

Mirković-Vilonen Polytopes and Measures

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Abstract. In the framework of measures, we relate the affine Mirković-Vilonen polytopes defined by Baumann, Kamnitzer and Tingley in 2014 to Littelmann path model. Thanks to the retractions sending the measure onto the standard apartment, we recover the bottom part of the affine MV polytope. We are also able to read on the paths the partitions decorating these polytopes in the case of $\widehat{\mathfrak{sl}}_2$.

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1. Introduction

Given a finite type Lie algebra \mathfrak{g} , Mirković-Vilonen and Anderson [1] have defined the so-called *MV polytopes* as the image by the moment map of cycles in the affine Grassmannian. They provide a realization of the combinatorial Kashiwara crystals associated with \mathfrak{g} . These polytopes have been recovered in several ways: via preprojective algebras (Baumann-Kamnitzer [6]), Poincaré-Birkhoff-Witt (PBW) bases of the half quantum group $U_q^+(\mathfrak{g})$ (Lusztig, Kamnitzer [14]), diagrammatic Khovanov-Lauda-Rouquier (KLR) algebras (Tingley-Webster [25]), or through affine buildings and galleries (Ehrig [9]). It is interesting to wonder how these constructions extend to the affine setting and more generally to the Kac-Moody setting.

Outside the finite dimensional world, few is known. The affine Grassmannian does not really exist as an algebraic-geometry object, neither does the moment map. However, in [10], using the measure (the analogue of the Bruhat-Tits building in the Kac-Moody setting), the second author and Rousseau introduce a finite dimensional quasi-projective variety associated to each LS path that should play the role of the MV cycle. These varieties can be seen as varieties of segments $[x, \mu]$ in the measure that retract onto the given LS path (between 0 and μ), via the retraction coming from the parallel class of the negative fundamental Weyl chamber.

On the polytope side, Baumann-Kamnitzer-Tingley [7] used the point of view of preprojective algebras to naturally define *affine MV polytopes*, recovered later by PBW (resp. KLR) methods by Muthiah-Tingley [21] (resp. Tingley-Webster [25]).

The present paper explains how to define a pseudo-Weyl polytope from any LS path π , using the measure, more precisely, using the retractions associated to the (parallel coset of) negative Weyl chambers in the Tits' cone, inside the standard

apartment. In practice, one takes a generic segment $[x, \mu]$ in the variety mentioned above and retracts it from all possible negative Weyl chambers. Since these segments are defined by finitely many parameters, only a finite numbers of paths $\Upsilon'_w(\pi)$ are reached, indexed by an element of the Weyl group, all ending at μ . Taking the origins of each of these paths gives the vertices of the polytope.

We prove in Theorem 4.3 that the paths $\Upsilon'_w(\pi)$ can also be obtained by constructions in terms of the crystal associated to the starting LS path. The proof relies on the existence of an extended tree associated to any real root in the measure and of the associated parabolic retraction that factors through that tree.

In the finite dimensional case, the origins of all those paths are the vertices of the MV polytope. Hence, we recover Ehrig's main result [9, Theorem 8.1], using a rather different approach in the proof as he uses galleries but does not use parabolic retractions.

In the affine Kac-Moody case, the MV polytopes defined by Baumann-Kamnitzer-Tingley can be divided into 3 parts, the top, the bottom and the faces parallel to the smallest imaginary root δ which come decorated by partitions. In this affine case, we show that the origins of the paths $\Upsilon'_w(\pi)$ give the bottom part of the polytope.

While the top part can then be recovered using Kashiwara's involution, we find a method to recognize directly from paths the partitions decorating the affine MV polytopes in the case of $\widehat{\mathfrak{sl}}_2$.

In the first section, we recall the notions we need about measures. Then in the second one, we give some definitions regarding LS-paths and prove a few combinatorial lemmas about their crystal structure. In the third section, we obtain the first main result of this paper, Theorem 4.3, relating these crystal-type data to retractions of paths in measures. We prove this statement by induction using the parabolic retractions introduced in [11]. In section 4, we introduce affine MV polytopes, in the $\widehat{\mathfrak{sl}}_2$ case, essentially following [4].

In § 4.2, we link this section to the previous ones, stating that we recover the bottom part of the undecorated polytope associated to a path using retractions. We then prove combinatorial and technical lemmas used in the last section 5 to prove our second main result Theorem 6.8: these decorations, which are partitions, can be recognized on a specific class of paths, by simple examination. The combination of our results allows one to recover these decorations for *any* path in the crystal.

It is of course natural to wonder how to retrieve the decorations defined in [7] for any affine type, as has been done for instance in the above mentioned [25]. We believe that one could achieve this program by using the $\widehat{\mathfrak{sl}}_2$ case as an elementary step since the associated polytopes are the 2-faces of arbitrary affine MV polytopes. To this end, the notion of zigzag defined in 6.1 should easily be generalized.

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2. Recollections

2.1. The vectorial data

Let $\mathbb{M} = (\alpha_j(\alpha_i^\vee))_{i,j \in I}$ be a Kac-Moody matrix, *i.e.* a generalized Cartan matrix, meaning a matrix with non positive integers coefficients, 2's on the diagonal and with the symmetry of 0's.

We consider a root generating system $(\mathbb{M}, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$ where \mathbb{M} is the Kac-Moody matrix, X and Y are two dual free \mathbb{Z} -modules of finite rank, I a finite set, $(\alpha_i^\vee)_{i \in I}$ a family in Y and $(\alpha_i)_{i \in I}$ a family in the dual X . We suppose these families free, *i.e.* the sets $\{\alpha_i \mid i \in I\}$ and $\{\alpha_i^\vee \mid i \in I\}$ are linearly independent. Further, we assume that $\mathbb{M}_{i,j} = \alpha_j(\alpha_i^\vee)$.

Set $V = Y \otimes \mathbb{R}$, then every element of X defines a linear form on V and the formula $s_i(v) = v - \alpha_i(v)\alpha_i^\vee$ defines a linear involution in V . The subgroup generated by the s_i is W^v , the Weyl group of the corresponding Kac-Moody Lie algebra $\mathfrak{g}_{\mathbb{M}}$ and the associated real root system is

$$\Phi = \{w(\alpha_i) \mid w \in W^v, i \in I\} \subset Q = \bigoplus_{i \in I} \mathbb{Z}.\alpha_i.$$

We consider also the dual action of W^v on V^* .

We set $\Phi^\pm = \Phi \cap Q^\pm$ where $Q^\pm = \pm(\bigoplus_{i \in I} (\mathbb{Z}_{\geq 0}).\alpha_i)$. Also $Q^\vee := \bigoplus_{i \in I} \mathbb{Z}.\alpha_i^\vee$ and $Q_\pm^\vee = \pm(\bigoplus_{i \in I} (\mathbb{Z}_{\geq 0}).\alpha_i^\vee)$. We have $\Phi = \Phi^+ \cup \Phi^-$ and, for $\alpha = w(\alpha_i) \in \Phi$, $s_\alpha = ws_iw^{-1}$ and $s_\alpha(v) = v - \alpha(v)\alpha^\vee$, where the coroot $\alpha^\vee = w(\alpha_i^\vee)$ depends only on α . The set Φ is an (abstract) reduced real root system in the sense of [18], [19] or [2].

The fundamental positive chamber is $C_f^v = \{v \in V \mid \alpha_i(v) > 0, \forall i \in I\}$. Its closure $\overline{C_f^v}$ is the disjoint union of the vectorial faces:

$$F^v(J) = \{v \in V \mid \alpha_i(v) = 0, \forall i \in J, \alpha_i(v) > 0, \forall i \in I \setminus J\} \text{ for } J \subset I.$$

We set $V_0 = F^v(I) = V^{W^v}$.

The positive (resp. negative) vectorial faces are the sets $w.F^v(J)$ (resp. $-w.F^v(J)$) for $w \in W^v$ and $J \subset I$. The support of such a face is the vector space it generates. The set J or the face $w.F^v(J)$ or an element of this face is called spherical if the group $W^v(J)$ generated by $\{s_i \mid i \in J\}$ is finite. An element of a vectorial chamber $\pm w.C_f^v$ is called regular.

The Tits cone \mathcal{T} is the (disjoint) union of the positive vectorial faces. It is a W^v -stable convex cone in V . Actually W^v permutes the vectorial walls $M^v(\alpha) = \ker(\alpha)$ (for $\alpha \in \Phi$) ; it acts simply transitively on the positive (resp. negative) vectorial chambers.

2.2. The model apartment

As in [22, 1.4] the model apartment \mathbb{A} is V considered as an affine space and endowed with a family \mathcal{M} of walls. These walls are the affine hyperplanes directed by $\ker(\alpha)$:

$$M(\alpha, k) = \{v \in V \mid \alpha(v) + k = 0\} \quad \text{for } \alpha \in \Phi \text{ and } k \in \mathbb{Z}.$$

For $\alpha = w(\alpha_i) \in \Phi$, $k \in \mathbb{Z}$ and $M = M(\alpha, k)$, the reflection $s_{\alpha, k} = s_M$ with respect to M is the affine involution of \mathbb{A} with fixed points the wall M and associated linear involution s_α . In equation, this gives for any $x \in \mathbb{A}$,

$$s_{\alpha, k}(x) = x - (\alpha(x) + k)\alpha^\vee.$$

The affine Weyl group W^a is the group generated by the reflections s_M for $M \in \mathcal{M}$; actually, W^a stabilizes \mathcal{M} . We know that $W^a = W^v \rtimes Q^\vee$; here Q^\vee can be understood as the group of translations.

An automorphism of \mathbb{A} is an affine bijection $\varphi: \mathbb{A} \rightarrow \mathbb{A}$ stabilizing the set of pairs (M, α^\vee) of a wall M and the coroot α^\vee associated with $\alpha \in \Phi$ such that $M = M(\alpha, k)$, $k \in \mathbb{Z}$. We write $\vec{\varphi}: V \rightarrow V$ for the linear application associated to φ . The group $\text{Aut}(\mathbb{A})$ of these automorphisms contains W^a and normalizes it. We consider also the group $\text{Aut}^W(\mathbb{A}) = \{\varphi \in \text{Aut}(\mathbb{A}) \mid \vec{\varphi} \in W^v\}$ of vectorially-Weyl automorphisms. One has $\text{Aut}^W(\mathbb{A}) = W^v \rtimes P^\vee$, where $P^\vee = \{v \in V \mid \alpha(v) \in \mathbb{Z}, \forall \alpha \in \Phi\}$. For $\alpha \in \Phi$ and $k \in \mathbb{R}$, $D(\alpha, k) = \{v \in V \mid \alpha(v) + k \geq 0\}$ is a half-space. It is called a half-apartment if $k \in \mathbb{Z}$. We write $D(\alpha, \infty) = \mathbb{A}$.

The Tits cone \mathcal{T} is convex and W^v -stable, therefore, we can define a W^v -invariant preorder relation on \mathbb{A} : $x \leq y \Leftrightarrow y - x \in \mathcal{T}$.

2.3. The faces and sectors

The faces in \mathbb{A} are associated to the above systems of walls and half-apartments. As the set of walls might be dense in \mathbb{A} , seen as a finite dimensional real vector space, the faces are no longer subsets, but filters of subsets of \mathbb{A} . For the definition of that notion and its properties, we refer to [8] or [10].

If F is a subset of \mathbb{A} containing an element x in its closure, the *germ* of F in x is the filter $\text{germ}_x(F)$ consisting of all subsets of \mathbb{A} which contain intersections of F and neighbourhoods of x . We say that x is the *origin* of this germ. In particular, if $x \neq y \in \mathbb{A}$, we denote the germ in x of the segment $[x, y]$ by $[x, y)$. For $y \neq x$, the segment germ $[x, y)$ is called of sign \pm if $y - x \in \pm\mathcal{T}$. The segment $[x, y)$ or the segment germ $[x, y)$ is called preordered if $x \leq y$ or $y \leq x$.

Given F a filter of subsets of \mathbb{A} , its *enclosure* $\text{cl}_{\mathbb{A}}(F)$ (resp. *closure* \overline{F}) is the filter made of the subsets of \mathbb{A} containing an element of F of the shape $\bigcap_{\alpha \in \Phi} D(\alpha, k_\alpha)$, where $k_\alpha \in \mathbb{Z} \cup \{\infty\}$ (resp. containing the closure \overline{S} of some $S \in F$). Note that this definition is the definition of an enclosure used by Hébert in [12] which is simpler than the original one from [10].

A *local face* F in the apartment \mathbb{A} is associated to a point $x \in \mathbb{A}$, its vertex or origin, and a vectorial face $F^v =: \vec{F}$ in V , its direction. It is defined as $F = \text{germ}_x(x + F^v)$. Its closure is $\text{germ}_x(x + \overline{F^v})$. Its sign is the sign of F^v .

There is an order on the local faces: the assertions “ F is a face of F' ”, “ F' covers F ” and “ $F \leq F'$ ” are by definition equivalent to $F \subset \overline{F'}$. The *dimension* of a local face F is the smallest dimension of an affine space generated by some $S \in F$. The (unique) such affine space E of minimal dimension is the *support* of F ; if $F = \text{germ}_x(x + F^v)$, $\text{supp}(F) = x + \text{supp}(F^v)$. A local face is spherical if the direction of its support meets the open Tits cone (i.e. if F^v is spherical), then its pointwise stabilizer W_F in W^a is finite.

A *local chamber* is a maximal local face, i.e. a local face $\text{germ}_x(x \pm w.C_f^v)$ for $x \in \mathbb{A}$ and $w \in W^v$. The *fundamental local chamber* of sign \pm is $C_0^\pm = \text{germ}_0(\pm C_f^v)$.

A (local) *panel* is a spherical local face maximal among local faces which are not chambers, or, equivalently, a spherical face of dimension $n - 1$. Its support is a *wall*. Sometimes the adjective “local” will be dropped out the notation.

A *sector* in \mathbb{A} is a V -translate $\mathfrak{s} = x + C^v$ of a vectorial chamber $C^v = \pm w.C_f^v$, $w \in W^v$. The point x is its *base point* and $C^v =: \vec{\mathfrak{s}}$ its direction. Two sectors have the same direction if, and only if, they are conjugate by V -translation, and if, and only if, their intersection contains another sector.

The *sector-germ* of a sector $\mathfrak{s} = x + C^v$ in \mathbb{A} is the filter \mathfrak{S} of subsets of \mathbb{A} consisting of the sets containing a V -translate of \mathfrak{s} , it is well determined by the direction $C^v = \vec{\mathfrak{s}} =: \vec{\mathfrak{S}}$. So, the set of translation classes of sectors in \mathbb{A} , the set of vectorial chambers in V and the set of sector-germs in \mathbb{A} are in canonical bijection. We denote the sector-germ associated to the fundamental vectorial chamber $\pm C_f^v$ by $\mathfrak{S}_{\pm\infty}$.

A *sector-face* in \mathbb{A} is a V -translate $\mathfrak{f} = x + F^v$ of a vectorial face $F^v = \pm w.F^v(J)$. The *sector-face-germ* of \mathfrak{f} is the filter \mathfrak{F} of subsets containing a translate \mathfrak{f}' of \mathfrak{f} by an element of F^v (*i.e.* $\mathfrak{f}' \subset \mathfrak{f}$). If F^v is spherical, then \mathfrak{f} and \mathfrak{F} are also called spherical. The sign of \mathfrak{f} and \mathfrak{F} is the sign of F^v .

2.4. The measure

In this section, we recall some properties of the measure as defined in [23] and in [13].

An apartment of type \mathbb{A} is a set A endowed with a set $\text{Isom}^W(\mathbb{A}, A)$ of bijections, called *Weyl-isomorphisms*, such that, if $f_0 \in \text{Isom}^W(\mathbb{A}, A)$, then $f \in \text{Isom}^W(\mathbb{A}, A)$ if, and only if, there exists $w \in W^a$ satisfying $f = f_0 \circ w$. If $A = \mathbb{A}$, one has $\text{Isom}^W(\mathbb{A}, A) = \text{Aut}^W(\mathbb{A})$.

A Weyl-isomorphism between two apartments $\varphi : A \rightarrow A'$ is a bijection such that, for any $f \in \text{Isom}^W(\mathbb{A}, A)$, $f' \in \text{Isom}^W(\mathbb{A}, A')$, we have $f'^{-1} \circ \varphi \circ f \in W^a$. The set of these isomorphisms is written $\text{Isom}^W(A, A')$. Thanks to these isomorphisms, faces, local faces, sectors, Tits cone ... are defined in any apartment of type \mathbb{A} .

A *measure* of type \mathbb{A} is a set \mathcal{S} endowed with a covering \mathcal{A} of subsets called apartments, each endowed with a structure of an apartment of type \mathbb{A} . We do not recall here the precise definition, which was simplified by Hébert [13]. We indicate some of its main properties:

- (a) If \mathfrak{F} is the germ of a spherical sector face and if F is a face or a germ of a spherical sector face, then there exists an apartment that contains \mathfrak{F} and F .
- (b) Given two apartments A, A' , their intersection $A \cap A'$ is a finite intersection of half-apartments. Further, there exists a Weyl-isomorphism from A onto A' fixing $A \cap A'$.
- (c) We consider the relation \leq on \mathcal{S} defined as follows:

$$x \leq y \iff \exists A \in \mathcal{A} \text{ such that } x, y \in A \text{ and } x \leq_A y$$

then \leq is a well defined preorder, in particular transitive, that extends the preorder on \mathbb{A} given by the Tits cone, see [22].

A filter or subset in \mathcal{S} is called a preordered segment, a preordered segment germ, a local face, a spherical sector face or a spherical sector face germ if it is included in some apartment A and is called like that in A .

2.5. The group

Set $\mathcal{K} = \mathbb{C}((t))$ and denote by $\mathcal{O} = \mathbb{C}[[t]]$ its ring of integers and by val the associated discrete valuation. Let $G = G(\mathcal{K})$ be a split minimal Kac-Moody group over \mathcal{K} , with Lie algebra \mathfrak{g}_M , defined by Tits, see [17, 7]. We denote by $T = T(\mathcal{K})$ the maximal torus such that the \mathbb{Z} -lattice of coweights $\text{Hom}(\mathcal{K}^*, T)$ is Y , and the dual \mathbb{Z} -lattice of weights $\text{Hom}(T, \mathcal{K}^*)$ is X . Let $B = B^+ = B(\mathcal{K})$ be the Borel subgroup associated to the choice of simple roots made in 2.1 and let B^- be the opposite Borel subgroup in G . The set of real roots of (G, T) is Φ and the \mathbb{Z} -lattice of coroots is Q^\vee . Finally the Weyl group of (G, T) identifies with W^v .

To each real root α corresponds a subgroup $U_\alpha = U_\alpha(\mathcal{K})$ isomorphic to $(\mathcal{K}, +)$, via $x_\alpha : \mathcal{K} \rightarrow U_\alpha$. Then the Borel subgroup decomposes as $B^\pm = TU^\pm$, for U^\pm generated by the subgroups U_α in G , for $\alpha \in \Phi^\pm$. And to any $k \in \mathbb{Z}$, one defines $U_{\alpha,k} = x_\alpha(\{u \in \mathcal{K}, \text{val}(u) \geq k\})$, a subgroup of U_α .

We consider now the measure $\mathcal{I} = \mathcal{I}(G, \mathcal{K})$ as defined in [23]. It satisfies the properties listed in 2.4. And even better, the group G acts upon it in a such a way that all apartments are given by this action, for any $A \in \mathcal{A}$, there exists $g \in G$, $A = g \cdot \mathbb{A}$. Further, one has the following properties for some fixators (pointwise stabilizers):

- $\text{Fix}_G(\mathbb{A}) = T(\mathcal{O})$;
- $\text{Fix}_G(D(\alpha, k)) = T(\mathcal{O})U_{\alpha,k}$ and all the apartments containing $D(\alpha, k)$ are given by the action of an element of the form $x_\alpha(u)$, $\text{val}(u) \geq k$;
- $\text{Fix}_G(\mathfrak{S}_{\pm\infty}) = T(\mathcal{O})U^\pm$;
- $\text{Fix}_G(\{0\})$ should be thought of as the minimal Kac-Moody group over \mathcal{O} . Even though $\text{Fix}_G(\{0\}) = G(\mathcal{O})$ for G finite dimensional and in the untwisted affine case [11, 3.4], it is not known in general.

For any subset or filter of subsets Ω of \mathcal{I} , we denote the fixator of Ω by G_Ω , in the case of $\Omega = \{x\}$, we just write G_x .

If N is the stabilizer of \mathbb{A} in G , there exists an homomorphism $\nu : N \rightarrow \text{Aut}(\mathbb{A})$ such that the image group $W_Y = W^v \rtimes Y$ permutes the walls, local faces, sectors, sector-faces... and contains the affine Weyl group $W^a = W^v \rtimes Q^\vee$ [24, 4.13.1]. The group T acts by translations as follows: for $t \in T$, $\nu(t)$ is defined by $\chi(\nu(t)) = -\text{val}\chi(t)$, for all $\chi \in X$. For example, an element $\mu(t) \in T$, for $\mu \in Q^\vee$ will act by the translation by $-\mu$.

2.6. The local behaviour of the action

Let x be a point in the standard apartment \mathbb{A} . Let Φ_x be the set of all roots α such that $\alpha(x) \in \mathbb{Z}$. It is a closed subsystem of roots. Its associated Weyl group W_x^v is a Coxeter group.

We have twinned buildings \mathcal{I}_x^+ and \mathcal{I}_x^- whose elements are segment germs $[x, y)$ for $y \in \mathcal{I}$, $y \neq x$, $y \geq x$ and $y \leq x$, respectively. We consider their unrestricted structure, so the associated Weyl group is W^v and the chambers (resp. closed chambers) are the local chambers $C = \text{germ}_x(x + C^v)$ (resp. local closed chambers $\overline{C} = \text{germ}_x(x + \overline{C^v})$), where C^v is a vectorial chamber, cf. [10, 4.5] or [22, § 5].

To \mathbb{A} is associated a twinned apartment $\mathbb{A}_x = (\mathbb{A}_x^-, \mathbb{A}_x^+)$ so that the walls and half-apartments of \mathbb{A} give walls and half-apartments in each apartment \mathbb{A}_x^- and \mathbb{A}_x^+ .

Let \mathcal{I}_x be the union of \mathcal{I}_x^+ and \mathcal{I}_x^- in \mathcal{I} . The group $\overline{G}_x = G_x/G_{\mathcal{I}_x}$ acts on \mathcal{I}_x^+ and \mathcal{I}_x^- . For any root $\alpha \in \Phi_x$ with $\alpha(x) = k \in \mathbb{Z}$, the group $\overline{U}_\alpha = U_{\alpha,k}/U_{\alpha,k+1}$ is a subgroup of \overline{G}_x that can be identified set-theoretically with the elements $x_\alpha(at^k)$, with $a \in \mathbb{C}$. Furthermore, in the twinned buildings \mathcal{I}_x^+ and \mathcal{I}_x^- , elements of the form $x_\alpha(at^k)$, $a \in \mathbb{C}$, act transitively on the apartments containing the half-apartment given by $D(\alpha, k)$. See [11], Section 4.1, for more details.

2.7. The paths

The elements of Y , through the identification $Y = N.0$, are called *vertices* of type 0 in \mathbb{A} .

We consider piecewise linear continuous paths $\pi : [0, 1] \rightarrow \mathbb{A}$ such that each (existing) tangent vector $\pi'(t)$ belongs to an orbit $W^v.\lambda$ for some $\lambda \in \overline{C}_f^v$. Such a path is called a λ -*path*; it is increasing with respect to the preorder relation \leq on \mathbb{A} , see [10, 3.1].

If in addition, we ask that the endings of the path $\pi(0)$ and $\pi(1)$ are vertices of type 0, then it called a λ -path in Y .

For any $t \neq 0$ (resp. $t \neq 1$), we let $\pi'_-(t)$ (resp. $\pi'_+(t)$) denote the derivative of π at t from the left (resp. from the right). These derivatives are identified with elements of the twinned apartments $(\mathbb{A}_x^-, \mathbb{A}_x^+)$. For $x = \pi(t)$, we will also identify them with $-\pi(t - \varepsilon)$ (resp. $\pi(t + \varepsilon)$), for a small real number $\varepsilon > 0$, considering that $-\pi(t - \varepsilon)$ (resp. $\pi(t + \varepsilon)$) is the germ at x of the segment $-[x, \pi(t - \varepsilon)]$ (resp. $[x, \pi(t + \varepsilon)]$).

Let π be a λ -path in \mathbb{A} and consider $w \in W^v$, $x = \pi(t)$. A $w.\mathfrak{S}_{-\infty}$ -chain from $\pi'_-(t)$ to $\pi'_+(t)$ is a finite sequence $(\pi'_-(t) = \xi_0, \xi_1, \dots, \xi_s = \pi'_+(t))$ such that $r_i(\xi_{i-1}) = \xi_i$ for some reflection $r_i = r_{M_i}$ in W_x^v , and such that the wall M_i separates ξ_i from ξ_{i-1} and $w.\mathfrak{S}_{-\infty}$. This condition means that there exists a V -translate of $-w.C_f^v$ separated from ξ_i by M_i .

A λ -path π is a *Hecke path* with respect to $w.\mathfrak{S}_{-\infty}$ if for all $t \in [0, 1] \setminus \{0, 1\}$ there exists a $w.\mathfrak{S}_{-\infty}$ -chain from $\pi'_-(t)$ to $\pi'_+(t)$. If $w = id$, one recovers the definition of a Hecke path from [10, 3.1.1].

For a λ -path π , a wall M is left positively by the opposite path $\bar{\pi} : s \mapsto \pi(1 - s)$, with respect to $w.\mathfrak{S}_{-\infty}$, if there exists $t \in [0, 1]$, such that $\bar{\pi}(t) \in M$ and M separates $-\pi'_-(1 - t)$ from $w.\mathfrak{S}_{-\infty}$.

For any $\beta \in \Phi^-$, consider

$$\text{pos}_\beta^w(\pi) = \#\{M = M(\beta, k), M \text{ is left positively by } \bar{\pi} \text{ with respect to } w.\mathfrak{S}_{-\infty}\}.$$

Further, set
$$\text{ddim}^w(\pi) = \sum_{\beta \in \Phi^-} \text{pos}_\beta^w(\pi).$$

Thanks to this statistic on paths, we define an LS path of shape λ with respect to $w.\mathfrak{S}_{-\infty}$ to be a path π in Y such that $\text{ddim}^w(\pi) = \rho(\lambda - \mu)$, for $\mu = \pi(1) - \pi(0)$. Here ρ is any solution of $\rho(\alpha_i^\vee) = 1$, for all i .

As shown in [10], Section 5.3, this definition of LS paths is equivalent to the Littelmann's one for $w = id$, and we recover the classical notion of the LS paths. We will use the term LS paths for the LS paths with respect to $\mathfrak{S}_{-\infty}$.

3. Sections on paths

In this section, we translate the definitions of [9, §5] into the world of paths. Particularly, we introduce the paths $\Upsilon'_w(\pi)$ in terms of crystals alluded to in the introduction. Their origins are the vertices of a pseudo-Weyl polytope.

3.1. The definition of sections

Definition 3.1. Let π be an LS path of shape λ and let us fix a real positive root α , not necessarily simple. An interval $[t_i, t_j] \subset [0, 1]$ is

- an α -zero section at $m \in \mathbb{Z}$ if

$$\alpha(\pi(t_i)) = m = \alpha(\pi(t_j)), \quad \forall t \in]t_i, t_j[, \quad \alpha(\pi(t)) = m;$$

- an α -stable section at $m \in \mathbb{Z}$ if

$$\alpha(\pi(t_i)) = m = \alpha(\pi(t_j)), \quad \forall t \in]t_i, t_j[, \quad \alpha(\pi(t)) > m$$

and $[t_i, t_j]$ is maximal with respect to the existence of such an m ;

- an α -directed section at $m \in \mathbb{Z}$ if

$$\alpha(\pi(t_i)) = m, \quad \alpha(\pi(t_j)) = m + 1 \quad \text{and} \quad \forall t \in]t_i, t_j[, \quad m < \alpha(\pi(t)) < m + 1 ;$$

- a $-\alpha$ -directed section at $m \in \mathbb{Z}$ if

$$\alpha(\pi(t_i)) = m, \quad \alpha(\pi(t_j)) = m - 1 \quad \text{and} \quad \forall t \in]t_i, t_j[, \quad m - 1 < \alpha(\pi(t)) < m.$$

Proposition 3.2. *To any LS path π with respect to $\mathfrak{S}_{-\infty}$, and any simple root α , there exists a unique partition $0 = t_1 < \dots < t_\ell = 1$ of $[0, 1]$ such that each interval $[t_i, t_{i+1}]$, $1 \leq i \leq \ell - 1$, is either an α -zero, α -stable, α -directed or $-\alpha$ -directed section.*

Proof. If $\alpha(\pi'_+([0, 1])) = 0$, the whole path lies in the wall $\ker(\alpha)$ and is an α -zero section. Otherwise there exists a minimal $t \in [0, 1]$ such that $\alpha(\pi'_+(t)) \neq 0$, which we denote t_1 if it is 0, t_2 if it is > 0 . Let us assume the latter without loss of generality.

Set $\pi(0) = k$. If $\alpha(\pi'_+(t_2)) > 0$ and if there exists a $t_3 > t_2$ such that $\alpha(\pi(t_3)) = k$, then for the smallest one, we have an α -stable section. If the path does not come back to the level $\alpha = k$, there must be a t_3 such that $\alpha(\pi(t_3)) = k + 1$ since our path is LS and $\pi(1) \in Y$, so for the smallest one, we have an α -directed section.

If $\alpha(\pi'_+(t_2)) < 0$, let $t_3 > t_2$ be the smallest possible such that $\alpha(\pi(t_2)) = k - 1$, which exists since our path is LS, see [16, 4.5]. Then $[t_2, t_3]$ is a $-\alpha$ -directed section. We conclude by iterating this process. ■

Remark 3.3. Note that a $(-\alpha)$ -directed section can not occur after an α -directed one, because such a sequence would then be part of an α -stable section.

3.2. The root operators and the sections

Sections are compatible with the Littelmann's operators. Let π be an LS path in Y with respect to $\mathfrak{S}_{-\infty}$ and let α be a simple root. Set $Q = \min\{\alpha(\pi(t)), t \in [0, 1]\}$: as π is an LS path, this minimum is an integer, see [16, 4.5].

Now fix $q = \min\{t \in [0, 1], \alpha(\pi(t)) = Q\}$ and $p = \max\{t \in [0, 1], \alpha(\pi(t)) = Q\}$.

If $Q = 0$ one sets $e_\alpha\pi = 0$, whereas if $Q \leq -1$, take $y \in [0, 1]$ such that $\alpha(\pi(y)) = Q + 1$ and for $y < t < q$, $Q < \alpha(\pi(t)) < Q + 1$. Then one cuts the path π into three pieces: $\pi = \pi_1 * \pi_2 * \pi_3$, with $\pi_1(t) = \pi(ty)$, $\pi_2(t) = \pi(y + t(q - y)) - \pi(y)$ and $\pi_3(t) = \pi(q - t(1 - q)) - \pi(q)$. And $e_\alpha\pi = \pi_1 * s_\alpha(\pi_2) * \pi_3$.

For f_α , the definition is analogous. If $\alpha(\pi(1)) - Q = 0$, one sets $f_\alpha\pi = 0$, but if $\alpha(\pi(1)) - Q \geq 1$, take $x \in [p, 1]$ such that $\alpha(\pi(x)) = Q + 1$ and for $p < t < x$, $Q < \alpha(\pi(t)) < Q + 1$. Then one cuts the path π into three pieces: $\pi = \pi_1 * \pi_2 * \pi_3$, with $\pi_1(t) = \pi(tp)$, $\pi_2(t) = \pi(p + t(x - p)) - \pi(p)$ and $\pi_3(t) = \pi(x - t(1 - x)) - \pi(x)$. And $f_\alpha\pi = \pi_1 * s_\alpha(\pi_2) * \pi_3$.

By definition, $[y, q]$ is a $-\alpha$ -directed section for π and an α -directed one for $e_\alpha\pi$. Likewise, $[p, x]$ is an α -directed section for π and a $-\alpha$ -directed one for $f_\alpha\pi$. So the partition of π , defined in Proposition 3.2, is preserved by the operators; directed sections might have a change of sign and the stable sections are eventually translated.

Further, one of the main result in the theory of Littelmann paths is that the set of LS paths with respect to $\mathfrak{S}_{-\infty}$ is stable under the root operators.

Next let us define the flip $(\pi)_{-\alpha}$ of π with respect to α . Cut π according to the partition: $\pi = \pi_0 * \pi_1 * \dots * \pi_\ell$, where, for $0 \leq i \leq \ell$, $\pi_i(t) = \pi(t_i - t(t_{i+1} - t_i)) - \pi(t_i)$. Then $(\pi)_{-\alpha}$ is obtained by concatenating the π_i if $[t_i, t_{i+1}]$ is not stable and the $s_\alpha(\pi_i)$ if $[t_i, t_{i+1}]$ is stable. The condition of being LS is not preserved by this operation.

As usual, given π and α , we set $e_\alpha^{\max}\pi$ to be $e_\alpha^m\pi$, where m is the maximal integer such that this quantity is nonzero. Same with $f_\alpha^{\max}\pi$.

Proposition 3.4. *For any LS path π and any real positive root α , $(e_\alpha^{\max}\pi)_{-\alpha} = s_\alpha f_\alpha^{\max}\pi$.*

Proof. The operation $f_\alpha^{\max}\pi$ turns all the α -directed sections into $-\alpha$ ones such that this path has only zero, stable and $-\alpha$ -directed sections. Then by taking the reflection s_α , the α -directed sections are coming back to their positions and the stables are also reflected. This is exactly what the operation $(e_\alpha^{\max}\pi)_{-\alpha}$ does. ■

Now, let $w = s_{i_1} \dots s_{i_n}$ be a reduced expression of an element w in the Weyl group. Let us define two sequences of paths. First, set $\Theta_0 = \pi$ an LS path of shape λ with $\pi(0) = 0$. And, for $1 \leq k \leq n$, set

$$\Theta_k = f_{\alpha_{i_k}}^{\max}(\Theta_{k-1}) = f_{\alpha_{i_k}}^{\max} f_{\alpha_{i_{k-1}}}^{\max} \dots f_{\alpha_{i_1}}^{\max}(\pi).$$

Second, set $\Upsilon_0(\pi) = \pi$ and, for $1 \leq k \leq n$,

$$\Upsilon_k(\pi) = w(k) f_{\alpha_{i_k}}^{\max}(w(k-1)^{-1} \Upsilon_{k-1}(\pi)),$$

where $w(k) = s_{i_1} \dots s_{i_k}$. A result of Kashiwara, in Section 7.2 of [15], on crystals implies that these paths do not depend on the reduced expression of w . So we set

$$\Upsilon_w(\pi) = w\Theta_w = w f_w^{\max}(\pi),$$

where $f_w^{\max} = f_{\alpha_{i_n}}^{\max} f_{\alpha_{i_{n-1}}}^{\max} \dots f_{\alpha_{i_1}}^{\max}$ and $\Theta_w := \Theta_n$. One can note that Θ_k is LS for any k and that $\Upsilon_w(\pi)$ is LS with respect to $w\mathfrak{S}_{-\infty}$. So if s_α , for a simple root α , is such that $\ell(ws_\alpha) > \ell(w)$ then $\Upsilon_{ws_\alpha}(\pi) = \Upsilon_{s_w(\alpha)}(\Upsilon_w(\pi))$.

Consider now, for any $w \in W^v$, $\Upsilon'_w(\pi) = \Upsilon_w(\pi) + (\pi(1) - \Upsilon_w(\pi)(1))$, the path $\Upsilon_w(\pi)$ translated by the vector $\pi(1) - \Upsilon_w(\pi)(1)$. As the partition of π is preserved by the operators (some sections might have a change of sign) and by the translations, we will consider sections of $\Upsilon'_w(\pi)$ which are $\pm w(\alpha)$ -directed or stable with respect to walls $M(-w(\alpha), k)$, for α simple.

3.3. A pseudo-Weyl polytope in terms of paths

Recall that a pseudo-Weyl polytope is a convex polytope in \mathbb{A} such that all edges are parallel to roots, see for example [25, 1.3]. Our goal is to define a pseudo-Weyl polytope associated to a given LS path. This is achieved by the next two propositions.

Proposition 3.5. *Take $w \in W^v$ and let $w = s_{i_1} \cdots s_{i_n}$ be a reduced expression. Set $\beta_n^\vee = w(n-1)(\alpha_{i_n}^\vee)$. Then*

$$\Upsilon'_{ws_{i_n}}(\pi)(0) - \Upsilon'_w(\pi)(0) = \varepsilon_{\alpha_{i_n}}(\Theta_{n-1})\beta_n^\vee,$$

where $\varepsilon_{\alpha_{i_n}}$ applied to an LS path is the number of times one can apply the root operator $e_{\alpha_{i_n}}$ to that path.

Proof. Let us compute :

$$\begin{aligned} &\Upsilon'_{ws_{i_n}}(\pi)(0) - \Upsilon'_w(\pi)(0) \\ &= \Upsilon_{ws_{i_n}}(\pi)(0) + (\pi(1) - \Upsilon_{ws_{i_n}}(\pi)(1)) - \Upsilon_w(\pi)(0) - (\pi(1) - \Upsilon_w(\pi)(1)). \end{aligned}$$

Since $\Upsilon_{ws_{i_n}}(\pi)(0) = \Upsilon_w(\pi)(0) = 0$, we get :

$$\Upsilon'_{ws_{i_n}}(\pi)(0) - \Upsilon'_w(\pi)(0) = \Upsilon_w(\pi)(1) - \Upsilon_{ws_{i_n}}(\pi)(1).$$

If we multiply this by $w(n-1)^{-1}$, we obtain :

$$\begin{aligned} &w(n-1)^{-1}(\Upsilon'_{ws_{i_n}}(\pi)(0) - \Upsilon'_w(\pi)(0)) \\ &= (w(n-1)^{-1}\Upsilon_w(\pi))(1) - (w(n-1)^{-1}\Upsilon_{ws_{i_n}}(\pi))(1) \\ &= (s_{i_n}\Theta_n)(1) - \Theta_{n-1}(1) = (s_{i_n}f_{\alpha_{i_n}}^{\max}\Theta_{n-1})(1) - \Theta_{n-1}(1) \\ &= (e_{\alpha_{i_n}}^{\max}\Theta_{n-1})_{-\alpha_{i_n}}(1) - \Theta_{n-1}(1) = (e_{\alpha_{i_n}}^{\max}\Theta_{n-1})(1) - \Theta_{n-1}(1) \\ &= \varepsilon_{\alpha_{i_n}}(\Theta_{n-1})\alpha_{i_n}^\vee. \end{aligned} \quad \blacksquare$$

Now, we consider the crystal $\mathcal{B}(\infty)$ associated to the Langlands dual G^\vee and identify it with the union of all the crystals \mathcal{B}_λ , where \mathcal{B}_λ is the set of all the LS paths of shape λ starting at 0. Recall that in [7] are defined affine Mirković-Vilonen polytopes P_b attached to any crystal element $b \in \mathcal{B}(\infty)$, as some pseudo-Weyl polytopes. Thanks to [5], Proposition 3.5 implies the following analog of [9, Proposition 6.7].

Proposition 3.6. *In the affine case, the convex hull of $\{\Upsilon'_w(\pi)(0) \mid w \in W^v\}$ is the bottom part of the affine Mirković-Vilonen polytope P_b , if b corresponds to π .*

Let us give here some results on the paths $\Upsilon'_w(\pi)$ for later use.

Lemma 3.7. *Set $m = \min_{t \in [0,1]} (w(\alpha)(\Upsilon'_w(\pi)(t)))$, then we have*

$$m = w(\alpha)(\pi(1) - \Upsilon_w(\pi)(1)) - \varepsilon_\alpha(\Theta_w).$$

Proof. Indeed, with $\mu = \pi(1)$,

$$\begin{aligned} \min\{w(\alpha)(\Upsilon'_w(\pi)(t)), t \in [0, 1]\} &= \min\{w(\alpha)(\Upsilon_w(\pi)(t) + (\mu - \Upsilon_w(\pi)(1))), t \in [0, 1]\} \\ &= w(\alpha)(\mu - \Upsilon_w(\pi)(1)) + \min\{w(\alpha)(\Upsilon_w(\pi)(t), t \in [0, 1]\} \\ &= w(\alpha)(\mu - \Upsilon_w(\pi)(1)) + \min\{w(\alpha)(w\Theta_w(t), t \in [0, 1]\} \\ &= w(\alpha)(\mu - \Upsilon_w(\pi)(1)) + \min\{\alpha(\Theta_w(t)), t \in [0, 1]\} \\ &= w(\alpha)(\mu - \Upsilon_w(\pi)(1)) - \varepsilon_\alpha(\Theta_w). \end{aligned} \quad \blacksquare$$

Lemma 3.8. Set $t = \min\{q \in [0, 1] \mid w(\alpha)(\Upsilon'_w(\pi))(q) = m\}$.

Then $\Upsilon'_{ws_\alpha}(\pi)(t) = s_{-w(\alpha),m}(\Upsilon'_w(\pi)(t))$, where $s_{-w(\alpha),m}$ is the reflection along the wall $M(-w(\alpha), m)$ in \mathbb{A} .

Proof. On the one hand we have

$$\begin{aligned} s_{-w(\alpha),m}(\Upsilon'_w(\pi)(t)) &= \Upsilon'_w(\pi)(t) - w(\alpha)\Upsilon'_w(\pi)(t)w(\alpha)^\vee + mw(\alpha)^\vee \\ &\stackrel{3.7}{=} \Upsilon'_w(\pi)(t) - w(\alpha)\Upsilon'_w(\pi)(t)w(\alpha)^\vee + (w(\alpha)(\mu - \Upsilon_w(\pi)(1)) - \varepsilon_\alpha(\Theta_w))w(\alpha)^\vee \\ &\stackrel{3.5}{=} \Upsilon'_w(\pi)(t) - w(\alpha)\Upsilon'_w(\pi)(t)w(\alpha)^\vee + w(\alpha)(\mu - \Upsilon_w(\pi)(1))w(\alpha)^\vee + \Upsilon_w(\pi)(1) \\ &\quad - \Upsilon_{ws_\alpha}(\pi)(1). \end{aligned}$$

On the other hand

$$\begin{aligned} \Upsilon_{ws_\alpha}(\pi)(t) &= \Upsilon_{ws_\alpha}(\pi)(t) = s_{w(\alpha)}f_{w(\alpha)}^{\max}(\Upsilon_w(\pi)(t)) \stackrel{3.4}{=} e_{w(\alpha)}^{\max}(\Upsilon_w(\pi)(t))_{-w(\alpha)} \\ &= s_{-w(\alpha)}(\Upsilon_w(\pi)(t)), \end{aligned}$$

which amounts to

$$\begin{aligned} \Upsilon'_{ws_\alpha}(\pi)(t) &= \mu - \Upsilon_{ws_\alpha}(\pi)(1) + s_{-w(\alpha)}(\Upsilon'_w(\pi)(t) - \mu + \Upsilon_w(\pi)(1)) \\ &= \mu - \Upsilon_{ws_\alpha}(\pi)(1) + \Upsilon'_w(\pi)(t) - \mu + \Upsilon_w(\pi)(1) \\ &\quad - w(\alpha)(\Upsilon'_w(\pi)(t) - \mu + \Upsilon_w(\pi)(1))w(\alpha)^\vee \end{aligned}$$

which is exactly what we want. \blacksquare

4. Retractions

In this section, we introduce the parabolic retractions in the measure and show how to compute retractions of a point and of a path. We recover the paths $\Upsilon'_w(\pi)$ as images of the retractions indexed by $w \in W$ of a generic segment $[x, \mu]$.

4.1. The definitions

Let \mathfrak{S} be any sector germ. For any $x \in \mathcal{I}$ there is an apartment containing x and \mathfrak{S} [22, 5.1] and this apartment is obtained from \mathbb{A} by the action an element of G fixing \mathfrak{S} . So, by the usual arguments, using [10, 4.4], we can define the retraction $\rho_{\mathfrak{S}}$ of \mathcal{I} onto the apartment \mathbb{A} with center the sector germ \mathfrak{S} .

For any such retraction $\rho_{\mathfrak{S}}$, the image of any segment $[x, y]$ with $x \leq y$ is a λ -path, for some $\lambda \in \overline{C_f^\vee}$. Actually, the image by $\rho_{\mathfrak{S}}$ of any segment $[x, y]$ with $x \leq y$ and x, y in Y is a Hecke path of shape λ with respect to \mathfrak{S} .

From now on we will denote by ρ^w the retraction associated to the sector germ $w \cdot \mathfrak{S}_{-\infty}$ for $w \in W^v$. We need a closer description of this retraction. Take $x \in \mathcal{I}$, and set $a = \rho^w(x) \in \mathbb{A}$. According to the properties of the measure, there exists an apartment A containing $w \cdot \mathfrak{S}_{-\infty}$ and x ; A is given by $A = g \cdot \mathbb{A}$, for some $g \in G$, and $x = g \cdot a$. But we have the Iwasawa decomposition $G = U_w^- N K_a$, where $U_w^- = wU^-w^{-1}$, N is the stabiliser of \mathbb{A} and K_a is the stabiliser of a .

Hence, there exists an element $z = n \cdot a \in \mathbb{A}$, for some $n \in N$ such that $x = \prod x_{\beta_j}(a_j t^{n_j}) \cdot z$, where $a_j \in \mathbb{C}$ and the product runs over some roots $\beta_j = w(\eta_j)$ for η_j a negative root. A priori, this product is infinite. But thanks to [3, Section 3], the product can be assumed to be finite. In the sequel of the paper, see Section 4.3, the point x is given as so from a finite number of parameters.

Let $\gamma_w : \mathbb{C}^\times \rightarrow G$ be a regular one-parameter subgroup such that $w^{-1} \cdot \gamma_w$ is antidominant. Let us compute $\gamma_w(s) \cdot x$. We have:

$$\gamma_w(s) \cdot x = \prod x_{\beta_j}(s^{\beta_j(\gamma_w(s))} a_j t^{n_j}) \cdot z,$$

since $\gamma_w(s) \cdot a = a$, as $\gamma_w(s) \in T(\mathbb{C}) \subset T(\mathcal{O})$. But

$$\beta_j(\gamma_w(s)) = w(\eta_j)(\gamma_w(s)) = \eta_j(w^{-1} \cdot \gamma_w(s)),$$

where η_j is a negative root and $w^{-1} \cdot \gamma_w(s)$ is in the antidominant chamber. So all the powers of s are positive in the finite product. Therefore, we can let $s = 0$ and compute the “limit”:

$$\lim_{s \rightarrow 0} \gamma_w(s) \cdot x = \prod x_{\beta_j}(s^{\beta_j(\gamma_w(s))} a_j t^{n_j})|_{s=0} \cdot z = z.$$

Let us record that as a lemma.

Lemma 4.1. *Consider $w \in W^v$. Let $\gamma_w : \mathbb{C}^\times \rightarrow G$ be a regular one-parameter subgroup such that $w^{-1} \cdot \gamma_w$ is antidominant. Let x be a point in \mathcal{I} , then $\rho^w(x) = \lim_{s \rightarrow 0} \gamma_w(s) \cdot x$.*

Let $w(\alpha)$ be a positive root, with $w \in W^v$ and $\alpha = \alpha_i$ a simple root. The measure \mathcal{I} contains the extended tree \mathcal{T}^w associated to $(\mathbb{A}, w(\alpha))$ that was defined in [11] under the name $\mathcal{I}(M_\infty)$. Its standard apartment is \mathbb{A} as affine space, the standard apartment of the measure, but with only walls the walls directed by $\ker w(\alpha)$. There, it is also proven that the retraction ρ^w factorizes through \mathcal{T}^w and equals the composition

$$\rho^w : \mathcal{I} \xrightarrow{\rho_1^w} \mathcal{T}^w \xrightarrow{\rho_2^{\pm, w}} \mathbb{A},$$

where ρ_1^w is the parabolic retraction defined in 5.6 of [11], associated to the panel germ \mathfrak{F}^w , germ of the panel $w(-F^v(\{i\}))$, and $\rho_2^{\pm, w}$ is the retraction with center the end $\pm\infty_{\mathcal{T}^w}$, i.e. the class of half-apartments in \mathcal{T}^w containing $w\mathfrak{S}_{\pm\infty}$.

Adapting the definitions and the proof of Lemma 4.1, we express the parabolic retraction in terms of the corresponding one-parameter subgroup.

Lemma 4.2. *Let $\gamma_1^w : \mathbb{C}^\times \rightarrow G$ be an antidominant one-parameter subgroup such that, for all s , $\alpha(w^{-1}\gamma_1^w(s)) = 0$ and, for a simple root $\alpha_j \neq \alpha$, $\alpha_j(w^{-1}\gamma_1^w(s)) < 0$. Let x be a point in \mathcal{I} , then $\rho_1^w(x) = \lim_{s \rightarrow 0} \gamma_1^w(s) \cdot x$.*

4.2. Computing the retraction through the extended tree

For $w \in W^v$, let Υ be an LS path with respect to $w.\mathfrak{S}_{-\infty}$ and $\gamma_1^w : \mathbb{C}^\times \rightarrow G$ be an antidominant one-parameter subgroup as in the previous Lemma.

Let us consider x a point in the measure \mathcal{I} given as

$$x = \prod_{j=1}^{\ell} x_{\beta_j}(a_j t^{n_j}) \cdot a,$$

for $a = \Upsilon(\theta) \in \mathbb{A}$, $\theta \in [0, 1]$, $a_j \in \mathbb{C}$, and the β_j 's are negative real roots. Let us compute $\gamma_1^w(s) \cdot x$. We have:

$$\gamma_1^w(s) \cdot x = \prod_{j=1}^{\ell} x_{\beta_j}(s^{\beta_j(\gamma_1^w(s))} a_j t^{n_j}) \cdot a,$$

since $\gamma_1^w(s) \cdot a = a$, as $\gamma_1^w(s) \in T(\mathbb{C}) \subset T(\mathcal{O})$. But $\beta_j(\gamma_1^w(s))$ is positive except when $\beta_j = -w(\alpha)$, where in that case it is 0. Hence

$$\rho_1^w(x) = \lim_{s \rightarrow 0} \gamma_1^w(s) \cdot x = x_{-w(\alpha)}(b_1 t^{m_1}) \cdots x_{-w(\alpha)}(b_p t^{m_p}) \cdot a,$$

where $b_i = a_{j_i}$, for $\beta_{j_i} = -w(\alpha)$ and among the integers m_i , denote by m_q the minimum. Hence, we can write

$$\rho_1^w(x) = x_{-w(\alpha)}(ct^{m_q} D) \cdot a,$$

where $D = 1 + d_1 t + \cdots + d_r t^r$ is a polynomial in t , for some $r \geq 0$, and $c = b_{q_1} + \cdots + b_{q_h}$, where $\{q_1, \dots, q_h\} = \{1 \leq j \leq p, m_j = m_q\}$.

Now we assume that a belongs to $D(w(\alpha), -m_q)$ and $c \neq 0$. Further, using the SL_2 relation $x_{-w(\alpha)}(A) = x_{w(\alpha)}(A^{-1})s_{w(\alpha)}(-A^{-1})^{-w(\alpha)^\vee} x_{w(\alpha)}(A^{-1})$, we get

$$\rho_1^w(x) = x_{w(\alpha)}((ct^{m_q} D)^{-1})s_{w(\alpha)}(-(ct^{m_q} D)^{-1})^{-w(\alpha)^\vee} x_{w(\alpha)}((ct^{m_q} D)^{-1}) \cdot a.$$

As the valuation of $(ct^{m_q} D)^{-1}$ is $-m_q$, the last term stabilizes $a \in D(w(\alpha), -m_q)$, so we get

$$\begin{aligned} \rho_1^w(x) &= x_{w(\alpha)}((ct^{m_q} D)^{-1})s_{w(\alpha)}(-c)^{w(\alpha)^\vee} t^{m_q w(\alpha)^\vee} D^{w(\alpha)^\vee} \cdot a \\ &= x_{w(\alpha)}((ct^{m_q} D)^{-1})s_{w(\alpha)} t^{m_q w(\alpha)^\vee} \cdot a \\ &= x_{w(\alpha)}((ct^{m_q} D)^{-1})s_{-w(\alpha), m_q}(a). \end{aligned}$$

The first step is the fact that $D^{w(\alpha)^\vee}$ and $(-c)^{w(\alpha)^\vee}$ stabilize a and then $s_{w(\alpha)} t^{m_q w(\alpha)^\vee} \cdot a = s_{-w(\alpha), m_q}(a)$, for any $a \in \mathbb{A}$. In conclusion, from the two writings of $\rho_1^w(x)$ we see that

$$\rho_2^{-,w}(\rho_1^w(x)) = a \quad \text{and} \quad \rho_2^{+,w}(\rho_1^w(x)) = s_{-w(\alpha), m_q}(a).$$

4.3. The retractions of a segment

Let π be an LS path of shape λ (with respect to $\mathfrak{S}_{-\infty}$), denote $\mu = \pi(1)$. Let $[x, \mu]$ be a segment in the measure such that $\rho_{\mathfrak{S}_{-\infty}}([x, \mu]) = \pi$. In particular, this implies that

$$x = \prod_{j=1}^{\ell} x_{\beta_j}(a_j t^{n_j}) \cdot \pi(0),$$

where ℓ is the number of walls $M(\beta_j, n_j)$, with multiplicity, left in the positive direction by the reverse path $\bar{\pi}$, the β_j 's being distinct negative roots. We record the times where this is happening: $1 \geq r_1 \geq \dots \geq r_\ell \geq 0$ (several roots can have the same time and a wall can appear several time). Further, as shown in [10, 6.2], the set of segments $[x, \mu]$ in \mathcal{S} that retract onto π is nonempty and is parameterised by $\ell = \text{ddim}(\pi)$ parameters. Further, the set of parameters $(a_j)_{1 \leq j \leq \ell}$ is a finite product of \mathbb{C} and \mathbb{C}^* . Let us denote this set by $P_w(\pi, \pi(1))$. In the following, we sometimes identify x , $[x, \mu]$ and the set of parameters $(a_j)_{1 \leq j \leq \ell}$.

In this section, we want to show the following theorem.

Theorem 4.3. *Let π be an LS path of shape λ , with $\pi(0) = 0$. For any $w \in W^v$, there exists a dense subset of the parameters $\mathcal{O}_w \subset P_w(\pi, \pi(1))$, such that for all $x \in \mathcal{O}_w$,*

$$\rho^w([x, \mu]) = \Upsilon'_w(\pi),$$

the translation of $\Upsilon_w(\pi) = wf_w^{\max}(\pi)$ so that the path ends at $\mu = \pi(1)$.

The proof proceeds by induction on the length of w . If $w = 1$, we are done as, by assumption, $\rho_{\mathfrak{S}_{-\infty}}([x, \mu]) = \pi$.

Consider a simple root α and an element $w \in W^v$ such that $\ell(ws_\alpha) > \ell(w)$, recall that $\Upsilon_{ws_\alpha}(\pi) = \Upsilon_{s_{w(\alpha)}}\Upsilon_w(\pi)$. Assume that $\rho^w([x, \mu]) = \Upsilon'_w(\pi)$, for x in a dense subset $\mathcal{O}_w \subset P_w(\pi, \pi(1))$. We want to show that $\rho^{ws_\alpha}([x, \mu]) = \Upsilon'_{ws_\alpha}(\pi)$, for x in a dense subset $\mathcal{O}_{ws_\alpha} \subset P_w(\pi, \pi(1))$.

Recall the retractions introduced in Section 4.1:

$$\rho^w : \mathcal{S} \xrightarrow{\rho_1^w} \mathcal{S}^w \xrightarrow{\rho_2^{\pm, w}} \mathbb{A},$$

where the panel germ \mathfrak{F}^w separating $ws_\alpha\mathfrak{S}_{-\infty}$ and $w\mathfrak{S}_{-\infty}$. So we have, for any $z \in \mathcal{S}$, $\rho^w(z) = \rho_2^{-, w} \circ \rho_1^w(z)$ and $\rho^{ws_\alpha}(z) = \rho_2^{+, w} \circ \rho_1^w(z)$.

We consider the path $\eta = \rho_1^w([x, \mu])$ in the extended tree \mathcal{S}^w , for $x \in \mathcal{O}_w$. The reverse path $\bar{\eta}$ starts at μ in \mathbb{A} and then changes apartment only at walls parallel to $\ker \alpha$, via elements of the group like $x_{-w(\alpha)}(at^m)$, with a a complex number and m an integer. Since we assume that $\rho^w([x, \mu]) = \Upsilon'_w(\pi)$, for $x \in \mathcal{O}_w$, we have $\rho_2^{-, w}(\eta) = \Upsilon'_w(\pi)$.

Let $[r, s] \subset [0, 1]$ be a $w(\alpha)$ -stable section of $\Upsilon'_w(\pi)$ at a wall $M(-w(\alpha), k)$, for some $k \in \mathbb{Z}$. Suppose that $\eta(s) \in M(-w(\alpha), k) \subset \mathbb{A}$. Then $\eta(r)$ also belongs to $M(-w(\alpha), k)$ and $\eta(s - \varepsilon)$, for $\varepsilon > 0$, is in an apartment obtained by applying to \mathbb{A} an element of the group of the form $x_{-w(\alpha)}(ct^k)$.

Lemma 4.4. *Suppose that $c \neq 0$, then $(\rho_2^{+, w} \circ \eta)[r, s]$ is the image of the $w(\alpha)$ -stable section of $\Upsilon'_w(\pi)$ by the reflection $s_{-w(\alpha), k}$ along the wall $M(-w(\alpha), k)$.*

Proof. Because the parameter c is not zero, the path $\bar{\eta}$ leaves the standard apartment at $\eta(s - \varepsilon)$ and will be back at $\eta(r)$. Now the $w(\alpha)$ -stable section $[r, s]$ of $\Upsilon'_w(\pi)$ may cross several walls parallel to $M(-w(\alpha), k)$, so that the path $\eta([r, s])$ may lie in an apartment given by an element of the form

$$x_{-w(\alpha)}(ct^k + c_{k+1}t^{k+1} + \dots + c_mt^m), \text{ for some integer } m > k.$$

On one side, by the induction assumption, the retraction $\rho_2^{-, w}$ sends $\eta([r, s])$ to the $w(\alpha)$ -stable section of $\Upsilon'_w(\pi)$. But on the other side, performing the SL_2 change

of variables associated to the extended tree \mathcal{T}^w to compute the retraction $\rho_2^{+,w}$, as in the previous section 4.2. Then we apply the results of that section to the points $x \in \eta([r, s])$ that are given as

$$x = x_{-w(\alpha)}(ct^k + c_{k+1}t^{k+1} + \dots + c_m t^m) \cdot a,$$

for some m and a $a \in \Upsilon = \Upsilon'_w(\pi)([r, s])$. The integer m_q is then k and the retraction $\rho_2^{+,w}$ sends $\eta([r, s])$ to the reflection of the $w(\alpha)$ -stable section by $s_{-w(\alpha),k}$. ■

Recall that, in the statement of Lemma 3.7 we fixed

$$m = \min_{t \in [0,1]} \left(w(\alpha)(\Upsilon'_w(\pi)(t)) \right)$$

and set $q = \min\{t \in [0, 1], w(\alpha)(\Upsilon'_w(\pi)(t)) = m\}$. We know that $\rho_2^{-,w}(\eta(q))$ belongs to the wall $M(-w(\alpha), m)$.

Lemma 4.5. *Suppose that $\eta(q) \in M(-w(\alpha), m)$ and that $\eta(q - \varepsilon)$, $\varepsilon > 0$, is in an apartment obtained by applying to \mathbb{A} an element of the group of the form $x_{-w(\alpha)}(dt^m)$, with $d \neq 0$. Then $\rho_2^{+,w}(z) = s_{-w(\alpha),m}(\rho_2^{-,w}(z))$, for any $z = \eta(t)$, with $0 \leq t \leq q$.*

Proof. Thanks to the assumptions, the apartment containing $\eta(q - \varepsilon)$ shares a half-apartment with \mathbb{A} along the wall $M(-w(\alpha), m)$. So computing in the extended tree, using the same techniques as in the proof of the Lemma 4.4, we see that $\rho_2^{+,w}(z) = s_{-w(\alpha),m}(\rho_2^{-,w}(z))$, for any $z = \eta(t)$, with $0 \leq t \leq q$. ■

We are now ready to conclude our induction step. We want to prove the equality $\rho^{ws_\alpha}([x, \mu]) = \Upsilon'_{ws_\alpha}(\pi)$ for x in a dense subset of the set of parameters, assuming $\rho^w([x, \mu]) = \Upsilon'_w(\pi)$, for x in \mathcal{O}_w .

Now, we compute the retraction $\rho_2^{+,w}(\eta)$ stepwise by retracting some $w(\alpha)$ -section at a time and starting backwards from $\mu = \pi(1) = \eta(1) \in \mathbb{A}$. As ε is a small real number, as small as necessary, we know that $\eta(1 - \varepsilon)$ is also in \mathbb{A} . Let $s_1 \in [0, 1]$ be the biggest real such that $\eta(s_1) \in \mathbb{A}$ and $\eta(s_1 - \varepsilon) \notin \mathbb{A}$. Then $\eta(s_1)$ belongs to a wall $M(-w(\alpha), k)$ and $\eta(s_1 - \varepsilon)$ is in an apartment given by acting by $x_{-w(\alpha)}(ct^k)$ on \mathbb{A} . If there exists $r_1 \in [0, s_1]$ such that $\eta(r_1) \in M(-w(\alpha), k)$, then the path $\Upsilon'_w(\pi)$ admits a $w(\alpha)$ -stable section $[r_1, s_1]$ at that wall. And we are in the position of applying 4.4.

But there might be several $w(\alpha)$ -stable sections $[r_1, s_1], \dots, [r_j, s_j]$ at that wall $M(-w(\alpha), k)$ such that $\Upsilon'_w(\pi)([r_j, s_1]) \subset D(-w(\alpha), k)$, the positive half-apartment bounded by $M(-w(\alpha), k)$. Let c_1, \dots, c_j be the parameters associated to those stable sections at times s_1, \dots, s_j respectively, meaning that $\eta(s_i - \varepsilon)$ is in an apartment given by acting by $x_{-w(\alpha)}(c_i t^k)$ on \mathbb{A} , for $1 \leq i \leq j$. The conditions $c_1 + c_2 + \dots + c_h \neq 0$, for all $1 \leq h \leq j$, will impose by applying 4.4, that the retraction of those stable sections will form a path Υ_1 such that $\Upsilon_1(t) = \eta(t)$ for $t \in [0, r_j]$ and $\Upsilon_1(t) = \Upsilon'_{ws_\alpha}(\pi)(t)$ for $t \in [r_j, 1]$.

We see such a example in Figure 1, where two stable sections occur at two different walls.

$$\begin{aligned}\bar{\mu}_\infty &= \lim_{\infty} \bar{\mu}_k \\ \mu^\infty &= \lim_{\infty} \mu^k = \mu_\infty + |\lambda|\delta \\ \bar{\mu}^\infty &= \lim_{\infty} \bar{\mu}^k = \bar{\mu}_\infty + |\bar{\lambda}|\delta \\ \mu^{k-1} - \mu^k &= a^k(\alpha_0 + (k-1)\delta) \\ \bar{\mu}^{k-1} - \bar{\mu}^k &= \bar{a}^k(\alpha_1 + (k-1)\delta).\end{aligned}$$

where the limits make sense thanks to the finite support condition. We obtain a polytope if $\bar{\mu}^0 = \mu^0$, in which case we say that our initial Lusztig data have same weight. We call $(\bar{a}_k, \bar{\lambda}_k, \bar{a}^k)$ the left datum, and (a_k, λ_k, a^k) the right one. We say that the polytope is decorated by the partitions $\bar{\lambda}$ and λ .

Definition 5.1. Such a polytope is called an *affine MV polytope* if

- (i) for each $k \geq 2$, $(\bar{\mu}_k - \mu_{k-1}, \omega_1) \leq 0$ and $(\mu_k - \bar{\mu}_{k-1}, \omega_0) \leq 0$, with at least one of these being an equality ;
- (ii) for each $k \geq 2$, $(\bar{\mu}^k - \mu^{k-1}, \omega_0) \geq 0$ and $(\mu^k - \bar{\mu}^{k-1}, \omega_1) \geq 0$, with at least one of these being an equality ;
- (iii) If $(\mu_\infty, \bar{\mu}_\infty)$ and $(\mu^\infty, \bar{\mu}^\infty)$ are parallel then $\lambda = \bar{\lambda}$. Otherwise, one is obtained from the other by removing a part of size $(\mu_\infty - \bar{\mu}_\infty, \alpha_1)/2$;
- (iv) $\lambda_1, \bar{\lambda}_1 \leq (\mu_\infty - \bar{\mu}_\infty, \alpha_1)/2$.

We will denote by MV the set of (affine) MV polytopes (up to translation).

It is proved in [4, Theorem 3.11] that for any given Lusztig datum \mathbf{a} , there is a unique MV polytope $P_{\mathbf{a}}$ whose right datum is \mathbf{a} . We reproduce the example given there as Figure 2.

There we have $a_1 = 2$, $a_2 = a_3 = 1$, $a_k = 0$ for $k \geq 4$, $\lambda = (9, 2, 1^2)$, $a^k = 0$ for $k \geq 4$, $a^3 = 1$, $a^2 = 0$, $a^1 = 1$ for the right Lusztig datum. Bold diagonals correspond to a choice, for each $k \geq 2$, of an *active* diagonal, that is satisfying equality in (i) or (ii) in 5.1. Such a choice form a *complete system* of active diagonals. Thanks to [4, Remark 3.7], we have the following.

Proposition 5.2. *Any active diagonal cuts an MV polytope in two MV polytopes.*

We use a different convention than in [4] and define a crystal structure on the set MV of MV polytopes by setting $f_0(P_{\mathbf{a}}) = P_{f_0(\mathbf{a})}$ where $f_0(\mathbf{a})$ just adds 1 to \bar{a}_1 , and $f_1(P_{\mathbf{a}}) = P_{f_1(\mathbf{a})}$ where $f_1(\mathbf{a})$ just adds 1 to a_1 (here we simply use f_i instead of f_{α_i} for $i \in \mathbb{Z}/2\mathbb{Z}$). We finally recall that [4, Theorem 4.5] proves that MV realizes the crystal $\mathcal{B}(\infty)$.

5.2. Link with retractions

Note here that we have $G^\vee = G$. Consider $b \in \mathcal{B}(\infty)$. We will denote by P_b the corresponding MV polytope. Thanks to 3.6, the lower vertices of the MV polytope associated to b are given, up to a common translation, by the weights of

$$s_{i_1} \dots s_{i_k} f_{i_k}^{\max} \dots f_{i_1}^{\max}(b)$$

where $i_p \in \mathbb{Z}/2\mathbb{Z}$ and $i_{p+1} = i_p + 1$. More precisely, we get $\bar{\mu}_k$ (resp. μ_k) when $i_1 = 0$ (resp. $i_1 = 1$). Thanks to 3.6 and 4.3, this bottom part can hence be recovered using retractions on line segments in the measure that retract on paths.

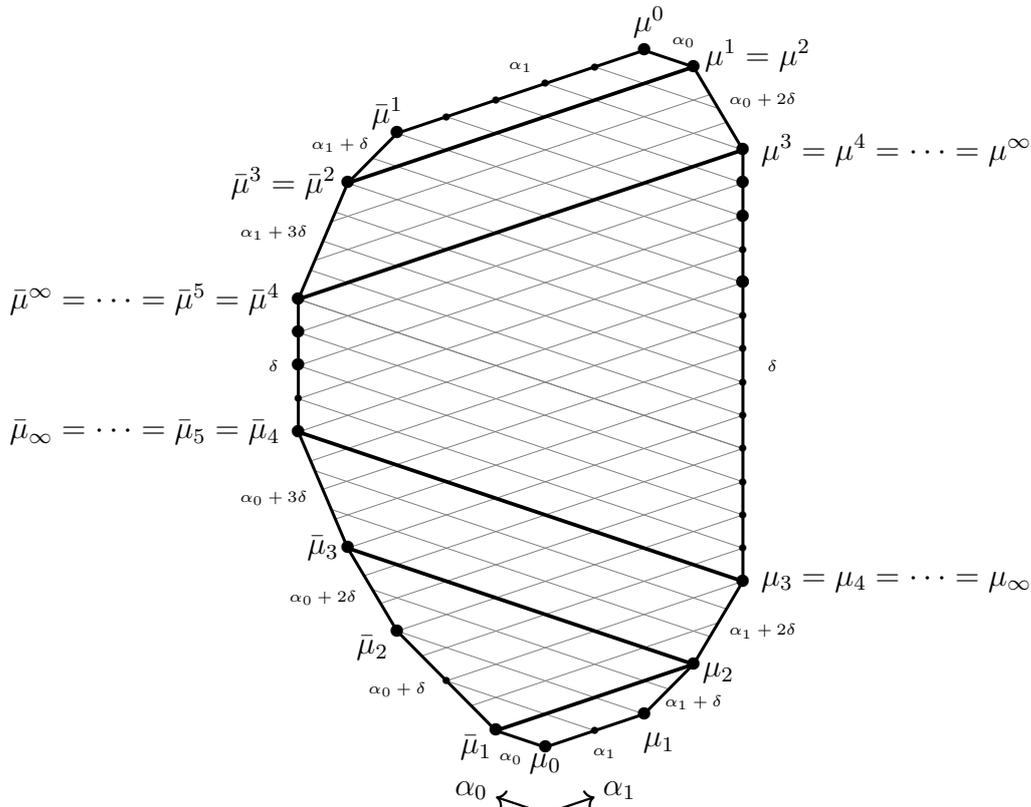


Figure 2

Remark 5.3. As a consequence, vertices $\bar{\mu}^k, \mu^k$ of the upper part of the polytope are given by the opposite of the weights of

$$s_{i_1} \dots s_{i_k} f_{i_k}^{\max} \dots f_{i_1}^{\max}(b^*)$$

where $*$ denotes Kashiwara’s involution. Indeed $*$ simply transforms an MV polytope in its opposite, see [4, Definition 4.3 & Theorem 4.5]

5.3. Top polytopes

Definition 5.4. We call *top polytope* an MV polytope P such that $\lambda = \bar{\lambda} = 0$ and $a_{\bullet} = \bar{a}_{\bullet} = 0$ except a_1 or \bar{a}_1 . We denote by MV^t the set of top polytopes.

Thanks to 5.2, the piece of an MV polytope P above the diagonal $[\bar{\mu}^\infty, \mu^\infty]$ is itself an MV polytope, that we denote by P^t . Denote by b_0 the highest weight element of $\mathcal{B}(\infty)$, which corresponds to the unit of the quantum group. Seen in some \mathcal{B}_λ , the associated path is the segment of shape λ .

Proposition 5.5. Consider $b \in \mathcal{B}(\infty)$. We have $P_b \in MV^t$ if and only if

$$b = f_{i_1}^{k_1} \dots f_{i_t}^{k_t}(b_0)$$

where $i_{k+1} = i_k + 1 \in \mathbb{Z}/2\mathbb{Z}$ and $k_1 > \dots > k_t$.

Proof. Recall that in general, $\epsilon_i(b)$ is the maximal integer k such that $e_i^k(b) \neq 0$. Assume that $\lambda = \bar{\lambda} = \bar{a}_\bullet = a_{\geq 2} = 0$ and $a_1 = N > 0$ so that $\epsilon_1(b) = N$. By induction on the weight (that is on the height $= (-, \omega_0 + \omega_1)$), we need to prove that $\epsilon_0(e_1^N(b)) = N' < N$ since clearly $P_{e_1^N(b)} \in MV^t$.

Consider a complete system of active diagonals for P_b and assume that it contains one directed by α_0 . Thanks to [4, Proposition 4.10], we can assume that our polytope is the one contained between the lowest diagonal $N\alpha_1$ and the lower active diagonal $n\alpha_0$ directed by α_0 :

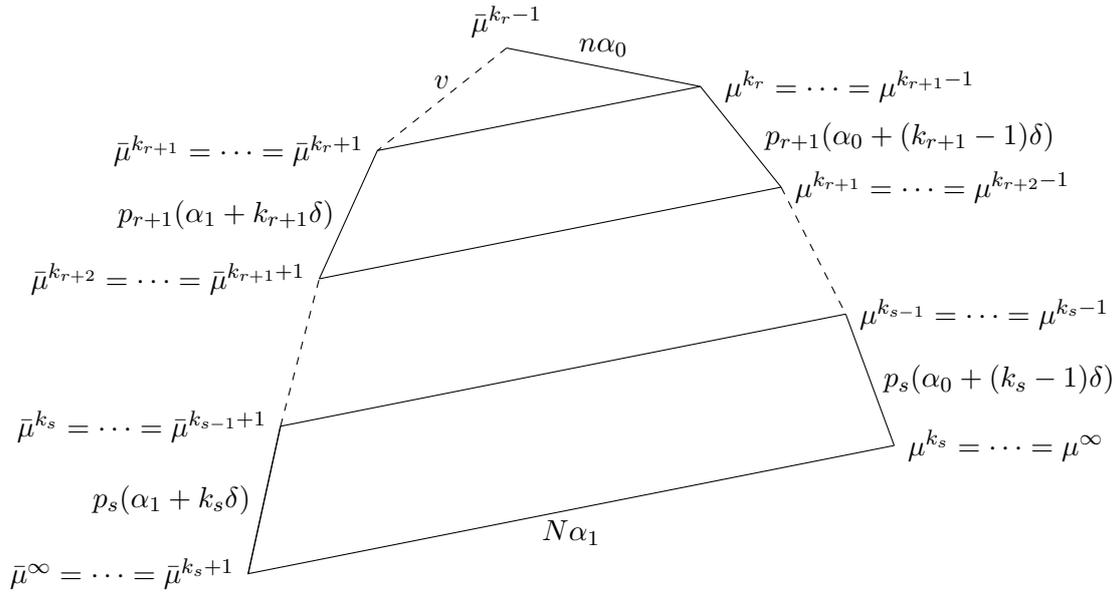


Figure 3 (1)

where $v = g\alpha_1 + h\delta$ for some $g > 0, h \geq 0$, and the p_i 's are positive integers. Using [4, Proposition 4.10], applying e_1^N yields

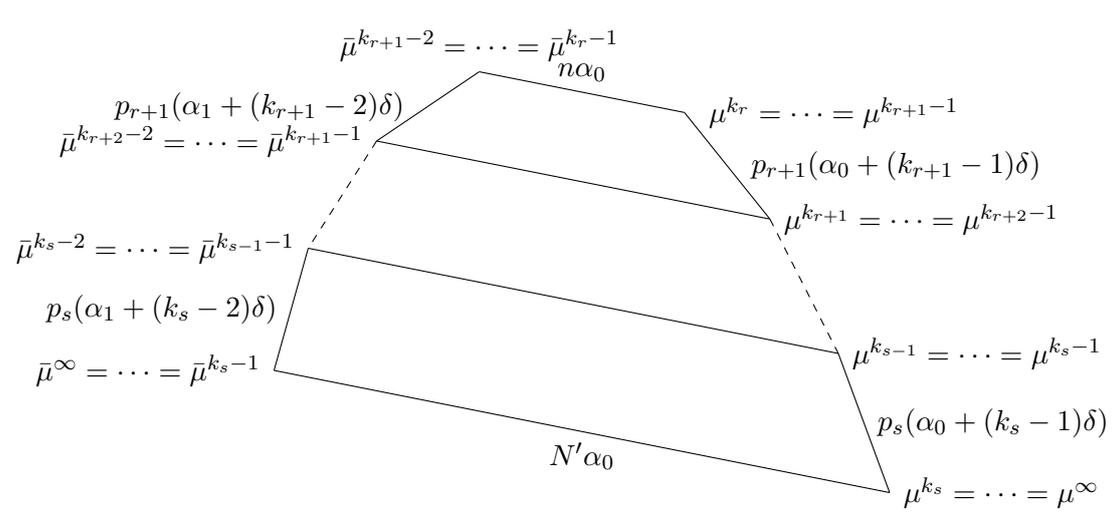


Figure 4 (2)

and we want to prove that $N' = \epsilon_0(e_1^N(b)) < N$.

But we have

$$\begin{aligned}
 N'\alpha_0 &= n\alpha_0 + \underbrace{\sum_{j=r+1}^s p_j(\alpha_0 + (k_j - 1)\delta)}_{=:u} - \sum_{j=r+1}^s p_j(\alpha_1 + (k_j - 2)\delta) \\
 &= n\alpha_0 + u - \underbrace{\sum_{j=r+1}^s p_j(\alpha_1 + k_j\delta)}_{=:v-N\alpha_1} + \underbrace{\left(\sum_{j=r+1}^s 2k_j\right)\delta}_{=:d}
 \end{aligned}$$

where we recall that $v = g\alpha_1 + h\delta$ for some $g > 0, h \geq 0$. Thus $N' = d + h$ and $N = d + g + h = N' + g > N'$ as expected. Computations are similar if all diagonals are directed by α_1 . We deal symmetrically with the case $\lambda = \bar{\lambda} = a_\bullet = \bar{a}_{\geq 2} = 0$ and $\bar{a}_1 > 0$.

We also prove the other way around by induction. Assume that P_b looks like the previous figure 2, with $\bar{a}_1 = N'$ (once again, the case $a_1 \neq 0$ is dealt with similarly). We want to prove that if $N > N', P_{f_1^N(b)}$ satisfies $\lambda = \bar{\lambda} = \bar{a}_\bullet = a_{\geq 2} = 0$ and $a_1 = N > 0$. If not, it looks like

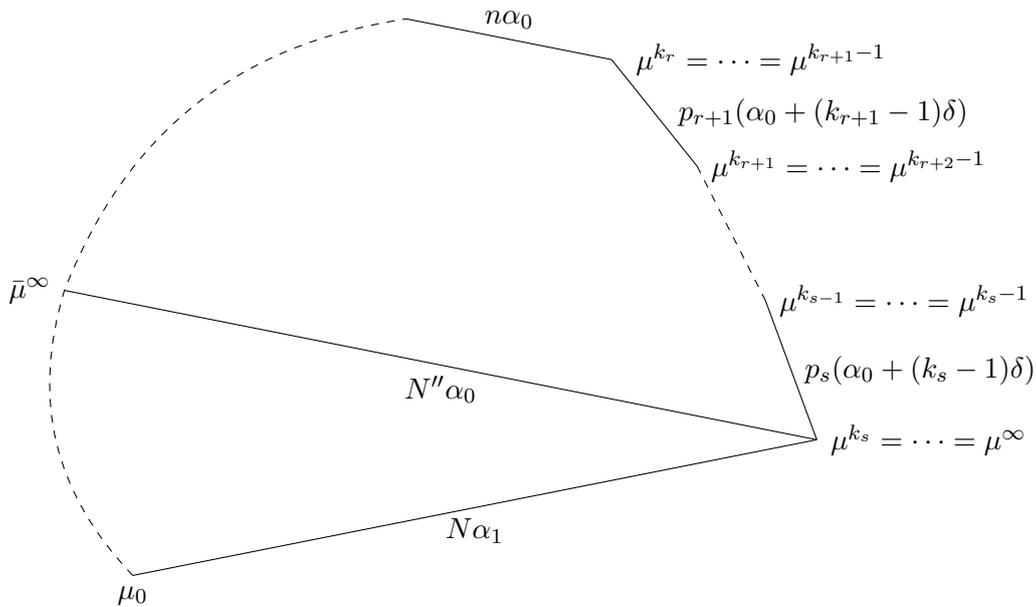


Figure 5

with $N'' \geq N$. But then the MV polytope between the two diagonals oriented by α_0 has the same right Lusztig datum than P_b represented by 2, but a different left one as $N'' > N'$, which is absurd thanks to [4, Theorem 3.11]. ■

Lemma 5.6. *Assume that $b = f_{i_1}^{k_1} \dots f_{i_t}^{k_t}(b_0)$ where $i_{k+1} = i_k + 1 \in \mathbb{Z}/2\mathbb{Z}$ and $k_1 > \dots > k_t$, and $i_1 = 1$. Then for every $K \leq k_1$, we have $\epsilon_0(e_1^K(b)) < K$.*

Proof. We can assume that P_b looks like 1, with $N = k_1$. Then $P_{e_1^N(b)}$ looks like 2 with $N' = k_2$. If $K < N - N'$, we have $b' = e_1^K(b) = f_1^{N-K} f_0^{N'} \dots f_{i_t}^{k_t}(b_0)$ with $N - K > N'$. Thus b' satisfies the conditions of 5.5 and a fortiori $\epsilon_0(b') = 0$.

If $K = N - N'$ then $P_{b'}$ looks like

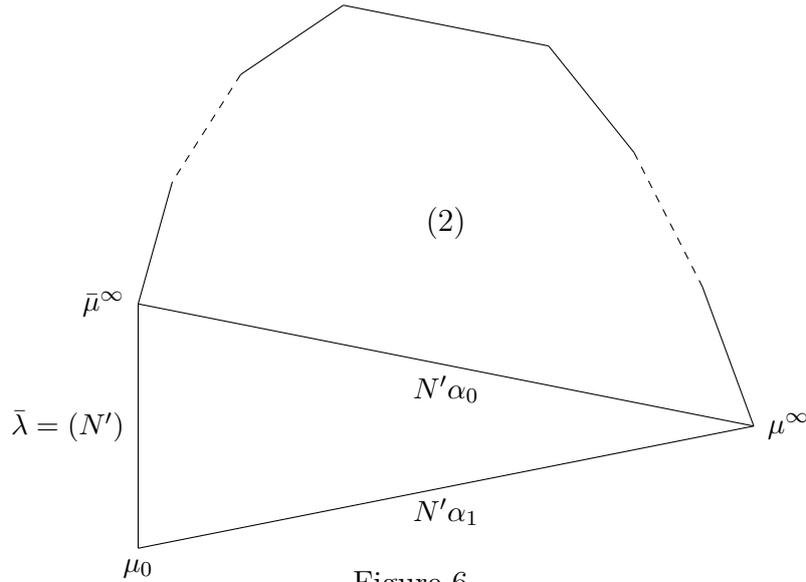


Figure 6

thus again $\epsilon_0(b') = 0$. Finally assume that $K > N - N'$ and set $h = N - K$. Then $P_{b'}$ looks like

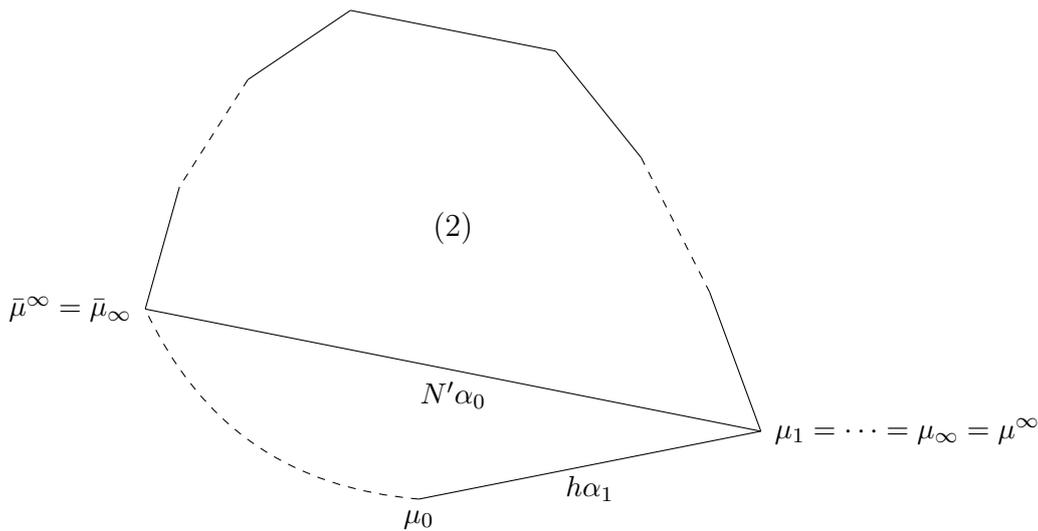


Figure 7

where we necessarily have $\epsilon_0(b') < N' - h = N' - N + K < K$ as wished. ■

5.4. δ -top polytopes

Definition 5.7. We call δ -top polytope an MV polytope P such that $a_\bullet = \bar{a}_\bullet = 0$ except a_1 or \bar{a}_1 . We denote by $MV^{\delta-t}$ the set of δ -top polytopes.

Proposition 5.8. Consider $b \in \mathcal{B}(\infty)$ and its MV polytope P_b . We have $P_b \in MV^{\delta-t}$ if and only if

$$b = f_{i_1}^{k_1} \dots f_{i_t}^{k_t}(b_0)$$

where $i_{k+1} = i_k + 1 \in \mathbb{Z}/2\mathbb{Z}$ and $k_1 \geq \dots \geq k_t$.

Proof. We use induction on the weight. Consider $P \in MV^{\delta-t}$. First assume that $\bar{\lambda} \neq \lambda$, say for instance $\lambda = N \cup \bar{\lambda}$. Then $P = f_0^N(P')$ where P' is simply the part of P which is above the bottom α_1 -active diagonal. We conclude by induction hypothesis since $\epsilon_1(P') = N$. Now assume that $\bar{\lambda} = \lambda$ in P , with, say, $a_1 = N \neq 0$. If $N = \lambda_1$, then $e_1^N(P) = P'$ with $(P')^t = P^t$, $\lambda' = \lambda$ and $\bar{\lambda}' = \lambda \setminus N$. We conclude by induction hypothesis. Otherwise, $N > \lambda_1$. Write $P' = e_1^N(P)$. If $\bar{\lambda}' \neq \lambda' = \lambda$, we have $\epsilon_0(P') = \lambda_1 < N$ and we conclude by induction hypothesis. If $\bar{\lambda}' = \lambda' = \lambda$, we have $f_1^N(P'^t) = P^t$ by [4, Theorem 3.11]. By 5.5, we necessarily have $N > \epsilon_0(P'^t) = \epsilon_0(P')$ and we can again conclude by induction hypothesis. ■

5.5. General polytopes

Thanks to 5.2, there is a natural map $MV \rightarrow MV^{\delta-t}$, $P \mapsto P^{\delta-t}$ which is the removal of the part of a polytope which is below the active diagonal $[\bar{\mu}_\infty, \mu_\infty]$.

Proposition 5.9. *For every $P \in MV$ there exist $k_1, \dots, k_r > 0$ and $h \geq 0$ such that $f_{i_0}^h e_{i_1}^{k_1} \dots e_{i_r}^{k_r}(P) = P^{\delta-t}$, where $i_{j+1} = i_j + 1 \in \mathbb{Z}/2\mathbb{Z}$.*

Proof. Consider the case where $\bar{a}_1 \neq 0$ in $P^{\delta-t}$. Thanks again to [4, Proposition 4.10] we may assume that the bottom part of our polytope is a stack of p subpolytopes of the following type

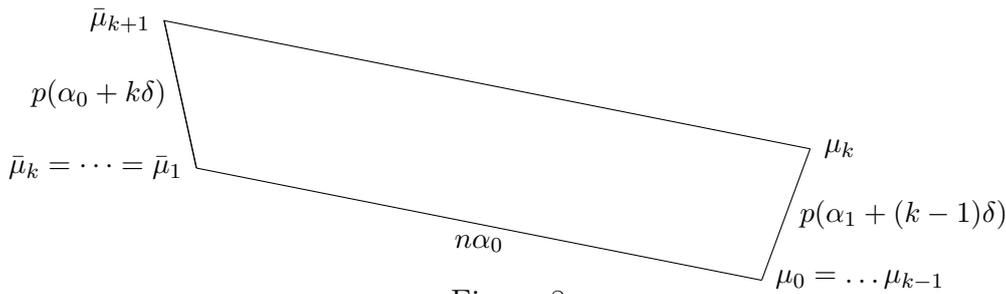


Figure 8

where by definition $(\mu_k - \bar{\mu}_{k-1}, \omega_0) \leq 0$. Thus the line through $\bar{\mu}_1$ directed by α_1 meets the edge $[\bar{\mu}_{k+1}, \mu_k]$:

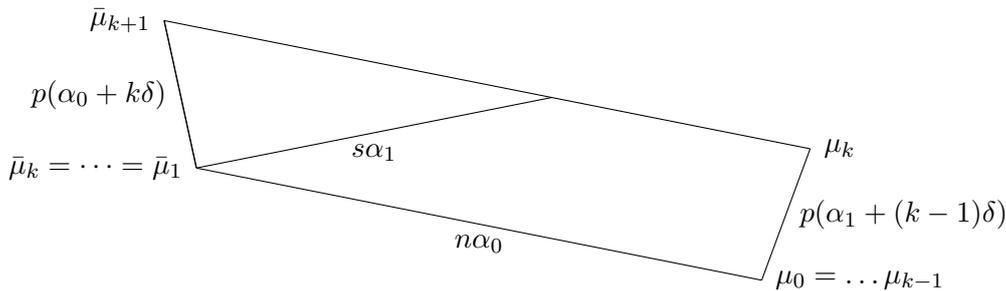


Figure 9

for some $s > 0$. But then the bottom part of $e_1^s e_0^n(P)$ is a stack of $p - 1$ polytopes of the previous type and $e_1^s e_0^n(P)^{\delta-t}$ and $P^{\delta-t}$ share the same left Lusztig datum except \bar{a}_1 . By induction on p , there exist $k_1, \dots, k_r > 0$ such that $e_{i_1}^{k_1} \dots e_{i_r}^{k_r}(P)$ and $P^{\delta-t}$ share the same left Lusztig datum except \bar{a}_1 and we can conclude since the left Lusztig datum determines the polytope. ■

Example 5.10. If the bottom part of P looks like

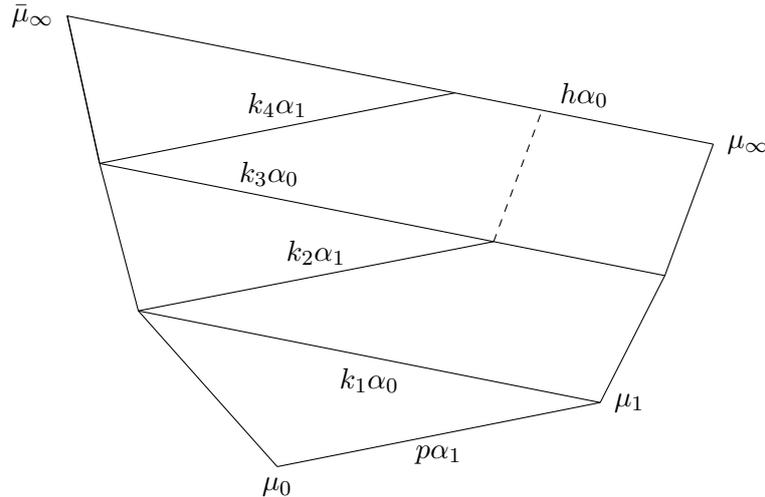


Figure 10

then $P^{\delta-t} = f_0^h e_1^{k_4} e_0^{k_3} e_1^{k_2} e_0^{k_1} e_1^p(P)$. The dashed segment is part of the right side of $e_0^{k_1} e_1^p(P)$.

6. Decorations and paths

In this section, we stay in the $\widehat{\mathfrak{sl}}_2$ case. For a given LS path π we want to be able to recognize the partitions decorating the associated affine MV polytope. Since we have seen in 5.2 that retractions allow one to recover the bottom part of this polytope, we may thanks to 5.9 stick to the case of paths associated to δ -top polytopes, which we treat in this section.

Definition 6.1. Consider an LS path π and $k \in \mathbb{Z}$. An interval $[s, v] \subseteq [0, 1]$ is said to be an α_i -zigzag if there exists $[t, u] \subset [s, v]$ such that $[s, u]$ is an α_i -stable section at k and $[t, v]$ is an α_{i+1} -stable section at $-k$. Moreover, we want $\alpha_i([s, v]) \geq k$ and $\alpha_{i+1}([s, v]) \geq -k$.

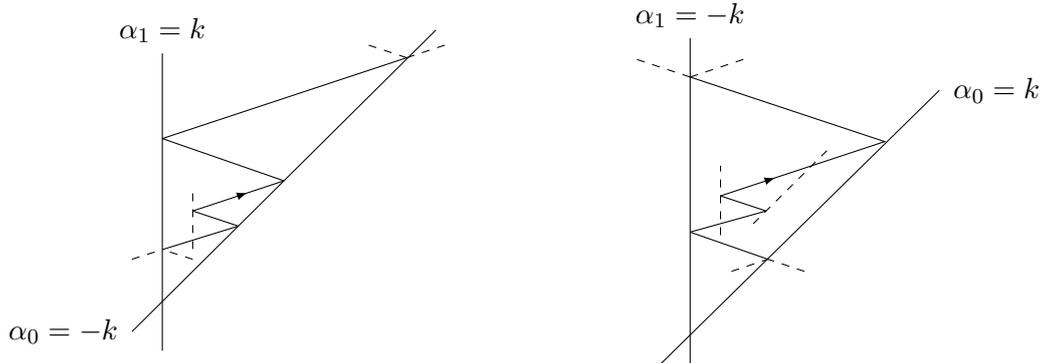


Figure 11: An α_1 -zigzag at k on the left, an α_0 -zigzag at k on the right.

We index partitions by finitely supported sequences of nonnegative integers $(m_k)_{k \geq 1}$, the *multiplicities*. The partition associated to such a sequence is denoted in its exponential form by (k^{m_k}) , which means that k appears m_k times, thus giving a partition of $\sum_{k \geq 1} m_k k$.

Definition 6.2. Let π be an LS path and k an integer. Denote by $m_{i,k}(\pi)$ the number of α_i -zigzags of π at k .

- (i) Assume that $\epsilon_0(\pi) = N > 0 = \epsilon_1(\pi)$. The *left partition* $\bar{\lambda}(\pi)$ of π is defined by the multiplicities $m_{1,k}(\pi)$, $k > 0$. The *right partition* is $\lambda(\pi) = N \cup \bar{\lambda}(\pi)$ if π reaches $\min \alpha_0(\pi) = -N$ within an α_1 -stable section at N . Otherwise $\lambda(\pi) = \bar{\lambda}(\pi)$.
- (ii) Assume that $\epsilon_1(\pi) = N > 0 = \epsilon_0(\pi)$. The *right partition* $\lambda(\pi)$ of π is defined by the multiplicities $m_{0,k}(\pi)$, $k > 0$. The *left partition* is $\bar{\lambda}(\pi) = N \cup \lambda(\pi)$ if π reaches $\min \alpha_1(\pi) = -N$ within an α_0 -stable section at N . Otherwise $\bar{\lambda}(\pi) = \lambda(\pi)$.

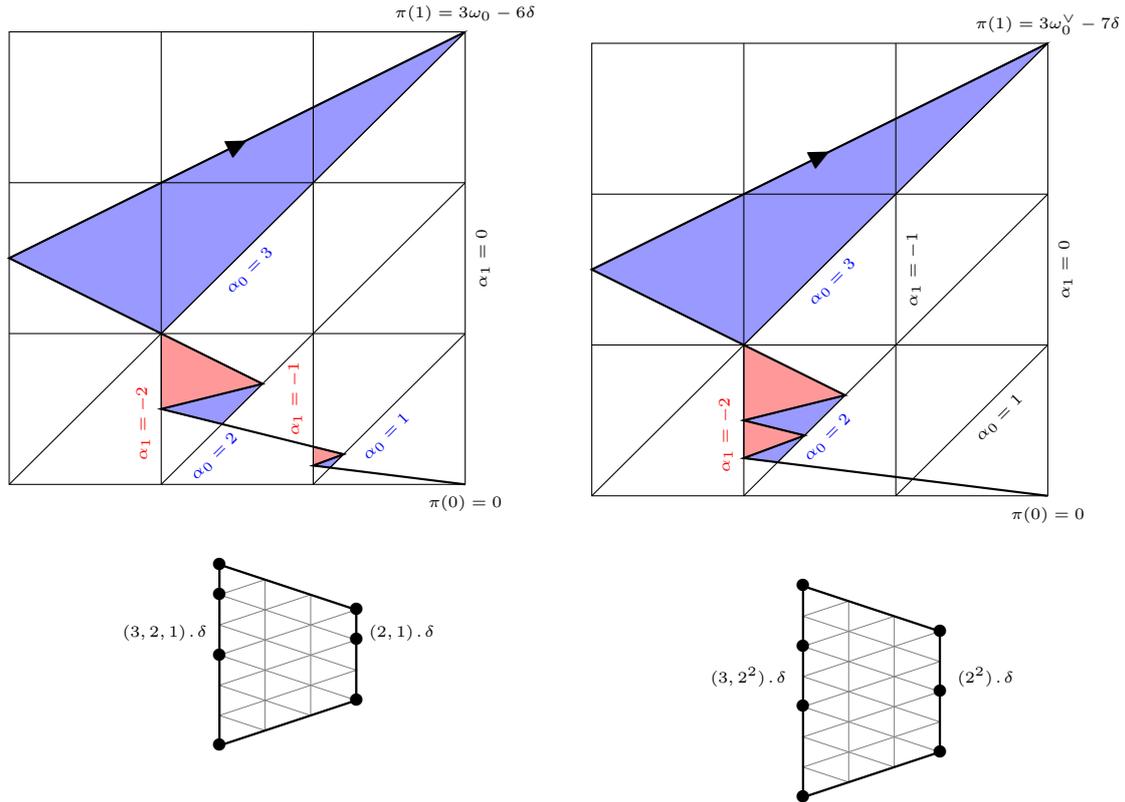


Figure 12

Remark 6.3. Such assumptions are typically (but not only) satisfied by δ -top polytopes.

Lemma 6.4. If $\lambda(\pi) = N \cup \bar{\lambda}(\pi)$ (resp. $\bar{\lambda}(\pi) = N \cup \lambda(\pi)$), then π is in the image of $f_0^N f_1^N$ (resp. $f_1^N f_0^N$).

Proof. Consider π with an α_1 -stable section at N , during which π reaches $\alpha_0 = -N = \min \alpha_0(\pi)$. Then π is equal to $f_0^N(\pi')$ for π' reaching $\alpha_1 = -N$. Thus π' is in the image of f_1^N . ■

Example 6.5. Consider $\pi = f_1^{\lambda_1} f_0^{\lambda_1} \dots f_1^{\lambda_r} f_0^{\lambda_r}(\pi_\Lambda)$ for a partition of the form $\lambda = (\lambda_1 \geq \dots \geq \lambda_r)$, Λ a large enough dominant integral weight, and π_Λ the corresponding highest weight path. We get $\bar{\lambda}(\pi) = \lambda$ and $\lambda(\pi) = \lambda \setminus \lambda_1$.

In Figure 12 on the preceding page the examples of $\lambda = (3, 2, 1)$ are shown on the left and of $\lambda = (3, 2^2)$ on the right side. Red zones correspond to α_1 -stable sections, blue ones to α_0 -stable sections. A red area succeeding a blue one hence gives an α_0 -zigzag. Below each path shows the associated decorated polytope.

Lemma 6.6. *Let π be an LS path and consider integers $1 \leq p \leq n$ such that $f_1^n(\pi)$ exists. We have the following.*

- (i) *Assume that $\epsilon_0(\pi) = p$, so that $\min \alpha_0(\pi) = -p$. If π reaches this minimum during an α_1 -stable section at p , then $f_1^n(\pi)$ has an α_0 -zigzag at p .*
- (ii) *Assume that $\epsilon_0(\pi) = n$. Then $f_1^n(\pi)$ has an α_0 -stable section at n , during which $\alpha_1(f_1^n(\pi))$ reaches its minimum $-n$.*
- (iii) *Assume that $\epsilon_0(\pi) = p$, and consider a positive integer q . Then π has an α_1 -zigzag at q if and only if $f_1^n(\pi)$ has an α_0 -zigzag at q (in which case $q \leq p$).*

Proof. For (i), π locally looks like

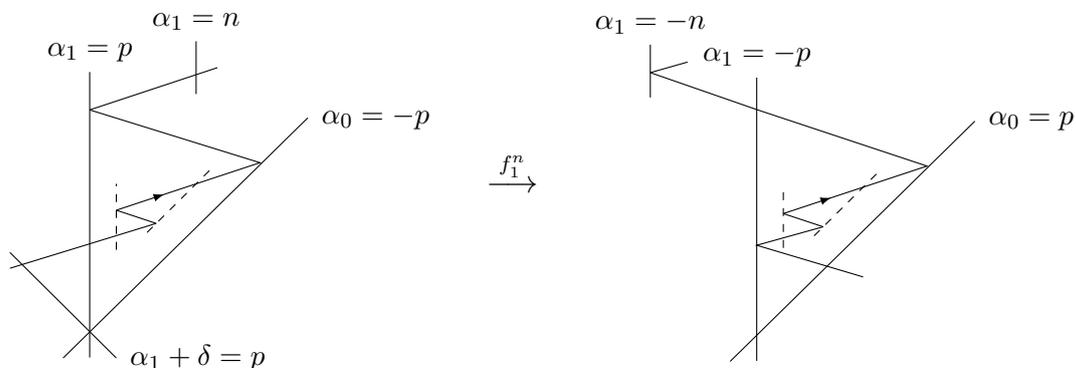


Figure 13

and we have an α_0 -zigzag at p : indeed, it can not be included in a larger α_0 -stable one because if π hits $\alpha_0 = -p$ after $\alpha_1 = n$, then after applying f_1^n this hit is at $\alpha_0 = 2n - p \geq p$. The same picture (without α_1 -stable sections) proves (ii), merging the two walls $\alpha_1 = p$ and $\alpha_1 = n$. For (iii), π locally looks like

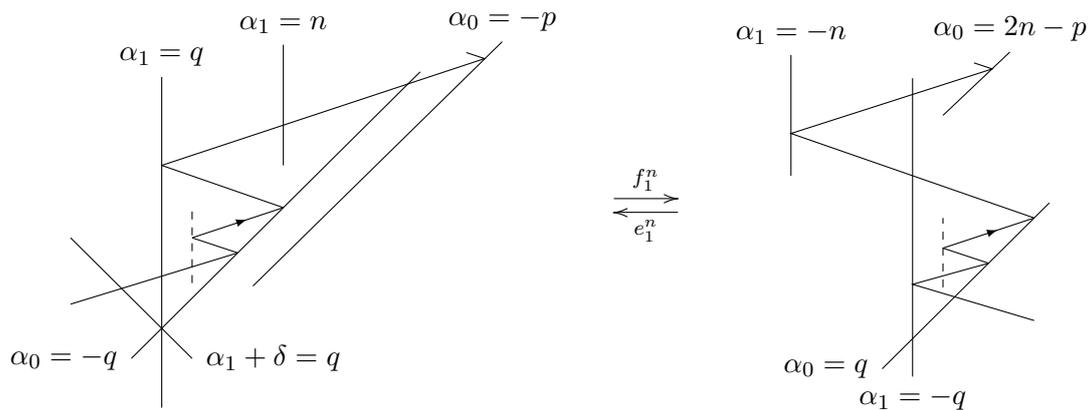


Figure 14

where the dashed wall might be at $\alpha_1 = q$ ($-q$ after applying f_1^n). Since $2n - p \geq q$, we get what we need. ■

Corollary 6.7. *Consider b such that $P_b \in MV^t$. Then the corresponding path π satisfies $\bar{\lambda}(\pi) = \lambda(\pi) = 0 = \bar{\lambda} = \lambda$.*

Proof. We use induction on the weight. Assume that $\lambda(\pi)$ or $\bar{\lambda}(\pi)$ is nonzero. Then by 6.4 and 5.6 we have $\lambda(\pi) = \bar{\lambda}(\pi)$. Assume then that π has an α_0 -zigzag at $k > 0$ (the case of an α_1 -zigzag is dealt with symmetrically). By induction hypothesis, if $k_1 = \epsilon_1(\pi)$, $e_1^{k_1}(\pi) = e_1^{\max}(\pi)$ does not have any α_1 -zigzag at k . It implies that all $(-\alpha_0)$ -directed sections in $e_1^{\max}(\pi)$ occur before the α_1 -stable section at k induced by the α_1 -stable section at $-k$ in the α_0 -zigzag of π . On the right hand side here:

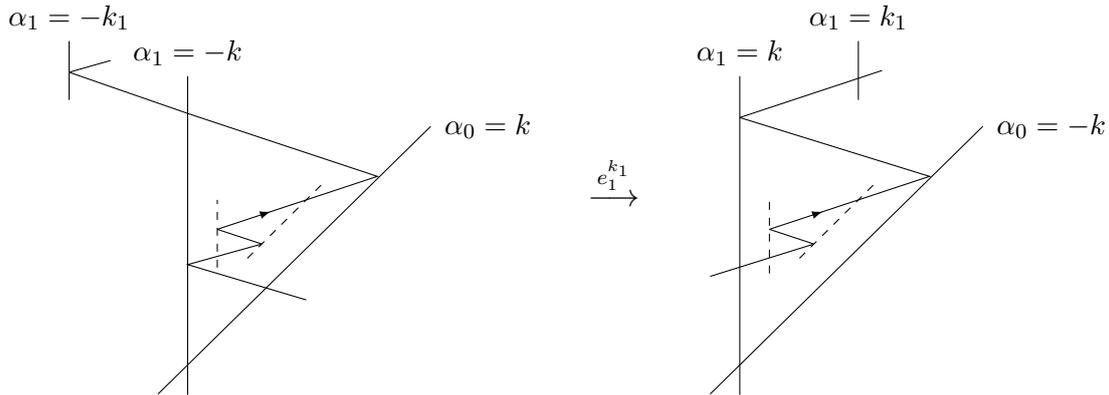


Figure 15

the wall $\alpha_0 = -k$ cannot be reached after the wall $\alpha_1 = k_1$, and therefore $\min \alpha_0(e_1^{\max}(\pi)) \leq -k$ is reached before or during the α_1 -stable section at k . But then $e_0^k e_1^{\max}(\pi)$ reaches $\alpha_1 = -k$ and $e_1^k e_0^k(e_1^{\max}(\pi))$ exists, which contradicts 5.6 applied to $e_1^{\max}(\pi)$. ■

Theorem 6.8. *Consider b such that $P_b \in MV^{\delta-t}$, and π the associated path. Then $\bar{\lambda} = \bar{\lambda}(\pi)$ and $\lambda = \lambda(\pi)$. More precisely:*

- (i) if $a_1 \neq 0$, then $k \in \bar{\lambda} \cap \lambda$ if and only if π has an α_0 -zigzag at k ;
- (ii) if $\bar{a}_1 \neq 0$, then $k \in \bar{\lambda} \cap \lambda$ if and only if π has an α_1 -zigzag at k ;
- (iii) $n \in \bar{\lambda} \setminus \lambda$ if and only if $-n = \min \alpha_1(\pi)$ is reached during an α_0 -stable section at n ;
- (iv) $n \in \lambda \setminus \bar{\lambda}$ if and only if $-n = \min \alpha_0(\pi)$ is reached during an α_1 -stable section at n .

Proof. We proceed by induction on the weight using 6.6. Assume $a_1 = n \neq 0$ so that we may write $\pi = f_1^n(\pi')$, and $P' = P_{\pi'}$. The first case is $\bar{\lambda} = n \cup \lambda$. Then $\bar{\lambda}' = \lambda' = \lambda$, $\bar{a}'_1 = n$ and $a'_1 = 0$. Then the induction hypothesis (ii) and 6.6(ii,iii) prove (i,iii) here. Then we consider the second case where $\bar{\lambda} = \lambda$. There are two subcases. If $\bar{\lambda}' = \lambda' = \lambda$, we prove (i) thanks to the induction hypothesis (ii) and 6.6(iii). Otherwise $\lambda = p \cup \bar{\lambda}'$, where $p \leq n$ thanks to 5.8, $\lambda' = \lambda$ and $\bar{a}'_1 = p$. The induction hypothesis (ii) allows us to use 6.6(iii) to prove that $\bar{\lambda}'$ and $\bar{\lambda}(\pi)$ differ by at most one element. The induction hypothesis (iv) together with 6.6(i) (applicable since $p \leq n$) proves that $p \in \bar{\lambda}(\pi)$. Thus $\bar{\lambda} = \bar{\lambda}(\pi)$. We have proved (i) and (iii) assuming $a_1 \neq 0$. Similarly, we prove (ii) and (iv) assuming $\bar{a}_1 \neq 0$ using the analog of 6.6 where the roles of α_0 and α_1 are exchanged. ■

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