On Homogeneous Hermite-Lorentz Spaces
of Low Dimension

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Abstract. We classify irreducible homogeneous almost Hermite-Lorentz spaces of complex dimension 3, and prove in particular they are geodesically complete.

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1. Introduction

An almost Hermite-Lorentz space (M, J, g) is an almost pseudo-Hermitian space of signature $-+\ldots+$. So, J is an almost complex structure, and g is a pseudo-Riemannian metric compatible with J having a signature $--+\ldots+$. Equivalently, each tangent space (T_xM, J_x, g_x) is isomorphic to $(\mathbb{C}^n, -|z_1|^2 + |z_2|^2 + \ldots + |z_n|^2)$, where $n = \dim_{\mathbb{C}} M$. We will use Hermite-Lorentz to mean that J is integrable.

One can think that the complexity of a pseudo-Riemannian metric grows with signature, and accordingly, its tractability from the algebraic, geometric, and dynamical point of views decreases with signature. This explains the special interest of Riemannian and Lorentzian structures whether in mathematics or physics. On the other hand, from a complex point of view, Hermite-Lorentz framework looks like a Lorentzian one in the real case. This motivates the study of Hermite-Lorentz spaces and leads to expect some rigidity of them because of the moderate signature and the cohabitation of the metric with the almost complex structure.

We are interested here in homogeneous almost Hermite-Lorentz spaces. Any Lie group L of even dimension $2n$ can be endowed with such a structure. Indeed, identify arbitrarily its Lie algebra $\mathfrak{l}$ to $(\mathbb{C}^n, -|z_1|^2 + |z_2|^2 + \ldots + |z_n|^2)$, and left translate this linear Hermite-Lorentz structure to all tangent spaces of L. The L-left action on itself preserves this structure, and therefore L becomes a homogeneous almost Hermite-Lorentz space. This convinces us that the important homogeneous spaces are those having a big isotropy group. (Note that this is rarely the case for left invariant structures on Lie groups). The symmetric spaces are of this kind, and they are the most homogeneous spaces after the constant curvature spaces. In [4], the authors prove that an almost Hermite-Lorentz space with big isotropy is necessarily symmetric (meaning in particular that the almost complex structure is

integrable and the Hermitian metric is Kählerian). Here, the word "big" should be understood in the sense that the isotropy is $\mathbb{C}$-irreducible.

The fact that big isotropy implies symmetry is also true in the (real) Lorentz case, as well as in the quaternionic case (cf. [8, 9]). However, this is not true in the Riemannian case and of course in the general pseudo-Riemannian case.

There is also a restriction on the dimension. More precisely, the (complex) dimension of an almost Hermite-Lorentz space should be ≥ 4 (cf. [4]). A similar restriction holds in the quaternionic case (cf. [8, 9]). It seems to be natural and worthwhile to understand what happens for the special dimension 3.

Let us first observe that the results proved in [4] extend well to the 2-dimensional case. Our purpose here is to investigate the remaining 3-dimensional case. Roughly speaking, the technical difficulty leading to the restriction to dim $\neq 3$ is that, in the real dimension 3, there exist equivariant cross products $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. Our results here show that, in fact, there are many more homogeneous almost Hermite-Lorentz spaces with $\mathbb{C}$ irreducible isotropy than the symmetric ones. Fortunately, we were able to classify them all.

1.1. Main results

The following two theorems give a complete description of all 3-dimensional complex homogeneous almost Hermite-Lorentz spaces with $\mathbb{C}$-irreducible isotropy.

Theorem 1.1. *Let $M = G/H$ be a homogeneous almost complex Hermite-Lorentz space of complex dimension 3 with an isotropy acting $\mathbb{C}$-irreducibly. Then, up to a central cover (i.e. quotienting by a discrete central subgroup of the universal covering $\tilde{G}$ of G), either M is symmetric or M is a 6-dimensional real Lie group L endowed with a left invariant almost complex Hermite-Lorentz structure and which corresponds to one of the following four cases:*

1. *$L = \mathsf{SL}(2, \mathbb{C})$.*

2. *$L = \mathsf{SL}(2, \mathbb{R}) \times \mathsf{SL}(2, \mathbb{R})$.*

3. *$L = \mathsf{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^3$, where $\mathsf{SL}(2, \mathbb{R})$ acts on $\mathbb{R}^3$ via its 3-dimensional irreducible representation (with image $\mathsf{SO}^0(1, 2)$).*

4. *$L = N_{\mathfrak{sl}_2}$, whose Lie algebra $\mathfrak{n}_{\mathfrak{sl}_2}$ is the 2-step 6-dimensional nilpotent algebra with underlying space $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ and bracket $[(u, u'), (v, v')] = (0, b(u, v))$, where b is the bracket of the Lie algebra*

$$\mathfrak{sl}(2, \mathbb{R}), \ b : \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{R}).$$

In each of these cases, and up to automorphism, there is a unique left invariant almost Hermite-Lorentz structure on L with $\mathbb{C}$-irreducible isotropy.

Moreover, the identity component of the isotropy group (of $1 \in L$) in the automorphism group of the almost Hermite-Lorentz structure is $\mathsf{SO}^0(1, 2)$, acting by automorphisms of L.

The almost complex structure is integrable only in the symmetric and the $\mathsf{SL}(2, \mathbb{C})$ cases.

Theorem 1.2. *Alternatively, we have the following description of the left invariant almost Hermite-Lorentz metrics with $\mathbb{C}$-irreducible isotropy on 6-dimensional real Lie groups L, seen as homogeneous spaces of the form G/H:*

1. *For $L = \mathsf{SL}(2,\mathbb{C})$, $G = \mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{C})$, and H is diagonal in the product $\mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R}) \subset \mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{C})$.*

2. *For $L = \mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$, $G = \mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$ and H is a diagonal embedding of $\mathsf{SL}(2,\mathbb{R})$.*

3. *For $L = \mathsf{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^3$, $G = \mathsf{SL}(2,\mathbb{R}) \times (\mathsf{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^3)$ and H is a diagonal embedding of $\mathsf{SL}(2,\mathbb{R})$ in $\mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$.*

4. *For $L = N_{\mathfrak{sl}_2}$, $G = \mathsf{SL}(2,\mathbb{R}) \ltimes N_{\mathfrak{sl}_2}$, $H = \mathsf{SL}(2,\mathbb{R})$.*

Let us notice here that instead of the semi-direct product $\mathsf{SL}(2,\mathbb{R}) \ltimes N_{\mathfrak{sl}_2}$ one could consider $\mathsf{SL}(2,\mathbb{R}) \ltimes \mathbb{C}^3$, which turns however to be flat.

1.2. Results in the integrable case

With Theorem 1.1 in hand and the classification of symmetric Hermite-Lorentz spaces with $\mathbb{C}$-irreducible isotropy given in [4], we obtain the following corollaries.

Corollary 1.3. *Up to a scaling and covering, there are exactly six 3-dimensional complex homogeneous Hermite-Lorentz spaces with $\mathbb{C}$-irreducible isotropy:*

1. $\mathsf{SL}(2,\mathbb{C})$ *endowed with a left invariant Hermite-Lorentz metric that is also invariant under the $Ad_{\mathsf{SL}(2,\mathbb{R})}$-action.*

2. *The universal (flat Hermite-Lorentz) complex Minkowski space $\mathsf{Mink}_3(\mathbb{C})$ (usually denoted by $\mathbb{C}^{1,2}$).*

3. *The complex de Sitter space $\mathsf{dS}_3(\mathbb{C}) = \mathsf{SU}(1,3)/\mathsf{U}(1,2)$.*

4. *The complex anti de Sitter space $\mathsf{AdS}_3(\mathbb{C}) = \mathsf{SU}(2,2)/\mathsf{U}(1,2)$.*

5. $\mathbb{C}\mathsf{dS}_3 = \mathsf{SO}^0(1,4)/\mathsf{SO}^0(1,2) \times \mathsf{SO}(2)$.

6. $\mathbb{C}\mathsf{AdS}_3 = \mathsf{SO}(3,2)/\mathsf{SO}(2) \times \mathsf{SO}^0(1,2)$.

All of these space are symmetric except for the first one.

Corollary 1.4. *Let M be an n-dimensional complex homogeneous and non-Kählerian Hermite-Lorentz space with $\mathbb{C}$-irreducible isotropy. Then, $n = 3$ and M is necessarily $\mathsf{SL}(2,\mathbb{C})$ endowed with a left invariant Hermite-Lorentz metric which is invariant under the action of $Ad_{\mathsf{SL}(2,\mathbb{R})}$.*

1.3. Completeness

Homogeneous pseudo-Riemannian spaces are not automatically complete! For instance, left invariant pseudo-Riemannian metrics on Lie groups are generically thought to be incomplete. In [12], it has been shown that incomplete left-invariant Lorentzian metrics on $\mathsf{SL}(2,\mathbb{R})$ exist, and by taking co-compact quotients we obtain locally homogeneous compact Lorentzian manifolds which are incomplete. However, homogeneous compact pseudo-Riemannian manifolds turn out to be complete [18]. We will prove here the following result.

Theorem 1.5. *Any homogeneous almost complex Hermite-Lorentz space (of any dimension) with $\mathbb{C}$-irreducible isotropy is geodesically complete.*

This remarkable fact of course let us dare ask the following question.

Question 1.6.   *Is in general any arbitrary homogeneous pseudo-Riemannian space with irreducible isotropy complete? Perhaps, a hypothesis of weak irreducibility suffices, such as the $\mathbb{C}$-irreducibility here.*

1.4. Overview and steps of the proof

Let $M = G/H$ be a 3-dimensional complex homogeneous almost Hermite-Lorentz space with $\mathbb{C}$-irreducible isotropy. The first step is to identify H and its infinitesimal representation in $\mathbb{C}^3$, identified to the tangent space of G/H at a base point.

1.4.1. The isotropy, the representation $\mathfrak{hl}$.   Consider the irreducible 3-dimensional real representation of $\mathsf{SL}(2,\mathbb{R})$, which has $\mathsf{SO}^0(1,2)$ as image. It is more convenient to work at the Lie algebra level, and define W_3 to be this infinitesimal representation of $\mathfrak{sl}_2(\mathbb{R})$. We note its complexification acting on $\mathbb{C}^3 = \mathbb{R}^3 + \sqrt{-1}\mathbb{R}^3$ by $\mathfrak{hl}$. This will denote the representation $\mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(3,\mathbb{C})$ as well as its image, which is nothing but $W_3 \oplus W_3$. It is in fact the unique $\mathbb{C}$-irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$ preserving a Hermite-Lorentz product.

Fact 1.7. [4, 22] *Any sub-algebra $\mathfrak{h} \subset \mathfrak{u}(1,2)$ which is $\mathbb{C}$-irreducible must contain a conjugate of $\mathfrak{hl}$. More precisely, $\mathfrak{h}$ equals $\mathfrak{hl}$, $\mathfrak{hl} \oplus \mathfrak{so}(2)$, $\mathfrak{su}(1,2)$, or $\mathfrak{u}(1,2)$.*

1.4.2. The symmetric case.   Observe that for both $\mathfrak{hl} \oplus \mathfrak{so}(2)$ and $\mathfrak{u}(1,2)$, the associated group contains the symmetry $u \to -u$ (of $\mathbb{C}^3$). The space G/H is thus symmetric in this case.

Remark 1.8.   Pseudo-Riemannian symmetric spaces have been extensively studied and partially classified. We refer to the following references [6], [13], [21], [25], [26] for generalities on symmetric spaces, and to [5] for their classification in the reductive case.

1.4.3. Case $\mathfrak{h} = \mathfrak{su}(1,2)$.   We will prove in Section 3 that $\mathfrak{h} = \mathfrak{su}(1,2)$ cannot occur. That is, if the isotropy contains $\mathfrak{su}(1,2)$, then it is in fact equal to $\mathfrak{u}(1,2)$.

1.4.4. Case $\mathfrak{h} = \mathfrak{hl}$.   This is the case that will occupy all the remaining sections. Observe here that $\dim G = \dim_{\mathbb{R}} M + \dim \mathfrak{h} = 6 + 3 = 9$.

– In Section 4, we will identify the semi-simple part of G.

– In Section 5, we will explicit the isometry group G.

– In Section 6, we will find the normal Lie subgroup L identified with G/H.

– In Section 7, we will make the necessary computations to decide in which case the almost complex structure is integrable and when the Hermite-Lorentz metric is Kählerian, as well as other curvature descriptions. This will, in particular, allow us to prove Fact 7.5 on the full isometry group.

– Finally, Section 8 will be devoted to the proof of the geodesic completeness of a homogeneous almost complex Hermite-Lorentz space with $\mathbb{C}$-irreducible isotropy (Theorem 1.5).

2.   Further remarks and constructions

In the present section we will see that most of the spaces that we found in our classification has become part of a more general setting. This gives them more

importance, and also raises more natural questions! We hope returning elsewhere to partially answer some of these questions.

2.1. Product metric for $L = \mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$

In the case $\mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$ the metric is just the product one where each factor is equipped with the Anti de Sitter structure given by the Killing form. So, as a pseudo-Riemannian space $\mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$ is symmetric. It is however not symmetric as an almost Hermite-Lorentz space. The automorphism group of the last structure is essentially $\mathsf{SL}(2,\mathbb{R})^3$, whereas the isometry group of the pseudo-Riemannian structure is essentially $\mathsf{SL}(2,\mathbb{R})^4$.

2.2. A hidden direct product in the case $L = \mathsf{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^3$

The construction of the left invariant metric on $\mathsf{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^3$ can be generalized as follows. Let A be any Lie group with a Lie algebra $\mathfrak{a}$, and consider the semi-direct product $L = A \ltimes \mathfrak{a}$. Let κ be the Killing form of $\mathfrak{a}$, and endow $\mathfrak{l} = \mathfrak{a} \oplus \mathfrak{a}$ with $\kappa \oplus \kappa$. This is always Ad_A-invariant, and non-degenerate exactly when A is semi-simple. It is in fact pseudo-Hermitian in the general semi-simple case, and of Hermite-Lorentz type exactly for $\mathfrak{a} = \mathfrak{sl}(2,\mathbb{R})$.

Endowed with this left invariant metric, $A \ltimes \mathfrak{a}$ can be seen as a homogeneous space $M = A \times (A \ltimes \mathfrak{a})/\Delta$ where Δ is the diagonal in $A \times A$. Observe that the direct product $L' = A \times \mathfrak{a}$ acts freely transitively on M. So, our left invariant metric on $A \ltimes \mathfrak{a}$ is isometric to a left invariant metric on $A \times \mathfrak{a}$. This is in fact nothing but the direct metric product $(A, \kappa) \times (\mathfrak{a}, \kappa)$. Observe that, in this case, the isotropy does not act by automorphisms of $A \times \mathfrak{a}$.

2.3. 2-step nilpotent construction

The definition of the 2-step nilpotent Lie algebra $\mathfrak{n}_{\mathfrak{sl}_2}$ is a particular case of the following construction. Let $\mathfrak{a}$ and $\mathfrak{b}$ two linear spaces and $c : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{b}$ an anti-symmetric bilinear map. On $\mathfrak{n} = \mathfrak{a} \times \mathfrak{b}$, define a bracket by $[u,v] = c(u,v)$, for $u,v \in \mathfrak{a}$, and 0 for all other possibilities.

2.4. Complexification of semi-simple groups

The construction of the Hermite-Lorentz metric on $\mathsf{SL}(2,\mathbb{C})$ can be generalized as follows. Let P be a semi-simple real Lie group with Lie algebra $\mathfrak{p}$. Its complexification $P^{\mathbb{C}}$ has $\mathfrak{p}^{\mathbb{C}} = \mathfrak{p} + \sqrt{-1}\mathfrak{p} \sim \mathfrak{p} \oplus \mathfrak{p}$ as a Lie algebra. Endow it with the scalar product $\kappa \oplus \kappa$, where κ is the Killing form of $\mathfrak{p}$. This gives rise to a left invariant pseudo-Hermitian metric on $P^{\mathbb{C}}$ that is invariant under Ad_P.

In our case, that is the case $\mathfrak{p}^{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$, we know that $\mathfrak{sl}(2,\mathbb{C})$ is the complexification of each of the two non isomorphic real Lie algebras $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$. It follows that we have two different constructions. The first one corresponds to the case $\mathfrak{p} = \mathfrak{sl}(2,\mathbb{R})$, where we obtain a left invariant pseudo-Hermitian metric on $\mathsf{SL}(2,\mathbb{C})$ that is invariant under the adjoint action $Ad_{\mathsf{SL}(2,\mathbb{R})}$. This is in fact a Hermite-Lorentz metric.

Whereas, in the case $\mathfrak{p} = \mathfrak{su}(2)$, we obtain a left invariant pseudo-Hermitian metric on $\mathsf{SL}(2,\mathbb{C})$ that is now invariant under the adjoint action $Ad_{\mathsf{SU}(2)}$, and which turns out to be a Riemannian metric (i.e. positive definite).

2.5. Almost Hermite-Lorentz structures on frame bundles

Let N be a 3-Lorentz manifold. The space of its (positively oriented) orthonormal frames is a principal $\mathsf{SO}^0(1,2)$-bundle, $P_N \to N$. It turns out that P_N has a natural almost Hermite-Lorentz structure. Indeed, TP_N splits into a horizontal and a vertical bundles, $TP_N = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{H}$ is given by the connection. Both $\mathcal{H}$ and $\mathcal{V}$ are identified to TM. This allows us to define the almost complex structure J sending $\mathcal{H}$ to $\mathcal{V}$. This also allows us to define Lorentz metrics on $\mathcal{H}$ and $\mathcal{V}$ which makes each isometric to TM. We also assume that $\mathcal{H}$ is orthogonal $\mathcal{V}$ in order to obtain an almost Hermite-Lorentz structure on P_N.

The isometry group of N acts freely on P_N, and acts transitively exactly when N is, up to scaling, one of the universal Lorentz spaces of constant sectional curvature: the Minkowski space Mink_3, the de Sitter space $\mathsf{dS}_3 = \mathsf{SO}^0(1,3)/\mathsf{SO}^0(1,2)$, or a quotient by a central cyclic group of the anti de Sitter space $\widetilde{\mathsf{SL}(2,\mathbb{R})}$ So, we have the following identifications:

- $P_{\mathsf{Mink}_3} = \mathsf{Iso}^0(\mathsf{Mink}_3) = \mathsf{SO}^0(1,2) \ltimes \mathbb{R}^3$.

- $P_{\mathsf{dS}_3} = \mathsf{Iso}^0(\mathsf{dS}_3) = \mathsf{PSL}(2,\mathbb{C})$.

- $P_{\mathsf{PSL}(2,\mathbb{R})} = \mathsf{Iso}^0(\mathsf{PSL}(2,\mathbb{R})) = \mathsf{PSL}(2,\mathbb{R}) \times \mathsf{PSL}(2,\mathbb{R})$.

2.5.1. Integrability. A similar construction in the Riemannian case is discussed in [10] (Section 6) where it is proved that the almost complex structure is integrable exactly if the Riemannian 3-manifold is hyperbolic, i.e. it has constant negative curvature, say -1. In this case, the frame bundle of the universal cover is identified to $\mathsf{SO}^0(1,3)$, which is isomorphic to $\mathsf{PSL}(2,\mathbb{C})$. We think that a similar characterization is also true in our (Lorentz) case: the almost complex structure is integrable exactly when N has, up to scaling, a constant curvature equal to -1, and so it is locally isometric to dS_3, in which case the frame bundle has the complex structure of $\mathsf{PSL}(2,\mathbb{C})$.

2.5.2. On the tangent bundle. The tangent bundle TN, too, has a natural almost Hermite-Lorentz structure. The isometry group $\mathsf{Iso}(N)$ does not act transitively in this case, since the subsets $T^r N$ of vectors of a given square length r are preserved. In the constant curvature case, the isometry group acts transitively on these subsets. But TN may have extra-isometries which do not come from $\mathsf{Iso}(N)$. For example, $T\mathsf{Mink}_3$ is $\mathbb{C}^3$ with its usual Hermite-Lorentz structure (and hence homogeneous).

It turns out that the Kähler form of TN is nothing but the push-forward by the dual isomorphism $TN \to T^*N$ of the canonical symplectic form of T^*N. Therefore, TN is always almost Kählerian [24].

2.6. Compact models

When we have a geometry (in Klein sense) $(G, G/H)$, we are interested to see if it admits a compact model. Say, in our four cases where G/H can be identified to a Lie group L one can ask if L has a compact quotient that locally has the same almost Hermite-Lorentz structure? Formally, one can ask if L has a co-compact lattice Γ?

It turns out that this is the case for $\mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$ and $\mathsf{SL}(2,\mathbb{C})$, because of classical constructions, or thanks to Borel's Theorem saying that semi-simple Lie groups admit co-compact lattices (see for instance [27]). It is also the case for $N_{\mathfrak{sl}_2}$, because it is a rational nilpotent Lie group (see for instance [23], Theorem 2.12). As for $\mathsf{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^3$, it turns out that one can classify its lattices. They are, up to finite index, isomorphic to $\mathsf{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^3$, and so they are not co-compact. Fortunately, as we have observed above, this space has also the product presentation $\mathsf{SL}(2,\mathbb{R}) \times \mathbb{R}^3$ and so admits co-compact lattices.

The left isometric action of $\mathsf{SL}(2,\mathbb{R})$ on L passes to the quotients L/Γ. This is an interesting isometric action with strong dynamics. For instance, it is ergodic by Moore's Theorem (see for instance [27]).

Observe that these Γ's can be deformed in the full isometry group $L \times \mathsf{SL}(2,\mathbb{R})$ and one then obtains "non-standard" examples (compare for instance with [10]).

2.6.1. The symmetric case. The symmetric cases listed in Corollary 1.3 do not admit a L-representation", i.e. they are not identified to left invariant almost Hermite-Lorentz structures on Lie groups. It is worthwhile to investigate the existence (and probably the non-existence) of compact models in each of these cases.

3.   Case of isotropy $\mathfrak{h} = \mathfrak{su}(1,2)$

Here, we want to prove that if the isotropy algebra contains $\mathfrak{su}(1,2)$, then the space is Kählerian with a constant holomorphic sectional curvature, and in fact its isotropy algebra is rather $\mathfrak{u}(1,2)$.

A Lie-theoretic proof seems to be difficult, since $\dim G$ would be equal to 14 and there will be many cases to consider. Our proof here is rather geometrical. So, let $(M = G/H, J, g)$ be an almost Hermite-Lorentz G-homogeneous space for which the isotropy algebra $\mathfrak{h}$ is isomorphic to $\mathfrak{su}(1,2)$ that is acting by its standard representation on $\mathfrak{g}/\mathfrak{h} \equiv \mathbb{C}^3$. Let $\omega(.,.) = g(., J.)$ be the Kähler form of (M, J, g).

We will prove that ω is closed, and then J is integrable. The idea is to observe that the H-orbits of M-points are "isomorphic" to the H-orbits in $\mathbb{C}^3$. This allows one to understand the restriction of ω and J to these H-orbits in M.

3.1. The $\mathfrak{su}(1,2)$-action on $\mathbb{C}^3$

Endow $\mathbb{C}^3$ with the usual Hermite-Lorentz form $q = -|z_1|^2 + |z_2|^2 + |z_3|^2$. So, $\mathsf{U}(1,2)$ and $\mathsf{SU}(1,2)$ both act transitively on each level of q in $\mathbb{C}^3 - \{0\}$. A negative level $\{q = -\epsilon^2\}$ corresponds to the $\mathsf{SU}(1,2)$-orbit of some point $(\epsilon, 0, 0) \in \mathbb{C}^3$. All of these ponits have the same compact isotropy $\mathsf{SU}(2)$). So, as homogeneous spaces, all these orbits are isomorphic to $\mathsf{SU}(1,2)/\mathsf{SU}(2)$. Therefore, we get a Hopf fibration $\mathsf{SU}(1,2)/\mathsf{SU}(2) \to \mathsf{SU}(1,2)/\mathsf{S}(\mathsf{U}(1) \times \mathsf{U}(2))$. The latter space (also denoted $\mathsf{SU}(1,2)/\mathsf{U}(2)$) is nothing but the complex hyperbolic plane $\mathbb{CH}^2$.

Fact 3.1.   *Up to scaling, there exists exactly one $\mathsf{SU}(1,2)$-invariant differential 2-form ω^0 on $\mathsf{SU}(1,2)/\mathsf{SU}(2)$. This is the restriction of the Kähler form of $\mathbb{C}^3$, or equivalently, the pull back of the Kähler form on $\mathbb{CH}^2$. In particular, such a form is closed.*

Proof.   The tangent space at $(\epsilon, 0, 0)$ of the level $\{q = -\epsilon^2\}$ is $\{0\} \times \mathbb{R} \times \mathbb{C}^2$. The isotropy $\mathsf{SU}(2)$-action is the product of the trivial one on $\mathbb{R}$ by the usual one

on $\mathbb{C}^2$. Let α be an anti-symmetric 2-form on $\mathbb{R} \times \mathbb{C}^2$ that is invariant under the $\mathsf{SU}(2)$-action. Its kernel is $\mathsf{SU}(2)$-invariant and consequently it equals $\mathbb{R}$ or $\mathbb{C}^2$. It cannot be $\mathbb{C}^2$ unless $\alpha = 0$. Hence, it is $\mathbb{R}$. It follows that α projects to a $\mathsf{SU}(2)$-invariant form on $\mathbb{C}^2$, which is known to be unique (up to scaling). ∎

3.2. The isotropy-action on G/H via the exponential map

Let p be any point of M, say p is the base point $1H$, and consider the exponential map $\exp_p$ defined on an open subset of T_pM to M. Then, $\exp_p$ is equivariant with respect to the H-derivative action on T_pM and the isometric H-action on M.

Fact 3.2. *Let ω be the Kähler form of $M = G/H$. Let $u \in T_pM$ be a timelike vector (small enough, say, lies in a small neighbourhood of $0 \in T_pM$ where $\exp_p$ is a diffeomorphism onto its image), and let $O_u = H.\exp_p u$ be the orbit of $\exp_p u$ under the isotropy H. Then ω is closed on O_u.*

Proof. From the equivariance of $\exp_p$, it follows in particular that u and $\exp_p u$ have the same isotropy in $H = \mathsf{SU}(1,2)$. Therefore, as a $\mathsf{SU}(1,2)$-homogeneous space, the orbit O_u is identified to $\mathsf{SU}(1,2)/\mathsf{SU}(2)$. We conclude from Fact 3.1 that the restriction of ω to O_u is closed. ∎

3.3. Proof that M is almost (pseudo)-Kählerian

Let u be, as above, a small timelike vector in T_pM, and $\gamma_u(t)$ the geodesic with $\dot{\gamma}_u(0) = u$. By Gauss lemma, the orthogonal $\gamma_u(t)^\perp$ coincides with the tangent space at $\gamma_u(t)$ of the image by $\exp_p$ of the level $\{q = t^2 q(u)\}$ in T_pM (q is the Hermite-Lorentz form on T_pM). Because these levels in T_pM correspond to H-orbits in T_pM, their images by $\exp_p$ correspond to the H-orbits in M. Therefore, the tangent space of the orbit $H.\gamma_u(t)$ at $\gamma_u(t)$ is $\dot{\gamma}_u^\perp(t)$. It follows from the fact above that $d\omega$ vanishes on $\dot{\gamma}_u^\perp(t)$. When $t \to 0$, we get that $d\omega$ vanishes on the real hyperplane $u^\perp \subset T_pM$.

This applies for any timelike vector $u \in T_pM$. It follows that $d\omega(u_1, u_2, u_3) = 0$ for any $(u_1, u_2, u_3) \in T_pM$ such that $(\mathbb{R}u_1 \oplus \mathbb{R}u_2 \oplus \mathbb{R}u_3)^\perp$ contains a timelike vector. But this subset of $T_pM \times T_pM \times T_pM$ is open and non-empty. We conclude that $d\omega$ vanishes on T_pM, and that ω is closed on M by homogeneity.

3.4. Integrability of the almost-complex structure

On the homogeneous space $\mathsf{SU}(1,2)/\mathsf{SU}(2)$, there is also an invariant CR-structure inherited from its realization as a real hypersurface in $\mathbb{C}^3$. This is a $\mathsf{SU}(1,2)$-invariant codimension one distribution $\mathcal{C}$ that is endowed with an almost complex structure J_0. As above, the distribution $\mathcal{C}$ is given at a point $(\epsilon, 0, 0)$ by the factor $\mathbb{C}^2$ in the tangent space identified to $\mathbb{R} \times \mathbb{C}^2$.

Just as for the Kähler form, this invariant CR-structure on $\mathsf{SU}(1,2)/\mathsf{SU}(2)$ is unique. In fact, $\mathcal{C}$ is the unique $\mathsf{SU}(1,2)$- invariant codimension one distribution, and the almost complex structure J_0 on it is unique.

Recall that, by definition of a CR structure (see for instance [15], Chapter 1), the almost complex structure J_0 on $\mathcal{C}$ is integrable in the sense that, for any vector fields X and Y tangent to $\mathcal{C}$, $A = [J_0X, Y] + [X, J_0Y]$ and $B = [J_0X, J_0Y] - [X, Y]$ are tangent to $\mathcal{C}$ and $J_0A = B$.

As above, consider an orbit $O_u = H.\exp_p u$, and remember that J denotes the almost complex structure of M. The distribution $\mathcal{D} = TO \cap J(TO)$ is $\mathsf{SU}(1,2)$-invariant and hence, by uniqueness, it coincides with the CR structure $\mathcal{C}$. In particular it satisfies the integrability condition $0 = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$, where X and Y are vector fields on O tangent to $\mathcal{D}$. By definition of the Nijenhuis tensor N_J of M (see Section 7), this exactly means that $N_J(X, Y) = 0$ for X and Y tangent to $\mathcal{D}$.

By considering a limit process as previously, we get that N_J vanishes on $(\mathbb{C}u)^\perp$ for any $u \in T_p M$ timelike. Again, as above, the non-empty space of elements $(v, w) \in (T_p M)^2$ such that $(\mathbb{C}v \oplus \mathbb{C}w)^\perp$ contains a timelike vector is open in $(T_p M)^2$. It follows that N_J vanishes at p (and hence vanishes everywhere), that is J is integrable.

3.5. End of the proof

In conclusion, (M, J, g) is pseudo-Kählerian, that is J is integrable and the Kähler form of g is closed. Observe now that M has a constant holomorphic sectional curvature, since $\mathsf{SU}(1,2)$ acts transitively on vectors of the same causal character in $\mathbb{C}^3$. Therefore, up to scaling, M is locally isomorphic to one of the universal Kähler-Lorentz spaces of constant holomorphic sectional curvature: $\mathsf{Mink}_3(\mathbb{C})$ (i.e. $\mathbb{C}^3$ endowed with the form q), $\mathsf{dS}_3(\mathbb{C}) = \mathsf{SU}(1,3)/\mathsf{U}(1,2)$, or $\mathsf{AdS}_3(\mathbb{C}) = \mathsf{SU}(1,3)/\mathsf{U}(1,2)$.

Exactly, as in [4], thanks to the high homogeneity of M (especially, its irreducible isotropy), we see that, up to a central cover, M equals one of these universal spaces. In particular, the infinitesimal isotropy is $\mathfrak{u}(1,2)$. Therefore, as we noticed at the beginning of the current section, the isotropy algebra cannot be equal to $\mathfrak{su}(1,2)$. ∎

4. Structure of the semi-simple part of $\mathfrak{g}$

Let G/H be an almost Hermite-Lorentz homogeneous space of complex dimension 3, with a $\mathbb{C}$-irreducible isotropy.

From paragraphs 1.4 and 3, we just need to consider the case $\mathfrak{h} = \mathfrak{so}(1,2)$. We have $\dim_\mathbb{R} G = 3 \times 2 + \dim_\mathbb{R} \mathfrak{so}(1,2) = 9$. So our problem reduces to finding a real Lie group G having dimension 9 and containing a subgroup H which is locally isomorphic to $\mathsf{SL}(2,\mathbb{R})$ and such that the H-action of $\mathfrak{g}/\mathfrak{h}$ is isomorphic to $\mathfrak{hl}$.

In what follows, for convenience, we will use the symbol "$\oplus_S$" to denote a semi direct sum of Lie algebras.

Consider a Levi decomposition $\mathfrak{g} = \mathfrak{s} \oplus_S \mathfrak{r}$, where $\mathfrak{s}$ is semi-simple and $\mathfrak{r}$ is the radical of $\mathfrak{g}$.

Lemma 4.1. $\mathfrak{s}$ *is isomorphic to either* $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{C})$, *or* $\oplus_{i=1}^k \mathfrak{sl}(2,\mathbb{R})$ *with* $k = 1, 2$, *or* 3.

Proof. $\mathfrak{s}$ is a real semi-simple Lie algebra of $\dim \leq 9$. Since it contains $\mathfrak{h}$, $\mathfrak{s}$ contains a nonzero simple factor of noncompact type, say $\mathfrak{s}_1$.

We will show that $\mathfrak{s}_1$ is $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{su}(1,2)$, or $\mathfrak{sl}(3,\mathbb{R})$. Indeed, assume that $\mathfrak{s}_1$ has real rank 1. Then, it is the Killing algebra of a real, complex, or quaternionic hyperbolic space, or a Cayley hyperbolic plane. But only $\mathbb{R}H^2$, $\mathbb{R}H^3$ and $\mathbb{C}H^3$ have Killing algebra of dimension ≤ 9. They are $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$, and $\mathfrak{su}(1,2)$, having dimensions 3, 6, and 8, respectively.

If $\mathfrak{s}_1$ has rank ≥ 2, then it contains $\mathfrak{sl}(3,\mathbb{R})$, or $\mathfrak{sp}(4,\mathbb{R})$ (see [17], Proposition I.1.6.2). They have dimensions 8 and 10, respectively.

Assume that $\mathfrak{g}$ contains $\mathfrak{sl}(3,\mathbb{R})$. Then $\mathfrak{sl}(3,\mathbb{R})$ has codimension 1 in $\mathfrak{g}$. It follows that $\mathfrak{g}$ is not semi-simple, since codimension one subalgebras of semisimple Lie algebras come only from $\mathfrak{sl}(2,\mathbb{R})$-factors. Therefore $\mathfrak{g}$ has a non-trivial radical which is isomorphic to $\mathbb{R}$ on which $\mathfrak{sl}(3,\mathbb{R})$ acts trivially, that is $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{R}$. The isotropy $\mathfrak{h}$ being simple is thus contained in $\mathfrak{sl}(3,\mathbb{R})$. This contradicts the fact that $\mathfrak{h}$ acts $\mathbb{C}$-irreducibly on $\mathfrak{g}/\mathfrak{h}$.

The same argument applies equally to $\mathfrak{s}_1 = \mathfrak{su}(1,2)$. Thus, we have proved that any factor of $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{sl}(2,\mathbb{C})$.

If $\mathfrak{s}$ contains $\mathfrak{sl}(2,\mathbb{C})$, then it equals either $\mathfrak{sl}(2,\mathbb{C})$ or $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{R})$. In the first case, $\mathfrak{g}$ is a semi-direct sum $\mathfrak{sl}(2,\mathbb{C}) \oplus_S \mathfrak{r}$. But $\mathfrak{sl}(2,\mathbb{C})$ cannot act non-trivially on a Lie algebra of real dimension 3. So $\mathfrak{g}$ is a direct sum. But, this contradicts the fact that the isotropy is $\mathbb{C}$-irreducible. ∎

5. Identification of $\mathfrak{g}$

By Lemma 4.1, $\mathfrak{g}$ is either semi-simple and isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{C})$, or $\mathfrak{g}$ is a semi-direct sum $(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})) \oplus_S \mathfrak{r}$ with $\dim \mathfrak{r} = 3$, or $\mathfrak{sl}(2,\mathbb{R}) \oplus_S \mathfrak{r}$ with $\dim \mathfrak{r} = 6$.

Let us analyze each of these cases, detremine explicitly the radical $\mathfrak{r}$ and the semi-direct product, and see how we can embed the isotropy $\mathfrak{h}$.

5.1. Case $\mathfrak{s} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$

Of course we have $\mathfrak{g} = \mathfrak{s}$ in this case. The isotropy $\mathfrak{h}$ embeds by the mapping $a \in \mathfrak{h} \to (\rho_1(a), \rho_2(a), \rho_3(a)) \in \mathfrak{g}$. Each ρ_i is either trivial or an isomorphism. If one ρ_i is trivial, then the $\mathfrak{h}$-representation on $\mathfrak{s}/\mathfrak{h}$ will have a trivial space corresponding to the projection of the i-factor (of $\mathfrak{s} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$), but $\mathfrak{hl}$ has no trivial space. Therefore all the ρ_i's are isomorphisms, and after conjugacy in each factor and identification of $\mathfrak{h}$ with $\mathfrak{sl}(2,\mathbb{R})$, we can assume that all the ρ_i are the identity. Thus, $\mathfrak{h} \equiv \mathfrak{sl}(2,\mathbb{R})$ embeds diagonally in $\mathfrak{s} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. Hence $\mathfrak{h} = \{(a,a,a) \in \mathfrak{sl}(2,\mathbb{R})\}$. Any element (b,c,d) is equivalent mod $\mathfrak{h}$ to $(b-d, c-d, 0)$. The quotient space $\mathfrak{s}/\mathfrak{h}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \{0\}$ and the $\mathfrak{h}$-action is the sum of two copies of the adjoint representation of $\mathfrak{sl}(2,\mathbb{R})$, so it is $\mathfrak{hl}$.

5.2. Case $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{C})$

Just as in the previous case, one shows that, up to an automorphism, $\mathfrak{h}$ embeds diagonally in $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{C})$.

5.3. Case $\mathfrak{s} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$

The isotropy $\mathfrak{h}$ is simple in $\mathfrak{g}$, we can assume it contained (up to conjugacy) in $\mathfrak{s}$. As in the previous case, it must be, up to conjugacy, the diagonal of $\mathfrak{sl}(2,\mathbb{R})^2$.

The radical $\mathfrak{r}$ has dimension 3. It projects injectively in $\mathfrak{g}/\mathfrak{h}$ (since $\mathfrak{h} \subset \mathfrak{s}$). Thus, $\mathfrak{r}$ is a factor of $\mathfrak{hl}$, i.e. it is isomorphic to the Ad-action of $\mathfrak{sl}(2,\mathbb{R})$. Only $\mathbb{R}^3$, as a 3-dimensional solvable group, can admit such an action. It follows that $\mathfrak{r} \cong \mathbb{R}^3$. The isotropy $\mathfrak{h}$ is the diagonal of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. The actions on $\mathbb{R}^3$ of the factors in the direct sum $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ commute. One can show this happens in dimension

3 only when one factor acts trivially. In this case, we deduce that the other factor acts like $\mathfrak{h}$ by Ad. Summarizing, $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus (\mathfrak{sl}(2, \mathbb{R}) \oplus_S \mathbb{R}^3)$, $\mathfrak{sl}(2, \mathbb{R})$ acts by Ad on $\mathbb{R}^3$ and $\mathfrak{h}$ embeds diagonally in $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$.

5.4. Case $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{R})$

In this case, we have $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus_S r$, with $\dim \mathfrak{r} = 6$. As previously, we can identify $\mathfrak{h}$ with a subset of the Levi factor $\mathfrak{sl}(2, \mathbb{R})$, and hence it equals this factor. Therefore $\mathfrak{r} \sim \mathfrak{g}/\mathfrak{h}$ and the $\mathfrak{sl}(2, \mathbb{R})$-action on $\mathfrak{r}$ is $\mathfrak{hl}$.

Let us prove that $\mathfrak{r}$ is nilpotent. Indeed, if $\mathfrak{n}$ is the nilradical of $\mathfrak{r}$, then $\mathsf{Aut}(\mathfrak{r})$ acts trivially on $\mathfrak{r}/\mathfrak{n}$ (see for instance [16]), but this cannot happen if $\mathfrak{r}/\mathfrak{n} \neq 0$.

Now, if $\mathfrak{r}$ is not abelian, its center $\mathfrak{c}$ is $\mathfrak{hl}$-invariant. Hence $\mathfrak{c}$ is isomorphic to $\mathbb{R}^3$, on which $\mathfrak{sl}(2, \mathbb{R})$ acts by Ad. The same thing happens for $[\mathfrak{r}, \mathfrak{r}]$, and consequently $\mathfrak{c} = [\mathfrak{r}, \mathfrak{r}]$. In particular, we deduce that $\mathfrak{r}$ is 2-step nilpotent. Also, since $\mathfrak{sl}(2, \mathbb{R})$ acts by $\mathfrak{hl}$, we see that there exists an invariant complement subspace $\mathfrak{c}'$ of $\mathfrak{c}$ on which $\mathfrak{sl}(2, \mathbb{R})$ acts by Ad. The bracket $\mathfrak{c}' \times \mathfrak{c}' \to \mathfrak{c}$ is $\mathfrak{sl}(2, \mathbb{R})$-equivariant. But the bracket $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{sl}(2, \mathbb{R})$ is the unique such an equivariant skew-symmetric bilinear map.

By its correct definition, this means that $\mathfrak{r}$ is the Lie algebra of $N_{\mathfrak{sl}_2}$.

6. Identification to a left invariant metric on a Lie group L

In each of the four cases considered in the previous section, we will find a normal subgroup L which acts simply transitively on G/H so that the latter space can be identified to a left invariant almost complex Hermite-Lorentz structure on L having an isotropy group $H \subset \mathsf{Aut}(L)$ that acts via $\mathfrak{hl}$. Of course, we can alternatively work with Lie algebras.

6.1. Case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$. Take $\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \{0\}$. The infinitesimal isotropy $\mathfrak{h}$ being the diagonal, it follows that $\mathfrak{l}$ is transversal to $\mathfrak{h}$. More precisely, at the group level, we have $G = LH$ and $L \cap H = 1$.

6.2. Case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{C})$. Take $\mathfrak{l} = \mathfrak{sl}(2, \mathbb{C})$.

6.3. Case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus (\mathfrak{sl}(2, \mathbb{R}) \oplus_S \mathbb{R}^3)$. Here $\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R}) \oplus_S \mathbb{R}^3$.

6.4. Case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus_S \mathfrak{n}_{\mathfrak{sl}_2}$. Here $\mathfrak{l} = \mathfrak{n}_{\mathfrak{sl}_2}$.

7. Integrability of the almost complex structure and curvature computations

In what follows, we will make use of the following well known formulas regarding the Lie group L (see for instance [3]). We first note that since $\mathfrak{l}$ is a subalgebra of $\mathfrak{g}$, then we can easily compute the Nijenhuis tensor of almost complex structure J. For $X, Y \in \mathfrak{l}$, we can use the formula

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] \tag{1}$$

To check that $M \cong L$ is Kählerian or not, we compute the differential of the Kähler form ω on L, which (we recall) is given by $\omega(X, Y) = \langle JX, Y \rangle$. We may use the formula

$$d\omega(X, Y, Z) = \frac{1}{3} \left\{ -\omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X) \right\}, \tag{2}$$

for all $X, Y, Z \in \mathfrak{l}$. Here, we have used the fact that on a the Lie group L the function $\langle X, Y \rangle$ is constant for all $X, Y \in \mathfrak{l}$, where $\langle ., . \rangle$ is the (indefinite) inner product on $\mathfrak{l}$ induced from the left-invariant pseudo-Riemannian metric on L.

To compute the Levi-Civita connection ∇ on L, we may use the formula

$$\nabla_{E_i} E_j = \sum_{k=1}^{6} \frac{1}{2} \left(\alpha_{ijk} - \epsilon_i \epsilon_k \alpha_{jki} + \epsilon_j \epsilon_k \alpha_{kij} \right) E_k, \tag{3}$$

where the α_{ijk} are the constant structures of $\mathfrak{l}$ with respect to the orthonormal basis $\{E_1, \ldots, E_6\}$ of $\mathfrak{l}$, and $\epsilon_i = \langle E_i, E_i \rangle$ (cf. [1] and [19]). We notice here that (3) can be deduced from the Koszul formula, using the fact that each function $\langle E_i, E_j \rangle$ is constant (The E_i are seen here as left-invariant vector fields on L).

We may compute ∇J according to the formula

$$(\nabla_X J) Y = \nabla_X (JY) - J (\nabla_X Y). \tag{4}$$

We will also make use of the basis $\{X_1, X_2, X_3\}$ of $\mathfrak{sl}(2, \mathbb{R})$ given by

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5}$$

for which the brackets are $[X_1, X_2] = 2X_3$, $[X_1, X_3] = 2X_2$ and $[X_2, X_3] = -2X_1$. This is an orthonormal basis for Lorentzian inner product

$$\langle X, Y \rangle = \frac{1}{2} \, trace \, (XY), \tag{6}$$

for which X_3 is timelike. This coincides with the Killing form of $\mathfrak{sl}(2, \mathbb{R})$ which is of signature $(+, +, -)$, and it is well known that the corresponding left-invariant (in fact, bi-invariant) Lorentz metric makes of $\mathsf{SL}(2, \mathbb{R})$ a Lorentz space of constant curvature -1 (cf. [20]). It is nothing but the so-called anti-de-Sitter space $\mathsf{AdS}_3(\mathbb{R})$.

The Levi-Civita connection of $\mathsf{SL}(2, \mathbb{R})$ endowed with the bi-invariant metric resulting from $\langle, \rangle$ is simply given by the formula

$$\nabla_X Y = \frac{1}{2} [X, Y]. \tag{7}$$

7.1. Case $\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$

With the notations above, let $\{X_1, X_2, X_3\}$ be the orthonormal basis of $\mathfrak{sl}(2, \mathbb{R})$ given by (5), and set

$$E_i = (X_i, 0) \quad \text{and} \quad E_{i+3} = (0, X_i), \quad i = 1, 2, 3.$$

This is an orthonormal basis of $\mathfrak{l}$ with E_3 and E_6 timelike vectors, for which the Lie brackets are given by

$$[E_1, E_2] = 2E_3, \quad [E_1, E_3] = 2E_2, \quad [E_2, E_3] = -2E_1,$$
$$[E_4, E_5] = 2E_6, \quad [E_4, E_6] = 2E_6, \quad [E_5, E_6] = -2E_4,$$

and all others are zeros or can be deduced from the above ones by antisymmetry. The almost complex structure J is given by

$$J(E_1) = E_4, \quad J(E_2) = E_5, \quad J(E_3) = E_6,$$
$$J(E_4) = -E_1, \quad J(E_5) = -E_2, \quad J(E_6) = -E_3.$$

In this case, the almost complex structure J is not integrable. Using (1), we have for example $N_J(E_1, E_2) = 2E_6 - 2E_3$.

We also see that M is not almost-Kählerian. Using (2), we have for example

$$d\omega\left(E_1, E_2, E_6\right) = \frac{2}{3}.$$

To compute the Levi-Civita connection on $\mathfrak{l}$, it suffices to remember here that $L = \mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$, where each factor $\mathsf{SL}(2,\mathbb{R})$ is endowed with the bi-invariant metric resulting from $\langle,\rangle$ for which the Levi-Civita connection may be computed using (7). In other words, we have

$$\begin{aligned}
\nabla_{E_1} E_2 &= E_3, & \nabla_{E_1} E_3 &= E_2, & \nabla_{E_2} E_3 &= -E_1, \\
\nabla_{E_4} E_5 &= E_6, & \nabla_{E_4} E_6 &= E_5, & \nabla_{E_5} E_6 &= -E_4,
\end{aligned}$$

and all the other terms are zero or can easily be obtained by using the formula $\nabla_Y X = \nabla_X Y - [X, Y]$.

Remark 7.1. Since L is a product of two Lorentz spaces of constant curvature, we deduce that M is a symmetric Pseudo-Riemannian manifold. However, M is not a locally symmetric Hermite-Lorentz space.

7.2. Case $\mathfrak{l} = \mathfrak{sl}(2,\mathbb{C})$

In this case, we look at $\mathfrak{sl}(2,\mathbb{C})$ as the complexification of $\mathfrak{sl}(2,\mathbb{R})$, that is, we have $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{R}) + \sqrt{-1}\mathfrak{sl}(2,\mathbb{R})$, where $\mathfrak{sl}(2,\mathbb{R})$ is endowed with the inner product (6). Of course, this is not a direct sum, and note that the inner product (6) is extended in a sesquilinear way to $\mathfrak{sl}(2,\mathbb{C})$. Consider the basis $\{X_1, X_2, X_3\}$ of $\mathfrak{sl}(2,\mathbb{R})$ given by (5), and set

$$E_i = (X_i, 0) \quad \text{and} \quad E_{i+3} = \sqrt{-1}\,(0, X_i), \quad i = 1, 2, 3.$$

This is an orthonormal basis of $\mathfrak{l}$ with E_3 and E_6 timelike vectors for which

$$\begin{aligned}
[E_1, E_2] &= 2E_3, & [E_1, E_3] &= 2E_2, & [E_2, E_3] &= -2E_1, \\
[E_4, E_5] &= -2E_3, & [E_4, E_6] &= -2E_2, & [E_5, E_6] &= 2E_1, \\
[E_1, E_5] &= 2E_6, & [E_1, E_6] &= 2E_5, & [E_2, E_4] &= -2E_6, \\
[E_2, E_6] &= -2E_4, & [E_3, E_4] &= -2E_5, & [E_3, E_5] &= 2E_4,
\end{aligned}$$

and all other brackets are zeros or can be deduced by antisymmetry.
The almost complex structure J is given by

$$\begin{aligned}
J\left(E_1\right) &= E_4, & J\left(E_2\right) &= E_5, & J\left(E_3\right) &= E_6, \\
J\left(E_4\right) &= -E_1, & J\left(E_5\right) &= -E_2, & J\left(E_6\right) &= -E_3.
\end{aligned}$$

Of course, the almost complex structure J is integrable in this case and coincides with the natural complex structure of $\mathsf{SL}(2,\mathbb{C})$.

The homogeneous space M is not almost Kählerian, since using (2) we have for example

$$d\omega\left(E_1, E_2, E_6\right) = -\frac{2}{3}.$$

The Levi-Civita connection on L may be computed by using (3). We obtain

$$\nabla_{E_1} E_2 = E_3, \qquad \nabla_{E_1} E_3 = E_2, \qquad \nabla_{E_2} E_3 = -E_1,$$
$$\nabla_{E_4} E_6 = -E_2, \qquad \nabla_{E_5} E_6 = -E_2, \qquad \nabla_{E_1} E_5 = 3E_6,$$
$$\nabla_{E_1} E_6 = 3E_5, \qquad \nabla_{E_2} E_4 = -3E_6, \qquad \nabla_{E_2} E_6 = -3E_4,$$
$$\nabla_{E_3} E_4 = -3E_5, \qquad \nabla_{E_3} E_5 = -3E_4, \qquad \nabla_{E_4} E_5 = -E_3,$$

and all the other $\nabla_{E_i} E_j$ are zeros or can be obtained by using the simple formula $\nabla_Y X = \nabla_X Y - [X, Y]$.

Remark 7.2. We can check that the Riemannian tensor $\mathcal{R}$ is not parallel by computing for example $\nabla \mathcal{R} (E_1, E_2, E_5, E_6) = 6E_2$.

We deduce from this that M is not locally symmetric as a pseudo-Riemannian space (and a fortiori as an almost Hermite-Lorentz space). ∎

7.3. Case $\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R}) \oplus_S \mathbb{R}^3$

Now, the Hermite-Lorentz inner product $\langle, \rangle$ on $\mathfrak{l} = \mathfrak{sl}(2, \mathbb{R}) \oplus_S \mathbb{R}^3$ is obtained as follows. For $i = 1, 2, 3$, we set

$$E_i = (X_i, 0) \quad \text{and} \quad E_{i+3} = (0, e_i),$$

where $\{e_1, e_2, e_3\}$ is the standard orthonormal Lorentz basis of $\mathbb{R}^3$, with e_3 is timelike. By letting $\mathfrak{sl}(2, \mathbb{R})$ act on $\mathbb{R}^3$ by its adjoint representation, we obtain an orthonormal basis of $\mathfrak{l}$ with E_3 and E_6 timelike vectors and for which the Lie brackets are given by the following identities

$$[E_1, E_2] = 2E_3, \qquad [E_1, E_3] = 2E_2, \qquad [E_2, E_3] = -2E_1,$$
$$[E_1, E_5] = 2E_6, \qquad [E_1, E_6] = 2E_5, \qquad [E_2, E_4] = -2E_6,$$
$$[E_2, E_6] = -2E_4, \qquad [E_3, E_4] = -2E_5, \qquad [E_3, E_5] = 2E_4,$$

and all other brackets are zeros or can be deduced from the latter by antisymmetry. The almost complex structure J is given by

$$J(E_1) = E_4, \qquad J(E_2) = E_5, \qquad J(E_3) = E_6,$$
$$J(E_4) = -E_1, \qquad J(E_5) = -E_2, \qquad J(E_6) = -E_3.$$

The complex structure J is not integrable given that, using (1), we have for example

$$N_J(E_1, E_2) = 2E_3.$$

To deduce that the Hermite-Lorentz space $M = G/H$ is not almost-Kähler, we may use (2) to find for example

$$d\omega(E_2, E_3, E_4) = -\frac{2}{3}.$$

We may also use (4) to get for example that $(\nabla_{E_1} J) E_2 = E_6$.

Regarding the connection, by using (3), we obtain

$$\nabla_{E_1} E_2 = E_3, \qquad \nabla_{E_1} E_3 = E_2, \qquad \nabla_{E_1} E_5 = 2E_6,$$
$$\nabla_{E_1} E_6 = 2E_5, \qquad \nabla_{E_2} E_3 = -E_1, \qquad \nabla_{E_2} E_4 = -2E_6,$$
$$\nabla_{E_2} E_6 = -2E_4, \qquad \nabla_{E_3} E_4 = -2E_5, \qquad \nabla_{E_3} E_5 = 2E_4,$$

and all the others are zero or can easily be obtained by using again the formula $\nabla_Y X = \nabla_X Y - [X, Y]$.

Remark 7.3. Of course, M is not a symmetric Hermite-Lorentz space. However, it turns out that M is a symmetric pseudo-Riemannian manifold as we can easily show that $\nabla \mathcal{R} (E_i, E_j, E_k, E_l) = 0$, for all $i, j, k, l \in \{1, \ldots, 6\}$, where $\mathcal{R}$ is the Riemannian tensor.

7.4. Case $\mathfrak{l} = \mathfrak{n}_{\mathfrak{sl}_2}$

With the notations of the previous case, let $\{X_1, X_2, X_3\}$ be the orthonormal basis of $\mathfrak{sl}(2, \mathbb{R})$ given by (5). Setting

$$E_i = (X_i, 0) \quad \text{and} \quad E_{i+3} = (0, X_i), \quad i = 1, 2, 3,$$

we get an orthonormal basis of $\mathfrak{l} = \mathfrak{n}_{\mathfrak{sl}_2}$ with E_3 and E_6 timelike vectors and for which the Lie bracket are given by the following identities

$$[E_1, E_2] = 2E_6, \quad [E_1, E_3] = 2E_5, \quad [E_2, E_3] = -2E_4,$$

and all other brackets are zeros or can be deduced from the latter by antisymmetry. The almost complex structure J is given by

$$J(E_1) = E_4, \quad J(E_2) = E_5, \quad J(E_3) = E_6,$$
$$J(E_4) = -E_1, \quad J(E_5) = -E_2, \quad J(E_6) = -E_3.$$

It is not integrable, as we can see for example from $N_J(E_1, E_2) = -E_6$. It also turns out that the Hermite-Lorentz space $M = G/H$ is not almost-Kähler. We have for example $d\omega(E_1, E_2, E_3) = -2$.

The Levi-Civita connection on L is computed using (3). We obtain

$$\nabla_{E_1} E_2 = E_6, \quad \nabla_{E_1} E_3 = E_5, \quad \nabla_{E_1} E_5 = E_3,$$
$$\nabla_{E_1} E_6 = E_2, \quad \nabla_{E_2} E_3 = -E_4, \quad \nabla_{E_2} E_4 = -E_3,$$
$$\nabla_{E_2} E_6 = -E_1, \quad \nabla_{E_3} E_4 = -E_2, \quad \nabla_{E_3} E_5 = E_1,$$

and all the other term are zero or can be obtained by using again the formula $\nabla_Y X = \nabla_X Y - [X, Y]$.

Remark 7.4. We can check that the Riemannian tensor $\mathcal{R}$ is not parallel by computing for example $\nabla \mathcal{R}(E_1, E_2, E_3, E_6) = E_3$.

We deduce from this that M is not locally symmetric as a pseudo-Riemannian space (and a fortiori as an almost Hermite-Lorentz space). ∎

7.5. Full isometry group

Fact 7.5. *In each of the four examples of spaces G/H or alternatively the Lie groups L for which $\mathfrak{h} = \mathfrak{so}(1, 2)$ (described in Sections 5 and 6), G is exactly the identity component of the full isometry group $\mathsf{Iso}(G/H)$ (to mean the automorphism group of the almost complex Hermite-Lorentz structure). Equivalently, H is the identity component of the full isotropy group.*

Furthermore, none of these spaces is (locally) symmetric (as a Hermite-Lorentz space).

Proof. If the identity component $\mathsf{Iso}^0(G/H)$ of the isometry group of G/H is bigger than G, that is if $\dim \mathsf{Iso}(G/H) > \dim G$, then the isotropy algebra contains

strictly $\mathfrak{so}(1,2)$. By Fact 1.7, the isotropy algebra $\mathfrak{h}$ is then $\mathfrak{so}(1,2)\oplus\mathfrak{so}(2)$, $\mathfrak{su}(1,2)$ or $\mathfrak{u}(1,2)$. By Section 3, $\mathfrak{h}$ can not be $\mathfrak{su}(1,2)$. So, if $\mathfrak{h}$ is bigger than $\mathfrak{so}(1,2)$, then it equals $\mathfrak{so}(1,2)\oplus\mathfrak{so}(2)$ or $\mathfrak{u}(1,2)$. As stated in 1.4.2, the case $\mathfrak{h} = \mathfrak{so}(1,2)\oplus\mathfrak{so}(2)$ implies the space is symmetric. Indeed, the $\mathsf{SO}(2)$-action on $\mathbb{C}^3$ is that given by the homotheties $Z \to e^{it}Z$. In particular $Z \to e^{i\pi}Z = -Z$, that is the isotropy contains an element inducing $-\mathsf{Id}$ on the tangent space, and therefore G/H is symmetric.

Observe now that a (locally) symmetric almost Hermite-Lorentz space is necessarily Kähler-Lorentz, that is, its almost complex structure is integrable and it is Kählerian. From the previous computations in the present Section 7, none of our four spaces satisfies all these conditions. Therefore, the isotropy algebra for all these spaces must be $\mathfrak{so}(1,2)$ and $\mathsf{Iso}^0(G/H) = G$. ∎

8. Geodesic completeness. Proof of Theorem 1.5

Recall first that a pseudo-Riemannian symmetric space M is always geodesically complete, for the simple reason that any global symmetry at any point p of M reverses geodesics through p. (An interval of $\mathbb{R}$ which is invariant under reflection about all its points is necessarily $\mathbb{R}$ itself).

According to [4], almost complex Hermite-Lorentz homogeneous spaces of dimension ≥ 4 with $\mathbb{C}$-irreducible isotropy are symmetric.

On the other hand, the left invariant metric on $L = \mathsf{SL}(2,\mathbb{R}) \times \mathsf{SL}(2,\mathbb{R})$ is in fact bi-invariant as we have noticed in Section 2.1.

Similarly, as explained in Section 2.2, the left invariant metric on $L = \mathsf{SL}(2,\mathbb{R})\ltimes\mathbb{R}^3$ is isometric to a left invariant metric on $\mathsf{SL}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ which is in fact nothing but the direct metric product $(\mathsf{SL}(2,\mathbb{R}),\kappa) \times (\mathfrak{sl}(2,\mathbb{R}),\kappa)$, where is the Killing form of $\mathfrak{sl}(2,\mathbb{R})$. So, this is also a symmetric space.

It follows that in all the above mentioned cases, we have geodesically complete spaces.

The case $L = N_{\mathfrak{sl}_2}$ is so special here. Indeed, it turns out that for a two-step nilpotent Lie group, any left invariant pseudo-Riemannian metric is geodesically complete (see [11]).

Finally, and in order to achieve the proof of Theorem 1.5, it remains to study completeness for the case $L = \mathsf{SL}(2,\mathbb{C})$. In fact, we will do it in a more general context, namely, we will prove the completeness of a certain class of left invariant pseudo-Riemannian metrics on semi-simple Lie groups of which our considered metric on $\mathsf{SL}(2,\mathbb{C})$ appears as a special case.

To do this, we need first to recall some general facts about the geodesic flow of left invariant pseudo-Riemannian metrics on Lie groups.

Let L be a Lie group with Lie algebra $\mathfrak{l}$. It is well known that giving a left-invariant pseudo-Riemannian metric $\langle.,.\rangle$ on L is equivalent to that of a nondegenerate quadratic form (that we also denote by $\langle.,.\rangle$) on $\mathfrak{l}$.

Since $\langle X, Y \rangle$ is constant for any vectors $X, Y \in \mathfrak{l}$ (seen as left-invariant vector fields on L), it follows from the Koszul formula that the Levi-Civita connection of $\langle.,.\rangle$ is given by the formula (cf. [7])

$$\nabla_X Y = \frac{1}{2}\left\{[X,Y] - ad_X^* Y - ad_Y^* X\right\}, \tag{8}$$

where ad_X^* is the adjoint of ad_X with respect to inner product $\langle.,.\rangle$ on $\mathfrak{l}$.

Every curve $c(t)$ in L gives rise to the curve $\gamma(t) = \left(dL_{c(t)}\right)^{-1}(\dot{c}(t))$ in $\mathfrak{l}$, where for $g \in L$, L_g denotes the left-invariant translation by g in L. We have

$$\left(dL_{c(t)}\right)^{-1}\left(\nabla_{\dot{c}(t)}\dot{c}(t)\right) = \dot{\gamma}(t) + \nabla_{\gamma(t)}\gamma(t) \tag{9}$$

By using (8), we deduce that $c(t)$ is a geodesic in L if and only if $\dot{\gamma}(t) = ad^*_{\gamma(t)}\gamma(t)$. In other words, we have shown that the curves of $\mathfrak{l}$ associated to the geodesics of L are solutions of the equation (see also [2])

$$\dot{x} = ad^*_x x \tag{10}$$

In the case where L is semi-simple with Lie algebra $\mathfrak{l}$ and Killing form κ, it turns out that giving a left-invariant pseudo-Riemannian metric $\langle .,. \rangle$ on L is also equivalent to that of a self-adjoint isomorphism A of $\mathfrak{l}$ such that

$$\langle X, Y \rangle = \kappa(X, A(Y)), \quad \text{for all } X, Y \in \mathfrak{l}. \tag{11}$$

By substituting (10) into (11) and using the bi-invariance of κ, we obtain

$$\kappa(A(\dot{x}), y) = \langle \dot{x}, y \rangle = \langle ad^*_x x, y \rangle = \langle [x, y], x \rangle = \kappa([x, y], A(x))$$
$$= -\kappa([x, A(x)], y) = \kappa([A(x), x], y), \quad \text{for all } y \in \mathfrak{l}.$$

Since y is arbitrary and κ is nondegenerate, we deduce that equation (10) translates to the following equation (cf. [14])

$$A(\dot{x}) = [A(x), x] \tag{12}$$

We are now in the position to give a proof of Theorem 1.5 in the case of semi-simple Lie groups of which $L = \mathsf{SL}(2, \mathbb{C})$ appears as special case.

For this, and as we have mentioned in Section 2.4, recall that the construction of the Hermite-Lorentz metric on $\mathsf{SL}(2, \mathbb{C})$ is a general fact. Indeed, let P be a semi-simple (real) Lie group with Lie algebra $\mathfrak{p}$, and let $L = P^{\mathbb{C}}$ be its complexification with Lie algebra $\mathfrak{l} = \mathfrak{p}^{\mathbb{C}} = \mathfrak{p} + \sqrt{-1}\mathfrak{p} \sim \mathfrak{p} \oplus \mathfrak{p}$. By endowing $\mathfrak{l}$ with $\kappa^P \oplus \kappa^P$, where κ^P is the Killing form of $\mathfrak{p}$, we obtain a left invariant pseudo-Hermitian metric h on L that is invariant under the adjoint action Ad_P. Denote by h_e its restriction to the Lie algebra $\mathfrak{l}$. This can be expressed in terms of the Killing form κ^L of $\mathfrak{l}$, seen as a real semi-simple Lie algebra. More precisely, we have

$$h_e(.,.) = \kappa^L(., A(.)),$$

where A is the self-adjoint isomorphism of $\mathfrak{l} = \mathfrak{p}^{\mathbb{C}} = \mathfrak{p} \oplus \mathfrak{p}$ defined by the formula $A(u, v) = (u, -v)$. Thus, the geodesic equation (12) becomes:

$$\dot{u}(t) = 0, \quad \dot{v}(t) = 2[v(t), u(t)]$$

Therefore $u(t)$ is constant, and thus we get a linear equation for $v(t)$, and so solutions are defined on the whole $\mathbb{R}$. Hence, the left invariant pseudo-Hermitian metric h on $L = P^{\mathbb{C}}$ is geodesically complete. ∎

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