

Elliptic Coadjoint Orbits of Holomorphic Type

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Communicated by M. Pevzner

Abstract. This article proves that any elliptic coadjoint orbit of a semisimple Lie group carries a holomorphic bundle structure over a flag variety if the polarization is given by a θ -stable parabolic subalgebra of holomorphic type. An application to the Penrose transform is given.

Mathematics Subject Classification: Primary 32M15; Secondary 53C65, 53C35, 17B20.

Key Words: Reductive Lie groups, coadjoint orbits, Borel embedding, Harish-Chandra decomposition, indefinite Kähler manifold.

1. Statement of main results

Let G be a connected semisimple Lie group, θ a Cartan involution, K the fixed point subgroup of θ , and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition of the Lie algebra \mathfrak{g} of G .

Given $H \in \mathfrak{k}$, and we denote by L the centralizer of H in G . Then the homogeneous space G/L is identified with an elliptic coadjoint orbit via the Killing form, and carries a G -invariant complex structure as follows. We write $\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}^- + \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ for the Gelfand–Naimark decomposition where \mathfrak{u}^- , $\mathfrak{l}_{\mathbb{C}}$ and \mathfrak{u} denote the sum of eigenspaces of $-\sqrt{-1}\operatorname{ad}(H)$ with negative, zero, and positive eigenvalues, respectively. We refer to $\mathfrak{q} := \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ as a θ -stable parabolic subalgebra in $\mathfrak{g}_{\mathbb{C}}$. Let Q be the normalizer of \mathfrak{q} in $G_{\mathbb{C}} := \operatorname{Int}(\mathfrak{g}_{\mathbb{C}})$. Then $L = Q \cap G$ and the natural homomorphism $G \rightarrow G_{\mathbb{C}}$ induces an open embedding (a generalized Borel embedding) $G/L \hookrightarrow G_{\mathbb{C}}/Q$ from which G/L carries a complex manifold structure (see *e.g.*, [9]).

Similarly, the submanifold $K/L \cap K$ carries a complex manifold structure such that

$$\iota: K/L \cap K \hookrightarrow G/L \text{ is holomorphic.}$$

On the other hand, G/L has a *smooth* fiber bundle structure

$$\pi: G/L \rightarrow K/L \cap K$$

induced by the K -isomorphism $K \times_{L \cap K} (\mathfrak{l}^{\perp} \cap \mathfrak{p}) \xrightarrow{\sim} G/L$, $(k, X) \mapsto ke^XL$, where \mathfrak{l}^{\perp} stands for the orthogonal complement of \mathfrak{l} in \mathfrak{g} with respect to the Killing form (see [4] in a more general setting). However, the projection π cannot be holomorphic even when $G = SL(2, \mathbb{R})$ and $K = L = SO(2)$.

In this paper, we find yet another fiber bundle structure $\varpi: G/L \rightarrow K/L \cap K$ such that ϖ is K -equivariant and holomorphic when \mathfrak{q} is holomorphic (Definition 1.1).

In order to state our main results, we assume further that G is a simple Lie group of Hermitian type. This means that the Lie algebra \mathfrak{k} has a one-dimensional center $\mathfrak{c}(\mathfrak{k})$, which gives a three graded structure on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^- \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+$. We recall from [1] the terminology in algebraic representation theory.

Definition 1.1. A θ -stable parabolic subalgebra \mathfrak{q} is *holomorphic* if $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}} \subset \mathfrak{p}^+$. We say the corresponding manifold G/L is *of holomorphic type*.

See [10, Table C.1] for the classification of holomorphic parabolic subalgebras.

We shall prove the following:

Theorem 1.2. *If \mathfrak{q} is of holomorphic type, then there exists an $(L \cap K)$ -invariant bounded open subset Ω in $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}$ such that G/L is biholomorphic to the homogeneous holomorphic fiber bundle $K \times_{L \cap K} \Omega$.*

Corollary 1.3. *If \mathfrak{q} is of holomorphic type, then there exists a K -equivariant holomorphic map $\varpi: G/L \rightarrow K/L \cap K$ such that $\varpi \circ \iota = \text{id}$.*

This article is organized as follows. Section 2 proves that the flag variety $G_{\mathbb{C}}/Q$ contains an open subset which is biholomorphic to a K -equivariant holomorphic vector bundle over $K/L \cap K$. In Section 3 we prove an analog of the Harish-Chandra decomposition, and realizes G/L in this K -equivariant vector bundle, from which Theorem 1.2 is derived. An application to a generalized Radon-Penrose transform is provided in Section 4.

2. Embedding of $K \times_{L \cap K} (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}})$ into $G_{\mathbb{C}}/Q$

Since the complex subgroup $Q \cap K_{\mathbb{C}}$ leaves $\mathfrak{q} \cap \mathfrak{p}_{\mathbb{C}}$ invariant, it induces an action on $\mathfrak{p}_{\mathbb{C}}/(\mathfrak{q} \cap \mathfrak{p}_{\mathbb{C}}) \simeq \mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}$. We form a $K_{\mathbb{C}}$ -equivariant holomorphic vector bundle

$$K_{\mathbb{C}} \times_{Q \cap K_{\mathbb{C}}} (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}) \simeq K \times_{L \cap K} (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}) \quad (1)$$

over the compact complex manifold $K_{\mathbb{C}}/(Q \cap K_{\mathbb{C}}) \simeq K/(L \cap K)$.

In this section we prove the following:

Theorem 2.1. *If \mathfrak{q} is of holomorphic type, then the map $K_{\mathbb{C}} \times (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}) \rightarrow G_{\mathbb{C}}$, $(k, X) \mapsto ke^X$ induces a $K_{\mathbb{C}}$ -equivariant injective holomorphic map*

$$i: K_{\mathbb{C}} \times_{Q \cap K_{\mathbb{C}}} (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}) \rightarrow G_{\mathbb{C}}/Q.$$

Remark 2.2. In light of the K -equivariant diffeomorphism

$$K \times_{L \cap K} (\mathfrak{l}^{\perp} \cap \mathfrak{p}) \xrightarrow{\sim} G/L$$

in Section 1, one might surmise that two complex manifolds $K \times_{L \cap K} (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}})$ and G/L are biholomorphic via the $(L \cap K)$ -isomorphisms

$$\mathfrak{l}^{\perp} \cap \mathfrak{p} \xrightarrow{\sim} \mathfrak{p}_{\mathbb{C}}/\mathfrak{q} \cap \mathfrak{p}_{\mathbb{C}} \xleftarrow{\sim} \mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}.$$

However, such a statement is false even when \mathfrak{q} is of holomorphic type.

Proof of Theorem 2.1. Without loss of generality, we may assume that we have $G \subset G_{\mathbb{C}} = \text{Int}(\mathfrak{g}_{\mathbb{C}})$. Since $(k\ell)e^{\text{Ad}(\ell^{-1})X}Q = ke^X\ell Q = ke^XQ$ for any $\ell \in L \cap K$, the

map $i: K \times_{L \cap K} (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}) \rightarrow G_{\mathbb{C}}/Q$ is well-defined. The K -equivariance is obvious. In light of the isomorphism (1), one sees that i is holomorphic.

For the injectivity of i , it suffices to prove the following proposition. ■

Proposition 2.3. *Suppose $X, Y \in \mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}$, $k \in K$, and $q \in Q$ satisfy $ke^X = e^Yq$. Then one has $k \in L \cap K$ and $Y = \text{Ad}(k)X$.*

Before entering the proof, we prepare some basic lemmas.

Lemma 2.4. *If \mathfrak{q} is of holomorphic type, one has a direct sum decomposition*

$$\mathfrak{p}^- = (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}) \oplus (\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{p}^-).$$

Proof. Since $\mathfrak{c}(\mathfrak{k}) \subset \mathfrak{l}$, one has $\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}} = (\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{p}^-) \oplus (\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{p}^+)$. Thus the lemma follows because $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}} \subset \mathfrak{p}^-$. ■

Lemma 2.5. *Suppose $k \in K$, $Y \in \mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}$ and $q \in Q$ satisfy $k = e^Yq$. Then one has $Y = 0$ and $k = q \in L \cap K$.*

We extend the Cartan involution θ to a holomorphic involution of $G_{\mathbb{C}} = \text{Int}(\mathfrak{g}_{\mathbb{C}})$, and use the same letter θ .

Proof of Lemma 2.5. Applying θ to the equation $k = e^Yq$, one has $k = e^{-Y}\theta q$, hence $e^{2Y} = (\theta q)q^{-1}$. Then $\theta Q = Q$ implies $Y = 0$ because $\exp(\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}) \cap Q = \{e\}$. In turn, one has $k = q \in Q \cap K$, which coincides with $L \cap K$. ■

Proof of Proposition 2.3. We set $Z := Y - \text{Ad}(k)X \in \mathfrak{p}^-$, and write $Z = Z_1 + Z_2$ according to the decomposition in Lemma 2.4. Then one has $k = e^{Z_1}(e^{Z_2}q)$ because \mathfrak{p}^- is abelian. Since $e^{Z_2}q \in Q$, Lemma 2.5 tells us that $Z_1 = 0$ and $k \in L \cap K$. In turn, $\text{Ad}(k)X \in \mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}$, which implies $Z \in \mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}$, that is, $Z = Z_1$. Therefore, one has $Z = 0$, namely, $Y = \text{Ad}(k)X$. ■

3. A generalized Harish-Chandra decomposition

This section completes the proof of Theorem 1.2. The main machinery is as an analog of the Harish-Chandra decomposition $G \subset \exp(\mathfrak{p}^-)K_{\mathbb{C}}\exp(\mathfrak{p}^+)$, which is stated as follows:

Theorem 3.1. *If \mathfrak{q} is of holomorphic type, then one has*

$$G \subset K \exp(\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}})Q. \tag{2}$$

Moreover there exists an $(L \cap K)$ -invariant bounded domain Ω in $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}$ such that $G \subset K \exp(\Omega)Q$.

Proof. We take a maximal torus T in $L \cap K$ which is also a maximal torus in K . We choose a positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ such that $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \supset \Delta(\mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and write $B_{K_{\mathbb{C}}}$ for the corresponding Borel subgroup of $K_{\mathbb{C}}$.

Lemma 3.2. *There exists a bounded K -invariant open subset $\tilde{\Omega}$ in \mathfrak{p}^- such that*

$$G \subset K \exp(\tilde{\Omega})B_{K_{\mathbb{C}}} \exp(\mathfrak{p}^+).$$

Proof. We take a set of maximal strongly orthogonal roots $\{\gamma_1, \dots, \gamma_\ell\}$ in $\Delta(\mathfrak{p}^+, \mathfrak{t}_{\mathbb{C}})$ where ℓ is the real rank of \mathfrak{g} , and consider the Cayley transform via the corresponding homomorphism $\varphi: \mathfrak{sl}(2, \mathbb{R}) \oplus \dots \oplus \mathfrak{sl}(2, \mathbb{R}) \hookrightarrow \mathfrak{g}$ for the \mathfrak{sl}_2 -triples $\{Y_j, H_j, X_j\}$ associated to γ_j ($1 \leq j \leq \ell$). We set $\mathfrak{a} := \mathbb{R}H_1 \oplus \dots \oplus \mathbb{R}H_\ell$ in \mathfrak{p} and consider a Cartan decomposition $G = KAK$. Then the proof of the Harish-Chandra decomposition (see *e.g.*, [3, Chap. VII]) tells us that

$$A \subset \exp(\Omega')T_{\mathbb{C}}\exp(\mathfrak{p}^+), \quad (3)$$

where $\Omega' := \{\sum_{j=1}^{\ell} t_j Y_j : -1 < t_j < 1 \text{ for all } j = 1, \dots, \ell\}$. We then define $\tilde{\Omega} := \text{Ad}(K)\Omega' (\subset \mathfrak{p}^-)$. Since $KB_{K_{\mathbb{C}}} = K_{\mathbb{C}}$, one has

$$K \exp(\tilde{\Omega})B_{K_{\mathbb{C}}} = K \exp(\Omega)KB_{K_{\mathbb{C}}} = K \exp(\Omega)K_{\mathbb{C}}.$$

Since $T_{\mathbb{C}}\exp(\mathfrak{p}^+)K \subset K_{\mathbb{C}}\exp(\mathfrak{p}^+)$ one concludes

$$KAK \subset K \exp(\Omega')T_{\mathbb{C}}\exp(\mathfrak{p}^+)K \subset K \exp(\tilde{\Omega})B_{K_{\mathbb{C}}}\exp(\mathfrak{p}^+).$$

Thus Lemma 3.2 is proved. \blacksquare

Lemma 3.3. *One has the inclusion relation $B_{K_{\mathbb{C}}}\exp(\mathfrak{p}^+) \subset Q$.*

Proof. Since $\Delta(\mathfrak{u}^- \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \subset \Delta^-(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ and $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}} \subset \mathfrak{p}^-$, one has

$$\begin{aligned} \Delta(\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) &= \Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \setminus \Delta(\mathfrak{u}^- \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \supset \Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}), \\ \Delta(\mathfrak{q} \cap \mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) &= \Delta(\mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \setminus \Delta(\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \supset \Delta(\mathfrak{p}^+, \mathfrak{t}_{\mathbb{C}}). \end{aligned}$$

Hence the lemma follows. \blacksquare

Proof. [Proof of Theorem 3.1] By Lemmas 3.2 and 3.3, one has $G \subset K \exp(\tilde{\Omega})Q$. Let Ω be the image of $\tilde{\Omega}$ under the projection $\mathfrak{p}^- \rightarrow \mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}$ with respect to the decomposition in Lemma 2.4. It follows from $\exp(\mathfrak{p}^-) = \exp(\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}})\exp(\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{p}^-)$ that $\exp(\tilde{\Omega})Q = \exp(\Omega)Q$. Now the theorem is proved. \blacksquare

Proof. [Proof of Theorem 1.2] By Theorems 2.1 and 3.1, one has open embeddings

$$G/L \subset K \times_{L \cap K} (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}) \subset G_{\mathbb{C}}/Q,$$

where both of the embeddings are holomorphic and K -equivariant. Hence G/L is a K -equivariant holomorphic subbundle of the bundle $K \times_{L \cap K} (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}})$ over $K/L \cap K$. Thus the theorem follows. \blacksquare

4. Application of the Penrose transform

This section discusses an application of our main results to Dolbeault cohomologies on non-compact complex manifolds G/L . It is noteworthy that our results are applicable to line bundles of which the parameters are outside the “good range”, or even outside the “weakly fair range” in the terminology of algebraic representation theory [16] for which the existing algebraic techniques for cohomological induction are well developed. On the other hand, such singular parameters are more mysterious, and of particular interest in a geometric context such as the twistor transform (Remark 4.2 (2)). Further applications will be discussed in a subsequent paper.

We begin with a general setting. Let G be a connected linear reductive Lie group in a complexification $G_{\mathbb{C}}$, K a maximal compact subgroup of G , $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ a θ -stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and $L = N_G(\mathfrak{q})$. Then \mathfrak{q} defines a complex structure on G/L via the generalized Borel (open) embedding $G/L \hookrightarrow G_{\mathbb{C}}/Q$, and one has a holomorphic closed embedding $\iota: K_{\mathbb{C}}/Q \cap K_{\mathbb{C}} \simeq K/L \cap K \hookrightarrow G/L$. Given a character \mathbb{C}_{λ} of L we form a holomorphic line bundle $\mathcal{L}_{\lambda} := G \times_L \mathbb{C}_{\lambda}$ over G/L .

Take a maximal torus T of $L \cap K$, which is also a maximal torus of K . Fix a positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ containing $\Delta(\mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. For a dominant character μ of T , we write V_{μ} for the finite-dimensional irreducible K -module with highest weight μ , and form a G -equivariant vector bundle $\mathcal{V}_{\mu} := G \times_K V_{\mu}$ over the Riemannian symmetric space G/K .

We set $S := \dim_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}})$, and $\mu_{\lambda} := \mathbb{C}_{\lambda} \otimes \Lambda^S(\mathfrak{k}_{\mathbb{C}}/\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}) \in \widehat{L \cap K}$. By the Borel-Weil-Bott theorem, the K -module $H_{\bar{\partial}}^S(K/L \cap K, \iota^* \mathcal{L}_{\lambda})$ is isomorphic to $V_{\mu_{\lambda}}$ as far as μ_{λ} is $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ -dominant. We write ℓ_g for the action of $g \in G$ on the line bundle $\mathcal{L}_{\lambda} \rightarrow G/L$. The natural map

$$\tilde{\mathcal{R}}: \mathcal{E}^{0,S}(G/L, \mathcal{L}_{\lambda}) \times G \rightarrow \mathcal{E}^{0,S}(K/L \cap K, \iota^* \mathcal{L}_{\lambda}), (\alpha, g) \mapsto \iota^* \ell_g^* \alpha$$

yields a G -intertwining operator between the Dolbeault cohomologies ([12, Thm. 2.6])

$$\mathcal{R}: H_{\bar{\partial}}^S(G/L, \mathcal{L}_{\lambda}) \rightarrow \mathcal{E}(G/K, \mathcal{V}_{\mu_{\lambda}})$$

if $\mu_{\lambda}|_T$ is $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ -dominant. We refer to \mathcal{R} as a generalized *Radon–Penrose transform*, or simply, as the *Penrose transform*.

We note that if the line bundle parameter λ is outside the “good range”, such an intertwining operator may vanish ([13, Ex. 1.5]). As an application of Theorem 1.2, we give a non-vanishing result when \mathfrak{q} is of holomorphic type.

Theorem 4.1. *Assume \mathfrak{q} is of holomorphic type. Then for any L -character \mathbb{C}_{λ} such that μ_{λ} is $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ -dominant, the Penrose transform \mathcal{R} is non-zero. To be more precise, $\text{Image } \mathcal{R}$ contains the K -type $V_{\mu_{\lambda}}$.*

Proof of Theorem 4.1. By Corollary 1.3, the pull-back of $\varpi: G/L \rightarrow K/L \cap K$ induces a K -intertwining operator

$$\varpi^*: \mathcal{E}^{0,S}(K/L \cap K, \iota^* \mathcal{L}_{\lambda}) \rightarrow \mathcal{E}^{0,S}(G/L, \mathcal{L}_{\lambda}),$$

which sends $\bar{\partial}$ -closed forms on $K/L \cap K$ to those on G/L .

Since $\varpi \circ \iota = \text{id}$, one has $\tilde{\mathcal{R}}(\cdot, e) \circ \varpi^* = \text{id}$ because $\iota^* = \tilde{\mathcal{R}}(\cdot, e)$. Hence the evaluation of the Penrose transform at the identity e induces an isomorphism

$$\mathcal{R}_e \circ \varpi^*: H_{\bar{\partial}}^S(K/L \cap K, \iota^* \mathcal{L}_{\lambda}) \longrightarrow V_{\mu_{\lambda}}.$$

Thus \mathcal{R} is non-zero. ■

Remark 4.2. As we have already mentioned, Theorem 4.1 is applicable to holomorphic line bundles for which the parameters are outside the weakly fair range [16].

(1) Our assumption on μ_{λ} is much weaker than that \mathbb{C}_{λ} is in the good range. In this generality, \mathcal{R} may have a kernel, see [12] for $G = Sp(n, \mathbb{R})$.

(2) The twistor transform between two elliptic coadjoint orbits involves parameters outside the weakly fair range [13, Thm. 4.1], but our theorem is still applicable as far as μ_{λ} is $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ -dominant.

(3) Theorem 4.1 was proved earlier when (G, L) is a symmetric pair for some classical Lie groups G . In [11] the proof of such partial results relies on the Stiefel coordinates for type A groups, and in [12] the proof relies on the theory of discretely decomposable restrictions [5, 7, 8] for type C groups.

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Received December 9, 2022
and in final form February 10, 2023