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The First and Second Homotopy Groups of a Homogeneous Space of a Complex Linear Algebraic Group

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Abstract. Let X be a homogeneous space of a connected linear algebraic group G defined over the field of complex numbers $\mathbb{C}$. Let $x \in X(\mathbb{C})$ be a point. We denote by H the stabilizer of x in G. When H is connected, we compute the topological fundamental group $\pi_1^{\mathrm{top}}(X(\mathbb{C}), x)$. Moreover, we compute the second homotopy group $\pi_2^{\mathrm{top}}(X(\mathbb{C}), x)$.

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1. Introduction

The topological fundamental group of a *connected linear algebraic group* over $\mathbb{C}$ was defined algebraically by Merkurjev [15, Section 10.1] and by the author [2, Definition 1.3]. A third definition was proposed by Colliot-Thélène [7, Proposition-Definition 6.1]. The definition of Colliot-Thélène was generalized to the reductive group schemes by González-Avilés [11, Definition 3.7] and by the author and González-Avilés [5, Definition 2.11]. We wish to compute algebraically the topological fundamental group of a *homogeneous space* G/H of a connected linear algebraic group G over $\mathbb{C}$. In general, by the result of the author and Cornulier [3], the topological fundamental group of the homogeneous space G/H cannot be computed algebraically. In this article we compute algebraically the topological fundamental group of G/H under a certain connectedness condition on H, in particular, in the case when H is connected. Moreover, we compute algebraically the second homotopy group of G/H (without connectedness assumptions on H).

In this article, by a variety over an algebraically closed field k of characteristic 0 we mean a separated integral scheme of finite type over k.

Let X be a variety defined over $\mathbb{C}$. Let $x \in X(\mathbb{C})$. Consider the pointed topological space $(X(\mathbb{C}), x)$ and its (topological) homotopy groups $\pi_1^{\mathrm{top}}(X(\mathbb{C}), x)$ and $\pi_2^{\mathrm{top}}(X(\mathbb{C}), x)$. We write $\pi_1^{\mathrm{top}}(X, x)$ for $\pi_1^{\mathrm{top}}(X(\mathbb{C}), x)$, and we write $\pi_2^{\mathrm{top}}(X, x)$ for $\pi_2^{\mathrm{top}}(X(\mathbb{C}), x)$. Set

$$\mathbb{Z}(1) = \pi_1^{\mathrm{top}}(\mathbb{G}_{\mathrm{m},\mathbb{C}}(\mathbb{C}), 1) = \pi_1^{\mathrm{top}}(\mathbb{C}^\times, 1)$$

where $\mathbb{G}_{\mathrm{m},\mathbb{C}}$ denotes the multiplicative group over $\mathbb{C}$.

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Choose an element $\boldsymbol{i} \in \mathbb{C}$ such that $\boldsymbol{i}^2 = -1$. There is a generator $\xi_{\boldsymbol{i}}$ (depending on the choice of $\boldsymbol{i}$) of $\mathbb{Z}(1) = \pi_1^{\rm top}(\mathbb{C}^\times, 1)$ given by the loop

$$t \mapsto \exp 2\pi \boldsymbol{i} t\colon \quad [0,1] \to \mathbb{C}^\times.$$

We obtain an isomorphism (depending on the choice of $\boldsymbol{i}$)

$$\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}(1)\colon m \mapsto m\xi_{\boldsymbol{i}} \quad \text{for } m \in \mathbb{Z}.$$

We assume that the group $\pi_1^{\rm top}(X,x)$ is abelian. Consider the abelian groups

$$\pi_n^{\rm top}(X,x)(-1) := \operatorname{Hom}\big(\mathbb{Z}(1), \pi_n^{\rm top}(X,x)\big) \quad \text{for } n = 1,2. \tag{1}$$

There are isomorphisms (depending on the choice of $\boldsymbol{i}$)

$$\pi_n^{\rm top}(X,x)(-1) = \operatorname{Hom}\big(\mathbb{Z}(1), \pi_n^{\rm top}(X,x)\big) \xrightarrow{\sim} \pi_n^{\rm top}(X,x), \quad \phi \mapsto \phi(\xi_{\boldsymbol{i}}).$$

Notation. Let k be an algebraically closed field of characteristic 0. Let H be a linear algebraic group defined over k. We use the following notation:

- H^0 is the identity component of H;
- $\pi_0(H) = H/H^0$, which is a finite k-group;
- $H^{\rm u}$ is the unipotent radical of H^0;
- $H^{\rm red} = H^0/H^{\rm u}$, which is a connected reductive group;
- $H^{\rm ss} = [H^{\rm red}, H^{\rm red}]$ (the commutator subgroup of $H^{\rm red}$), which is semisimple;
- $H^{\rm sc}$ is the universal cover of $H^{\rm ss}$; it is simply connected;
- $H^{\rm tor} = H^{\rm red}/H^{\rm ss}$, which is a torus;
- $H^{\rm ssu} = \ker\big[H^0 \to H^{\rm tor}\big]$; it fits into a short exact sequence

$$1 \to H^{\rm u} \to H^{\rm ssu} \to H^{\rm ss} \to 1. \tag{2}$$

Observe that $H^{\rm tor}$ is the largest toric quotient of H^0 and that $H^{\rm ssu}$ is connected and has no nontrivial characters.
If T is a torus over k, we write T_* for the cocharacter group of T, that is,

$$T_* = \operatorname{Hom}_k(\mathbb{G}_{{\rm m},k}, T)$$

where $\mathbb{G}_{{\rm m},k}$ denotes the multiplicative group over k. Then $T_* \cong \operatorname{Hom}\big(\widehat{T}, \mathbb{Z}\big)$ where $\widehat{T} = \operatorname{Hom}_k(T, \mathbb{G}_{{\rm m},k})$ is the character group of T. Let $T_H^{\rm red} \subseteq H^{\rm red}$ be a maximal torus. Consider the composite homomorphism

$$\rho\colon H^{\rm sc} \twoheadrightarrow H^{\rm ss} \hookrightarrow H^{\rm red}$$

and set $T_H^{\rm sc} = \rho^{-1}(T_H^{\rm red})$, which is a maximal torus of $H^{\rm sc}$. Following [2, Section 1], we define the *algebraic fundamental group*

$$\pi_1^{\rm alg}(H) := \operatorname{coker}\big[(T_H^{\rm sc})_* \to (T_H^{\rm red})_*\big].$$

This group is well defined (does not depend on the choice of T_H^{red} up to a transitive system of isomorphisms); see [2, Lemma 1.2].

Let X be a homogeneous space of a connected linear algebraic group G defined over an algebraically closed field k of characteristic 0. Choose a k-point $x \in X(k)$ and set $H = \mathrm{Stab}_G(x)$. Denote by H^{mult} the largest quotient group of H of multiplicative type. Set $H^{\mathrm{ker.char}} := \ker\big[H \to H^{\mathrm{mult}}\big]$; then $H^{\mathrm{ker.char}}$ is the intersection of the kernels of all characters $\chi\colon H \to \mathbb{G}_{\mathrm{m},k}$ of H. We consider the following conditions on G and H:

(a) $\mathrm{Pic}(G) = 0$,

(b) $H^{\mathrm{ker.char}}$ is connected.

Observe that (a) is satisfied if and only if G^{ss} is simply connected (see Corollary 3.4 below), and that (b) is satisfied if H is connected or abelian.

Denote $\widehat{G} := \mathrm{Hom}(G, \mathbb{G}_{\mathrm{m},k})$ and $\widehat{H} := \mathrm{Hom}(H, \mathbb{G}_{\mathrm{m},k})$; these groups are clearly abelian. We write

$$\mathrm{Ext}^0\big(\widehat{G} \to \widehat{H}, \mathbb{Z}\big) := \mathrm{Ext}^0\big(\big[\widehat{G} \xrightarrow{i^*} \widehat{H}\big\rangle, \mathbb{Z}\big)$$

where $\big[\widehat{G} \xrightarrow{i^*} \widehat{H}\big\rangle$ is a complex with $\widehat{G}$ in degree 0 and $\widehat{H}$ in degree 1. The homomorphism i^* is induced by the inclusion $i\colon H \hookrightarrow G$. See Section 2 for the definition of Ext^0.

Theorem 1.1. *Let X be a homogeneous space of a connected linear algebraic group G over $\mathbb{C}$. Let $x \in X(\mathbb{C})$, and set $H = \mathrm{Stab}_G(x)$. Assume that $\mathrm{Pic}(G) = 0$ and that $H^{\mathrm{ker.char}}$ is connected. Then the group $\pi_1^{\mathrm{top}}(X, x)$ is abelian and there exists a canonical isomorphism of abelian groups*

$$\mathrm{Ext}^0\big(\widehat{G} \to \widehat{H}, \mathbb{Z}\big) \xrightarrow{\sim} \pi_1^{\mathrm{top}}(X, x)(-1).$$

Corollary 1.2. *Under the hypotheses of Theorem 1.1:*

(i) *There is a canonical exact sequence*

$$\mathrm{Hom}\big(\widehat{H}, \mathbb{Z}\big) \xrightarrow{i_*} \mathrm{Hom}\big(\widehat{G}, \mathbb{Z}\big) \longrightarrow \pi_1^{\mathrm{top}}(X, x)(-1) \longrightarrow \mathrm{Hom}\big(\widehat{H}_{\mathrm{tors}}, \mathbb{Q}/\mathbb{Z}\big)$$

where $i\colon H \hookrightarrow G$ is the inclusion homomorphism, and $\widehat{H}_{\mathrm{tors}}$ denotes the torsion subgroup of $\widehat{H}$.

(ii) *If, moreover, the subgroup H is connected, then the exact sequence of (i) induces a canonical isomorphism*

$$\mathrm{coker}\big[H_*^{\mathrm{tor}} \xrightarrow{i_*} G_*^{\mathrm{tor}}\big] \xrightarrow{\sim} \pi_1^{\mathrm{top}}(X, x)(-1).$$

Observe that Corollary 1.2(ii) is a more explicit version of [6, Theorem 8.5(i)].

Theorem 1.3. *Let X be a homogeneous space of a connected linear algebraic group G over $\mathbb{C}$. We do not assume that $\mathrm{Pic}(G) = 0$. Let $x \in X(\mathbb{C})$, and set $H = \mathrm{Stab}_G(x)$. Let $i\colon H \hookrightarrow G$ denote the inclusion homomorphism.*

(i) *If H is connected, then the group $\pi_1^{\mathrm{top}}(X, x)$ is abelian and there is a canonical isomorphism*

$$\mathrm{coker}\big[\pi_1^{\mathrm{alg}}(H) \xrightarrow{i_*} \pi_1^{\mathrm{alg}}(G)\big] \xrightarrow{\sim} \pi_1^{\mathrm{top}}(X, x)(-1).$$

(ii) *Even without the assumption that H is connected, we have a canonical isomorphism*

$$\pi_2^{\rm top}(X,x)(-1) \xrightarrow{\sim} \ker\left[\pi_1^{\rm alg}(H) \xrightarrow{i_*} \pi_1^{\rm alg}(G)\right].$$

Observe that Theorem 1.3(i) is stronger than Corollary 1.2(ii) because in Theorem 1.3(i) we do not assume that $\mathrm{Pic}(G) = 0$. Moreover, our Theorem 1.3(ii) is stronger than Theorem 8.5(ii) of [6], where only $\pi_2^{\rm top}(X,x)(-1)$ modulo torsion was computed.

Remark 1.4. Assume that our pair (G,X) comes from some pair (G_0,X_0) defined over the field of real numbers $\mathbb{R}$: $(G,X) = (G_0,X_0)\times_{\mathbb{R}}\mathbb{C}$. Here G_0 is a connected linear algebraic group over $\mathbb{R}$, and X_0 is a real algebraic variety on which G_0 acts over $\mathbb{R}$. Let $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1,\gamma\}$ denote the Galois group of $\mathbb{C}$ over $\mathbb{R}$ where γ denotes complex conjugation. Assume that X_0 has an $\mathbb{R}$-point x_0, that is, a point $x_0 \in X_0(\mathbb{C}) = X(\mathbb{C})$ such that ${}^\gamma x_0 = x_0$. Since Γ acts on $X_0(\mathbb{C}) = X(\mathbb{C})$ continuously and preserves x_0, it naturally acts on the abelian groups $\pi_1^{\rm top}(X,x_0)$ and $\pi_2^{\rm top}(X,x_0)$. Write $\mathbb{Z}(1) = \pi_1^{\rm top}(\mathbb{G}_{\mathrm{m},\mathbb{R}}(\mathbb{C}),1)$ and define abelian groups $\pi_n^{\rm top}(X,x)(-1)$ for $n = 1,2$ by formula (1). Then Γ acts (nontrivially) on $\mathbb{Z}(1)$, and so it acts on $\pi_n^{\rm top}(X,x)(-1)$. Note that $\pi_n^{\rm top}(X,x_0)(-1)$ and $\pi_n^{\rm top}(X,x_0)$ are isomorphic as abelian groups, but in general they are not isomorphic as Γ-modules.

Let $H_0 = \mathrm{Stab}_{G_0}(x_0)$; then H_0 is a linear algebraic $\mathbb{R}$-group. Set $H = H_0\times_{\mathbb{R}}\mathbb{C}$. The Galois group Γ naturally acts on $\widehat{H}$ and on $\widehat{G}$. We can choose the torus $T_H^{\rm red}$ to be defined over $\mathbb{R}$; then Γ acts on $\pi_1^{\rm alg}(H)$, and similarly it acts on $\pi_1^{\rm alg}(G)$. Now the canonical isomorphisms of abelian groups in Theorem 1.1, Corollary 1.2, and Theorem 1.3 are actually isomorphisms of Γ-modules. ■

Related results were obtained by Demarche [9] over an algebraically closed field k *of arbitrary characteristic.*

In the case $k = \mathbb{C}$, a result of [9] gives the profinite completion of the group $\pi_1^{\rm top}(X,x)(-1)$. To be more precise, let $\mathbb{Z}^\wedge$ denote the profinite completion of $\mathbb{Z}$. Let k be an algebraically closed field *of characteristic* 0. Let X be a k-variety and $x \in X(k)$ be a k-point. We write $\pi_1^{\rm \acute{e}t}(X,x)$ for the étale fundamental group of (X,x). Set

$$\mathbb{Z}^\wedge(1) = \pi_1^{\rm \acute{e}t}(\mathbb{G}_{\mathrm{m},k},1), \qquad \pi_1^{\rm \acute{e}t}(X,x)(-1) = \mathrm{Hom}\big(\mathbb{Z}^\wedge(1), \pi_1^{\rm \acute{e}t}(X,x)\big).$$

Then Theorem 1.4 of [9] in the case of characteristic 0 says that under the hypotheses of our Theorem 1.1, there is a canonical isomorphism

$$\pi_1^{\rm \acute{e}t}(X,x)(-1) \xrightarrow{\sim} \mathrm{Ext}^0\big(\widehat{G}\to\widehat{H},\mathbb{Z}\big)\otimes_{\mathbb{Z}}\mathbb{Z}^\wedge.$$

When $k = \mathbb{C}$, the profinite group $\pi_1^{\rm \acute{e}t}(X,x)(-1)$ is isomorphic to the profinite completion $\pi_1^{\rm top}(X,x)(-1)^\wedge$ of the finitely generated abelian group $\pi_1^{\rm top}(X,x)(-1)$, and we obtain an isomorphism

$$\pi_1^{\rm top}(X,x)(-1)^\wedge \;\simeq\; \mathrm{Ext}^0\big(\widehat{G}\to\widehat{H},\mathbb{Z}\big)\otimes_{\mathbb{Z}}\mathbb{Z}^\wedge. \tag{3}$$

Since $\pi_1^{\rm top}(X,x)(-1)$ and $\mathrm{Ext}^0\big(\widehat{G}\to\widehat{H},\mathbb{Z}\big)$ are finitely generated abelian groups, it follows from (3) that they are isomorphic. However, from (3) we do not immediately obtain the canonical isomorphism of our Theorem 1.1. Thus in our case $k=\mathbb{C}$, we obtain a stronger result by elementary methods (we use only the exact sequence of homotopy groups associated to a fibration).

Observe that the constructions of the corresponding isomorphisms in Theorem 1.1 of the present article and in [9] are based on the same principles (these constructions first appeared in the preprint [4] by Demarche and the author). The difference is that we use technical tools of the classical (topological) homotopy theory, whereas Demarche [9] uses tools of the étale homotopy theory.

The plan of the article is as follows. In Section 2 we discuss the functor Ext^0. In Section 3, for a connected $\mathbb{C}$-group G we compare the conditions $\mathrm{Pic}(G)=0$ and $\pi_1^{\mathrm{top}}(G^{\mathrm{ssu}})=1$. In Section 4 we recall constructions of auxiliary groups and homogeneous spaces. In Section 5 we prove Theorem 1.1 and Corollary 1.2. In Section 6 we prove Theorem 1.3.

2. The functor Ext^0

In this section we consider the functor $\mathrm{Ext}^0(A^0 \xrightarrow{\alpha} A^1, \mathbb{Z})$ where $\alpha\colon A^0\to A^1$ is a homomorphism of abelian groups.

Let $K^\bullet$ be a bounded complex in an abelian category $\mathscr{A}$, and let B be an object of $\mathscr{A}$. Define

$$\mathrm{Ext}^i_{\mathscr{A}}(K^\bullet, B) := \mathrm{Hom}_{D^b(\mathscr{A})}(K^\bullet, B[i])$$

where $D^b(\mathscr{A})$ is the derived category of bounded complexes in $\mathscr{A}$, and $B[i]$ is the complex consisting of one object B in degree $-i$. If A is an object of $\mathscr{A}$, we have

$$\mathrm{Ext}^i_{\mathscr{A}}(A[0], B) = \mathrm{Hom}_{D^b(\mathscr{A})}(A[0], B[i]) =: \mathrm{Ext}^i_{\mathscr{A}}(A, B);$$

see Gelfand and Manin [10, Definition III.5.3]. By definition, $\mathrm{Ext}^0_{\mathscr{A}}(A,B) = \mathrm{Hom}_{\mathscr{A}}(A,B)$.

We consider the category $\mathcal{A}b$ of abelian groups and write Ext^i for $\mathrm{Ext}^i_{\mathcal{A}b}$. Let A be a finitely generated abelian group. We write A_{tors} for the torsion subgroup of A, and we set $A_{\mathrm{t.f.}} := A/A_{\mathrm{tors}}$; then $A_{\mathrm{t.f.}}$ is a torsion-free finitely generated abelian group, and therefore, it is a free abelian group. It is clear that

$$\mathrm{Ext}^0(A,\mathbb{Z}) = \mathrm{Hom}(A,\mathbb{Z}) = \mathrm{Hom}(A_{\mathrm{t.f.}},\mathbb{Z}).$$

Lemma 2.1. (well-known) *For a finitely generated abelian group A, we have*

$$\mathrm{Ext}^1(A,\mathbb{Z}) \cong \mathrm{Hom}(A_{\mathrm{tors}}, \mathbb{Q}/\mathbb{Z}).$$

Proof. From the injective resolution of $\mathbb{Z}$

$$0\to\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}\to 0$$

we obtain an exact sequence

$$\mathrm{Hom}(A,\mathbb{Q}) \to \mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}^1(A,\mathbb{Z}) \to \mathrm{Ext}^1(A,\mathbb{Q}) = 0,$$

whence

$$\mathrm{Ext}^1(A,\mathbb{Z}) \cong \mathrm{coker}\big[\mathrm{Hom}(A,\mathbb{Q}) \to \mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z})\big].$$

We have

$$\operatorname{coker}\bigl[\operatorname{Hom}(A,\mathbb{Q})\to\operatorname{Hom}(A,\mathbb{Q}/\mathbb{Z})\bigr]=\operatorname{coker}\bigl[\operatorname{Hom}(A_{\text{t.f.}},\mathbb{Q})\to\operatorname{Hom}(A,\mathbb{Q}/\mathbb{Z})\bigr]$$
$$=\operatorname{coker}\bigl[\operatorname{Hom}(A_{\text{t.f.}},\mathbb{Q}/\mathbb{Z})\to\operatorname{Hom}(A,\mathbb{Q}/\mathbb{Z})\bigr]\cong\operatorname{Hom}(A_{\text{tors}},\mathbb{Q}/\mathbb{Z}),$$

which proves the lemma. ■

Corollary 2.2. *For a homomorphism* $\alpha\colon A^0\to A^1$ *of finitely generated abelian groups, the abelian group* $\operatorname{Ext}^0(A^0\to A^1,\mathbb{Z})$ *fits into a canonical exact sequence*

$$\operatorname{Hom}(A^1,\mathbb{Z})\xrightarrow{\alpha^*}\operatorname{Hom}(A^0,\mathbb{Z})\to\operatorname{Ext}^0(A^0\to A^1,\mathbb{Z})$$
$$\to\operatorname{Hom}(A^1_{\text{tors}},\mathbb{Q}/\mathbb{Z})\xrightarrow{\alpha^*}\operatorname{Hom}(A^0_{\text{tors}},\mathbb{Q}/\mathbb{Z}).$$

Proof. The short exact sequence of complexes

$$0\to(0\to A^1)\to(A^0\to A^1)\to(A^0\to 0)\to 0$$

gives rise to an exact sequence

$$\operatorname{Hom}(A^1,\mathbb{Z})\xrightarrow{\alpha^*}\operatorname{Hom}(A^0,\mathbb{Z})\to\operatorname{Ext}^0(A^0\to A^1,\mathbb{Z})\to\operatorname{Ext}^1(A^1,\mathbb{Z})\to\operatorname{Ext}^1(A^0,\mathbb{Z}).$$

Applying Lemma 2.1, we obtain the exact sequence of the corollary. ■

Let $\alpha\colon A^0\to A^1$ be a homomorphism of finitely generated abelian groups where A^0 is torsion-free. We wish to compute $\operatorname{Ext}^0(A^0\to A^1,\mathbb{Z})$.

Choose a surjective homomorphism $\varphi^1\colon B^1\to A^1$ where B^1 is a finitely generated *torsion-free* abelian group. Set

$$B^0=A^0\times_{A^1}B^1=\bigl\{(a_0,b_1)\in A^0\times B^1\ \big|\ \alpha(a_0)=\varphi^1(b_1)\bigr\}.$$

We have a canonical homomorphism

$$\beta\colon B^0\to B^1,\quad (a_0,b_1)\mapsto b_1.$$

Lemma 2.3. $\operatorname{Ext}^0(A^0\to A^1,\mathbb{Z})\cong\operatorname{coker}\bigl[\operatorname{Hom}(B^1,\mathbb{Z})\xrightarrow{\beta^*}\operatorname{Hom}(B^0,\mathbb{Z})\bigr]$.

Proof. We have a commutative diagram

$$\begin{array}{ccc} B^0 & \xrightarrow{\beta} & B^1\\ {\scriptstyle\varphi^0}\big\downarrow & & \big\downarrow{\scriptstyle\varphi^1}\\ A^0 & \xrightarrow{\alpha} & A^1 \end{array}$$

where $\varphi^0(a_0,b_1)=a_0$. We obtain a morphism of complexes

$$\varphi=(\varphi^0,\varphi^1)\colon\ (B^0\to B^1)\ \longrightarrow\ (A^0\to A^1).$$

One can easily check that φ is a quasi-isomorphism, and therefore the induced homomorphism

$$\varphi^*\colon\operatorname{Ext}^0(A^0\to A^1,\mathbb{Z})\ \longrightarrow\ \operatorname{Ext}^0(B^0\to B^1,\mathbb{Z})$$

is an isomorphism. Since both A^0 and B^1 are torsion-free, we see that B^0 is torsion-free as well. By Corollary 2.2 we have an exact sequence

$$\operatorname{Hom}(B^1,\mathbb{Z})\xrightarrow{\alpha^*}\operatorname{Hom}(B^0,\mathbb{Z})\longrightarrow\operatorname{Ext}^0(B^0\to B^1,\mathbb{Z})\to 0,$$

and the lemma follows. ■

3. The Picard group and the topological fundamental group

In this section we show that for a connected linear algebraic group G over $\mathbb{C}$, the condition $\mathrm{Pic}(G)=0$ is equivalent to $\pi_1^{\mathrm{top}}(G^{\mathrm{ssu}})=1$.

Lemma 3.1. *Let G be a connected linear algebraic group over an algebraically closed field k of characteristic 0. Then there are canonical isomorphisms*

$$\mathrm{Pic}(G) \xrightarrow{\sim} \mathrm{Pic}(G^{\mathrm{ssu}}) \xrightarrow{\sim} \mathrm{Pic}(G^{\mathrm{ss}}).$$

Proof. A short exact sequence of connected linear algebraic groups over an algebraically closed field of characteristic 0

$$1 \to G' \to G \to G'' \to 1$$

gives rise to an exact sequence

$$\widehat{G'} \to \mathrm{Pic}(G'') \to \mathrm{Pic}(G) \to \mathrm{Pic}(G') \to 0; \tag{4}$$

see Sansuc [16, (6.11.2)]. Applying (4) to the short exact sequence

$$1 \to G^{\mathrm{ssu}} \to G \to G^{\mathrm{tor}} \to 1,$$

we obtain an exact sequence

$$\mathrm{Pic}(G^{\mathrm{tor}}) \to \mathrm{Pic}(G) \to \mathrm{Pic}(G^{\mathrm{ssu}}) \to 0.$$

Taking into account that $\mathrm{Pic}(G^{\mathrm{tor}})=0$ (see, for instance, [16, Lemma 6.9(ii)]), we obtain an isomorphism $\mathrm{Pic}(G) \xrightarrow{\sim} \mathrm{Pic}(G^{\mathrm{ssu}})$. Applying (4) to the short exact sequence (2), we obtain an exact sequence

$$\widehat{G^{\mathrm{u}}} \to \mathrm{Pic}(G^{\mathrm{ss}}) \to \mathrm{Pic}(G^{\mathrm{ssu}}) \to \mathrm{Pic}(G^{\mathrm{u}}).$$

Taking into account that $\mathrm{Pic}(G^{\mathrm{u}})=0$ (because in characteristic 0 the variety of the unipotent group G^{u} is isomorphic to the affine space $\mathrm{Lie}(G^{\mathrm{u}})$ via the logarithm map) and that $\widehat{G^{\mathrm{u}}}=0$, we obtain an isomorphism $\mathrm{Pic}(G^{\mathrm{ss}}) \xrightarrow{\sim} \mathrm{Pic}(G^{\mathrm{ssu}})$, which completes the proof. ∎

Lemma 3.2. *For G as in Lemma* 3.1*, we have* $\mathrm{Pic}(G)=0$ *if and only if G^{ss} is simply connected.*

Proof. By Lemma 3.1, we have $\mathrm{Pic}(G)\cong\mathrm{Pic}(G^{\mathrm{ss}})$. Set $\varkappa = \ker\left[G^{\mathrm{sc}} \to G^{\mathrm{ss}}\right]$, which is a finite abelian group. By [16, Lemma 6.9(iii)] we have

$$\mathrm{Pic}(G^{\mathrm{ss}}) \cong \mathrm{Hom}(\varkappa_G, \mathbb{C}^\times).$$

It follows that $\mathrm{Pic}(G)=0$ if and only if $\varkappa=\{1\}$, that is, G^{ss} is simply connected, as required. ∎

Lemma 3.3 (well-known). *Let G be a connected semisimple $\mathbb{C}$-group. Then the Lie group $G(\mathbb{C})$ is simply connected if and only if the algebraic group G is simply connected.*

Proof. Let G be a connected semisimple $\mathbb{C}$-group. Recall that G is called *simply connected* if any isogeny $G' \to G$, where G' is a semisimple group, is an isomorphism. Let $T \subset G$ be a maximal torus, and consider the root datum $\mathrm{RD}(G,T) = (X, X^\vee, R, R^\vee)$; see Springer [17, Sections 1 and 2]. A root datum is called *simply connected* if $X = P$ where $X = \widehat{T}$ is the character group of T, and P is the weight lattice. Then the algebraic group G is simply connected if and only if $\mathrm{RD}(G,T)$ is simply connected; see [17, Section 2.15]. On the other hand, also the Lie group $G(\mathbb{C})$ is simply connected if and only if the root datum $\mathrm{RD}(G,T)$ is simply connected; see [12, Chapter 3, Section 2.4, Theorem 2.6] or Conrad [8, Proposition D.4.1]. The lemma follows. ■

Corollary 3.4. *For a connected linear algebraic group G over $\mathbb{C}$, the following assertions are equivalent:*

(i) $\mathrm{Pic}(G) = 0$*;*

(ii) *the algebraic group G^{ss} is simply connected;*

(iii) $\pi_1^{\mathrm{top}}(G^{\mathrm{ss}}) = 1$*;*

(iv) $\pi_1^{\mathrm{top}}(G^{\mathrm{ssu}}) = 1$.

Proof. By Lemma 3.2, we have (i) ⇔ (ii).

By Lemma 3.3, the Lie group $G^{\mathrm{ss}}(\mathbb{C})$ is simply connected if and only if the algebraic group G^{ss} is simply connected. Thus (ii) ⇔ (iii).

From the short exact sequence (2) we obtain an isomorphism

$$\pi_1^{\mathrm{top}}(G^{\mathrm{ssu}}) \xrightarrow{\sim} \pi_1^{\mathrm{top}}(G^{\mathrm{ss}}).$$

Thus (iii) ⇔ (iv), which completes the proof of the lemma. ■

4. Auxiliary pairs

In this section we recall the constructions of auxiliary groups and homogeneous spaces that we shall need for our proof of Theorem 1.1. For a homogeneous space X of an algebraic k-group G satisfying the hypotheses of Theorem 1.1, we construct homogeneous spaces Y, Z, and W of certain k-groups (G_Y, G_Z, and G_W, respectively), with morphisms of pairs

$$(G, X) \leftarrow (G_Y, Y) \to (G_Z, Z) \to (G_W, W)$$

that will permit us to prove Theorem 1.1 successively for W, Z, Y, and finally for X. These constructions already appeared in the preprint [4] and were published in [9]; for the reader's convenience we repeat them here.

Construction of the homogeneous space Y. Let X be a homogeneous space of a connected linear algebraic k-group defined over an algebraically closed field k of characteristic 0. Assume that $\mathrm{Pic}(G) = 0$.

Choose a k-point $x \in X(k)$. We denote by H the stabilizer of x in G. We do not assume that H is connected.

Let H^{mult} denote the largest quotient group of H that is a group of multiplicative type. Set $H^{\mathrm{ker.char}} = \ker\left[H \to H^{\mathrm{mult}}\right]$. We have a canonical homomorphism $H^{\mathrm{mult}} \to G^{\mathrm{tor}}$, which in general is not injective.

We choose an embedding $j\colon H^{\text{mult}} \hookrightarrow Q$ of H^{mult} into a k-torus Q. Consider the embedding

$$j_*\colon H \to G \times_k Q, \quad h \mapsto (h, j(m_H(h))) \text{ for } h \in H$$

where $m_H\colon H \to H^{\text{mult}}$ is the canonical epimorphism. Set

$$G_Y = G \times_k Q, \quad H_Y = j_*(H) \subset G_Y, \quad Y = G_Y/H_Y, \quad y = 1 \cdot H_Y \in Y(k).$$

The projection $\pi\colon G_Y = G \times Q \to G$ satisfies $\pi(H_Y) = H$, and it induces a G_Y-equivariant map $\pi_*\colon Y \to X$ such that $\pi_*(y) = x$. One can easily see that Y is a torsor over X under the torus Q. We obtain a morphism of pairs

$$(G_Y, Y) \to (G, X).$$

Observe that the homomorphism $H_Y^{\text{mult}} \to G_Y^{\text{tor}}$ is injective, and therefore

$$H_Y \cap G_Y^{\text{ssu}} = (H_Y)^{\text{ker.char}} \cong H^{\text{ker.char}}.$$

Construction of the homogeneous space Z. Set

$$G_Z = G_Y^{\text{tor}} = G_Y/G_Y^{\text{ssu}} \quad \text{where} \quad G_Y^{\text{ssu}} := \ker\left[G_Y \to G_Y^{\text{tor}}\right].$$

We have a canonical homomorphism $\mu\colon G_Y \to G_Z$. Then G_Z is a k-torus and we have $\widehat{G_Z} = \widehat{G_Y}$.

The inclusion $i\colon H \hookrightarrow G$ induces a homomorphism $i^{\text{mult}}\colon H^{\text{mult}} \to G^{\text{mult}} = G^{\text{tor}}$. We obtain an embedding

$$\iota\colon H^{\text{mult}} \hookrightarrow G^{\text{tor}} \times_k Q, \quad h \mapsto (i^{\text{mult}}(h), j(h)) \text{ for } h \in H^{\text{mult}}.$$

Set

$$Z = Y/G_Y^{\text{ssu}} = (G^{\text{tor}} \times_k Q)/\iota(H^{\text{mult}});$$

then we have a G_Y-equivariant map $\mu_*\colon Y \to Z$ whose fiber over the k-point $z := \mu_*(y) \in Z(k)$ is isomorphic to

$$G_Y^{\text{ssu}}/(H_Y \cap G_Y^{\text{ssu}}) \cong G^{\text{ssu}}/H^{\text{ker.char}}.$$

The variety Z is a homogeneous space of G_Z with stabilizer

$$H_Z = H_Y^{\text{mult}} \subset G_Y^{\text{tor}} = G_Z.$$

Note that

$$\widehat{H_Z} = \widehat{H_Y^{\text{mult}}} = \widehat{H_Y}.$$

We have a natural morphism of pairs

$$(G_Y, Y) \to (G_Z, Z).$$

Construction of the homogeneous space W. We set $G_W = G_Z/H_Z$, $W = Z$, $w = z$; then W is a principal homogeneous space of the torus G_W. There is a natural morphism of pairs

$$(G_Z, Z) \to (G_W, W).$$

5. Proof of Theorem 1.1

In this section we prove Theorem 1.1 and Corollary 1.2. We use lemmas of Section 3 and the constructions of Section 4.

Step 1. First we treat the case of a principal homogeneous space W of a k-torus G_W. Let $w \in W(\mathbb{C})$ be a $\mathbb{C}$-point. The map $G_W \to W$ defined by $g \mapsto g \cdot w$ is an isomorphism of $\mathbb{C}$-varieties, and we have an induced isomorphism of groups

$$\pi_1^{\mathrm{top}}(G_W, 1) \xrightarrow{\sim} \pi_1^{\mathrm{top}}(W, w).$$

Since G_W is a torus, the group $\pi_1^{\mathrm{top}}(G_W)$ is abelian, and therefore the group $\pi_1^{\mathrm{top}}(W, w)$ is abelian as well. We have an induced isomorphism of abelian groups

$$\pi_1^{\mathrm{top}}(G_W, 1)(-1) \xrightarrow{\sim} \pi_1^{\mathrm{top}}(W, w)(-1).$$

Since $$\pi_1^{\mathrm{top}}(G_W, 1)(-1) = (G_W)_* \cong \mathrm{Hom}\big(\widehat{G_W}, \mathbb{Z}\big) = \mathrm{Ext}^0\big(\widehat{G_W}, \mathbb{Z}\big),$$

we obtain a canonical isomorphism

$$\pi_1^{\mathrm{top}}(W, w)(-1) \xrightarrow{\sim} \mathrm{Ext}^0\big(\widehat{G_W}, \mathbb{Z}\big).$$

This proves Theorem 1.1 for (G_W, W).

Step 2. Assume that we have a homomorphism of $\mathbb{C}$-tori $\gamma_\alpha\colon G_{W'} \to G_W$ and a γ_α-equivariant map of principal homogeneous spaces $\alpha\colon W' \to W$ sending a $\mathbb{C}$-point $w' \in W'(\mathbb{C})$ to a $\mathbb{C}$-point $w \in W(\mathbb{C})$. Then the following diagram clearly commutes:

$$\begin{array}{ccc} \pi_1^{\mathrm{top}}(W', w')(-1) & \xrightarrow{\alpha_*} & \pi_1^{\mathrm{top}}(W, w)(-1) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Ext}^0\big(\widehat{G_{W'}}, \mathbb{Z}\big) & \xrightarrow{\gamma_{\alpha *}} & \mathrm{Ext}^0\big(\widehat{G_W}, \mathbb{Z}\big) \end{array} \tag{5}$$

where the vertical arrows are canonical isomorphisms of Step 1.

We have $Z = W$, whence $\pi_1^{\mathrm{top}}(Z)$ is an abelian group. Moreover, $G_Z/H_Z = G_W$, and the evident morphism of complexes

$$\widehat{G_W} \to \big[\widehat{G_Z} \to \widehat{H_Z}\big\rangle$$

is a quasi-isomorphism, whence

$$\pi_1^{\mathrm{top}}(Z, z)(-1) = \pi_1^{\mathrm{top}}(W, w)(-1) \cong \mathrm{Ext}^0\big(\widehat{G_W}, \mathbb{Z}\big) \cong \mathrm{Ext}^0\big(\big[\widehat{G_Z} \to \widehat{H_Z}\big\rangle, \mathbb{Z}\big),$$

and we obtain a canonical isomorphism

$$\pi_1^{\mathrm{top}}(Z, z)(-1) \xrightarrow{\ \sim\ } \mathrm{Ext}^0\big(\big[\widehat{G_Z} \to \widehat{H_Z}\big\rangle, \mathbb{Z}\big).$$

This proves 1.1 for (G_Z, Z).

Step 3. There is a fibration $G^{\mathrm{ssu}}(\mathbb{C}) \to G^{\mathrm{ssu}}(\mathbb{C})/H^{\mathrm{ker.char}}(\mathbb{C})$ *with connected fiber* $H^{\mathrm{ker.char}}(\mathbb{C})$, and hence a fibration exact sequence

$$1 = \pi_1^{\mathrm{top}}(G^{\mathrm{ssu}}) \longrightarrow \pi_1^{\mathrm{top}}(G^{\mathrm{ssu}}/H^{\mathrm{ker.char}}) \longrightarrow \pi_0(H^{\mathrm{ker.char}}) = 1.$$

In this sequence, we have $\pi_1^{\rm top}(G^{\rm ssu})=1$ because $\mathrm{Pic}(G)=0$; see Corollary 3.4. We see that $\pi_1^{\rm top}(G^{\rm ssu}/H^{\rm ker.char})=1$. But we have a fibration $\mu_*\colon Y(\mathbb{C})\to Z(\mathbb{C})$ with fiber $G^{\rm ssu}(\mathbb{C})/H^{\rm ker.char}(\mathbb{C})$, and hence a fibration exact sequence

$$1=\pi_1^{\rm top}(G^{\rm ssu}/H^{\rm ker.char})\longrightarrow \pi_1^{\rm top}(Y,y)\xrightarrow{\mu_*}\pi_1^{\rm top}(Z,z)\longrightarrow \pi_0(G^{\rm ssu}/H^{\rm ker.char})=1.$$

It follows that the homomorphism $\pi_1^{\rm top}(Y,y)\xrightarrow{\mu_*}\pi_1^{\rm top}(Z,z)$ is an isomorphism, whence the group $\pi_1^{\rm top}(Y,y)$ is abelian and we have an isomorphism of abelian groups

$$\pi_1^{\rm top}(Y,y)(-1)\xrightarrow{\mu_*}\pi_1^{\rm top}(Z,z)(-1).$$

Since $\widehat{G_Y}=\widehat{G_Z}$ and $\widehat{H_Y}=\widehat{H_Z}$, we deduce Theorem 1.1 for (G_Y,Y) from Theorem 1.1 for (G_Z,Z).

Step 4. We have a torsor $\pi_*\colon Y\to X$ under the torus Q, whence we obtain an exact sequence of groups

$$\pi_1^{\rm top}(Q,1)\xrightarrow{\lambda_*}\pi_1^{\rm top}(Y,y)\xrightarrow{\pi_*}\pi_1^{\rm top}(X,x)\to 1$$

where the arrow λ_* is induced by the map

$$\lambda\colon Q\to Y,\quad q\mapsto q\cdot y\ \text{ for }\ q\in Q.$$

Since the group $\pi_1^{\rm top}(Y,y)$ is abelian, so is the group $\pi_1^{\rm top}(X,x)$, and we obtain an exact sequence of abelian groups

$$\pi_1^{\rm top}(Q,1)(-1)\xrightarrow{\lambda_*}\pi_1^{\rm top}(Y,y)(-1)\xrightarrow{\pi_*}\pi_1^{\rm top}(X,x)(-1)\to 0.$$

We have a short exact sequence of complexes

$$0\to\big(\widehat{G}\to\widehat{H}\big)\longrightarrow\big(\widehat{G_Y}\to\widehat{H}\big)\longrightarrow\big(\widehat{Q}\to 0\big)\to 0,$$

whence we obtain an exact sequence

$$\mathrm{Ext}^0\big(\widehat{Q},\mathbb{Z}\big)\longrightarrow\mathrm{Ext}^0\big(\widehat{G_Y}\to\widehat{H},\mathbb{Z}\big)\longrightarrow\mathrm{Ext}^0\big(\widehat{G}\to\widehat{H},\mathbb{Z}\big)\longrightarrow\mathrm{Ext}^1\big(\widehat{Q},\mathbb{Z}\big)=0$$

(because by Lemma 2.1 we have $\mathrm{Ext}^1\big(\widehat{Q},\mathbb{Z}\big)=\mathrm{Hom}\big(\widehat{Q}_{\rm tors},\mathbb{Q}/\mathbb{Z}\big)=0$). We obtain a diagram with exact rows

$$\begin{array}{ccccccc}
\pi_1^{\rm top}(Q,1)(-1) & \xrightarrow{\lambda_*} & \pi_1^{\rm top}(Y,y)(-1) & \xrightarrow{\pi_*} & \pi_1^{\rm top}(X,x)(-1) & \longrightarrow & 0\\
\downarrow\cong & \boxed{1} & \downarrow\cong & \boxed{2} & \vdots & & \\
\mathrm{Ext}^0\big(\widehat{Q},\mathbb{Z}\big) & \longrightarrow & \mathrm{Ext}^0\big(\widehat{G_Y}\to\widehat{H_Y},\mathbb{Z}\big) & \longrightarrow & \mathrm{Ext}^0\big(\widehat{G}\to\widehat{H},\mathbb{Z}\big) & \longrightarrow & 0\,.
\end{array}\tag{6}$$

We show that rectangle $\boxed{1}$ commutes. Consider the diagram

$$\begin{array}{ccccccc}
\pi_1^{\rm top}(Q,1)(-1) & \xrightarrow{\lambda_*} & \pi_1^{\rm top}(Y,y)(-1) & \xrightarrow{\cong} & \pi_1^{\rm top}(Z,z)(-1) & \xrightarrow{\cong} & \pi_1^{\rm top}(W,w)(-1)\\
\downarrow\cong & \boxed{1} & \downarrow\cong & \boxed{3} & \downarrow\cong & \boxed{4} & \downarrow\cong\\
\mathrm{Ext}^0\big(\widehat{Q},\mathbb{Z}\big) & \longrightarrow & \mathrm{Ext}^0\big(\widehat{G_Y}\to\widehat{H_Y},\mathbb{Z}\big) & \xrightarrow{\cong} & \mathrm{Ext}^0\big(\widehat{G_Z}\to\widehat{H_Z},\mathbb{Z}\big) & \xrightarrow{\cong} & \mathrm{Ext}^0\big(\widehat{G_W},\mathbb{Z}\big).
\end{array}$$

By construction, rectangles $\boxed{3}$ and $\boxed{4}$ commute. The commutative diagram (5) shows that the big rectangle $\boxed{1}\cup\boxed{3}\cup\boxed{4}$ commutes. It follows that rectangle $\boxed{1}$ commutes.

In the diagram with exact rows (6), rectangle $\boxed{1}$ commutes, which permits one to define the dotted arrow so that rectangle $\boxed{2}$ commutes. Thus we obtain an isomorphism

$$\pi_1^{\mathrm{top}}(X,x)(-1) \xrightarrow{\sim} \mathrm{Ext}^0\big(\widehat{G}\to\widehat{H},\mathbb{Z}\big), \tag{7}$$

which a priori might depend on the embedding $j\colon H^{\mathrm{mult}}\hookrightarrow Q$.

When constructing the torsor $Y\to X$ in Section 4, we constructed it from an embedding $j\colon H^{\mathrm{mult}}\hookrightarrow Q$. If we choose another embedding $j'\colon H^{\mathrm{mult}}\hookrightarrow Q'$, we obtain another torsor $Y'\to X$ under Q'. Set $Q''=Q\times_k Q'$ and denote by $j''\colon H^{\mathrm{mult}}\hookrightarrow Q''$ the diagonal embedding. We obtain a torsor $Y''\to X$ under Q'' dominating both Y and Y', and from this fact we can easily see that the isomorphism (7) does not depend on the choice of the embedding $j\colon H^{\mathrm{mult}}\hookrightarrow Q$. This completes the proof of Theorem 1.1. ∎

Proof of Corollary 1.2. The short exact sequence of complexes

$$0\to\big[0\to\widehat{H}\big\rangle\longrightarrow\big[\widehat{G}\to\widehat{H}\big\rangle\longrightarrow\big[\widehat{G}\to 0\big\rangle\to 0$$

induces a long exact sequence

$$\begin{aligned}\mathrm{Ext}^0\big(\widehat{H},\mathbb{Z}\big)\to\mathrm{Ext}^0\big(\widehat{G},\mathbb{Z}\big)&\to\mathrm{Ext}^0\big(\widehat{G}\to\widehat{H},\mathbb{Z}\big)\\ &\to\mathrm{Ext}^1\big(\widehat{H},\mathbb{Z}\big)\to\mathrm{Ext}^1\big(\widehat{G},\mathbb{Z}\big).\end{aligned}\tag{8}$$

We have $\mathrm{Ext}^0\big(\widehat{H},\mathbb{Z}\big)=\mathrm{Hom}\big(\widehat{H},\mathbb{Z}\big)$ and $\mathrm{Ext}^0\big(\widehat{G},\mathbb{Z}\big)=\mathrm{Hom}\big(\widehat{G},\mathbb{Z}\big)$. By Lemma 2.1, we have $\mathrm{Ext}^1\big(\widehat{G},\mathbb{Z}\big)\cong\mathrm{Hom}\big(\widehat{G}_{\mathrm{tors}},\mathbb{Q}/\mathbb{Z}\big)=0$ and

$$\mathrm{Ext}^1\big(\widehat{H},\mathbb{Z}\big)\cong\mathrm{Hom}\big(\widehat{H}_{\mathrm{tors}},\mathbb{Q}/\mathbb{Z}\big).$$

By Theorem 1.1 we may write $\pi_1^{\mathrm{top}}(X,x)(-1)$ instead of $\mathrm{Ext}^0\big(\widehat{G}\to\widehat{H},\mathbb{Z}\big)$ in (8). Thus we obtain from (8) an exact sequence

$$\mathrm{Hom}\big(\widehat{H},\mathbb{Z}\big)\xrightarrow{i_*}\mathrm{Hom}\big(\widehat{G},\mathbb{Z}\big)\to\pi_1^{\mathrm{top}}(X,x)(-1)\to\mathrm{Hom}\big(\widehat{H}_{\mathrm{tors}},\mathbb{Q}/\mathbb{Z}\big)\to 0,$$

which proves assertion (i) of Corollary 1.2. If H is connected, then $\widehat{H}=\widehat{H^{\mathrm{tor}}}$ is torsion-free, and assertion (ii) follows from assertion (i). ∎

6. Proof of Theorem 1.3.

Let G and H be as in Theorem 1.3. Let $T_H^{\mathrm{red}}\subseteq H^{\mathrm{red}}$ and $T_H^{\mathrm{sc}}\subseteq H^{\mathrm{sc}}$ be compatible maximal tori. We recall the construction of an isomorphism

$$\mathrm{ev}_H\colon\ \pi_1^{\mathrm{alg}}(H):=\pi_1^{\mathrm{alg}}(H^{\mathrm{red}}):=\mathrm{coker}\big[(T_H^{\mathrm{sc}})_*\to(T_H^{\mathrm{red}})_*\big]\xrightarrow{\sim}\pi_1^{\mathrm{top}}(H)(-1)\tag{9}$$

from [2, Prop. 1.11]. Here we write $\pi_1^{\mathrm{top}}(H)(-1)$ for $\pi_1^{\mathrm{top}}(H^0)(-1)$. A cocharacter $\nu\in(T_H^{\mathrm{red}})_*$ gives a homomorphism

$$\mathbb{G}_{\mathrm{m},\mathbb{C}}\to T_{H^{\mathrm{red}}}\to H^{\mathrm{red}}$$

and induces a homomorphism

$$\nu_*\colon\pi_1^{\mathrm{top}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})\to\pi_1^{\mathrm{top}}(H^{\mathrm{red}})\cong\pi_1^{\mathrm{top}}(H).$$

If ν comes from $(T_H^{\rm sc})_*$, then ν_* factors via $\pi_1^{\rm top}(H^{\rm sc})=1$ and hence is trivial. Thus we obtain the desired homomorphism (9)

$$\pi_1^{\rm alg}(H^{\rm red}) := \operatorname{coker}\big[(T_H^{\rm sc})_*\to (T_H^{\rm red})_*\big] \\ \longrightarrow \operatorname{Hom}\big(\pi_1^{\rm top}(\mathbb{G}_{{\rm m},\mathbb{C}}),\pi_1^{\rm top}(H)\big) =: \pi_1^{\rm top}(H)(-1).$$

By [2, Prop. 1.11] this homomorphism is an isomorphism. Similarly we construct an isomorphism

$$\mathrm{ev}_G\colon \pi_1^{\rm alg}(G^{\rm red}) \xrightarrow{\sim} \pi_1^{\rm top}(G)(-1). \tag{10}$$

Since the group $\pi_1^{\rm top}(G)$ is isomorphic (non-canonically) to the abelian group $\pi_1^{\rm alg}(G)$, it is abelian. Note that the following diagram commutes:

$$\begin{array}{ccc} \pi_1^{\rm alg}(H) & \xrightarrow{i_*} & \pi_1^{\rm alg}(G) \\ {\scriptstyle \mathrm{ev}_H}\downarrow\cong & & \cong\downarrow{\scriptstyle \mathrm{ev}_G} \\ \pi_1^{\rm top}(H)(-1) & \xrightarrow{i_*} & \pi_1^{\rm top}(G)(-1). \end{array}$$

It is well known that for a connected linear algebraic group G over $\mathbb{C}$, we have $\pi_2^{\rm top}(G)=0$. Indeed, we may assume that G is simple and simply connected. Let K be a maximal compact subgroup of $G(\mathbb{C})$. We have $\pi_2^{\rm top}(K)=0$ by Cartan's theorem; see Borel [1, Theorem 1(2)]. The Cartan decomposition for $G(\mathbb{C})$ shows that $G(\mathbb{C})$ is homotopically equivalent to K; see [12, Chapter 7, Section 3.2, Corollaries 1 and 5 of Theorem 3.2] or Knapp [14, Theorem 6.31(c),(g)]. It follows that $\pi_2^{\rm top}(G)\cong\pi_2^{\rm top}(K)=0$.

We have a fibration $\quad G\to X,\quad g\mapsto g\cdot x\;$ for $g\in G$

with fiber H, where we write G for $G(\mathbb{C})$, H for $H(\mathbb{C})$, and so on. This fibration gives rise to a homotopy exact sequence

$$0=\pi_2^{\rm top}(G)\to\pi_2^{\rm top}(X,x)\to\pi_1^{\rm top}(H)\to\pi_1^{\rm top}(G)\to\pi_1^{\rm top}(X,x)\to\pi_0^{\rm top}(H); \tag{11}$$

see, for instance, Hatcher [13, Theorem 4.41].

First assume that H is connected, that is, $\pi_0^{\rm top}(H)=1$. Since in the sequence (11) the group $\pi_1^{\rm top}(G)$ is abelian, so is $\pi_1^{\rm top}(X,x)$. Applying to a part of sequence (11) the twisting functor $\operatorname{Hom}(\pi_1^{\rm top}(\mathbb{G}_{{\rm m},\mathbb{C}}),\,\cdot\,)$, we obtain an exact sequence

$$\pi_1^{\rm top}(H)(-1)\xrightarrow{i_*}\pi_1^{\rm top}(G)(-1)\to\pi_1^{\rm top}(X,x)(-1)\to 0,$$

which by (9) and (10) gives Theorem 1.3(i).

Now we do not assume that H is connected. Applying to a part of sequence (11) the twisting functor $\operatorname{Hom}(\pi_1^{\rm top}(\mathbb{G}_{{\rm m},\mathbb{C}}),\,\cdot\,)$, we obtain an exact sequence

$$0\to\pi_2^{\rm top}(X,x)(-1)\to\pi_1^{\rm top}(H)(-1)\xrightarrow{i_*}\pi_1^{\rm top}(G)(-1),$$

which by (9) and (10) gives Theorem 1.3(ii). ■

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References

[1] A. Borel: *Groupes d'homotopie des groupes de Lie I*, in: Séminaire Henri Cartan 2 (1949-1950), exp. no. 12.

[2] M. Borovoi: *Abelian Galois Cohomology of Reductive Groups*, Mem. Amer. Math. Soc. 132/626 (1998).

[3] M. Borovoi, Y. Cornulier: *Conjugate complex homogeneous spaces with non-isomorphic fundamental groups*, C. R. Math. Acad. Sci. Paris 353 (2015) 1001–1005.

[4] M. Borovoi, C. Demarche: *Le groupe fondamental d'un espace homogène d'un groupe algébrique linéaire*, arXiv: 1301.1046 (2013).

[5] M. Borovoi, C. D. González-Avilés: *The algebraic fundamental group of a reductive group scheme over an arbitrary base scheme*, Cent. Eur. J. Math. 12 (2014) 545–558.

[6] M. Borovoi, J. van Hamel: *Extended equivariant Picard complexes and homogeneous spaces*, Transform. Groups 17 (2012) 51–86.

[7] J.-L. Colliot-Thélène: *Résolutions flasques des groupes linéaires connexes*, J. Reine Angew. Math. 618 (2008) 77–133.

[8] B. Conrad: *Reductive group schemes*, in: *Autour des schémas en groupes, Vol. I*, Panoramas et Synthèses 42-43, Soc. Math. France, Paris (2014) 93–444.

[9] C. Demarche: *Le groupe fondamental étale d'un espace homogène d'un groupe algébrique linéaire*, Math. Ann. 368 (2017) 339–365.

[10] S. I. Gelfand, Yu. I. Manin: *Methods of Homological Algebra*, Springer, Berlin (1996).

[11] C. D. González-Avilés: *Flasque resolutions of reductive group schemes*, Cent. Eur. J. Math. 11 (2013) 1159–1176.

[12] V. V. Gorbatsevich, A. L. Onishchik, E. B. Vinberg: *Lie Groups and Lie Algebras. III: Structure of Lie Groups and Lie Algebras*, Encyclopaedia of Mathematical Sciences 41, Springer, Berlin (1994).

[13] A. Hatcher: *Algebraic Topology*, Cambridge University Press, Cambridge (2002).

[14] A. W. Knapp: *Lie Groups Beyond an Introduction*, 2nd ed., Progress in Mathematics 140, Birkhäuser, Boston (2002).

[15] A. S. Merkurjev: *K-theory and algebraic groups*, in: European Congress of Mathematics, Vol. II, Budapest 1996, Progress in Mathematics 169, Birkhäuser, Basel (1998) 43–72.

[16] J.-J. Sansuc: *Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres*, J. Reine Angew. Math. 327 (1981) 12–80.

[17] T. A. Springer: *Reductive groups*, in: *Automorphic Forms, Representations and L-Functions*, Sympos. Pure Math. Oregon State Univ., Corvallis 1977, Proc. Sympos. Pure Math. 33, Part 1, American Mathematical Society, Providence (1979) 3–27.

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