Four-Dimensional Lie Algebras Revisited

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Abstract. The projective variety of Lie algebra structures on a 4-dimensional complex vector space has four irreducible components of dimension 11. We compute their prime ideals in the polynomial ring in 24 variables. By listing their degrees and Hilbert polynomials, we correct an earlier publication and we solve a problem raised by Kirillov and Neretin in 1987.

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1. Introduction

We examine the projective variety Lie_4 whose points are the Lie algebra structures on the vector space $\mathbb{C}^4$, with standard basis $\{e_1, e_2, e_3, e_4\}$. Each point in Lie_4 is given by a matrix

$$
A \;=\; \begin{bmatrix} a_{121} & a_{131} & a_{141} & a_{231} & a_{241} & a_{341} \\ a_{122} & a_{132} & a_{142} & a_{232} & a_{242} & a_{342} \\ a_{123} & a_{133} & a_{143} & a_{233} & a_{243} & a_{343} \\ a_{124} & a_{134} & a_{144} & a_{234} & a_{244} & a_{344} \end{bmatrix}. \tag{1}
$$

The 24 matrix entries are homogeneous coordinates on $\mathbb{P}^{23}$. They represent the structure constants of a Lie algebra, by setting $[e_i, e_j] = a_{ij1}e_1 + a_{ij2}e_2 + a_{ij3}e_3 + a_{ij4}e_4$ for $1 \le i \le j \le 4$. Since $[e_i, e_j] = -[e_j, e_i]$, we fix the conventions $a_{iik} = 0$ and $a_{jik} = -a_{ijk}$. The Jacobi identity

$$
[e_i, [e_j, e_k]] \;+\; [e_j, [e_k, e_i]] \;+\; [e_k, [e_i, e_j]] \;\;=\;\; 0 \tag{2}
$$

yields 16 quadratic equations in the 24 unknowns a_{ijk}, shown explicitly in Section 2. These equations cut out the variety $\mathrm{Lie}_4 \subset \mathbb{P}^{23}$. Our discussion revolves around the following result.

Theorem 1.1. *The variety* Lie_4 *has four irreducible components* C_1, C_2, C_3, C_4 *in the ambient space* $\mathbb{P}^{23}$. *Each of these components has dimension* 11. *Their degrees are* $55, 361, 121, 295$.

The article [7] had reported the degrees $660, 57, 121, 195$, where the computation used methods from intersection theory. However, three of those numbers are incorrect. Thus the present paper also serves as an erratum for [7]. The corrections will be discussed in Section 5.

Our study grew out of a student summer project by the third author at MPI Leipzig. When applying the software `HomotopyContinuation.jl` [2] to the 16 quadrics defining Lie_4, she discovered numerically that the degree equals 832 and not $1033 = 660 + 57 + 121 + 195$.

We here use computer algebra to determine the prime ideals for C_1, C_2, C_3, C_4. Their generators are presented in Sections 3 and 4. The latter incorporates the action of $\mathrm{GL}(4, \mathbb{C})$. Theorem 3.1 answer a question raised at the end of [6] by presenting the Hilbert polynomials.

2. Sixteen quadrics

Two Lie algebra structures on $\mathbb{C}^4$ are isomorphic if they are in the same orbit under the action of the general linear group $G = \mathrm{GL}(4, \mathbb{C})$. In this paper we use representation theory of the group G at the level of [8, §10.2]. The matrix entries in (1) generate the polynomial ring $\mathbb{C}[A] = \mathbb{C}[a_{121}, a_{122}, \ldots, a_{344}]$. We write the G-action by matrix multiplication as follows:

$$G \times \mathbb{C}[A] \to \mathbb{C}[A], \ (g, A) \mapsto g^{-1} \cdot A \cdot \wedge_2(g). \tag{3}$$

The entries of the 6×6 matrix $\wedge_2(g)$ are the 2×2 minors of g, with rows and columns labeled to match (1). The $\mathbb{Z}^4$-grading of the polynomial ring $\mathbb{C}[A]$ induced by (3) equals

$$\deg(a_{ijk}) \quad = \quad e_i + e_j - e_k.$$

The space of linear forms in $\mathbb{C}[A]$ decomposes into two irreducible representations:

$$\mathbb{C}[A]_1 \quad = \quad S_{(1,0,0,0)}(\mathbb{C}^4) \ \oplus \ S_{(1,1,0,-1)}(\mathbb{C}^4) \quad \simeq \quad \mathbb{C}^4 \oplus \mathbb{C}^{20}. \tag{4}$$

Explicitly, with highest weight vectors underlined, the two G-invariant subspaces are

$S_{(1,0,0,0)}(\mathbb{C}^4) = \mathbb{C}\{ \underline{a_{122} + a_{133} + a_{144}}, -a_{121} + a_{233} + a_{244}, -a_{131} - a_{232} + a_{344}, -a_{141} - a_{242} - a_{343} \},$

$S_{(1,1,0,-1)}(\mathbb{C}^4) = \mathbb{C}\{ \underline{a_{124}}, a_{123}, a_{132}, a_{134}, a_{142}, a_{143}, a_{231}, a_{234}, a_{241}, a_{243}, a_{341}, a_{342}, a_{122} - a_{133},$

$\qquad a_{133} - a_{144}, a_{121} + a_{233}, a_{233} - a_{244}, a_{131} - a_{232}, a_{232} + a_{344}, a_{141} - a_{242}, a_{242} - a_{343} \}.$

The 16 quadrics in $\mathbb{C}[A]_2$ that encode the Jacobi identity (2) are organized into a 4×4 matrix $\Theta = (\theta_{ij})$. The columns are labeled e_4, e_3, e_2, e_1, in this order. The rows, labeled $123, -\mathbf{124}, 134, -234$, contain the coefficients of (2). For instance, in the second row of Θ,

$$-[e_{\mathbf{1}}, [e_{\mathbf{2}}, e_{\mathbf{4}}]] - [e_2, [e_4, e_1]] - [e_4, [e_1, e_2]] \quad = \quad \theta_{21}e_4 + \theta_{22}e_3 + \theta_{23}e_2 + \theta_{24}e_1.$$

The entries in the upper left of our 4×4 matrix Θ are

$$\theta_{11} = a_{124}a_{131} - a_{121}a_{134} + a_{124}a_{232} + a_{134}a_{233} - a_{122}a_{234} - a_{133}a_{234} + a_{144}a_{234}$$
$$- a_{134}a_{244} + a_{124}a_{344},$$
$$\theta_{12} = a_{123}a_{131} - a_{121}a_{133} + a_{123}a_{232} - a_{122}a_{233} + a_{143}a_{234} - a_{134}a_{243} + a_{124}a_{343},$$
$$\theta_{21} = -a_{124}a_{141} + a_{121}a_{144} + a_{143}a_{234} - a_{124}a_{242} - a_{134}a_{243} + a_{122}a_{244} + a_{123}a_{344}, \ \text{etc.}$$

Viewed invariantly, our matrix represents the linear map given by the following composition:

$$\Theta : \wedge_3\mathbb{C}^4 \hookrightarrow \wedge_2\mathbb{C}^4 \otimes \mathbb{C}^4 \to \mathbb{C}^4 \otimes \mathbb{C}^4 \to \wedge_2\mathbb{C}^4 \to \mathbb{C}^4.$$

The second and fourth map are given by Lie algebra multiplication, i.e. by the matrix A. The Jacobi identity states $\Theta = 0$. Hence $\mathrm{Lie}_4 \subset \mathbb{P}^{23}$ is defined by the 16 quadrics $\theta_{ij} \in \mathbb{C}[A]_2$.

The space $\mathbb{C}[A]_2 \simeq \mathbb{C}^{300}$ decomposes into eight irreducible representations. The quadrics θ_{ij} account for two of these Schur modules. After relabeling and matrix inversion, the group $G = \mathrm{GL}(4, \mathbb{C})$ acts on the 4×4 matrix of quadrics by congruence. In symbols, $\Theta \mapsto g^T \Theta g$. This action is compatible with the decomposition of 4×4 matrices into their symmetric and skew-symmetric parts. Our space of quadrics $\mathbb{C}\{\text{entries of } \Theta\} \simeq \mathbb{C}^{16}$ is the direct sum of

$$S_{(1,1,1,-1)}(\mathbb{C}^4) \;=\; \mathbb{C}\{\text{entries of } \Theta + \Theta^T\} \;\simeq\; \mathbb{C}^{10}$$

$$\text{and} \qquad S_{(1,1,0,0)}(\mathbb{C}^4) \;=\; \mathbb{C}\{\text{entries of } \Theta - \Theta^T\} \;\simeq\; \mathbb{C}^6. \tag{5}$$

Highest weight vectors for these two G-modules are θ_{11} and $\theta_{12} - \theta_{21}$ respectively. The ideal of $\mathbb{C}[A]$ generated by (5) is not radical. Its radical is the intersection of four prime ideals. These four prime ideals and their varieties in $\mathbb{P}^{23}$ will be described in the next section. Explicit lists of all ideal generators and `Macaulay2` code verifying Theorems 1.1 and 3.1 are available at the repository website `MathRepo` [4] of MPI-MiS via the link https://mathrepo.mis.mpg.de/Lie4.

3. Four components

The irreducible components of Lie_n were described by Kirillov and Neretin [6] for $n \leq 6$ and by Carles and Diakité [3] for $n = 7$. Here we revisit the basic case $n = 4$. The variety Lie_4 has four irreducible components. We denote these by C_1, C_2, C_3, C_4 as in [7, Proposition 2.1]. This differs from the orderings used in [1, 6]. We begin by presenting a parametrization for each component of Lie_4, starting from a generic G-orbit. For this we use the matrices

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6}$$

The variety C_i is the closure of the G-orbit of A_i where x and y range over $\mathbb{C}$. For instance,

$$C_2 \;=\; \text{closure of } \left\{ g^{-1} \cdot A_2(x) \cdot \wedge_2(g) \;:\; x \in \mathbb{C}, \, g \in G \right\} \;\subset\; \mathbb{P}^{23}. \tag{7}$$

The number of free parameters is consistent with the table in [6, page 25]. We see there that generic G-orbits are dense in the varieties C_1 and C_4, and they have codimension 1 in C_2 and codimension 2 in C_3. Given our rational parametrizations, such as (7), we can try to find the prime ideals $I_{C_1}, \ldots, I_{C_4}$ by performing implicitization [8, §4.2] in a computer algebra system, such as `Macaulay2` [5]. While this works in theory, it is not easy to do in practise.

We next recall the Lie-theoretic descriptions of the general point on each component:

C_1 : The Lie algebra $\mathfrak{gl}_2$ of the general linear group $\mathrm{GL}(2,\mathbb{C})$. Its derived algebra is $\mathfrak{sl}_2$.

C_2 : Lie algebras whose derived algebra is the Heisenberg algebra $\mathfrak{he}_3$, of dimension 3.

C_3 : Lie algebras whose derived algebra is abelian and three-dimensional.

C_4 : The Lie algebra $2\mathfrak{aff}_2$. Its derived algebra is abelian and two-dimensional.

The derived algebra of a Lie algebra is generated by all the commutators. As a subspace of $\mathbb{C}^4$, the derived algebra is simply the image of our 4×6 matrix A. This means that the 4×4-minors of A vanish on all components, while the 3×3-minors of A vanish only on C_4. We now present the prime ideals $I_{C_1}, I_{C_2}, I_{C_3}$ and I_{C_4} in $\mathbb{C}[A]$ that define the components.

Theorem 3.1. *The ideal I_{C_1} is generated by 4 linear forms, 10 quadrics and 20 cubics. The linear forms and quadrics are $S_{(1,0,0,0)}(\mathbb{C}^4)$ and $S_{(1,1,1,-1)}(\mathbb{C}^4)$ in Section 2. The ideal I_{C_2} is generated by 16 quadrics and 44 cubics, I_{C_3} is generated by 26 quadrics and 40 cubics, and I_{C_4} is generated by 16 quadrics and 60 cubics. The Hilbert polynomials of the varieties are*

$$
\begin{aligned}
\mathrm{Hilb}_{C_1} &= \mathbf{55}\,[\mathbb{P}^{11}] - 120\,[\mathbb{P}^{10}] + 86\,[\mathbb{P}^9] - 20\,[\mathbb{P}^8], \\
\mathrm{Hilb}_{C_2} &= \mathbf{361}\,[\mathbb{P}^{11}] - 1184\,[\mathbb{P}^{10}] + 1526\,[\mathbb{P}^9] - 964\,[\mathbb{P}^8] + 298\,[\mathbb{P}^7] - 36\,[\mathbb{P}^6], \\
\mathrm{Hilb}_{C_3} &= \mathbf{121}\,[\mathbb{P}^{11}] - 284\,[\mathbb{P}^{10}] + 220\,[\mathbb{P}^9] - 56\,[\mathbb{P}^8], \\
\mathrm{Hilb}_{C_4} &= \mathbf{295}\,[\mathbb{P}^{11}] - 920\,[\mathbb{P}^{10}] + 1114\,[\mathbb{P}^9] - 652\,[\mathbb{P}^8] + 184\,[\mathbb{P}^7] - 20\,[\mathbb{P}^6].
\end{aligned}
$$

Here we write $[\mathbb{P}^d]$ for the Hilbert polynomial $n \mapsto \binom{n+d}{d}$ of the projective space $\mathbb{P}^d$. The leading coefficients are the degrees of the varieties. Theorem 1.1 is a corollary to Theorem 3.1.

Proof. We found the quadrics, cubics and quartics that vanish on each variety by solving linear systems of equations over $\mathbb{Q}$. Namely, we substituted the parametrizations presented in (6) into a polynomial with unknown coefficients, and we solved for these coefficients. In order for the linear systems to have small sizes, we organized the computations by $\mathbb{Z}^4$-degrees.

In this manner, we identified the polynomials of degree at most four in each ideal I_{C_i}. We verified that the dimension of each projective variety defined by the equations we found equals 11. We also checked that the degree matches that predicted by the numerical approach with `homotopyContinuation.jl`. The Hilbert polynomials listed above were computed in `Macaulay2` by applying the command `hilbertPolynomial` to the proposed ideal generators.

What remains to be shown is that, in each of the four cases, our polynomials do indeed generate a prime ideal. For the ideals I_{C_1} and I_{C_3}, this was done by simply running the command `isPrime` in `Macaulay2`. The computation was harder for I_{C_2} and I_{C_4}. To show that I_{C_4} is generated by the 26 quadrics and 40 cubics, we ran the commands `# minimalPrimes C4` and `radical C4 == C4`. Their outputs are `1` and `true`, respectively. This proves the claim.

It remains to prove the claim for the variety C_2. We perform implicitization as follows:

```
R = QQ[f1,f2,f3,f4,f5,k1,k2,k3,k4,k5,k6,m, a121,a122,a123,a124,
     a131,a132,a133,a134,a141,a142,a143,a144,a231,a232,a233,a234,
     a241,a242,a243,a244,a341,a342,a343,a344, MonomialOrder=>Eliminate 12];
I = ideal(a121-f1*f4*k5-f2*f5*k5+f1*k1+f2*k3+f3*k5, a122+f4*k5-k1,
   a132+f4*k6-k2, a142+f4^2*k5+f4*f5*k6+f4*k4-f5*k2,
   a232+f1*f4*k6-f2*f4*k5-f1*k2+f2*k1+f4*m,
   a242+f1*f4^2*k5+f1*f4*f5*k6+f1*f4*k4-f1*f5*k2-f3*f4*k5+f4*f5*m+f3*k1,
   a342+f2*f4^2*k5+f2*f4*f5*k6+f2*f4*k4-f2*f5*k2-f3*f4*k6-f4^2*m+f3*k2,
   a123+f5*k5-k3, a133+f5*k6-k4, a143+f4*f5*k5+f5^2*k6-f4*k3+f5*k1,
   a131-f1*f4*k6-f2*f5*k6+f1*k2+f2*k4+f3*k6,
   a233+f1*f5*k6-f2*f5*k5-f1*k4+f2*k3+f5*m,
   a243+f1*f4*f5*k5+f1*f5^2*k6-f1*f4*k3+f1*f5*k1-f3*f5*k5+f5^2*m+f3*k3,
   a343+f2*f4*f5*k5+f2*f5^2*k6-f2*f4*k3+f2*f5*k1-f3*f5*k6-f4*f5*m+f3*k4,
   a144-f4*k5-f5*k6-k1-k4, a244-f1*f4*k5-f1*f5*k6-f1*k1-f1*k4+f3*k5-f5*m,
   a234-f1*k6+f2*k5-m, a344-f2*f4*k5-f2*f5*k6-f2*k1-f2*k4+f3*k6+f4*m,
   a141-f1*f4^2*k5-f1*f4*f5*k6-f2*f4*f5*k5-f2*f5^2*k6-f1*f4*k4+f1*f5*k2
        +f2*f4*k3-f2*f5*k1+f3*f4*k5+f3*f5*k6+f3*k1+f3*k4,
   a231-f1^2*f4*k6+f1*f2*f4*k5-f1*f2*f5*k6+f2^2*f5*k5+f1^2*k2-f1*f2*k1
        +f1*f2*k4+f1*f3*k6-f1*f4*m-f2^2*k3-f2*f3*k5-f2*f5*m+f3*m,
   a124-k5, a134-k6,
   a241-f1^2*f4^2*k5-f1^2*f4*f5*k6-f1*f2*f4*f5*k5-f1*f2*f5^2*k6-f1^2*f4*k4
        +f1^2*f5*k2+f1*f2*f4*k3-f1*f2*f5*k1+2*f1*f3*f4*k5+f1*f3*f5*k6
        -f1*f4*f5*m+f2*f3*f5*k5-f2*f5^2*m+f1*f3*k4-f2*f3*k3-f3^2*k5
        +f3*f5*m,
   a341-f1*f2*f4^2*k5-f1*f2*f4*f5*k6-f2^2*f4*f5*k5-f2^2*f5^2*k6
        -f1*f2*f4*k4+f1*f2*f5*k2+f1*f3*f4*k6+f1*f4^2*m+f2^2*f4*k3
        -f2^2*f5*k1+f2*f3*f4*k5+2*f2*f3*f5*k6+f2*f4*f5*m-f1*f3*k2
        +f2*f3*k1-f3^2*k6-f3*f4*m);
C2 = ideal selectInSubring( 1, gens gb(I) )
codim C2, degree C2, betti mingens C2
```

This `Macaulay2` code represents the birational parametrization of C_2 introduced in [7, Section 3.2] and revisited in Section 5 below. The ideal `I` defines the graph of that parametrization. It is prime because each of its 24 generators is equal to a_{ijk} minus a polynomial in the 12 parameters. The ideal `C2` is obtained by eliminating the 12 parameters $f_1, f_2, \ldots, k_6, m$, so it is prime as well. The last line yields that it is generated by 16 quadrics and 44 cubics.

The parametrization is a modification of (7). Namely, our `Macaulay2` code says that $A = \wedge_2 g \cdot B \cdot \det(g) \cdot g^{-1}$ where

$$
g = \begin{bmatrix} 1 & f_1 & f_2 & f_3 \\ 0 & 1 & 0 & f_4 \\ 0 & 0 & 1 & f_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ k_1 & k_2 & 0 & 0 & 0 & 0 \\ k_3 & k_4 & 0 & 0 & 0 & 0 \\ k_5 & k_6 & k_1+k_4 & m & 0 & 0 \end{bmatrix}.
$$

The five parameters f_i seen in g are local coordinates on the flag variety $\mathrm{Fl}(1, 3, \mathbb{C}^4)$. The flag amounts to the inclusion of the second derived algebra, spanned by the last column of the matrix g, into the derived algebra, which is spanned by the last three columns of g.

The matrix B depends linearly on the parameters $k_1, \ldots, k_6, m$. It defines a rank seven vector bundle F over $\mathrm{Fl}(1, 3, \mathbb{C}^4)$. The structure of B ensures that the derived algebra is the Heisenberg algebra. In symbols, $\mathrm{image}(B) \simeq \mathfrak{h}_3$. The projective bundle $\mathbb{P}(F)$ is a nonsingular variety of dimension $11 = 6 + 5$ that maps birationally onto C_2. The map is given by $\mathtt{I}$. ∎

We close this section by showing that the 16 quadrics in (5) do not give a radical ideal.

Corollary 3.2. *The radical ideal of* Lie_4 *is minimally generated by* 16 *quadrics and* 15 *quartics. The quadrics are those in* (5) *and the quartics are the* 4×4 *minors of the matrix* A *in* (1).

Proof. The radical ideal of Lie_4 is the intersection of the prime ideals I_{C_1}, I_{C_2}, I_{C_3}, I_{C_4}. We computed this intersection using $\mathtt{Macaulay2}$, and we verified that it has the asserted minimal generators. It can also be shown directly that the ideal generated by the 16 quadrics in (1) is not radical. Namely, the 4×4 minors of A are not in this ideal, but their squares are. ∎

4. Polynomials: Explicit versus invariant

We now examine our ideal generators through the lens of the G-action. We identify the Schur modules $S_\lambda(\mathbb{C}^4)$ that generate the ideals I_{C_i}, and we identify polynomials that serve as highest weight vectors. The space $\mathbb{C}[A]_2 \simeq \mathbb{C}^{300}$ decomposes into five isotypical components:

λ	$(3, 1, -1, -1)$	$(2, 1, 0, -1)$	$(2, 0, 0, 0)$	$(1, 1, 1, -1)$	$(1, 1, 0, 0)$
dim	126	64	10	10	6
mult	1	2	2	2	1

The last two columns were seen already in (5). The space of cubics $\mathbb{C}[A]_3 \simeq \mathbb{C}^{2600}$ decomposes into 11 isotypical components. We display the four components that are relevant for us:

λ	$(2, 1, 1, -1)$	$(3, 0, 0, 0)$	$(2, 1, 0, 0)$	$(1, 1, 1, 0)$	$\cdots$
dim	36	20	20	4	$\cdots$
mult	5	3	4	3	$\cdots$

One ingredient in an invariant description of these G-modules is the *adjoint* $\mathrm{ad}(u)$ of an element u in our Lie algebra. This is the endomorphism $\mathbb{C}^4 \to \mathbb{C}^4$ given by $v \mapsto [u, v]$. For any index $i \in \{1, 2, 3, 4\}$, the adjoint of e_i is represented by the 4×4 matrix $(a_{ijk})_{1 \leq j,k \leq 4}$. The traces of the matrices $\mathrm{ad}(e_i)$ are the four linear forms that span $S_{(1,0,0,0)}(\mathbb{C}^4) \subset I_{C_1}$. To be explicit, $\mathrm{ad}(e_1)$ is obtained by prepending a zero column to the left three columns of A.

We begin with the module $S_{(3,0,0,0)}(\mathbb{C}^4)$. This has dimension 20 and occurs with multiplicity three in the space of cubics $\mathbb{C}[A]_3$. A highest weight vector for one embedding is

$$f_{3000} = a_{122}^3 - a_{122}^2 a_{133} - a_{122}^2 a_{144} + 4a_{122}a_{123}a_{132} + 4a_{122}a_{124}a_{142} - a_{122}a_{133}^2$$
$$+ 2a_{122}a_{133}a_{144} - 4a_{122}a_{134}a_{143} - a_{122}a_{144}^2 + 4a_{123}a_{132}a_{133} - 4a_{123}a_{132}a_{144}$$
$$+ 8a_{123}a_{134}a_{142} + 8a_{124}a_{132}a_{143} - 4a_{124}a_{133}a_{142} + 4a_{124}a_{142}a_{144} + a_{133}^3$$
$$- a_{133}^2 a_{144} + 4a_{133}a_{134}a_{143} - a_{133}a_{144}^2 + 4a_{134}a_{143}a_{144} + a_{144}^3.$$

This module generates the ideal I_{C_1}, together with the linear forms and quadrics seen in Theorem 3.1. The module Gf_{3000} is also contained in I_{C_2}. But this ideal has 24 additional cubic generators. These additional cubics for C_2 are given by two irreducible G-modules:

$$S_{(2,1,0,0)}(\mathbb{C}^4) \oplus S_{(1,1,1,0)}(\mathbb{C}^4) \simeq \mathbb{C}^{20} \oplus \mathbb{C}^4.$$

The highest weight vectors for these two irreducible G-modules in $\mathbb{C}[A]_3$ are

$$\text{trace}\big(\text{ad}(e_1) \cdot \text{ad}(e_2) \cdot \text{ad}(e_3)\big) - \text{trace}\big(\text{ad}(e_2) \cdot \text{ad}(e_1) \cdot \text{ad}(e_3)\big)$$

and $\quad \text{trace}\big(\text{ad}(e_1)\big) \cdot \text{trace}\big(\text{ad}(e_1) \cdot \text{ad}(e_2)\big) - \text{trace}\big(\text{ad}(e_2)\big) \cdot \text{trace}\big(\text{ad}(e_1)^2\big).$

These are cubic polynomials in the 24 unknowns a_{ijk}, having 51 and 39 terms respectively.

We also consider the trace of the third power of the adjoint of e_1. This is the cubic

$$g_{3000} = \text{trace}\big(\text{ad}(e_1)^3\big) = a_{122}^3 + a_{133}^3 + a_{144}^3 + 3(a_{122}a_{123}a_{132} + a_{122}a_{124}a_{142}$$
$$+ a_{123}a_{132}a_{133} + a_{123}a_{134}a_{142} + a_{124}a_{132}a_{143} + a_{124}a_{142}a_{144}$$
$$+ a_{133}a_{134}a_{143} + a_{134}a_{143}a_{144}).$$

This is the highest weight vector of another embedding of $S_{(3,0,0,0)}(\mathbb{C}^4)$ into I_{C_1}. This G-module can also serve to generate I_{C_1}. We note that $g_{3000} - f_{3000}$ lies in the ideal generated by the four linear forms in I_{C_1}. However, unlike f_{3000}, the cubic g_{3000} does not lie in I_{C_2}.

We now turn to the two components C_3 and C_4. Lie algebras in these components have the property that their derived algebra is abelian. Hence the second derived algebra is zero.

The ideal I_{C_3} requires 10 additional quadrics and 40 cubics. These 10 quadrics span a module $S_{(1,1,1,-1)}(\mathbb{C}^4)$ inside $\mathbb{C}[A]_2$, with highest weight vector $a_{124}a_{344} - a_{134}a_{244} + a_{144}a_{234}$. Hence both embeddings of $S_{(1,1,1,-1)}(\mathbb{C}^4)$ into the space of quadrics $\mathbb{C}[A]_2$ vanish on C_3.

The second derived algebra of a Lie algebra is the column space of the matrix $A \cdot (\wedge_2 A)$. This is the product of a 4×6 matrix with a 6×15 matrix. The 60 entries of the 4×15 matrix $A \cdot (\wedge_2 A)$ are cubics. These polynomials lie in both I_{C_3} and I_{C_4} because the second derived algebra is zero. Twenty of the 60 matrix entries are already accounted for: they are in the ideal of (5). The remaining 40 cubics are generators of I_{C_3} and they form a G-module

$$S_{(2,1,1,-1)}(\mathbb{C}^4) \oplus S_{(1,1,1,0)}(\mathbb{C}^4) \simeq \mathbb{C}^{36} \oplus \mathbb{C}^4. \tag{8}$$

This module also gives 40 of the minimal generators of I_{C_4}. The remaining 20 cubic generators of I_{C_4} form a Schur module $S_{(3,0,0,0)}(\mathbb{C}^4)$. They arise from the constraint that the matrix A has rank two. A highest weight vector is the lower left 3×3-minor of the matrix A, which equals

$$a_{122}a_{133}a_{144} - a_{122}a_{134}a_{143} - a_{123}a_{132}a_{144} + a_{123}a_{134}a_{142} + a_{124}a_{132}a_{143} - a_{124}a_{133}a_{142}.$$

In conclusion, the polynomials in a G-invariant ideal can be described either invariantly, using the language of representation theory, or explicitly as linear combinations of monomials. This section straddles between these two paradigms. We explored the four associated prime ideals I_{C_i} of Lie_4. While some readers favor the invariant description, others will prefer to work with the explicit polynomials which we posted at https://mathrepo.mis.mpg.de/Lie4.

5. From vector bundles to degrees

The approach in [7] is based on desingularization of each component C_i, following Basili [1]. The key idea is to *linearize the Jacobi identity*, i.e. to express (2) by linear equations in A over a variety derived from $\mathrm{GL}(4, \mathbb{C})$. This yields alternative parametrizations which allow for the computation of degrees using Chern classes. We now correctly derive the numbers in Theorem 1.1 by this method. In what follows we present this for C_2 and then for C_1. The derivation for C_4 is similar to that for C_2, and that for C_3 was correct in [7, Section 3.3].

The linearized parametrization for C_2 was shown in the `Macaulay2` code in the proof of Theorem 3.1. It is given by a vector bundle F of rank 7 over the 5-dimensional flag variety $\mathrm{Fl}(1, 3, \mathbb{C}^4)$. The degree of C_2 in $\mathbb{P}^{23}$ is computed using Chern classes and Segre classes:

$$\deg(C_2) = \int_{C_2} c_1(O(1))^{11} = \int_{\mathbb{P}(F)} c_1(O_F(1))^{11} = \int_{\mathrm{Fl}(1,3,V)} s_5(F). \qquad (9)$$

From now set $V = \mathbb{C}^4$. The pair $(L, U) \in \mathrm{Fl}(1, 3, V)$ gives the second and first derived algebra. We compute the Segre class in (9), as in [7, Section 3.2]. This uses the exact sequence

$$0 \longrightarrow K \simeq \mathrm{Hom}(V/U, \mathrm{End}_L^0(U)) \longrightarrow F \longrightarrow M = \mathrm{Hom}(\wedge^2(U/L), L) \longrightarrow 0.$$

Here, M is a line bundle and K is a rank six vector bundle. Note the matching parameters m and $k_1, k_2, k_3, k_4, k_5, k_6$ in the proof of Theorem 3.1. The kernel K fits into the exact sequence

$$0 \longrightarrow K \longrightarrow \mathrm{Hom}(V/U, \mathrm{End}(U)) \longrightarrow \mathrm{Hom}(V/U, \mathrm{Hom}(L, U)) \longrightarrow 0.$$

From the convention that the Segre class and Chern class are inverse to each other we get

$$s(F) = s(M) \cdot s(K) \qquad (10)$$
$$= s(M) \cdot s(\mathrm{Hom}(V/U, \mathrm{End}(U)) \cdot c(\mathrm{Hom}(V/U, \mathrm{Hom}(L, U))).$$

This is a rational generating function $s(x, y)$ in $x = -c_1(L)$ and $y = -c_1(U) = c_1(V/U)$. These classes satisfy $x^4 = y^4 = 0$ because they are induced from $\mathbb{P}(V)$ and $\mathbb{P}(V^\vee)$ respectively.

Let $s_5(x, y)$ denote the component of degree 5 in $s(x, y)$. We need to integrate it on the flag variety. Since $\mathrm{Fl}(1, 3, V)$ is a divisor of type $(1, 1)$ in $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$, we can write

$$\deg(C_2) = \int_{F(1,3,V)} s_5(F) = \int_{\mathbb{P}(V) \times \mathbb{P}(V^\vee)} (x + y) s_5(x, y).$$

This says that the desired degree is the coefficient of $x^3 y^3$ in the polynomial $(x + y) s_5(x, y)$.

To find $s(x, y)$ we compute the three factors on the right of (10). Since M is a line bundle, we have $s(M) = 1/(1 + y - 2x)$. The second factor is pulled back from $\mathbb{P}(V^\vee)$. We obtain

$$s\big(\mathrm{Hom}(V/U, \mathrm{End}(U))\big) \;=\; \frac{(1 - 2y)^4}{(1 - y)^{17}}.$$

A twist of the tautological sequence from $\mathbb{P}(V^\vee)$ finally gives

$$c\big(\mathrm{Hom}(V/U, \mathrm{Hom}(L, U))\big) \;=\; \frac{(1 + x - y)^4}{1 + x}.$$

Multiplying all these rational functions and extracting the coefficient of $x^3 y^3$ leads to the result **361**.

We next consider the first component C_1 and we show degree(C_1) $= 55$. This corrects [7, Section 3.1]. The mistake was due to the fact that the model in [7, Proposition 3.1] was only birational, with non-empty indeterminacy locus which was not properly taken into account.

To correct this we will use a slightly different, better behaved model. We recall from [1, §2] or [7, §2.1] that, for $U \simeq \mathbb{C}^3$, a Lie bracket $\theta \in \mathrm{Hom}(\wedge^2 U, U)$ defining a Lie algebra structure isomorphic to $\mathfrak{sl}_2$ must belong to the space $\mathrm{Hom}_s(\wedge^2 U, U)$ of symmetric matrices. This relies on the fact that $\wedge^2 U \simeq U^\vee \otimes \det(U)$, since U is 3-dimensional, and therefore

$$\mathrm{Hom}(\wedge^2 U, U) \;\simeq\; U \otimes U \otimes \det(U)^\vee \;=\; (S^2 U \oplus \wedge^2 U) \otimes \det(U)^\vee.$$

The first component $S^2 U \otimes \det(U)^\vee =: \mathrm{Hom}_s(\wedge^2 U, U)$ is a 6-dimensional subspace of $\mathrm{Hom}(\wedge^2 U, U) \simeq \mathbb{C}^9$. Explicitly, if u is a vector in U and $\Omega \in \det(U)^\vee$ is a volume form on U, then the Lie bracket defined by $u^2 \otimes \Omega$ is given by the formula

$$\theta(x, y) \;=\; \Omega(x, y, u) \cdot u \qquad \text{for all } x, y \in U.$$

By linearity, this formula extends to any element $q = u_1^2 + u_2^2 + u_3^2$ of $S^2 U$, and the resulting Lie algebra structure on U is isomorphic to $\mathfrak{sl}_2$ exactly when q is non-degenerate.

Now suppose that $V = \mathbb{C}^4$ has a Lie bracket θ such that the Lie algebra structure is isomorphic to $\mathfrak{gl}_2$. The center L is 1-dimensional. We have $\theta(L \wedge V) = 0$ since elements of L commute with all vectors in V. Therefore θ descends to an element of $\mathrm{Hom}(\wedge^2 Q, V)$, where $Q = V/L$. Composing with the projection to Q, we get a Lie bracket $\bar{\theta} \in \mathrm{Hom}(\wedge^2 Q, Q)$ which, by the previous paragraph, must be a non-degenerate element of $\mathrm{Hom}_s(\wedge^2 Q, Q)$. In other words, we can describe θ as a generic matrix in the subspace of $\mathrm{Hom}(\wedge^2 Q, V) \subset \mathrm{Hom}(\wedge^2 V, V)$ consisting of elements whose projection to $\mathrm{Hom}(\wedge^2 Q, Q)$ is symmetric.

When the line L varies, the construction above defines a rank nine vector bundle E over $\mathbb{P}(V) = \mathbb{P}^3$. This fits into the following exact sequence, where L and Q, with a slight abuse of notation, now denote the tautological line bundle and the quotient vector bundle:

$$0 \longrightarrow \mathrm{Hom}(\wedge^2 Q, L) \longrightarrow E \longrightarrow \mathrm{Hom}_s(\wedge^2 Q, Q) \longrightarrow 0. \tag{11}$$

Since E was constructed as a subbundle of the trivial vector bundle with fiber $\mathrm{Hom}(\wedge^2 V, V)$ there is a natural morphism from the total space of $\mathbb{P}(E)$ to the projective space $\mathbb{P}(\mathrm{Hom}(\wedge^2 V, V)) = \mathbb{P}^{23}$. This is a resolution of singularities of the irreducible component C_1. This allows us to compute the degree of C_1 in its embedding into $\mathbb{P}^{23}$ exactly as before:

$$\deg(C_1) \ = \ \int_{C_1} c_1(O(1))^{11} = \int_{\mathbb{P}(E)} c_1(O_E(1))^{11} = \int_{\mathbb{P}(V)} s_3(E). \qquad (12)$$

The exact sequence (11) lets us compute the Segre class of E in terms of $x = -c_1(L)$. We get

$$s(E) = \frac{1 - 3x}{(1 - x)^{10}} = 1 + 7x + 25x^2 + 55x^3.$$

This establishes the desired equation $\deg(C_1) = \mathbf{55}$.

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References

[1] R. Basili: *Resolutions of singularities of varieties of Lie algebras of dimensions 3 and 4*, J. Lie Theory 12 (2002) 397–407.

[2] P. Breiding, S. Timme: *HomotopyContinuation.jl: a package for homotopy continuation in Julia*, in: Math. Software – ICMS 2018, Springer, Berlin (2018) 458–465.

[3] R. Carles, Y. Diakité: *Sur les variétés d'algèbres de Lie de dimension ≤ 7*, J. Algebra 91 (1984) 53–63.

[4] C. Fevola, C. Görgen: *The mathematical research-data repository MathRepo*, Computeralgebra Rundbrief 70 (2022) 16–20.

[5] D. Grayson, M. Stillman: *Macaulay2, a software system for research in algebraic geometry*, available at http://www.math.uiuc.edu/Macaulay2/.

[6] A. Kirillov, Y. Neretin: *The variety A_n of n-dimensional Lie algebras structures*, Transl., II. Ser., American Mathematical Society 137 (1987) 21–30.

[7] L. Manivel: *On the variety of four dimensional Lie algebras*, J. Lie Theory 26 (2016) 1–10.

[8] M. Michałek, B. Sturmfels: *Invitation to Nonlinear Algebra*, Graduate Studies in Mathematics 211, American Mathematical Society, Providence (2021).

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