Kronecker's Method and Complete Systems of Functions in Bi-Involution on Classical Lie Algebras

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Abstract.   We use Kronecker's method to construct systems of functions in bi-involution with respect to two Poisson brackets: the canonical one and the bracket with frozen argument $A \in \mathfrak{g}$. For the Lie algebras $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$, we construct complete systems of functions in bi-involution for any $A \in \mathfrak{g}$. For the Lie algebras $\mathfrak{so}_{2n+1}$ and $\mathfrak{so}_{2n}$, we describe elements A such that we can construct a complete system of functions in bi-involution and the elements A such that we can construct the Kronecker part of a complete system of functions in bi-involution. Also, we prove that the constructed functions freely generate some limits of Mishchenko-Fomenko subalgebras. Finally, for the Lie algebras $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$, we show that the Kronecker indices are the same for all elements A in any given sheet, while for the Lie algebras $\mathfrak{so}_{2n}$ and $\mathfrak{so}_{2n+1}$, we give examples of sheets such that this is not true.

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Introduction

The symmetric algebra $\mathcal{S}(\mathfrak{g})$ of a complex Lie algebra $\mathfrak{g}$ can be identified with the algebra of polynomials on the dual space: $\mathcal{S}(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$. There is the canonical Poisson structure on $\mathcal{S}(\mathfrak{g})$. Namely, the Poisson bracket of elements $X, Y \in \mathfrak{g}$ coincides with their commutator: $\{X, Y\} = [X, Y]$, and then the bracket extends to $\mathcal{S}(\mathfrak{g})$ by the Leibniz rule.

For each $\gamma \in \mathfrak{g}^*$ a Poisson structure "with frozen argument" is defined: for $X, Y \in \mathfrak{g}$, by definition, $\{X, Y\}_\gamma$ is constant as a function on $\mathfrak{g}^*$ and equal to $\gamma([X, Y])$, and then the bracket extends to $\mathcal{S}(\mathfrak{g})$ by the Leibniz rule.

Polynomials $\varphi, \psi \in \mathcal{S}(\mathfrak{g})$ *Poisson-commute* or *are in involution* with respect to the bracket $\{\, , \}$ if $\{\varphi, \psi\} = 0$. If φ and ψ Poisson-commute with respect to both brackets $\{\, , \}$ and $\{\, , \}_\gamma$, then they are in *bi-involution.* An important problem in Hamiltonian mechanics is to find complete systems of functions in bi-involution. In our case, all functions are polynomials. Completeness of the system means that all functions in it are algebraically independent and that their number is maximal, i.e., equal to $\frac{1}{2}(\operatorname{ind} \mathfrak{g} + \dim \mathfrak{g})$, where the index of the Lie algebra $\operatorname{ind} \mathfrak{g}$ is defined as the codimension of a generic coadjoint orbit in $\mathfrak{g}^*$.

Now let $\mathfrak{g}$ be reductive. Then $\mathfrak{g}^*$ can be identified with $\mathfrak{g}$ by means of an invariant scalar product. Thus, for each $A \in \mathfrak{g}$ we have a bracket $\{\, , \}_A$ on $\mathbb{C}[\mathfrak{g}] = \mathcal{S}(\mathfrak{g}^*) \simeq \mathcal{S}(\mathfrak{g})$. If an element $A \in \mathfrak{g}$ is regular, then the complete system of functions in

bi-involution can be obtained by the Mishchenko-Fomenko argument shift method [12]. Namely, the directional derivatives (of all orders) of all basic invariants along the element A form a complete system of functions in bi-involution. In the case of a singular element A, the system of functions in bi-involution obtained in this way is not complete. In this paper, we will apply algebraic methods to construct a complete system of functions in bi-involution in the case of a singular element A.

In [1] Bolsinov and Zhang describe an approach to work with Poisson structures in terms of linear algebra. Poisson brackets $\{\ ,\ \}_A$ and $\{\ ,\ \}$ are considered as skew-symmetric bilinear forms $\mathcal{B}_A$ and $\mathcal{B}$ over the field $\mathbb{K} = \mathbb{C}(\mathfrak{g})$ on the space $\mathfrak{g} \otimes \mathbb{K}$ of rational vector fields on $\mathfrak{g}$. Namely, if $\varphi, \psi \in \mathbb{C}[\mathfrak{g}]$, then $d\varphi$ and $d\psi$ can be considered as elements of $\mathfrak{g} \otimes \mathbb{K}$ and then $\{\varphi, \psi\} = \mathcal{B}(d\varphi, d\psi)$ and $\{\varphi, \psi\}_A = \mathcal{B}_A(d\varphi, d\psi)$.

Polynomials $\varphi_1, \ldots, \varphi_s$ form a complete system of functions in bi-involution with respect to $\{\ ,\ \}_A$ and $\{\ ,\ \}$ if and only if their differentials $d\varphi_1, \ldots, d\varphi_s$ form a basis of a bilagrangian subspace (i.e., a maximal subspace which is isotropic with respect to both bilinear forms). Thus, to obtain a complete system of functions in bi-involution, it is sufficient to find a basis of a bilagrangian subspace and "integrate" it.

Suppose, we found a basis (let us call it *canonical*) such that the matrices of both forms $\mathcal{B}_A$ and $\mathcal{B}$ are simultaneously reduced to the Jordan-Kronecker canonical form (with blocks of two types: Jordan and Kronecker, see [1]). Then the second halves of the bases of each block form a basis of the bilagrangian subspace. The basis of the constructed bi-Lagrangian subspace splits into two parts: the Jordan part and the Kronecker one. Let us give the same names to the corresponding parts of the full system of functions in bi-involution.

In [7, 8], for the Lie algebras $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$, the author constructed the Kronecker part of a canonical basis and the corresponding part of a complete system of functions in bi-involution. We describe these and other known results in more detail in Section 1.

In this paper, first, we construct a complete system of functions in bi-involution for an arbitrary element $A \in \mathfrak{g}$ for $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{sp}_{2n}$. Namely, we show that some elements of the Kronecker part constructed in [7, 8] generate $\mathcal{S}(\mathfrak{g}_A)^{\mathfrak{g}_A}$, where $\mathfrak{g}_A$ is the centralizer of A. From this we deduce that the Jordan part of the complete system of functions in bi-involution can be obtained by applying the argument shift method to these invariants.

Next, we describe elements $A \in \mathfrak{so}_n$ such that the Kronecker part of a complete system of functions in bi-involution can be constructed in the same way as in the case of $\mathfrak{sl}_n$ (we call such elements *good*). We give the description of good elements in terms of the Jordan normal form of the matrix A. Then we show that for all semisimple elements $A \in \mathfrak{so}_{2n+1}$ and for some semisimple elements $A \in \mathfrak{so}_{2n}$ the Kronecker part constructed in this way can be extended to a complete system of functions in bi-involution exactly as in the case of $\mathfrak{sl}_n$.

Then we describe some nilpotent elements $A \in \mathfrak{so}_n$ such that it is possible to obtain the Kronecker part of a complete system of functions in bi-involution by "modifying" the answer for $\mathfrak{sl}_n$. Namely, based on the work of Charbonnel and Moreau [5], we give explicit formulas for the elements of the Kronecker part.

Another approach to construct commutative subalgebras of maximal transcendence degree is described in [15, 16, 17]. The idea is to consider a formal limit of

Mishchenko-Fomenko subalgebras $\lim_{s\to 0} F_{A(s)}$ for regular elements $A(s)$, see Section 6. For the classical Lie algebras $\mathfrak{g}$ we show that in the case $A(s) = A + sB$, where A, B belong to a fixed Cartan subalgebra, the resulting limit subalgebra is freely generated by the elements of the complete system of functions in bi-involution constructed in Sections 1-4. In the case $\mathfrak{g} = \mathfrak{so}_{2n}$, we impose an additional condition that A is *good* (see Definition 3.1 below).

Finally, we describe how the Kronecker indices of the forms $(\mathcal{B}, \mathcal{B}_A)$, see Definition 1.2, are related to the sheets of $\mathfrak{g}$. We show that, for $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$, the Kronecker indices of the pair $(\mathcal{B}, \mathcal{B}_A)$ are the same for all elements A in any given sheet, while for the Lie algebras $\mathfrak{so}_{2n}$ and $\mathfrak{so}_{2n+1}$, we give examples of sheets such that this is not true.

1. Notation and known results

1.1. Notation

As a rule, we consider and denote by $\mathfrak{g}$ one of the classical simple complex Lie algebras. However, in the following constructions some subalgebras may not be reductive. Therefore, we first give the basic definitions in terms of an arbitrary complex Lie algebra and then specify them for the case of reductive Lie algebras.

Let $\mathfrak{g}$ be a complex Lie algebra. Denote by G a connected Lie group such that $\operatorname{Lie} G = \mathfrak{g}$. In the introduction we defined two Poisson brackets on the symmetric algebra $\mathcal{S}(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]$: the canonical Poisson bracket $\{\,,\,\}$ and the bracket "with frozen argument" $\{\,,\,\}_\gamma$ for each $\gamma \in \mathfrak{g}^*$.

Let $\mathfrak{g}_\gamma$ denote the stabilizer of $\gamma \in \mathfrak{g}^*$ in the Lie algebra $\mathfrak{g}$ in the sense of the coadjoint representation: $\mathfrak{g}_\gamma = \{\xi \in \mathfrak{g}|\ \operatorname{ad}^*(\xi)\gamma = 0\}$. The index of a Lie algebra is the minimal dimension of such stabilizers: $\operatorname{ind}\mathfrak{g} = \min_{\gamma\in\mathfrak{g}^*} \dim\mathfrak{g}_\gamma$. In other words, the index is equal to the codimension of a generic coadjoint orbit in $\mathfrak{g}^*$.

Let us denote by G_γ the stabilizer of γ in G in the sense of the coadjoint action. Then $\mathfrak{g}_\gamma = \operatorname{Lie} G_\gamma$. The elements $\gamma \in \mathfrak{g}^*$ such that the dimension of the stabilizer is minimal ($\dim\mathfrak{g}_\gamma = \operatorname{ind}\mathfrak{g}$) are called *regular*. They form a dense open subset in $\mathfrak{g}^*$. All other elements are called *singular*; we denote their set by $\mathfrak{g}^*_{\mathrm{sing}}$.

For any Lie subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$ we denote by $\mathcal{S}(\mathfrak{g})^{\mathfrak{l}}$ the Poisson centralizer of $\mathfrak{l}$:

$$\mathcal{S}(\mathfrak{g})^{\mathfrak{l}} = \{F \in \mathcal{S}(\mathfrak{g})|\ \{\xi, F\} = 0 \text{ for all } \xi \in \mathfrak{l}\}.$$

Similarly, for any connected Lie subgroup $L \subseteq G$ we denote by $\mathcal{S}(\mathfrak{g})^L$ the subalgebra of L-invariants:

$$\mathcal{S}(\mathfrak{g})^L = \{F \in \mathcal{S}(\mathfrak{g})\,|\, gF = F \text{ for all } g \in L\}.$$

If a subalgebra $\mathfrak{l}$ is the Lie algebra of a connected subgroup $L \subseteq G$, then we have $\mathcal{S}(\mathfrak{g})^{\mathfrak{l}} = \mathcal{S}(\mathfrak{g})^L$.

Let $F_\gamma \subseteq \mathcal{S}(\mathfrak{g})$ denote the Mishchenko-Fomenko subalgebra generated by all derivatives (of all orders) of all invariants along the element γ, that is

$$F_\gamma = \mathbb{C}[\partial_\gamma^k f \,|\, f \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}, k = 0, \ldots, \deg f - 1].$$

The algebra constructed in this way is necessarily commutative with respect to both Poisson brackets $\{\,,\,\}$ and $\{\,,\,\}_\gamma$ [12]. But it may be not complete, i.e., it may not contain a complete system of functions in bi-involution. For example, if $\gamma = 0$, then the algebra F_γ coincides with $\mathcal{S}(\mathfrak{g})^\mathfrak{g}$. In the case when $\operatorname{trdeg}\mathcal{S}(\mathfrak{g})^\mathfrak{g} = \operatorname{ind}\mathfrak{g}$ and the element γ is regular, the completeness criterion for F_γ is given in [2]: the algebra F_γ is complete if and only if $\operatorname{codim}(\mathfrak{g}_\gamma^*)_{\mathrm{sing}} \geq 2$.

Now let $\mathfrak{g}$ be reductive. Then it can be identified with the dual space $\mathfrak{g}^*$ by an invariant scalar product $(\,,\,)$. Let $d_X\varphi$ denote the differential of a polynomial $\varphi \in \mathbb{C}[\mathfrak{g}]$ at a point $X \in \mathfrak{g}$. Now the Poisson structures can be defined on the Lie algebra $\mathfrak{g}$ itself. For an arbitrary element $X \in \mathfrak{g}$ we have:

$$\{\varphi,\psi\}(X) = (X, [d_X\varphi, d_X\psi]), \qquad \{\varphi,\psi\}_A(X) = (A, [d_X\varphi, d_X\psi]),$$

where $A \in \mathfrak{g}$, φ and ψ are polynomials on $\mathfrak{g}$, and $d_X\varphi$ and $d_X\psi$ are considered as elements of the algebra $\mathfrak{g}$.

Moreover, for $A \in \mathfrak{g}$ we have $\operatorname{ind}\mathfrak{g} = \operatorname{rk}\mathfrak{g}$ and $\mathfrak{g}_A = \mathfrak{z}(A)$, where $\mathfrak{z}(A)$ is the centraliser of A in $\mathfrak{g}$. We will use the identifications $\mathcal{S}(\mathfrak{g}) \simeq \mathcal{S}(\mathfrak{g}^*)$ and $\mathcal{S}(\mathfrak{g})^\mathfrak{g} \simeq \mathbb{C}[\mathfrak{g}]^\mathfrak{g}$. For the reductive Lie algebra $\mathfrak{g}$ the algebra $\mathcal{S}(\mathfrak{g})^\mathfrak{g}$ of its invariants is freely generated by some homogeneous polynomials $f_1, \ldots, f_n$, where $n = \operatorname{rk}\mathfrak{g}$. These polynomials are called basic symmetric invariants, further we will omit the word "symmetric". Therefore the Mishchenko-Fomenko subalgebra is generated by all derivatives of the basic invariants along the element A:

$$F_A = \mathbb{C}[\,\partial_A^k f_i \,|\, i = 1, \ldots, n,\ k = 0, \ldots, \deg f_i - 1\,].$$

We set $\mathbb{K} = \mathbb{C}(\mathfrak{g})$ and extend the scalar product from $\mathfrak{g}$ to $\mathfrak{g} \otimes \mathbb{K}$. Denote by X the generic element of $\mathfrak{g} \otimes \mathbb{K}$, i.e., an element such that for all $C \in \mathfrak{g}$ we have $(X, C) = (C, \cdot) \in \mathfrak{g}^* \subseteq \mathbb{K}$. In the case of classical simple Lie algebras, the matrix X consists of elements of $\mathbb{K}$ which correspond to coordinates on $\mathfrak{g}$. Then the skew-symmetric forms $\mathcal{B}$ and $\mathcal{B}_A$ are expressed as follows:

$$\mathcal{B}(B,C) = (X, [B,C]), \qquad \mathcal{B}_A(B,C) = (A, [B,C]),$$

where $B, C \in \mathfrak{g} \otimes \mathbb{K}$.

1.2. Jordan-Kronecker theorem

In this paper we use the Jordan-Kronecker theorem below for a pair of forms $(\mathcal{B}, \mathcal{B}_A)$ over a non-algebraically closed field $\mathbb{K} = \mathbb{C}(\mathfrak{g})$. Nevertheless, in this case it is valid. The reason is that, in the case of a reductive Lie algebra, the Jordan blocks can only be of size 2×2 and with eigenvalue $\lambda = \infty$ (see [1, p. 10]), i.e. the Jordan part of the canonical form of $\mathcal{B}_A$ is a zero matrix, and that of $\mathcal{B}$ is a canonical form of a skew-symmetric bilinear form.

Let us now formulate the classical Jordan-Kronecker theorem on the canonical form of a pair of skew-symmetric bilinear forms.

Theorem 1.1. (Jordan-Kronecker; see [6] Ch. 12/5, Thm. 6, and also [11]) *For any pair of skew-symmetric bilinear forms $(\mathcal{A}, \mathcal{B})$ on a finite-dimensional vector space over an algebraically closed field, there exists a basis in which the matrices of both forms $\mathcal{A}$ and $\mathcal{B}$ have a block-diagonal form with blocks A_i and B_i of the following types:*

1. Jordan block with eigenvalue $\lambda \in \Bbbk$:

$$A_i = \begin{pmatrix} & & & & \lambda & 1 & & \\ & 0 & & & & \lambda & \ddots & \\ & & & & & & \ddots & 1 \\ & & & & & & & \lambda \\ \hline -\lambda & & & & & & & \\ -1 & -\lambda & & & & & 0 & \\ & \ddots & \ddots & & & & & \\ & & -1 & -\lambda & & & & \end{pmatrix}, \quad B_i = \begin{pmatrix} & & & & 1 & & & \\ & 0 & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ \hline -1 & & & & & & & \\ & -1 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & -1 & & & & \end{pmatrix}.$$

2. Jordan block with eigenvalue $\lambda = \infty$:

$$A_i = \begin{pmatrix} & & & & 1 & & & \\ & 0 & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ \hline -1 & & & & & & & \\ & -1 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & -1 & & & & \end{pmatrix}, \quad B_i = \begin{pmatrix} & & & & 0 & 1 & & \\ & 0 & & & & 0 & \ddots & \\ & & & & & & \ddots & 1 \\ & & & & & & & 0 \\ \hline 0 & & & & & & & \\ -1 & 0 & & & & & 0 & \\ & \ddots & \ddots & & & & & \\ & & -1 & 0 & & & & \end{pmatrix}.$$

3. Kronecker block:

$$A_i = \begin{pmatrix} & & & & 1 & 0 & & \\ & 0 & & & & \ddots & \ddots & \\ & & & & & & 1 & 0 \\ \hline -1 & & & & & & & \\ 0 & \ddots & & & & & 0 & \\ & \ddots & -1 & & & & & \\ & & 0 & & & & & \end{pmatrix}, \quad B_i = \begin{pmatrix} & & & & 0 & 1 & & \\ & 0 & & & & \ddots & \ddots & \\ & & & & & & 0 & 1 \\ \hline 0 & & & & & & & \\ -1 & \ddots & & & & & 0 & \\ & \ddots & 0 & & & & & \\ & & -1 & & & & & \end{pmatrix}.$$

Definition 1.2. If the dimensions of the Kronecker blocks are $2m_0+1, \ldots, 2m_k+1$, then the numbers $m_0, \ldots, m_k$ are called *the Kronecker indices* of a pair of forms $(\mathcal{B}, \mathcal{B}_A)$.

Remark 1.3. From the canonical form of $\mathcal{B}_A$ one can deduce that

$$m_0 + \ldots + m_k = \frac{1}{2} \operatorname{rk} \mathcal{B}_A.$$

1.3. Kronecker's method

To find the Kronecker part of a basis of a bilagrangian subspace, one can use the Kronecker method. Namely, given a pair of skew-symmetric bilinear forms $(\mathcal{A}, \mathcal{B})$ on a vector space V over a field $\Bbbk$ we consider the module $V[t] := V \otimes \Bbbk[t]$ over $\Bbbk[t]$. Let $Z = \operatorname{Ker}(\mathcal{A} - t\mathcal{B})$. Then we find a *minimal basis* of Z, i.e., a basis such that the leading coefficients of its elements are linearly independent.

Theorem 1.4. (Kronecker; see [6] Ch. 12, §5) *The degrees of the polynomials of a minimal basis of Z are equal to the Kronecker indices of $(\mathcal{A}, \mathcal{B})$, and the coefficients of these polynomials form the Kronecker part of a basis of a bilagrangian subspace.*

We have $Z = \text{Ker}(\mathcal{B} - t\mathcal{B}_A) \subset \mathfrak{g} \otimes \mathbb{K}[t]$. Let $f_1, \ldots, f_n$ denote the basic invariants of $\mathfrak{g}$; let us extend them on $\mathfrak{g} \otimes \mathbb{K}[t]$. Then the differentials $d_Y f_i$ at a point $Y \in \mathfrak{g} \otimes \mathbb{K}[t]$ are elements of $\mathfrak{g} \otimes \mathbb{K}[t]$. By [7, Lemma 1], if $\mathfrak{g} = \mathfrak{gl}_n$, then Z is generated as $\mathbb{K}[t]$-module by differentials of basic invariants at $X - tA$:

$$Z = \langle d_{X-tA} f_1, \ldots, d_{X-tA} f_{n-1} \rangle_{\mathbb{K}[t]}.$$

The proof of this statement is based on the following three properties of the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$: $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is freely generated by some homogeneous polynomials $f_1, \ldots, f_n$, $\text{codim}(\mathfrak{g}^*)_{\text{sing}} \geq 2$ and $d_B f_1, \ldots d_B f_n$ form a basis of $\mathfrak{z}(B)$ for any regular B. But all these properties are satisfied for any reductive $\mathfrak{g}$. The first property is well-known while the second and the third were proved in [10]. Thus, this result is valid for any reductive Lie algebra.

If some polynomials generate Z, then they form a minimal basis if and only if the sum of their degrees equals half the rank of $\mathcal{B}_A$ (see [7, §3]).

1.4. Kronecker indices of a pair of forms on $\mathfrak{g}$

Let $\mathfrak{g}$ be a reductive Lie algebra and $(\, , \,)$ be an invariant scalar product on $\mathfrak{g}$. Given $A \in \mathfrak{g}$ we denote by b_A the skew-symmetric form on $\mathfrak{g}$ defined by the formula $b_A(X, Y) = (A, [X, Y])$. If we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of $(\, , \,)$, then the form b_A corresponds to the operator $\text{ad}(A)$.

Unlike the operator $\text{ad}(A)$, the form b_A has no invariants (except the rank). But non-trivial invariants will appear as soon as we consider not one but two forms b_A and b_B. If the element A is regular and the element B is in general position (from an open dense set), then the canonical form of (b_A, b_B) is known. Namely, there are no Jordan blocks and the Kronecker indices coincide with the exponents of $\mathfrak{g}$. If A is singular then the Jordan-Kronecker invariants of (b_A, b_B) are unknown.

The Kronecker indices can be expressed in terms of the ranks of some matrices (see [11] Thm. 8). But the rank of a matrix over $\mathbb{K}$ is the same as the rank of the value of this matrix at a generic point. Therefore, if B is in general position, then the Kronecker indices of (b_A, b_B) on $\mathfrak{g}$ over $\mathbb{C}$ are equal to the Kronecker indices of $(\mathcal{B}_A, \mathcal{B})$ on $\mathfrak{g} \otimes \mathbb{K}$ over $\mathbb{K}$.

1.5. Kronecker part of a canonical basis for $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$

For the Lie algebras $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$, in [13] Molev and Yakimova constructed some basis of the vector space spanned by the Kronecker part of a canonical basis. Namely, they constructed the corresponding functions in bi-involution. However, this basis is not canonical. In the same paper they showed that some elements of this basis generate $\mathcal{S}(\mathfrak{g}_A)^{\mathfrak{g}_A}$. The same results were obtained for some "good" elements A of $\mathfrak{so}_{2n+1}$ and $\mathfrak{so}_{2n}$.

In [7, 8], for $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$, the author constructed the Kronecker part of a canonical basis and the corresponding functions in bi-involution. Let us formulate these results.

Let N denote the size of matrices in $\mathfrak{g}$. That is, $N = n$ for $\mathfrak{g} = \mathfrak{sl}_n$, $N = 2n$ for $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n}$, $N = 2n + 1$ for $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Suppose that the distinct eigenvalues of $A \in \mathfrak{g}$ are $\{\lambda_1, \ldots, \lambda_s\}$. Let the eigenvalue λ_k correspond to the Jordan blocks of sizes $n_{k,1} \geq \cdots \geq n_{k,i_k}$. Denote by m the maximum of all i_k: $m = \max_k \{i_k, k = 1, \ldots, s\}$. For each $i \in \{1, \ldots, m\}$ we set $l_i = \sum_{j=1}^{s} n_{j,i}$, where $n_{j,i} = 0$ if $i > i_j$.

For any matrix A we construct the corresponding Young diagram with the rows of lengths $l_1 \geq \ldots \geq l_m$, the top row is of length l_1. Let us label the boxes of this diagram with numbers $0, \ldots, N-1$ consequently from left to right and from top to bottom. We denote by p_i the number of rows above the box with a label i.

Let $\{\mu_0, \ldots, \mu_{N-1}\}$ be the eigenvalues of A, ordered according to [7]: first we write down the eigenvalues in blocks of sizes $n_{1,1}, n_{2,1}, \ldots$, then in blocks of sizes $n_{1,2}, n_{2,2}, \ldots$, etc.

$$\{\mu_0, \ldots, \mu_{n-1}\} = \{\underbrace{\lambda_1, \ldots, \lambda_1}_{n_{1,1}}, \ldots, \underbrace{\lambda_s, \ldots, \lambda_s}_{n_{s,1}}, \ldots, \underbrace{\lambda_1, \ldots, \lambda_1}_{n_{1,m}}, \ldots, \underbrace{\lambda_s, \ldots, \lambda_s}_{n_{s,m}}\}. \quad (1)$$

Finally, we define the polynomials $q_N, \ldots, q_0 \in \mathbb{C}[\mathfrak{g}][t, z]$ and $r_{N-1}, \ldots, r_0 \in \mathbb{C}[\mathfrak{g}][t]$ by the formulas

$$q_N(z) = \chi_{X-tA}(z), \qquad q_{k+1}(z) = (z + \mu_k t)q_k(z) + r_k, \quad (2)$$

where χ_C is the characteristic polynomial of C. Thus, $q_k(z)$ is the partial quotient of a division $\chi_{X-tA}(z)$ by $(z + \mu_k t) \cdot \ldots \cdot (z + \mu_{N-1} t)$. If A is nilpotent, then the elements $r_k(t)$ coincide with the coefficients of the characteristic polynomial of $X - tA$.

Let us substitute the matrix $X - tA$ for variable z in the polynomials $q_k(z)$. We obtain the elements $q_k(X - tA) \in \mathfrak{gl}_N \otimes \mathbb{K}[t]$. Note that

$$\mathfrak{gl}_N \otimes \mathbb{K} \simeq \mathfrak{gl}_N(\mathbb{K}).$$

Consider the scalar product on $\mathfrak{gl}_N(\mathbb{K})$ given by the formula $(A, B) = \operatorname{tr} AB$. Let π denote the orthogonal projection of $\mathfrak{gl}_N(\mathbb{K})$ onto $\mathfrak{g} \otimes \mathbb{K}$ in the sense of this scalar product. Namely, if $\mathfrak{g} = \mathfrak{sl}_n$ then $\pi(X^k) = X^k - \frac{\operatorname{tr} X^k}{N} E$, if $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$ then $\pi(X^{2k+1}) = X^{2k+1}$ and $\pi(X^{2k}) = 0$. We set

$$h_k(X - tA) = \pi(q_k(X - tA)). \quad (3)$$

Thus, $h_k(X - tA) \in \mathfrak{g} \otimes \mathbb{K}[t]$. Note that the polynomials h_i and r_i depend on the ordering of the eigenvalues of A.

Theorem 1.5. [7] *For any matrix $A \in \mathfrak{sl}_n$ we have:*

- *The Kronecker indices $m_1, \ldots, m_{n-1}$ of $(\mathcal{B}, \mathcal{B}_A)$ are equal to $m_i = i - p_i$.*
- *The polynomials $h_1(X - tA), \ldots, h_{n-1}(X - tA)$ form a minimal basis of Z.*
- *The coefficients of the polynomials $h_1(X - tA), \ldots, h_{n-1}(X - tA)$ in a variable t form the Kronecker part of a canonical basis of $(\mathcal{B}, \mathcal{B}_A)$.*
- *The coefficients of the polynomials $r_1(t), \ldots, r_{n-1}(t)$ form the Kronecker part of a complete system of functions in bi-involution; and $dr_k = -h_k(X - tA)$.*

For the Lie algebra $\mathfrak{sp}_{2n}$, a similar result was obtained in [8].

Theorem 1.6. [8] *For any matrix $A \in \mathfrak{sp}_{2n}$ we have:*

- *The Kronecker indices $m_1, m_3, \ldots, m_{2n-1}$ of $(\mathcal{B}, \mathcal{B}_A)$ are equal to $m_i = i - p_i$.*
- *The polynomials $h_1(X - tA), h_3(X - tA), \ldots, h_{2n-1}(X - tA)$ form a minimal basis of Z.*

- *The coefficients of the polynomials $h_1(X - tA), h_3(X - tA), \ldots, h_{2n-1}(X - tA)$ in a variable t form the Kronecker part of a canonical basis of $(\mathcal{B}, \mathcal{B}_A)$.*

- *The coefficients of the polynomials $r_1(t), r_3(t), \ldots, r_{2n-1}(t)$ form the Kronecker part of a complete system of functions in bi-involution; and $dr_k = -h_k(X - tA)$.*

From the Theorems 1.5 and 1.6 and Section 1.4 one can deduce the following.

Corollary 1.7. *For any matrix $A \in \mathfrak{g}$ and a matrix $B \in \mathfrak{g}$ in general position, the Kronecker indices of (b_A, b_B) are equal to $m_i = i - p_i$, where $i = 1, 2, 3, \ldots, n-1$ if $\mathfrak{g} = \mathfrak{sl}_n$ and $i = 1, 3, 5, \ldots, 2n - 1$ if $\mathfrak{g} = \mathfrak{sp}_{2n}$.*

2. The Jordan part of a basis for the Lie algebras $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$

In this section we construct the Jordan part of a complete system of functions in bi-involution for the Lie algebras $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$. Combining this and Theorems 1.5 and 1.6 we obtain a complete system of functions in bi-involution with respect to $\{\,,\}$ and $\{\,,\}_A$ for any given A.

Further we will work with the Lie algebra $\mathfrak{g}_A \subseteq \mathfrak{g}$, which may not be reductive. Therefore, in this section, we identify $\mathcal{S}(\mathfrak{g}) \simeq \mathcal{S}(\mathfrak{g}^*) = \mathbb{C}[\mathfrak{g}]$ and assume that the polynomials $r_k(t)$ defined in (2) belong to $\mathcal{S}(\mathfrak{g})[t]$.

In this section the reasoning for Lie algebras $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$ is the same. But the Kronecker indices are numbered differently in these two cases. We set

$$I = \begin{cases} 1, 2, \ldots, n-1, & \mathfrak{g} = \mathfrak{sl}_n, \\ 1, 3, \ldots, 2n-1, & \mathfrak{g} = \mathfrak{sp}_{2n}. \end{cases}$$

Now the set of Kronecker indices in both cases can be written as $\{m_i,\ i \in I\}$. We denote by $f_{k,i} \in \mathcal{S}(\mathfrak{g})$ ($k \in I, i = 0, \ldots, m_k$) the coefficients of polynomials $r_k(t)$ in a variable t:

$$r_k(t) = f_{k,0}(X) + f_{k,1}(X)t + \ldots + f_{k,m_k}(X)t^{m_k}. \tag{4}$$

Recall that $f_{k,i}$ form the Kronecker part of a complete system of functions in bi-involution. Also recall that G_A denotes the stabilizer of A in the group G. Let us prove an analogue of Lemma 1.5 from [13].

Lemma 2.1. *For all $k \in I$ we have $f_{k,m_k} \in \mathcal{S}(\mathfrak{g}_A)^{\mathfrak{g}_A}$.*

Proof. The polynomials $r_k(t)$ are G_A-invariant by construction, this means that the polynomials f_{k,m_k} are also G_A-invariant. Thus, $f_{k,m_k} \in \mathcal{S}(\mathfrak{g})^{G_A}$.

Now let us show that $f_{k,m_k} \in \mathcal{S}(\mathfrak{g}_A)$. For any $B \notin \mathfrak{g}_A$ let $\{Y_1, \ldots Y_{d-1}\}$ be a basis of the orthogonal complement of $[A, B]$: $[A, B]^\perp = \langle Y_1, \ldots Y_{d-1} \rangle$, where $d = \dim \mathfrak{g}$. Then for any $Y \notin [A, B]^\perp$ the elements $Y, Y_1, \ldots, Y_{d-1}$ form a basis of $\mathfrak{g}$. Let

$$f_{k,m_k} = Y^s u_s + Y^{s-1} u_{s-1} + \ldots + u_0, \text{ where } u_i \in \mathbb{C}[Y_1, \ldots, Y_{d-1}],\ u_s \neq 0.$$

By definition we have $\{B, Y_i\}_A = (A, [B, Y_i]) = ([A, B], Y_i) = 0$ and $\{B, Y\}_A \neq 0$. Without loss of generality we may assume that $\{B, Y\}_A = 1$. Thus,

$$\{B, f_{k,m_k}\}_A = (A, [B, df_{k,m_k}]) = sY^{s-1}u_s + \ldots + u_1.$$

The Theorems 1.5 and 1.6 imply that $dr_k(t) \in Z = \text{Ker}(\mathcal{B} - t\mathcal{B}_A)$. Considering the leading coefficient in t one can see that $df_{k,m_k} \in \text{Ker}\,\mathcal{B}_A$. Thus, for any $\varphi \in \mathcal{S}(\mathfrak{g})$ we have $\{\varphi, f_{k,m_k}\}_A = 0$. Then for any $B \notin \mathfrak{g}_A$ and $Y \notin [A, B]^\perp$ the polynomial f_{k,m_k} does not dependent on Y. So, for any $B \notin \mathfrak{g}_A$ we have $f_{k,m_k} \in \mathcal{S}([A, B]^\perp)$. But $\bigcap_{B \notin \mathfrak{g}_A} [A, B]^\perp = \mathfrak{g}_A$, so $f_{k,m_k} \in \mathcal{S}(\mathfrak{g}_A)$. ∎

Lemma 2.2. *The polynomials f_{k,m_k} generate the algebra $\mathcal{S}(\mathfrak{g}_A)^{\mathfrak{g}_A}$.*

Proof. In [14, Thms. 4.2, 4.4] it was shown that $\mathcal{S}(\mathfrak{g}_A)^{\mathfrak{g}_A}$ is freely generated by homogeneous polynomials $G_i, i \in I$ with $\deg G_k = p_k + 1$. Note that the polynomials $f_{k,m_k} \in \mathcal{S}(\mathfrak{g}_A)^{\mathfrak{g}_A}$ are algebraically independent, since their differentials are the leading coefficients of the minimal basis (and therefore they are linearly independent). Next, one can see that $\deg f_{k,m_k} = k + 1 - m_k = \deg G_k$. Thus, the polynomials f_{k,m_k} are algebraically independent, homogeneous, and have "right" degrees, so they are basic invariants of $\mathcal{S}(\mathfrak{g}_A)^{\mathfrak{g}_A}$. ∎

Now let us apply the argument shift method to the elements $f_{k,m_k}, k \in I$. We define the polynomials $g_{k,j} \in \mathcal{S}(\mathfrak{g})$ as follows:

$$f_{k,m_k}(X - sB) = g_{k,0}(X) + g_{k,1}(X)s + \ldots + g_{k,l_k}(X)s^{d_k}, \tag{5}$$

where $B \in \mathfrak{g} \simeq \mathfrak{g}^*$ is such that $B|_{\mathfrak{g}_A^*}$ is regular in $\mathfrak{g}_A^*$. In [14] one can find examples of such elements.

Let $b(\mathfrak{g}_A)$ denote the maximal transcendence degree of a Poisson-commutative sub-algebra in $\mathcal{S}(\mathfrak{g}_A)$, i.e., $b(\mathfrak{g}_A) = \frac{1}{2}(\dim \mathfrak{g}_A + \text{ind}\,\mathfrak{g}_A)$, see [16, s. 0.2] for details. By Elashvili's conjecture (proved in [4]), $\text{ind}\,\mathfrak{g}_A = \text{ind}\,\mathfrak{g} = \text{rk}\,\mathfrak{g}$ for semisimple Lie algebras.

Theorem 2.3. *The functions $\{f_{k,l}\,|\,k \in I,\, l = 0, \ldots, m_k\}$ defined in (4) together the functions $\{g_{i,j}\,|\,i \in I,\, j = 1, \ldots, d_i - 1\}$ defined in (5), form a complete system of functions in bi-involution with respect to brackets $\{\,,\,\}$ and $\{\,,\,\}_A$.*

Proof. The polynomials f_{k,m_k} generate $\mathcal{S}(\mathfrak{g}_A)^{\mathfrak{g}_A}$ and $\{g_{i,j}\,|\,i \in I,\, j = 0, ..., d_i - 1\}$ generate the Mishchenko-Fomenko subalgebra $F_B \subseteq \mathcal{S}(\mathfrak{g}_A)$.

By [2, Thm. 1] the algebra F_B is complete (i.e. $\text{trdeg}\,F_B = b(\mathfrak{g}_A)$) if two conditions are satisfied: $\text{codim}(\mathfrak{g}_A^*)_{\text{sing}} \geq 2$ and $\text{ind}\,\mathfrak{g}_A = \text{trdeg}\,\mathcal{S}(\mathfrak{g}_A)^{\mathfrak{g}_A}$. In [14, Thms. 3.4, 3.11] it was shown that $\text{codim}(\mathfrak{g}_A^*)_{\text{sing}} \geq 2$ for any $A \in \mathfrak{g}$, where $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{sp}_{2n}$. By [14, Thms. 4.2, 4.4], we have $\text{trdeg}\,\mathcal{S}(\mathfrak{g}_A)^{\mathfrak{g}_A} = \text{ind}\,\mathfrak{g}_A$.

Next $\sum \deg G_i = b(\mathfrak{g}_A)$ by [14, Thm. 4.2, 4.4]. Thus, $\sum \deg f_{k,m_k} = b(\mathfrak{g}_A)$ and the polynomials $g_{i,j}$ are algebraically independent. Moreover, they lie in the kernel of $\{\,,\,\}_A$ and are in bi-involution with respect to the brackets $\{\,,\,\}$ and $\{\,,\,\}_B$.

The canonical form of two skew-symmetric bilinear forms implies that $\text{Ker}\,\mathcal{B}_A$ is a span of the Jordan blocks and the elements $df_{k,m_k}, k \in I$. Thus, $\text{Ker}\,\mathcal{B}_A$ is orthogonal (w.r.t. the both forms $\mathcal{B}$ and $\mathcal{B}_A$) to the Kronecker part of a basis of a bi-lagrangian subspace. Therefore, the polynomials $g_{i,j} \in \mathcal{S}(\mathfrak{g}_A)$ commute with all polynomials $f_{i,j}$ with respect to both Poisson structures $\{\,,\,\}$ and $\{\,,\,\}_A$. But the polynomials $\{g_{i,0}\,|\,i \in I\}$ already lie in the Kronecker part. So, the remaining polynomials extend the Kronecker part to a complete system of functions in bi-involution. ∎

The differentials of these functions in bi-involution form a basis of a bi-lagrangian subspace. Now let us prove that this basis can be completed to the canonical basis of a pair of skew-symmetric bilinear forms.

Lemma 2.4. *The elements*

$$\{df_{k,l}|\, k \in I,\, l = 0, \ldots, m_k\} \ \text{ and } \ \{dg_{i,j}|\, i \in I,\, j = 1, \ldots, d_i - 1\}$$

can be completed to a canonical basis for forms $\mathcal{B}$ *and* $\mathcal{B}_A$.

Proof. For brevity reasons, we set $v_{k,l} = df_{k,l}$ and $u_{i,j} = dg_{i,j}$. We will construct elements $\{\tilde{v}_{k,l}|\, k \in I,\, l = 1, \ldots, m_k\}$ and $\{\tilde{u}_{i,j}|\, i \in I,\, j = 1, \ldots, d_i - 1\}$ that complete the above vectors to a basis of Kronecker and Jordan blocks respectively. Note that the vectors are numbered from right to left in each half of each block.

Let us first complete $\{v_{k,l}\}$ to the basis of Kronecker blocks as described in [6]. Thus, we construct elements $\{\tilde{v}_{k,l}|\, k \in I,\, l = 1, \ldots, m_k\}$. Next, we construct elements $\tilde{u}_{i,j} \in \operatorname{Ker} \mathcal{B}_A$ that complete $u_{i,j}$ to the canonical basis of Jordan blocks. For $\mathcal{B}_A$ to have a canonical form it is sufficient that $\tilde{u}_{i,j} \in \operatorname{Ker} \mathcal{B}_A$. For the form $\mathcal{B}$ we want the following 10 conditions to be fulfilled:

$$(c1) \quad \mathcal{B}(\tilde{u}_{i,j}, \tilde{u}_{k,l}) = 0, \qquad (c2) \quad \mathcal{B}(\tilde{u}_{i,j}, u_{k,l}) = \delta_{ij}^{kl}, \qquad (c3) \quad \mathcal{B}(u_{i,j}, u_{k,l}) = 0,$$

$$(c4) \quad \mathcal{B}(\tilde{u}_{i,j}, \tilde{v}_{k,l}) = 0, \qquad (c5) \quad \mathcal{B}(u_{i,j}, \tilde{v}_{k,l}) = 0, \qquad (c6) \quad \mathcal{B}(\tilde{v}_{i,j}, \tilde{v}_{k,l}) = 0,$$

$$(c7) \quad \mathcal{B}(\tilde{u}_{i,j}, v_{k,l}) = 0, \qquad (c8) \quad \mathcal{B}(u_{i,j}, v_{k,l}) = 0, \qquad (c9) \quad \mathcal{B}(\tilde{v}_{i,j}, v_{k,l}) = \delta_{ij}^{kl},$$

$$(c10) \quad \mathcal{B}(v_{i,j}, v_{k,l}) = 0,$$

where $\delta_{ij}^{kl} = \delta_i^k \delta_j^l$ and δ_i^j is the Kronecker delta.

By the definition of $\{v_{k,l}\}$, $\{u_{k,l}\}$ and $\{\tilde{v}_{k,l}\}$ the conditions $(c3)$, $(c6)$, $(c8)$–$(c10)$ are satisfied. Let $W = \operatorname{Ker} \mathcal{B}_A$. Note that W is the span of all basis vectors of the Jordan part and the vectors $\{v_{k,m_k}, k \in I\}$. So,

$$\mathcal{B}(W, v_{k,l}) = 0, \quad \text{ for } k \in I, l = 0, \ldots, m_k,$$

$$\mathcal{B}(W, \tilde{v}_{k,l}) = 0, \quad \text{ for } k \in I, l = 1, \ldots, m_k - 1.$$

Let $V = \langle v_{k,m_k}, k \in I \rangle \subseteq W$. Consider a subspace $U \subseteq W$ such that

$$W = V \oplus U, \qquad u_{i,j} \in U, \ \text{ for all } i \in I, j = 1, \ldots, d_i - 1.$$

The canonical form of a pair of skew-symmetric bilinear forms implies $\operatorname{Ker} \mathcal{B}|_W = V$. Thus, the form $\mathcal{B}|_U$ is non-degenerate. Hence the elements $u_{i,j}$ can be completed by some elements to the canonical basis of the skew-symmetric bilinear form. Let us denote these elements by $\{\tilde{u}_{i,j}, i \in I, j = 1 \ldots, d_i - 1\}$. Thus, the conditions $(c1)$ and $(c2)$ are satisfied.

Since $u_{i,j}, \tilde{u}_{i,j} \in W$, the condition $(c7)$ is satisfied. Also if $l \neq m_k$ then the conditions $(c4)$ and $(c5)$ are satisfied. Thus, it remains to achieve $\mathcal{B}(u_{i,j}, \tilde{v}_{k,m_k}) = 0$ and $\mathcal{B}(\tilde{u}_{i,j}, \tilde{v}_{k,m_k}) = 0$. Let us subtract from $\tilde{u}_{i,j}$ the elements v_{k,m_k} with suitable coefficients. Then we have $\mathcal{B}(\tilde{u}_{i,j}, \tilde{v}_{k,m_k}) = 0$. Now subtract from $\tilde{v}_{ij}$ the elements $\tilde{u}_{k,l}$ with suitable coefficients. So, we have $\mathcal{B}(u_{i,j}, \tilde{v}_{k,m_k}) = 0$. Note that we only subtracted the elements of $\operatorname{Ker} \mathcal{B}_A$, so the matrix of the form $\mathcal{B}_A$ remained the same. ∎

Example 2.5. Let us construct a complete system of functions in bi-involution for the matrix $A = \mathrm{diag}\{\lambda, \lambda, \mu\} \in \mathfrak{sl}_3$, where $2\lambda + \mu = 0$.

In [7, Example 2], the author constructed the Kronecker part of a complete system of functions in bi-involution for the matrix $\mathrm{diag}\{\lambda, \lambda, \mu\} \in \mathfrak{gl}_3$. Recall that X denotes the generic element of $\mathfrak{g} \otimes \mathbb{K}$, i.e., the matrix X consists of elements of $\mathbb{K}$ which correspond to coordinates on $\mathfrak{g}$. To obtain the Kronecker part of a complete system of functions in bi-involution for $A \in \mathfrak{sl}_3$, one can take all polynomials except $f_{0,0}$ and take into account that $\mathrm{tr}\, X = \mathrm{tr}\, A = 0$. Thus, we obtain:

$$f_{1,1} = -\mathrm{tr}\, AX, \qquad f_{1,0} = -\frac{1}{2}\,\mathrm{tr}\, X^2,$$

$$f_{2,1} = -\frac{(\lambda+\mu)}{2}\,\mathrm{tr}\, X^2 + \mathrm{tr}\, X^2 A, \qquad f_{2,0} = -\frac{1}{3}\,\mathrm{tr}\, X^3.$$

To construct the Jordan part, we apply the argument shift method to the polynomials $f_{1,1}, f_{2,1}$. The polynomial $f_{1,1}$ has degree 1, so it does not contribute to the Jordan part. To get the only polynomial that belongs to the Jordan part, one can take a matrix B such that $B|_{\mathfrak{g}_A}$ is regular and take the coefficient of s in $f_{2,1}(X - sB)$. We obtain

$$g_{2,1} = (\lambda+\mu)\,\mathrm{tr}\, XB - \mathrm{tr}\, XBA - \mathrm{tr}\, BXA.$$

Thus, we constructed $5 = \frac{1}{2}(\mathrm{ind}\,\mathfrak{sl}_3 + \dim \mathfrak{sl}_3)$ algebraically independent functions in bi-involution.

3. Good elements for the Lie algebras $\mathfrak{so}_{2n+1}$ and $\mathfrak{so}_{2n}$

In this section we describe some elements of the Lie algebras $\mathfrak{so}_{2n+1}$ and $\mathfrak{so}_{2n}$ such that the polynomials defined by the formulas (2) form the Kronecker part of a complete system of functions in bi-involution.

It is well known that for any matrix $A \in \mathfrak{so}_N$ its eigenvalues are divided into pairs of opposite $\{\lambda, -\lambda\}$ with the same block structure. And for the eigenvalue $\lambda = 0$ the number of blocks of each even size must be even. Let us construct the Young diagram Λ corresponding to the matrix A with the rows of lengths $l_1, \ldots, l_m$ as described in Subsection 1.5.

Note that if exactly r consecutive rows of Λ are of even length, then r is even. Indeed, if A is nilpotent then it is obvious, since the number of blocks of each even size must be even. But the parities of the row lengths remain the same after the adding of Jordan blocks with nonzero eigenvalues, since such blocks are divided into pairs with opposite eigenvalues.

Definition 3.1 (see [14], Lemma 4.5). An element $A \in \mathfrak{so}_N$ is called *good* if the following conditions are satisfied:

(1) l_1 is odd; $\hfill (6)$

(2) if l_{i-1} is odd and $i \leq m$ is odd, then l_i is odd.

Thus, the Young diagram of a good element is as follows: the first row has an odd length, and all the following rows, except maybe the last one, are divided into pairs of rows with the same parity of lengths. That is, $l_1 \equiv 1$, $l_2 \equiv l_3$, $l_4 \equiv l_5, \ldots$, where $\equiv$ stands for congruence modulo 2. If $\mathfrak{g} = \mathfrak{so}_{2n}$ then the last row has an odd length, $l_m \equiv 1$, and if $\mathfrak{g} = \mathfrak{so}_{2n+1}$ we have $l_{m-1} \equiv l_m$.

Examples of good and not good diagrams are shown in Figures 1 and 2.

Theorem 3.2. *For any good element $A \in \mathfrak{so}_{2n+1}$ we have:*

- *The Kronecker indices $m_1, m_3, \ldots, m_{2n-1}$ of $(\mathcal{B}, \mathcal{B}_A)$ are equal to $m_i = i - p_i$.*
- *The polynomials $h_1(X - tA), h_3(X - tA), \ldots, h_{2n-1}(X - tA)$ form a minimal basis of Z.*
- *The coefficients of the polynomials $h_1(X - tA), h_3(X - tA), \ldots, h_{2n-1}(X - tA)$ in a variable t form the Kronecker part of a canonical basis of $(\mathcal{B}, \mathcal{B}_A)$.*
- *The coefficients of the polynomials $r_1(t), r_3(t), \ldots, r_{2n-1}(t)$ form the Kronecker part of a complete system of functions in bi-involution; and $dr_k = -h_k(X - tA)$.*

Proof. By Section 1.3, it is sufficient to check that

$$\sum \deg_t h_i(X - tA) = \tfrac{1}{2} \operatorname{rk} \mathcal{B}_A = \tfrac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{g}_A).$$

We know that $\deg_t h_i(X - tA) = i - p_i$. By [14, Lemma 4.5] for any good nilpotent $A \in \mathfrak{so}_{2n+1}$ we have $\sum(p_i + 1) = \tfrac{1}{2}(\dim \mathfrak{g}_A + \operatorname{rk} \mathfrak{g})$. The numbers p_i depend only on the Young diagram, so that this equality is true for arbitrary good elements $A \in \mathfrak{so}_{2n+1}$. Then we have

$$\sum_{i \, \text{odd}} \deg_t h_i(X - tA) = \sum_{i \, \text{odd}}(i - p_i) = n^2 - \sum_{i \, \text{odd}}(p_i + 1) + n$$

$$= n^2 - \tfrac{1}{2}(\dim \mathfrak{g}_A + n) + n = \tfrac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{g}_A). \qquad \blacksquare$$

In the case $\mathfrak{g} = \mathfrak{so}_{2n}$, one of the basic invariants is the Pfaffian. Therefore the polynomials $h_1(X - tA), \ldots, h_{2n-1}(X - tA)$ cannot form a minimal basis of Z even for a regular A, since the degree of the last polynomial is twice as large as necessary. Let us show that if A is good then one can always extract the root of the polynomial $r_{2n-1}(t)$.

As mentioned above, for good A, the last row of the Young diagram is of odd length. Thus, we may assume that $\mu_{2n-1} = 0$, since 0 is one of the eigenvalues contributing to l_m (otherwise l_m would be even). By the formula (2), r_{2n-1} is the constant term of the characteristic polynomial of $X - tA$. Thus, r_{2n-1} is the determinant of a skew-symmetric matrix and we can extract the root of it, the Pfaffian of the matrix $X - tA$. Denote this Pfaffian by $\tilde{r}_{2n-1}$ and set $\tilde{h}_{2n-1}(X - tA) = -d\tilde{r}_{2n-1}(t)$ and $\tilde{m}_{2n-1} = m_{2n-1}/2$. For $i < 2n - 1$ we set $\tilde{r}_i(t) = r_i(t)$, $\tilde{h}_i = h_i$, $\tilde{m}_i = m_i$.

Theorem 3.3. *For any good element $A \in \mathfrak{so}_{2n}$ we have:*

- *The Kronecker indices $\tilde{m}_1, \tilde{m}_3, \ldots, \tilde{m}_{2n-1}$ of $(\mathcal{B}, \mathcal{B}_A)$ are equal to $\tilde{m}_i = i - p_i$ for $i = 1, 3, \ldots, 2n - 3$ and $\tilde{m}_i = \tfrac{1}{2}(i - p_i)$ for $i = 2n - 1$.*
- *The polynomials $\tilde{h}_1(X - tA), \tilde{h}_3(X - tA), \ldots, \tilde{h}_{2n-1}(X - tA)$ form a minimal basis of Z.*
- *The coefficients of the polynomials $\tilde{h}_1(X - tA), \tilde{h}_3(X - tA), \ldots, \tilde{h}_{2n-1}(X - tA)$ in a variable t form the Kronecker part of a canonical basis of $(\mathcal{B}, \mathcal{B}_A)$.*
- *The coefficients of the polynomials $\tilde{r}_1(t), \tilde{r}_3(t), \ldots, \tilde{r}_{2n-1}(t)$ form the Kronecker part of a complete system of functions in bi-involution; and $d\tilde{r}_k = -\tilde{h}_k(X - tA)$.*

Proof. As in the previous theorem, it suffices to check that

$$\sum \deg_t \tilde{h}_i(X - tA) = \tfrac{1}{2} \operatorname{rk} \mathcal{B}_A = \tfrac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{g}_A).$$

We know that $\deg_t h_i(X - tA) = i - p_i$ for $i = 1, 3, \ldots, 2n - 3$, whereas we have $\deg_t h_i(X - tA) = \tfrac{1}{2}(i - p_i)$ for $i = 2n - 1$. By [14, Lemma 4.5], for good nilpotent elements $A \in \mathfrak{so}_{2n}$, we have

$$\sum_{i \, \mathrm{odd}}^{2n-3} (p_i + 1) + \tfrac{1}{2}(p_{2n-1} + 1) = \tfrac{1}{2}(\dim \mathfrak{g}_A + \operatorname{rk} \mathfrak{g}).$$

Just as in the previous theorem, this equality is true for arbitrary good elements $A \in \mathfrak{so}_{2n}$. Then, we have

$$\sum_{i \, \mathrm{odd}} \deg_t h_i(X - tA) = \sum_{i \, \mathrm{odd}}^{2n-3} (i - p_i) + \tfrac{1}{2}(2n - 1 - p_{2n-1})$$

$$= n^2 - n - \sum_{i \, \mathrm{odd}}^{2n-3} (p_i + 1) + n - \tfrac{1}{2}(p_{2n-1} + 1) = n^2 - \sum_{i \, \mathrm{odd}}^{2n-3} (p_i + 1) - \tfrac{1}{2}(p_{2n-1} + 1)$$

$$= n^2 - \tfrac{1}{2}(\dim \mathfrak{g}_A + n) = \tfrac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{g}_A). \qquad \blacksquare$$

4. Complete system of functions in bi-involution for semisimple good elements of $\mathfrak{so}_{2n+1}$ and $\mathfrak{so}_{2n}$

In this section, for $\mathfrak{g} = \mathfrak{so}_{2n}$, we describe all good semisimple elements, while for $\mathfrak{g} = \mathfrak{so}_{2n+1}$, we show that all semisimple elements are good. Then we show that for semisimple good elements $A \in \mathfrak{g}$ one can construct a complete system of functions in bi-involution in the same way as for $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$. Namely, the Kronecker part consists of the coefficients of the polynomials defined by (2). And the Jordan part can be obtained by applying the argument shift method to some polynomials of the Kronecker part.

Let $A \in \mathfrak{so}_N$ be semisimple with eigenvalues $\{0, \pm\lambda_1, \ldots, \pm\lambda_s\}$. Let n_i denote the multiplicity of eigenvalues $\pm\lambda_i$, and n_0 denote the multiplicity of $\lambda = 0$.

Denote by Λ the Young diagram corresponding to A. The first n_0 rows of Λ have odd lengths, and all the rows below have even lengths. So, if $\mathfrak{g} = \mathfrak{so}_{2n+1}$ then the condition (6) is satisfied (since n_0 is odd) and the element A is good.

If $\mathfrak{g} = \mathfrak{so}_{2n}$ then n_0 is even. Thus, A is good if and only if Λ has exactly n_0 rows. In other words, $n_0 \geq n_i$ for any $i \neq 0$. Thus, the following lemma is proved:

Lemma 4.1. *Good semisimple elements in $\mathfrak{so}_N$ are described as follows:*
- *For $\mathfrak{g} = \mathfrak{so}_{2n+1}$, all semisimple elements are good.*
- *For $\mathfrak{g} = \mathfrak{so}_{2n}$, a semisimple element is good iff $n_0 \geq n_i$ for all $i \neq 0$.*

Lemma 4.2. *Let $A \in \mathfrak{so}_N$ be a good semisimple element. Then the algebra $S(\mathfrak{g}_A)^{\mathfrak{g}_A}$ is freely generated by homogeneous polynomials of the following degrees:*
- $p_i + 1$, *where* $i = 1, 3, 5, \ldots, 2n - 1$, *for* $\mathfrak{g} = \mathfrak{so}_{2n+1}$,
- $p_i + 1$, *where* $i = 1, 3, 5, \ldots 2n - 3$, *and* $\tfrac{1}{2}(p_{2n-1} + 1)$, *for* $\mathfrak{g} = \mathfrak{so}_{2n}$.

Proof.   If $A \in \mathfrak{so}_N$ is semisimple then $\mathfrak{g}_A \approx \mathfrak{gl}_{n_1} \oplus \ldots \oplus \mathfrak{gl}_{n_s} \oplus \mathfrak{so}_{n_0}$.
Recall that the boxes of Λ are labeled with the numbers $0, \ldots, N-1$ (see Subsection 1.5). We call the box *marked* if its label is odd. Let us fill each marked box with the number of the row containing this box. In other words, if a box with the label i is marked then it is filled with the number $p_i + 1$.

Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$. Then n_0 is odd and $\mathcal{S}(\mathfrak{g}_A)^{\mathfrak{g}_A}$ is freely generated by homogeneous polynomials of the following degrees:

$$1, 2, \ldots, n_1, 1, 2, \ldots, n_2, \ldots, 1, 2, \ldots, n_s, 2, 4, 6, \ldots, n_0 - 1. \tag{7}$$

In order to prove the lemma, it suffices to check that the numbers written in Λ coincide with (7) up to permutation. Indeed, the diagram Λ consists of "doubled" columns of height n_i, where $i > 0$, and a single column of height n_0. The doubled column of height n_i is filled with the numbers $1, 2, \ldots, n_i$, since each row of a doubled column has exactly one marked box. And the single column of height n_0 is filled with the numbers $2, 4, \ldots, n_0 - 1$, since there are only doubled columns to the left and to the right of this column.

Now let $\mathfrak{g} = \mathfrak{so}_{2n}$. Then the algebra $\mathcal{S}(\mathfrak{g}_A)^{\mathfrak{g}_A}$ is freely generated by homogeneous polynomials of the following degrees:

$$1, 2, \ldots, n_1, 1, 2, \ldots, n_2, \ldots, 1, 2, \ldots, n_s, 2, 4, 6, \ldots, n_0 - 2, \frac{n_0}{2}. \tag{8}$$

In the case $\mathfrak{g} = \mathfrak{so}_{2n}$, we have to divide by 2 the number $p_{2n-1} + 1$. Obviously, this number lies in the bottom row of Λ and all the numbers written in this row are equal. So, to prove the lemma, it suffices to divide by 2 one of the numbers in the bottom row and check that the numbers written in Λ coincide with (8) up to permutation. Since A is good, the bottom row of Λ contains a box of the single column and this box is filled with $n_0 = p_{2n+1} - 1$. So, dividing by 2 the number in this box, we obtain the required equality. ∎

Theorem 4.3.   *Let $A \in \mathfrak{so}_N$ be a good semisimple element. Then the polynomials $\{g_{i,j}\}$ defined by the formulas* (5) *form the Jordan part of a complete system of functions in bi-involution with respect to the brackets* $\{\,,\,\}$ *and* $\{\,,\,\}_A$.

Proof.   By Lemma 2.1, $f_{k,m_k} \in S(\mathfrak{g}_A)^{\mathfrak{g}_A}$. Combining the proof of Lemma 2.2 and the results of Lemma 4.2, we obtain that the elements f_{k,m_k} are basic invariants of $S(\mathfrak{g}_A)^{\mathfrak{g}_A}$.

Since A is semisimple, the algebra $\mathfrak{g}_A$ is reductive. Thus, the argument shift method gives a complete system of functions in involution on $\mathfrak{g}_A$. Therefore, the elements $\{g_{i,j} \,|\, i \in I, j = 1 \ldots d_i - 1\}$ form the Jordan part of a complete system of functions in bi-involution. ∎

5.   Correctable nilpotent elements in $\mathfrak{so}_N$

In Section 3 we showed that for good nilpotent elements of $\mathfrak{so}_N$ the differentials of the coefficients of the characteristic polynomial form a minimal basis of Z. In this section, we describe the nilpotent elements $A \in \mathfrak{so}_N$ that are not good, but that are *correctable*. Namely, for correctable A, we show how to "modify" some of the coefficients of the characteristic polynomial in order to obtain the minimal basis of Z.

In [5], Charbonnel and Moreau studied the question: when is the algebra $S(\mathfrak{g}_A)^{\mathfrak{g}_A}$ a polynomial algebra? Particularly, for some nilpotent elements $A \in \mathfrak{so}_N$ they implicitly constructed basic invariants $\hat{a}_1, \ldots, \hat{a}_n$ of $\mathfrak{so}_N$ such that the polynomials $\hat{a}_i(X - tA)$ have sufficiently small degrees in t.

In this section, we give explicit formulas for the polynomials $\hat{a}_i$, based on the proof from the above paper. Thus, for correctable elements, we obtain the Kronecker part of a complete system of functions in bi-involution.

Denote by $e_1, \ldots e_n$ the elementary symmetric polynomials and by $p_1, \ldots, p_n$ the power sums. Recall that, by Newton's identities, $ke_k = \sum_{i=1}^{k}(-1)^{i-1}p_ie_{k-i}$ (we set $e_0 = 1$). Next, let P_k and E_k denote the polynomials expressing the power sums in terms of the elementary symmetric polynomials and vice versa:

$$p_k = P_k(e_1, \ldots, e_k), \qquad e_k = E_k(p_1, \ldots, p_k).$$

We define symmetric polynomials $\tilde{e}_i$ by formulas $\tilde{e}_0 = 2$ and $\tilde{e}_i = E_i(2p_1, \ldots, 2p_i)$ for $i = 1, \ldots, n$. Let us prove a technical lemma.

Lemma 5.1. *For $k = 1, \ldots, n$ we have*

$$\tilde{e}_k = e_0e_k + e_1e_{k-1} + \ldots + e_ke_0 = \sum_{i=0}^{k} e_ie_{k-i}.$$

Proof. We argue by induction on k. If $k = 1$, then $\tilde{e}_k = \tilde{e}_1 = 2e_1 = e_0e_1 + e_1e_0$. Assuming that for all $i < k$ the statement is true, we prove it for k. Set $\tilde{p}_i = 2p_i$. Applying Newton's identities to the polynomials $\tilde{e}_k$, we obtain

$$k\tilde{e}_k = \sum_{i=1}^{k}(-1)^{i-1}\tilde{p}_i\tilde{e}_{k-i} = 2\sum_{i=1}^{k}(-1)^{i-1}p_i\tilde{e}_{k-i}.$$

Let us apply the induction hypothesis, change the order of summation, apply Newton's identities and combine similar terms:

$$k\tilde{e}_k = 2\sum_{i=1}^{k}(-1)^{i-1}p_i\sum_{j=0}^{k-i}e_je_{k-i-j} = 2\sum_{j=0}^{k-1}e_j\sum_{i=1}^{k-j}(-1)^{i-1}p_ie_{k-i-j} =$$

$$= 2\sum_{j=0}^{k-1}e_j \cdot (k-j)e_{k-j} = k\sum_{j=0}^{k}e_ie_{k-i}. \qquad \blacksquare$$

Let us denote by a_i the nonzero coefficients of the characteristic polynomial of $X - tA$: $\chi_{X-tA}(z) = z^m + a_1z^{m-2} + \ldots + a_nz^{m-2n}$.

In the case of the Lie algebra $\mathfrak{so}_{2n}$, we redefine a_n as the Pfaffian of the matrix $X - tA$, i.e., $a_n = \mathrm{Pf}(X - tA)$.

Consider separately the cases $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{g} = \mathfrak{so}_{2n}$.

5.1. The case $\mathfrak{g} = \mathfrak{so}_{2n+1}$

Definition 5.2. Let $A \in \mathfrak{so}_{2n+1}$ be nilpotent and correspond to the partition $l_1, \ldots l_s$. We call A *correctable* if for some $k > 0$ the following conditions are satisfied:

(1) $l_1 = \ldots = l_{2k} = 2l$, where $l \in \mathbb{N}$ (9)

(2) l_{2k+1} is odd,

(3) for all $i > k$ if l_{2i} is odd, then l_{2i+1} is also odd.

In other words, partitions corresponding to correctable elements are obtained by adding an even rectangle on the top of a good partition.
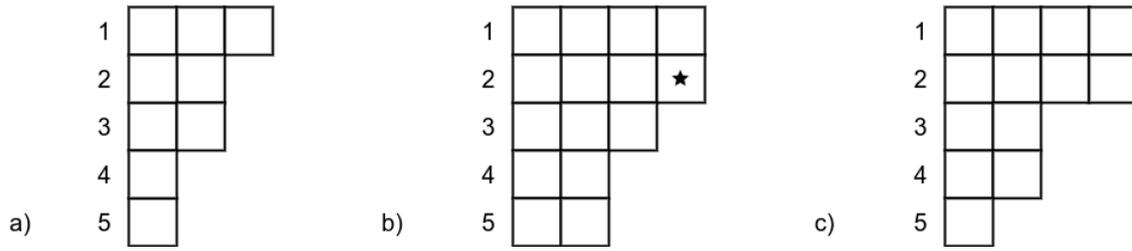

Figure 1: a) good element; b) correctable (but not good) element; c) neither good nor correctable element. In figure b) an asterisk marks the box where we "modify" the coefficient of the characteristic polynomial.

For correctable A we set:

$$\begin{cases} \tilde{a}_{il} = E_i\left(\dfrac{P_1(a_l)}{2}, \dfrac{P_2(a_l, a_{2l})}{2}, \ldots, \dfrac{P_i(a_l, \ldots a_{il})}{2}\right), & k < i \leq 2k, \\ \tilde{a}_j = a_j, \text{ if } j \notin \{(k+1)l, \ldots, 2kl\}. \end{cases} \tag{10}$$

Let us define the numbers m_i as before: $m_i = i - p_i$ and set

$$\begin{cases} \tilde{m}_j = m_j - 1, & \text{if } j = 2il - 1 \text{ and } k < i \leq 2k, \\ \tilde{m}_j = m_j, & \text{otherwise.} \end{cases} \tag{11}$$

Theorem 5.3. *If $A \in \mathfrak{so}_{2n+1}$ is a correctable nilpotent element, then the coefficients of polynomials $\tilde{a}_1(X - tA), \ldots, \tilde{a}_n(X - tA)$ form the Kronecker part of a complete system of functions in bi-involution. The Kronecker indices of forms $(\mathcal{B}, \mathcal{B}_A)$ are equal to $\tilde{m}_1, \tilde{m}_3, \ldots, \tilde{m}_{2n-1}$; and $\deg \tilde{a}_i(X - tA) = \tilde{m}_{2i-1}$.*

Proof. For $i = 1, \ldots, n$ we denote by c_i the leading coefficient of the polynomial a_{li}. In [5], the authors define the polynomials $\bar{c}_i$ as the restrictions of polynomials c_i to some subalgebra $\bar{\mathfrak{g}}_A \subseteq \mathfrak{g}$. Lemma 5.14 of the cited work proves that an algebraic dependence between $\bar{c}_i$ implies the same algebraic dependence between c_i.

We will not give the exact definition of the polynomials $\bar{c}_i$, since Proposition 5.18 of the same paper gives its explicit description. Namely, for $i = 1, \ldots, 2k$, where k is the same as in Definition 5.2, we have:

$$\begin{cases} -\bar{c}_1 = e_0 e_1 + e_1 e_0, \\ \ldots \\ (-1)^i \bar{c}_i = e_0 e_i + e_1 e_{i-1} + \ldots + e_i e_0, \\ \ldots \\ \bar{c}_{2k} = e_0 e_{2k} + e_1 e_{2k-1} + \ldots + e_{2k} e_0, \end{cases} \tag{12}$$

where e_i are the elementary symmetric polynomials in k variables. Thus, if $i > k$ then $e_i = 0$.

Notice that the polynomials e_i can be expressed in terms of $\bar{c}_i$. Namely, there exist polynomials $R_i \in \mathbb{C}[x_1, \ldots, x_i]$ such that:

$$
\begin{cases}
e_1 = -\frac{1}{2}\bar{c}_1 = R_1(\bar{c}_1), \\
e_2 = \frac{1}{2}\bar{c}_2 - \frac{1}{8}\bar{c}_1^2 = R_2(\bar{c}_1, \bar{c}_2), \\
\cdots \\
e_{2k} = R_{2k}(\bar{c}_1, \ldots, \bar{c}_{2k}).
\end{cases}
$$

Let us give the explicit formulas for the polynomials R_i. Lemma 5.1 implies that $\bar{c}_i = (-1)^i \tilde{e}_i$. Hence

$$
\begin{aligned}
R_i(x_1, \ldots, x_i) &= E_i \left(\frac{P_1(-x_1)}{2}, \frac{P_2(-x_1, x_2)}{2}, \ldots, \frac{P_i(-x_1, x_2, \ldots, (-1)^i x_i)}{2} \right) \\
&= E_i \left(\frac{-P_1(x_1)}{2}, \frac{P_2(x_1, x_2)}{2}, \ldots, \frac{(-1)^i P_i(x_1, \ldots, x_i)}{2} \right) \\
&= (-1)^i E_i \left(\frac{P_1(x_1)}{2}, \ldots, \frac{P_i(x_1, \ldots, x_i)}{2} \right).
\end{aligned}
$$

Since $e_i = 0$ for $i > k$, it follows that $R_i(\bar{c}_1, \ldots, \bar{c}_i) = 0$ for $i = k+1, \ldots, 2k$. So, $R_i(c_1, \ldots, c_i) = 0$. Thus, the leading coefficients of polynomials $\tilde{a}_{jl}$ defined by (10) are reduced. This means that $\deg_t \tilde{a}_{jl} < \deg_t a_{jl}$ for $j = k+1, \ldots, 2k$.

By Remark [5, Rem. 5.24] we have

$$
2 \sum_{i=1}^{n} (2i - \deg_t a_i) = \dim \mathfrak{g}_A + n - 2k.
$$

Hence, for the degrees of polynomials $\tilde{a}_i$ we obtain the following inequality:

$$
\begin{aligned}
2 \sum_{i=1}^{n} \deg_t \tilde{a}_i &\leq 2 \cdot \left(\sum_{i=1}^{n} \deg_t a_i - k \right) = 2 \sum_{i=1}^{n} 2i - \dim \mathfrak{g}_A - n + 2k - 2k \\
&= \dim \mathfrak{g} - \dim \mathfrak{g}_A = \operatorname{rk} \mathcal{B}_A.
\end{aligned}
$$

Thus, the polynomials $\tilde{a}_1, \ldots, \tilde{a}_n$ form the minimal basis of the module Z and the theorem is proved. ∎

5.2. The case $\mathfrak{g} = \mathfrak{so}_{2n}$

Definition 5.4. Let $A \in \mathfrak{so}_{2n}$ be nilpotent and correspond to the partition $l_1, \ldots l_s$. We call A *correctable* if it satisfies one of the following conditions:

(a) for some k (maybe, $k = n$) the condition (9) holds   (13)

(b) $s = 4$, all l_i are even.

Let $A \in \mathfrak{so}_{2n}$ be correctable of type (a). We define the polynomials $\tilde{a}_i$ as in (10) and the numbers $\tilde{m}_i$ as in (11). As we already noted, if $\mathfrak{g} = \mathfrak{so}_{2n}$ then one of the basic invariants is Pfaffian (instead of the determinant).
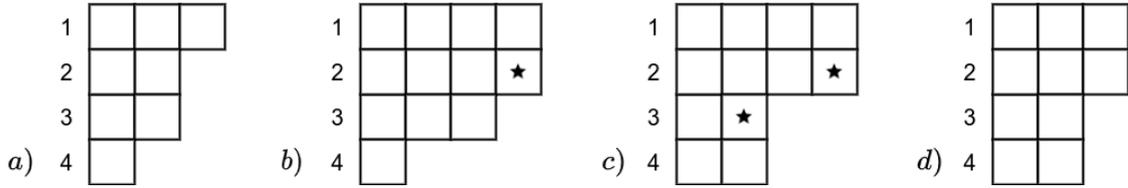
Figure 2: a) good element; b) correctable element of type a); c) correctable element of type b; d) neither good nor correctable element. In Figures b) and c), the asterisks mark the boxes where the coefficients of the characteristic polynomial must be "modified".

So, let us redefine the polynomial $\tilde{a}_n$:

$$\tilde{a}_n = \begin{cases} a_n - (-1)^k E_k \left(\dfrac{P_1(a_l)}{2}, \dfrac{P_2(a_l, a_{2l})}{2}, \ldots, \dfrac{P_k(a_l, \ldots a_{kl})}{2} \right), & \text{if } k = n \\ a_n, \text{ otherwise.} \end{cases} \quad (14)$$

Here k is the same as in the Definition 5.4. Also we set $\tilde{m}_{2n-1} = \frac{1}{2}m_{2n-1} - 1$ if $k = n$ and $\tilde{m}_{2n-1} = \frac{1}{2}m_{2n-1}$ otherwise.

Now let $A \in \mathfrak{so}_{2n}$ be correctable of type $b)$. Then all four rows of the Young diagram are of even lengths, i.e., $l_1 = l_2 = 2d_1$ and $l_3 = l_4 = 2d_2$. We set

$$\begin{cases} \tilde{a}_{2d_1} = a_{2d_1} - \frac{1}{4}a_{d_1}^2, \\ \tilde{a}_{2d_1+d_2} = a_{2d_1+d_2} - a_{d_1}a_n, \\ \tilde{a}_i = a_i, \text{ for all other } i. \end{cases} \quad (15)$$

And let, in this case, $\tilde{m}_{2d_1-1} = m_{2d_1-1} - 1$ and $\tilde{m}_{2d_2+d_1-1} = m_{2d_2+d_1-1} - 1$.

Theorem 5.5. *Let $A \in \mathfrak{so}_{2n}$ be a correctable nilpotent element. Then the coefficients of the polynomials $\tilde{a}_1(X - tA), \ldots, \tilde{a}_n(X - tA)$ form the Kronecker part of a complete system of functions in bi-involution. The Kronecker indices of $(\mathcal{B}, \mathcal{B}_A)$ are $\tilde{m}_1, \tilde{m}_3, \ldots, \tilde{m}_{2n-1}$, and $\deg \tilde{a}_i(X - tA) = \tilde{m}_{2i-1}$.*

Proof. First, let $A \in \mathfrak{so}_{2n}$ be a correctable nilpotent element of type $a)$. The last line of (12) implies that $\bar{c}_{2k} = e_k^2$. This means that $\bar{c}_{2k} = R_k^2(\bar{c}_1, \ldots, \bar{c}_k)$. Thus, for $k = n$ we have $\deg_t \tilde{a}_n < \deg_t a_n$, and again

$$\sum_{i=1}^n \deg_t \tilde{a}_i \leq \sum_{i=1}^n \deg_t a_i - k. \quad (16)$$

As in the proof of Theorem 5.3, the result follows from [5, Rem. 5.24].
For elements of type $b)$ the inequality (16) was proved in [5, Thm. 5.23] (in this case, $k = 2$). The result follows again from [5, Rem. 5.24] and (16). ∎

Example 5.6. Let $A \in \mathfrak{so}_{2n+1}$ be a correctable nilpotent element such that $l_1 = \ldots = l_{2k} = 2$. All nonzero eigenvalues of the matrix $X - tA$ are divided into pairs of opposite $\pm\nu_i, i = 1, \ldots, n$. Hence the coefficients of the characteristic polynomial are expressed as follows: $a_i = (-1)^i e_i(\nu_1^2, \ldots, \nu_n^2)$. Therefore $(-1)^i P_i(a_1, \ldots, a_i) = p_i(\nu_1^2, \ldots, \nu_n^2) = \frac{1}{2}\operatorname{tr} X^{2i}$. For example, for the nilpotent matrix $A \in \mathfrak{so}_9$ corresponding to the partition $(2, 2, 2, 2, 1)$ we have the following system of functions:

$\tilde{a}_1 = a_1 = \operatorname{tr} X^2,$

$\tilde{a}_2 = a_2 = \frac{1}{2}(\operatorname{tr}^2 X^2 - \operatorname{tr} X^4),$

$\tilde{a}_3 = E_3\left(\frac{\operatorname{tr} X^2}{4}, \frac{\operatorname{tr} X^4}{4}, \frac{\operatorname{tr} X^6}{4}\right) = \frac{1}{12}\left(\operatorname{tr} X^6 - \frac{3}{8}\operatorname{tr} X^4 \operatorname{tr} X^2 + \frac{1}{32}\operatorname{tr}^3 X^2\right),$

$\tilde{a}_4 = -\frac{1}{16}\left(\operatorname{tr} X^8 - \frac{1}{3}\operatorname{tr} X^6 \operatorname{tr} X^2 - \frac{1}{8}\operatorname{tr}^2 X^4 + \frac{1}{16}\operatorname{tr} X \operatorname{tr}^2 X^2 - \frac{1}{384}\operatorname{tr}^4 X^2\right).$

6. Limits of Mishchenko-Fomenko subalgebras

In this section we will define a formal limit of Mischenko-Fomenko subalgebras and explicitly give a system of generators for some of the limit algebras.

Recall that if $\mathfrak{g}$ is reductive, then the Mishchenko-Fomenko subalgebra F_A is generated by the directional derivatives (of all orders) of all basic invariants along the element A. In the case of a regular element A, this algebra is free and complete, i.e., it has the maximal transcendence degree $\frac{1}{2}(\operatorname{rk} \mathfrak{g} + \dim \mathfrak{g})$ [2].

By considering limits of Mishchenko-Fomenko subalgebras one can obtain other commutative (with respect to the Poisson bracket) subalgebras. In [15], Shuvalov studied the limits $\lim_{s \to 0} F_{A(s)}$, where $A(s) = A_0 + A_1 s + \ldots + A_r s^r$ is regular for small enough $s \in \mathbb{C} \setminus \{0\}$ and all A_i belong to a fixed Cartan subalgebra $\mathfrak{h}$. In [17], it was shown that algebras constructed in this way have the maximal transcendence degree.

Let us clarify what we mean by $\lim_{s \to 0} F_{A(s)}$. For any s the subalgebra $F_{A(s)}$ is graded: $F_{A(s)} = \bigoplus_{k=0}^{\infty} (F_{A(s)})_k$, where k stands for the degree of a homogeneous polynomial, $(F_{A(s)})_k = F_{A(s)} \cap \mathcal{S}^k(\mathfrak{g})$. The dimension of the grading subspace $(F_{A(s)})_k$ does not depend on s, since the element $A(s)$ is regular. Thus, in each Grassmanian the limit $F_k = \lim_{s \to 0} (F_{A(s)})_k$ is defined. Adding all these limit subspaces we obtain the limit subalgebra: $\lim_{s \to 0} F_{A(s)} = \bigoplus_{k=0}^{\infty} F_k$.

Set $\mathfrak{z}_{-1} = \mathfrak{g}$ and $\mathfrak{z}_k = \mathfrak{z}(A_0) \cap \ldots \cap \mathfrak{z}(A_k)$ for $k = 0, \ldots, r$. Since $A(s)$ is regular for small $s \neq 0$, we have $\mathfrak{z}_r = \mathfrak{h}$.

Theorem 6.1. ([15], Thm.1) *The algebra $\lim_{s \to 0} F_{A(s)}$ is free. As its free generators we can take the union of an arbitrary basis of $\mathfrak{h}$ and some of directional derivatives of the basic invariants of $\mathfrak{z}_{k-1}$ along A_k $(k = 0, \ldots, r)$.*

In this paper, for classical simple Lie algebras, we explicitly construct free generators for linear $A(s)$, i.e., for $A(s) = A + Bs$. In the case $\mathfrak{g} = \mathfrak{so}_{2n}$, we impose an additional condition that A is good. Note that the element $A + Bs$ is regular for small $s \neq 0$ if and only if $\bar{B} = B|_{\mathfrak{g}_A}$ is regular in $\mathfrak{g}_A^* \simeq \mathfrak{g}_A$. Denote by $f_1, \ldots, f_n$ basic invariants of $\mathfrak{g}$ and by $g_1, \ldots, g_n$ we denote basic invariants of $\mathfrak{g}_A = \mathfrak{z}(A)$. In this case, Theorem 6.1 can be reformulated as follows:

Corollary 6.2. *The limit subalgebra $\lim_{s \to 0} F_{A+Bs}$ is freely generated by some of the directional derivatives (of all orders) of f_i along A and of g_i along B.*

Proof. Let us consider the elements $\{\partial_B^{\deg g_i - 1} g_i \,|\, i = 1, \ldots, n\}$, the last non-constant directional derivatives of basic invariants of $\mathfrak{g}_A$. In order to prove the corollary it suffices to show that these elements form a basis of $\mathfrak{h}$.

Indeed, these elements are the differentials of basic invariants at the point $\bar{B}$, $\partial_B^{\deg g_i - 1} g_i = (\deg g_i - 1)! \cdot d_{\bar{B}} g_i$. Since $\mathfrak{g}_A$ is reductive, the differentials of its basic

invariants at a regular point generate the centralizer of this point (by Kostant's criterion, see [13, Def. 1.6]). But the element $\bar{B}$ is regular at $\mathfrak{g}_A$ and therefore the elements $\{\partial_B^{\deg g_i - 1} g_i \mid i = 1, \ldots, n\}$ generate $\mathfrak{z}_{\mathfrak{g}_A}(\bar{B}) = \mathfrak{z}(A) \cap \mathfrak{z}(B) = \mathfrak{h}$. Thus, these elements form a basis of $\mathfrak{h}$ (since $\dim \mathfrak{h} = n$). ∎

The element $B|_{\mathfrak{g}_A}$ is regular in $\mathfrak{g}_A$. Therefore, one can use $B|_{\mathfrak{g}_A}$ as a regular element of $\mathfrak{g}_A$ when constructing the Jordan part of a complete system of functions in bi-involution with respect to the brackets $\{\,,\,\}$ and $\{\,\}_A$. Let us prove that the elements of this complete system of functions in bi-involution are free generators of the limit subalgebra $\lim_{s \to 0} F_{A+Bs}$.

Lemma 6.3. *The Kronecker part of a complete system of functions in bi-involution $\{f_{i,j}\}$, constructed in Theorems 1.5, 1.6, 3.2, 3.3, generates the Mishchenko-Fomenko subalgebra F_A.*

Proof. In [7, Lemma 4] the explicit formulas for the polynomials $r_k(t)$ are given.

Namely,
$$r_k(t) = (-1)^k E_{k+1}(y_1, \ldots, y_{k+1}), \tag{17}$$

where $y_i = \operatorname{tr}(X - tA)^i - (\mu_k^i + \ldots + \mu_{n-1}^i)(-t)^i$. Since $(-1)^k E_{k+1}(y_1, \ldots, y_{k+1}) = \frac{1}{k+1} y_{k+1} + \tilde{E}_{k+1}(y_1, \ldots, y_k)$, the polynomials $\operatorname{tr}(X - tA)^k$ are expressed as polynomials in $r_0(t), \ldots r_{k-1}(t)$ over $\mathbb{C}[t]$. Thus, the coefficients of $\operatorname{tr}(X - tA)^k$ can be expressed in terms of the coefficients of $r_k(t)$.

Note that for the Lie algebras $\mathfrak{sl}_n, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n+1}$, all non-zero polynomials $\operatorname{tr} X, \ldots, \operatorname{tr} X^N$ are basic invariants. Thus, the coefficients of $\operatorname{tr}(X - tA)^k$ generate the subalgebra F_A. In the case of the Lie algebra $\mathfrak{so}_{2n}$, one of the basic invariants is Pfaffian. But the element A is good, so the Pfaffian coincides with the polynomial $\tilde{r}_{2n-1}$ used in Theorem 3.3. Thus, we proved that $F_A \subseteq \mathbb{C}[\,f_{i,j} \mid i \in I, j = 0, \ldots, m_k\,]$.

The inverse inclusion is obvious since the polynomials $f_{i,j}$ are expressed in terms of the coefficients $\operatorname{tr}(X - tA)^k$ by the formulas (17). ∎

Theorem 6.4. *Let $\mathfrak{g}$ be a classical simple Lie algebra, the elements $A, B \in \mathfrak{g}$ lie in the Cartan subalgebra $\mathfrak{h}$, and the element $A + Bs$ be regular for small enough $s \neq 0$. In the case $\mathfrak{g} = \mathfrak{so}_{2n}$, we impose an additional condition that A is good. Then the functions constructed in Theorems 1.5–4.3 freely generate the limit subalgebra $\lim_{s \to 0} F_{A+Bs}$.*

Proof. Recall that $f_1, \ldots, f_n$ denote the basic invariants of $\mathfrak{g}$, and $g_1, \ldots, g_n$ denote the basic invariants of $\mathfrak{g}_A = \mathfrak{z}(A)$. By Corollary 6.2 the limit subalgebra $\lim_{s \to 0} F_{A+Bs}$ is freely generated by some of the directional derivatives

$$\{\partial_A^k f_i \mid i = 1, \ldots, n,\, k = 0, \ldots, \deg f_i\} \cup \{\partial_B^k g_i \mid i = 1, \ldots, n,\, k = 0, \ldots, \deg g_i\}.$$

Note that the elements $\{\partial_A^k f_i\}$ generate $F_A \subseteq \mathcal{S}(\mathfrak{g})$ and the elements $\{\partial_B^k g_i\}$ generate $F_B \subseteq \mathcal{S}(\mathfrak{g}_A)$. By Lemma 6.3 the Kronecker part of the complete system of functions in bi-involution $\{f_{i,j}\}$ freely generates $F_A \subseteq \mathcal{S}(\mathfrak{g})$. By Lemma 2.2 the Jordan part $\{g_{i,j}\}$ together with the elements f_{k,m_k} freely generate $F_B \subseteq \mathcal{S}(\mathfrak{g}_A)$.

Moreover, all elements of the complete system of functions in bi-involution are algebraically independent by construction. So, this complete system of functions in bi-involution freely generates the Mishchenko-Fomenko limit subalgebra. ∎

7.   Kroneker indices and the sheets of $\mathfrak{g}$

Recall that the sheets of a Lie algebra $\mathfrak{g}$ are the irreducible components of the subsets $X^{(d)} = \{x \in \mathfrak{g} : \dim Gx = d\}$. In other words, the sheets are the maximal irreducible subsets consisting of G-orbits of fixed dimension. It is known [3, Sect.5.8 Kor.(a)] that each sheet contains exactly one nilpotent orbit. The sheets containing a semi-simple orbit are called *Dixmier sheets.* If a sheet contains a semi-simple element, then semisimple elements are dense in the sheet.

For example, for $\mathfrak{g} = \mathfrak{sl}_n$, the sheets do not intersect and each sheet is defined by a partition $l_1, \ldots, l_s$. That is, the sheets consist of matrices with the same Young diagram. So, in this case, all the sheets are Dixmier sheets.

In this section we show that, for $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$, the Kronecker indices of the pair $(\mathcal{B}, \mathcal{B}_A)$ are the same for all elements A in any given sheet. For $\mathfrak{g} = \mathfrak{so}_{2n}$ and $\mathfrak{so}_{2n+1}$, we prove the same statement for the sheets containing a good semisimple element. On the other hand, we give examples of sheets in $\mathfrak{so}_{2n}$ and $\mathfrak{so}_{2n+1}$ such that this is not true.

7.1. The Kronecker indices within the sheets of $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$

Let us first describe how to find a nilpotent element A_n in a sheet containing a given matrix A.

For $\mathfrak{g} = \mathfrak{sl}_n$, the sheets are represented by Young diagrams. So, one can construct the Young diagram Λ corresponding to A and take the nilpotent matrix $A_n \in \mathfrak{sl}_n$ corresponding to Λ.

For $\mathfrak{g} = \mathfrak{sp}_{2n}$, we again construct the Young diagram Λ corresponding to $A \in \mathfrak{sp}_{2n}$. In the diagram Λ some rows of odd length can occur odd number of times, so not every diagram corresponds to a nilpotent element. Let us modify Λ to obtain a diagram Λ' that corresponds to some nilpotent orbit in $\mathfrak{sp}_{2n}$.

Consider the first (starting from the top) row of odd length q that occurs an odd number of times. Move one box from the bottom row of length q to the next row. This is possible because the next row necessarily has odd length. Repeating this procedure (we call it *crumbling*), we obtain the diagram Λ' corresponding to some nilpotent orbit. In [9, §2], it was shown that the described nilpotent orbit belongs to the same sheet as A.

Lemma 7.1.   *For Lie algebras $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$, the Kronecker indices of the pair $(\mathcal{B}, \mathcal{B}_A)$ are the same for all elements A in any given sheet.*

Proof.   For the Lie algebra $\mathfrak{sl}_n$ this is obvious, since any sheet is represented by Young diagram Λ and the Kronecker indices depend only on Λ.

For $\mathfrak{g} = \mathfrak{sp}_{2n}$, different Young diagrams can occur within a sheet. Nevertheless, it suffices to prove that the Kronecker indices do not change during crumbling. Indeed, only boxes with even labels crumble, since there are an even number of odd-length rows above them (by the definition of crumbling). So the boxes with odd labels remain in the same rows (though they are shifted along the row). Thus, the Kronecker indices do not change during crumbling.   ∎

7.2. The Kronecker indices and the sheets of $\mathfrak{so}_N$ containing a good semisimple element

In this section we consider the sheets of $\mathfrak{so}_N$ that contain good semisimple elements. We show that all elements in such sheets are good and that the Kronecker indices are constant within such sheets.

Recall that if a sheet contains a semisimple element then the semisimple elements are dense in this sheet. Thus, any element A from this sheet can be obtained as a limit of semisimple elements: $A = \lim_{s \to 0} A_s$, where A_s are semisimple and lie in the same sheet. By [9] all the semisimple elements in a sheet correspond to the same Young diagram, let us denote it by Λ.

Let us prove first that Young diagrams do not increase when taking the limit. Namely, recall that on the set of Young diagrams one can define a partial order.

Definition 7.2. Let Λ and M be Young diagrams with rows of lengths $l_1, \ldots, l_s$ and $m_1, \ldots, m_t$ respectively, and $l_1 + \ldots + l_s = m_1 + \ldots + m_t$. Then $\Lambda \succeq M$ if and only if $l_1 + \ldots + l_k \geq m_1 + \ldots + m_k$ for $k = 1, \ldots, s$.

Lemma 7.3. *Suppose $A = \lim_{s \to 0} A_s$, where all A_s correspond to the same Young diagram Λ. Then the Young diagram M corresponding to A is less than or equal to Λ, $M \preceq \Lambda$.*

Proof. Given $C \in \mathfrak{so}_N$ denote by $l_1, \ldots, l_m$ the row lengths of the corresponding Young diagram. Then $l_1, \ldots, l_m$ are the degrees of the invariant factors of C, i.e., the invariant factors of the finitely generated module $\mathbb{C}^N$ (with the multiplication given by C) over the principal ideal domain $\mathbb{C}[t]$. Thus, $l_k = \deg(\Delta_{n-k+1}/\Delta_{n-k})$, where Δ_k denotes the greatest common divisor of all minors of order k of the matrix $C - tE$ (where E stands for the identity matrix). So, $l_1 + \ldots + l_k = \deg \Delta_n/\Delta_{n-k} = n - \deg \Delta_{n-k}$. Thus, it is sufficient to prove that the degrees of polynomials Δ_{n-k} do not decrease when taking the limit.

Let us fix k and denote by q_s the polynomial Δ_k for the matrix $A_s - tE$, $s \neq 0$. We assume that q_s is monic. Let p_s denote the principal minor obtained by deleting the last $n - k$ rows and columns of the matrix $A_s - tE$. Note that the polynomial p_s is monic and $\deg p_s = k$. Recall that the roots of a monic polynomial depend continuously on its coefficients. So, the roots of p_s tend to the roots of p_0 as s tends to 0.

Obviously, each root of q_s is a root of p_s and there are only a finite number of ways to choose $\deg q_s$ elements from the set with k elements. Thus, one of these ways corresponds to a dense subset of $\{s \in \mathbb{C} \mid s \neq 0\}$. So, there exists a limit $\lim_{s \to 0} q_s = q$ and $\deg q = \deg q_s$. But q is still a common divisor of all minors. So, the degree of the greatest common divisor is not less than $\deg q_s$. $\blacksquare$

Lemma 7.4. *If a sheet contains a good semisimple element, then all the elements in this sheet are good.*

Proof. For $\mathfrak{g} = \mathfrak{so}_N$, a nilpotent element in a sheet can be obtained in the same way as for $\mathfrak{g} = \mathfrak{sp}_{2n}$, namely, by means of crumbling. But in this case we crumble the rows of even length.

Let N denote the Young diagram corresponding to a nilpotent element of the sheet.

Note that the Young diagrams decrease during crumbling. Thus, for any element of a sheet its Young diagram M is greater than or equal to N. By Lemma 7.3, we obtain that $\Lambda \succeq M \succeq N$.

Note that the diagram Λ is good if and only if the sums $s_k(\Lambda) = l_1 + \ldots + l_k$ are odd for $k = 1, 3, 5, \ldots$. In order to prove that all elements of a sheet are good it suffices to show that $s_k(\Lambda) = s_k(N)$, since $s_k(\Lambda) \geq s_k(M) \geq s_k(N)$.

In the case $\mathfrak{g} = \mathfrak{so}_{2n}$ all rows of Λ are of odd length, so the diagram requires no crumbling, and thus $\Lambda = N$.

In the case $\mathfrak{g} = \mathfrak{so}_{2n+1}$ there is an odd number of odd-length rows at the top of Λ and some even-length rows below. Only the even-length rows with even index $2k$ can be crumbled. If $l_{2k} = l_{2k+1}$ then there is no crumbling, and if $l_{2k} \neq l_{2k+1}$ then these rows are replaced by the rows of lengths $l_{2k} - 1$ and $l_{2k+1} + 1$. So, for all k we have $s_{2k+1}(\Lambda) = s_{2k+1}(N)$. ∎

Theorem 7.5. *If a sheet contains a good semisimple element then the Kronecker indices of the pair $(\mathcal{B}, \mathcal{B}_A)$ are the same for all elements A in this sheet.*

Proof. In the case $\mathfrak{g} = \mathfrak{so}_{2n}$ all the elements of the sheet are good and correspond to the same Young diagram. Therefore, the Kronecker indices are constant within the sheet.

In the case $\mathfrak{g} = \mathfrak{so}_{2n+1}$ it is sufficient to check that the Kronecker indices do not change when crumbling. The proof is similar to the proof of Lemma 7.1. In this case, only boxes with even labels are crumbled. Indeed, above them there is an odd number of odd-length rows, and the rows themselves are of even length. So, the boxes with odd labels remain in the same rows (though they are shifted along the row). Thus, the Kronecker indices remain the same when crumbling. ∎

7.3. Example of a sheet in $\mathfrak{so}_{2n+1}$ with different Kronecker indices

Consider $\mathfrak{g} = \mathfrak{so}_7$ and suppose that $A \in \mathfrak{so}_7$ has eigenvalue $\lambda = 0$ with blocks of sizes $2, 2, 1$ and two opposite sign eigenvalues $\lambda = \pm 1$, each of multiplicity 1. Denote by $A_n \in \mathfrak{so}_7$ the nilpotent element that corresponds to the partition $3, 3, 1$. Note that A and A_n lie in the same sheet.

The Kronecker indices of the pair $(\mathcal{B}, \mathcal{B}_{A_n})$ are $1, 2, 4$, since the element A_n is good. Let us prove that the Kronecker indices of the pair $(\mathcal{B}, \mathcal{B}_A)$ are $1, 3, 3$. Indeed, the module $Z = \mathrm{Ker}(\mathcal{B} - t\mathcal{B}_A)$ is generated by $X - tA, (X - tA)^3, (X - tA)^5$ and their leading coefficients are A, A^3, A^5. The matrices A and A^3 are linearly independent and by Remark 1.3 we have $m_1 + m_3 + m_5 = 7$. Thus, $m_1 = 1$, $m_3 = 3$ and $m_5 = 3$.

7.4. Example of a Dixmier sheet in $\mathfrak{so}_{2n}$ with different Kronecker indices

Consider $\mathfrak{g} = \mathfrak{so}_8$ and let $A \in \mathfrak{so}_8$ be a semisimple element with eigenvalues $\lambda = \pm 1$, each of multiplicity 3 and $\lambda = \pm 2$, each of multiplicity 1. Denote by $A_n \in \mathfrak{so}_8$ the nilpotent element that corresponds to the partition $3, 3, 1, 1$. The elements A and A_n again belong to the same sheet.

The Kronecker indices of the pair $(\mathcal{B}, \mathcal{B}_{A_n})$ are $1, 2, 2, 4$, since the element A_n is good. We prove that the Kronecker indices of the pair $(\mathcal{B}, \mathcal{B}_A)$ are $1, 2, 3, 3$. Indeed, the module $Z = \mathrm{Ker}(\mathcal{B} - t\mathcal{B}_A)$ is generated by the elements $X - tA$, $(X - tA)^3$, $(X - tA)^5$, $d\,\mathrm{pf}(X - tA)$ and their leading coefficients are $A, A^3, A^5, 5A - A^3$ (the last coefficient is obtained by computer calculations). Let us "modify" the

polynomial f_7 corresponding to the Pfaffian: $\tilde{f}_7 = f_7 + 5t^2(X - tA) - (X - tA)^3$. Now, the leading coefficients of $f_1, f_3, \tilde{f}_7$ are linearly independent and their degrees are $1, 3, 2$ (also checked by computer calculations). By Remark 1.3 we then have $m_1 + m_3 + m_5 + m_7 = 9$, so, $m_1 = 1$, $m_3 = 3$, $m_5 = 3$ and $m_7 = 2$.

In particular, we obtained that for $\mathfrak{so}_{2n}$ even in a Dixmier sheet there can be different Kronecker indices.

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