Classification of Irreducible Integrable Modules for Extended Affine Lie Algebras with Center Acting Trivially

Santanu Tantubay, Punita Batra

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Abstract. We classify the irreducible integrable modules for the twisted toroidal extended affine Lie algebras with finite dimensional weight spaces where the finite dimensional center acts trivially. S. E. Rao, S. S. Sharma and P. Batra [*Integrable modules for twisted toroidal extended affine Lie algebras*, J. Algebra 556 (2020) 1057–1072] have classified such modules for twisted toroidal extended affine Lie algebras (with $n \geq 3$), where the finite dimensional center acts nontrivially. C. Fulin, L. Zhiqiang and S. Tan [*Classification of integrable representations for toroidal extended affine Lie algebras*, J. Algebra 514 (2021) 1–37], classified such modules for non twisted toroidal extended affine algebras only for two variables. Our results are valid for any n in case of twisted toroidal extended affine Lie algebras with the finite dimensional center acting trivially. We prove that the universal central extension part acts trivially on the modules and our modules turn out to be the highest weight modules.

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1. Introduction

The representation theory of infinite dimensional Lie algebras is an important topic in both mathematics and physics. Extended affine Lie algebras (EALAs) form a category of important Lie algebras consisting of finite dimensional simple Lie algebras, affine Lie algebras and some other classes of Lie algebras. Twisted toroidal extended affine Lie algebras are examples of EALA. The finite dimensional simple Lie algebras are EALAs of rank 0 and affine Lie algebras are EALAs of rank 1. The structure theory of EALAs has been developed by several mathematicians like Allison, Azam, Berman, Gao, Neher, Pianzola and Yoshii (see [1], [14], [20] and references therein). With an extended affine Lie algebra, we associate a graded ideal, the core. The cores of EALAs may have infinite dimensional centers. Hence the representation theory of EALA is getting attention by several mathematicians.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ and A_n be the Laurent polynomial ring in $n \geq 2$ commuting variables $t_1, \ldots, t_n$. Let $L(\mathfrak{g}) = \mathfrak{g} \otimes A_n$ be a multi-loop algebra. Now $L(\mathfrak{g}) \oplus \Omega_{A_n}/dA_n$ carries a Lie algebra structure of the the universal central extension of $L(\mathfrak{g})$. This is called the toroidal Lie algebra (TLA). Let $Der(A_n)$ be the Lie algebra of derivations of A_n which acts naturally on $L(\mathfrak{g}) \oplus \Omega_{A_n}/dA_n$. So $L(\mathfrak{g}) \oplus \Omega_{A_n}/dA_n \oplus Der(A_n)$ is a Lie algebra called the full

toroidal Lie algebra (FTLA). Neither TLA nor FTLA is an example of an EALA, as none of them has a non-degenerate symmetric invariant bilinear form. So instead of $Der(A_n)$, one takes S_n the Lie subalgebra of $Der(A_n)$, consisting of divergence zero vector fields. The Lie algebra $L(\mathfrak{g}) \oplus \Omega_{A_n}/dA_n \oplus S_n$ is called a toroidal extended affine Lie algebra. It admits a non-degenerate symmetric bilinear form and hence is an example of an EALA.

Here we consider more general EALAs. We take $\sigma_1, \ldots, \sigma_n$ to be finite order commuting automorphisms of $\mathfrak{g}$ and consider the multi-loop algebra $\oplus_{k \in \mathbb{Z}^n} \mathfrak{g}(\bar{k}) \otimes t^k$. The finite dimensional representations of multiloop algebras have been classified by Lau in [11]. With some assumptions this multi-loop algebra is a Lie torus. Now we consider the universal central extension of multi-loop algebra and add S_n, the Lie algebra consisting of divergence zero vector fields on n dimensional torus. This Lie algebra is called twisted toroidal extended affine Lie algebra. We denote it by τ. In [19], Rao, Sharma and Batra classified irreducible integrable modules for τ with finite dimensional weight spaces, where the zero degree central operators act nontrivially. In [7] Fulin, Zhiqiang and Tan classified irreducible integrable modules with finite dimensional weight space for $n = 2$, both zero and non zero case. In this paper we will classify irreducible integrable modules with finite dimensional weight spaces for τ, where the zero degree central operators act trivially for any $n \geq 2$. In [7], the authors considered the non twisted case only for two variables. We make use of some important results from [3] and [8] in order to classify our modules.

The paper is organized as follows. In Section 2, we define the twisted toroidal extended affine Lie algebra τ as above. This Lie algebra has a natural triangular decomposition given by $\tau = \tau^- \oplus \tau^0 \oplus \tau^+$ (see Section 3). In Section 3, we prove the existence of an highest weight vector with respect to this triangular decomposition. We show that the highest weight space is an irreducible module for τ^0 as well as it is Γ-graded. In Section 4, we prove that if the zero degree center $\mathbb{C}$-span$\{K_1, \ldots K_n\}$ acts trivially on V, then the whole central extension part $Z = \Omega_{A_n(m)}/d_{A_n(m)}$ acts trivially on V. In Section 5, in order to classify the modules, we prove as an important result Proposition 5.6(2) using a new method. In Section 6, we prove our main theorem, Theorem 6.7 using a result of [3].

2. Notation and preliminaries

Throughout the paper we fix $n \geq 2$ and let $\mathbb{C}^n$ denote the n-dimensional complex vector space. Let $(,)$ be the standard bilinear form of $\mathbb{C}^n$. All vector spaces, Lie algebras, tensor products are over the field of complex numbers $\mathbb{C}$. For any Lie algebra L, let $U(L)$ be the universal enveloping algebra of L. Let $A_n = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1} \ldots, t_n^{\pm 1}]$ be the Laurent polynomial ring in n commuting variables $t_1, \ldots, t_n$ over $\mathbb{C}$. Let $L(\mathfrak{g}) = \mathfrak{g} \otimes A_n$ be the corresponding multi-loop algebra, where $\mathfrak{g}$ is a finite dimensional simple Lie algebra over $\mathbb{C}$ with a Cartan subalgebra $\mathfrak{h}$. Let Ω_{A_n} be a vector space spanned by the symbols $t^k K_i, 1 \leq i \leq n, k \in \mathbb{Z}^n$. Let dA_n be the subspace of Ω_{A_n} spanned by $\sum_{i=1}^n k_i t^k K_i$. It is well known that $\tilde{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \Omega_{A_n}/dA_n$ is the universal central extension of $L(\mathfrak{g})$ with the following brackets:

$$[x(p), y(q)] = [x, y](p + q) + (x|y) \sum_{i=1}^n p_i t^{p+q} K_i,$$

where $x(p) = x \otimes t^p$.

The basis for the Lie algebra $Der(A_n)$ is $\{d_i, t^r d_i | 1 \leq i \leq n, 0 \neq r \in \mathbb{Z}^n\}$. Now $Der(A_n)$ acts on Ω_{A_n}/dA_n by

$$t^p d_a(t^q K_b) = q_a t^{p+q} K_b + \delta_{ab} \sum_{c=1}^{n} p_c t^{p+q} K_c.$$

There are two non-trivial 2-cocycle of $Der(A_n)$ with values in Ω_{A_n}/dA_n:

$$\phi_1(t^p d_a, t^q d_b) = -q_a p_b \sum_{i=1}^{n} p_i t^{p+q} K_i$$

$$\phi_2(t^p d_a, t^q d_b) = -p_a q_b \sum_{i=1}^{n} p_i t^{p+q} K_i.$$

Let ϕ be any linear combination of ϕ_1 and ϕ_2. Then $\tau = L(g) \oplus \Omega_{A_n}/dA_n \oplus Der(A_n)$ is a Lie algebra with the following brackets and is called a full toroidal Lie algebra:

$$[t^p d_a, X(q)] = q_a X(p+q),$$

$$[t^p d_a, t^q K_b] = q_a t^{p+q} K_b + \delta_{ab} \sum_{i=1}^{n} p_i t^{(p+q)} K_i,$$

$$[t^p d_a, t^q d_b] = q_a t^{p+q} d_b - p_b t^{p+q} d_a + \phi(t^p d_a, t^q d_b).$$

Now consider the Lie subalgebra of divergence zero vector fields S_n of $Der(A_n)$. One can define

$$S_n = \{D(u, r) = \sum_{1 \leq i \leq n} u_i t^r t_i \tfrac{d}{dt_i} : (u|r) = 0, u \in \mathbb{C}^n, r \in \mathbb{Z}^n\}.$$

Let $d_i = t_i \tfrac{d}{dt_i}$ be the zero degree derivation for $1 \leq i \leq n$. Now consider the subalgebra $\tau_{div} = L(\mathfrak{g}) \oplus \Omega_{A_n}/dA_n \oplus S_n$ of τ. Note that when we restrict to S_n, the cocycle ϕ_2 becomes trivial. Therefore any cocycle of S_n on Ω_{A_n}/dA_n will become a scalar multiple of ϕ_1. It is well known that unlike τ, τ_{div} possesses a non-degenerate symmetric, invariant bilinear form and is called as toroidal extended affine Lie algebra. The form on τ_{div} is defined as follows:

$$(X(r)|Y(s)) = \delta_{r,-s}(X|Y), \quad \forall X, Y \in \mathfrak{g}, r, s \in \mathbb{Z}^n;$$

$$\left(\sum_{i=1}^{n} a_i t^r d_i | t^s K_d \right) = \delta_{r,-s} a_d.$$

All other brackets of bilinear form are zero.

Now let $\mathfrak{g}_1$ be any arbitrary finite dimensional simple Lie algebra over $\mathbb{C}$ with a Cartan subalgebra $\mathfrak{h}_1$. Let $\Delta(\mathfrak{g}_1, \mathfrak{h}_1) = \mathrm{supp}_{\mathfrak{h}_1}(\mathfrak{g}_1)$. Then

$$\Delta_1^\times = \Delta^\times(\mathfrak{g}_1, \mathfrak{h}_1) = \Delta(\mathfrak{g}_1, \mathfrak{h}_1) - \{0\}$$

is an irreducible reduced finite root system with at most two root lengths. Define

$$\Delta_{1,en}^\times = \begin{cases} \Delta_1^\times \cup 2\Delta_{1,sh}^\times & \text{if } \Delta_1^\times = B_l \text{ types} \\ \Delta_1^\times & \text{otherwise.} \end{cases}$$

Definition 2.1. A finite dimensional $\mathfrak{g}_1$-module V is said to satisfy condition (M) if

1. V is irreducible with dimension greater than 1.
2. The weights of V relative to $\mathfrak{h}_1$ are contained in $\Delta_{1,en}^{\times}$.

Now recall that $\mathfrak{g}$ is a finite dimensional simple Lie algebra with a Cartan subalgebra $\mathfrak{h}$ and let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the commuting automorphisms of $\mathfrak{g}$ of order $m_1, m_2, \ldots, m_n$ respectively. Let $m = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$. Then define $\Gamma = m_1\mathbb{Z} \oplus \cdots \oplus m_n\mathbb{Z}$ and $\Lambda := \mathbb{Z}^n/\Gamma$. Then we have $\mathfrak{g} = \oplus_{\bar{k}\in\Lambda}\mathfrak{g}(\bar{k})$, where $\mathfrak{g}(\bar{k}) = \{X \in \mathfrak{g} | \sigma_i(X) = \zeta_i^{k_i}X, 1 \leq i \leq n\}$ and ζ_i are m_i-th primitive roots of unity for $i = 1, \ldots, n$.

Definition 2.2. A multiloop algebra $\oplus_{k\in\mathbb{Z}^n}\mathfrak{g}(\bar{k}) \otimes t^k$ is called a *Lie torus LT* if

(1) $\mathfrak{g}(\bar{0})$ is a finite dimensional simple Lie algebra.
(2) For $\bar{k} \neq 0$ and $\mathfrak{g}(\bar{k}) \neq 0$, $\mathfrak{g}(\bar{k}) \cong U(\bar{k}) \oplus W(\bar{k})$, where $U(\bar{k})$ is trivial as $\mathfrak{g}(\bar{0})$-module and either $W(\bar{k})$ is zero or satisfy condition (M).
(3) The order of the group generated by $\sigma_i, 1 \leq i \leq n$ is equal to the product of the orders of each σ_i, for $1 \leq i \leq n$.

Let $\mathfrak{h}(\bar{0})$ denote a Cartan subalgebra of $\mathfrak{g}(\bar{0})$. Then by Lemma 3.1.3 of [13], $\mathfrak{h}(\bar{0})$ is ad-diagonalizable on $\mathfrak{g}$ and $\triangle^{\times} = \triangle^{\times}(\mathfrak{g}, \mathfrak{h}(\bar{0}))$ is an irreducible finite root system in $\mathfrak{h}(\bar{0})$ (Proposition 3.3.5,[13]).

Let $\triangle_0 := \triangle(\mathfrak{g}(\bar{0}), \mathfrak{h}(\bar{0}))$. One of the main property of Lie torus is that $\triangle := \triangle_{0,en}$ (Proposition 3.2.5,[1]). Let $A_n(m) = \mathbb{C}[t_1^{\pm m_1}, \ldots, t_n^{\pm m_n}]$ and

$$S_n(m) = \{D(u,r) | (u|r) = 0, u \in \mathbb{C}^n, r \in \Gamma\}.$$

Now take $\tau = LT \oplus \Omega_{A_n(m)}/d_{A_n(m)} \oplus S_n(m)$. Here $\Omega_{A_n(m)}$ be the vector space spanned by $t^k K_i, 1 \leq i \leq n, k \in \Gamma$ and $dA_n(m)$ be the subspace spanned by $\sum_{1\leq i\leq n} k_i t^k K_i$. One can easily check that if $x \in \mathfrak{g}(\bar{k})$, $y \in \mathfrak{g}(\bar{l})$ with $(x|y) \neq 0$, then $\bar{k} + \bar{l} \in \Gamma$. Therefore we can see that τ is a well defined Lie algebra called *twisted toroidal extended affine Lie algebra*. The aim of this paper is to classify the irreducible integrable modules of twisted toroidal EALAs where K_i's are acting trivially $\forall i \in \{1, \ldots, n\}$.

3. Existence of highest weight space

In this section we will give a root space decomposition of τ. Let

$$H = \mathfrak{h}(\bar{0}) \oplus \sum_{1\leq i\leq n} \mathbb{C}K_i \oplus \sum_{1\leq i\leq n} \mathbb{C}d_i$$

be a Cartan subalgebra for the root space decomposition of τ. Define $\delta_i, w_i \in H^*$ for $1 \leq i \leq n$ by setting

$$\delta_i(\mathfrak{h}(\bar{0})) = 0, \delta_i(K_j) = 0 \text{ and } \delta_i(d_j) = \delta_{ij};$$

$$w_i(\mathfrak{h}(\bar{0})) = 0, \ w_i(K_j) = \delta_{ij} \text{ and } w_i(d_j) = 0.$$

Take $\delta_\beta = \sum_{1\leq i\leq n} \beta_i\delta_i$ for $\beta \in \mathbb{C}^n$. For $k \in \mathbb{Z}^n$, we shall refer to the vector $\delta_{k+\gamma}$ as the translate of δ_k by the vector δ_γ where $\gamma \in \mathbb{C}^n$. Define

$$\mathfrak{g}(\bar{k}, \alpha) := \{x \in \mathfrak{g}(\bar{k}) | [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}(\bar{0})\}.$$

Then we have $\tau = \oplus_{\beta\in\triangle}\tau_\beta$, where $\triangle \subseteq \{\alpha + \delta_k | \alpha \in \triangle_{0,en}, k \in \mathbb{Z}^n\}$.

Here
$$\tau_{\alpha+\delta_k} = \mathfrak{g}(\bar{k}, \alpha) \otimes t^k \; for \; \alpha \neq 0,$$

$$\tau_{\delta_k} = \mathfrak{g}(\bar{k}, 0) \otimes t^k \oplus \bigoplus_{i=1}^{n} \mathbb{C}t^k K_i \oplus \bigoplus_{i=1}^{n} \mathbb{C}t^k d_i$$

where $0 \neq k \in \mathbb{Z}^n$ and $\tau_0 = H$.

In order to get a non-degenerate form on H^*, we extend $\alpha \in \mathfrak{h}(\bar{0})^*$ to H by defining $\alpha(K_i) = \alpha(d_i) = 0, \forall \; 1 \leq i \leq n$. Then $(\mathfrak{h}(\bar{0})|K_i) = 0$, $(\delta_k|\delta_l) = (w_i|w_j) = 0$ and $(\delta_i|w_j) = \delta_{ij}$ and form on $\mathfrak{h}(\bar{0})$ is the restriction of the form of $\mathfrak{g}$. One can easily check that this form is non-degenerate on H^*. A root $\beta = \alpha + \delta_k$ is called a *real root* if $\alpha \neq 0$.

Let $\triangle^{re}$ denote the set of all real roots and $\beta^\vee = \alpha^\vee + \frac{2}{(\alpha|\alpha)} \sum_{1 \leq i \leq n} k_i K_i$ be the co-root of β, where $\alpha^\vee$ is the co-root of $\alpha \in \triangle_{0,en}$ (see 1.3 of [18]). For $\gamma \in \triangle^{re}$, define $r_\gamma(\lambda) = \lambda - \lambda(\gamma^\vee)\gamma$ for $\lambda \in H^*$. Let W be the Weyl group of τ generated by $r_\gamma, \forall \gamma \in \triangle^{re}$.

Definition 3.1. A τ-module V is called *integrable* if

(1) $V = \bigoplus_{\lambda \in H^*} V_\lambda$, where $V_\lambda = \{v \in V \mid h.v = \lambda(h)v \; \forall \; h \in H\}$ and $\dim(V_\lambda) < \infty$

(2) All the real root vectors act locally nilpotently on V, i.e., $\mathfrak{g}(\bar{k}, \alpha) \otimes t^k$ acts locally nilpotently on V for all $0 \neq \alpha \in \triangle_{0,en}$.

Let $P(V) = \{\lambda \in H^* : V_\lambda \neq 0\}$ be the set of weights of V. Then we can see that W acts on $P(V)$.

Proposition 3.2. *Let V be an irreducible integrable module for τ. Then*

(1) $P(V) = \{\gamma \in H^* | V_\gamma \neq 0\}$ *is W-invariant.*

(2) $\dim(V_\gamma) = \dim(V_{w\gamma}), \forall w \in W$.

(3) *If $\lambda \in P(V)$ and $\gamma \in \triangle^{re}$, then $\lambda(\gamma^\vee) \in \mathbb{Z}$.*

(4) *If $\lambda \in P(V)$ and $\gamma \in \triangle^{re}$, and $\lambda(\gamma^\vee) > 0$, then $\lambda - \gamma \in P(V)$.*

Proof. The proof of this Proposition follows as in Lemma (2.3) of [18]. ∎

Now we consider the natural triangular decomposition of τ

$$\tau^+ = \bigoplus_{\alpha>0, k\in\mathbb{Z}^n} \mathfrak{g}(\bar{k}, \alpha) \otimes t^k, \quad \tau^- = \bigoplus_{\alpha<0, k\in\mathbb{Z}^n} \mathfrak{g}(\bar{k}, \alpha) \otimes t^k,$$

$$\tau^0 = \bigoplus_{k\in\mathbb{Z}^n} \mathfrak{g}(\bar{k}, 0) \otimes t^k \oplus Z \oplus S_n(m),$$

where $Z = \Omega_{A_n(m)}/d_{A_n(m)}$.

We assume V to be an integrable module for τ. Let $\widetilde{LT} = LT \bigoplus \Omega_{A_n}/d_{A_n} \bigoplus D$, where D is the $\mathbb{C}$ linear span of $d_i, 1 \leq i \leq n$. Then $\widetilde{LT}$ will become a Lie subalgebra of τ with the same Cartan subalgebra H. Therefore V will be an integrable module for $\widetilde{LT}$.

Let us recall a result about integrable module for $\widetilde{LT}$ from [16], which will be used in order to find highest weight vector for τ.

Lemma 3.3. (Lemma 3.6,[16]) *Let V be an integrable $\widetilde{LT}$-module . Then there exists some $\lambda \in P(V)$ such that $\lambda + \eta + \delta_k \notin P(V)$, for all $\eta \in Q \setminus \{0\}$ and $k \in \mathbb{Z}^n$, where Q denotes the non-negative root lattice of $\mathfrak{g}(\bar{0})$.*

Theorem 3.4. *The highest weight space $V_+ = \{v \in V | \tau^+.v = 0\}$ is non-zero.*

Proof. $\widetilde{LT}$ is a Lie subalgebra of τ. So an integrable module for τ with finite dimensional weight spaces will be an integrable module for $\widetilde{LT}$ with finite dimensional weight spaces. Now take $\lambda \in P(V)$ as in Lemma 3.3. Using Proposition 3.2, it can be proved that $\lambda|_{\mathfrak{h}(\bar{0})}$ is a dominant integral weight of $\mathfrak{g}(\bar{0})$. Using Lemma 3.3, we can find $\mu \in P(V)$ such that $V_{\mu+\alpha+\delta_k} = 0$, $\forall \alpha \in \triangle_0^+, k \in \mathbb{Z}^n$. Again using Proposition 3.2, we will get $V_{\mu+2\alpha+\delta_k} = 0$, $\forall \alpha \in \triangle_0^+, k \in \mathbb{Z}^n$. ∎

We can see that V_+ is a τ^0 module.

Lemma 3.5. (1) V_+ *is irreducible module over τ^0.* (2) $V = U(\tau^-)V_+$.

Proof. Using PBW theorem and some weight arguments it is easy to see (1). Using (1) and PBW theorem we get (2). ∎

Lemma 3.6. (1) *There exists unique $\bar{\lambda} \in \mathfrak{h}(\bar{0})^*$ and a $\beta \in \mathbb{C}^n$, such that the weights of V_+ will be of the form $\bar{\lambda} + \delta_{r+\beta}$ where $r \in \mathbb{Z}^n$, here β need not be unique.*
(2) $\bar{\lambda}$ *is a dominant integral.*

Proof. (1) We can see that $\mathfrak{h}(\bar{0})$ commutes with τ^0. Now using Lemma 3.5, we will get $\mathfrak{h}(\bar{0})$ acts by scalars on V_+ and hence it will act by a single linear functional on V_+. Let us denote it by $\bar{\lambda}$. If $D(u,r) \in S_n(m)$, then $D(u,r).V_\mu \subseteq V_{\mu+\delta_r}$ for $\mu \in P(V)$. Since $[d_i, t^r K_j] = r_i t^r K_j$ for $1 \le i, j \le n$, therefore $t^r K_j.V_\mu \subseteq V_{\mu+\delta_r}$ for $\mu \in P(V)$.
(2) The proof follows from Proposition 3.2. ∎

If $\bar{\lambda} = 0$ in Lemma 3.6, then using Proposition 3.2, we can check that only possible weights of V are $\delta_\beta, \beta \in \mathbb{C}^n$ (see Remark 3.7 of [17]). Now one can prove that $LT + \Omega_{A_n(m)}/d_{A_n(m)}$ acts trivially on V. Then V will be an irreducible module for $S_n(m)$ with finite dimensional weight spaces with respect to the Cartan subalgebra $D = \oplus_{1 \le i \le n} \mathbb{C} d_i$. It is still unknown about the irreducible modules for $S_n(m)$ with finite-dimensional weight spaces. So we will take $\bar{\lambda} \ne 0$ in Lemma 3.6.

Take $\lambda \in P(V)$ from Lemma 3.3. Then by our assumption there exists $\alpha \in \triangle_0$ such that $\lambda(h_\alpha) \ne 0$. Then using Lemma 3.5, we get

$$V_+ = \bigoplus_{r \in \mathbb{Z}^n} V_+(r),$$

where $V_+(r) = \{v \in V_+ | d_i.v = (\lambda(d_i) + r_i)v, 1 \le i \le n\}$. So V_+ is $\mathbb{Z}^n$-graded. Now for $r \in \Gamma$, we define $V'(r) = \underset{r_i \le k_i < m_i + r_i}{\oplus} V_+(k)$. Then $V_+ = \underset{r \in \Gamma}{\oplus} V'_+(r)$ is Γ-graded.

4. Action of the central extension

In this section, we will prove that when K_i for $1 \leq i \leq n$ acts trivially on V, then $\Omega_{A_n(m)}/d_{A_n(m)}$ acts trivially on V. In order to prove this let us fix some i, $1 \leq i \leq n$ and consider the extended loop algebra $\mathfrak{g}(\bar{0}) \otimes \mathbb{C}[t_i^{\pm m_i}] \oplus \mathbb{C}d_i$. Let θ be the highest root of $\mathfrak{g}(\bar{0})$, $\theta^{\vee}$ be its co-root and W_0 be the Weyl group of $\mathfrak{g}(\bar{0})$. Also assume that W_i be the Weyl group of the loop algebra $\mathfrak{g}(\bar{0}) \otimes \mathbb{C}[t_i^{\pm m_i}] \oplus \mathbb{C}d_i$. We can see that $h.v = \bar{\lambda}(h)v$ for all $v \in V_+$ and $h \in \mathfrak{h}(\bar{0})$. Let

$$r_\lambda = \min_{h \in \mathbb{Z}(W_0\theta^{\vee})}\{\lambda(h) : \lambda(h) > 0\} \in \mathbb{N}.$$

Define $V_{\bar{\lambda}} = \{v \in V_+ \mid d_i.v = \beta_i v, \ 1 \leq i \leq n\}$. Therefore from 2.4 of [5], we have the following Lemma.

Lemma 4.1.   *For all $s_i \in \mathbb{Z}$, there exists $w_i \in W_i$ such that $w_i(\bar{\lambda}+s_i\delta_i) = \bar{\lambda}+\bar{s}_i\delta_i$, where $0 \leq \bar{s}_i < r_{\bar{\lambda}}$.*

Now we consider the Lie algebra $\tau_{div} = \mathfrak{g}(\bar{0}) \otimes A_n(m) \oplus \Omega_{A_n(m)}/d_{A_n(m)} \oplus S_n(m)$. Let W be the Weyl group of τ_{div}, then $W_i \subseteq W$ for $1 \leq i \leq n$.

Lemma 4.2. (Corollary 3.5, [19]) *Let $\delta_r = \sum_{1 \leq i \leq n} r_i\delta_i$, where $r = (r_1, ..., r_n) \in \mathbb{Z}^n$. Let $r_i = k_i + s_im_i$, where $0 \leq k_i < m_i$. Then there exists $w \in W$ such that $w(\bar{\lambda} + \delta_r) = \bar{\lambda} + \delta_k + \sum_{1 \leq i \leq n} \bar{s}_im_i\delta_i$, where $0 \leq \bar{s}_i < r_\lambda$.*

Let $M = \{V_+(r) \mid r_i = k_i + s_im_i, \ 0 \leq k_i \lneq m_i, \ 0 \leq s_i \lneq r_\lambda, \ 1 \leq i \leq n\}$.

Consider $M' = \bigoplus_{V_+(r) \in M} V_+(r)$, therefore M' is a finite dimensional vector subspace of V_+. Now we consider the toroidal Lie subalgebra of τ,

$$\mathfrak{L} = \mathfrak{g}(\bar{0}) \otimes A_n(m) \oplus \Omega_{A_n(m)}/d_{A_n(m)} \oplus D,$$

where $D = \mathrm{span}\{d_i \mid 1 \leq i \leq n\}$. Let us assume $\mathfrak{L}^{\pm} = \mathfrak{L} \cap \tau^{\pm}$ and $\mathfrak{L}^0 = \mathfrak{L} \cap \tau^0$, therefore $\mathfrak{L} = \mathfrak{L}^- \oplus \mathfrak{L}^0 \oplus \mathfrak{L}^+$ is a triangular decomposition of $\mathfrak{L}$. Using Lemma 4.1 and Lemma 4.2 we can prove the following proposition.

Proposition 4.3.   *The weight spaces of V_+ are uniformly bounded.*

Proposition 4.4.   *Any strictly decreasing sequence of $\mathfrak{L}$ submodule of V intersecting V_+ non trivially has finite length.*

Proof.   Let $V_1 \supsetneq V_2$ be two $\mathfrak{L}$-submodules of V such that $V_i \cap V_+ \neq 0$ for $i = 1, 2$. Then we know that there exist subspaces W_1, W_2 of M' such that $V_i = U(\mathfrak{L})W_i \ \forall \ i = 1, 2$. Put $W_i' = U(\mathfrak{L}^0)W_i$, $\widetilde{W}_i = W_i' \cap M' \ \forall \ i = 1, 2$. Since we assume that $V_1 \supsetneq V_2$, therefore $W_1' \supseteq W_2'$. This will imply $\widetilde{W}_1 \supseteq \widetilde{W}_2$.

Now it is easy to see that $U(\mathfrak{L}^0)W_i = U(\mathfrak{L}^0)\widetilde{W}_i$ for $i = 1, 2$. Therefore by PBW theorem $U(\mathfrak{L})W_i = U(\mathfrak{L})\widetilde{W}_i$ for $i = 1, 2$. In order to prove the proposition, it is enough to prove that $\widetilde{W}_2 \subsetneq \widetilde{W}_1$. If not, then we have $\widetilde{W}_2 = \widetilde{W}_1$. This will give us $U(\mathfrak{L})W_2 = U(\mathfrak{L})W_1$. But this implies $V_2 = V_1$, a contradiction.   ∎

Proposition 4.5.   ([18], Proposition 4.13) *Let W be an irreducible integrable module for $\mathfrak{L}$ with finite dimensional weight spaces, where all K_i acts trivially on W, then $\Omega_{A_n(m)}/d_{A_n(m)}$ acts trivially on W.*

Proposition 4.6. $\Omega_{A_n(m)}/d_{A_n(m)}$ *acts trivially on V.*

Proof.   Using Proposition 4.4, we take a minimal $\mathfrak{L}$-submodule of V which intersects V_+ non trivially, assume W. Now W need not be an irreducible module for $\mathfrak{L}$, but every proper submodule of W intersects V_+ trivially. Therefore sum of all proper $\mathfrak{L}$-submodule of W is again proper and intersect V^+ trivially and it will be maximal. Let $\bar{W}$ be the quotient module of W by this maximal submodule. Therefore it will be irreducible $\mathfrak{L}$-module where $W \cap V_+$ goes injectively. Therefore by Proposition 4.5, $\Omega_{A_n(m)}/d_{A_n(m)}$ acts trivially on $W \cap V_+$. It is easy to check that $S = \{v \in V | t^r K_i.v = 0, 1 \leq i \leq n\}$ is a non-zero τ-submodule of V. Hence we are done by irreducibility of V.   ∎

5.   Action of $h_\alpha \otimes t^k$ on highest weight space

Lemma 5.1.   *Let $h \otimes t^k \in \tau^0$, where $k \in \Gamma \setminus \{0\}$ and $h \in h(\bar{0})$. If there exists a nonzero vector of V_+ on which $h \otimes t^k$ acts trivially, then $h \otimes t^k$ is locally nilpotent on V_+.*

Proof.   Consider the set $S = \{v \in V_+ | \text{ there is a } n \in \mathbb{N} \text{ s.t. } (h \otimes t^k)^n.v = 0\}$. By assumption clearly this set is nonzero. We know that $h \otimes t^k$ commutes with $\bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(\bar{k}, 0) \otimes t^k$. So $(\bigoplus_{k \in \mathbb{Z}^n} \mathfrak{g}(\bar{k}, 0) \otimes t^k)S \subseteq S$. Now using the equation

$$(h \otimes t^k)^n.D(u, r) = D(u, r).(h \otimes t^k)^n - n(u.k)(h \otimes t^k)^{n-1}.h \otimes t^{k+r}$$

we can prove that $S_n(m).S \subseteq S$. Therefore S is a nonzero τ^0-submodule of V_+. Hence by irreducibility of V_+, we can say that $S = V_+$. Hence we have the result.   ∎

Now we can not proceed as [10], since we don't have the full derivation space here. In this paper we give new proof of the following two Lemmas (Lemma 5.2, Lemma 5.3).

Lemma 5.2.   *Let $h \otimes t^k$ acts locally nilpotently on V_+, for every $k \in \Gamma \setminus \{0\}$, where $h \in \mathfrak{h}(\bar{0})$. Then $h \otimes t^k$ acts trivially on V_+, for every $k \in \Gamma \setminus \{0\}$.*

Proof.   Let $p \in \Gamma \setminus \{0\}$. Then by Proposition 4.3, we will get $(h \otimes t^{-p}.h \otimes t^p)^N = 0$. Therefore $(h \otimes t^{-p})^N.(h \otimes t^p)^N = 0$ (Since $h \otimes t^a$ and $h \otimes t^b$ commutes for any $a, b \in \Gamma$).

Let $r_1 \in \Gamma \setminus \{0\}$ be such that $r_1 \notin \mathbb{Q}.p$. Then there exists $u \in \mathbb{C}^n$ such that $(u, p) \neq 0$ and $(u, r_1) = 0$. Therefore we have the following equation:

$$D(u, r_1).(h \otimes t^{-p})^N.(h \otimes t^p)^N = 0.$$

This equation gives us

$$N(u, p)h \otimes t^{r_1+p}.(h \otimes t^{-p})^N.(h \otimes t^p)^{N-1}$$
$$- N(u, p)h \otimes t^{r_1-p}.(h \otimes t^{-p})^{N-1}.(h \otimes t^p)^{N-1} = 0.$$

Now applying $h \otimes t^p$ to the above equation we get

$$h \otimes t^{r_1-p}.(h \otimes t^{-p})^{N-1}.(h \otimes t^p)^{N+1} = 0.$$

Using induction on j, we want to prove

$$h \otimes t^{r_1-p}.\ldots.h \otimes t^{r_j-p}.(h \otimes t^{-p})^{N-j}.(h \otimes t^p)^{N+j} = 0,$$

where $1 \le j \le N$ and $r_1, \ldots, r_j \in \Gamma$ with $\sum_{i=1}^{j} \epsilon_i r_i \notin \mathbb{Q}.p$ and $\epsilon_i \in \{0, 1\}$.
For $j = 1$, we have proved this already. Now assume the result holds for $1 \le j < N$.
Let $r \in \Gamma$ with $r + \sum_{i=1}^{j} \epsilon_i r_i \notin \mathbb{Q}.p$. So we can find $u \in \mathbb{C}^n$ such that $(u, p) \ne 0$ and $(u, r) = 0$. Therefore we have

$$D(u, r).h \otimes t^{r_1-p}.\ldots.h \otimes t^{r_j-p}.(h \otimes t^{-p})^{N-j}.(h \otimes t^p)^{N+j} = 0.$$

From this equation we get

$$(N+j)(u,p)h \otimes t^{r_1-p}.\ldots.h \otimes t^{r_j-p}.h \otimes t^{r+p}.(h \otimes t^{-p})^{N-j}.(h \otimes t^p)^{N+j-1}$$
$$- (N-j)(u,p)h \otimes t^{r_1-p}.\ldots.h \otimes t^{r_j-p}.h \otimes t^{r-p}.(h \otimes t^{-p})^{N-j-1}.(h \otimes t^p)^{N+j} = 0.$$

Applying $h \otimes t^p$, we have

$$h \otimes t^{r_1-p}.\ldots.h \otimes t^{r_j-p}.h \otimes t^{r-p}.(h \otimes t^{-p})^{N-j}.(h \otimes t^p)^{N+j} = 0.$$

Now taking $j = N$ implies $h \otimes t^{r_1-p}.\ldots.h \otimes t^{r_N-p}.(h \otimes t^p)^{2N} = 0$.
From this equation, using induction again, we can prove

$$h \otimes t^{r_1-p}.\ldots.h \otimes t^{r_{N+j}-p}.(h \otimes t^p)^{2N-j} = 0,$$

for all $0 \le j \le 2N$ with $\sum_{i=1}^{N+j} \epsilon_i r_i \notin \mathbb{Q}.p$ and $\epsilon_i \in \{0, 1\}$.
In this case instead of taking $D(u, r)$ we take $D(u, r - 2p)$, it is easy to find such $u \in \mathbb{C}^n$. Hence we will have

$$h \otimes t^{r_1-p}.\ldots.h \otimes t^{r_{3N}-p} = 0,$$

with all $r_i \in \Gamma$ and $\sum_{i=1}^{3N} \epsilon_i r_i \notin \mathbb{Q}.p$.
Let $s_1, \ldots s_{3N} \in \Gamma \setminus \{0\}$ be arbitrary. Take $p \in \Gamma \setminus \cup_{\epsilon_i \in \{0,1\}} \mathbb{Q}.(\sum_{i=1}^{3N} \epsilon_i s_i)$. Now taking $r_i = s_i + p$ in the above equation we obtain

$$h \otimes t^{s_1} \ldots h \otimes t^{s_{3N}} = 0. \tag{1}$$

Take $W = \{v \in V_+ | h \otimes t^p.v = 0, \forall p \in \Gamma \setminus \{0\}\}$. By equation (1), W is nonzero. We can easily check that W is a τ^0-submodule of V_+. Hence by the irreducibility of V_+, we have the Lemma. ∎

Lemma 5.3. $h \otimes t^k$ *acts injectively or trivially on* V_+, *for all* $k \in \Gamma \setminus \{0\}$.

Proof.   Let $h \otimes t^p$ acts injectively on V_+, for some $p \in \Gamma \setminus \{0\}$.

Claim 1. For any $q \in \Gamma \setminus \mathbb{Q}.p$, if $h \otimes t^q$ acts locally nilpotently, then it will act nilpotently.

Assume $h \otimes t^q$ acts locally nilpotently. Since $\dim(V_+(r)) \le N$ for all $r \in \Gamma$, we have $(h \otimes t^{-q})^N.(h \otimes t^q)^N = 0$.
Suppose there exists $1 \le k \le N$ and N_k such that $(h \otimes t^{-q})^k.(h \otimes t^q)^{N_k} = 0$.

We assume $q \notin \mathbb{Q}.p$. So there exists $u \in \mathbb{C}^n$ such that $(u, p+q) = 0$ and $(u, q) \neq 0$. Now applying $D(u, p+q)$, to the above equation we obtain

$$N_k(u,q)h \otimes t^{p+2q}.(h \otimes t^{-q})^k(h \otimes t^q)^{N_k - 1} - k(u,q)h \otimes t^p.(h \otimes t^{-q})^{k-1}.(h \otimes t^q)^{N_k} = 0.$$

Now applying $h \otimes t^q$, we have $h \otimes t^p.(h \otimes t^{-q})^{k-1}.(h \otimes t^q)^{N_k+1} = 0$. Since $h \otimes t^p$ acts injectively on V_+, so $(h \otimes t^{-q})^{k-1}.(h \otimes t^q)^{N_k+1} = 0$. Now repeating the process we can find $N_0 \in \mathbb{N}$ such that $(h \otimes t^q)^{N_0} = 0$.

Claim 2. If $q \in \Gamma \setminus \{0\} \cap \mathbb{Q}.p$, then $h \otimes t^q$ acts injectively on V_+.

First we will prove that $h \otimes t^{\frac{a}{b}p}$ acts injectively, if $a \in \mathbb{Z}_-$, $b \in Z_+$.

If for some such $a, b \in \mathbb{Z}$, $h \otimes t^{\frac{a}{b}p}$ is not acting injectively then by Lemma 5.1, it will act locally nilpotently. Therefore $(h \otimes t^p)^{-a}.(h \otimes t^{\frac{a}{b}p})^b$ acts nilpotently on each homogeneous spaces. Hence $(h \otimes t^p)^{-aN}.(h \otimes t^{\frac{a}{b}p})^{bN} = 0$. Therefore $(h \otimes t^{\frac{a}{b}p})^{bN} = 0$, since $h \otimes t^p$ acts injectively on V_+. Assume $N_0 \in \mathbb{N}$ be smallest such that $(h \otimes t^{\frac{a}{b}p})^{N_0} = 0$. Let $q \in \Gamma \setminus \mathbb{Q}.p$, then we can find $u \in \mathbb{C}^n$ such that, $(u, q - \frac{a}{b}p) = 0$ and $(u, p) \neq 0$. Now take

$$0 = D(u, q - \frac{a}{b}p).(h \otimes t^{\frac{a}{b}p})^{N_0} = N_0(u, \frac{a}{b}p)(h \otimes t^{\frac{a}{b}p})^{N_0 - 1}.h \otimes t^q.V_+.$$

From the minimality of N_0, we can say from the above equation and Lemma 5.1, that $h \otimes t^q$ acts locally nilpotently. Therefore by claim (1), $h \otimes t^q$ acts nilpotently on V_+. Choose N' be the smallest such that $(h \otimes t^q)^{N'} = 0$. Choose $u \in \mathbb{C}^n$ such that $(u, p - q) = 0$ and $(u, q) \neq 0$. Then

$$0 = D(u, p-q)(h \otimes t^q)^{N'} = N'(u,q)(h \otimes t^q)^{N'-1}.h \otimes t^p.$$

Now by the minimality of N', we see that $h \otimes t^p$ is not acting injectively, which is a contradiction. So our claim (2) is true for $a \in \mathbb{Z}_-, b \in \mathbb{Z}_+$. Now let both a, b are positive and $h \otimes t^{\frac{a}{b}p}$ do not act injectively. We know by above $h \otimes t^{-ap}$ acts injectively. By our assumption $h \otimes t^{-ap}.(h \otimes t^{\frac{a}{b}p})^b$ acts nilpotently on each homogeneous space. Now repeating the same process as above we will get the contradiction, $h \otimes t^p$ not acting injectively on V_+. So our claim (2) is true. We are now going to prove that $h \otimes t^q$ acts injectively for all $q \in \Gamma \setminus \{0\}$. We already proved it for $q \in \mathbb{Q}.p$. Let $q \notin \mathbb{Q}.p$ and $h \otimes t^q$ does not act injectively on V_+. Therefore by Lemma 5.1 and claim(1), we can say that $h \otimes t^q$ acts nilpotently on V_+. Using the minimal power of $h \otimes t^q$, for which it acts trivially, we will arrive at $h \otimes t^p$ not acting injectively on V_+, a contradiction to our assumption. Hence using Lemma 5.1 and Lemma 5.2, we have the Lemma. ∎

Now we have chosen $\alpha \in \triangle_0$ such that $\bar{\lambda}(h_\alpha) \neq 0$. Now if there is a $k \in \Gamma \setminus \{0\}$ such that $h_\alpha \otimes t^k$ is not injective on V_+, then by Lemma 5.3, we can say that $h_\alpha \otimes t^m$ acts trivially on V_+ for all $m \in \Gamma \setminus \{0\}$. Then using [6] (see also [19]), we can say that h_α acts trivially on V_+, which is a contradiction to our assumption.

We know $V_+ = \oplus_{r \in \Gamma} V_+(r)$. Let $\{v_0, \ldots, v_k\}$ be a basis of $V_+(0)$. Therefore by above discussion, $\{h_\alpha \otimes t^p.v_0, \ldots, h_\alpha \otimes t^p.v_k\}$ will be a basis of $V_+(p)$ for all $p \in \Gamma$. Now assume, $h_\alpha \otimes t^p.h_\alpha \otimes t^q.(v_1, \ldots, v_k) = h_\alpha \otimes t^{p+q}.(v_1, \ldots, v_k)A_{p,q}$, where $A_{p,q} \in M_k(\mathbb{C})$.

Since $\{h_\alpha \otimes t^p | \ p \in \Gamma\}$ is a commuting family, we can easily see that $\{A_{p,q} | p, q \in \Gamma\}$ is a commuting family. Therefore we can find an invertible matrix $S \in M_k(\mathbb{C})$ such that $\{B_{p,q} = S^{-1}A_{p,q}S | p, q \in \Gamma\}$ is an upper triangular family.

Set $(w_1, \ldots, w_k) = (v_1, \ldots, v_k).S$, then we have

$$h_\alpha \otimes t^p.h_\alpha \otimes t^q.(w_1, \ldots, w_k) = h_\alpha \otimes t^{p+q}.(w_1, \ldots, w_k)B_{p,q}.$$

So without loss of generality we can assume that $A_{p,q}$ is upper triangular for all $p, q \in \Gamma$.

Lemma 5.4. *Let $p, q \in \Gamma$, then there exists $\lambda_{p,q}$ and $v_0 \in V_+(0)$ such that $(h_\alpha \otimes t^p.h_\alpha \otimes t^q - \lambda_{p,q}h_\alpha \otimes t^{p+q})v_0 = T_{p,q}v_0 = 0$.*

Proof. Since for $\lambda \in \mathbb{C}$,

$$(h_\alpha \otimes t^p.h_\alpha \otimes t^q - \lambda h_\alpha \otimes t^{p+q}).(v_1, \ldots v_k) = h_\alpha \otimes t^{p+q}.(v_1, \ldots, v_k)(A_{p,q} - \lambda.I_k).$$

Now the Lemma follows from the fact $A_{p,q}$ has eigenvalues. ∎

Lemma 5.5. *For $p, q \in \Gamma$, $\lambda \in \mathbb{C}$, if there exists $v \in V_+(0) \setminus \{0\}$ such that $T_\lambda.v = 0$, then T_λ acts locally nilpotently on V_+.*

Proof. The proof will be same as Lemma 5.1. ∎

Now using Lemma 5.4 and Lemma 5.5, we can say that for every $p, q \in \Gamma$, $A_{p,q}$ has exactly one eigenvalue (say, $\lambda_{p,q}$).

Proposition 5.6. (1) $h_\alpha \otimes t^k$ *acts injectively on V_+ for every $k \in \Gamma$.*

(2) $h_\alpha \otimes t^r.h_\alpha \otimes t^s = \lambda_{r,s}h_\alpha \otimes t^{r+s}$ *on V_+, where $\lambda_{r,s} = \lambda$ for all $r \neq 0$, $s \neq 0, r + s \neq 0$, $\lambda_{r,-r} = \mu$ for all $r \neq 0$ and $\lambda_{0,r} = \bar{\lambda}(h_\alpha)$ for all $r \in \Gamma$. Further we have $\mu\lambda_{0,r} = \lambda^2 \neq 0$.*

(3) $\dim(V'_+(r)) = \dim(V'_+(s))$ *for all $r, s \in \Gamma$.*

Proof. (1) is a direct consequence of above discussion. (3) also follows from (1). Now we prove (2). Set $V_0 = \sum_{i=0}^{k-1} \mathbb{C}v_i$, where $v_0 = 0$. Since $A_{p,q}$ is an upper triangular, we have $T_{p,q}.v_l \in \sum_{i=0}^{l-1} h_\alpha \otimes t^{p+q}.v_i$. We can easily see that $T_{p,q}.V_+(-p-q) \subseteq V_0$. Now for any $p, q, r \in \Gamma$, with $u \in \mathbb{C}^n$ such that $(u, r) = 0$, we define

$$S^r_{p,q} = (u, p)h_\alpha \otimes t^{p+r}.h_\alpha \otimes t^q + (u, q)h_\alpha \otimes t^p.h_\alpha \otimes t^{q+r} - \lambda_{p,q}(u, p + q)h_\alpha \otimes t^{p+q+r}.$$

Therefore

$$h_\alpha \otimes t^{-(p+q+r)}.S^r_{p,q}.v_k$$
$$= \lambda_{-(p+q+r),p+q+r}((u, p)\lambda_{p+r,q} + (u, q)\lambda_{p,q+r} - (u, p + q)\lambda_{p,q}).v_k + w',$$

where $w' \in V_0$. One can also check that $h_\alpha \otimes t^{-(p+q+r)}.S^r_{p,q}.V_0 \subseteq V_0$. Now we can see that $[D(u, r), T_{p,q}] = S^r_{p,q}$. Given any $p, q, r \in \Gamma$, and $u \in \mathbb{C}^n$ with $(u, r) = 0$ we have

$$0 = (h_\alpha \otimes t^{-(p+q+r)})^k.(D(u, r))^k.(T_{p,q})^k.v_k = T_{p,q}.w + k!(h_\alpha \otimes t^{-(p+q+r)})^k.(S^r_{p,q})^k.v_k,$$

where $w \in V_+(-p - q)$.

From the above equation, we obtain

$$(u, p)\lambda_{p+r,q} + (u, q)\lambda_{p,q+r} - (u, p+q)\lambda_{p,q} = 0, \tag{2}$$

for any $p, q, r \in \Gamma$. Take $W = \{v \in V_+ | T_{p,q}.v = 0\}$, then $v_1 \in W$. Hence using the above equation we can check that W is a nonzero τ^0-submodule of V_+, therefore by irreducibility of V_+, $W = V_+$. Now taking $p = jq$, with $j \in \mathbb{Q} \setminus \{0, -1\}$ in equation (2) , we get

$$j(u, q)\lambda_{jq+r,q} + (u, q)\lambda_{jq,q+r} - (j+1)(u, q)\lambda_{p,q} = 0.$$

Taking, $r \in \Gamma \setminus \mathbb{Q}.q$, we can see that

$$j\lambda_{jq+r,q} + \lambda_{jq,q+r} - (j+1)\lambda_{q,q} = 0.$$

Putting $j = 1$, we have $\lambda_{q,q+r} = \lambda_{q,q}$ and hence $\lambda_{jq+r,q} = \lambda_{jq,q+r} = \lambda_{q+r,q+r} = \lambda_{q,q}$ for all $r \in \Gamma \setminus \mathbb{Q}.q$. Therefore $\lambda_{p,q} = \lambda_{q,q}(= \lambda$, say$)$, if $p \neq 0$, $q \neq 0$, $p + q \neq 0$.

Taking $r = -p - q$ with $p \in \Gamma \setminus \mathbb{Q}.q$ in equation (2), we get $\lambda_{p,-p} = \lambda_{q,-q}$ $(= \mu$, say$)$ and hence for every $p, q \in \Gamma$. It is easy to see that $\lambda_{r,0} = \bar{\lambda}(h_\alpha)$.

Let us assume $p, q \in \Gamma$ be such that $p \neq 0$, $q \neq 0$, $p + q \neq 0$, then for any $v \in V_+$, we have

$$(h_\alpha \otimes t^p.h_\alpha \otimes t^{-p}).(h_\alpha \otimes t^q.h_\alpha \otimes t^{-q}).v = (\mu)^2.(\bar{\lambda}(h_\alpha))^2 v = \lambda_{p,q}\lambda_{-p,-q}\lambda_{p+q,-p-q}\bar{\lambda}(h_\alpha).v.$$

hence we have the identity of (2). ∎

6. Classification of modules

Now recall that our Lie algebra reduces to $\tau = LT \oplus S_n(m)$ with

$$\tau^0 = \oplus_{k \in \mathbb{Z}^n} \mathfrak{g}(\bar{k}, 0) \otimes t^k \oplus S_n(m).$$

Take $\mathfrak{g}' = \{x \in \mathfrak{g} | [h, x] = 0, \forall h \in \mathfrak{h}(\bar{0})\}$. Now since $\mathfrak{g}'$ is invariant under σ_i's, therefore $\mathfrak{g}'$ is Λ graded. It is easy to see that $L(\mathfrak{g}', \sigma) = LT^0$. Let us take

$$S_n'(m) = \mathrm{span}\{D(u, r) - D(u, 0) \,|\, u \in \mathbb{C}^n, \; r \in \Gamma, \; (u.r) = 0\}.$$

We can easily check that $S_n'(m)$ is a Lie subalgebra of $S_n(m)$. Furthermore let us set $L = S_n'(m) \ltimes L(\mathfrak{g}', \sigma)$ and $W = \mathrm{span}\{h_\alpha \otimes t^r.v - v | r \in \Gamma, v \in V_+\}$. We can see that W is a L-module.

Lemma 6.1.   (1) W is a proper L-submodule of V_+.

(2) $\widetilde{V} = V_+/W$ is a finite dimensional L-module.

Proof.   The proof is parallel to Lemma 7.2 of [17]. Let $z_i = h_\alpha \otimes t_i^m$ for each $i = 1, \ldots, n$. Without loss of generality we can assume that $\lambda_{r,s} = 1$ for $r \neq 0$, $s \neq 0$, $r + s \neq 0$. Therefore we can say that $W = \mathrm{span}\{z_i.v - v | v \in V_+, \; 1 \leq i \leq n\}$. Now the proof will be same as in [17] ∎

Let $\beta \in \mathbb{C}^n$ be as in Lemma 3.6. Then for any L module V' we can give a τ^0 module structure on $L(V')$ by

$$x \otimes t^k.(v_1 \otimes t^s) = ((x \otimes t^k).v_1) \otimes t^{k+s}.$$

$$D(u, r).(v_1 \otimes t^s) = ((D(u, r) - D(u, 0)).v_1) \otimes t^{r+s} + (u, s + \beta)(v_1 \otimes t^{r+s})$$

for all $v_1 \in V', x \in g(\bar{k}, 0)$ and $D(u, r) \in S_n(m)$.

For $v \in V_+$, let $\bar{v}$ be the image of v in $\widetilde{V}$. Now define

$$\phi : V_+ \to L(\widetilde{V}) \ \text{ by } \ v \mapsto \bar{v} \otimes t^k \ \text{ for } \ v \in V_+(k).$$

This map is clearly a nonzero τ^0-module homomorphism. Hence by irreducibility of V_+, it follows that $V_+ \cong \phi(V_+)$ is a τ^0 submodule of $L(\widetilde{V})$.

Clearly L is naturally Λ graded. Now since V_+ and W is Z^n graded, therefore they are naturally Λ graded and so is $\widetilde{V}$. Therefore $\widetilde{V} = \oplus_{\bar{p} \in \Lambda} \widetilde{V}(\bar{p})$.

Now for $\bar{p} \in \Lambda$, we set

$$L(\widetilde{V})(\bar{p}) = \{v \otimes t^{k+r+p} | v \in \widetilde{V}(\bar{k}), r \in \Gamma, k \in \mathbb{Z}^n\}$$

It can be easily verified that $L(\widetilde{V})(\bar{p})$ is a τ^0 submodule of $L(\widetilde{V})$. The following result can be deduced similarly as in [2].

Proposition 6.2. (1) $V_+ \cong L(\widetilde{V})(\bar{0})$ as τ^0-modules.

(2) $\widetilde{V}$ is Λ-graded-irreducible module over L.

(3) $\widetilde{V}$ is completely reducible module over L and all its irreducible components are mutually isomorphic as $S_n'(m) \ltimes h(\bar{0}) \otimes A(m)$-modules.

Now we will concentrate on irreducible representation of L. Let (W, π) be a finite dimensional representation of L. Let $\pi(L(\mathfrak{g}', \sigma)) = \mathfrak{g}^1$, then $\pi(L) = \mathfrak{g}^1 \oplus \mathfrak{g}^2$, where $\mathfrak{g}^2$ is the unique complement of $\mathfrak{g}^1$ in $\mathfrak{gl}(W)$ (Proposition 19.1(b) of [9]). So W will be an irreducible module for $\mathfrak{g}^1 \oplus \mathfrak{g}^2$. Therefore $W \cong W_1 \otimes W_2$, where W_1 and W_2 are irreducible modules for $\mathfrak{g}^1$ and $\mathfrak{g}^2$ respectively (see [12]). Let $\mathfrak{g}' = \mathfrak{g}'_{ss} \oplus R$, where $\mathfrak{g}'$ and R are Levi and radical part of $\mathfrak{g}'$. Then as $\sigma_i(\mathfrak{g}') = \mathfrak{g}'$ and $\sigma_i(R) = R$ for $1 \leq i \leq n$, we have $L(\mathfrak{g}', \sigma) = L(\mathfrak{g}'_{ss}, \sigma) \oplus L(R, \sigma)$. Now W_1 is irreducible module for $L(\mathfrak{g}', \sigma)$. As R is solvable ideal, it follows that $\pi(L(R, \sigma))$ lies in the center of $\pi(L)$, which is at most one dimensional, hence $L(R, \sigma)$ acts as a scalar on W. So W_1 will be a irreducible module for $L(\mathfrak{g}'_{ss}, \sigma)$.

Fix a positive integer l. For each i, let $a_i = (a_{i,1}, \ldots, a_{i,l})$ such that $a_{i,j}^{m_i} \neq a_{i,t}^{m_i}$ for $j \neq t$ (*). Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Let $\sigma_1, \ldots \sigma_n$ be finite order automorphisms on g of order $m_1, \ldots m_n$ respectively. Let $L(\mathfrak{g}, \sigma)$ be the corresponding multiloop algebra. Let $I = \{(i_1, i_2, \ldots, i_n) | 1 \leq i_j \leq l\}$. Now for $S = (i_1, i_2, \ldots, i_n) \in I$ and $r = (r_1, r_2, \ldots r_n) \in \mathbb{Z}^n$, $a_S^r = a_{1,i_1}^{r_1} a_{2,i_2}^{r_2} \cdots a_{n,i_n}^{r_n}$. Now consider the evaluation map

$$\phi : \mathfrak{g} \otimes A \to \bigoplus \mathfrak{g} \ (l^n \text{ copies}), \quad \phi(X \otimes t^r) = (a_{I_1}^r X, a_{I_2}^r X, \ldots, a_{I_{l^n}}^r X),$$

where $I_1, I_2, \cdots I_{l^n}$ is some order on I. Now consider the restriction of ϕ to $L(\mathfrak{g}, \sigma)$.

Theorem 6.3. (see [15]) *Let W' be a finite dimensional irreducible representation of $L(\mathfrak{g}, \sigma)$. Then the representation factors through $\bigoplus \mathfrak{g}$ (l^n copies).*

Proposition 6.4. *Let W_1 be an irreducible module for $L(\mathfrak{g}'_{ss}, \sigma)$ as above. Then the representation of $L(\mathfrak{g}'_{ss}, \sigma)$ factors through only one copy of $\bigoplus \mathfrak{g}'_{ss}$. So $\mathfrak{g}_{ss}^1 \cong \mathfrak{g}'_{ss}$.*

Proof. We know by Theorem 6.3, that the representation factors through l^n copies, for some positive integer l. We will prove here that $l = 1$. The proof

is similar to that of [19]. Choose the i-th piece of $\mathfrak{g}'_{ss}$ and choose the projection of the map π, say π_i onto it. Doing the same calculation as in [19], we will get $\pi_i(S'_n(m)) = 0$ and $a^r_{I_i} = 1$ for all $r \in \Gamma$. Now suppose there are at least two pieces, say i-th and j-th piece. Therefore I_i and I_j are two different elements of I with $a^r_{I_i} = 1 = a^r_{I_j}$ for all $r \in \Gamma$. Let $I_i = (i_1, i_2, \ldots, i_n)$ and $I_j = (j_1, j_2, \ldots j_n)$. Therefore there is k with $1 \le k \le n$ such that $i_k \ne j_k$. Now if we take $r = (0, \ldots, m_k, \ldots, 0)$, then $a^r_{I_i} = 1 = a^r_{I_j}$ will give $a^{m_k}_{k,i_k} = a^{m_k}_{k,j_k}$, a contradiction to (*). So there is at most one piece. ∎

Now we know that $\pi_i(S'_n(m)) = 0$, therefore $\mathfrak{g}^2 \subseteq \pi(S'_n(m))$. Now our aim is to understand finite dimensional irreducible modules for $S'_n(m)$. There is a relation between finite dimensional S'_n modules and $S_n \ltimes A$-modules with finite dimensional weight spaces.

Theorem 6.5. (see [3]) *Suppose W is a finite dimensional irreducible sl_n-module (extend W to gl_n by letting I acting trivially). Let $\alpha, \beta \in \mathbb{C}^n$. Take $L(W) = W \otimes A$ and the action $D(u,r)(w \otimes t^k) = (u.k + \beta)w \otimes t^{k+r} + \sum_{i,j} u_i r_j E_{ji} w \otimes t^{k+r}$, for $r \ne 0$ and $D(u,0)(w \otimes t^k) = (u.\alpha + k)w \otimes t^k$ and $t^r(w \otimes t^k) = w \otimes t^{k+r}$. And all irreducible representations of $S_n \ltimes A$ with finite dimensional weight spaces occur in this way.*

Theorem 6.6. *Let W be finite dimensional irreducible module for $\mathfrak{sl}_n$ and extend it trivially as before. Let E_{ij} be generators of $\mathfrak{gl}_n$. Then W can be made into S'_n-module by the action:*

$$(D(u,r) - D(u,0)).w = \sum_{i,j} u_i r_j E_{ji} w + (u|\zeta)w,$$

where $\zeta \in \mathbb{C}^n$. In this case W will be irreducible module for S'_n. Every irreducible finite dimensional S'_n module occur in this way.

Proof. See Theorem 4.5 of [19] and discussion thereafter. ∎

We proved in Proposition 6.2 that $\widetilde{V}$ is completely reducible L module. Therefore assume $\widetilde{V} = \oplus_{i=1}^K \widetilde{V}_i$ for some $K \in \mathbb{N}$. Then by the previous discussion each $\widetilde{V}_i \cong W_1^i \otimes W_2^i$ as $\mathfrak{g}'_{ss} \oplus \mathfrak{sl}_n$-module, where W_1^i, W_2^i are irreducible module for $\mathfrak{g}'_{ss}$ and $\mathfrak{sl}_n$ respectively. Since each component $\widetilde{V}_i$ is isomorphic as $S'_n(m) \ltimes (\mathfrak{h}(\bar{0}) \otimes A(m))$-module, we can take $W_2^i \cong W_2^1 (= W_2$,say) as $\mathfrak{sl}_n$-module for each $i \in \{1, \ldots, K\}$. Now consider $W_1 = \sum_{i=1}^K W_1^i$, which is a $L(\mathfrak{g}'_{ss}, \sigma)$-module, in particular a $\mathfrak{g}'_{ss}$-module. Since each W_1^i is irreducible, without loss of generality we can assume that the above sum is direct. It is easy to see that L is Λ-graded with zero-th component $S'_n(m) \ltimes (\mathfrak{h}(\bar{0}) \otimes A(m))$ and since $\widetilde{V}$ is Λ graded irreducible module (Proposition 6.2), we can take W_1 as Λ-graded irreducible $L(\mathfrak{g}'_{ss}, \sigma)$ module and W_2 will be zero graded as $S'_n(m)$ lies inside the zero-th graded component of L.

We now define a τ^0-module structure on $W_1 \otimes W_2 \otimes A_n$ by

$$X \otimes t^k(w_1 \otimes w_2 \otimes t^l) = Xw_1 \otimes w_2 \otimes t^{k+l}, \quad \text{for} \quad k, l \in \mathbb{Z}^n \quad \text{and} \quad X \in \mathfrak{g}'_{ss}(\bar{k}).$$

$$D(u,r)(w_1 \otimes w_2 \otimes t^k) = (u.k + \beta)w_1 \otimes w_2 \otimes t^{k+r} + w_1 \otimes \left(\left(\sum_{i,j}\right) u_i r_j E_{ji} w_2\right) \otimes t^{k+r}$$

for $r \ne 0$ and $D(u,r) \in S_n(m)$.

$$D(u,0)(w_1 \otimes w_2 \otimes t^k) = (u.k + \alpha)w_1 \otimes w_2 \otimes t^k.$$

Now take any one dimensional representation of $L(R,\sigma)$ say ψ. Then for $y \in R(\bar{k})$ we take $y \otimes t^k(w_1 \otimes w_2 \otimes t^l) = \psi(y)(w_1 \otimes w_2 \otimes t^{k+l})$. Since W_1 is Λ graded which is compatible with Λ-gradation of $\mathfrak{g}'_{ss}$, so the submodule $V' = \bigoplus_{k \in \mathbb{Z}^n} W_{1,k} \otimes W_2 \otimes t^k$ will be irreducible module for τ^0. One can easily check that $L(\widetilde{V})(\bar{0}) \cong V'$ as τ^0-module.

Now as in [17], consider the triangular decomposition of LT given by

$$LT = \tau^- \oplus L(\mathfrak{g}'_{ss}, \sigma) \oplus L(R, \sigma) \oplus \tau^+.$$

Let V_1 be the unique Λ-graded irreducible quotient of the induced module for W_1 and let (V_1^i, ρ_i) denote the irreducible quotient of the induced representation of W_1^i, for each $1 \le i \le K$. Now by [17], we can say that $V_1 = \oplus_{i=1}^K V_1^i$ is Λ-graded irreducible module for LT.

Define a τ-module structure on $V_1 \otimes W_2 \otimes A$ by

$$(x \otimes t^l).\Big(\sum_{i=1}^K v_1^i \otimes v_2 \otimes t^k \Big) = \sum_{i=1}^K (\rho_i(x \otimes t^l)v_1^i) \otimes v_2 \otimes t^{k+l},$$

$$D(u,r).\Big(\sum_{i=1}^K v_1^i \otimes v_2 \otimes t^k \Big) =$$

$$= (u, k+\beta)\Big(\sum_{i=1}^K v_1^i \otimes v_2 \otimes t^{k+r} \Big) + \sum_{i=1}^K v_1^i \otimes \Big(\sum_{l,j}(u_l r_j E_{jl}).v_2 \Big) \otimes t^{k+r}.$$

Let $V_1 = \oplus_{\bar{k} \in \Lambda} V_{1,\bar{k}}$. Take $\bigoplus_{k \in \mathbb{Z}^n} V_{1,\bar{k}} \otimes W_2 \otimes t^k$, which is a τ-submodule of $V_1 \otimes W_2 \otimes A$. One can prove that $\bigoplus_{k \in \mathbb{Z}^n} V_{1,\bar{k}} \otimes W_2 \otimes t^k$ is irreducible τ module.

Theorem 6.7. *Let V be an irreducible integrable τ module with finite dimensional weight spaces, with all K_i acting trivially for $1 \le i \le n$. Then we have $V \cong \bigoplus_{k \in \mathbb{Z}^n} V_{1,\bar{k}} \otimes W_2 \otimes t^k$.*

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Santanu Tantubay, Punita Batra, Harish-Chandra Research Institute, A CI of Homi Bhabha National Institute, Chhatnag Road, Jhunsi, Prayagraj 211 019, India; santanutantubay@hri.res.in, batra@hri.res.in.