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Representations and Cohomologies of Differential Lie-Yamaguti Algebras with any Weights

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Abstract. The goal of the present paper is to provide representation and cohomological theory of differential Lie-Yamaguti algebras with any weights. We introduce the notion of a differential Lie-Yamaguti algebra and its representation. We also consider matched pairs and Manin triples of Lie-Yamaguti algebras. Furthermore, we discuss cohomology theory of a differential Lie-Yamaguti algebra. The deformations and abelian extensions of differential Lie-Yamaguti algebras are also investigated.

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Key Words: Differential Lie-Yamaguti algebra, representation, matched pair, Manin triple, cohomology, deformation.

1. Introduction

Lie-Yamaguti algebras (or generalized Lie triple systems) are closely connected to reductive homogeneous spaces, which is a natural abstraction made by Yamaguti [29] of Nomizu's studies. Yamaguti termed these systems general Lie triple systems, while Kikkawa [14] named them Lie triple algebras. The concept of Lie-Yamaguti algebra arose initially in Kinyon and Weinstein's study of Courant algebroids [15]. Numerous works have been devoted to various aspects of Lie-Yamaguti algebras, see [3, 4, 5, 17, 18, 22, 27, 31] and their references.

Studies on differential algebras has a long history, which includes in various mathematical branches from differential Galois theory, differential algebraic geometry to differential algebraic groups. Originally, a differential algebra is a commutative algebra together with a linear operator satisfying the Leibniz rule. In the recent decades, the area has been developed to the noncommutative case such as associative algebra, Lie algebra and path algebra [12, 21], when operator identity has a weight in order to include difference operators and difference algebras [16]. In another direction, the notion of a differential algebra of weight λ was considered [1, 11] by generalizing the Leibniz rule.

As is well known, deformation is a very important subject both in mathematics and physics. Algebraic deformation theory was studied by Gerstenhaber for rings

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and algebras in [9, 10]. Furthermore, Nijenhuis and Richardson considered the deformation theory of Lie algebras [19, 20]. More generally, Balavoine investigated deformation theory of quadratic operads [2]. It is vital to develop cohomology theory of differential algebras with any weights. Cohomological and deformation theories of differential Lie algebras with weight zero were studied in [28]. This results have been extended to the case of associative algebras, Leibniz algebras, pre-Lie algebras, Lie triple systems and n-Lie algebras [6, 7, 24, 25, 26]. In [13], the cohomologies, extensions and deformations of differential associative algebras with any weights were studied.

In this paper, we would like to study the representation and cohomology theory of differential Lie-Yamaguti algebras with any weights.

The paper is organized as follows. In Section 2, we introduce the notion of a differential Lie-Yamaguti algebra (T, d_T) and its representation (V, d_V). Matched pairs and Manin triples of Lie-Yamaguti algebras are also considered. In Section 3, we investigate the cohomology theory of differential Lie-Yamaguti algebras. In Section 4, we discuss the deformations of differential Lie-Yamaguti algebras. Finally, we study the abelian extensions of differential Lie-Yamaguti algebras.

2. Differential Lie-Yamaguti algebras and representations

A Lie-Yamaguti algebra is a vector space T with a bilinear map $[\cdot,\cdot] : \wedge^2 T \longrightarrow T$ and a trilinear map $\{\ ,\ ,\ \} : \wedge^2 T \otimes T \longrightarrow T$, satisfying

$$[x_1, x_2] + [x_2, x_1] = 0, \tag{1}$$

$$\{x_1, x_2, x_3\} + \{x_2, x_1, x_3\} = 0, \tag{2}$$

$$\begin{aligned}&[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2]\\ &\quad +\{x_1, x_2, x_3\} + \{x_2, x_3, x_1\} + \{x_3, x_1, x_2\} = 0,\end{aligned} \tag{3}$$

$$\{[x_1, x_2], x_3, y_1\} + \{[x_2, x_3], x_1, y_1\} + \{[x_3, x_1], x_2, y_1\} = 0, \tag{4}$$

$$\{x_1, x_2, [y_1, y_2]\} = [\{x_1, x_2, y_1\}, y_2] + [y_1, \{x_1, x_2, y_2\}], \tag{5}$$

$$\begin{aligned}\{x_1, x_2, \{y_1, y_2, y_3\}\} &= \{\{x_1, x_2, y_1\}, y_2, y_3\}\\ &\quad +\{y_1, \{x_1, x_2, y_2\}, y_3\} + \{y_1, y_2, \{x_1, x_2, y_3\}\},\end{aligned} \tag{6}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3 \in T$.

A representation of a Lie-Yamaguti algebra T consists of a vector space V together with a linear map $\rho : T \longrightarrow gl(V)$ and a bilinear map $\theta : T \wedge T \longrightarrow gl(V)$ satisfying

$$\theta([x_1, x_2], x_3) = \theta(x_1, x_3)\rho(x_2) - \theta(x_2, x_3)\rho(x_1), \tag{7}$$

$$[D_{\rho,\theta}(x_1, x_2), \rho(y_1)] = \rho(\{x_1, x_2, y_1\}), \tag{8}$$

$$\theta(x_1, [y_1, y_2]) = \rho(y_1)\theta(x_1, y_2) - \rho(y_2)\theta(x_1, y_1), \tag{9}$$

$$[D_{\rho,\theta}(x_1, x_2), \theta(y_1, y_2)] = \theta(\{x_1, x_2, y_1\}, y_2) + \theta(y_1, \{x_1, x_2, y_2\}), \tag{10}$$

$$\theta(x_1, \{y_1, y_2, y_3\}) = \theta(y_2, y_3)\theta(x_1, y_1) - \theta(y_1, y_3)\theta(x_1, y_2) + D_{\rho,\theta}(y_1, y_2)\theta(x_1, y_3), \tag{11}$$

for all $x_i, y_i \in T$, where

$$D_{\rho,\theta}(x_1, x_2) - \theta(x_2, x_1) + \theta(x_1, x_2) + \rho([x_1, x_2]) - [\rho(x_1), \rho(x_2)] = 0. \tag{12}$$

We denote the representation of the Lie-Yamaguti algebra T by $(V, \rho, \theta, D_{\rho,\theta})$ or simply by (V, ρ, θ).

When (V,ρ,θ) is a representation of the Lie-Yamaguti algebra T, by a direct calculation, the following conditions are also satisfied:

$$D_{\rho,\theta}([x_1,x_2],x_3)+D_{\rho,\theta}([x_2,x_3],x_1)+D_{\rho,\theta}([x_3,x_1],x_2)=0, \tag{13}$$

$$D_{\rho,\theta}(\{x_1,x_2,x_3\},x_4)+D_{\rho,\theta}(x_3,\{x_1,x_2,x_4\})=[D_{\rho,\theta}(x_1,x_2),D_{\rho,\theta}(x_3,x_4)], \tag{14}$$

$$\theta(\{y_1,y_2,y_3\},x_1)=\theta(y_1,x_1)\theta(y_3,y_2)-\theta(y_2,x_1)\theta(y_3,y_1)-\theta(y_3,x_1)D_{\rho,\theta}(y_1,y_2). \tag{15}$$

A *semidirect product* $T\ltimes V$ is a Lie-Yamaguti algebra with

$$\{x+u,y+v,z+w\}=\{x,y,z\}+\theta(y,z)u-\theta(x,z)v+D_{\rho,\theta}(x,y)w,$$

and $$[x+u,y+v]=[x,y]+\rho(x)v-\rho(y)u,\ \forall\ x,y,z\in T,u,v,w\in V$$

if and only if (V,ρ,θ) is a representation of T [31]. For more details about Lie-Yamaguti algebras and representations, we refer to [4, 5, 15, 29, 31]. ■

Definition 2.1. A differential Lie-Yamaguti algebra of weight λ is a *Lie-Yamaguti algebra* T with a linear map $d_T:T\longrightarrow T$ such that

$$\begin{aligned}d_T(\{x,y,z\})=&\{d_T(x),y,z\}+\{x,d_T(y),z\}+\{x,y,d_T(z)\}+\lambda(\{d_T(x),d_T(y),z\}\\&+\{d_T(x),y,d_T(z)\}+\{x,d_T(y),d_T(z)\})+\lambda^2\{d_T(x),d_T(y),d_T(z)\},\end{aligned} \tag{16}$$

and $$d_T([x,y])=[d_T(x),y]+[x,d_T(y)]+\lambda[d_T(x),d_T(y)], \tag{17}$$

for any $x,y,z\in T$. One denotes it by (T,d_T). ■

Definition 2.2. Let (T,d_T) be a differential Lie-Yamaguti algebra of weight λ. A *representation* of (T,d_T) is a quadruple (V,ρ,θ,d_V), where (V,ρ,θ) is a representation of the Lie-Yamaguti algebra T and $d_V\in gl(V)$, and for any $x,y\in T$, the following conditions are satisfied:

$$d_V\rho(x)=\rho(d_Tx)+\rho(x)d_V+\lambda\rho(d_Tx)d_V, \tag{18}$$

$$d_V\theta(x,y)=(\theta(d_Tx,y)+\theta(x,d_Ty)+\lambda\theta(d_Tx,d_Ty))(id+\lambda d_V)+\theta(x,y)d_V. \tag{19}$$

Based on (18) and (19), we get

$$\begin{aligned}&d_VD_{\rho,\theta}(x,y)-D_{\rho,\theta}(x,y)d_V\\&=(D_{\rho,\theta}(d_Tx,y)+D_{\rho,\theta}(x,d_Ty)+\lambda D_{\rho,\theta}(d_Tx,d_Ty))(id+\lambda d_V).\end{aligned} \tag{20}$$

Example 2.3. Let (T,d_T) be a differential Lie-Yamaguti algebra T with any weight λ and define $ad:T\longrightarrow gl(T)$, $R,L:T\wedge T\longrightarrow gl(T)$ respectively by $ad(x)(y)=[x,y]$, $R(x,y)(z)=\{z,x,y\}$ and $L(x,y)(z)=\{x,y,z\}$. Then (T,ad,R,L,d_T) is a representation of the differential Lie-Yamaguti algebra (T,d_T), which is called the *adjoint representation.*
In view of [22], if (V,ρ,θ) is a representation of Lie-Yamaguti algebra T, then $(V^*,\rho^*,-\theta^*\tau)$ is a dual representation of T with $D_{\rho^*,-\theta^*\tau}(x,y)=\theta^*(y,x)-\theta^*(x,y)+[\rho^*(x),\rho^*(y)]-\rho^*([x,y])$, where $\tau:T\wedge T\longrightarrow T\wedge T$ is the switch map given by $\tau(x,y)=(y,x)$ for any $x,y\in T$. In particular, $(T^*,\mathrm{ad}^*,-R^*\tau)$ is the dual representation of T. But it does not hold in the case of differential Lie-Yamaguti algebras. ■

In the sequel, we study how to construct a new differential Lie-Yamaguti algebra from two given differential Lie-Yamaguti algebras.

Theorem 2.4. *Let T_1, T_2 be two Lie-Yamaguti algebras. Suppose that there are linear maps $\rho_1 : T_1 \longrightarrow gl(T_2)$, $\rho_2 : T_2 \longrightarrow gl(T_1)$ and bilinear maps $\theta_1 : \wedge^2 T_1 \longrightarrow gl(T_2)$, $\theta_2 : \wedge^2 T_2 \longrightarrow gl(T_1)$ such that (T_2, ρ_1, θ_1) is a representation of T_1 and (T_1, ρ_2, θ_2) is a representation of T_2 and they satisfy the following conditions:*

$$\rho_2(a)([x,y]) + [\rho_2(a)y, x] - [\rho_2(a)x, y] + \rho_2(\rho_1(x)a)y - \rho_2(\rho_1(y)a)x = 0, \tag{21}$$

$$\rho_1(x)([a,b]) + [\rho_1(x)b, a] - [\rho_1(x)a, b] - \rho_1(\rho_2(b)x)a + \rho_1(\rho_2(a)x)b = 0, \tag{22}$$

$$\{\rho_2(a)y, x, z\} - \{\rho_2(a)x, y, z\} = 0, \tag{23}$$

$$\{\rho_1(x)a, b, c\} - \{\rho_1(x)b, a, c\} = 0, \tag{24}$$

$$\theta_2(\rho_1(x)a, b)y - \theta_2(\rho_1(y)a, b)x + \theta_2(a,b)[x,y] = 0, \tag{25}$$

$$\theta_1(\rho_2(a)x, y)b - \theta_1(\rho_2(b)x, y)a + \theta_1(x,y)[a,b] = 0, \tag{26}$$

$$\{x, y, \rho_2(a)z\} - \rho_2(a)\{x,y,z\} - \rho_2(D_1(x,y)a)z = 0, \tag{27}$$

$$\{a, b, \rho_1(x)c\} - \rho_1(x)\{a,b,c\} - \rho_1(D_2(a,b)x)c = 0, \tag{28}$$

$$\rho_2(\theta_1(x,y)b)z - \rho_2(\theta_1(x,z)a)y = 0, \tag{29}$$

$$\rho_1(\theta_2(a,b)x)c - \rho_1(\theta_2(a,c)x)b = 0, \tag{30}$$

$$\theta_2(a, \rho_1(y)b)x - [y, \theta_2(a,b)x] = 0, \tag{31}$$

$$\theta_1(x, \rho_2(b)y)a - [b, \theta_1(x,y)a] = 0, \tag{32}$$

$$\{x, y, \theta_2(a,b)z\} = \theta_2(a,b)\{x,y,z\} + \theta_2(D_1(x,y)a, b)z + \theta_2(a, D_1(x,y)b)z, \tag{33}$$

$$\{a, b, \theta_1(x,y)c\} = \theta_1(x,y)\{a,b,c\} + \theta_1(D_2(a,b)x, y)c + \theta_1(x, D_2(a,b)x)c, \tag{34}$$

$$\{y, z, \theta_2(a,b)x\} = \theta_2(\theta_1(x,z)a, b)y - \theta_2(\theta_1(x,y)a, b)z + \theta_2(a, D_1(y,z)b)x, \tag{35}$$

$$\{b, c, \theta_1(x,y)a\} = \theta_1(\theta_2(a,c)x, y)b - \theta_1(\theta_2(a,b)x, y)c + \theta_1(x, D_2(b,c)y)a, \tag{36}$$

$$\theta_2(a, \theta_1(y,z)b)x = D_2(\theta_1(x,y)a, b)z - \{y, \theta_2(a,b)x, z\} + \theta_2(b, \theta_1(x,z)a)y, \tag{37}$$

$$\theta_1(x, \theta_1(b,c)y)a = D_1(\theta_2(a,b)x, y)c - \{b, \theta_1(x,y)a, c\} + \theta_1(y, \theta_2(a,c)x)b, \tag{38}$$

for any $x, y, z \in T_1$ and $a, b, c \in T_2$, where $D_1 = D_{\rho_1,\theta_1}, D_2 = D_{\rho_2,\theta_2}$. Then there is a Lie-Yamaguti algebra structure on $T_1 \oplus T_2$ (as the direct sum of vector spaces) given by

$$\begin{aligned}\{x+a, y+b, z+c\} = {} & \{x,y,z\} + \theta_2(b,c)x - \theta_2(a,c)y \\ & + D_2(a,b)z + [a,b,c] + \theta_1(y,z)a - \theta_1(x,z)b + D_1(x,y)c,\end{aligned} \tag{39}$$

and $$[x+a, y+b] = [x,y] + \rho_2(a)y - \rho_2(b)x + [a,b] + \rho_1(x)b - \rho_1(y)a \tag{40}$$

for any $x, y, z \in T_1$, $a, b, c \in T_2$. We denote this Lie-Yamaguti algebra by $T = T_1 \bowtie T_2$. And $(T_1, T_2, \rho_1, \theta_1, D_1, \rho_2, \theta_2, D_2)$ satisfying the above conditions is called a matched pair of Lie-Yamaguti algebras.

Moreover, suppose (T_1, d_{T_1}) and (T_2, d_{T_2}) are differential Lie-Yamaguti algebras with the same weights λ, $(T_2, \rho_1, \theta_1, d_{T_2})$ is a representation of the differential Lie-Yamaguti algebra (T_1, d_{T_1}) and $(T_1, \rho_2, \theta_2, d_{T_1})$ is a representation of the differential Lie-Yamaguti algebra (T_2, d_{T_2}). Define

$$d_T(x+a) = d_{T_1}(x) + d_{T_2}(a), \quad \forall\ x \in T_1, a \in T_2.$$

Then (T, d_T) is a differential Lie-Yamaguti algebra.

$((T_1, d_{T_1}), (T_2, d_{T_2}), \rho_1, \theta_1, D_1, \rho_2, \theta_2, D_2)$ is called a matched pair of the differential Lie-Yamaguti algebras *(T_1, d_{T_1}) and (T_2, d_{T_2}).*

Before giving the proof, we first give a short remark. Based on (21)–(38), by a direct computation, we have the following identities:

$$D_1(\rho_2(a)x,y)b - D_1(\rho_2(a)y,x)b = 0, \tag{41}$$

$$D_2(\rho_1(x)a,b)y - D_2(\rho_1(y)b,a)x = 0, \tag{42}$$

$$D_1(x,x)[a,b] - [D_1(x,y)a,b] - [a,D_1(x,y)b] = 0, \tag{43}$$

$$D_2(a,b)[x,y] - [D_2(a,b)x,y] - [x,D_2(a,b)y] = 0, \tag{44}$$

$$\{x,y,D_2(a,b)z\} = (D_2(D_1(x,y)a,b) + D_2(a,D_1(x,y)b))z + D_2(a,b)(\{x,y,z\}), \tag{45}$$

$$\{a,b,D_1(x,y)c\} = (D_1(D_2(a,b)x,y) + D_1(x,D_2(a,b)y))c + D_1(x,y)(\{a,b,c\}), \tag{46}$$

$$D_2(a,b)(\{x,y,z\}) = \{D_2(a,b)x,y,z\} + \{x,D_2(a,b)y,z\} + \{x,y,D_2(a,b)z\}, \tag{47}$$

$$D_1(x,y)(\{a,b,c\}) = \{D_1(x,y)a,b,c\} + \{a,D_1(x,y)b,c\} + \{a,b,D_1(x,y)c\}, \tag{48}$$

for any $x,y,z \in T_1, a,b,c \in T_2$.

Next we give the Proof of Theorem 2.4.

Proof. Clearly, (1) and (2) hold for $\{\ ,\ \}$ and $[\ ,\ ]$. We only need to verify that (3)–(6) hold. For the identity (3), we consider it by the following cases:

(I) when two of the three elements x_1, x_2, x_3 belong to T_1, (3) holds if and only if (12), (21) hold.

(II) when one of the three elements x_1, x_2, x_3 belong to T_2, (3) holds if and only if (12), (22) hold.

For the identity (4), we discuss it by the following cases:

(I) When one of the four elements x_1, x_2, x_3, y_1 belongs to T_2:
- (i) the fourth element belongs to T_2, (4) holds if and only if (13) holds.
- (ii) except the case (i), (4) holds if and only if (7), (23) hold.

In the rest of the proof, x, y, z denote the elements in T_1 and a, b, c denote the elements in T_2.

(II) When two of the four elements x_1, x_2, x_3, y_1 belong to T_2, that is, when x_1, x_2, x_3, y_1 equal to:
- (i) x, y, a, b or x, a, y, b or a, x, y, b respectively, (4) holds if and only if (25), (41) hold.
- (ii) a, b, x, y or a, x, b, y or x, a, b, y respectively, (4) holds if and only if (26), (42) hold.

(III) When three of the four elements x_1, x_2, x_3, y_1 belong to T_2:
- (i) the fourth element belongs to T_1, (4) holds if and only if (13) holds.
- (ii) except the case (i), (4) holds if and only if (7), (24) hold.

For the identity (5), we consider it by the following cases:

(I) When one of the four elements x_1, x_2, y_1, y_2 belongs to T_2, that is, when x_1, x_2, y_1, y_2 equal to:
- (i) x, y, z, a or x, y, a, z respectively, (5) holds if and only if (8), (27) hold.
- (ii) x, a, y, z or a, x, y, z respectively, (5) holds if and only if (9), (29) hold.

(II) When two of the four elements x_1, x_2, y_1, y_2 belong to T_2, that is, when x_1, x_2, y_1, y_2 equal to:
- (i) x, y, a, b or a, b, x, y respectively, (5) holds if and only if (43) holds.
- (ii) x, a, y, b or a, x, y, b or x, a, b, y or a, x, b, y respectively, (5) holds if and only if (31), (32) hold.
- (iii) a, b, x, y, (5) holds if and only if (44) holds.

(III) When three of the four elements x_1, x_2, y_1, y_2 belong to T_2, that is, when x_1, x_2, y_1, y_2 equal to:

(i) a, b, c, x or a, b, x, c respectively, (5) holds if and only if (8), (28) hold.

(ii) a, x, b, c or x, a, b, c respectively, (5) holds if and only if (9), (30) hold.

For the identity (6), we discuss it for the following cases:

(I) when only one of the five elements x_1, x_2, y_1, y_2, y_3 belongs to T_1 or T_2, (6) holds if and only if (10) or (11) or (14) holds.

(II) when x_1, x_2, y_1, y_2, y_3 equal to:

(i) x, y, z, a, b or x, y, a, z, b respectively, (6) holds if and only if (33) holds.

(ii) a, x, y, z, b or x, a, y, z, b respectively, (6) holds for if and only if (35) holds.

(iii) x, a, y, b, z or a, x, y, b, z or x, a, b, y, z or a, x, b, y, z respectively, (6) holds if and only if (37) holds.

(iv) x, y, a, b, z, (6) holds if and only if (45) holds.

(v) a, b, x, y, z, (6) holds if and only if (47) holds.

(vi) a, b, c, x, y or a, b, x, c, y respectively, (6) holds if and only if (34) holds.

(vii) x, a, b, c, y or a, x, b, c, y respectively, (6) holds if and only if (36) holds.

(viii) a, x, b, y, c or a, x, y, b, c or x, a, y, b, c or x, a, b, y, c respectively, (6) holds if and only if (38) holds.

(ix) a, b, x, y, c, (6) holds if and only if (46) holds.

(x) x, y, a, b, c, (6) holds if and only if (48) holds.

By direct computation, we can prove that the above statements hold and d_T is a differential operator of the Lie-Yamaguti algebra T. Therefore, (T, d_T) is a differential Lie-Yamaguti algebra. ■

If $(T, T^*, \mathrm{ad}_1^*, -{R_1}^*\tau, {L_1}^*, \mathrm{ad}_2^*, -{R_2}^*\tau, {L_2}^*)$ is a matched pair of the Lie-Yamaguti algebras T and T^*, then the operations on $T \bowtie T^*$ are given by

$$\begin{aligned}\{x+a, y+b, z+c\}_{T\bowtie T^*} = \{x, y, z\} - {R_2}^*(c, b)x + {R_2}^*(c, a)y \\ + {L_2}^*(a, b)z + \{a, b, c\} - {R_1}^*(z, y)a + {R_1}^*(z, x)b + {L_1}^*(x, y)c,\end{aligned} \tag{49}$$

and

$$[x+a, y+b]_{T\bowtie T^*} = [x, y] + \mathrm{ad}_2^*(a)y - \mathrm{ad}_2^*(b)x + [a, b] + \mathrm{ad}_1^*(x)b - \mathrm{ad}_1^*(y)a, \tag{50}$$

for any $x, y, z \in T, a, b, c \in T^*$.

Definition 2.5. A *Manin triple* of Lie-Yamaguti algebras is a triple (T, T_1, T_2) with a nondegenerate symmetric bilinear form $S(.,.) \in T^* \otimes T^*$ on T such that

(i) $S(.,.)$ is invariant, i.e., for any $x_i \in T$, it satisfies:
$S(\{x_1, x_2, x_3\}, x_4) = S(x_1, \{x_4, x_3, x_2\})$, $S([x_1, x_2], x_3) = S(x_1, [x_2, x_3])$.

(ii) $T = T_1 \oplus T_2$ as vector spaces, where T_1, T_2 are subalgebras of T.

(iii) T_1, T_2 are isotropic.

(iv) For any $x, y \in T_1$ and $a, b \in T_2$, it satisfies $\{x, y, a\}, \{a, x, y\}, \{x, a, y\} \in T_2$ and $\{x, a, b\}, \{a, b, x\}, \{a, x, b\} \in T_1$. ■

Theorem 2.6. *Let T and T^* be Lie-Yamaguti algebras. Then the following two conditions are equivalent:*

(i) $(T\oplus T^*, S(.,.), T, T^*)$ *is a Manin triple with* $S(.,.)$ *given by*

$$S(x+a, y+b)=\langle x,b\rangle+\langle y,a\rangle,\quad \forall\ x,y\in T, a,b\in T^*.$$

(ii) $(T, T^*, ad_1^*, -{R_1}^*\tau, {L_1}^*, ad_2^*, -{R_2}^*\tau, {L_2}^*)$ *is a matched pair of Lie-Yamaguti algebras.*

Proof. On the one hand, if $(T\oplus T^*, S(.,.), T, T^*)$ is a Manin triple, for any $x,y,z\in T$ and $a\in T^*$, we have

$$S(x,\{a,y,z\})=S(\{x,z,y\},a)=\langle R_1(z,y)x,a\rangle=-\langle x,{R_1}^*(z,y)a\rangle,$$
$$S(x,\{y,a,z\})=-S(x,\{a,y,z\})=-S(\{x,z,y\},a)=\langle x,{R_1}^*(z,y)a\rangle,$$
$$S(x,\{y,z,a\})=-S(\{y,z,x\},a)=-\langle L_1(y,z)x,a\rangle=\langle x,{L_1}^*(y,z)a\rangle,$$

which imply that

$$\{a,y,z\}=-{R_1}^*(z,y)a,\quad \{y,a,z\}={R_1}^*(z,y)a,\quad \{y,z,a\}={L_1}^*(y,z)a.$$

Similarly, $$[x,a]=-\mathrm{ad}_2^*(a)x+\mathrm{ad}_1^*(x)a.$$

Thus, $T\oplus T^*$ is a Lie-Yamaguti algebra with the brackets given by (49) and (50), that is, $(T,T^*,\mathrm{ad}_1^*,-{R_1}^*\tau,{L_1}^*,\mathrm{ad}_2^*,-{R_2}^*\tau,{L_2}^*)$ is a matched pair of Lie-Yamaguti algebras. On the other hand, (ii)$\Rightarrow$(i) is obvious. ■

Proposition 2.7. *Let (V,ρ,θ,d_V) be a representation of a differential Lie-Yamaguti algebra (T,d_T) with weight λ. Define a linear map $\hat{\rho}: T\longrightarrow gl(V)$ and a bilinear map $\hat{\theta}: T\wedge T\longrightarrow gl(V)$ by*

$$\hat{\rho}(x)=\rho(x)+\lambda\rho(d_Tx), \tag{51}$$
$$\hat{\theta}(x,y)=\theta(x,y)+\lambda(\theta(d_Tx,y)+\theta(x,d_Ty)+\lambda\theta(d_Tx,d_Ty)), \tag{52}$$

for all $x,y\in T$. Then $(V,\hat{\rho},\hat{\theta},d_V)$ is also a representation of (T,d_T) denotes by V_λ.

Proof. It is routine to verify that (7)–(11) hold for $\hat{\rho},\hat{\theta}$. We only need to check that (18) and (19) hold. In fact, for any $x,y\in T$, by (19) and (52), we obtain

$$\begin{aligned}
&\hat{\theta}(d_Tx,y)+\hat{\theta}(x,d_Ty)+\lambda\hat{\theta}(d_Tx,d_Ty)+\hat{\theta}(x,y)d_V+\lambda(\hat{\theta}(d_Tx,y)\\
&\quad+\hat{\theta}(x,d_Ty)+\lambda\hat{\theta}(d_Tx,d_Ty))d_V-d_V\hat{\theta}(x,y)\\
&=\theta(d_Tx,y)+\lambda\theta(d_T^2x,y)+\lambda\theta(d_Tx,d_Ty)+\lambda^2\theta(d_T^2x,d_Ty)+\theta(x,d_Ty)+\lambda\theta(x,d_T^2y)\\
&\quad+\lambda\theta(d_Tx,d_Ty)+\lambda^2\theta(d_Tx,d_T^2y)+\lambda(\theta(d_Tx,d_Ty)+\lambda\theta(d_T^2x,d_Ty)+\lambda\theta(d_Tx,d_T^2y)\\
&\quad+\lambda^2\theta(d_T^2x,d_T^2y))+(\theta(x,y)+\theta(d_Tx,y)+\theta(x,d_Ty)+\lambda\theta(d_Tx,d_Ty))d_V\\
&\quad+\lambda(\theta(d_Tx,y)+\lambda\theta(d_T^2x,y)+\lambda\theta(d_Tx,d_Ty)+\lambda^2\theta(d_T^2x,d_Ty)+\theta(x,d_Ty)+\lambda\theta(x,d_T^2y)\\
&\quad+\lambda\theta(d_Tx,d_Ty)+\lambda^2\theta(d_Tx,d_T^2y)+\lambda\theta(d_Tx,d_Ty)+\lambda^2\theta(d_T^2x,d_Ty)+\lambda^2\theta(d_Tx,d_T^2y)\\
&\quad+\lambda^3\theta(d_T^2x,d_T^2y))d_V-d_V(\theta(x,y)+\theta(d_Tx,y)+\theta(x,d_Ty)+\lambda\theta(d_Tx,d_Ty))\\
&=0.
\end{aligned}$$

Analogously, $\hat{\rho}(d_Tx)+\hat{\rho}(x)d_V+\lambda\hat{\rho}(d_Tx)d_V-d_V\hat{\rho}(x)=0$. ■

Remark 2.8. In Proposition 2.7, if we further define $\hat{D}_{\rho,\theta}: T \wedge T \longrightarrow gl(V)$ by

$$\hat{D}_{\rho,\theta}(x,y) = D_{\rho,\theta}(x,y) + \lambda(D_{\rho,\theta}(d_T x, y) + D(x, d_T y) + \lambda D_{\rho,\theta}(d_T x, d_T y)). \quad (53)$$

Then we obtain by direct computation, $D_{\hat{\rho},\hat{\theta}} = \hat{D}_{\rho,\theta}$. ■

3. Cohomologies of differential Lie-Yamaguti algebras

This section is devoted to discussing the cohomologies of a differential Lie-Yamaguti algebra with coefficients in its representation. We begin with reviewing the cohomologies of Lie-Yamaguti algebras [30].

Let T be a Lie-Yamaguti algebra and (V, ρ, θ) be its representation.

Let $f : T \times \cdots \times T \longrightarrow V$ be an n-linear map of T into V such that:

$$f(x_1, \cdots, x_{2i-1}, x_{2i}, \cdots, x_n) = 0, \quad \text{if } x_{2i-1} = x_{2i}.$$

The set of all such linear maps is called an n-cochain of T, which is denoted by $C^n(T,V)$ for $n \geq 1$. For $n \geq 1$, the coboundary operator

$$d = (d_I, d_{II}) : C^{2n}(T,V) \times C^{2n+1}(T,V) \longrightarrow C^{2n+2}(T,V) \times C^{2n+3}(T,V)$$

is given as follows:

$$\begin{aligned}
(d_I f)(x_1, \cdots, x_{2n+2}) &= \rho(x_{2n+1}) g(x_1, \cdots, x_{2n}, x_{2n+2}) \\
&\quad - \rho(x_{2n+2}) g(x_1, \cdots, x_{2n+1}) - g(x_1, \cdots, x_{2n}, [x_{2n+1}, x_{2n+2}]) \\
&\quad + \sum_{k=1}^{n} (-1)^{n+k+1} D_{\rho,\theta}(x_{2k-1}, x_{2k}) f(x_1, \cdots, x_{2k-2}, x_{2k+1}, \cdots, x_{2n+2}) \\
&\quad + \sum_{k=1}^{n} \sum_{j=2k+1}^{2n+2} (-1)^{n+k} f(x_1, \cdots, x_{2k-2}, x_{2k+1}, \cdots, \{x_{2k-1}, x_{2k}, x_j\}, \cdots, x_{2n+2})
\end{aligned}$$

and

$$\begin{aligned}
&(d_{II} f)(x_1, \cdots, x_{2n+3}) \\
&= \theta(x_{2n+2}, x_{2n+3}) g(x_1, \cdots, x_{2n+1}) - \theta(x_{2n+1}, x_{2n+3}) g(x_1, \cdots, x_{2n}, x_{2n+2}) \\
&\quad + \sum_{k=1}^{n+1} (-1)^{n+k+1} D_{\rho,\theta}(x_{2k-1}, x_{2k}) g(x_1, \cdots, x_{2k+1}, \cdots, x_{2n+3}) \\
&\quad + \sum_{k=1}^{n+1} \sum_{j=2k+1}^{2n+3} (-1)^{n+k} g(x_1, \cdots, x_{2k+1}, \cdots, \{x_{2k-1}, x_{2k}, x_j\}, \cdots, x_{2n+3})
\end{aligned}$$

for any pair $(f, g) \in C^{2n}(T,V) \times C^{2n+1}(T,V)$ and $x_1, \cdots, x_{2n+3} \in T$. And when $n = 0$, the coboundary operator

$$d = (d_I, d_{II}) : C^1(T,V) \longrightarrow C^2(T,V) \times C^3(T,V)$$

is defined as follows:

$$\begin{aligned}
(d_I \omega)(x_1, x_2) &= \rho(x_1)\omega(x_2) - \rho(x_2)\omega(x_1) - \omega([x_1, x_2]), \\
(d_{II} \omega)(x_1, x_2, x_3) &= \theta(x_2, x_3)\omega(x_1) - \theta(x_1, x_3)\omega(x_2) \\
&\quad + D_{\rho,\theta}(x_1, x_2)\omega(x_3) - \omega(\{x_1, x_2, x_3\})
\end{aligned}$$

for any $\omega \in C^1(T,V)$ and $x_1, x_2, x_3 \in T$.

Denote the set of the $(2n, 2n+1)$-cocycles and the $(2n, 2n+1)$-coboundaries, respectively by $Z^{2n}(T,V) \times Z^{2n+1}(T,V)$ and $B^{2n}(T,V) \times B^{2n+1}(T,V)$. Define

$$H^{2n}(T,V) \times H^{2n+1}(T,V) = (Z^{2n}(T,V) \times Z^{2n+1}(T,V))/(B^{2n}(T,V) \times B^{2n+1}(T,V)),$$

which is called the *(2n,2n+1)-cohomology group* of T with coefficients in the representation V.

Remark 3.1. (i) For any $(\nu, \omega) \in C^2(T,V) \times C^3(T,V)$, (ν,ω) satisfies the following conditions:

$$\nu(x_1,x_2) + \nu(x_2,x_1) = 0, \tag{54}$$

$$\omega(x_1,x_2,x_3) + \omega(x_2,x_1,x_3) = 0. \tag{55}$$

(ii) For any pair $(\nu,\omega) \in Z^2(T,V) \times Z^3(T,V)$, besides $d(\nu,\omega) = 0$, we also claim that

$$\omega(x_1,x_2,x_3) + c.p. - \rho(x_1)\nu(x_2,x_3) - c.p. + \nu([x_1,x_2],x_3) + c.p. = 0, \tag{56}$$

$$\theta(x_1,y_1)\nu(x_2,x_3) + c.p. + \omega([x_1,x_2],x_3,y_1) + c.p. = 0, \tag{57}$$

for all $x_1, x_2, x_3, y_1 \in T$. ■

Next we begin to define cohomologies of differential Lie-Yamaguti algebras.

Let (V, ρ, θ, d_V) be a representation of the differential Lie-Yamaguti algebra (T, d_T). In view of Proposition 2.7, $(V, \hat{\rho}, \hat{\theta}, d_V)$ is also a representation of (T, d_T). Denote it by V_λ. Consider the cohomologies of the Lie-Yamaguti algebra (T, d_T) with coefficients in the representation V_λ. The corresponding $(2n, 2n+1)$-cochains group of T is denoted by $C^{2n}(T,V_\lambda) \times C^{2n+1}(T,V_\lambda)$. For $n \geq 1$, the coboundary operator

$$d^\lambda = (d_I^\lambda, d_{II}^\lambda) : C^{2n}(T,V_\lambda) \times C^{2n+1}(T,V_\lambda) \longrightarrow C^{2n+2}(T,V_\lambda) \times C^{2n+3}(T,V_\lambda)$$

is defined as follows:

$$\begin{aligned}
&(d_I^\lambda f)(x_1, \cdots, x_{2n+2}) = \hat{\rho}(x_{2n+1})g(x_1, \cdots, x_{2n}, x_{2n+2}) \\
&\quad - \hat{\rho}(x_{2n+2})g(x_1, \cdots, x_{2n+1}) - g(x_1, \cdots, x_{2n}, [x_{2n+1}, x_{2n+2}]) \\
&\quad + \sum_{k=1}^{n} (-1)^{n+k+1} D_{\hat{\rho},\hat{\theta}}(x_{2k-1}, x_{2k}) f(x_1, \cdots, x_{2k-2}, x_{2k+1}, \cdots, x_{2n+2}) \\
&\quad + \sum_{k=1}^{n} \sum_{j=2k+1}^{2n+2} (-1)^{n+k} f(x_1, \cdots, x_{2k-2}, x_{2k+1}, \cdots, \{x_{2k-1}, x_{2k}, x_j\}, \cdots, x_{2n+2})
\end{aligned}$$

and

$$\begin{aligned}
&(d_{II}^\lambda f)(x_1, \cdots, x_{2n+3}) \\
&= \hat{\theta}(x_{2n+2}, x_{2n+3})g(x_1, \cdots, x_{2n+1}) - \hat{\theta}(x_{2n+1}, x_{2n+3})g(x_1, \cdots, x_{2n}, x_{2n+2}) \\
&\quad + \sum_{k=1}^{n+1} (-1)^{n+k+1} D_{\hat{\rho},\hat{\theta}}(x_{2k-1}, x_{2k}) g(x_1, \cdots, x_{2k+1}, \cdots, x_{2n+3}) \\
&\quad + \sum_{k=1}^{n+1} \sum_{j=2k+1}^{2n+3} (-1)^{n+k} g(x_1, \cdots, x_{2k+1}, \cdots, \{x_{2k-1}, x_{2k}, x_j\}, \cdots, x_{2n+3})
\end{aligned}$$

for any pair $(f,g) \in C^{2n}(T,V_\lambda) \times C^{2n+1}(T,V_\lambda)$ and $x_1, \cdots, x_{2n+3} \in T$.

And $$d^\lambda = (d^\lambda_I, d^\lambda_{II}) : C^1(T, V_\lambda) \longrightarrow C^2(T, V_\lambda) \times C^3(T, V_\lambda)$$

is given by $$(d^\lambda_I\omega)(x_1, x_2) = \hat{\rho}(x_1)\omega(x_2) - \hat{\rho}(x_2)\omega(x_1) - \omega([x_1, x_2]),$$

$$(d^\lambda_{II}\omega)(x_1, x_2, x_3) = \hat{\theta}(x_2, x_3)\omega(x_1) - \hat{\theta}(x_1, x_3)\omega(x_2) + D_{\hat{\rho},\hat{\theta}}(x_1, x_2)\omega(x_3) - \omega(\{x_1, x_2, x_3\})$$

for any $\omega \in C^1(T, V_\lambda)$ and $x_1, x_2, x_3 \in T$.

For $n = 1$, the $(2, 3)$-cocycle (f, g) satisfies $d^\lambda(f, g) = 0$ and (f, g) satisfies also (54)–(57) – just replace $\rho, \theta, D_{\rho,\theta}$ by $\hat{\rho}, \hat{\theta}, D_{\hat{\rho},\hat{\theta}}$.

The corresponding $(2n, 2n+1)$-cohomology group $H^{2n}(T, V_\lambda) \times H^{2n+1}(T, V_\lambda)$ of the cochain complex $(C^*(T, V_\lambda) \times C^*(T, V_\lambda), d^\lambda)$ is called the $(2n, 2n+1)$-cohomology group of the differential operator d_T with coefficients in the representation V.

Clearly, $C^{2n}(T, V_\lambda) \times C^{2n+1}(T, V_\lambda) = C^{2n}(T, V) \times C^{2n+1}(T, V)$ as vector spaces, but they are not equal as cochain complexes except $\lambda = 0$.

Proposition 3.2. *Define a bilinear map*

$$\Delta = (\delta, \delta) : C^{2n}(T, V) \times C^{2n+1}(T, V) \longrightarrow C^{2n}(T, V_\lambda) \times C^{2n+1}(T, V_\lambda)$$

with $\delta : C^p(T, V) \longrightarrow C^p(T, V_\lambda)$ *given by*

$$\delta f(x_1, \cdots, x_p) = \sum_{k=1}^{p} \lambda^{k-1} f^k(x_1, \cdots, x_p) - d_V f(x_1, \cdots, x_p),$$

where $$f^k(x_1, \cdots, x_p) = f(\underbrace{I \otimes \cdots \otimes d_T \otimes \cdots \otimes I}_{d_T \text{ appears } k \text{ times}})(x_1, \cdots, x_p).$$

Then Δ *is a cochain map from the cochain complex* $(C^*(T, V) \times C^*(T, V), d)$ *to* $(C^*(T, V_\lambda) \times C^*(T, V_\lambda), d^\lambda)$*, that is,* $d^\lambda\Delta = \Delta d$.

We leave the long and technical proof of this result to the Appendix.

For $n \geq 1$, we define

$$C^{2n}_{DYA}(T, V) \times C^{2n+1}_{DYA}(T, V) = \{(C^{2n}(T, V) \times C^{2n+1}(T, V)) \times (C^{2n-2}(T, V_\lambda) \times C^{2n-1}(T, V_\lambda))\},$$

and $$C^0_{DYA}(T, V) \times C^1_{DYA}(T, V) = C^1(T, V).$$

Then a map

$$\partial : C^{2n}_{DYA}(T, V) \times C^{2n+1}_{DYA}(T, V) \longrightarrow C^{2n+2}_{DYA}(T, V) \times C^{2n+3}_{DYA}(T, V)$$

is defined by

$$\partial\{(f, g), (\bar{f}, \bar{g})\} = \{d(f, g), d^\lambda(\bar{f}, \bar{g}) + (-1)^{n+1}\Delta(f, g)\}, \ n \geq 1,$$

for any pair $\{(f, g), (\bar{f}, \bar{g})\} \in C^{2n}_{DYA}(T, V) \times C^{2n+1}_{DYA}(T, V)$ and

$$\partial : C^0_{DYA}(T, V) \times C^1_{DYA}(T, V) \longrightarrow C^2_{DYA}(T, V) \times C^3_{DYA}(T, V)$$

is given by $\partial(\omega) = \{(d_I\omega, d_{II}\omega), -\delta\omega\}$, for all $\omega \in C^0_{DYA}(T, V) \times C^1_{DYA}(T, V)$.

In view of Proposition 3.2, we have

Theorem 3.3. *∂ is a coboundary operator, that is, $\partial\partial = 0$.*

With the cochain complex $(C^*_{DYA}(T,V) \times C^*_{DYA}(T,V), \partial)$, denote the space of $(2n, 2n+1)$-cocycles and $(2n, 2n+1)$-coboundaries, respectively by

$$Z^{2n}_{DYA}(T,V) \times Z^{2n+1}_{DYA}(T,V) \text{ and } B^{2n}_{DYA}(T,V) \times B^{2n+1}_{DYA}(T,V).$$

The corresponding quotients

$$H^{2n}_{DYA}(T,V) \times H^{2n+1}_{DYA}(T,V) = Z^{2n}_{DYA}(T,V) \times Z^{2n+1}_{DYA}(T,V)/B^{2n}_{DYA}(T,V) \times B^{2n+1}_{DYA}(T,V),$$

are called the $(2n, 2n+1)$-cohomology of the differential Lie-Yamaguti algebra (T, d_T) with coefficients in the representation (V, ρ, θ, d_V).

Remark 3.4. The $(2,3)$-cocycle $\{(f,g),(\bar{g})\} \in Z^{2n}_{DYA}(T,V) \times Z^{2n+1}_{DYA}(T,V)$ means that (f,g) is the $(2,3)$-cocycle of T with coefficients in V, $d^{\lambda}_{I}\bar{g} + \delta f = 0$ and $d^{\lambda}_{II}\bar{g} + \delta g = 0$.

From the definitions of $C^{2n}_{DYA}(T,V) \times C^{2n+1}_{DYA}(T,V)$ and $C^{2n}(T,V_\lambda) \times C^{2n+1}(T,V_\lambda)$ we easily get an exact sequence of cochain complexes,

$$\begin{aligned} 0 \longrightarrow C^{2n}(T,V_\lambda) \times C^{2n+1}(T,V_\lambda) &\longrightarrow C^{2n+2}_{DYA}(T,V) \times C^{2n+3}_{DYA}(T,V) \\ &\longrightarrow C^{2n+2}(T,V) \times C^{2n+3}(T,V) \longrightarrow 0. \end{aligned}$$

Moreover, combining the Snake Lemma, we have the following long exact sequence of cohomology groups

$$\begin{aligned} \cdots \longrightarrow H^{2n}(T,V_\lambda) \times H^{2n+1}(T,V_\lambda) &\longrightarrow H^{2n+2}_{DYA}(T,V) \times H^{2n+3}_{DYA}(T,V) \\ \longrightarrow H^{2n+2}(T,V) \times H^{2n+3}(T,V) &\longrightarrow H^{2n+2}(T,V_\lambda) \times H^{2n+3}(T,V_\lambda) \longrightarrow \cdots. \end{aligned}$$

4. Deformations of differential Lie-Yamaguti algebras

Let $(T, \pi = [\ ,\ ], \omega = \{\ ,\ ,\ \}, d_T)$ be a differential Lie-Yamaguti algebra with any weight λ, $\pi_i : T \wedge T \longrightarrow T$ be a bilinear map, $\omega_i : T \wedge T \otimes T \longrightarrow T$ a trilinear map and $\varphi_i : T \longrightarrow T$ a linear map respectively.

Consider the space $T[[t]]$ of formal power series in t with coefficients in T and a t-parametrized family of trilinear operations $\omega_t(x,y,z) = \sum_{i=0}^{\infty} t^i \omega_i(x,y,z)$, bilinear operations $\pi_t(x,y) = \sum_{i=0}^{\infty} t^i \pi_i(x,y)$, and linear operations $\varphi_t(x) = \sum_{i=0}^{\infty} t^i \varphi_i(x)$, where $\varphi_0 = d_T, \pi_0 = \pi$ and $\omega_0 = \omega$.

If all $(T[[t]], \pi_t, \omega_t, \varphi_t)$ are differential Lie-Yamaguti algebras, we say that $(\pi_t, \omega_t, \varphi_t)$ generates a deformation of the differential Lie-Yamaguti algebra $(T, \pi = [\ ,\ ], \omega = \{\ ,\ ,\ \}, d_T)$.

If all $(T[[t]], \pi_t, \omega_t, \varphi_t)$ are differential Lie-Yamaguti algebras with weight λ, then we have

$$\sum_{n\geq 0} \pi_n(x,y) + \pi_n(y,x) = 0, \quad \sum_{i\geq 0} \omega_n(x,y,z) + \omega_n(y,x,z) = 0, \tag{58}$$

$$\begin{aligned} \sum_{i,j\geq 0}^{i+j=n} \pi_i(\pi_j(x,y),z) + \pi_i(\pi_j(y,z),x) + \pi_i(\pi_j(z,x),y) \\ + \sum_{n\geq 0} \omega_n(x,y,z) + \omega_n(y,x,z) + \omega_n(z,x,y) = 0, \end{aligned} \tag{59}$$

$$\sum_{i,j\geq 0}^{i+j=n} \omega_i(\pi_j(x,y),z,w) + \omega_i(\pi_j(y,z),x,w) + \pi_i(\pi_j(z,x),y,w) = 0, \tag{60}$$

$$\sum_{i,j\geq 0}^{i+j=n} \omega_i(x,y,\pi_j(z,w)) = \sum_{i,j\geq 0}^{i+j=n} \pi_i(\omega_j(x,y,z),w) + \pi_i(z,\omega_j(x,y,w)), \tag{61}$$

$$\sum_{i,j\geq 0}^{i+j=n} (\omega_i(\omega_j(v,w,x),y,z) + \omega_i(x,\omega_j(v,w,y),z) + \omega_i(x,y,\omega_j(v,w,z)) - \omega_i(v,w,\omega_j(x,y,z))) = 0, \tag{62}$$

$$\sum_{i,j\geq 0}^{i+j=n} \varphi_i\pi_j(x,y) - \sum_{i,j\geq 0}^{i+j=n} \pi_i(\varphi_j(x),y) + \pi_i(\varphi_j(x),y) + \pi_i(x,\varphi_j(y)) = \sum_{i,j,k\geq 0}^{i+j+k=n} \lambda\pi_i(\varphi_j(x),\varphi_k(y)), \quad \text{and} \tag{63}$$

$$\sum_{i,j\geq 0}^{i+j=n} (\omega_i(\varphi_j(x),y,z) + \omega_i(x,\varphi_j(y),z) + \omega_i(x,y,\varphi_j(z))) + \sum_{i,j,k\geq 0}^{i+j+k=n} \lambda(\omega_i(\varphi_j(x),\varphi_k(y),z) + \omega_i(x,\varphi_j(y),\varphi_k(z)) + \omega_i(\varphi_j(x),y,\varphi_k(z))) + \sum_{i,j,k,l\geq 0}^{i+j+k+l=n} \lambda^2\omega_i(\varphi_j(x),\varphi_k(y),\varphi_l(z)) - \sum_{i,j\geq 0}^{i+j=n} \varphi_i\omega_j(x,y,z) = 0. \tag{64}$$

Proposition 4.1. *$(T[[t]],\pi_t,\omega_t,\varphi_t)$ generates a deformation of the differential Lie-Yamaguti algebra $(T,\pi = [\ ,\ ],\omega = \{\ ,\ ,\ \},d_T)$. Then $\{(\pi_1,\omega_1),(\varphi_1)\}$ is a $(2,3)$-cocycle of (T,d_T) with coefficients in the adjoint representation (T,ad,R,d_T).*

Proof. If $(T[[t]],\pi_t,\omega_t,\varphi_t)$ generates a deformation of the differential Lie-Yamaguti algebra (T,π,ω,d_T), then (58)–(64) hold. For the identities (58)–(62), letting $n=1$, which implies that (π_1,ω_1) is a (2,3)-cocycle of T with coefficients in the adjoint representation (T,ad,R).
On the other hand, for any pair $\{(\pi_1,\omega_1),(\varphi_1)\} \in C^2_{DYA}(T,T) \times C^3_{DYA}(T,T)$, $\{(\pi_1,\omega_1),(\varphi_1)\}$ is a $(2,3)$-cocycle if (π_1,ω_1) is a $(2,3)$-cocycle of T with coefficients in the adjoint representation and

$$d^\lambda(\varphi_1) + (-1)^2\Delta(\pi_1,\omega_1) = (d^\lambda_I\varphi_1 + \delta\pi_1, d^\lambda_{II}\varphi_1 + \delta\omega_1) = 0. \tag{65}$$

By computation, when $n=1$, (63) and (64) are equivalent to (65). ∎

Definition 4.2. A deformation of the differential Lie-Yamaguti algebra (T,d_T) is said to be *trivial* if there is a linear map $N: T \longrightarrow T$ such that $K_t = I + tN$ $(\forall\ t)$ satisfies

$$K_t d_T = d_T K_t,\ \ K_t[x,y]_t = [K_t x, K_t y] \ \text{ and } \ K_t\{x,y,z\}_t = \{K_t x, K_t y, K_t z\}.$$

Similar to the case of Lie-Yamaguti algebras [22, 23], we give a definition of Nijenhuis operators and relative Rota-Baxter operators of differential Lie-Yamaguti algebras.

Definition 4.3. Let (T, d_T) be a differential Lie-Yamaguti algebra. A linear map $N : T \longrightarrow T$ is called a *Nijenhuis operator* on (T, d_T) if $Nd_T = d_T N$ and N is a Nijenhuis operator on T, that is, for any $x,\ y,\ z \in T$,

$$[Nx, Ny] = N([Nx, y] + [x, Ny] - N[x, y]),$$
$$\{Nx, Ny, Nz\} = N(\{Nx, Ny, z\} + \{x, Ny, Nz\} + \{Nx, y, Nz\}) \\ -N^2(\{Nx, y, z\} + \{x, Ny, z\} + \{x, y, Nz\}) + N^3\{x, y, z\}$$

Remark 4.4. $(T, [\ ,\ ]_N, \{\ ,\ ,\}_N, d_T)$ is also a differential Lie-Yamaguti algebra,

where
$$[x, y]_N = [Nx, y] + [x, Ny] - N[x, y] \tag{66}$$

and
$$\{x, y, z\}_N = \{Nx, Ny, z\} + \{x, Ny, Nz\} + \{Nx, y, Nz\} - N(\{Nx, y, z\} \\ +\{x, Ny, z\} + \{x, y, Nz\}) + N^2\{x, y, z\}. \tag{67}$$

Definition 4.5. A linear map $K : V \longrightarrow T$ is called a *relative Rota-Baxter operator* on the differential Lie-Yamaguti algebra (T, d_T) associated to the representation (V, ρ, θ, d_V) if $Kd_V = d_T K$ and K is a relative Rota-Baxter operator on T, that is, for all $u, v, w \in V$,

$$[Ku, Kv] = K(\rho(Ku)v - \rho(Kv)u),$$

and $\{Ku, Kv, Kw\} = K(\theta(Kv, Kw)u - \theta(Ku, Kw)v + D_{\rho,\theta}(Ku, Kv)w)$.

Proposition 4.6. *Let $K : V \longrightarrow T$ be a relative Rota-Baxter operator on the differential Lie-Yamaguti algebra (T, d_T) associated to the representation (V, ρ, θ, d_V). Then $(V, [\ ,\ ]_K, \{\ ,\ ,\ \}_K, d_V)$ is a differential Lie-Yamaguti algebra, where*

$$[u, v]_K = K(\rho(Ku)v - \rho(Kv)u),$$
$$\{u, v, w\}_K = \theta(Kv, Kw)u - \theta(Ku, Kw)v + D_{\rho,\theta}(Ku, Kv)w$$

for all $u, v, w \in V$.

Proof. Based on [22], $(V, [\ ,\ ]_K, \{\ ,\ ,\ \}_K)$ is a Lie-Yamaguti algebra. We only need to verify that d_V is a differential operator. In fact, for all $u, v, w \in V$

$$\begin{aligned}
& d_V(\{u, v, w\}_K) \\
= & \ d_V(\theta(Kv, Kw)u - \theta(Ku, Kw)v + D_{\rho,\theta}(Ku, Kv)w) \\
= & \ \theta(d_T Kv, Kw)u + \theta(Kv, d_T Kw)u + \lambda\theta(d_T Kv, d_T Kw)u + \theta(Kv, Kw)d_V u \\
& +\lambda(\theta(d_T Kv, Kw) + \theta(Kv, d_T Kw) + \lambda\theta(d_T Kv, d_T Kw))d_V u \\
& -\theta(d_T Ku, Kw)v - \theta(Ku, d_T Kw)v - \lambda\theta(d_T Ku, d_T Kw)v \\
& -\theta(Ku, Kw)d_V v - \lambda(\theta(d_T Ku, Kw) + \theta(Ku, d_T Kw) \\
& +\lambda\theta(d_T Ku, d_T Kw))d_V v + D_{\rho,\theta}(d_T Ku, Kv)w + D_{\rho,\theta}(Ku, d_T Kv)w \\
& +\lambda D_{\rho,\theta}(d_T Ku, d_T Kv)w + D_{\rho,\theta}(Ku, Kv)d_V w + \lambda(D_{\rho,\theta}(d_T Ku, Kv)w \\
& +D_{\rho,\theta}(Ku, d_T Kv) + \lambda D_{\rho,\theta}(d_T Ku, d_T Kv))d_V w \\
= & \ \{u, d_V v, w\}_K + \{d_V u, v, w\}_K + \{u, v, d_V w\}_K + \lambda\{d_V u, d_V v, w\}_K \\
& +\lambda\{d_V u, v, d_V w\}_K + \lambda\{u, d_V v, d_V w\}_K + \lambda^2\{d_V u, d_V v, d_V w\}_K.
\end{aligned}$$

Similarly, $d_V([u, v]_K) = [d_V u, v]_K + [u, d_V v]_K + \lambda[d_V u, d_V v]_K$.

Thus, d_V is a differential operator of $(V, [\ ,\ ]_K,\ \{\ ,\ ,\ \}_K)$. ∎

Proposition 4.7. *$K:V\longrightarrow A$ is a relative Rota-Baxter operator on a differential Lie-Yamaguti algebra (T,d_T) associated to the representation $(V,\hat{\rho},\hat{\theta},d_V)$ if and only if $\hat{K}=K+\lambda d_T K$ is a relative Rota-Baxter operator on the differential Lie-Yamaguti algebra (T,d_T) associated to the representation (V,ρ,θ,d_V), where $\hat{\rho},\hat{\theta}$ are defined in Section 2.*

Proof. Notice that

$$\begin{aligned}
&\hat{\theta}(Ku,Kv)\\
=\ &\theta(Ku,Kv)+\lambda\theta(d_TKu,Kv)+\lambda\theta(Ku,d_TKv)+\lambda^2\theta(d_TKu,d_TKv)\\
=\ &\theta(\hat{K}u,\hat{K}v).
\end{aligned}$$

Analogously, $D_{\hat{\rho},\hat{\theta}}(Ku,Kv)=D_{\rho,\theta}(\hat{K}u,\hat{K}v)$, $\hat{\rho}(Ku)=\rho(\hat{K}u)$.
This completes the proof. ∎

Proposition 4.8. *Let $K:V\longrightarrow T$ be a relative Rota-Baxter operator on a differential Lie-Yamaguti algebra (T,d_T) associated to the representation (V,ρ,θ,d_V). Define a linear map $\tilde{\rho}:V\longrightarrow gl(T)$ and a bilinear map $\tilde{\theta}:V\wedge V\longrightarrow gl(T)$ respectively by*

$$\tilde{\rho}(u)x=[Ku,x]+K(\rho(x)u),\ \forall\ x\in T,u\in V,$$

$$\tilde{\theta}(u,v)x=\{x,Ku,Kv\}+K(\theta(x,Kv)u-D_{\rho,\theta}(x,Ku)v),\ \forall\ x\in T,u,v\in V.$$

Then $(T,\tilde{\rho},\tilde{\theta},d_T)$ is a representation of $(V,[\ ,\ ]_K,\ \{\ ,\ ,\ \}_K,d_V)$.

Proof. Define $\tilde{K}:T\oplus V\longrightarrow T\oplus V$ by

$$\tilde{K}(x+u)=Ku$$

for any $x\in T$ and $u\in V$. Then $\tilde{K}$ is a Nijenhuis operator on $T\oplus V$. So there is a Lie-Yamaguti algebra structure on $T\oplus V$. On the one hand, using (39), (40) and (4.4), we obtain

$$\begin{aligned}
&\{x+u,y+v,z+w\}_{\tilde{K}}\\
&=\{\tilde{K}(x+u),\tilde{K}(y+v),z+w\}+\{\tilde{K}(x+u),y+v,\tilde{K}(z+w)\}\\
&\quad+\{x+u,\tilde{K}(y+v),\tilde{K}(z+w)\}-\tilde{K}(\{\tilde{K}(x+u),y+v,z+w\}\\
&\quad+\{x+u,\tilde{K}(y+v),z+w\}+\{x+u,y+v,\tilde{K}(z+w)\})\\
&\quad+\tilde{K}^2\{x+u,y+v,z+w\}\\
&=\{Ku,Kv,z+w\}+\{Ku,y+v,Kw\}+\{x+u,Kv,Kw\}\\
&\quad-\tilde{K}(\{Ku,y+v,z+w\}+\{x+u,Kv,z+w\}+\{x+u,y+v,Kw\})\\
&=\{Ku,Kv,z\}+D_{\rho,\theta}(Ku,Kv)w+\{Ku,y,Kw\}-\theta(Ku,Kw)v+\{x,Kv,Kw\}\\
&\quad+\theta(Kv,Kw)u+K(\theta(Ku,z)v-D_{\rho,\theta}(Ku,y)w-\theta(Kv,z)u\\
&\quad-D_{\rho,\theta}(x,Kv)w-\theta(y,Kw)u+\theta(x,Kw)v)\\
&=\{u,v,w\}_K+\{Ku,Kv,z\}+\{Ku,y,Kw\}+\{x,Kv,Kw\}+K(\theta(Ku,z)v\\
&\quad-D_{\rho,\theta}(Ku,y)w-\theta(Kv,z)u-D_{\rho,\theta}(x,Kv)w-\theta(y,Kw)u+\theta(x,Kw)v),
\end{aligned}\tag{68}$$

and
$$\begin{aligned}
&[x+u,y+v]_{\tilde{K}}\\
&=[\tilde{K}(x+u),y+v]+[x+u,\tilde{K}(y+v)]-\tilde{K}([x+u,y+v])\\
&=[Ku,y+v]+[x+u,Kv]-\tilde{K}([x,y]+\rho(x)v-\rho(y)u))\\
&=[u,v]_K+[Ku,y]+[x,Kv]-K(\rho(x)v-\rho(y)u), \qquad (69)
\end{aligned}$$

On the other hand, $T\oplus V$ is a semidirect product, and thus
$$\{x+u,y+v,z+w\}_{\tilde{K}}=\{u,v,w\}_K+\tilde{\theta}(v,w)x-\tilde{\theta}(u,w)y+D_{\tilde{\rho},\tilde{\theta}}(u,v)z, \qquad (70)$$
and
$$[x+u,y+v]_{\tilde{K}}=[u,v]_K+\tilde{\rho}(u)y-\tilde{\rho}(v)x. \qquad (71)$$

Combining the equations (68)–(71), we obtain that $(T,\tilde{\rho},\tilde{\theta})$ is a representation of $(V,[\,,\,]_K,\ \{\,,\,,\,\}_K)$. At the same time,

$$\begin{aligned}
&d_T\tilde{\theta}(u,v)x\\
&=d_T(\{x,Ku,Kv\}+K(\theta(x,Kv)u-D_{\rho,\theta}(x,Ku)v)\\
&=\{d_Tx,Ku,Kv\}+\{x,d_TKu,Kv\}+\{x,Ku,d_TKv\}\\
&\quad+\lambda(\{d_Tx,d_TKu,Kv\}+\{x,d_TKu,d_TKv\}+\{d_Tx,Ku,d_TKv\}\\
&\quad+\lambda\{d_Tx,d_TKu,d_TKv\})+Kd_V(\theta(x,Kv)(u)-Kd_V(D_{\rho,\theta}(x,Ku)(v)\\
&=\{d_Tx,Ku,Kv\}+\{x,d_TKu,Kv\}+\{x,Ku,d_TKv\}\\
&\quad+\lambda(\{d_Tx,d_TKu,K(v)\}+\{x,d_TKu,d_TKv\}+\{d_Tx,Ku,d_TKv\}\\
&\quad+\lambda\{d_Tx,d_TKu,d_TKv\})+K(\theta(d_Tx,Kv)u+\theta(x,d_TKv)u+\lambda\theta(d_Tx,d_TKv)u\\
&\quad+\theta(x,Kv)d_Vu)+\lambda K(\theta(d_Tx,Kv)+\theta(x,d_TKv)+\lambda\theta(d_Tx,d_TKv))d_V(u)\\
&\quad-K(D_{\rho,\theta}(d_Tx,Ku)v+D_{\rho,\theta}(x,d_TKu)v+\lambda D_{\rho,\theta}(d_Tx,d_TKu)v+D_{\rho,\theta}(x,Ku)d_Vv)\\
&\quad-\lambda K(D_{\rho,\theta}(d_Tx,Ku)v+D_{\rho,\theta}(x,d_TKu)v+\lambda D_{\rho,\theta}(d_Tx,d_TKu)d_Vv)\\
&=\{d_Tx,Ku,Kv\}+\{x,Kd_Vu,Kv\}+\{x,Ku,Kd_Vv\}\\
&\quad+\lambda(\{d_Tx,Kd_Vu,Kv\}+\{x,Kd_Vu,Kd_Vv\}+\{d_Tx,Ku,Kd_Vv\}\\
&\quad+\lambda\{d_Tx,Kd_Vu,Kd_Vv\})+K(\theta(d_Tx,Kv)u+\theta(x,Kd_Vv)u+\lambda\theta(d_Tx,Kd_Vv)u\\
&\quad+\theta(x,Kv)d_Vu)+\lambda K(\theta(d_Tx,Kv)+\theta(x,Kd_Vv)+\lambda\theta(d_Tx,Kd_Vv))d_V(u)\\
&\quad-K(D_{\rho,\theta}(d_Tx,Ku)v+D_{\rho,\theta}(x,Kd_Vu)v+\lambda D_{\rho,\theta}(d_Tx,Kd_Vu)v+D_{\rho,\theta}(x,Ku)d_Vv)\\
&\quad-\lambda K(D_{\rho,\theta}(d_Tx,Ku)+D_{\rho,\theta}(x,Kd_Vu)+\lambda D_{\rho,\theta}(d_Tx,Kd_Vu))d_Vv\\
&=\tilde{\theta}(d_Vu,v)x+\tilde{\theta}(u,d_Vv)x+\lambda\tilde{\theta}(d_Vu,d_Vv)x+\tilde{\theta}(u,v)d_T(x)\\
&\quad+\lambda(\tilde{\theta}(d_Vu,v)+\tilde{\theta}(u,d_Vv)x+\lambda\tilde{\theta}(d_Vu,d_Vv))d_T(x).
\end{aligned}$$

Analogously, $d_T\tilde{\rho}(u)x=\tilde{\rho}(d_Vu)x+\tilde{\rho}(u)d_T(x)+\lambda\tilde{\rho}(d_Vu)d_T(x)$.
Hence, $(T,\tilde{\rho},\tilde{\theta},d_T)$ is a representation of $(V,[\,,\,]_K,\{\,,\,,\,\}_K)$. ∎

In the light of Proposition 2.7 we induce another representation of $(V,[\,,\,]_K,\{\,,\,,\,\}_K)$ from the representation $(T,\tilde{\rho},\tilde{\theta},d_T)$ of $(V,[\,,\,]_K,\{\,,\,,\,\}_K)$. We denote it by T_λ.
Consider the cohomologies of the differential Lie-Yamaguti algebra $(V,[\,,\,]_K,$ $\{\,,\,,\,\}_K,d_V)$ with coefficients in the representation $(T,\tilde{\rho},\tilde{\theta},d_T)$ as in Section 3.

For $n\geq 1$, we define the set of $(2n,2n+1)$-cochains by

$$\begin{aligned}&C^{2n}_{DYA}(V,T)\times C^{2n+1}_{DYA}(V,T)\\&=\{(C^{2n}(V,T)\times C^{2n+1}(V,T))\times(C^{2n-2}(V,T_\lambda)\times C^{2n-1}(V,T_\lambda,))\},\end{aligned}$$

and $$C^{0}_{DYA}(V,T)\times C^{1}_{DYA}(V,T)=C^1(V,T).$$

The coboundary operator $\partial\colon C^{2n}_{DYA}(V,T)\times C^{2n+1}_{DYA}(V,T)\to C^{2n+2}_{DYA}(V,T)\times C^{2n+3}_{DYA}(V,T)$ is given by

$$\partial\{(f,g),(\bar f,\bar g)\}=\{d(f,g),d^\lambda(\bar f,\bar g)+(-1)^{n+1}\Delta(f,g)\}$$

for any $\{(f,g),(\bar f,\bar g)\}\in C^{2n}_{DYA}(V,T)\times C^{2n+1}_{DYA}(V,T)$ and

$$\partial: C^{0}_{DYA}(V,T)\times C^{1}_{DYA}(V,T)\longrightarrow C^{2}_{DYA}(V,T)\times C^{3}_{DYA}(V,T)$$

is given by $\partial(\omega)=\{(d_I\omega,d_{II}\omega),-\delta\omega\},\ \forall\,\omega\in C^{0}_{DYA}(V,T)\times C^{1}_{DYA}(V,T)$.

Associated to the representation $(T,\tilde\rho,\tilde\theta,d_T)$ is the cochain complex product $(C^{*}_{DYA}(V,T)\times C^{*}_{DYA}(V,T),\partial)$. Denote the $(2n,2n+1)$-cohomology group of this cochain complex by $H^{2n}_{DYA}(V,T)\times H^{2n+1}_{DYA}(V,T)$, which is called the cohomology of the differential Lie-Yamaguti algebra (V,d_V) with coefficients in the representation $(T,\tilde\rho,\tilde\theta,d_T)$.

Notice that (2,3)-cocycle of the cochain complex $(C^{2n}_{DYA}(V,T)\times C^{2n+1}_{DYA}(V,T),\partial)$ is also defined as that of cochain complex $(C^{2n}_{DYA}(T,V)\times C^{2n+1}_{DYA}(T,V),\partial)$ in Section 3.

We compute the (0,1)-cocycle. For any $f\in C^1(V,T)=\mathrm{Hom}(V,T)$ and $v_i\in V$,

$$\begin{aligned}&d_I(f)(v_1,v_2)\\&=\tilde\rho(v_1)f(v_2)-\tilde\rho(v_2)f(v_1)-f([v_1,v_2]_K)\\&=[Kv_1,f(v_2)]+K(\rho(f(v_2))v_1)-[Kv_2,f(v_1)]-K(\rho(f(v_1))v_2)\\&\quad-\rho(Kv_1)v_2+\rho(Kv_2)v_1,\\&d_{II}(f)(v_1,v_2,v_3)\\&=\tilde\theta(v_2,v_3)f(v_1)-\tilde\theta(v_1,v_3)f(v_2)+D_{\tilde\rho,\tilde\theta}(v_1,v_2)f(v_3)-f(\{v_1,v_2,v_3\}_K)\\&=\{f(v_1),Kv_2,Kv_3\}+K\theta(f(v_1),Kv_3)v_2-KD_{\rho,\theta}(f(v_1),Kv_2)v_3\\&\quad-\{f(v_2),Kv_1,Kv_3\}-K\theta(f(v_2),Kv_3)v_1+KD_{\rho,\theta}(f(v_2),Kv_1)v_3\\&\quad+\{K(v_1),Kv_2,f(v_3)\}+K\theta(f(v_3),Kv_1)v_2-K\theta(f(v_3),Kv_2)v_1-f(\{v_1,v_2,v_3\}_K),\end{aligned}$$

and $\delta(f)(v)=fd_V(v)-d_Tf(v)$.

Obviously $f:V\longrightarrow T$ is a (0,1)-cocycle if and only if

$$[Kv_1,f(v_2)]+K(\rho(f(v_2))v_1)-[Kv_2,f(v_1)]-K(\rho(f(v_1))v_2)-f([v_1,v_2]_K)=0,$$

$$\begin{aligned}f(\{v_1,v_2,v_3\}_K)=&\{f(v_1),Kv_2,Kv_3\}+K\theta(f(v_1),Kv_3)v_2-KD_{\rho,\theta}(f(v_1),Kv_2)v_3\\&-\{f(v_2),Kv_1,Kv_3\}-K\theta(f(v_2),Kv_3)v_1+KD_{\rho,\theta}(f(v_2),Kv_1)v_3\\&+\{K(v_1),Kv_2,f(v_3)\}+K\theta(f(v_3),Kv_1)v_2-K\theta(f(v_3),Kv_2)v_1,\end{aligned}$$

and $fd_V=d_Tf$. Thus, we have

Proposition 4.9. *The relative Rota-Baxter operator $K:V\longrightarrow T$ on a differential Lie-Yamaguti algebra (T,d_T) associated to the representation (V,ρ,θ,d_V) is a (0,1)-cocycle of $(V,[\ ,\ ]_K,\{\ ,\ ,\ \}_K,d_V)$ with coefficients in the representation $(T,\tilde\rho,\tilde\theta,d_T)$.*

5. Abelian extensions of differential Lie-Yamaguti algebras

In this section, we study abelian extensions of differential Lie-Yamaguti algebras.

Definition 5.1. An *abelian extension* of a differential Lie-Yamaguti algebra (T, d_T) by a differential Lie-Yamaguti algebra (V, d_V) is an exact sequence of differential Lie-Yamaguti algebras:

$$0 \longrightarrow (V, d_V) \stackrel{i}{\longrightarrow} (\hat{T}, d_{\hat{T}}) \stackrel{p}{\longrightarrow} (T, d_T) \longrightarrow 0,$$

where $\{V, V, \hat{T}\}_{\hat{T}} = \{V, \hat{T}, V\}_{\hat{T}} = \{\hat{T}, V, V\}_{\hat{T}} = 0$ and $[V, V]_{\hat{T}} = 0$.
A section of an abelian extension $(\hat{T}, d_{\hat{T}})$ of (T, d_T) by (V, d_V) is a linear map $s : T \longrightarrow \hat{T}$ such that $ps = id$. ■

Definition 5.2. Let $(\hat{T}_1, d_{\hat{T}_1})$ and $(\hat{T}_2, d_{\hat{T}_2})$ be two abelian extensions of (T, d_T) by (V, d_V). They are said to be equivalent if there is a homomorphism of differential Lie-Yamaguti algebras $\varphi : (\hat{T}_1, d_{\hat{T}_1}) \longrightarrow (\hat{T}_2, d_{\hat{T}_2})$ such that the following commutative diagram holds:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (V, d_V) & \stackrel{i}{\longrightarrow} & (\hat{T}_1, d_{\hat{T}_1}) & \stackrel{p}{\longrightarrow} & (T, d_T) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & (V, d_V) & \stackrel{i}{\longrightarrow} & (\hat{T}_2, d_{\hat{T}_2}) & \stackrel{p}{\longrightarrow} & (T, d_T) & \longrightarrow & 0. \end{array}$$

Suppose $(\hat{T}, d_{\hat{T}})$ is an abelian extension of (T, d_T) by (V, d_V). Let $s : T \longrightarrow \hat{T}$ be any section of p. Define linear maps $\rho : T \longrightarrow gl(V)$, $\chi : T \longrightarrow V$, bilinear maps $\theta : \wedge^2 T \longrightarrow gl(V)$, $\phi : \wedge^2 T \longrightarrow V$ and a trilinear map $\psi : \wedge^3 T \longrightarrow V$ respectively by

$$\rho(x)v = [s(x), v]_{\hat{T}}, \quad \chi(x) = d_{\hat{T}}(s(x)) - s(d_T(x)),$$
$$\theta(x, y)v = \{v, s(x), s(y)\}_{\hat{T}},$$
$$\phi(x, y) = [s(x), s(y)] - s[x, y], \quad \psi(x, y, z) = \{s(x), s(y), s(z)\} - s\{x, y, z\}.$$

Theorem 5.3. *With the above notations, (V, ρ, θ, d_V) is a representation of (T, d_T) and it is independent on the choice of section s. Furthermore, equivalent abelian extensions give the same representation.*

Proof. Based on the deformation theory of Lie-Yamaguti algebras [31], we only need to check that (18)–(19) hold. For any $x, y, z \in T$, $v \in V$, since

$$s[x, y] - [s(x), s(y)],\ d_{\hat{T}}(s(x)) - s(d_T(x)),\ s\{x, y, z\} - \{s(x), s(y), s(z)\} \in V$$

and V is abelian, which imply that (18) and (19) hold. ■

Proposition 5.4. *The vector space $(T \oplus V, [\ ,\ ]_\phi, \{\ ,\ ,\ \}_\psi, d_\chi)$ is a differential Lie-Yamaguti algebra if and only if $\{(\phi, \psi), (\chi)\}$ is a $(2, 3)$-cocycle of (T, d_T) with coefficients in the representation (V, ρ, θ, d_V), where*

$$[x + a, y + b]_\phi = [x, y] + \phi(x, y) + \rho(x)b - \rho(y)a,$$
$$\{x + a, y + b, z + c\}_\psi = \{x, y, z\} + \psi(x, y, z) + \theta(b, c)x - \theta(a, c)y + D_{\rho,\theta}(a, b)z,$$

and
$$d_\chi(x + a) = d_T(x) + d_V(a) + \chi(x),$$

for any $x, y, z \in T$ and $a, b, c \in V$.

Proof. On the one hand, d_χ is a differential operator of $T\oplus V$ if and only if

$$\begin{aligned}&d_\chi(\{x+a,y+b,z+c\}_\psi)\\&=\{d_\chi(x+a),y+b,z+c\}_\psi+\{x+a,d_\chi(y+b),z+c\}_\psi+\{x+a,y+b,d_\chi(z+c)\}_\psi\\&\quad+\lambda(\{d_\chi(x+a),d_\chi(y+b),z+c\}_\psi+\{x+a,d_\chi(y+b),d_\chi(z+c)\}_\psi\\&\quad+\{d_\chi(x+a),y+b,d_\chi(z+c)\}_\psi+\lambda\{d_\chi(x+a),d_\chi(y+b),d_\chi(z+c)\}_\psi),\end{aligned}$$

and

$$d_\chi([x+a,y+b]_\phi)=[d_\chi(x+a),y+b]_\phi+[x+a,d_\chi(y+b)]_\phi+\lambda[d_\chi(x+a),d_\chi(y+b)]_\phi.$$

These equations induce that

$$\begin{aligned}&\chi(\{x,y,z\})+d_V\psi(x,y,z)\\&=\psi(d_Tx,y,z)+\psi(x,d_Ty,z)+\psi(x,y,d_Tz)+\lambda(\psi(d_Tx,d_Ty,z)+\psi(x,d_Ty,d_Tz)\\&\quad+\psi(d_Tx,y,d_Tz)+\lambda\psi(d_Tx,d_Ty,d_Tz))+\lambda(\theta(y,z)\chi(x)-\theta(x,z)\chi(y)\\&\quad+D_{\rho,\theta}(x,y)\chi(z)+\theta(d_Ty,z)\chi(x)-\theta(d_Tx,z)\chi(y)-\theta(x,d_Tz)\chi(y)\\&\quad+D_{\rho,\theta}(x,d_Ty)\chi(z)+\theta(y,d_Tz)\chi(x)+D_{\rho,\theta}(d_Tx,y)\chi(z))\\&\quad+\lambda^2(\theta(d_Ty,d_Tz)\chi(x)+D_{\rho,\theta}(d_Tx,d_Ty)\chi(z)-\theta(d_Tx,d_Tz)\chi(y))\end{aligned}\tag{72}$$

and
$$\chi([x,y])+d_V\phi(x,y)=\phi(d_Tx,y)+\phi(x,d_Ty)+\lambda\phi(d_Tx,d_Ty).\tag{73}$$

On the other hand, if $\{(\phi,\psi),(\chi)\}\in(C^2_{DYA}(T,V)\times C^3_{DYA}(T,V)$ is a (2,3)-cocycle, then (ϕ,ψ) is a (2,3)-cocycle of T with coefficients in the representation (V,ρ,θ,d_V) and $d_I^\lambda\chi+\delta\phi=0$, $d_{II}^\lambda\chi+\delta\psi=0$. By a direct calculation, (ϕ,ψ) is a (2,3)-cocycle of T with coefficients in the representation V if and only if $T\oplus V$ is a Lie-Yamaguti algebra, and $d_I^\lambda\chi+\delta\phi=0$ is equivalent to (73), $d_{II}^\lambda\chi+\delta\psi=0$ is equivalent to (72). This implies the assertion. ■

6. Appendix: Proof of Proposition 3.2

Proof. Define $f_{i,j}^k$ by

$$f_{i,j}^k(x_1,\cdots,,x_n)=f(\underbrace{id\otimes\cdots\otimes d_T\otimes\cdots\otimes id}_{\substack{d_T\text{ appears }k\text{ times but}\\ \text{not in the }i\text{th, }j\text{th places}}})(x_1,\cdots,x_n).$$

And f_j^k is just the $f_{i,j}^k$ for no i.

For any pair $(f,g)\in C^{2n}(T,V)\times C^{2n+1}(T,V)$, since

$$\Delta d(f,g)=(\delta,\delta)(d_I,d_{II})(f,g)=(\delta d_If,\delta d_{II}g)$$

and
$$d^\lambda\Delta(f,g)=(d_I^\lambda,d_{II}^\lambda)(\delta,\delta)(f,g)=(d_I^\lambda\delta f,d_{II}^\lambda\delta g),$$

we only need to prove $d_I^\lambda\delta=\delta d_I$ and $d_{II}^\lambda\delta=\delta d_{II}$. In fact, for all $x_1,\cdots,x_{2n+2}\in T$,

$$\begin{aligned}&(\delta d_If)(x_1,\cdots,x_{2n+2})\\&=\sum_{k=1}^{2n+2}\lambda^{k-1}(d_If)^k(x_1,\cdots,x_{2n+2})-d_V(d_If)(x_1,\cdots,x_{2n+2})\\&=A-B-C+E+F-G,\end{aligned}$$

where

$$
\begin{aligned}
A = &\sum_{k=0}^{2n+1} \lambda^k \rho(d_T x_{2n+1}) g^k(x_1, \cdots, x_{2n}, x_{2n+2}) \\
&+ \sum_{k=1}^{2n+1} \lambda^{k-1} \rho(x_{2n+1}) g^k(x_1, \cdots, x_{2n}, x_{2n+2}),
\end{aligned}
$$

$$
\begin{aligned}
B = &\sum_{k=0}^{2n+1} \lambda^k \rho(d_T x_{2n+2}) g^k(x_1, \cdots, x_{2n+1}) \\
&+ \sum_{k=1}^{2n+1} \lambda^{k-1} \rho(x_{2n+2}) g^k(x_1, \cdots, x_{2n+1}),
\end{aligned}
$$

$$
\begin{aligned}
C = &\sum_{k=0}^{2n} \lambda^{k+1} g^k_{2n+1}(x_1, \cdots, x_{2n}, [d_T x_{2n+1}, d_T x_{2n+2}]) \\
&+ \sum_{k=0}^{2n} \lambda^k g^k_{2n+1}(x_1, \cdots, x_{2n}, [d_T x_{2n+1}, x_{2n+2}]) \\
&+ \sum_{k=0}^{2n} \lambda^k g^k_{2n+1}(x_1, \cdots, x_{2n}, [x_{2n+1}, d_T x_{2n+2}]) \\
&+ \sum_{k=0}^{2n} \lambda^{k-1} g^k_{2n+1}(x_1, \cdots, x_{2n}, [x_{2n+1}, x_{2n+2}]),
\end{aligned}
$$

$$
\begin{aligned}
E = &\sum_{j=1}^{n} (-1)^{n+j+1} \sum_{k=0}^{2n} \lambda^{k+1} D_{\rho,\theta}(d_T x_{2j-1}, d_T x_{2j}) f^k(x_1, \cdots, x_{2j-2}, x_{2j+1}, \cdots, x_{2n+2}) \\
&+ \sum_{j=1}^{n} (-1)^{n+j+1} \sum_{k=0}^{2n} \lambda^{k} D_{\rho,\theta}(d_T x_{2j-1}, x_{2j}) f^k(x_1, \cdots, x_{2j-2}, x_{2j+1}, \cdots, x_{2n+2}) \\
&+ \sum_{j=1}^{n} (-1)^{n+j+1} \sum_{k=0}^{2n} \lambda^{k} D_{\rho,\theta}(x_{2j-1}, d_T x_{2j}) f^k(x_1, \cdots, x_{2j-2}, x_{2j+1}, \cdots, x_{2n+2}) \\
&+ \sum_{j=1}^{n} (-1)^{n+j+1} \sum_{k=1}^{2n} \lambda^{k-1} D_{\rho,\theta}(x_{2j-1}, x_{2j}) f^k(x_1, \cdots, x_{2j-2}, x_{2j+1}, \cdots, x_{2n+2}),
\end{aligned}
$$

$$
\begin{aligned}
F = &\sum_{i=1}^{n} \sum_{j=2i+1}^{2n+2} (-1)^{n+i} \Big(\sum_{k=0}^{2n-1} \lambda^{k+2} f^k_{2i}\big(x_1, \cdots, x_{2i-2}, x_{2i+1}, \cdots, \\
&\qquad \{d_T x_{2i-1}, d_T x_{2i}, d_T x_j\}, \cdots, x_{2n+2}\big) \\
&+ \sum_{k=0}^{2n-1} \lambda^{k+1} f^k_{2i}(x_1, \cdots, x_{2i-2}, x_{2i+1}, \cdots, \{d_T x_{2i-1}, d_T x_{2i}, x_j\}, \cdots, x_{2n+2}) \\
&+ \sum_{k=0}^{2n-1} \lambda^{k+1} f^k_{2i}(x_1, \cdots, x_{2i-2}, x_{2i+1}, \cdots, \{x_{2i-1}, d_T x_{2i}, d_T x_j\}, \cdots, x_{2n+2}) \\
&+ \sum_{k=0}^{2n-1} \lambda^{k+1} f^k_{2i}(x_1, \cdots, x_{2i-2}, x_{2i+1}, \cdots, \{d_T x_{2i-1}, x_{2i}, d_T x_j\}, \cdots, x_{2n+2})
\end{aligned}
$$

$$+\sum_{k=0}^{2n-1}\lambda^k f_{2i}^k(x_1,\cdots,x_{2i-2},x_{2i+1},\cdots,\{d_T x_{2i-1},x_{2i},x_j\},\cdots,x_{2n+2})$$
$$+\sum_{k=0}^{2n-1}\lambda^k f_{2i}^k(x_1,\cdots,x_{2i-2},x_{2i+1},\cdots,\{x_{2i-1},d_T x_{2i},x_j\},\cdots,x_{2n+2})$$
$$+\sum_{k=0}^{2n-1}\lambda^k f_{2i}^k(x_1,\cdots,x_{2i-2},x_{2i+1},\cdots,\{x_{2i-1},x_{2i},d_T x_j\},\cdots,x_{2n+2})$$
$$+\sum_{k=0}^{2n-1}\lambda^k f_{2i}^k(x_1,\cdots,x_{2i-2},x_{2i+1},\cdots,\{x_{2i-1},x_{2i},x_j\},\cdots,x_{2n+2})\Big),$$

and $G = d_V(d_I f)(x_1,\cdots,x_{2n+2}) = G_1 - G_2 - G_3 + G_4 + G_5$ with

$$G_1 = d_V(\rho(x_{2n+1})g(x_1,\cdots,x_{2n},x_{2n+2}),$$
$$G_2 = \rho(x_{2n+2})g(x_1,\cdots,x_{2n+1}),$$
$$G_3 = g(x_1,\cdots,x_{2n},[x_{2n+1},x_{2n+2}]),$$
$$G_4 = \sum_{k=1}^{n}(-1)^{n+k+1}D_{\rho,\theta}(x_{2k-1},x_{2k})f(x_1,\cdots,x_{2k-2},x_{2k+1},\cdots,x_{2n+2}),$$
$$G_5 = \sum_{k=1}^{n}\sum_{j=2k+1}^{2n+2}(-1)^{n+k}f(x_1,\cdots,x_{2k-2},x_{2k+1},\cdots,\{x_{2k-1},x_{2k},x_j\},\cdots,x_{2n+2})).$$

At the same time, $(d_I^\lambda \delta f)(x_1,\cdots,x_{2n+2}) = H - I - J + K + L$ where

$$H = \hat{\rho}(x_{2n+1})\delta g(x_1,\cdots,x_{2n},x_{2n+2}),$$
$$I = \hat{\rho}(x_{2n+2})\delta g(x_1,\cdots,x_{2n+1}),$$
$$J = \delta g(x_1,\cdots,x_{2n},[x_{2n+1},x_{2n+2}]),$$
$$K = \sum_{j=2k+1}^{2n+2}(-1)^{n+k}\delta f(x_1,\cdots,x_{2k-2},x_{2k+1},\cdots,\{x_{2k-1},x_{2k},x_j\},\cdots,x_{2n+2}),$$
$$L = \sum_{k=1}^{n}(-1)^{n+k+1}D_{\hat{\rho},\hat{\theta}}(x_{2k-1},x_{2k})\,\delta f(x_1,\cdots,x_{2k-2},x_{2k+1},\cdots,x_{2n+2}),$$

Using (53), we have $L = E - G_4$. Analogously we have $H = A - G_1$, $I = B - G_2$, $J = C - G_3$, $K = F - G_5$. Hence, $d_I^\lambda \delta = \delta d_I$. Similarly, $d_{II}^\lambda \delta = \delta d_{II}$. Thus, $d^\lambda \Delta = \Delta d$. ■

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