$
31	$\{5,3,2,1\}, \{6,4,3,2,1\}$
32	$\{5,3,2,1\}, \{6,4,3,2\}$
33	$\{5,3,2,1\}, \{6,4,2,1\}$
34	$\{5,3,2,1\}, \{6,4,1\}$
35	$\{6,3,2,1\}, \{6,5,4,3,1\}$
36	$\{6,3,2,1\}, \{5,4,2,1\}$
37	$\{5,4,2,1\}, \{6,5,3,2,1\}$
38	$\{5,4,2,1\}, \{6,4,3,2,1\}$
39	$\{6,4,2,1\}, \{6,5,3,2,1\}$
40	$\{6,5,2,1\}, \{6,4,3,2\}$
41	$\{5,4,3,1\}, \{6,5,4,3,2\}$
42	$\{6,4,3,1\}, \{6,5,4,2,1\}$
43	$\{6,4,3,1\}, \{6,5,3,2,1\}$
44	$\{6,4,3,1\}, \{6,5,2,1\}$
45	$\{6,4,3,1\}, \{5,3,2,1\}$
46	$\{6,4,3,1\}, \{5,2,1\}$
47	$\{6,5,4,1\}, \{6,5,3,2,1\}$
48	$\{6,5,4,1\}, \{6,4,3,2,1\}$
49	$\{6,5,4,1\}, \{6,3,2,1\}$
50	$\{6,5,3,2\}, \{6,5,4,3,1\}$
51	$\{6,5,3,2\}, \{6,5,4,2,1\}$
52	$\{6,5,3,2\}, \{6,4,3,2,1\}$
53	$\{6,5,3,2\}, \{6,5,4,1\}$
54	$\{6,5,3,2\}, \{6,4,2,1\}$
55	$\{6,5,3,2\}, \{5,4,2,1\}$
56	$\{6,5,3,2\}, \{6,4,1\}$
57	$\{5,4,3,2\}, \{6,5,3,1\}$
58	$\{5,4,3,2\}, \{6,5,1\}$

59	$\{5,4,3,2\},\{6,3,1\}$
60	$\{6,5,4,2\},\{5,4,3,2,1\}$
61	$\{6,5,4,3\},\{5,4,3,2,1\}$
62	$\{5,2,1\},\{6,4,3,2\}$
63	$\{6,4,1\},\{6,5,3,2,1\}$
64	$\{5,3,2\},\{6,5,4,2,1\}$
65	$\{5,3,2\},\{6,4,3,2,1\}$
66	$\{5,3,2\},\{6,4,2,1\}$

67	$\{5,3,2\},\{6,4,1\}$
68	$\{6,3,2\},\{6,5,4,3,1\}$
69	$\{6,3,2\},\{6,5,4,1\}$
70	$\{6,3,2\},\{5,4,2,1\}$
71	$\{6,4,2\},\{5,4,3,2,1\}$
72	$\{6,5,2\},\{5,4,3,2,1\}$
73	$\{6,1\},\{5,4,3,2\}$
74	$\{6,2\},\{5,4,3,2,1\}$

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Received February 13, 2022
and in final form December 18, 2022