Journal of Lie Theory
Volume **33** (2023) 799–830

On the Classification of 2-Solvable Frobenius Lie Algebras

André Diatta, Bakary Manga, Ameth Mbaye

Communicated by R. Avdeev

Abstract. We prove that every 2-solvable Frobenius Lie algebra splits as a semidirect sum of an n-dimensional vector space V and an n-dimensional maximal Abelian subalgebra (MASA) of the full space of endomorphisms of V. We supply a complete classification of 2-solvable Frobenius Lie algebras corresponding to nonderogatory endomorphisms, as well as those given by maximal Abelian nilpotent subalgebras (MANS) of class 2, hence of Kravchuk signature $(n-1,0,1)$. In low dimensions, we classify all 2-solvable Frobenius Lie algebras in general up to dimension 8. We correct and complete the classification list of MASAs of $\mathfrak{sl}(4,\mathbb{R})$ by Winternitz and Zassenhaus. As a biproduct, we give a simple proof that every nonderogatory endormorphism of a real vector space admits a Jordan form and also provide a new characterization of Cartan subalgebras of $\mathfrak{sl}(n,\mathbb{R})$.

Mathematics Subject Classification: 17B05, 17B08, 15A27, 53A15, 53D15, 22E60, 17B60, 70G45, 16W25, 13B25.

Key Words: Frobenius Lie algebra, 2-step solvable exact symplectic Lie algebra, symplectic Lie group, maximal Abelian subalgebra, nonderogatory endomorphism, cyclic matrix, companion matrix, Kravchuk signature, Cartan subalgebra, Jordan form.

1. Introduction

Throughout this work, V stands for a real vector space of dimension n. We are mainly interested in the classification of 2-solvable Frobenius Lie algebras. It turns out that such a classification is closely related to that of n-dimensional maximal Abelian subalgebras (MASAs) of the space $\mathfrak{gl}(V)$ of linear operators (endomorphisms) on V. The sudy of MASAs traces back to Frobenius and Schur, it has been a vibrant and vivid subject these last decades, in relation with several subjects in mathematics and physics such as the classification of Lie algebras and the study of dynamical systems ([5, 10, 12, 13, 14, 20, 23, 24, 25, 26]). Our interest in 2-solvable Frobenius Lie algebras is mainly related to their applications in symplectic ([3, 7, 9, 15]) and information geometry ([16]), as well as to the investigation of Lie algebras in general. Frobenius Lie algebras have gained popularity mainly since the work of Ooms ([19, 18]). Among other results, a characterization of 2-solvable Frobenius Lie algebras is given in [19] (Theorem 4.1, Section 4, p. 43). On the other hand, the family of 2-solvable Frobenius Lie algebras with an Abelian nilradical is studied in [1]. The main results of the present paper are the following.

- We show that, if $\mathfrak{B} \subset \mathfrak{gl}(V)$ is Abelian and the semidirect sum $\mathfrak{B} \ltimes V$ is a Frobenius Lie algebra, then $\mathfrak{B}$ is a MASA of $\mathfrak{gl}(V)$ (Theorem 3.1). Moreover,

ISSN 0949–5932 / \$2.50

every 2-solvable Frobenius Lie algebra is of that form. Two 2-solvable Frobenius Lie algebras $\mathfrak{B}_1 \ltimes V$ and $\mathfrak{B}_2 \ltimes V$ are isomorphic if and only if there exists some ϕ in the group $\mathrm{GL}(V)$ of invertible linear operators of V, such that $\mathfrak{B}_2 = \phi\mathfrak{B}_1\phi^{-1}$ (Proposition 3.3).

- In Theorem 3.6, we classify 2-solvable Frobenius Lie algebras $\mathfrak{B} \ltimes V$, where $\mathfrak{B} \cap \mathfrak{sl}(V)$ is a maximal Abelian nilpotent subalgebra (MANS) of class 2 and hence of Kravchuk signature $(n-1, 0, 1)$. Here $\mathfrak{sl}(V) \subset \mathfrak{gl}(V)$ stands for the subspace of traceless linear operators.
- We fully classify, in Theorem 4.3, the family of 2-solvable Frobenius Lie algebras $\mathfrak{B} \ltimes V$, where $\mathfrak{B}$ is the algebra $\mathbb{K}[\phi]$ of the polynomials in some nonderogatory $\phi \in \mathfrak{gl}(V)$. We will simply write $\mathcal{G}_\phi = \mathbb{K}[\phi] \ltimes V$ instead of $\mathfrak{B} \ltimes V$.
- We supply a full classification list of all 2-solvable Frobenius Lie algebras in general, up to and including dimension 8 (Theorem 5.1).
- Theorems 3.11, 4.19 give a new characterization of Cartan subalgebras of $\mathfrak{sl}(V)$.
- We correct the classification list of MASAs of $\mathfrak{sl}(4,\mathbb{R})$ of [26] and further complete it with a missing item.
- We prove that every nonderogatory real matrix admits a Jordan form (Theorem 6.1).

The paper is organized as follows. Section 2, is devoted to some preliminaries and notations. In Section 3, we discuss the classification of 2-solvable Frobenius Lie algebras, in general. General examples of MASAs and 2-solvable Frobenius Lie algebras are discussed in details and a new characterization of Cartan subalgebras of $\mathfrak{sl}(n,\mathbb{R})$ is given. We supply a full classification of 2-solvable Frobenius Lie algebras given by nonderogatory maps in Section 4. We fully classify all 2-solvable Frobenius Lie algebras up to dimension 8 in Section 5 and furthermore, we correct and complete the classification list of MASAs of $\mathfrak{sl}(4,\mathbb{R})$ of [26] with a missing item. The paper ends by a simple direct proof of the Jordanization of nonderogatory real matrices in Appendix (Section 6).

2. Preliminaries

Throughout this work, if $\mathfrak{F}$ is a vector space, $\mathfrak{F}^*$ will stand for its (linear) dual, $\mathfrak{gl}(\mathfrak{F})$ the space of linear maps $\psi : \mathfrak{F} \to \mathfrak{F}$ and $\mathrm{GL}(\mathfrak{F})$ the subspace of $\mathfrak{gl}(\mathfrak{F})$ consisting of invertible maps. The symmetric bilinear form $\langle , \rangle$ stands for the duality pairing between vectors and linear forms: $\langle u, f\rangle := f(u)$, for $u \in \mathfrak{F}$, $f \in \mathfrak{F}^*$. If (e_j) is a basis of $\mathfrak{F}$, we denote by (e_j^*) its dual basis. Let $\mathfrak{B}$ be a Lie subalgebra of $\mathfrak{gl}(V)$, where V is a vector space. We consider the natural action $\mathfrak{B} \times V \to V$, $(a, x) \mapsto \rho(a)x := ax$. For any $(a, f) \in \mathfrak{B} \times V^*$, let $\rho^*(a)f$ be the element of V^* defined on elements x of V by $\langle \rho^*(a)f, x\rangle := -f(ax)$. We recall that ρ^* is called the contragredient representation (or just action) of ρ.

Definition 2.1. By abuse of language, we say that the orbit of $\alpha \in V^*$, under ρ^*, is *open*, if $V^* = \{\rho^*(a)\alpha, \ a \in \mathfrak{B}\}$. We say that α has a *trivial isotropy* if the equality $\rho^*(a)\alpha = 0$ implies $a = 0$. ∎

The vector space $\mathfrak{B} \oplus V$ endowed with the Lie bracket defined, for every $a, b \in \mathfrak{B}$ and $x, y \in V$, as $[a, b] = [a, b]_{\mathfrak{B}}$, $[x, y] = 0$, and $[a, x] = \rho(a)x = ax$, will be termed the semidirect sum of $\mathfrak{B}$ and V via ρ and denoted by $\mathfrak{B} \ltimes_\rho V$, or simply $\mathfrak{B} \ltimes V$ if no confusion is to be made. In the present work, $\mathfrak{B}$ is Abelian, $[,]_{\mathfrak{B}} = 0$.

A Lie algebra $\mathcal{G}$ is called a *Frobenius Lie algebra* if there exists $\alpha \in \mathcal{G}^*$, called a Frobenius functional, such that $\partial\alpha$ is nondegenerate, where, for any $u, v \in \mathcal{G}$,

$$\partial\alpha(u, v) := -\langle \alpha, [u, v] \rangle. \tag{1}$$

Every Frobenius Lie algebra is a codimension 1 subalgebra of some contact Lie algebra and could be used to construct the latter. The converse is true under some conditions ([6]). Any Lie group G with Lie algebra $\mathcal{G}$, has a left invariant symplectic form ω^+ with value $\omega^+_\epsilon = \partial\alpha$ at the unit (neutral) element ϵ of G. Here $\mathcal{G}$ is identified with the tangent space to G at ϵ. See for example [7], [9], for more details.

Definition 2.2. A Lie algebra $\mathcal{G}$ is said to be 2*-step solvable* (*2-solvable*, for short) if its derived ideal $[\mathcal{G}, \mathcal{G}]$ is Abelian. A Lie algebra is said to be *indecomposable* if it cannot be written as the direct sum of two of its ideals. ■

Remark 2.3. Note that two direct sums $\mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_k$ and $\mathcal{G}'_1 \oplus \cdots \oplus \mathcal{G}'_k$ of 2-solvable Frobenius Lie algebras, are isomorphic if and only if, up to rearranging the order of the indices, each $\mathcal{G}_j$ is isomorphic to each $\mathcal{G}'_j$. So the classification of 2-solvable Frobenius Lie algebras boils down to that of the indecomposable ones.

Definition 2.4. Let $\mathcal{G}$ be a Lie algebra, $\mathcal{A}$ a subalgebra of $\mathcal{G}$. The *normalizer* of $\mathcal{A}$ in $\mathcal{G}$ is $\mathcal{N}_{\mathcal{G}}(\mathcal{A}) := \{a \in \mathcal{G}, [a, \mathcal{A}] \subset \mathcal{A}\}$. The *centralizer* of $\mathcal{A}$ in $\mathcal{G}$ is $\mathrm{cent}(\mathcal{A}, \mathcal{G}) := \{a \in \mathcal{G}, [a, \mathcal{A}] = 0\}$. We say that $\mathcal{A}$ is a *maximal Abelian subalgebra* (MASA) of $\mathcal{G}$, if $\mathcal{A} = \mathrm{cent}(\mathcal{A}, \mathcal{G})$. Equivalently, $\mathcal{A}$ is Abelian and is contained in no Abelian subalgebra of higher dimension of $\mathcal{G}$. For $\phi \in \mathfrak{gl}(V)$, we use the convention $\phi^0 = \mathbb{I}_V =$ identity map of V and $\phi^{p+1}(x) = \phi(\phi^p(x))$, for every $x \in V$ and p an integer. We let $\mathbb{K}[\phi]$ stand for the space of polynomials in ϕ with coefficients in $\mathbb{K}$ and denote by $cent(\phi, \mathfrak{gl}(V))$ the set of endomorphisms commuting with ϕ. Recall that ϕ (resp. an $n \times n$ matrix M) is said to be nonderogatory (or cyclic), if its characteristic polynomial χ_ϕ and its minimal polynomial $\chi_{\min,\phi}$ coincide. ■

Lemma 2.5. (see e.g. [3]) *Let V be a vector space of dimension n over a field $\mathbb{K}$ of characteristic zero and $\phi \in \mathfrak{gl}(V)$. The following assertions are equivalent:*

(1) *there exists $\bar\alpha \in V^*$ such that $(\bar\alpha, \bar\alpha \circ \phi, ..., \bar\alpha \circ \phi^{n-1})$ is a basis of V^*,*
(2) *there exists $\bar x \in V$ such that $(\bar x, \phi(\bar x), ..., \phi^{n-1}(\bar x))$ is a basis of V,*
(3) $\dim(cent(\phi, \mathfrak{gl}(V))) = n$ *and* $cent(\phi, \mathfrak{gl}(V))$ *is commutative,*
(4) $cent(\phi, \mathfrak{gl}(V)) = \mathbb{K}[\phi]$,
(5) *the characteristic and the minimal polynomials of ϕ are the same,*
(6) *In every extension $\bar{\mathbb{K}}$ of $\mathbb{K}$ where ϕ admits a Jordan form, this latter has only one Jordan bloc for each eigenvalue.*

In Theorem 6.1, we go beyond Lemma 2.5 to prove that, for every n-dimensional real vector space V, every nonderogatory $\phi \in \mathfrak{gl}(V)$ admits a Jordan form. Throughout this work, $E_{i,j}$ stands for the $n \times n$ matrix with zero in all entries except the (i, j) entry which is equal to 1.

3. On the structure of 2-solvable Frobenius Lie algebras and n-dimensional MASAs

3.1. MASAs of $\mathfrak{gl}(V)$ and maximal Abelian subgroups of $GL(V)$

In this section, we briefly discuss the 1-1 correspondence between MASAs of $\mathfrak{gl}(V)$ and maximal Abelian subgroups of $GL(V)$. Let $\mathfrak{B}$ be a MASA of $\mathfrak{gl}(V)$. Then $\mathfrak{B}$ is a commutative subalgebra of the associative algebra $\mathfrak{gl}(V)$, and it contains the unit $\mathbb{I}_V$. Hence its (open) subset $U(\mathfrak{B})$ consisting of invertible elements is an analytic Lie group and $\dim U(\mathfrak{B}) = \dim \mathfrak{B}$, (see e.g. J. P. Serre [21], p. 103, for the general case) which, by definition, is also an Abelian Lie subgroup of GL(V).

Furthermore, since $\mathfrak{B}$ is a MASA of $\mathfrak{gl}(V)$, it contains the exponential of all its elements. Thus $\exp \mathfrak{B}$ coincides with the connected component of the unit of $U(\mathfrak{B})$. In fact, $U(\mathfrak{B})$ is a maximal Abelian subgroup of $GL(V)$. Conversely, every maximal Abelian subgroup of $GL(V)$ is of that form $U(\mathfrak{B})$, for some MASA $\mathfrak{B}$ of $\mathfrak{gl}(V)$. Two such subgroups are conjugate in $GL(V)$ if and only if the corresponding MASAs are conjugate in $\mathfrak{gl}(V)$ (see [24]). So we deduce that a simple way to construct a 2-solvable Lie group with Lie algebra $\mathfrak{B} \ltimes V$ is to simply take $U(\mathfrak{B}) \ltimes V$.

3.2. Main result on MASAs and 2-solvable Frobenius Lie algebras

Let V be a vector space of dimension n and $\mathfrak{B}$ an n-dimensional Abelian Lie subalgebra of $\mathfrak{gl}(V)$. Then $\mathfrak{B} \ltimes V$ is a Frobenius Lie algebra if and only if $\mathfrak{B}$ has an open orbit in V^* and every 2-solvable Frobenius Lie algebra is of that form ([8]). Equivalently, the Lie group $U(\mathfrak{B})$ of invertible elements of $\mathfrak{B}$, has some open or dense orbit in V^*. Here we prove the following.

Theorem 3.1. *Let V be a vector space of dimension n and $\mathfrak{B}$ an n-dimensional Abelian Lie subalgebra of $\mathfrak{gl}(V)$. Suppose $\mathfrak{B} \ltimes V$ is a Frobenius Lie algebra. Then $\mathfrak{B}$ is a MASA of $\mathfrak{gl}(V)$. The Lie group $U(\mathfrak{B})$ of invertible elements of $\mathfrak{B}$, is a maximal Abelian subgroup of $GL(V)$. Conversely, every 2-solvable Frobenius Lie algebra is isomorphic to one as above.*

Proof. Without loss of generality, set $V = \mathbb{K}^n$. The Lie algebra $\mathfrak{B} \ltimes \mathbb{K}^n$ is Frobenius if and only if the action $\mathfrak{B} \times (\mathbb{K}^n)^* \to (\mathbb{K}^n)^*$, $(a, f) \mapsto -f \circ a$, has an open orbit ([8]). Consider some $\alpha \in (\mathbb{K}^n)^*$ with an open orbit. Thus, any basis $(a_1, \ldots, a_n)$ of $\mathfrak{B}$ gives rise to a basis $(\rho^*(a_1)\alpha, \ldots, \rho^*(a_n)\alpha)$ of $(\mathbb{K}^n)^*$ and the (linear) orbital map $Q : \mathfrak{B} \to (\mathbb{K}^n)^*$, $Q(a) = \rho^*(a)\alpha$ is an isomorphism between the vector spaces $\mathfrak{B}$ and $(\mathbb{K}^n)^*$. Just for convenience, we will let $(\hat{e}_1, \ldots, \hat{e}_n)$ stand for the basis of $\mathbb{K}^n$ whose dual basis is $(\hat{e}_1^* := \rho^*(a_1)\alpha, \ldots, \hat{e}_n^* := \rho^*(a_n)\alpha)$. Suppose $\tilde{a} \in \mathfrak{gl}(n, \mathbb{K})$ is such that $[\tilde{a}, a] = 0$ for any $a \in \mathfrak{B}$ and $\tilde{a} \neq 0$. Assume $\tilde{a}$ is not an element of $\mathfrak{B}$, then $\tilde{\mathfrak{B}} := \mathbb{K}\tilde{a} \oplus \mathfrak{B}$ is an $(n+1)$-dimensional Abelian subalgebra of $\mathfrak{gl}(n, \mathbb{K})$. Because $\dim \tilde{\mathfrak{B}} = \dim(\mathbb{K}^n)^* + 1$, the orbital map $\tilde{Q} : \tilde{\mathfrak{B}} \to (\mathbb{K}^n)^*$, also given by $\tilde{Q}(a) = \tilde{\rho}^*(a)\alpha := -\alpha \circ a$, must have a 1-dimensional kernel. So, there exists some $\tilde{b} = k\tilde{a} + a_0 \neq 0$, with $k \in \mathbb{K}$ and $a_0 \in \mathfrak{B}$ such that $\tilde{\rho}^*(\tilde{b})\alpha = -\alpha \circ \tilde{b} = 0$. Since $\tilde{\rho}^*(a_0)\alpha = \rho^*(a_0)\alpha = Q(a_0) \neq 0$ if $a_0 \neq 0$, we must have $k \neq 0$. We then must have $\tilde{b}x = 0$, for any $x \in \mathbb{K}^n$, or equivalently $\tilde{b} = 0$. Indeed, for any $x \in \mathbb{K}^n$, the expression of $\tilde{b}x$ in the above basis is $\tilde{b}x = \sum_{j=1}^{n} \langle \hat{e}_j^*, \tilde{b}x \rangle \hat{e}_j$. But the components $\langle \hat{e}_j^*, \tilde{b}x \rangle$ are all equal to zero for any $j = 1, \ldots, n$, since the following holds true

$$\langle \hat{e}_j^*, \tilde{b}x \rangle = \langle \rho^*(a_j)\alpha, \tilde{b}x \rangle = -\langle \alpha, a_j\tilde{b}x \rangle = -\langle \alpha, \tilde{b}a_jx \rangle = \langle \tilde{\rho}^*(\tilde{b})\alpha, a_jx \rangle = 0.$$

Now, the equality $\tilde{b} = k\tilde{a} + a_0 = 0$, contradicts the assumption that $\tilde{a}$ is not in $\mathfrak{B}$. So $\tilde{a}$ must necessary be in $\mathfrak{B}$. This proves that $\mathfrak{B}$ is a MASA of $\mathfrak{gl}(n,\mathbb{K})$. Due to the correspondence between MASAs of $\mathfrak{gl}(n,\mathbb{K})$ and maximal Abelian subgroups of $\mathrm{GL}(n,\mathbb{K})$, see [24] and Section 3.1, $\mathfrak{B}$ is a MASA if and only if $U(\mathfrak{B})$ is a maximal Abelian subgroup of $\mathrm{GL}(n,\mathbb{K})$. ■

Remark 3.2. From Lemma 2.5, if $\phi \in \mathfrak{gl}(V)$ is nonderogatory, then $\mathbb{K}[\phi]$ is an n-dimensional MASA of $\mathfrak{gl}(V)$, there is $\bar{\alpha} \in V^*$ with an open orbit for the action of $\mathbb{K}[\phi]$. The 2-solvable Lie algebra $\mathcal{G}_\phi := \mathbb{K}[\phi] \ltimes V$ is a Frobenius Lie algebra (Theorem 3.1). Frobenius Lie algebras of type $\mathcal{G}_\phi$ are studied in details and fully classified in Section 4. Obviously, if $\phi, \psi \in \mathfrak{gl}(V)$ satisfy $\phi = P\psi P^{-1}$, for some $P \in GL(V)$, then $\mathbb{K}[\phi]$ and $\mathbb{K}[\psi]$ are conjugate, $\mathcal{G}_\phi$ and $\mathcal{G}_\psi$ are isomorphic via the map $\xi : \mathcal{G}_\psi \to \mathcal{G}_\phi$, $\xi(a) = PaP^{-1}$, $\xi(x) = Px$, for any $a \in \mathbb{K}[\psi]$ and $x \in V$.

Proposition 3.3. *Let V be a vector space and $\mathfrak{B}_1$, $\mathfrak{B}_2$ two MASAs of $\mathfrak{gl}(V)$, such that $\dim V = \dim \mathfrak{B}_1 = \dim \mathfrak{B}_2$. Then $\mathfrak{B}_1 \ltimes V$ and $\mathfrak{B}_2 \ltimes V$ are isomorphic if and only if $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are conjugate. A linear map $\psi : \mathfrak{B}_1 \ltimes V \to \mathfrak{B}_2 \ltimes V$ is an isomorphism if and only if there exists $(\phi_\psi, x_\psi) \in GL(V) \times V$ such that $\phi_\psi \circ \mathfrak{B}_1 \circ \phi_\psi^{-1} = \mathfrak{B}_2$ and for any $(a,x) \in \mathfrak{B}_1 \ltimes V$:*

$$\psi(a,x) = (\phi_\psi \circ a \circ \phi_\psi^{-1}\ ,\ \phi_\psi \circ a \circ \phi_\psi^{-1}(x_\psi) + \phi_\psi(x)).$$

Proof. Suppose $\mathcal{G}_1 := \mathfrak{B}_1 \ltimes V$ and $\mathcal{G}_2 := \mathfrak{B}_2 \ltimes V$ are isomorphic and $\psi : \mathcal{G}_1 \to \mathcal{G}_2$ is an isomorphism. As the derived ideal $[\mathcal{G}_1, \mathcal{G}_1] = V$ must be mapped to $[\mathcal{G}_2, \mathcal{G}_2] = V$, there are linear maps $\psi_{1,1} : \mathfrak{B}_1 \to \mathfrak{B}_2$, $\psi_{1,2} : \mathfrak{B}_1 \to V$, $\phi_\psi : V \to V$, with $\psi_{1,1}$ and ϕ_ψ invertible, such that $\psi(a) = \psi_{1,1}(a) + \psi_{1,2}(a)$ and $\psi(x) = \phi_\psi(x)$ for any $a \in \mathfrak{B}_1$, $x \in V$. We deduce the equality $\psi_{1,1}(a) = \phi_\psi \circ a \circ \phi_\psi^{-1}$ from the following

$$\phi_\psi(ax) = \psi([a,x]) = [\psi_{1,1}(a) + \psi_{1,2}(a), \phi_\psi(x)] = \Big(\psi_{1,1}(a) \circ \phi_\psi\Big)(x).$$

In particular $\phi_\psi \circ a \circ \phi_\psi^{-1} \in \mathfrak{B}_2$, for any $a \in \mathfrak{B}_1$, equivalently $\phi_\psi \circ \mathfrak{B}_1 \circ \phi_\psi^{-1} = \mathfrak{B}_2$. Now set $x_\psi := \psi_{1,2}(b)$, with $b = \mathbb{I}_V \in \mathfrak{B}_1$. We have $\psi_{1,1}(\mathbb{I}_V) = \mathbb{I}_V$, since ψ is an isomorphism. The equalities

$$0 = \psi([a,b]) = [\psi(a), \psi(b))] = \psi_{1,1}(a)\psi_{1,2}(b) - \psi_{1,1}(b)\psi_{1,2}(a)$$

yield $\psi_{1,2}(a) = \phi_\psi \circ a \circ \phi_\psi^{-1} x_\psi$, for any $a \in \mathfrak{B}_1$. Conversely, suppose there exists some $\phi \in GL(V)$ such that $\mathfrak{B}_2 = \phi \mathfrak{B}_1 \phi^{-1}$. Then the map $\psi : \mathcal{G}_1 \to \mathcal{G}_2$ given for any $a \in \mathfrak{B}_1$ and $x \in V$ by $\psi(a + x) = \phi \circ a \circ \phi^{-1} + \phi \circ a \circ \phi^{-1}(x_0) + \phi(x)$ is an isomorphism, for any $x_0 \in V$. ■

3.3. Different types of MASAs

A MASA of $\mathfrak{gl}(V)$ is said to be decomposable, if V can be written as the direct sum of subspaces which are all preserved by the MASA. Otherwise, the MASA is said to be indecomposable. Every MASA of $\mathfrak{gl}(V)$ is a direct sum of $\mathbb{K}\mathbb{I}_V$ and a MASA of $\mathfrak{sl}(V)$. Without loss of generality, we will sometimes simply take $V = \mathbb{K}^n$. A MASA of $\mathfrak{sl}(n,\mathbb{R})$ is decomposable into a direct sum of indecomposable MASAs. The latter can either be absolutely indecomposable (AID), or indecomposable but not absolutely indecomposable (ID & NAID).

The NAID ones become decomposable after complexification. See e.g. [17]. Let us recall that a set of matrices is decomposable if and only if it commutes with some idempotent $Z \neq \mathbb{I}_{\mathbb{K}^n} =: \mathbb{I}_n$. Further recall that, an idempotent of a $\mathbb{K}$-algebra, is an element Z satisfying $Z^2 = Z$. The NAID MASAs of $\mathfrak{sl}(n,\mathbb{R})$ are those MASAs all of whose elements commute with M_s, given in (8). Upon complexification, M_s becomes $M_{\mathbb{C}} :=\mathrm{diag}(i\mathbb{I}_{\frac{n}{2}}, -i\mathbb{I}_{\frac{n}{2}})$, so that $Z := \frac{1}{2}(\mathbb{I}_n + iM_{\mathbb{C}})$ and $Z' := \frac{1}{2}(\mathbb{I}_n - iM_{\mathbb{C}})$ are idempotents that commute with the elements of the NAID MASAs. They exist only for n even and have the form $\mathbb{R}M_s \oplus \mathrm{MASA}(\mathfrak{sl}(\frac{n}{2},\mathbb{C}))$, and in our case, we only need MASAs of $\mathfrak{sl}(\frac{n}{2},\mathbb{C})$ of real dimension $n-2$. An indecomposable MASA of $\mathfrak{sl}(n,\mathbb{C})$ is always a maximal Abelian nilpotent subalgebra (MANS). The AID MASAs are MANS and they remain indecomposable after field extension. A MANS is represented by nilpotent matrices in any finite-dimensional representation. Following [5], a MANS $\mathcal{A}$ of $\mathfrak{sl}(n,\mathbb{K})$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is characterized by a Kravchuk signature (ν, m, μ), where $1 \leq \nu = n - \dim_{\mathbb{K}} \mathcal{A}\mathbb{K}^n$, $1 \leq \mu = \dim\ker\mathcal{A}$, where $\ker\mathcal{A} := \{x \in \mathbb{K}^n, ax = 0, \ \forall a \in \mathcal{A}\}$ and $m = n - \mu - \nu$.

Definition 3.4. We say that $\mathcal{A}$ is *of class* $p \geq 2$ if $a_1 a_2 \cdots a_p = 0$ for any p elements $a_1, \ldots, a_p \in \mathcal{A}$ and there exists some $a \in \mathcal{A}$ such that $a^{p-1} \neq 0$. We simply write $\mathcal{A}^{p-1} \neq 0$ and $\mathcal{A}^p = 0$. Set $\mathrm{Im}(\mathcal{A}^k) := \mathcal{A}^k\mathbb{K}^n$, $k \geq 1$.

3.4. Classification of 2-solvable Frobenius Lie algebras corresponding to class 2 MANS

In this section, we treat one of the extreme cases, namely the case of MANS of class 2. The other extreme case, the class n, will be treated together with the more general case of MASAs given by nonderogatory maps, in Section 4.

Lemma 3.5. *Let $\mathcal{A}$ be an $(n-1)$-dimensional MASA of $\mathfrak{sl}(n,\mathbb{R})$ such that $\mathcal{A}^{p+1} = 0$ and $\mathcal{A}^p \neq 0$, for some $p \geq 1$. If $\mathcal{G} := (\mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus \mathcal{A}) \ltimes \mathbb{R}^n$ is a Frobenius Lie algebra, then* $\dim$ *Im($\mathcal{A}^p$)= 1. In particular if $\mathcal{A}^2 = 0$ and $(\mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus \mathcal{A}) \ltimes \mathbb{R}^n$ is Frobenius, then $\nu = n-1$ and* $\dim$ *$Im(\mathcal{A}) = 1$.*

Proof. If $\dim \mathrm{Im}(\mathcal{A}^p) \geq 2$, then we would have $Im(\mathcal{A}^p) \cap \ker\alpha \neq 0$ for any $\alpha \in \mathcal{G}^*$. Choose $x \in Im(\mathcal{A}^p) \cap \ker\alpha$ such that $x \neq 0$. As any $v \in \mathcal{G}$ is of the form $v = k\mathbb{I}_{\mathbb{R}^n} + a + y$, where $k \in \mathbb{R}$, $a \in \mathcal{A}$, $y \in \mathbb{R}^n$, we would have $\partial\alpha(x,v) = \langle \alpha, kx + ax\rangle = 0$, $\forall v \in \mathcal{G}$. This would mean that $\partial\alpha$ is degenerate, for any $\alpha \in \mathcal{G}^*$. Thus $\mathcal{G}$ would not be Frobenius. In particular if $p = 1$ and $(\mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus \mathcal{A}) \ltimes \mathbb{R}^n$ is Frobenius, then $\dim Im(\mathcal{A}) = 1$ and hence $\nu = n-1$. ■

Now we are ready to prove the following classification theorem.

Theorem 3.6. *Up to conjugation, $\mathcal{A} := \mathcal{A}_{n,1}$ as in Example* 3.7*, is the unique MANS of class* 2*, of $\mathfrak{sl}(n,\mathbb{R})$, such that $(\mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus \mathcal{A}) \ltimes \mathbb{R}^n$ is Frobenius. Equivalently, it is the unique MANS of $\mathfrak{sl}(n,\mathbb{R})$ with Kravchuk signature $(n-1, 0, 1)$ such that $(\mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus \mathcal{A}) \ltimes \mathbb{R}^n$ is Frobenius. In other words, $\mathcal{G}_{n,1}$ as in Example* 3.7*, is the unique* 2*-solvable Frobenius Lie algebra given by a MANS of class* 2*.*

Proof. First, note that if $\mathcal{A} \subset \mathfrak{sl}(n,\mathbb{R})$ is an $(n-1)$-dimensional MASA such that $\mathcal{A}^2 = 0$ and $(\mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus \mathcal{A}) \ltimes \mathbb{R}^n$ is a Frobenius Lie algebra, then from Lemma 3.5, $\dim(Im(\mathcal{A})) = 1$. Hence $\nu = n-1$ and $\mu = 1$. Conversely, if $\nu = n-1$, since $\mathcal{A}$ is nilpotent, we necessarily have $\mathcal{A}^2 = 0$.

Now, under the assumption that $\mathcal{A}^2 = 0$ and $(\mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus \mathcal{A}) \ltimes \mathbb{R}^n$ is a Frobenius Lie algebra, let $\hat{e}_1 \in \mathbb{R}^n$ such that $\mathcal{A}\mathbb{R}^n = \mathbb{R}\hat{e}_1$. We have $\mathcal{A}\hat{e}_1 = 0$ and $\ker \mathcal{A} = \mathbb{R}\hat{e}_1$. Thus, in any basis of $\mathbb{R}^n$ whose first vector is $\hat{e}_1$, the elements of $\mathcal{A}$ are linear combinations of $E_{1,j}$, $j = 2, \dots, n$ and so $\mathcal{A}$ assumes the same form as the MASA $\mathcal{A}_{n,1}$ of $\mathfrak{sl}(n, \mathbb{R})$ of Example 3.7. Consequently, $\mathcal{A}$ is conjugate to $\mathcal{A}_{n,1}$ in $\mathfrak{gl}(n, \mathbb{R})$ and $(\mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus \mathcal{A}) \ltimes \mathbb{R}^n$ is isomorphic to the Lie algebra $\mathcal{G}_{n,1}$, see Example 3.7). ■

3.5. Some important examples

We propose several families of pairwise non-isomorphic (resp. non-conjugate) 2-solvable Frobenius Lie algebras (resp. MASAs).

Example 3.7. For any integer $n \geq 2$, we construct below a family of $(n-1)$ 2-solvable Frobenius Lie algebras $\mathcal{G}_{n,p} := \mathfrak{B}_{n,p} \ltimes \mathbb{R}^n$, $p = 1, \dots, n-1$. The $\mathfrak{B}_{n,p}$'s are pairwise non-conjugate commutative algebras of polynomials in $n-1, n-2, \dots, 2, 1$ $n \times n$ matrices, respectively. For a given number p, with $1 \leq p \leq n-1$, define $M_{n,p} := \sum_{l=1}^{p} E_{l,l+1}$, so that $M_{n,p}^{j-1} = \sum_{i=1}^{p-j+2} E_{i,i+j-1}$, for $2 \leq j \leq p+1$, and $M_{n,p}^{p+1} = 0$. Let $\mathfrak{B}_{n,p}$ be the Abelian subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ spanned by e_j, $j = 1, ..., n$, where $e_j := M_{n,p}^{j-1}$, $j = 1, \dots, p+1$, and for $j = p+2, \dots, n$, $e_j := E_{1,j}$. In the canonical basis $(\tilde{e}_1, \dots, \tilde{e}_n)$ of $\mathbb{R}^n$, with dual basis $(\tilde{e}_1^*, \dots, \tilde{e}_n^*)$, we have

$$\begin{aligned} M_{n,p}\tilde{e}_1 &= M_{n,p}\tilde{e}_q = 0, \quad q \geq p+2, \\ M_{n,p}^k \tilde{e}_q &= M_{n,p}^{k-1}\tilde{e}_{q-1} = M_{n,p}^{k-j}\tilde{e}_{q-j}, \quad 1 \leq j \leq \min(k, q-1), \forall q, \ 2 \leq q \leq p+1. \end{aligned}$$

So if $k \geq q$, then $M_{n,p}^k \tilde{e}_q = 0$, and if $k \leq q-1 \leq p$, then $M_{n,p}^k \tilde{e}_q = \tilde{e}_{q-k}$. As one can see, we have $\tilde{e}_1^* \circ e_j = \tilde{e}_1^* \circ M_p^{j-1} = \tilde{e}_j^*$, $j = 1, \dots, p+1$ and $\tilde{e}_1^* \circ e_j = \tilde{e}_1^* \circ E_{1,j} = \tilde{e}_j^*$, for $j = p+2, \dots, n$. This shows that $\tilde{e}_1^*$ has an open orbit for the contragredient action of $\mathfrak{B}_{n,p}$ on $(\mathbb{R}^n)^*$. For any $M := x_1 e_1 + \dots + x_n e_n \in \mathfrak{B}_{n,p}$, we have $\chi_M(\lambda) = (x_1 - \lambda)^n$. But the equality $(x_1 - M)^{p+1} = 0$ implies that $\chi_{\min,M}(\lambda)$ divides $(x_1 - \lambda)^{p+1}$. So $\mathfrak{B}_{n,p}$ is not the space of polynomials of a nonderogatory matrix, unless $p = n-1$. In the basis $(e_j, e_{n+j} = \tilde{e}_j, \ j = 1, \dots, n)$ the Lie bracket of the 2-solvable Frobenius Lie algebra $\mathcal{G}_{n,p} := \mathfrak{B}_{n,p} \ltimes \mathbb{R}^n$, is given by the following table, $j, k, q \in \{1, 2, \dots, n\}$,

$$\begin{aligned} &[e_1, e_{n+j}] = e_{n+j} && j = 1, \dots, n, \\ &[e_p, e_{n+q}] = 0, && \text{if } p \geq q+1, \\ &[e_k, e_{n+q}] = e_{n+q-k+1} && \text{if } 1 \leq k \leq q \leq p+1, \\ &[e_q, e_{n+q}] = e_{n+1}, && \text{for } q = p+2, \dots, n. \end{aligned} \tag{2}$$

The form e_{n+1}^* is a Frobenius functional on $\mathcal{G}_{n,p}$ and $\partial e_{n+1}^* = -\sum_{i=1}^{n} e_j^* \wedge e_{n+j}^*$. Note that $\mathcal{G}_{n,p}$ and $\mathcal{G}_{n,q}$ are isomorphic if and only if $p = q$. Indeed, although each $\mathcal{G}_{n,p}$ has a codimension 1 nilradical $\mathcal{N}_{n,p} = \operatorname{span}(e_2, \dots, e_{2n})$, the derived ideal $[\mathcal{N}_{n,p}, \mathcal{N}_{n,p}] = \operatorname{span}(e_{n+1}, \dots, e_{n+p})$ is p-dimensional and $\mathcal{N}_{n,p}$ is $(p+1)$-nilpotent. So $\mathcal{N}_{n,p}$ and $\mathcal{N}_{n,q}$ are not isomorphic whenever $p \neq q$ and hence, neither are the Lie algebras $\mathcal{G}_{n,p}$ and $\mathcal{G}_{n,q}$. Thus $\mathfrak{B}_{n,p}$ and $\mathfrak{B}_{n,q}$ are not conjugate. The family $(\mathcal{G}_{n,p})_{1 \leq p \leq n-1}$, has two special cases.

(1) The case $\mathfrak{B}_{n,n-1} = \mathbb{R}[M_0]$, where $M_0 = \sum_{j=1}^{n-1} E_{j,j+1}$ is nonderogatory. See Section 4 for a full classification of 2-solvable Frobenius Lie algebras $\mathbb{R}[M] \ltimes V$,

for nonderogatory $M \in \mathfrak{gl}(V)$. To keep the same notations as in Section 4, set $\mathfrak{D}_0^n = \mathcal{G}_{n,n-1}$.

(2) As regards the case $p = 1$, the nilradical $\mathcal{N}_{n,1}$ of $\mathcal{G}_{n,1}$ is the $(2n-1)$-dimensional Heisenberg Lie algebra $\mathcal{H}_{2n-1}$:=span$(e_j := E_{1,j}, j = 2, \dots, n, e_{n+k},\ k = 1, \dots, n)$, with Lie brackets $[e_j, e_{n+j}] = e_{n+1},\ j = 2, \dots, n$. Note also that the $(n-1)$ spaces $\mathcal{A}_{n,p}$ =span$(e_2, e_3, \dots, e_n)$ are all $(n-1)$-dimensional Abelian subalgebras of $\mathfrak{sl}(n, \mathbb{R})$, which according to Theorem 3.1 and Proposition 3.3, are pairwise non-conjugate MASAs of $\mathfrak{sl}(n, \mathbb{R})$. In particular, $\mathcal{A}_{n,1}$ is spanned by $E_{1,j}$, $j = 2, \dots, n$ and so $\mathcal{A}_{n,1}^2 = 0$.

Example 3.8. For $n \geq 3$, let $\mathcal{P}_{n,p}$ be the Abelian subalgebra of $\mathfrak{sl}(n, \mathbb{R})$ defined as

$$\mathcal{P}_{n,p} = \Big\{ \sum_{j=2}^{p} m_j (E_{1,j} + E_{j,n}) + \sum_{j=p+1}^{n} m_j E_{1,j},\ m_2, \dots, m_n \in \mathbb{R} \Big\},$$

$p = 2, 3, \dots, n-1$. On $\mathcal{C}_{n,p} := \mathcal{P}_{n,p} \oplus \mathbb{R}\mathbb{I}_{\mathbb{R}^n}$, set $e_1 := \mathbb{I}_{\mathbb{R}^n}$ and $e_j := E_{1,j} + E_{j,n}$, $j = 2, \dots, p$, $e_{p+i} = E_{1,p+i}$, $i = 1, \dots, n-p$. Let $(\tilde{e}_j)$ be the canonical basis of $\mathbb{R}^n$. For any $M = m_1 e_1 + \dots + m_n e_n$, we have

$$\chi_M(X) = (m_1 - X)^n,\ (m_1 - M)^2 = (m_2^2 + \dots + m_p^2) E_{1,n} \text{ and } (m_1 - M)^3 = 0.$$

So, up to a scaling, $\chi_{\min,M} = (m_1 - X)^3$. Thus for any $n \geq 4$, the algebra $\mathcal{C}_{n,p}$ contains no nonderogatory matrix. We have $\tilde{e}_1^* \circ e_1 = \tilde{e}_1^*$, $\tilde{e}_1^* \circ e_j = \tilde{e}_1^* \circ (E_{1,j} + E_{j,n}) = \tilde{e}_1^* \circ E_{1,j} = \tilde{e}_j^*$, $\tilde{e}_1^* \circ e_{p+i} = \tilde{e}_1^* \circ E_{1,p+i} = \tilde{e}_{p+i}^*$, $i = 1, \dots, n-p$, $j = 2, \dots, p$. So $\tilde{e}_1^*$ has an open orbit. Hence $\mathcal{C}_{n,p}$ and $\mathcal{P}_{n,p}$ are MASAs of $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{sl}(n, \mathbb{R})$, respectively (Theorem 3.1). The Lie bracket of $\mathfrak{h}_{n,p} := \mathcal{C}_{n,p} \ltimes \mathbb{R}^n$, in the basis $(e_j, e_{n+j} := \tilde{e}_j, j = 1, \dots, n)$, is

$$\begin{aligned} [e_1, e_{n+j}] &= e_{n+j}, \quad j = 1, \dots, n, \\ [e_j, e_{n+j}] &= e_{n+1}, \quad j = 2, \dots, n, \\ [e_k, e_{2n}] &= e_{n+k}, \quad k = 2, \dots, p, \end{aligned}$$

so $\partial e_{n+1}^* = -\sum_{j=1}^{n} e_j^* \wedge e_{n+j}^*$ is nondegenerate, $\mathfrak{h}_{n,p}$ is a Frobenius Lie algebra. The nilradical $\mathfrak{n}_{n,p}$:=span$(e_2, \dots, e_{2n})$ of $\mathfrak{h}_{n,p}$ is of codimension 1. It is the semidirect sum $\mathfrak{n}_{n,p} = \mathbb{R}e_{2n} \ltimes \big(\mathcal{H}_{2n-3} \oplus \mathbb{R}e_n\big)$ of the line $\mathbb{R}e_{2n}$ and the so-called Abbena Lie algebra $\mathcal{H}_{2n-3} \oplus \mathbb{R}e_n$ where the former acts on the latter by nilpotent derivations. Here, $\mathcal{H}_{2n-3}$ is the $(2n-3)$-dimensional Heisenberg Lie algebra $\mathcal{H}_{2n-3} = \text{span}(e_2, \dots, e_{n-1}, e_{n+1}, \dots, e_{2n-1})$. The derived ideal $[\mathfrak{n}_{n,p}, \mathfrak{n}_{n,p}]$ is p-dimensional and spanned by $(e_{n+1}, \dots, e_{n+p})$. So if $p \neq q$, then $\mathfrak{n}_{n,p}$ and $\mathfrak{n}_{n,q}$ are not isomorphic, hence $\mathcal{P}_{n,p}$ and $\mathcal{P}_{n,q}$ are not conjugate, $\mathfrak{h}_{n,p}$ and $\mathfrak{h}_{n,q}$ are not isomorphic. Note that $\mathfrak{n}_{n,p}$ is 3-step nilpotent. So, for any (n, p) with $n \geq 4$ and $3 \leq p \leq n-1$, $\mathfrak{n}_{n,p}$ is isomorphic to none of the nilradicals $\mathcal{N}_{n,q}$, $3 \leq q \leq n-1$, of Example 3.7, as they are all $(q+1)$-step nilpotent. When $p = 2$, we note that the linear map $\psi : \mathcal{G}_{n,2} \to \mathfrak{h}_{n,2}$,

$$\begin{aligned} \psi(e_m) &= e'_m,\ 1 \leq m \leq 2n,\ m \notin \{3, n, n+3, 2n\}, \\ \psi(e_3) &= e'_n,\ \psi(e_n) = e'_3,\ \psi(e_{n+3}) = e'_{2n},\ \psi(e_{2n}) = e'_{n+3}, \end{aligned} \tag{3}$$

is an isomorphism of Lie algebras, for any $n \geq 3$, where $\mathcal{G}_{n,2}$ is as in Example 3.7 and the above basis of $\mathfrak{h}_{n,2}$ has been renamed $(e'_1, \dots, e'_{2n})$. Hence, altogether, the

Lie algebras $\mathfrak{h}_{n,p}$ are not isomorphic to any of the Lie algebras $\mathcal{G}_{n,q}$ of Example 3.7, unless $p = q = 2$. Thus none of the MASAs $\mathfrak{B}_{n,q}$ of Example 3.7 is conjugate to $\mathcal{C}_{n,p}$, unless $p = q = 2$.

Example 3.9. Let $L'_{n,n}$ be the following Abelian subspace of $\mathfrak{sl}(n,\mathbb{R})$,

$$L'_{n,n} = \{M := \sum_{j=2}^{n} m_j(E_{1,j} + E_{n-j+1,n}) \ , \ m_j \in \mathbb{R}, j = 2, \dots, n\}. \tag{4}$$

On $\mathfrak{B}'_{n,n} := \mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus L'_{n,n}$, we set $e_1 := \mathbb{I}_{\mathbb{R}^n}$, $e_j := E_{1,j} + E_{n-j+1,n}$, $j = 2, \dots, n$. For any $M = m_1e_1 + \cdots + m_ne_n \in \mathfrak{B}$, we have $\chi(X) = (m_1 - X)^n$. We also have $\chi_{\min,M} = (m_1 - X)^3$, since $(m_1 - M)^3 = 0$ and $(m_1 - M)^2 \neq 0$. Thus, for any $n \geq 4$, $\mathfrak{B}'_{n,n}$ contains no nonderogatory matrix. We have $\tilde{e}_1^* \circ e_j = \tilde{e}_j^*$, $j = 1, \dots, n-1$, $\tilde{e}_1^* \circ e_n = 2\tilde{e}_n^*$. So $\tilde{e}_1$ has an open orbit. According to Theorem 3.1, $\mathfrak{B}'_{n,n}$ and $L'_{n,n}$ are MASAs of $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{sl}(n,\mathbb{R})$, respectively. In the basis $(e_j, e_{n+j} := \tilde{e}_j, j = 1, \dots, n)$, the Lie bracket of $\mathcal{G}'_{n,n} := \mathfrak{B}'_{n,n} \ltimes \mathbb{R}^n$, is

$$\begin{aligned}
[e_1, e_{n+j}] &= e_{n+j} \ , \ j = 1, \dots, n, \\
[e_j, e_{n+j}] &= e_{n+1}, \ [e_j, e_{2n}] = e_{2n-j+1}, \ j = 2, \dots, n-1, \ \ [e_n, e_{2n}] = 2e_{n+1} \ .
\end{aligned}$$

As one sees, $\partial e_{n+1}^* = -\sum_{j=1}^{n-1} e_j^* \wedge e_{n+j}^* - 2e_n^* \wedge e_{2n}^*$ is nondegenerate, so $\mathcal{G}'_{n,n}$ is a Frobenius Lie algebra. It also has a codimension 1 nilradical $\mathcal{N}'_{n,n} = span(e_2, \dots, e_{2n})$, which is the semidirect sum of $\mathbb{R}e_{2n}$ and $\mathcal{H}_{2n-3} \oplus \mathbb{R}e_n$, where the former acts on the latter by nilpotent derivations. However, $[\mathcal{N}'_{n,n}, \mathcal{N}'_{n,n}] = \mathrm{span}(e_{n+1}, \dots, e_{2n-1})$ is $(n-1)$-dimensional. So $\mathcal{G}'_{n,n}$ is not isomorphic to any of the Lie algebras $\mathcal{G}_{n,p}$ of Example 3.7 and $\mathfrak{h}_{n,p}$ of Example 3.8. From Theorem 3.1, none of the MASAs $\mathfrak{B}_{n,p}$ in Example 3.7 or $\mathcal{C}_{n,p}$ in Example 3.8, is conjugate to $\mathfrak{B}'_{n,n}$.

Remark 3.10. From Theorem 3.1, the classification of 2-solvable Frobenius Lie algebras is equivalent to that of n-dimensional MASAs of $\mathfrak{gl}(n,\mathbb{K})$ acting on $(\mathbb{K}^n)^*$ with an open an orbit. However, not all n-dimensional MASAs have open orbit in $(\mathbb{K}^n)^*$. Indeed, for $n \geq 3$, the algebra $\mathfrak{B}_n := \mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus L_n$, is a MASA of $\mathfrak{gl}(n,\mathbb{R})$, where $L_n := \left\{ \sum_{i=1}^{n-1} k_{i,n}E_{i,n}, \ k_{i,n} \in \mathbb{R}, \ i = 1, \dots, n-1 \right\}$ is a MASA of $\mathfrak{sl}(n,\mathbb{R})$. To see that, consider $b = \sum_{p,q=1}^{n} t_{p,q}E_{p,q} \in \mathfrak{gl}(n,\mathbb{R})$, with $t_{p,q} \in \mathbb{R}, \ p,q = 1 \dots, n$. We have $[E_{i,n}, b] = \sum_{l=1}^{n-1} t_{n,l}E_{i,l} + (t_{n,n} - t_{i,i})E_{i,n} - \sum_{1 \leq k \leq n, \ k \neq i} t_{k,i}E_{k,n}$, for any $i = 1, \dots, n-1$. So the relation $[b, a] = 0$, for any $a \in \mathfrak{B}_n$, is equivalent to the following, valid for any i with $1 \leq i \leq n-1$: $t_{i,i} = t_{n,n}$ and for any k with $1 \leq k \leq n$ and $k \neq i$, one has $t_{k,i} = 0$. In other words $b = t_{n,n}\mathbb{I}_{\mathbb{R}^n} + t_{1,n}E_{1,n} + \cdots + t_{n-1,n}E_{n-1,n}$, or equivalently, b is an element of $\mathfrak{B}_n$. This simply means that $\mathfrak{B}_n$ is a MASA of $\mathfrak{gl}(n,\mathbb{R})$. The orbit $\{\alpha \circ a, a \in \mathfrak{B}_n\}$ of any $\alpha \in (\mathbb{R}^n)^*$ is at most 2-dimensional and spanned by α and $\tilde{e}_n^*$. More precisely, let $(\tilde{e}_1, \dots, \tilde{e}_n)$ be the canonical basis of $\mathbb{R}^n$ and let $\alpha = s_1\tilde{e}_1^* + \cdots + s_n\tilde{e}_n^* \in (\mathbb{R}^n)^*$, where $s_1, \dots, s_n \in \mathbb{R}$, then for any $k_1, k_{1,n}, \dots, k_{n-1,n} \in \mathbb{R}$ and $a = k_1\mathbb{I}_{\mathbb{R}^3} + k_{1,n}E_{1,n} + \cdots + k_{n-1,n}E_{n-1,n} \in \mathfrak{B}_n$, one has $\tilde{e}_i^* \circ a = k_1\tilde{e}_i^* + k_{i,n}\tilde{e}_n^*$, $i = 1, \dots, n-1$ and $\tilde{e}_n^* \circ a = k_1\tilde{e}_n^*$, so that

$$\alpha \circ a = k_1\alpha + (k_{1,n}s_1 + k_{2,n}s_2 + \cdots + k_{n-1,n}s_{n-1})e_n^*.$$

For $n = 3$, L_3 coincides with $L_{2,4}$ in the list of MASA of $\mathfrak{sl}(n,\mathbb{R})$ supplied in [25].

3.6. Cartan subalgebras of $\mathfrak{sl}(n,\mathbb{R})$

Recall that a Cartan subalgebra $\mathfrak{h}$ of a Lie algebra $\mathcal{G}$ is a nilpotent subalgebra which is equal to its own normalizer $\mathcal{N}_{\mathcal{G}}(\mathfrak{h})$ in $\mathcal{G}$. Cartan subalgebras of semisimple Lie algebras must necessarily be Abelian, more precisely they are MASAs which contain only semisimple elements. They have been extensively studied by several authors amongst which Cartan, Harish-Chandra ([12]), Kostant ([14]), Sugiura ([23]), etc. Further recall that a Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathcal{G}$, splits into a direct sum $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$ of two subalgebras $\mathfrak{h}^+$ and $\mathfrak{h}^-$, respectively, called its toroidal and its vector parts, such that $\mathfrak{h}^+$ is only made of elements of $\mathfrak{h}$ whose adjoint operator (as a linear operator of $\mathcal{G}$) only has purely imaginary eigenvalues and adjoint operators of elements of $\mathfrak{h}^-$ have only real eigenvalues. See e.g. [23]. We propose the following characterization of Cartan subalgebras of $\mathfrak{sl}(n,\mathbb{R})$.

Theorem 3.11. *Let $\mathfrak{h} \subset \mathfrak{sl}(n,\mathbb{R})$ be a subalgebra. Set $\mathfrak{B}_{\mathfrak{h}} := \mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus \mathfrak{h}$. The following are equivalent.* (a) *$\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}(n,\mathbb{R})$, with toroidal and vector parts of respective dimensions p and q.* (b) *$\mathfrak{B}_{\mathfrak{h}} = \mathbb{R}[M]$, for some nonderogatory $M \in \mathfrak{gl}(n,\mathbb{R})$ with distinct $2p$ complex and q real eigenvalues, where $2p+q=n$.* (c) *$\mathfrak{B}_{\mathfrak{h}} \ltimes \mathbb{R}^n$ is the direct sum of p copies of $\mathfrak{aff}(\mathbb{C})$ and q copies of $\mathfrak{aff}(\mathbb{R})$, where $\mathfrak{aff}(\mathbb{R})$ and $\mathfrak{aff}(\mathbb{C})$ are as in Examples* 4.1.1 *and* 4.1.2.

Theorem 3.11 is proved in Section 4.6. Theorem 4.19 gives a complementary characterization.

4. Classification of 2-solvable Frobenius Lie algebras given by nonderogatory linear maps

4.1. Some key examples

4.1.1. The Lie algebras $\mathfrak{D}_0^n$. (1) Consider the simplest nonderogatory linear map $\psi := \mathbb{I}_{\mathbb{R}}$. One gets the Lie algebra $\mathcal{G}_\psi = \mathfrak{aff}(\mathbb{R})$ of the group of affine motions of $\mathbb{R}$. In the basis (e_1, e_2), with $e_1 = \psi^0$ and $e_2 \in \mathbb{R}$, the Lie bracket is $[e_1, e_2] = e_2$. The 2-form $\partial e_2^* = -e_1^* \wedge e_2^*$ is nondegenerate. (2) In the canonical basis $(\tilde{e}_j)$ of $\mathbb{R}^n$, let $M_0 \in \mathfrak{gl}(n,\mathbb{R})$ be the principal nilpotent matrix

$$M_0 = \sum_{i=1}^{n-1} E_{i,i+1}. \tag{5}$$

It is nonderogatory, as $\chi_{\min,M_0}(X) = \chi_{M_0}(X) = X^n$. We use the notation $e_j = M_0^{j-1}$, $j = 1,\dots,n$. Set $e_{n+j} := M_0^{j-1}\bar{x} = \tilde{e}_{n-j+1}$, where $\bar{x} := \tilde{e}_n$. Then (e_{n+j}) is a basis of $\mathbb{R}^n$. In the basis $(e_1, e_2, \dots, e_{2n})$, the Lie bracket of $\mathcal{G}_{M_0} = \mathbb{R}[M_0] \ltimes \mathbb{R}^n$, is

$$[e_i, e_{n+j}] = e_{n+j+i-1}, \ i,j = 1,\dots,n, \tag{6}$$

where we use the convention $e_{2n+k} = 0$, whenever $k \geq 1$. In particular we have $[e_1, e_{n+j}] = e_{n+j}$, $j = 1,\dots,n$, and $[e_j, e_{2n}] = 0$, $j = 2,\dots,n$. We write $\mathfrak{D}_0^n$ instead of $\mathcal{G}_{M_0}$. The following 2-form on $\mathfrak{D}_0^n$, is non-degenerate,

$$\partial e_{2n}^* = -\sum_{j=1}^{n} e_j^* \wedge e_{2n-j+1}^* = -\sum_{j=1}^{n} e_{n-j+1}^* \wedge e_{n+j}^* \ . \tag{7}$$

Note that $\mathfrak{D}_0^n$ has an n-nilpotent codimension 1 nilradical $\mathcal{N} = \text{span}(e_2, \dots, e_{2n})$.

4.1.2. The Lie algebra $\mathfrak{D}_{0,1}^n$ (A) Let $\psi \in \mathfrak{gl}(2,\mathbb{R})$ with $\psi(\tilde{e}_1) = \tilde{e}_2$, $\psi(\tilde{e}_2) = -\tilde{e}_1$. One has $\chi_{\min,\psi}(X) = \chi_\psi(X) = 1 + X^2$. So ψ has the 2 complex eigenvalues i, $-i$. Of course, $\mathbb{R}[\psi] = \mathbb{R}\mathbb{I}_{\mathbb{R}^2} \oplus \mathbb{R}\psi$ is a MASA of $\mathfrak{gl}(2,\mathbb{R})$ and $\tilde{e}_1^* \circ \psi^0 = \tilde{e}_1^*$, $\tilde{e}_1^* \circ \psi = -\tilde{e}_2^*$. In the basis $(e_1, \ldots, e_4)$ of $\mathcal{G}_\psi := \mathbb{R}[\psi] \ltimes \mathbb{R}^2$, with $e_1 = \psi^0$, $e_2 = \psi$, $e_3 := \tilde{e}_1$, $e_4 := \tilde{e}_2$, the Lie bracket reads $[e_1, e_3] = e_3, [e_1, e_4] = e_4, [e_2, e_3] = e_4, [e_2, e_4] = -e_3$. The 2-form $\partial e_3^* = -e_1^* \wedge e_3^* + e_2^* \wedge e_4^*$ is nondegenerate on $\mathcal{G}_\psi$. Note that $\mathcal{G}_\psi$ is the Lie algebra

$$\mathfrak{aff}(\mathbb{C}) = \left\{ N(z_1, z_2) := \begin{pmatrix} z_1 & z_2 \\ 0 & 0 \end{pmatrix}, z_j \in \mathbb{C},\ j = 1, 2 \right\}$$

of the group of affine motions of the complex line $\mathbb{C}$, looked at as a real Lie algebra, e.g. with the identifications $e_1 = N(1, 0)$, $e_2 = N(i, 0)$, $e_3 = N(0, 1)$, $e_4 = N(0, i)$. We set $\mathfrak{D}_{0,1}^2 := \mathfrak{aff}(\mathbb{C})$.

(B) Let $M_{0,1} = E_{2,1} - E_{1,2} + E_{4,3} - E_{3,4} + E_{1,3} + E_{2,4} \in \mathfrak{gl}(4,\mathbb{R})$. One has $\chi_{\min,M_{0,1}}(X) = \chi_{M_{0,1}}(X) = (X^2 + 1)^2$. So i and $-i$ are the only eigenvalues of $M_{0,1}$ and $\mathbb{R}[M_{0,1}]$ is a MASA of $\mathfrak{gl}(4,\mathbb{R})$. In the basis $(e_1, e_2, \ldots, e_8)$ of $\mathcal{G}_{M_{0,1}}$, with $e_1 := \mathbb{I}_{\mathbb{R}^4}$, $e_2 := M_{0,1}$, $e_3 := M_{0,1}^2$, $e_4 := M_{0,1}^3$, $e_{4+j} = \tilde{e}_j$, $j = 1, \ldots, 4$, the Lie bracket reads

$$\begin{array}{llll}
[e_1, e_l] = e_l, & l = 5, 6, 7, 8, & & \\
[e_2, e_5] = e_6, & [e_2, e_6] = -e_5, & [e_2, e_7] = e_5 + e_8, & [e_2, e_8] = e_6 - e_7, \\
[e_3, e_5] = -e_5, & [e_3, e_6] = -e_6, & [e_3, e_7] = 2e_6 - e_7, & [e_3, e_8] = -2e_5 - e_8, \\
[e_4, e_5] = -e_6, & [e_4, e_6] = e_5, & [e_4, e_7] = -3e_5 - e_8, & [e_4, e_8] = -3e_6 + e_7.
\end{array}$$

Note that both $(e_7, M_{0,1}e_7 = e_5 + e_8, M_{0,1}^2 e_7 = 2e_6 - e_7, M_{0,1}^3 e_7 = -3e_5 - e_8)$ and $(e_8, M_{0,1}e_8 = e_6 - e_7, M_{0,1}^2 e_8 = -2e_5 - e_8, M_{0,1}^3 e_8 = -3e_5 + e_7)$ are bases of $\mathbb{R}^4$. The 2-form $\partial e_5^* = -e_1^* \wedge e_5^* + e_2^* \wedge (e_6^* - e_7^*) + e_3^* \wedge (e_5^* + 2e_8^*) - e_4^* \wedge (e_6^* - 3e_7^*)$ is nondegenerate.

(C) This generalizes (see Section 4.4.3) to a $2n$-dimensional 2-solvable Frobenius Lie algebra $\mathfrak{D}_{0,1}^n := \mathcal{G}_{M_{0,1}} = \mathbb{R}[M_{0,1}] \ltimes \mathbb{R}^n$, for any even $n \geq 4$, where the nonderogatory $M_{0,1} \in \mathfrak{gl}(n,\mathbb{R})$ is defined, in the canonical basis $(\tilde{e}_1, \ldots, \tilde{e}_n)$ of $\mathbb{R}^n$, as

$$M_{0,1} = M_s + M_n, \quad M_s = -\sum_{j=0}^{\frac{n}{2}-1} (E_{2j+1,2j+2} - E_{2j+2,2j+1}), \quad M_n = \sum_{j=1}^{n-2} E_{j,j+2}. \tag{8}$$

Hence, $\chi_{\min,M_{0,1}}(X) = \chi_{M_{0,1}}(X) = (X^2 + 1)^{\frac{n}{2}}$, so i and $-i$ are the only ($\frac{n}{2}$ times repeated complex conjugate) eigenvalues of $M_{0,1}$ and $\mathbb{R}[M_{0,1}]$ is a MASA of $\mathfrak{gl}(n,\mathbb{R})$. In the basis $(e_1, e_2, \ldots, e_{2n})$ of $\mathfrak{D}_{0,1}^n$, with $e_j := (M_{0,1})^{j-1}$, $e_{n+j} = \tilde{e}_j$, $j = 1, \ldots, n$, the Lie bracket reads, for any $j, l = 1, \ldots, n$,

$$[e_j, e_{n+l}] = (M_{0,1})^{j-1} \tilde{e}_l\ . \tag{9}$$

4.2. The Classification Theorem

In Theorem 4.3, we completely classify all 2-solvable Frobenius Lie algebras of the form $\mathcal{G}_\phi := \mathbb{R}[\phi] \ltimes V$, where $\phi \in \mathfrak{gl}(V)$ is nonderogatory and V a real vector space of dimension n. In particular, we show that the Lie algebras $\mathfrak{D}_0^p$, $\mathfrak{D}_{0,1}^{2p}$, where $p \geq 1$ is an integer, discussed in Section 4.1, are the building blocks that make up, in a trivial way (direct sums), the Lie algebras $\mathcal{G}_\phi$. As in Examples 4.1.1, 4.1.2, the notations $\mathfrak{D}_0^1 := \mathfrak{aff}(\mathbb{R})$ and $\mathfrak{D}_{0,1}^2 := \mathfrak{aff}(\mathbb{C})$, are implicitly adopted. It is obvious that, if $z, \bar{z}$ are two complex conjugate eigenvalues of ϕ, then $(X - Re(z))^2 + Im(z)^2$ divides $\chi_\phi(X)$ (Lemma 4.2).

Definition 4.1. We say that 2 complex conjugate eigenvalues $z, \bar{z}$ of $\phi \in \mathfrak{gl}(V)$ are of *multiplicity* m, if m is the largest integer such that $\big((X - Re(z))^2 + Im(z)^2\big)^m$ is a factor of $\chi_\phi(X)$.

We call the following the Factorization Lemma.

Lemma 4.2. (Factorization Lemma) *Let V be a real vector space,* $\dim V = n$*. Suppose a nonderogatory $\phi \in \mathfrak{gl}(V)$ has p distinct real eigenvalues λ_j, with multiplicity k_j, $j = 1, \ldots, p$ and $2q$ complex eigenvalues $\lambda_{p+l}, \bar{\lambda}_{p+l}$, with multiplicity k_{p+l}, $l = 1, \ldots, q$, where the equality $n = k_1 + \cdots + k_p + 2(k_{p+1} + \cdots + k_{p+q})$ holds. Then the characteristic polynomial of ϕ factorizes as*

$$\chi_\phi(X) = \prod_{j=1}^{p} (X - \lambda_j)^{k_j} \prod_{l=1}^{q} \Big((X - Re(\lambda_{p+l}))^2 + Im(\lambda_{p+l})^2\Big)^{k_{p+l}}. \tag{10}$$

In particular, if ϕ has only 2 eigenvalues which are both complex λ, $\bar{\lambda}$, then

$$\chi_\phi(X) = \big((Re(\lambda) - X)^2 + Im(\lambda)^2\big)^{\frac{n}{2}}. \tag{11}$$

Proof. Suppose a nonderogatory $\phi \in \mathfrak{gl}(V)$ has only 2 eigenvalues which are both complex λ, $\bar{\lambda}$. Since $(X - \lambda)(X - \bar{\lambda}) = (X - Re(\lambda))^2 + Im(\lambda)^2$ divides $\chi_\phi(X)$, we therefore write $\chi_\phi(X) = \big((X - Re(\lambda))^2 + Im(\lambda)^2\big)P_1(X)$ where $P_1(X)$ is a polynomial of degree $n - 2$, with real coefficients. As $\mathbb{C}$ is a closed field, $P_1(X)$ admits some complex zeros, bound to be λ, $\bar{\lambda}$ since they are the only zeros of $\chi_\phi(X)$, by hypothesis. We now re-write $\chi_\phi(X)$ as $\chi_\phi(X) = \big((X - Re(\lambda))^2 + Im(\lambda)^2\big)^2 P_2(X)$ where $P_2(X)$ is a polynomial of degree $n-4$, with real coefficients. The result follows by inductively reapplying the same process to P_2. The proof of the general case immediately follows by applying the Primary Decomposition Theorem to $\chi_\phi(X)$ to reduce the problem to the cases where ϕ admits a unique eigenvalue or only two eigenvalues which are both complex and conjugate, as above. ∎

Note that Lemma 4.2 is still valid even if ϕ is not nonderogatory. In Theorem 4.3 below, V is a real vector space of dimension n.

Theorem 4.3. *Let $\phi \in \mathfrak{gl}(V)$ be nonderogatory with p real distinct eigenvalues λ_j with multiplicities k_j, $j = 1, \ldots, p$ and $2q$ distinct complex eigenvalues $z_l, \bar{z}_l$, with multiplicities m_l, $l = 1, \ldots, q$ where $n = k_1 + \cdots + k_p + 2(m_1 + \cdots + m_q)$. Then the Lie algebra $\mathcal{G}_\phi := \mathbb{R}[\phi] \ltimes V$ is isomorphic to the direct sum of the Lie algebras $\mathfrak{D}_0^{k_1}, \ldots, \mathfrak{D}_0^{k_p}, \mathfrak{D}_{0,1}^{2m_1}, \ldots, \mathfrak{D}_{0,1}^{2m_q}$. In particular,*

(a) *if $p = n$, then $\mathcal{G}_\phi$ is isomorphic to the direct sum of n copies of $\mathfrak{aff}(\mathbb{R})$,*

(b) *if $p = 1$ and $q = 0$, then $\mathcal{G}_\phi$ is isomorphic to $\mathfrak{D}_0^n$,*

(c) *if $2q = n$, then $\mathcal{G}_\phi$ is isomorphic to the direct sum of $\frac{n}{2}$ copies of $\mathfrak{aff}(\mathbb{C})$,*

(d) *if $p = 0$ and $q = 1$, then $\mathcal{G}_\phi$ is isomorphic to $\mathfrak{D}_{0,1}^n$.*

The following is a direct corollary of Theorem 4.3.

Corollary 4.4. *Let $\phi \in \mathfrak{gl}(V)$ be nonderogatory. The Lie algebra $\mathcal{G}_\phi$ is indecomposable if and only if one of the following holds true:*

(a) *ϕ has a unique eigenvalue, in which case $\mathcal{G}_\phi$ is completely solvable, or*

(b) *ϕ has only 2 eigenvalues which are both complex.*

The rest of this section and Sections 4.3, 4.4 are mainly concerned with discussions and the proof of Theorem 4.3. Lemma 4.5 allows us to split the proof into two main cases discussed in Propositions 4.9, 4.13, 4.14, 4.15. Part (a) of Theorem 4.3 is obtained by taking, in the general case, $p = n$, $q = 0$, with the identification $\mathfrak{D}_0^1 = \mathfrak{aff}(\mathbb{R})$. A direct proof (for nonzero eigenvalues) is also presented in Proposition 4.10. Part (b) can be directly found in Lemma 4.8, whereas parts (c) and (d) are directly proved in Propositions 4.13 and 4.14, respectively.

Lemma 4.5. *Suppose $V = \mathcal{E}_1 \oplus \mathcal{E}_2$ and $\phi\mathcal{E}_j \subset \mathcal{E}_j$, $j = 1, 2$, where $\phi \in \mathfrak{gl}(V)$ is nonderogatory. Let $\mathcal{G}_{\phi_j} := \mathbb{R}[\phi_j] \ltimes \mathcal{E}_j$, where ϕ_j is the restriction of ϕ to $\mathcal{E}_j$. Then both $\mathcal{G}_{\phi_1}$, $\mathcal{G}_{\phi_2}$ are ideals of $\mathcal{G}_\phi$ and, in fact, $\mathcal{G}_\phi = \mathcal{G}_{\phi_1} \oplus \mathcal{G}_{\phi_2}$.*

Proof. Let us extend ϕ_j to the linear map $\tilde{\phi}_j$ on V such that $\tilde{\phi}_j(\mathcal{E}_p) = 0$ if $p \neq j$. So $[\tilde{\phi}_1, \phi] = [\tilde{\phi}_2, \phi] = 0$ and hence $\tilde{\phi}_1$, $\tilde{\phi}_2 \in \mathbb{R}[\phi]$. More precisely $\mathbb{R}[\tilde{\phi}_j]$ is a subalgebra of $\mathbb{R}[\phi]$. We thus see that $\mathcal{G}_{\phi_j}$, identified with $\mathcal{G}_{\tilde{\phi}_j} = \mathbb{R}[\tilde{\phi}_j] \ltimes \mathcal{E}_j$, is a Lie subalgebra of $\mathcal{G}_\phi := \mathbb{R}[\phi] \ltimes V$. As a matter of fact, each $\mathcal{G}_{\phi_j}$ is an ideal of $\mathcal{G}_\phi$. This directly follows from the combination of the following properties $\phi = \tilde{\phi}_1 + \tilde{\phi}_2$, $\mathbb{R}^n = \mathcal{E}_1 \oplus \mathcal{E}_2$, $\phi\mathcal{E}_j \subset \mathcal{E}_j$, $j = 1, 2$ and $\tilde{\phi}_j(\mathcal{E}_p) = 0$ if $p \neq j$. Thus, as the ideals $\mathcal{G}_{\phi_j}$ form a direct sum (they only meet at $\{0\}$, unless they are identical), we get $\mathcal{G} := \mathcal{G}_{\tilde{\phi}_1} \oplus \mathcal{G}_{\tilde{\phi}_2}$. ■

Consider the general case where $\phi \in \mathfrak{gl}(V)$ admits p real and $2q$ complex eigenvalues. We write $\chi_\phi(X)$ as the product $\chi_\phi(X) = Q_1(X)Q_2(X)$ where $Q_1(X)$ has only complex (nonreal) zeros, whereas the zeros of $Q_2(X)$ are all real. Of course as $Q_1(X)$ and $Q_2(X)$ must be relatively prime, the Primary Decomposition Theorem combined with Cayley-Hamilton theorem imply the following

$$V = \ker(\chi_\phi(\phi)) = \ker(Q_1(\phi)) \oplus \ker(Q_2(\phi)). \tag{12}$$

As $\ker Q_1(\phi)$ and $\ker Q_2(\phi)$ are both stable by ϕ, Lemma 4.5 reduces the proof of Theorem 4.3 to two cases: the case where all the eigenvalues of ϕ are real and the case where all the eigenvalues of ϕ are complex.

4.3. Nonderogatory $\phi \in \mathfrak{gl}(V)$ with only real eigenvalues

If a nonderogatory $\phi \in \mathfrak{gl}(V)$ has only real eigenvalues λ_j of respective multiplicity k_j, $j = 1, \ldots, p$, then its restriction ϕ_j to each subspace $\mathcal{E}_j := \ker(\phi - \lambda_j)^{k_j}$ is again nonderogatory with a unique eigenvalue λ_j of multiplicity k_j and V splits as $V = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_p$.

Definition 4.6. When a nonderogatory $\phi \in \mathfrak{gl}(V)$ has a unique real eigenvalue λ of multiplicity n, we set $\mathfrak{D}_\lambda^n := \mathcal{G}_\phi = \mathbb{R}[\phi] \ltimes V$.

Lemma 4.7. *Suppose a nonderogatory $\phi \in \mathfrak{gl}(V)$ has p distinct eigenvalues $\lambda_1, \ldots, \lambda_p$, all of which are real and of respective multiplicity $k_1, \ldots, k_p$, where $k_1 + \cdots + k_p = n$. Then $\mathcal{G}_\phi$ is isomorphic to the direct sum $\mathfrak{D}_{\lambda_1}^{k_1} \oplus \cdots \oplus \mathfrak{D}_{\lambda_p}^{k_p}$.*

Proof. Applying Lemma 4.5 to (19), where $\ker \chi_j(\phi) = \ker(\phi - \lambda_j \mathbb{I}_V)^{k_j} =: \mathcal{E}_j$, $j = 1, \ldots, p$, leads to $\mathcal{G}_\phi = \mathcal{G}_{\phi_1} \oplus \cdots \oplus \mathcal{G}_{\phi_p}$, where ϕ_j is the restriction of ϕ to $\mathcal{E}_j$. Each ϕ_j is a nonderogatory element of $\mathfrak{gl}(\mathcal{E}_j)$, with a unique eigenvalue λ_j of multiplicity k_j, so $\mathcal{G}_{\phi_j} = \mathfrak{D}_{\lambda_j}^{k_j}$, by Definition 4.6. Thus $\mathcal{G}_\phi = \mathfrak{D}_{\lambda_1}^{k_1} \oplus \cdots \oplus \mathfrak{D}_{\lambda_p}^{k_p}$. ■

Lemma 4.8. *If a nonderogatory $\phi \in \mathfrak{gl}(V)$ has a unique real eigenvalue λ, then the Lie algebra $\mathcal{G}_\phi =: \mathfrak{D}^n_\lambda$ is isomorphic to $\mathfrak{D}^n_0$ as in Example* 4.1.1.

Proof. Following Theorem 6.1, consider a basis in which the matrix of ϕ has the form $M_\lambda = \lambda \mathbb{I}_V + M_0$, with M_0 as in (5). Both M_0 and M_λ being nonderogatory and M_λ being a polynomial in M_0, imply $\operatorname{cent}(M_0, \mathfrak{gl}(V)) = \operatorname{cent}(M_\lambda, \mathfrak{gl}(V)) = \mathbb{K}[M_0]$. Thus we have $\mathfrak{D}^n_0 = \mathcal{G}_{M_0} = \mathcal{G}_{M_\lambda} = \mathfrak{D}^n_\lambda$. ■

Lemmas 4.7 and 4.8 prove the following.

Proposition 4.9. *Suppose a nonderogatory $\phi \in \mathfrak{gl}(V)$ has p distinct eigenvalues $\lambda_1, \dots, \lambda_p$, all of which are real and of respective multiplicity $k_1, \dots, k_p$, where $k_1 + \cdots + k_p = n$. Then $\mathcal{G}_\phi$ is isomorphic to the direct sum $\mathfrak{D}^{k_1}_0 \oplus \cdots \oplus \mathfrak{D}^{k_p}_0$.*

Proof. Indeed, if a nonderogatory $\phi \in \mathfrak{gl}(V)$ has p real eigenvalues $\lambda_1, \dots, \lambda_p$, of respective multiplicity $k_1, \dots, k_p$, where $k_1 + \cdots + k_p = n$, then Lemma 4.7 ensures that $\mathcal{G}_\phi$ is isomorphic to the direct sum $\mathfrak{D}^{k_1}_{\lambda_1} \oplus \cdots \oplus \mathfrak{D}^{k_p}_{\lambda_p}$ and Lemma 4.8 further proves that each $\mathfrak{D}^{k_j}_{\lambda_j}$ is isomorphic to $\mathfrak{D}^{k_j}_0$, for $j = 1, \dots, p$. ■

This concludes the proof of Theorem 4.3 in the case of nonderogatory $\phi \in \mathfrak{gl}(V)$ with only real eigenvalues. Proposition 4.10 supplies another rather direct proof (valid only when all the eigenvalues are nonzero) of Theorem 4.3 (a).

Proposition 4.10. *Let $\mathcal{G}$ be the direct sum of n copies of the Lie algebra $\mathfrak{aff}(\mathbb{R})$ and $\lambda_1, \dots, \lambda_n$, n distinct nonzero real numbers. Suppose $\phi \in \mathfrak{gl}(\mathbb{R}^n)$ is nonderogatory with eigenvalues $\lambda_1, \dots, \lambda_n$. Then $\mathcal{G}$ and $\mathcal{G}_\phi$ are isomorphic.*

Proof. Choose a basis (a_i, b_i) of the ith copy of $\mathfrak{aff}(\mathbb{R})$, so that $[a_i, b_i] = b_i$ and set $\mathfrak{B} = \operatorname{span}(a_1, \dots, a_n)$. Let $V := [\mathcal{G}, \mathcal{G}] = \operatorname{span}(b_1, \dots, b_n)$. For $a \in \mathfrak{B}$, denote by $\rho(a)$ the restriction to V of the adjoint operator $[a, \cdot]$ of a. So one has $\mathcal{G} = \mathfrak{B} \ltimes_\rho V$. Further choose $a_0 = \sum_{i=1}^n \lambda_i a_i$, with λ_i as in the hypotheses, so that the λ_i's are the eigenvalues of $\rho(a_0)$. By hypothesis, the characteristic and minimal polynomials of $\rho(a_0)$ coincide. Hence $\mathbb{R}[\rho(a_0)]$ and $\mathbb{R}[\phi]$ are conjugate (with the identification $V = \mathbb{R}^n$).

Consider the linear map $\psi : \mathcal{G} \to \mathbb{R}[\rho(a_0)] \ltimes V = \mathcal{G}_\phi$ defined by $\psi(b) = b$, for every $b \in V$ and $\psi(a) = \sum_{s=1}^n v_s(\rho(a_0))^s$ for any $a = \sum_{i=1}^n k_i a_i$, where $v = (v_1, \dots, v_n)$ is the solution of the equation $Nv^T = K^T$, with $K = (k_1, \dots, k_n)$ and N is the $n \times n$ matrix with coefficients $N_{ij} = \lambda_i^j$. One sees that, for any $b = \sum_j^n t_j b_j$, $t_j \in \mathbb{R}$,

$$\psi[a,b] = [a,b] = \sum_{j=1}^n t_j k_j b_j = \sum_{j=1}^n t_j \sum_{s=1}^n N_{js} v_s b_j = \sum_{j=1}^n t_j \sum_{s=1}^n \lambda_j^s v_s b_j$$

$$[\psi(a), \psi(b)] = \sum_{j=1}^n t_j \sum_{s=1}^n v_s (\rho(a_0))^s b_j = \sum_{j=1}^n t_j \sum_{s=1}^n v_s \lambda_j^s b_j = \psi[a,b]$$

and
$$[\psi(a), \psi(a')] = \psi[a, a'] = [\psi(b), \psi(b')] = \psi[b, b'] = 0,$$

for any $a, a' \in \mathfrak{B}$, $b, b' \in V$. We further have

$$\det(N) = \left(\prod_{i=1}^n \lambda_i\right)\left(\prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)\right) \neq 0.$$

Thus ψ is a Lie algebra isomorphism. ■

Example 4.11. Let $\mathcal{G}$ be the 6-dimensional rank-three Kähler-Einstein solvable Lie algebra in [11], with Lie brackets $[e_4,e_1]=se_1$, $[e_4,e_2]=se_2$, $[e_4,e_3]=se_3$, $[e_5,e_1]=-s\frac{\sqrt{6}}{2}e_1$, $[e_5,e_2]=s\frac{\sqrt{6}}{2}e_2$, $[e_6,e_1]=\frac{\sqrt{2}}{2}e_1$, $[e_6,e_2]=s\frac{\sqrt{2}}{2}e_2$, $[e_6,e_3]=-s\sqrt{2}e_3$, in a basis $(e_1,\dots,e_6)$, with $s\in\mathbb{R}$.
Consider $\mathcal{B}=\mathfrak{aff}(\mathbb{R})\oplus\mathfrak{aff}(\mathbb{R})\oplus\mathfrak{aff}(\mathbb{R})$ and a basis $(a_1,b_1,a_2,b_2,a_3,b_3)$ in which its Lie bracket reads $[a_j,b_j]=b_j$, $j=1,2,3$. The linear map $\psi:\mathcal{G}\to\mathcal{B}$, $\psi(e_1)=b_1$, $\psi(e_2)=b_2$, $\psi(e_3)=b_3$, $\psi(e_4)=s(a_1+a_2+a_3)$, $\psi(e_5)=s\frac{\sqrt{6}}{2}(a_2-a_1)$, $\psi(e_6)=\frac{\sqrt{2}}{2}(a_1+sa_2-2sa_3)$, is a homomorphism and $\det(\psi)=-\frac{\sqrt{3}}{2}s^2(5s+1)$. So, ψ is an isomorphism between the Lie algebras $\mathcal{G}$ and $\mathcal{B}$, except when $s\in\{0,-\frac{1}{5}\}$. Thus, any $\phi\in\mathfrak{gl}(3,\mathbb{R})$ with 3 distinct real eigenvalues gives rise to $\mathcal{G}_\phi=\mathcal{G}$, when $s\notin\{0,-\frac{1}{5}\}$. Note that $\mathcal{G}$ is Frobenius if and only if $s\notin\{0,-\frac{1}{5}\}$.

4.4. Nonderogatory $\phi\in\mathfrak{gl}(V)$ all of whose eigenvalues are complex

4.4.1. Nonderogatory $\phi\in\mathfrak{gl}(V)$ diagonalizable in $\mathbb{C}$

Let $\phi\in\mathfrak{gl}(V)$ be nonderogatory. Suppose all the eigenvalues $\lambda_1,\bar\lambda_1,\dots,\lambda_{\frac{n}{2}},\bar\lambda_{\frac{n}{2}}$ of ϕ are complex (nonreal) and distinct, in which case n is even and ϕ is diagonalizable in $\mathbb{C}$. From Theorem 6.1 (B), there is a basis of V in which the matrix ϕ is of the form diag$(J_1,\dots,J_{\frac{n}{2}})$, where each block J_j is of the form

$$\begin{pmatrix} Re(\lambda_j) & -Im(\lambda_j)\\ Im(\lambda_j) & Re(\lambda_j)\end{pmatrix},\quad Im(\lambda_j)\neq 0.$$

Note that each J_j is nonderogatory with characteristic polynomial

$$\chi_{J_j}(X)=(X-Re(\lambda_j))^2+Im(\lambda_j)^2.$$

Let v_j be an eigenvector of ϕ with corresponding eigenvalue λ_j. Both $Re(v_j)$ and $Im(v_j)$ are in $\ker\chi_{J_j}(\phi)=:\mathcal{E}_j$. Obviously, one has $\phi\mathcal{E}_j\subset\mathcal{E}_j$, $j=1,\dots,\frac{n}{2}$.

Set $\phi_j:=\phi_{|\mathcal{E}_j}$. Of course, $\mathbb{R}[\phi_j]=\mathbb{R}\mathbb{I}_{\mathcal{E}_j}\oplus\mathbb{R}\phi_j$. In the basis $e_1=\mathbb{I}_{\mathcal{E}_j}$, $e_2=\phi_j$, $e_3=Re(v_j)$, $e_4=Im(v_j)$, the Lie bracket of $\mathcal{G}_{\phi_j}:=\mathbb{R}[\phi_j]\ltimes\mathcal{E}_j$, is $[e_1,e_3]=e_3$, $[e_1,e_4]=e_4$, $[e_2,e_3]=Re(\lambda_j)e_3+Im(\lambda_j)e_4$, $[e_2,e_4]=-Im(\lambda_j)e_3+Re(\lambda_j)e_4$.

Lemma 4.5 applied to the direct sum $V=\mathcal{E}_1\oplus\cdots\oplus\mathcal{E}_{\frac{n}{2}}$, leads to $\mathcal{G}_\phi=\mathcal{G}_{\phi_1}\oplus\cdots\oplus\mathcal{G}_{\phi_{\frac{n}{2}}}$. We identify $\mathcal{E}_j$ with $\mathbb{R}^2$ and show below that each Lie ideal $\mathcal{G}_{\phi_j}$ is isomorphic to $\mathfrak{aff}(\mathbb{C})$.

Lemma 4.12. *Let $\phi\in\mathfrak{gl}(2,\mathbb{R})$ with complex eigenvalues λ, $\bar\lambda$, where $\lambda=r-is$ and $r,s\in\mathbb{R}$, with $s\neq 0$. Then $\mathcal{G}_\phi$ is isomorphic to $\mathfrak{aff}(\mathbb{C})$, as in Example* 4.1.2.

Proof. Without loss of generality, set $\phi(\tilde e_1)=r\tilde e_1+s\tilde e_2$, $\phi(\tilde e_2)=-s\tilde e_1+r\tilde e_2$, in the canonical basis $(\tilde e_1,\tilde e_2)$ of $\mathbb{R}^2$. In the basis $e_1'=\mathbb{I}_{\mathbb{R}^2}$, $e_2'=\phi$, $e_3'=\tilde e_1$, $e_4'=\tilde e_2$, the Lie bracket of $\mathcal{G}_\phi$ reads $[e_1',e_3']=e_3'$, $[e_1',e_4']=e_4'$, $[e_2',e_3']=re_3'+se_4'$, $[e_2',e_4']=-se_3'+re_4'$. In the new basis $X_1:=e_1'$, $X_2:=-\frac{r}{s}e_1'+\frac{1}{s}e_2'$, $X_3:=pe_3'-qe_4'$, $X_4:=qe_3'+pe_4'$, with $p^2+q^2\neq 0$, we now have $[X_1,X_3]=X_3$, $[X_1,X_4]=X_4$, $[X_2,X_3]=X_4$, $[X_2,X_4]=-X_3$, which is the Lie bracket of $\mathfrak{aff}(\mathbb{C})$. In other words, the invertible linear map $\psi:\mathfrak{aff}(\mathbb{C})\to\mathcal{G}_\phi$, $\psi(e_j)=X_j$, $j=1,2,3,4$, is an isomorphism between the Lie algebras $\mathfrak{aff}(\mathbb{C})$ and $\mathcal{G}_\phi$. ■

Proposition 4.13. *Suppose a nonderogatory $\phi\in\mathfrak{gl}(V)$ has n distinct complex eigenvalues $\lambda_j,\bar\lambda_j$, $j=1,\dots,\frac{n}{2}$. Then the Lie algebra $\mathcal{G}_\phi:=\mathbb{R}[\phi]\ltimes V$ is isomorphic to the direct sum $\mathfrak{aff}(\mathbb{C})\oplus\cdots\oplus\mathfrak{aff}(\mathbb{C})$ of $\frac{n}{2}$ copies of the Lie algebra $\mathfrak{aff}(\mathbb{C})$.*

Proof. Suppose a nonderogatory $\phi \in \mathfrak{gl}(V)$ has n distinct complex (nonreal) eigenvalues $\lambda_j, \bar{\lambda}_j$, $j = 1, \dots, \frac{n}{2}$. Then $\mathcal{G}_\phi$ is isomorphic to the direct sum of the ideals $\mathcal{G}_{\phi_j} := \mathbb{R}[\phi_j] \ltimes \mathcal{E}_j$, $j = 1, \dots, \frac{n}{2}$, where $\ker \chi_{\phi_j}(\phi_j) =: \mathcal{E}_j$, as above. From Lemma 4.12, each $\mathcal{G}_{\phi_j}$ is isomorphic to $\mathfrak{aff}(\mathbb{C})$. ■

4.4.2. Example: the circular permutation of the vectors of a basis

Here is a typical example of a nonderogatory $\phi \in \mathfrak{gl}(n, \mathbb{R})$ with n real and complex eigenvalues, hence diagonalizable in $\mathbb{C}$. It is given by the circular permutation of the canonical basis $\phi(\tilde{e}_i) = \tilde{e}_{i+1}$, for $i = 1, \dots, n-1$ and $\phi(\tilde{e}_n) = \tilde{e}_1$. Any vector $\tilde{e}_i$ of this basis is such that $(\tilde{e}_i, \phi(\tilde{e}_i), \dots, \phi^{n-1}(\tilde{e}_i))$ is again a basis of $\mathbb{R}^n$. The map ϕ is nonderogatory and its matrix in the above basis reads $[\phi] = E_{1,n} + \sum_{i=1}^{n-1} E_{i+1,i}$. Its characteristic polynomial is $\chi_\phi(X) = X^n - 1$, up to a sign. So the eigenvalues are the complex nth roots of 1. They are $z_k = e^{ik\frac{2\pi}{n}}$, where $k = 1, 2, \dots, n$. When $n = 2$, then it reads $[\phi] = E_{1,2} + E_{2,1}$ and has the two distinct real eigenvalues $z_1 = -1$ and $z_2 = 1$. So $\mathcal{G}_\phi$ is isomorphic to $\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$.

For $n = 3$, the eigenvalues are $z_1 = 1$, $z_2 = -e^{i\frac{\pi}{3}}$, $z_3 = -e^{-i\frac{\pi}{3}}$. So $\mathcal{G}_\phi$ is isomorphic to $\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{C})$. For $n = 4$, the eigenvalues are $z_1 = -1$, $z_2 = 1$, $z_3 = i$, $z_4 = -i$, thus $\mathcal{G}_\phi$ is isomorphic to $\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{C})$. When $n = 5$, there are one real and 4 complex eigenvalues and hence we get $\mathcal{G}_\phi = \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{C}) \oplus \mathfrak{aff}(\mathbb{C})$. For $n = 7$, we have one real and six complex eigenvalues, namely $1, -e^{i\frac{\pi}{7}}$, $-e^{-i\frac{\pi}{7}}$, $e^{i\frac{2\pi}{7}}$, $e^{-i\frac{2\pi}{7}}$, $-e^{i\frac{3\pi}{7}}$, $-e^{-i\frac{3\pi}{7}}$. This gives rise to $\mathcal{G}_\phi = \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{C}) \oplus \mathfrak{aff}(\mathbb{C}) \oplus \mathfrak{aff}(\mathbb{C})$.

4.4.3. Nonderogatory $\phi \in \mathfrak{gl}(V)$ non-diagonalizable in $\mathbb{C}$

As above, V is a real vector space with $\dim V = n$. In this section, we discuss the case of nonderogatory $\phi \in \mathfrak{gl}(V)$ all of whose eigenvalues are complex (nonreal, in which case, n is even), but which are not diagonalizable in $\mathbb{C}$. First, consider the case where ϕ has only 2 eigenvalues, say $z = r + is$ and $r - is$, with $s \neq 0$. In the same basis in which the matrix of ϕ is in the form M_z as in (18), consider the nonderogatory $M_{0,1} = M_s + M_n$, where M_s, M_n are as in (8). We have

$$M_s M_n = -\sum_{j=0}^{\frac{n}{2}-2} \Big(E_{2j+1,2j+4} - E_{2j+2,2j+3} \Big) = M_n M_s, \tag{13}$$

and, of course $[M_{0,1}, M_s] = [M_n, M_s] = [M_n, M_{0,1}] = 0$. In particular, M_s and M_n are respectively the semisimple and the nilpotent parts of $M_{0,1}$. So, M_s and M_n are both polynomials in $M_{0,1}$. Thus, $M_z = r\mathbb{I}_{\mathbb{R}^n} + sM_s + M_n$ is also a polynomial in $M_{0,1}$. This induces the following equalities

$$\mathbb{R}[M_{0,1}] = \mathbb{R}[M_z] \text{ and } \mathcal{G}_{M_z} = \mathcal{G}_{M_{0,1}} = \mathfrak{D}^n_{0,1}. \tag{14}$$

We have thus proved the

Proposition 4.14. *If a nonderogatory $\phi \in \mathfrak{gl}(V)$ has only 2 eigenvalues which are complex (thus conjugate), and if ϕ is non-diagonalizable in $\mathbb{C}$, then $\mathcal{G}_\phi = \mathfrak{D}^n_{0,1}$.*

Finally, we deduce the following.

Proposition 4.15. *If all the eigenvalues $\lambda_1, \bar{\lambda}_1, \dots, \lambda_p, \bar{\lambda}_p$ of a nonderogatory $\phi \in \mathfrak{gl}(V)$, are complex, then $\mathcal{G}_\phi$ is isomorphic to the direct sum $\mathfrak{D}^{2k_1}_{0,1} \oplus \dots \oplus \mathfrak{D}^{2k_p}_{0,1}$ of copies of $\mathfrak{D}^{2k_j}_{0,1}$, where k_j is the multiplicity of the eigenvalues λ_j, $\bar{\lambda}_j$.*

Proof. Factorizing the characteristic polynomial of ϕ, via Lemma 4.2, then successively applying the Primary Decomposition Theorem, Lemma 4.5, together with Proposition 4.14, yield the result. ■

4.4.4. More on the Lie algebra $\mathfrak{D}^n_{0,1}$

Without loss of generality, we suppose that the form (18) is achieved in the canonical basis $(\tilde{e}_1, \ldots, \tilde{e}_n)$ of $\mathbb{R}^n$. Note that M_s and M_n in (8) satisfy $M_s^2 = -\mathbb{I}_{\mathbb{R}^n}$, $M_n^{\frac{n}{2}} = 0$, $(M_sM_n)^{\frac{n}{2}} = 0$ and for any $j = 1, \ldots, \frac{n}{2} - 1$,

$$\big(M_n\big)^j = \sum_{p=1}^{n-2j} E_{p,p+2j} \quad \text{and} \quad M_s\big(M_n\big)^j = -\sum_{p=0}^{\frac{n}{2}-j-1} \big(E_{2p+1,2p+2j+2} - E_{2p+2,2p+2j+1}\big).$$

Each $M^p_{0,1} = (M_s + M_n)^p = \sum_{j=0}^{p} \complement^j_p (M_s)^j (M_n)^{p-j}$, $p = 0, 1, 2, \ldots, n-1$, is a linear combination of $\mathbb{I}_{\mathbb{R}^n}$, M_s, $(M_n)^j$, $M_s(M_n)^j$, $j = 1, \ldots, \frac{n}{2} - 1$, where, as above, $\complement^j_p = \frac{p!}{j!(p-j)!}$, $p \geq j$. Thus $e_1 := \mathbb{I}_{\mathbb{R}^n}$, $e_2 := M_s$, $e_{2j+1} := (M_n)^j$, $e_{2j+2} := -M_s(M_n)^j$, $j = 1, \ldots, \frac{n}{2} - 1$, is another basis of $\mathbb{R}[M_{0,1}]$. In the basis $(e_s, \tilde{e}_s,\ s = 1, \ldots, n)$, the Lie bracket of $\mathfrak{D}^n_{0,1}$ reads, for any $q = 0, 1, \ldots, \frac{n}{2} - 1$ and $i = 1, \ldots, n$,

$$[e_{2q+1}, \tilde{e}_i] = \tilde{e}_{i-2q}, \quad \text{if } \ 2q+1 \leq i \leq n, \quad \text{and} \quad [e_{2q+1}, \tilde{e}_i] = 0 \ \text{ otherwise},$$

$$[e_{2q+2}, \tilde{e}_i] = \begin{cases} -\tilde{e}_{i-2q+1}, & \text{if } i \text{ odd and } 2q+1 \leq i \leq n-1, \\ \tilde{e}_{i-2q-1}, & \text{if } i \text{ even and } 2q+2 \leq i \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

So the 2-form $\partial \tilde{e}_1^* = -\sum_{k=1}^{n} e_k^* \wedge \tilde{e}_k^*$ is non-degenerate on $\mathfrak{D}^n_{0,1}$. The codimension 2 subspace $\mathcal{N} := span\big((M_n)^j, M_s(M_n)^j,\ j = 1, \ldots, \frac{n}{2} - 1\big) \ltimes \mathbb{R}^n$ is an ideal of $\mathfrak{D}^n_{0,1}$, as it contains the derived ideal $\mathbb{R}^n = [\mathfrak{D}^n_{0,1}, \mathfrak{D}^n_{0,1}]$. The equalities $(M_n)^{\frac{n}{2}} = (M_sM_n)^{\frac{n}{2}} = 0$, show that $\mathcal{N}$ is $\frac{n}{2}$-step nilpotent, and is in fact the nilradical of $\mathfrak{D}^n_{0,1}$. Indeed, if we set $\mathfrak{F} := \mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus \mathbb{R}M_{0,1}$, then the vector space underlying $\mathfrak{D}^n_{0,1}$ splits as the direct sum $\mathfrak{D}^n_{0,1} = \mathfrak{F} \oplus \mathcal{N}$, so that, any subspace of dimension higher than $n-2$, must meet $\mathfrak{F}$ non-trivially and hence cannot be a nilpotent subalgebra of $\mathfrak{D}^n_{0,1}$. Thus $\mathcal{N}$ is the biggest nilpotent ideal of $\mathfrak{D}^n_{0,1}$. Altogether, $\mathfrak{D}^n_{0,1}$ is an indecomposable, non-completely solvable (the adjoint of M_s has complex eigenvalues) 2-solvable Frobenius Lie algebra with a codimension 2 non-Abelian nilradical $\mathcal{N}$. One sees that $\mathfrak{D}^n_{0,1}$ is not isomorphic to $\mathfrak{D}^n_0$ which is completely solvable and has a codimension 1 non-Abelian nilradical (except for $\mathfrak{D}^1_0 = \mathfrak{aff}(\mathbb{R})$).

4.5. Derivations and automorphisms of $\mathcal{G}_\phi$

Proposition 4.16. *Let $\phi \in \mathfrak{gl}(V)$ be nonderogatory. Set*

$$\mathfrak{N} := \{N \in \mathfrak{gl}(V),\ \textit{such that } [N, \mathbb{K}[\phi]] \subset \mathbb{K}[\phi]\} = \mathcal{N}_{\mathfrak{gl}(V)}(\mathbb{K}[\phi]). \tag{15}$$

Up to isomorphism, the Lie algebra $Der(\mathcal{G}_\phi)$ of derivations of $\mathcal{G}_\phi$ is given by

$$Der(\mathcal{G}_\phi) = \mathfrak{N} \ltimes V. \tag{16}$$

More precisely, $D \in Der(\mathcal{G}_\phi)$ if and only if there are $x_D \in V$ and $h \in \mathfrak{N}$, such that D is of the form $D(a + x) = [h, a] + ax_D + h(x)$, for every $a \in \mathbb{K}[\phi]$ and $x \in V$.

Proof. Let $D \in Der(\mathcal{G}_\phi)$. As the derived ideal $V = [\mathcal{G}_\phi, \mathcal{G}_\phi]$ is preserved by D, we write $D(x) = h(x)$ and $D(a) = D_{1,1}(a) + D_{1,2}(a)$, for $a \in \mathbb{K}[\phi]$, $x \in V$, where $D_{1,1} \in \mathfrak{gl}(\mathbb{K}[\phi])$, $h \in \mathfrak{gl}(V)$ and $D_{1,2} : \mathbb{K}[\phi] \to V$ is linear. Now, setting $e_1 := \mathbb{I}_V$ and $x_D := D_{12}(e_1)$, the equality $D_{12}(a) = ax_D$, for any $a \in \mathbb{K}[\phi]$, follows:

$$0 = D([e_1, a]) = [D(e_1), a] + [e_1, D(a)] = [x_D, a] + [e_1, D_{12}(a)] = D_{12}(a) - ax_D.$$

We also have $h(ax) = D[a, x] = [D_{11}(a), x] + [a, h(x)] = D_{11}(a)x + ah(x)$, thus entailing $[h, a] = D_{11}(a) \in \mathbb{K}[\phi]$, for any $a \in \mathbb{K}[\phi]$. In consequence the equality $D_{11}(a) := [h, a]$ stands as the definition of $D_{11}(a)$ and implies $h \in \mathfrak{N}$. Thus for any D in $Der(\mathcal{G}_\phi)$, there are $x_D \in V$, $h \in \mathfrak{N}$ such that $D(a+x) = [h, a] + h(x) + ax_D$. Conversely, any $(h, \hat{x}) \in \mathfrak{N} \times V$ defines a unique $D \in Der(\mathcal{G}_\phi)$ by the formula $D(a + x) = [h, a] + h(x) + a\hat{x}$. We get the invertible linear $\psi : Der(\mathcal{G}_\phi) \to \mathfrak{N} \ltimes V$, $D \mapsto (h, x_D)$. As easily seen, for any $D_j \in Der(\mathcal{G}_\phi)$ given by (h_j, x_{D_j}), $j = 1, 2$, we have $\psi([D_1, D_2]) = ([h_1, h_2], h_1(x_{D_2}) - h_2(x_{D_1})) = [\psi(D_1), \psi(D_2)]$. Hence ψ is an isomorphism between the Lie algebras $Der(\mathcal{G}_\phi)$ and $\mathfrak{N} \ltimes V$. ■

Note that in Proposition 4.16, the inner derivations of $\mathcal{G}_\phi$ are those for which the component h belongs to $\mathbb{K}[\phi]$.

Example 4.17. The space $Der(\mathfrak{D}_0^n)$ of derivations of $\mathfrak{D}_0^n$.
For $n = 2$, the normalizer of $\mathbb{R}[E_{1,2}]$ in $\mathfrak{gl}(2, \mathbb{R})$, is the 3-dimensional real algebra spanned by $E_{1,1}, E_{1,2}, E_{22}$, which is 1 dimension higher than $\mathbb{R}[E_{1,2}]$. For example, $E_{2,2}$ is not in $\mathbb{R}[E_{1,2}]$ and will thus act as an outer derivation on $\mathfrak{D}_0^2$. Considering elements of $\mathbb{R}^2$ as inner derivations, we see that $Der(\mathfrak{D}_0^2)$ is spanned by $E_{1,1}, E_{1,2}, E_{22}, \tilde{e}_1, \tilde{e}_2$. For $n = 3$, the normalizer of $\mathbb{R}[E_{1,2} + E_{2,3}]$ in $\mathfrak{gl}(3, \mathbb{R})$, is the 5-dimensional algebra of matrices spanned by $E_{1,1} - E_{3,3}$, $E_{22} + 2E_{33}$, $E_{1,2}$, $E_{2,3}$, $E_{1,3}$. For example, $E_{1,1} - E_{3,3}$, $E_{22} + 2E_{33}$, are not in $\mathbb{R}[E_{1,2} + E_{2,3}]$, so they represent outer derivations of $\mathfrak{D}_0^3$. Counting elements of $\mathbb{R}^3$ in, as inner derivations, we get $\dim Der(\mathfrak{D}_0^3) = 8$. More generally, for a given $n \geq 4$, if we set $M_0 := E_{1,2} + E_{2,3} + \cdots + E_{n-1,n}$ as in (5), then the normalizer of $\mathbb{R}[M_0]$ in $\mathfrak{gl}(n, \mathbb{R})$, is of dimension $2n - 1$ and is spanned by

$$D_k := E_{1,k} - \sum_{j=3}^{n-k+1} (j-2)E_{j,j+k-1} \quad \text{and} \quad D'_k := E_{2,k+1} + \sum_{j=3}^{n-k+1} (j-1)E_{j,j+k-1},$$

for $k = 1, \ldots, n-2$ and $D_{n-1} := E_{1,n-1}$, $D'_{n-1} := E_{2,n}$, $D_n := E_{1,n}$. For example, D_1 and D'_1 are not in $\mathbb{R}[M_0]$ and will act as non-trivial outer derivations of $\mathfrak{D}_0^n$. In particular $\mathbb{R}[M_0]$ is not a Cartan subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ and $\dim Der(\mathfrak{D}_0^n) = 3n - 1$.

Example 4.18. On the space $Der(\mathfrak{D}_{0,1}^n)$ of derivations of $\mathfrak{D}_{0,1}^n$.
For $n = 4$, consider the $n \times n$ matrix $M_{0,1} = E_{2,1} - E_{1,2} + E_{4,3} - E_{3,4} + E_{1,3} + E_{2,4}$ as in Example 4.1.2. The normalizer of $\mathbb{R}[M_{0,1}]$ in $\mathfrak{gl}(4, \mathbb{R})$, is of dimension 6 and is spanned by the matrices $E_{1,1} + E_{2,2}$, $E_{3,3} + E_{4,4}$, $E_{1,2} - E_{2,1}$, $E_{3,4} - E_{4,3}$, $E_{1,3} + E_{2,4}$, $E_{2,3} - E_{1,4}$. In particular the two matrices $E_{1,1} + E_{2,2}$, $E_{3,3} + E_{4,4}$, are not elements of $\mathbb{R}[M_{0,1}]$. So they both represent non-trivial outer derivations of $\mathfrak{D}_{0,1}^4$. More generally, for any $n \geq 6$, we let again $M_{0,1}$ stand for the $n \times n$ matrix $M_{0,1} = M_s + M_n$ as in (8). The $n \times n$ matrix

$$Z_1 := E_{1,1} + E_{2,2} - \textstyle\sum_{j=2}^{\frac{n}{2}-1} (j-1)(E_{2j+1,2j+1} + E_{2j+2,2j+2})$$

is in the normalizer of $\mathbb{R}[M_{0,1}]$ in $\mathfrak{gl}(n, \mathbb{R})$, but not in $\mathbb{R}[M_{0,1}]$.

Indeed, for any $s \geq 1$, we have

$$[Z_1, (M_n)^s] = s \sum_{j=1}^{n-2s} E_{j,j+2s} = s(M_n)^s,$$

$$[Z_1, M_s] = 0$$

and $$[Z_1, M_s(M_n)^s] = M_s[Z_1, (M_n)^s] + [Z_1, M_s](M_n)^s = sM_s(M_n)^s.$$

So the linear map $M \mapsto [Z_1, M]$ preserves $\mathbb{R}[M_{0,1}]$ and has a diagonal matrix $\mathrm{diag}(0, 0, 1, 2, \ldots, \frac{n}{2} - 1, 1, 2, \ldots, \frac{n}{2} - 1)$ in the basis

$$(\mathbb{I}_{\mathbb{R}^n}, M_s, M_n, (M_n)^2, \ldots, (M_n)^{\frac{n}{2}-1}, M_sM_n, M_s(M_n)^2, \ldots, M_s(M_n)^{\frac{n}{2}-1})$$

and $Z_1 \notin \mathbb{R}[M_{0,1}]$, given that $\mathbb{R}[M_{0,1}]$ is Abelian. In particular $\mathbb{R}[M_{0,1}]$ is not a Cartan subalgebra of $\mathfrak{gl}(n, \mathbb{R})$. Also, Z_1 will act as an outer derivation of $\mathfrak{D}^n_{0,1}$.

We have the following.

Theorem 4.19. *Let $\phi \in \mathfrak{gl}(n, \mathbb{R})$ be nonderogatory. The following are equivalent.*

(1) *Every derivation of $\mathcal{G}_\phi$ is an inner derivation.*

(2) *$\mathbb{R}[\phi]$ is a Cartan subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.*

(3) *ϕ has n distinct (real or complex) eigenvalues.*

(4) *The nilradical of $\mathcal{G}_\phi$ is Abelian.*

(5) *$\mathcal{G}_\phi$ is the direct sum of only copies of $\mathfrak{aff}(\mathbb{R})$ and $\mathfrak{aff}(\mathbb{C})$.*

Proof. The equivalence between (1) and (2) directly follows from Proposition 4.16. Indeed, every derivation of $\mathcal{G}_\phi$ is inner, if and only if the normalizer of $\mathbb{R}[\phi]$ in $\mathfrak{gl}(n, \mathbb{R})$, coincides with $\mathbb{R}[\phi]$. The equivalence between (2), (3) and (5) has been shown in Theorem 3.11. The proof that (5) implies (4) directly follows from the fact that each copy, of either $\mathfrak{aff}(\mathbb{R})$ or $\mathfrak{aff}(\mathbb{C})$, has an Abelian nilradical. We now prove that (4) implies (5). In the decomposition of $\mathcal{G}_\phi$ (Theorem 4.3), the nilradical of $\mathcal{G}_\phi$ is the sum of the nilradicals of its ideals. From Examples 4.17, 4.18, the nilradicals of $\mathcal{D}^k_0$ and $\mathcal{D}^k_{0,1}$ are not Abelian, for any $k \geq 2$. So in order for $\mathcal{G}_\phi$ to have an Abelian nilradical, it must not contain an ideal isomorphic $\mathcal{D}^k_0$ or $\mathcal{D}^k_{0,1}$, $k \geq 2$. ∎

Note that Theorem 4.19 is in agreement with the classification of Cartan subalgebras of $\mathfrak{gl}(n, \mathbb{R})$ supplied by Kostant ([14]) and Sugiura ([23]). Theorem 4.19 also implies that when $\mathbb{R}[\phi]$ is a Cartan subalgebra of $\mathfrak{gl}(n, \mathbb{R})$, then $\mathcal{G}_\phi$ is a part of the 2-solvable Lie algebras studied in [1], [17].

4.6. Proof of Theorem 3.11

From Proposition 4.19, for a given n, the number of isomorphism classes of 2-solvable Frobenius Lie algebras of dimension $2n$ of the form $\mathcal{G}_\phi := \mathbb{R}[\phi] \ltimes \mathbb{R}^n$, where $\mathbb{R}[\phi]$ is a Cartan subalgebra of $\mathfrak{gl}(n, \mathbb{R})$, is exactly $[\frac{n}{2}] + 1$. Indeed, one can look at $[\frac{n}{2}] + 1$ as the number (counting from zero) of possible copies of $\mathfrak{aff}(\mathbb{C})$ that one can count in a decomposable Lie algebra containing only copies of either $\mathfrak{aff}(\mathbb{C})$ or $\mathfrak{aff}(\mathbb{R})$. On the other hand, from e.g. [23], there are exactly $[\frac{n}{2}] + 1$ non-conjugate Cartan subalgebras of $\mathfrak{gl}(n, \mathbb{R})$. So we have derived Theorem 3.11, in a simple and direct way. More precisely we prove it as follows.

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}(n,\mathbb{R})$ with a k-dimensional toroidal part. From [23], up to conjugacy under an element of the Weyl group, there is a basis $(\tilde{e}_1,\dots,\tilde{e}_n)$, considered here as the canonical basis of $\mathbb{R}^n$, in which $\mathfrak{h}$ is of the form

$$\mathfrak{h} = \left\{ \begin{pmatrix} D_1 & -D_2 & \mathbf{0} \\ D_2 & D_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_3 \end{pmatrix} \right\},$$

where $D_1 = \operatorname{diag}(h_1,\dots,h_k)$, $D_2 = \operatorname{diag}(h_{k+1},\dots,h_{2k})$, $D_3 = \operatorname{diag}(h_{2k+1},\dots,h_n)$ with $h_j \in \mathbb{R}$, $j = 1,\dots,n$. Hence, $\mathfrak{h}$ is conjugate to

$$\mathfrak{h}' = \left\{ \begin{array}{l} M := \operatorname{diag}(D_1',\dots,D_k',D_3), \text{ with } D_j' = \begin{pmatrix} h_j & -h_{k+j} \\ h_{k+j} & h_j \end{pmatrix} \\ \text{and } D_3 = \operatorname{diag}(h_{2k+1},\dots,h_n),\ h_j \in \mathbb{R},\ j = 1,\dots,n \end{array} \right\},$$

obtained by reordering the canonical basis of $\mathbb{R}^n$ into

$$(\tilde{e}_1, \tilde{e}_{k+1}, \tilde{e}_2, \tilde{e}_{k+2}, \dots, \tilde{e}_k, \tilde{e}_{2k}, \tilde{e}_{2k+1}, \tilde{e}_{2k+2}, \dots, \tilde{e}_n).$$

An M in $\mathfrak{h}'$ is nonderogatory if and only if $(h_i, h_{k+i}) \neq (h_j, h_{k+j})$, whenever $i \neq j$ and $h_{2k+s} \neq h_{2k+l}$ whenever $s \neq l$. But the existence of an (regular) element satisfying these conditions is guaranteed by the fact that $\mathfrak{h}$ is a Cartan subalgebra. Hence, there exists a nonderogatory M with the n distinct eigenvalues $h_1 + ih_{k+1}, h_1 - ih_{k+1}, \dots, h_k + ih_{2k}, h_k - ih_{2k}$, $h_{2k+1},\dots,h_n$, such that, up to a conjugation, $\mathbb{I}_{\mathbb{R}^n} \oplus \mathfrak{h} = \mathbb{R}[M] = \mathfrak{B}_{\mathfrak{h}}$. So (a) implies (b).

Furthermore, Theorem 4.3 ensures that the Lie algebra $\mathfrak{B}_{\mathfrak{h}} \ltimes \mathbb{R}^n$ is isomorphic to the direct sum $\mathfrak{aff}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{aff}(\mathbb{C}) \oplus \mathfrak{aff}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{aff}(\mathbb{R})$ of k copies of $\mathfrak{aff}(\mathbb{C})$ and $(n-k)$ of $\mathfrak{aff}(\mathbb{R})$. In fact, the equivalence between (b) and (c) has already been proved by Theorem 4.3. Conversely, suppose $M \in \mathfrak{gl}(n,\mathbb{R})$ is nonderogatory and has n distinct eigenvalues, 2k of which are complex. From Proposition 4.19, $\mathbb{R}[M]$ is a Cartan subalgebra of $\mathfrak{gl}(n,\mathbb{R})$ and so (b) implies (a). Furthermore, using the Primary Decomposition Theorem and results from Section 4.4, we put $\mathbb{R}[M]$ in the form $\mathfrak{h}'$. ■

5. Classification of low dimensional 2-solvable Frobenius Lie algebras

Applying Theorem 4.3 and the analysis carried out above, we get Theorem 5.1 below which provides, up to isomorphism, a complete list of all 2-solvable Frobenius Lie algebras of dimension 2, 4, 6 or 8. We note that $\mathfrak{D}_0^2$, $\mathfrak{aff}(\mathbb{C})$ and $\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$ correspond to the family PHC7 of [2] and to S11, S8 and S10 respectively in [22]. We also note that $\mathfrak{D}_0^2 \oplus \mathfrak{D}_0^2$ is missing from the list in [26], see Section 5.2. It is also worth recalling here, that the Lie algebras $\mathcal{G}_{4,2}$ of Example 3.7 and $\mathfrak{h}_{4,2}$ of Example 3.8 are isomorphic via the isomorphism (3).

Theorem 5.1. *A 2-solvable Frobenius Lie algebra of dimension ≤ 6 is either isomorphic to $\mathfrak{aff}(\mathbb{R})$, $\mathfrak{D}_0^2$, $\mathfrak{aff}(\mathbb{C})$, $\mathfrak{D}_0^3$, $\mathcal{G}_{3,1}$ as in Example 3.7, or to one of the direct sums $\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$, $\mathfrak{D}_0^2 \oplus \mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{C}) \oplus \mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$ of their copies.*

In dimension 8, there are 14 non-isomorphic 2-solvable Frobenius Lie algebras, 9 of which are of the form $\mathcal{G}_M$, for some nonderogatory $M \in \mathfrak{gl}(4,\mathbb{R})$, namely $\mathfrak{D}_{0,1}^4$, $\mathfrak{D}_0^4$, the direct sums $\mathfrak{D}_0^3 \oplus \mathfrak{aff}(\mathbb{R})$, $\mathfrak{D}_0^2 \oplus \mathfrak{D}_0^2$, $\mathfrak{D}_0^2 \oplus \mathfrak{aff}(\mathbb{C})$, $\mathfrak{D}_0^2 \oplus \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$,

$\mathfrak{aff}(\mathbb{C}) \oplus \mathfrak{aff}(\mathbb{C})$, $\mathfrak{aff}(\mathbb{C}) \oplus \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$, *and* 5 *of which are not given by a nonderogatory* $M \in \mathfrak{gl}(4,\mathbb{R})$, *namely* $\mathcal{G}_{4,1}$ *of Example* 3.7, $\mathfrak{h}_{4,2}$ *and* $\mathfrak{h}_{4,3}$ *of Example* 3.8, $\mathcal{G}'_{4,4}$ *of Example* 3.9, $\mathcal{G}_{3,1} \oplus \mathfrak{aff}(\mathbb{R})$.

5.1. Proof of Theorem 5.1

First, let us recall the classification list of nilpotent 3-dimensional associative real algebras $A_{4,j}$, $j = 1, 2, \ldots, 6$, given in [4]. We will need the commutative ones for the proof of Theorem 5.1. They have a basis (a, b, c) in which the (non-zero) products are as follows.

$$\begin{aligned}
A_{3,1}: &\quad (A_{3,1})^2 = 0,\\
A_{3,2}: &\quad a^2 = c,\\
A^s_{3,3}: &\quad a^2 = sc,\ \ b^2 = c,\ \ s \in \mathbb{R},\ \ s \neq 0,\\
A^s_{3,4}: &\quad a^2 = sc,\ \ b^2 = c,\ \ ab = c,\ \ s \in \mathbb{R},\\
A_{3,5}: &\quad ab = c,\ \ ba = -c,\\
A_{3,6}: &\quad a^2 = b,\ \ ab = ba = c.
\end{aligned} \tag{17}$$

Note that $A^s_{3,4}$ and $A_{3,5}$ are not commutative, since $ab \neq ba$. As noted in [4], $A^s_{3,3}$ and $A^t_{3,3}$ are isomorphic if and only if $t = \epsilon^2 s$ for some $\epsilon \neq 0$. In our case where the ground field is $\mathbb{R}$, there are only two isomorphism classes corresponding to $s > 0$, say, $A^1_{3,3}$ and $s < 0$, say $A^{-1}_{3,3}$.

The above being said, let us now dive into the proof of Theorem 3.11.

The case $n = 1$ is trivial, as every 2-dimensional non-Abelian Lie algebra is isomorphic to $\mathfrak{aff}(\mathbb{R}) = \mathcal{G}_\psi$, where $\psi = \mathbb{I}_\mathbb{R}$, as in Example 4.1.1. Let $\mathcal{G}$ be a 2-solvable Frobenius Lie algebra of dimension $2n$, with $2 \leq n \leq 3$. Write $\mathcal{G} = \mathfrak{B} \ltimes \mathbb{R}^n$ where $\mathfrak{B}$ is an n-dimensional MASA of $\mathfrak{gl}(n,\mathbb{R})$. Note that, when $n \leq 3$, then $[\frac{n^2}{4}] + 1 = n$, so that from Jacobson's theorem ([13]), every n-dimensional Abelian subalgebra of $\mathfrak{gl}(n,\mathbb{R})$ is a MASA. Further set $\mathfrak{B} = \mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus L$, where L is a MASA of $\mathfrak{sl}(n,\mathbb{R})$.

When $n = 2$, then $L = \mathbb{R}M$, for some nonzero $M \in \mathfrak{sl}(2,\mathbb{R})$. But every nonzero $M \in \mathfrak{sl}(2,\mathbb{R})$ is nonderogatory. Thus, $\mathfrak{B} = \mathbb{R}[M]$ and $\mathfrak{B} \ltimes \mathbb{R}^2 = \mathcal{G}_M$. From Theorem 4.3, $\mathcal{G}_M$ is isomorphic to $\mathfrak{D}^2_0$, $\mathfrak{aff}(\mathbb{C})$, or $\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$. For $n = 3$, using the Lie algebras of lower dimensions listed above, the list of decomposable 2-solvable Frobenius Lie algebras of dimension 6 simply reads $\mathfrak{D}^2_0 \oplus \mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{C}) \oplus \mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$. As regards the non-decomposable ones, they are all of the form $\mathfrak{g} = (\mathbb{R}\mathbb{I}_{\mathbb{R}^3} \oplus \mathcal{A}) \ltimes \mathbb{R}^3$ with $\mathcal{A}$ a 2-dimensional AID MASA of $\mathfrak{sl}(3,\mathbb{R})$, since n is odd. We have 2 possibilities. The first is $\mathcal{A}^2 \neq 0$, in which case $\mathbb{R}\mathbb{I}_{\mathbb{R}^3} \oplus \mathcal{A} = \mathbb{R}[M]$, yielding the only solution $M = M_0$ and $\mathfrak{g} = \mathfrak{D}^3_0$. The second possibility is $\mathcal{A}^2 = 0$.

From Theorem 3.6, $\mathcal{A}$ is unique and the corresponding Lie algebra is $\mathbb{R}\mathbb{I}_{\mathbb{R}^3} \oplus \mathcal{A} = \mathfrak{g}_{3,1}$, as in Example 3.7. For the case $n = 4$, again using the Lie algebras of lower dimensions, we get the following list of decomposable 2-solvable Frobenius Lie algebras of dimension 8: $\mathfrak{D}^3_0 \oplus \mathfrak{aff}(\mathbb{R})$, $\mathfrak{D}^2_0 \oplus \mathfrak{D}^2_0$, $\mathfrak{D}^2_0 \oplus \mathfrak{aff}(\mathbb{C})$, $\mathfrak{D}^2_0 \oplus \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{C}) \oplus \mathfrak{aff}(\mathbb{C})$, $\mathfrak{aff}(\mathbb{C}) \oplus \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$ and $\mathcal{G}_{3,1} \oplus \mathfrak{aff}(\mathbb{R})$. We put the non-decomposable ones in the form $\mathfrak{g} = (\mathbb{R}\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{A}) \ltimes \mathbb{R}^4$ where $\mathcal{A}$ is either an AID or NAID 3-dimensional MASA of $\mathfrak{sl}(4,\mathbb{R})$. First, in the case where $\mathcal{A}$ is a NAID MASA of $\mathfrak{sl}(4,\mathbb{R})$, we write $\mathcal{A} = \mathbb{R}M_s \oplus \mathcal{A}'$, where $\mathcal{A}'$ a 2-dimensional MANS of $\mathfrak{sl}(2,\mathbb{C})$. Using a direct approach and writing elements of $\mathcal{A}'$ as upper triangular 4×4 matrices commuting with M_s, one easily sees that

$\mathcal{A}' = \{m_1(E_{1,3}+E_{2,4}) + m_2(E_{2,3}-E_{1,4}), m_1, m_2 \in \mathbb{R}\}$. This leads to $\mathbb{R}\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{A} = \mathbb{R}[M_{0,1}]$ and $(\mathbb{R}\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{A}) \ltimes \mathbb{R}^4 = \mathfrak{D}^4_{0,1}$, as in Example 4.1.2. Note that using lower triangular matrices leads to $\mathbb{R}[(M_{0,1})^T]$ which is conjugate to $\mathbb{R}[M_{0,1}]$. Now in the case where $\mathcal{A}$ is an AID, there are 3 possibilities.

(A) $\mathcal{A}^3 \neq 0$ and $\mathcal{A}^4 = 0$, there exists $a \in \mathcal{A}$ such that $a^3 \neq 0$ and $a^4 = 0$, so that $\mathbb{R}\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{A} = \mathbb{R}[a]$. From Theorem 4.3, we have $\mathbb{R}\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{A} = \mathbb{R}[M_0]$ and $(\mathbb{R}\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{A}) \ltimes \mathbb{R}^4 = \mathfrak{D}^4_0$. This corresponds to the case $A_{3,6}$ in [4], as $A_{3,6} = span(a, a^2 = b, a^3 = c)$.

(B) The second possibility is $\mathcal{A}^3 = 0$ and $\mathcal{A}^2 \neq 0$. From Lemma 3.5, we have $\dim Im(\mathcal{A}^2) = 1$. This splits into 3 cases (17):

(a) $A_{3,2}$: with $(A_{3,2})^2 = \mathbb{R}c$. A representative of this class is $\mathcal{P}_{4,2}$ as in Example 3.8 with $a = E_{1,2}+E_{2,4}$, $b = E_{1,3}$, $c = E_{1,4}$, so the corresponding Lie algebra is $(\mathbb{R}\mathbb{I}_{\mathbb{R}^4} \oplus A_{3,2}) \ltimes \mathbb{R}^4 = \mathfrak{h}_{4,2}$.

(b) $A^1_{3,3}$, with $(A^1_{3,3})^2 = \mathbb{R}c$. A representative of this class is $\mathcal{P}_{4,3}$ as in Example 3.8 with $a = E_{1,2}+E_{2,4}$, $b = E_{1,3}+E_{3,4}$, $c = E_{1,4}$, and $(\mathbb{R}\mathbb{I}_{\mathbb{R}^4} \oplus A^1_{3,3}) \ltimes \mathbb{R}^4 = \mathfrak{h}_{4,3}$.

(c) $A^{-1}_{3,3}$: A representative of this class is $L'_{4,4}$ as in Example 3.9 with $a = E_{1,2}+E_{3,4} -E_{1,3}-E_{2,4}$, $b = E_{1,2}+E_{3,4}+E_{1,3}+E_{2,4}$, $c = 2E_{1,4}$, so that $(\mathbb{R}\mathbb{I}_{\mathbb{R}^4} \oplus A^{-1}_{3,3}) \ltimes \mathbb{R}^4 = \mathcal{G}'_{4,4}$.

(C) The last case is given by the condition $\mathcal{A}^2 = 0$, which, from Theorem 3.6, provides a unique $\mathcal{A}$ and $(\mathbb{R}\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{A}) \ltimes \mathbb{R}^4 = \mathfrak{g}_{4,1}$, as in Example 3.7. Note that this corresponds to $\mathcal{A} = A_{3,1}$ in [4]. ■

5.2. A discussion on the lists by Winternitz and Zassenhaus

We revisit the classification lists of MASAs of $\mathfrak{sl}(n,\mathbb{R})$, $n = 3,4$ provided in [25] and [26]. A systematic comparison shows a match with our classification, except for one missing item which we complete and correct some misprint on another item.

For $n = 3$, according to [25], there are six classes of non-mutually conjugate MASAs of $\mathfrak{sl}(3,\mathbb{R})$. With the same notation as in [25], we denote them by $L_{2,i}$, $i = 1,\dots,6$.

(1) The first is $L_{2,1} := \{diag(k_1+k_2, -k_1+k_2, -2k_2),\ k_1,k_2 \in \mathbb{R}\}$. We note that $L_{2,1} := \{(k_2-k_1)S^0_{2,1} + \frac{1}{2}(k_1+3k_2)S_{2,1} + \frac{3}{2}(k_1-k_2)S^2_{2,1},\ k_1,k_2 \in \mathbb{R}\} \subset \mathbb{R}[S_{2,1}]$, where $S_{2,1} = diag(1,0,-1)$, and $\mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus L_{2,1} = \mathbb{R}[S_{2,1}]$. So $(\mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus L_{2,1}) \ltimes \mathbb{R}^3$ is isomorphic to $\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R})$, as $S_{2,1}$ is nonderogatory with 3 eigenvalues.

(2) In $L_{2,2} := \big\{k_1(E_{1,1}+E_{2,2}-2E_{3,3}) + k_2(E_{1,2}-E_{2,1}), k_1,k_2 \in \mathbb{R}\big\}$, every element is a nonderogatory matrix with one real eigenvalue $-2k_1$ and two complex conjugate eigenvalues k_1+ik_2 and k_1-ik_2, except when $k_2 = 0$. For instance, $S_{2,2} := E_{1,2}-E_{2,1} \in L_{2,2}$ is nonderogatory, so $\mathbb{R}\mathbb{I}_{\mathbb{R}^3} \oplus L_{2,2} = \mathbb{R}[S_{2,2}]$. From Theorem 4.3, $(\mathbb{R}\mathbb{I}_{\mathbb{R}^3} \oplus L_{2,2}) \ltimes \mathbb{R}^3$ is isomorphic to $\mathfrak{aff}(\mathbb{C}) \oplus \mathfrak{aff}(\mathbb{R})$.

(3) In $L_{2,3} := \big\{k_1(E_{1,1}+E_{2,2}-2E_{3,3}) + k_2E_{1,2}, k_1,k_2 \in \mathbb{R}\big\}$, each element is of the form $k_1S^0_{2,3} + k_2S_{2,3} - (3k_1+k_2)S^2_{2,3}$ where $S_{2,3} := E_{1,2}+E_{3,3}$ is nonderogatory with the double eigenvalue 0 and the simple eigenvalue 1. Hence, $\mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus L_{2,3} = \mathbb{R}[S_{2,3}]$ and $(\mathbb{R}\mathbb{I}_{\mathbb{R}^n} \oplus L_{2,3}) \ltimes \mathbb{R}^3 = \mathcal{G}_{S_{2,3}}$ is isomorphic to $\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{D}^2_0$ (Theorem 4.3).

(4) As regards $L_{2,5} := \big\{k_1E_{1,2} + k_2E_{1,3}, k_1,k_2 \in \mathbb{R}\big\}$, the algebra $\mathbb{R}\mathbb{I}_{\mathbb{R}^3} \oplus L_{2,5}$ is the same as $\mathfrak{B}_{3,1}$ in Example 3.7 and $\mathfrak{B}_{3,1} \ltimes \mathbb{R}^3 = \mathcal{G}_{3,1}$ is indecomposable.

(5) $L_{2,6} := \big\{k_1(E_{1,2}+E_{2,3}) + k_2E_{1,3}, k_1,k_2 \in \mathbb{R}\big\}$ is such that $\mathbb{R}\mathbb{I}_{\mathbb{R}^3} \oplus L_{2,6} = \mathbb{R}[M_0]$, where the nonderogatory matrix $M_0 = E_{1,2}+E_{2,3}$ has 0 as a unique eigenvalue of multiplicity 3. Thus, $(\mathbb{R}\mathbb{I}_{\mathbb{R}^3} \oplus L_{2,6}) \ltimes \mathbb{R}^3 = \mathfrak{D}^3_0$, (Theorem 4.3).

(6) For the remaining algebra $L_{2,4} := \{k_{1,3}E_{1,3} + k_{2,3}E_{2,3}, k_{1,3}, k_{2,3} \in \mathbb{R}\}$ in the list in [25], it is easy to see that $(\mathbb{R}\mathbb{I}_{\mathbb{R}^3} \oplus L_{2,4}) \ltimes \mathbb{R}^3$ is not a Frobenius Lie algebra, as every linear form α satisfies $(\partial\alpha)^3 = 0$. See also Remark 3.10.

For $n = 4$, we treat the pair-wise non-conjugate 16 MASAs of $\mathfrak{sl}(4,\mathbb{R})$ in the order in which they appear in the classification list in [26] and name them $\mathcal{Y}_j$, $j = 1,\dots,16$. We correct the algebra $\mathcal{Y}_8$ from its original expression in [26] which was not commutative and complete the list of MASAs from which the MASA $\mathcal{Y}_{17}$ giving rise to $\mathfrak{D}_0^2 \oplus \mathfrak{D}_0^2$ is missing.

1. $\mathcal{Y}_1 := \{k_{1,3}E_{1,3} + k_{1,4}E_{1,4} + k_{2,3}E_{2,3} + k_{2,4}E_{2,4},\ k_{1,3}, k_{1,4}, k_{2,3}, k_{2,4} \in \mathbb{R}\}$, is 4-dimensional, thus it is not relevant to our study.
2. $\mathcal{Y}_2 := \{k_{1,2}E_{1,2} + k_{1,3}E_{1,3} + k_{1,4}E_{1,4},\ k_{1,2}, k_{1,3}, k_{1,4} \in \mathbb{R}\}$ is the algebra $L_{4,1}$ in Example 3.7, so the 2-solvable Frobenius Lie algebra $(\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{Y}_2) \ltimes \mathbb{R}^4$ is isomorphic to $\mathcal{G}_{4,1}$.
3. $\mathcal{Y}_3 := \{k_{1,4}E_{1,4} + k_{2,4}E_{2,4} + k_{3,4}E_{3,4},\ k_{1,4}, k_{2,4}, k_{3,4} \in \mathbb{R}\}$ is the algebra L_n in Remark 3.10, when $n = 4$. So $(\mathbb{R}\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{Y}_3) \ltimes \mathbb{R}^4$ is not a Frobenius Lie algebra.
4. $\mathcal{Y}_4 := \{k_{1,3}(E_{1,3}+E_{3,4})+k_{1,4}E_{1,4}+k_{2,4}E_{2,4},\ k_{1,3}, k_{1,4}, k_{2,4} \in \mathbb{R}\}$ yields a 2-solvable Lie algebra which is not a Frobenius Lie algebra. Indeed, if we set $e_1 := \mathbb{I}_{\mathbb{R}^4}$, $e_2 := E_{1,3} + E_{3,4}$, $e_3 := E_{1,4}$, $e_4 = E_{2,4}$, the Lie bracket of $(\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{Y}_4) \ltimes \mathbb{R}^4$ is $[e_1, \tilde{e}_j] = \tilde{e}_j$, $j = 1,\dots,4$, $[e_2, \tilde{e}_3] = \tilde{e}_1$, $[e_2, \tilde{e}_4] = \tilde{e}_3$, $[e_3, \tilde{e}_4] = \tilde{e}_1$, $[e_4, \tilde{e}_4] = \tilde{e}_2$. Any linear form $\alpha = k_1e_1^* + k_2e_2^* + k_3e_3^* + k_4e_4^* + s_1\tilde{e}_1^* + s_2\tilde{e}_2^* + s_3\tilde{e}_3^* + s_4\tilde{e}_4^*$ satisfies $(\partial\alpha)^4 = 0$, as we have $\partial\alpha = -e_1^* \wedge \eta_1 - e_2^* \wedge \eta_2 - \eta_3 \wedge \tilde{e}_4^*$, where $\eta_1^* := s_1\tilde{e}_1^* + s_2\tilde{e}_2^* + s_3\tilde{e}_3^* + s_4\tilde{e}_4^*$, $\eta_2 := s_1\tilde{e}_3^* + s_3\tilde{e}_4^*$, $\eta_3 := s_1e_3^* + s_2e_4^*$.
5. $\mathcal{Y}_5 := \{k_{1,2}(E_{1,2}+E_{2,4})+k_{1,3}E_{1,3}+k_{1,4}E_{1,4}, k_{1,2}, k_{1,3}, k_{1,4} \in \mathbb{R}\}$ is the MASA $\mathcal{P}_{4,2}$ of $\mathfrak{sl}(4,\mathbb{R})$ given in Example 3.8. So $(\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{Y}_5) \ltimes \mathbb{R}^4$ is the 2-solvable Frobenius Lie algebra $\mathfrak{h}_{4,2}$ in Example 3.8.
6. $\mathcal{Y}_{6,\varepsilon} := \{k_{1,2}(E_{1,2} + E_{2,4}) + k_{1,3}(E_{1,3} + \varepsilon E_{3,4}) + k_{1,4}E_{1,4},\ k_{1,2}, k_{1,3}, k_{1,4} \in \mathbb{R}\}$, $\varepsilon = \pm 1$, yields a 2-solvable Frobenius Lie algebra $(\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{Y}_{6,\varepsilon}) \ltimes \mathbb{R}^4$ with the following Lie brackets, in the basis $e_1 := \mathbb{I}_{\mathbb{R}^4}$, $e_2 := E_{1,2}+E_{2,4}$, $e_3 := E_{1,3}+\varepsilon E_{3,4}$, $e_4 = E_{1,4}$, $\tilde{e}_j$, $1 \le j \le 4$: $[e_1, \tilde{e}_j] = \tilde{e}_j$, $j = 1,\dots,4$, $[e_2, \tilde{e}_2] = \tilde{e}_1$, $[e_2, \tilde{e}_4] = \tilde{e}_2$, $[e_3, \tilde{e}_3] = \tilde{e}_1$, $[e_3, \tilde{e}_4] = \varepsilon\tilde{e}_3$, $[e_4, \tilde{e}_4] = \tilde{e}_1$. Furthermore we note that the form $\partial\tilde{e}_1^* = -e_1^*\wedge\tilde{e}_1^* - e_2^*\wedge\tilde{e}_2^* - e_3^*\wedge\tilde{e}_3^* - e_4^*\wedge\tilde{e}_4^*$ is nondegenerate. Note that $\mathcal{Y}_{6,1}$ coincides with $\mathcal{P}_{4,3}$ in Example 3.8. So $(\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{Y}_{6,1}) \ltimes \mathbb{R}^4$ is the same as $\mathfrak{h}_{4,3}$ in Example 3.8. When $\varepsilon = -1$, the linear map defined by $\psi : (\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{Y}_{6,-1}) \ltimes \mathbb{R}^4 \to \mathcal{G}'_{4,4}$, $\psi(e_1) = e'_1$, $\psi(e_2) = e'_2 - e'_3$, $\psi(e_3) = -e'_2 - e'_3$, $\psi(e_4) = -2e'_4$, $\psi(\tilde{e}_2) = -\tilde{e}'_2 + \tilde{e}'_3$, $\psi(\tilde{e}_3) = \tilde{e}'_2 + \tilde{e}'_3$, $\psi(\tilde{e}_1) = -4\tilde{e}'_1$, is a Lie algebra isomorphism, where $\mathcal{G}'_{4,4}$ is as in Example 3.9. So $\mathcal{Y}_{6,-1}$ is conjugate to $L'_{4,4}$.
7. $\mathcal{Y}_7 := \{k_{1,2}(E_{1,2} + E_{2,3} + E_{3,4}) + k_{1,3}(E_{1,3} + E_{2,4}) + k_{1,4}E_{1,4},\ k_{1,2}, k_{1,3}, k_{1,4} \in \mathbb{R}\}$ satisfies $\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_7 = \mathbb{R}[M_0]$, with $M_0 = E_{1,2}+E_{2,3}+E_{3,4}$. So $(\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_7)\ltimes\mathbb{R}^4 = \mathfrak{D}_0^4$.
8. $\mathcal{Y}_8 := \{k_{1,2}(E_{1,2} - E_{2,1} + E_{3,4} - E_{4,3}) + k_{1,3}(E_{1,3} + E_{2,4}) + k_{1,4}(E_{1,4} - E_{2,3})$, $k_{1,2}, k_{1,3}, k_{1,4} \in \mathbb{R}\}$ is such that $\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{Y}_8 = \mathbb{R}[M_{0,1}]$, where
$$M_{0,1} = E_{1,2} - E_{2,1} + E_{3,4} - E_{4,3} + E_{1,3} + E_{2,4}.$$
Thus $(\mathbb{I}_{\mathbb{R}^4} \oplus \mathcal{Y}_8) \ltimes \mathbb{R}^4 = \mathfrak{D}_{0,1}^4$. Note that, in the original list in [26], this item was listed with some mistakes as
$\{k_{1,2}(E_{1,2} - E_{2,1} + E_{2,4} - E_{4,3}) + k_{1,3}(E_{1,3} + E_{2,4}) + k_{1,4}E_{1,4},\ k_{1,2}, k_{1,3}, k_{1,4} \in \mathbb{R}\}$
which is not commutative.

9. $\mathcal{Y}_9 := \{k_{1,1}(E_{1,1}+E_{2,2}+E_{3,3}-3E_{4,4})+k_{1,2}E_{1,2}+k_{1,3}E_{1,3}\,,\ k_{1,1},k_{1,2},k_{1,3}\in\mathbb{R}\}$, consider the basis $e_1=\mathbb{I}_{\mathbb{R}^n}\,,\ e_2=E_{1,1}+E_{2,2}+E_{3,3}-3E_{4,4}\,,\ e_3=E_{1,2}\,,\ e_4:=E_{1,3}$ of $\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_9$ and change it into $e'_1:=\frac{1}{4}(3e_1+e_2),\ e'_2=e_3\,,\ e'_3=e_4,\ e'_4:=\frac{1}{4}(e_1-e_2)$. The Lie bracket of $(\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_9)\ltimes\mathbb{R}^4$ reads $[e'_1,\tilde e_1]=\tilde e_1\,,\ [e'_1,\tilde e_2]=\tilde e_2\,,\ [e'_1,\tilde e_3]=\tilde e_3$, $[e'_2,\tilde e_2]=\tilde e_1,\ [e_3,\tilde e_3]=\tilde e_1\,,\ [e'_4,\tilde e_4]=\tilde e_4$. That is clearly the Lie bracket of $\mathcal{G}_{3,1}\oplus\mathfrak{aff}(\mathbb{R})$, where $\mathcal{G}_{3,1}=span(e'_1,e'_2,e'_3,\tilde e_1,\tilde e_2,\tilde e_3)$ corresponds to $n=3,p=1$ in Example 3.7.

10. $\mathcal{Y}_{10}:=\{k_{1,1}(E_{1,1}+E_{2,2}+E_{3,3}-3E_{4,4})+k_{1,3}E_{1,3}+k_{2,3}E_{2,3},\ k_{1,1},k_{1,3},k_{2,3}\in\mathbb{R}\}$, a basis of $\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{10}$ is $e_1=\mathbb{I}_{\mathbb{R}^4}\,,\ e_2=E_{1,1}+E_{2,2}+E_{3,3}-3E_{4,4}\,,\ e_3=E_{1,3}\,,$ $e_4:=E_{2,3}$ which we change into $e'_1:=\frac{1}{4}(3e_1+e_2),\ e'_2=e_3\,,\ e'_3=e_4,\ e'_4:=\frac{1}{4}(e_1-e_2)$. The Lie bracket of $(\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{10})\ltimes\mathbb{R}^4$ reads $[e'_1,\tilde e_1]=\tilde e_1\,,\ [e'_1,\tilde e_2]=\tilde e_2\,,$ $[e'_1,\tilde e_3]=\tilde e_3,\ [e'_2,\tilde e_3]=\tilde e_1,\ [e'_3,\tilde e_3]=\tilde e_2\,,\ [e'_4,\tilde e_4]=\tilde e_4$, which is the same as that of $(\mathfrak{B}_3\ltimes\mathbb{R}^3)\oplus\mathfrak{aff}(\mathbb{R})$, where $\mathfrak{B}_3=$span(e'_1,e'_2,e'_3) is as in Remark 3.10, when $n=3$ and $\mathbb{R}^3=$span$(\tilde e_1,\tilde e_2,\tilde e_3)$. As $\mathfrak{B}_3\ltimes\mathbb{R}^3$ is not a Frobenius Lie algebra (Remark 3.10), neither is $(\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{10})\ltimes\mathbb{R}^4$.

11. $\mathcal{Y}_{11}:=\{k_{1,1}(E_{1,1}+E_{2,2}+E_{3,3}-3E_{4,4})+k_{1,2}(E_{1,2}+E_{2,3})+k_{1,3}E_{1,3}\,,\ k_{1,1}\,,$ $k_{1,2},k_{1,3}\in\mathbb{R}\}$, a basis of $\mathbb{R}\mathbb{I}_{\mathbb{R}^n}\oplus\mathcal{Y}_{11}$ is $e_1=\mathbb{I}_{\mathbb{R}^n}\,,\ e_2=E_{1,1}+E_{2,2}+E_{3,3}-3E_{4,4}\,,$ $e_3=E_{1,2}+E_{2,3}\,,\ e_4:=E_{1,3}=e_2^2$, which we change into $e'_1:=\frac{1}{4}(3e_1+e_2)$, $e'_2=e_3\,,\ e'_3=e_4,\ e'_4:=\frac{1}{4}(e_1-e_2)$, so that the Lie bracket of $(\mathbb{R}\mathbb{I}_{\mathbb{R}^n}\oplus\mathcal{Y}_{11})\ltimes\mathbb{R}^4$ reads $[e'_1,\tilde e_1]=\tilde e_1,\ [e'_1,\tilde e_2]=\tilde e_2,\ [e'_1,\tilde e_3]=\tilde e_3,\ [e'_2,\tilde e_2]=\tilde e_1,\ [e'_2,\tilde e_3]=\tilde e_2,$ $[e'_3,\tilde e_3]=\tilde e_1,\ [e'_4,\tilde e_4]=\tilde e_4$. This is the Lie bracket of $\mathfrak{D}_0^3\oplus\mathfrak{aff}(\mathbb{R})$, where $\mathfrak{D}_0^3=span(e'_1,e'_2,e'_3,\tilde e_1,\tilde e_2,\tilde e_3)=\mathbb{R}[e'_2]\ltimes\mathbb{R}^3$.

12. $\mathcal{Y}_{12}:=\{k_{1,1}(E_{1,1}+E_{2,2}-E_{3,3}-E_{4,4})+k_{1,2}(E_{1,2}-E_{2,1})+k_{3,4}(E_{3,4}-E_{4,3}),\ k_{1,1},k_{1,2},$ $k_{3,4}\in\mathbb{R}\}$, a basis of $\mathbb{R}\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{12}$ is $e_1=\mathbb{I}_{\mathbb{R}^n}\,,\ e_2=E_{1,1}+E_{2,2}-E_{3,3}-E_{4,4}\,,$ $e_3=E_{1,2}-E_{2,1}\,,\ e_4:=E_{3,4}-E_{4,3}$, we change it into $e'_1=\frac{1}{2}(e_1+e_2),\ e'_2=e_3\,,$ $e'_3=\frac{1}{2}(e_1-e_2),\ e'_4=e_4$. The Lie bracket of $(\mathbb{R}\mathbb{I}_{\mathbb{R}^n}\oplus\mathcal{Y}_{12})\ltimes\mathbb{R}^4$ is given by $[e'_1,\tilde e_1]=\tilde e_1\,,\ [e'_1,\tilde e_2]=\tilde e_2,\ [e'_2,\tilde e_1]=-\tilde e_2,\ [e'_2,\tilde e_2]=\tilde e_1,\ [e'_3,\tilde e_3]=\tilde e_3\,,\ [e'_3,\tilde e_4]=\tilde e_4,$ $[e'_4,\tilde e_3]=-\tilde e_4,\ [e'_4,\tilde e_4]=\tilde e_3$. This is the direct sum $\mathcal{I}_1\oplus\mathcal{I}_2$ of two ideals $\mathcal{I}_1=span(e'_1,e'_2,\tilde e_1,\tilde e_2)$ and $\mathcal{I}_2=span(e'_3,e'_4,\tilde e_3,\tilde e_4)$. Both $\mathcal{I}_1$ and $\mathcal{I}_2$ are isomorphic to $\mathfrak{aff}(\mathbb{C})$. So $(\mathbb{R}\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{12})\ltimes\mathbb{R}^4$ is isomorphic to $\mathfrak{aff}(\mathbb{C})\oplus\mathfrak{aff}(\mathbb{C})$.

13. $\mathcal{Y}_{13}:=\{k_{1,1}(E_{1,1}+E_{2,2}-E_{3,3}-E_{4,4})+k_{1,2}(E_{1,2}-E_{2,1})+k_{3,4}E_{3,4}\,,\ k_{1,1},k_{1,2},$ $k_{3,4}\in\mathbb{R}\}$, consider the following basis of $\mathbb{R}\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{13}$: $e_1=\mathbb{I}_{\mathbb{R}^n}\,,\ e_2=E_{1,1}+$ $E_{2,2}-E_{3,3}-E_{4,4}\,,\ e_3=E_{1,2}-E_{2,1}\,,\ e_4:=E_{3,4}$. In the new basis $e'_1=$ $\frac{1}{2}(e_1+e_2),\ e'_2=e_3\,,\ e'_3=\frac{1}{2}(e_1-e_2),\ e'_4=e_4$, we get the following Lie bracket of $(\mathbb{R}\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{13})\ltimes\mathbb{R}^4$: $[e'_1,\tilde e_1]=\tilde e_1\,,\ [e'_1,\tilde e_2]=\tilde e_2,\ [e'_2,\tilde e_1]=-\tilde e_2,\ [e'_2,\tilde e_2]=\tilde e_1,$ $[e'_3,\tilde e_3]=\tilde e_3\,,\ [e'_3,\tilde e_4]=\tilde e_4,\ [e'_4,\tilde e_4]=\tilde e_3$. This is the Lie bracket of $\mathfrak{aff}(\mathbb{C})\oplus\mathfrak{D}_0^2$, with $\mathfrak{aff}(\mathbb{C})=span(e'_1,e'_2,\tilde e_1,\tilde e_2)$ and $\mathfrak{D}_0^2=span(e'_3,e'_4,\tilde e_3,\tilde e_4)$.

14. $\mathcal{Y}_{14}:=\{k_{1,1}(E_{1,1}+E_{2,2}-E_{3,3}-E_{4,4})+k_{1,2}(E_{1,2}-E_{2,1})+k_{3,3}(E_{3,3}-E_{3,4}),$ $k_{1,1},\ k_{1,2},\ k_{3,3}\in\mathbb{R}\}$, consider the following basis of $\mathbb{R}\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{14}$: $e_1=\mathbb{I}_{\mathbb{R}^n}\,,$ $e_2=E_{1,1}+E_{2,2}-E_{3,3}-E_{4,4}\,,\ e_3=E_{1,2}-E_{2,1}\,,\ e_4:=E_{3,3}-E_{4,4}$. In the basis $e'_1=\frac{1}{2}(e_1+e_2),\ e'_2=e_3,\ e'_3=\frac{1}{4}(e_1-e_2+2e_4),\ e'_4=\frac{1}{4}(e_1-e_2-2e_4),\ \tilde e_j,$ $j=1,2,3,4$, the Lie bracket of $(\mathbb{R}\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{14})\ltimes\mathbb{R}^4$ is: $[e'_1,\tilde e_1]=\tilde e_1\,,\ [e'_1,\tilde e_2]=\tilde e_2,$ $[e'_2,\tilde e_1]=-\tilde e_2,\ [e'_2,\tilde e_2]=\tilde e_1,\ [e'_3,\tilde e_3]=\tilde e_3\,,\ [e'_4,\tilde e_4]=\tilde e_4$. This is the Lie bracket of $\mathfrak{aff}(\mathbb{C})\oplus\mathfrak{aff}(\mathbb{R})\oplus\mathfrak{aff}(\mathbb{R})$, where $\mathfrak{aff}(\mathbb{C})=$span$(e'_1,e'_2,\tilde e_1,\tilde e_2)$ and the two copies of $\mathfrak{aff}(\mathbb{R})$ are span$(e'_3,\tilde e_3)$, span$(e'_4,\tilde e_4)$.

15. $\mathcal{Y}_{15} := \{k_{1,1}(E_{1,1}+E_{2,2}-E_{3,3}-E_{4,4})+k_{1,2}E_{1,2}+k_{3,3}(E_{3,3}-E_{4,4}),\ k_{1,1},k_{1,2},k_{3,3}\in\mathbb{R}\}$, change the basis $e_1 = \mathbb{I}_{\mathbb{R}^n}$, $e_2 = E_{1,1}+E_{2,2}-E_{3,3}-E_{4,4}$, $e_3 = E_{1,2}$, $e_4 = E_{3,3}-E_{4,4}$ of $\mathbb{R}\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{15}$, into $e_1' = \frac{1}{2}(e_1+e_2)$, $e_2' = e_3$, $e_3' = \frac{1}{4}(e_1-e_2+2e_4)$, $e_4' = \frac{1}{4}(e_1-e_2-2e_4)$. The Lie bracket of $(\mathbb{R}\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{15})\ltimes\mathbb{R}^4$ is $[e_1',\tilde e_1] = \tilde e_1$, $[e_1',\tilde e_2] = \tilde e_2$, $[e_2',\tilde e_2] = \tilde e_1$, $[e_3',\tilde e_3] = \tilde e_3$, $[e_4',\tilde e_4] = \tilde e_4$. This is the Lie bracket of $\mathfrak{D}_0^2\oplus\mathfrak{aff}(\mathbb{R})\oplus\mathfrak{aff}(\mathbb{R})$, where $\mathfrak{D}_0^2 = \mathrm{span}(e_1', e_2, \tilde e_1, \tilde e_2)$.

16. $\mathcal{Y}_{16} := \{k_1(E_{1,1}+E_{2,2}+E_{3,3}-3E_{4,4})+k_2(E_{1,1}+E_{2,2}-2E_{3,3})+k_3(E_{1,1}-E_{2,2})$, $k_1, k_2, k_3\in\mathbb{R}\}$, we consider the basis $e_1 = \mathbb{I}_{\mathbb{R}^n}$, $e_2 = E_{1,1}+E_{2,2}+E_{3,3}-3E_{4,4}$, $e_3 = E_{1,1}+E_{2,2}-2E_{3,3}$, $e_4 := E_{1,1}-E_{2,2}$ of $\mathbb{R}\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{16}$, then change it to $e_1' = \frac{1}{12}(3e_1+e_2+2e_3+6e_4)$, $e_2' = \frac{1}{12}(3e_1+e_2+2e_3-6e_4)$, $e_3' = \frac{1}{4}(3e_1+e_2-4e_3)$, $e_4' = \frac{1}{4}(e_1-e_2)$. The Lie bracket of $(\mathbb{R}\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{16})\ltimes\mathbb{R}^4$ is: $[e_1',\tilde e_1] = \tilde e_1$, $[e_2',\tilde e_2] = \tilde e_2$, $[e_3',\tilde e_3] = \tilde e_3$, $[e_4',\tilde e_4] = \tilde e_4$. This is the Lie bracket of

$$\mathfrak{aff}(\mathbb{R})\oplus\mathfrak{aff}(\mathbb{R})\oplus\mathfrak{aff}(\mathbb{R})\oplus\mathfrak{aff}(\mathbb{R}),$$

where the copies of $\mathfrak{aff}(\mathbb{R})$ are $span(e_j',\tilde e_j)$, $j = 1,2,3,4$, respectively.

17. $\mathcal{Y}_{17} := \{k_{1,1}(E_{1,1}+E_{2,2}-E_{3,3}-E_{4,4})+k_{1,2}E_{1,2}+k_{3,4}E_{3,4},\ k_{1,1},k_{1,2},k_{3,4}\in\mathbb{R}\}$. Consider the basis $e_1 = \mathbb{I}_{\mathbb{R}^n}$, $e_2 = E_{1,1}+E_{2,2}-E_{3,3}-E_{4,4}$, $e_3 = E_{1,2}$, $e_4 := E_{3,4}$ of $\mathbb{R}\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{17}$ and further change it into $e_1' = \frac{1}{2}(e_1+e_2)$, $e_2' = e_3$, $e_3' = \frac{1}{2}(e_1-e_2)$, $e_4' = e_4$. The Lie bracket of $(\mathbb{R}\mathbb{I}_{\mathbb{R}^4}\oplus\mathcal{Y}_{17})\ltimes\mathbb{R}^4$ reads: $[e_1',\tilde e_1] = \tilde e_1$, $[e_1',\tilde e_2] = \tilde e_2$, $[e_2',\tilde e_2] = \tilde e_1$, $[e_3',\tilde e_3] = \tilde e_3$, $[e_3',\tilde e_4] = \tilde e_4$, $[e_4',\tilde e_4] = \tilde e_3$. This is the Lie bracket of the direct sum $\mathfrak{D}_0^2\oplus\mathfrak{D}_0^2$, where the copies of $\mathfrak{D}_0^2$ are $\mathrm{span}(e_1', e_2', \tilde e_1, \tilde e_2)$ and $span(e_3', e_4', \tilde e_3, \tilde e_4)$. The present item is missing in the list in [26].

6. Appendix: Jordan form of nonderogatory real Matrices

Theorem 6.1. (Jordan form of real endomorphisms) *Let V be a real vector space, $\dim V = n$. Let $\phi\in\mathfrak{gl}(V)$ be nonderogatory. There exists a basis of V in which the matrix $[\phi]$ of ϕ has a Jordan form, more precisely, the following hold.*

(A) *If ϕ has a unique eigenvalue λ with multiplicity n, then*

$$[\phi] = \lambda\mathbb{I}_V + \sum_{i=1}^{n-1} E_{i,i+1}.$$

(B) *If ϕ has n distinct complex eigenvalues λ_j, $\bar\lambda_j$, $j = 1,\dots,\frac{n}{2}$ then*

$$[\phi] = diag(J_1,\dots,J_{\frac{n}{2}}), \text{ where } J_j = \begin{pmatrix} Re(\lambda_j) & -Im(\lambda_j)\\ Im(\lambda_j) & Re(\lambda_j)\end{pmatrix},\ Im(\lambda_j)\neq 0.$$

(C) *If ϕ has only two eigenvalues which are both complex $z = r+is$ and $\bar z$, where $r,s\in\mathbb{R}$, $s\neq 0$, and if $n\geq 4$ (in which case ϕ is not diagonalizable in $\mathbb{C}$), then $[\phi]$ has the form, with M_s, M_n as in* (8):

$$[\phi] = r\mathbb{I}_V + sM_s + M_n =: M_z\,. \tag{18}$$

(D) *In the general case, $[\phi]$ is a block diagonal matrix with blocks in the form* (A), (B), (C) *or diagonal with distinct diagonal real entries.*

6.1. Proof of Theorem 6.1

Let $\phi \in \mathfrak{gl}(V)$ be nonderogatory, where V is a real n-dimensional vector space. We write $\chi_\phi(X) = Q_1(X)Q_2(X)$ where $Q_1(X)$ has only complex (nonreal) zeros and the zeros of $Q_2(X)$ are all real, so that $V = \ker(\chi_\phi(\phi)) = \ker(Q_1(\phi)) \oplus \ker(Q_2(\phi))$. The result follows by noting again that ϕ preserves both $\ker(Q_1(\phi))$ and $\ker(Q_2(\phi))$ and its restriction to each of them has a Jordan form, as discussed below. In fact, $Q_1(X)$ factorizes as $Q_1(X) = Q_{1a}(X)Q_{1b}(X)$ where all the zeros of $Q_{1a}(X)$ have multiplicity 1 and we can further factorize $Q_{1b}(X)$ into factors, each having only 2 (complex conjugate) zeros with multiplicity greater than 1. So the latter boils down to the case where ϕ has only two complex conjugate eigenvalues, as discussed in (C) below.

(A) Suppose ϕ has p distinct eigenvalues $\lambda_1, \dots, \lambda_p$, all of which are real and of respective multiplicity $k_1, \dots, k_p$, with $k_1 + \cdots + k_p = n$, so that its characteristic polynomial factorizes in the form $\chi_\phi(X) = \chi_1(X)\chi_2(X)\cdots\chi_p(X)$, where we have $\chi_j(X) = (X-\lambda_j)^{k_j}$ for $j = 1, \dots, p$. The polynomials $\chi_j(X)$ are pairwise relatively prime. By the Primary Decomposition Theorem and the Cayley-Hamilton theorem, we have

$$V = \ker \chi_\phi(\phi) = \ker \chi_1(\phi) \oplus \ker \chi_2(\phi) \oplus \cdots \oplus \ker \chi_p(\phi). \tag{19}$$

Of course the subspaces $\mathcal{E}_j := \ker(\phi - \lambda_j)^{k_j}$, $j = 1, \dots, p$, are all stable by ϕ and the restriction of ϕ to $\mathcal{E}_j$ is again a nonderogatory endomorphism of $\mathcal{E}_j$ with a unique eigenvalue, for every $j = 1, \dots, p$. So this reduces to the case where ϕ has n distinct real eigenvalues and is hence diagonalizable, or ϕ has a unique real eigenvalue λ with multiplicity n. We now assume the latter case. Up to a sign,

$$\chi_\phi(X) = (X-\lambda)^n = \sum_{j=0}^{n} \mathtt{C}_n^j (-1)^{n-j} \lambda^{n-j} X^j, \quad \text{with} \quad \mathtt{C}_p^q = \frac{p!}{q!(p-q)!}, \text{ for } p \geq q.$$

By Cayley-Hamilton's Theorem we have

$$\phi^n = \sum_{j=0}^{n-1} \mathtt{C}_n^j (-1)^{n-j+1} \lambda^{n-j} \phi^j.$$

Choose $\bar{x} \in V$ such that $(\bar{e}_1, \dots, \bar{e}_n) = (\phi^{n-1}\bar{x}, \ \phi^{n-2}\bar{x}, \ \dots, \ \phi^{n-j}\bar{x}, \dots, \ \bar{x})$ is a basis of V. We thus have

$$\phi\bar{e}_1 = \phi^n \bar{x} = \sum_{j=0}^{n-1} \mathtt{C}_n^j (-1)^{n-j+1} \lambda^{n-j} \phi^j \bar{x} = \sum_{j=1}^{n} \mathtt{C}_n^j (-1)^{j+1} \lambda^j \bar{e}_j$$

and $\phi\bar{e}_j = \bar{e}_{j-1}$, for $j = 2, \dots, n$. Therefore the matrix of ϕ has the form

$$\tilde{M}_\lambda = \textstyle\sum_{j=1}^{n} \tilde{k}_j E_{j,1} + \sum_{j=1}^{n-1} E_{j,j+1}, \quad \text{with} \quad \tilde{k}_j = \mathtt{C}_n^j (-1)^{j+1} \lambda^j.$$

If $\lambda = 0$, then we are done. If $\lambda \neq 0$, we use a direct approach by looking for the coefficients $p_{l,j}$ of a matrix P such that we have $\tilde{M}_\lambda P = P M_\lambda$, where $M_\lambda = \lambda \mathbb{I}_V + \sum_{l=1}^{n-1} E_{l,l+1}$. We get

$$p_{k,l} = \sum_{j=0}^{l-1} (-1)^{k-l+j} \lambda^{k-l+j} \mathtt{C}_{n-l+j}^{k-l+j} p_{1,j+1},$$

where the numbers $p_{1,j+1}$, $j = 0, \dots, n-1$, are seen as parameters.

In particular, the matrix P is a solution, whose coefficients in the above basis $(\bar{e}_s)$ are

$$p_{kl} = \begin{cases} (-1)^{k-l}\lambda^{k-l}\mathtt{C}_{n-l}^{k-l} & \text{if} \quad k \geq l, \\ 0 & \text{if} \quad k < l \end{cases} .$$

(B) Suppose all the eigenvalues of ϕ are complex (and nonreal) and ϕ is diagonalizable in $\mathbb{C}$. Equivalently, ϕ has n distinct complex eigenvalues, say, λ_j, $\bar{\lambda}_j$, $j = 1, \ldots, \frac{n}{2}$, where of course, here, $\bar{\lambda}_j$ is the complex conjugate of λ_j. Let $\lambda = \lambda_R - i\lambda_I$ be an eigenvalue of ϕ, where λ_R, λ_I are real numbers and $\lambda_I \neq 0$. Consider an eigenvector v (with complex components) of ϕ with corresponding eigenvalue λ. Further set $\mathcal{E}_\lambda := \{zv + \overline{zv}, z \in \mathbb{C}\}$. Note that, as $\bar{v}$ is also an eigenvector of ϕ with eigenvalue $\bar{\lambda}$, we also have $\mathcal{E}_\lambda = \mathcal{E}_{\bar{\lambda}}$. Furthermore, $\mathcal{E}_\lambda$ is a real 2-dimensional vector subspace of V which is stable by ϕ and the vectors $Re(v)$ and $Im(v)$ form a basis of $\mathcal{E}_\lambda$ in which the matrix of the restriction of ϕ is

$$M_\lambda := \begin{pmatrix} \lambda_R & -\lambda_I \\ \lambda_I & \lambda_R \end{pmatrix}.$$

Indeed, any element of $\mathcal{E}_\lambda$ is of the form $zv + \overline{zv} = 2Re(z)\ Re(v) - 2Im(z)\ Im(v)$ and taking the real and imaginary parts of both sides of the equation

$$\begin{aligned} \phi v &= \lambda v = \big(\lambda_R - i\,\lambda_I\big)\ \big(Re(v) + i\ Im(v)\big) \\ &= \big(\lambda_R\ Re(v) + \lambda_I\ Im(v)\big) + i\ \big(\lambda_R\ Im(v) - \lambda_I\ Re(v)\big), \end{aligned} \tag{20}$$

one gets

$$\phi Re(v) = \lambda_R\ Re(v) + \lambda_I\ Im(v)\ ,\quad \phi Im(v) = -\lambda_I\ Re(v) + \lambda_R\ Im(v). \tag{21}$$

Note that M_λ is nonderogatory with $\chi_{M_\lambda}(X) = (X - \lambda_R)^2 + \lambda_I^2$ and both $Re(v)$ and $Im(v)$ are in $\ker \chi_{M_\lambda}(\phi)$, more precisely, $\ker \chi_{M_\lambda}(\phi) = \mathcal{E}_\lambda$.

In the basis $(Re(v_j), Im(v_j))$, $j = 1, \ldots, \frac{n}{2}$, of V, ϕ assumes the Jordan form $\operatorname{diag}(M_{\lambda_1}, \ldots, M_{\lambda_{\frac{n}{2}}})$, where v_j is an eigenvector of ϕ, as above, with eigenvalue λ_j

(C) Suppose ϕ has only 2 eigenvalues which are complex $z = r + is$ and $\bar{z}$, and $n \geq 4$. So ϕ is not diagonalizable and

$$\chi_\phi(X) = \big((X - r)^2 + s^2\big)^{\frac{n}{2}} = \sum_{j=0}^{\frac{n}{2}} \mathtt{C}_{\frac{n}{2}}^{j}\ s^{n-2j}(X - r)^{2j}.$$

We write the latter as $\chi_\phi(X) = \sum_{k=0}^{n} D_{n,k}(z)X^k$, where, by a direct calculation, we obtain the coefficients $D_{n,j} : \mathbb{C} \to \mathbb{R}$ which are given by

$$D_{n,j}(z) = (-1)^j \sum_{p=\varepsilon_j}^{\frac{n}{2}} \mathtt{C}_{\frac{n}{2}}^{p}\ \mathtt{C}_{2p}^{j} \Big(\frac{z - \bar{z}}{2i}\Big)^{n-2p} \Big(\frac{z + \bar{z}}{2}\Big)^{2p-j}, \tag{22}$$

where $2\varepsilon_j = j + \frac{1}{2}(1 - (-1)^j)$. From the Cayley-Hamilton's Theorem we deduce that $\phi^n = -\sum_{j=0}^{n-1} D_{n,j}(z)\phi^j$. According to Lemma 2.5, let $\bar{x} \in V$ be such that $(\bar{e}_j := \phi^{n-j}\bar{x},\ j = 1, \ldots, n)$ is a basis of V. We have

$$\phi\bar{e}_j = \bar{e}_{j-1},\ 2 \leq j \leq n,\ \ \phi\bar{e}_1 = \phi^n\bar{x} = -\sum_{j=0}^{n-1} D_{n,j}(z)\phi^j\bar{x} = -\sum_{j=0}^{n-1} D_{n,j}(z)\bar{e}_{n-j},$$

which in matrix form, say $\tilde{M}_z$, reads

$$\tilde{M}_z = \sum_{j=1}^{n-1} E_{j,j+1} - \sum_{j=1}^{n} D_{n,n-j}(z) E_{j,1}.$$

Now one may use a direct approach by looking for the explicit expressions of the coefficients $p_{k,l}$ of a matrix P such that $\tilde{M}_z P = P M_z$, where $M_z = r\mathbb{I}_{\mathbb{R}^n} + sM_s + M_n$, with M_s, M_n as in (8). Before proceeding, it is interesting to note the following.

If P_1 and P_2 are invertible and satisfy the equation $\tilde{M}_z P = P M_z$, then $P_1^{-1}P_2$ is in the isotropy subgroup of M_z, that is, the subgroup of $\mathrm{GL}(V)$ consisting of those P_3 satisfying $P_3 M_z P_3^{-1} = M_z$. Since M_z is nonderogatory, its isotropy subgroup is a maximal subgroup of $\mathrm{GL}(V)$. In fact, it is the group of invertible elements of $\mathbb{R}[M_z]$, it is connected, as the intersection $\mathbb{R}[M_z] \cap \det^{-1}(]0, +\infty[)$ of two connected subsets and hence it coincides with $\exp\big(\mathbb{R}[M_z]\big)$. So, there exists $Q \in \mathbb{R}[M_z]$ such that $P_1^{-1}P_2 = \exp(Q)$. Thus, for a fixed invertible solution P_1, the map $P_2 \mapsto P_1^{-1}P_2$ is a 1-1 correspondence between the set of invertible solutions of the equation $\tilde{M}_z P = P M_z$ and $\exp\big(\mathbb{R}[M_z]\big)$. We summarize this as follows.

Lemma 6.2. *Let P_1 be an invertible solution of the equation $\tilde{M}_z P = P M_z$. Then any other invertible solution P is of the form $P = P_1 e^Q$, where $Q \in \mathbb{R}[M_z]$. The linear map $P_2 \mapsto P_1^{-1}P_2$ is a $1{-}1$ correspondence between the space of solutions and $\mathbb{R}[M_z]$, and invertible solutions are mapped to elements of the form e^Q, $Q \in \mathbb{R}[M_z]$.*

Now from the explicit expressions of the equation $\tilde{M}_z P = P M_z$, we extract the following linear recurrence relations, where of course, $s = \frac{z-\bar{z}}{2i}$ and $r = \frac{z+\bar{z}}{2}$:

$$\begin{aligned} p_{k+1,1} &= p_{1,1}D_{n,n-k} + rp_{k,1} + sp_{k,2}, \quad p_{k+1,2} = p_{1,2}D_{n,n-k} + rp_{k,2} - sp_{k,1}, \\ p_{k+1,2j+3} &= p_{1,2j+3}D_{n,n-k} + rp_{k,2j+3} + sp_{k,2j+4} + p_{k,2j+1}, \\ p_{k+1,2j+4} &= p_{1,2j+4}D_{n,n-k} + rp_{k,2j+4} - sp_{k,2j+3} + p_{k,2j+2}, \end{aligned} \tag{23}$$

where $j = 0, \ldots, \frac{n}{2} - 2$, $k = 1, \ldots, n-1$ and $D_{k,l} = D_{k,l}(z)$. Obviously, these linear recurrence relations admit explicit solutions $p_{k,l}$ which are linear combinations of the $p_{1,j}$, $j = 1, \ldots, n$, the coefficients being polynomials in z, $\bar{z}$. One easily checks that

$$\det(P) = \big(\frac{z-\bar{z}}{2i}\big)^{\frac{n^2}{4}} (p_{1,1}^2 + p_{1,2}^2)^{\frac{n}{2}} q_n,$$

where $q_4 = 4$ and for $n > 4$, $q_n = (-1)^{\frac{n}{2}} (\frac{n}{2} - 1)^n$.

This proves that invertible solutions, of the equation $\tilde{M}_z P = P M_z$, always exist, they correspond to $p_{1,1}^2 + p_{1,2}^2 \neq 0$. More precisely, from (23), the two first columns of P can be rewritten, for any $k = 1, \ldots, n-1$, as

$$\begin{pmatrix} p_{k+1,1} \\ p_{k+1,2} \end{pmatrix} = \begin{pmatrix} r & s \\ -s & r \end{pmatrix} \begin{pmatrix} p_{k,1} \\ p_{k,2} \end{pmatrix} + D_{n,n-k} \begin{pmatrix} p_{1,1} \\ p_{1,2} \end{pmatrix}$$

which gives rise to the explicit expression

$$\begin{pmatrix} p_{k+1,1} \\ p_{k+1,2} \end{pmatrix} = \Big[\sum_{l=0}^{k} D_{n,n-(k-l)} \begin{pmatrix} r & s \\ -s & r \end{pmatrix}^l \Big] \begin{pmatrix} p_{1,1} \\ p_{1,2} \end{pmatrix}. \tag{24}$$

Again from (23), the columns $2j+3$ and $2j+4$ of P read, for any $j = 0, \ldots, \frac{n}{2} - 2$ and $k = 1, \ldots, n-1$, as

$$\begin{aligned}\begin{pmatrix} p_{k+1,2j+3} \\ p_{k+1,2j+4} \end{pmatrix} &= D_{n,n-k} \begin{pmatrix} p_{1,2j+3} \\ p_{1,2j+4} \end{pmatrix} + \begin{pmatrix} r & s \\ -s & r \end{pmatrix} \begin{pmatrix} p_{k,2j+3} \\ p_{k,2j+4} \end{pmatrix} + \begin{pmatrix} p_{k,2j+1} \\ p_{k,2j+2} \end{pmatrix} \\ &= \Big(\sum_{l_1=0}^{k} D_{n,n-(k-l_1)} \begin{pmatrix} r & s \\ -s & r \end{pmatrix}^{l_1} \Big) \begin{pmatrix} p_{1,2j+3} \\ p_{1,2j+4} \end{pmatrix} \\ &\quad + \Big(\sum_{l_1=0}^{k-1} \begin{pmatrix} r & s \\ -s & r \end{pmatrix}^{l_1} \Big) \begin{pmatrix} p_{k-l_1,2j+1} \\ p_{k-l_1,2j+2} \end{pmatrix} . \end{aligned} \tag{25}$$

Further developing (25), leads to the final explicit expressions

$$\begin{pmatrix} p_{k+1,2j+3} \\ p_{k+1,2j+4} \end{pmatrix} = \sum_{q=0}^{j+1} U(k,j,q) \begin{pmatrix} p_{1,2(j-q)+3} \\ p_{1,2(j-q)+4} \end{pmatrix} , \tag{26}$$

some of the 2×2 matrices $U(k,j,q)$ have quite lengthy general expressions, some of which being zero (see cases $n = 4, 6, 10$ below). For example, one gets

$$\begin{aligned} p_{2,1} &= -r(n-1)p_{1,1} + sp_{1,2} \ , \ p_{2,2} = -r(n-1)p_{1,2} - p_{1,1}s, \\ p_{3,1} &= \frac{n-2}{2}\Big((s^2 + (n-1)r^2)\, p_{1,1} - 2rs\, p_{1,2} \Big), \\ p_{3,2} &= \frac{n-2}{2}\Big((s^2 + (n-1)r^2)\, p_{1,2} + 2rs\, p_{1,1} \Big) \ , \ \text{and for } j \geq 1, \\ p_{2,2j+1} &= -(n-1)rp_{1,2j+1} + sp_{1,2j+2} + p_{1,2j-1} \ , \\ p_{2,2j+2} &= -(n-1)rp_{1,2j+2} - sp_{1,2j+1} + p_{1,2j} \ , \\ &\vdots \\ p_{n,1} &= (sp_{1,2} - rp_{1,1})|z|^{n-2} \ , \ p_{n,2} = -(sp_{1,1} + rp_{1,2})|z|^{n-2} \ , \\ p_{n,3} &= \Big((-r^2 + s^2)p_{1,1} + 2rsp_{1,2} \Big)|z|^{n-4} + \Big(sp_{1,4} - rp_{1,3} \Big)|z|^{n-2} \ , \\ p_{n,4} &= \Big(2rsp_{1,1} + (r^2 - s^2)p_{1,2} \Big)|z|^{n-4} + \Big(rp_{1,4} + sp_{1,3} \Big)|z|^{n-2}, \\ p_{n,2j+3} &= (sp_{1,2j+4} - rp_{1,2j+3})|z|^{n-2} + (rp_{n,2j+1} - sp_{n,2j+2})|z|^{-2}, \\ p_{n,2j+4} &= -(rp_{1,2j+4} + sp_{1,2j+3})|z|^{n-2} - (rp_{n,2j+2} + sp_{n,2j+1})|z|^{-2} \ . \end{aligned} \tag{27}$$

6.2. Some particular examples in low dimensions

In low dimensions, one gets simple expressions for P by setting $p_{1,1} = 1$ and $p_{1,j} = 0$, if $j \geq 2$ or $p_{1,2} = 1$ and $p_{1,j} = 0$, if $j \neq 2$. Here are some examples.

- For $n = 4$, we get
  $p_{1,1} = 1,\ p_{1,2} = p_{1,3} = p_{1,4} = 0,$
  $p_{2,1} = -3r,\ p_{2,2} = -s,\ p_{2,3} = 1,\ p_{2,4} = 0,\ p_{3,1} = 3r^2 + s^2,$
  $p_{3,2} = 2rs,\ p_{3,3} = -2r,\ p_{3,4} = -2s,$
  $p_{4,1} = -r(r^2+s^2),\ p_{4,2} = -r^2 s - s^3,\ p_{4,3} = r^2 - s^2,\ p_{4,4} = 2rs$ and $\det(P) = 4s^4$.
- For $n = 6$, one gets
  $p_{1,1} = 1,\ p_{1,2} = p_{1,3} = p_{1,4} = p_{1,5} = p_{1,6} = 0,$
  $p_{2,1} = -5r,\ p_{2,2} := -s,\ p_{2,3} = 1,\ p_{2,4} = p_{2,5} = p_{2,6} = 0,$
  $p_{3,1} = 10r^2 + 2s^2,\ p_{3,2} = 4rs,\ p_{3,3} = -4r,\ p_{3,4} = -2s,\ p_{3,5} = 1,$

$p_{4,1} = -2r(5r^2+3s^2)$, $p_{4,2} = -2s(3r^2+s^2)$, $p_{4,3} = 6r^2$, $p_{4,4} = 6rs$, $p_{4,5} = -3r$, $p_{4,6} = -3s$,
$p_{5,1} := (5r^2+s^2)(r^2+s^2)$, $p_{5,2} := 4rs(r^2+s^2)$, $p_{5,3} = -4r^3$, $p_{5,4} := -2s(3r^2+s^2)$, $p_{5,5} = 3r^2-3s^2$, $p_{5,6} = 6rs$,
$p_{6,1} = -r(r^2+s^2)^2$, $p_{6,2} = -s(r^2+s^2)^2$, $p_{6,3} = r^4-s^4$, $p_{6,4} := 2rs(r^2+s^2)$, $p_{6,5} := -r(r^2-3s^2)$, $p_{6,6} = -s(3r^2-s^2)$, $\det(P) = -64s^9$.

- For $n = 10$, we choose $p_{1,1} = 1$ and $p_{1,j} = 0$, $\forall j \geq 2$, so that the nonzero coefficients of P are:
$p_{1,1} = 1$,
$p_{2,1} = -9r$, $p_{2,2} = -s$, $p_{2,3} = 1$, $p_{3,1} = 4(9r^2+s^2)$, $p_{3,2} = 8sr$,
$p_{3,3} = -8r$, $p_{3,4} = -2s$, $p_{3,5} = 1$, $p_{4,1} = -28r(3r^2+s^2)$,
$p_{4,2} = -4s(7r^2+s^2)$, $p_{4,3} = 28r^2+2s^(2)$, $p_{4,4} = 14sr$, $p_{4,5} = -7r$, $p_{4,6} = -3s$, $p_{4,7} = 1$,
$p_{5,1} = 6(21r^4+14r^2s^2+s^4)$, $p_{5,2} = 8sr(7r^2+3s^2)$, $p_{5,3} = -4r(14r^2+3s^2)$, $p_{5,4} = -6s(7r^2+s^2)$, $p_{5,5} = 21r^2-s^2$, $p_{5,6} = 18sr$, $p_{5,7} = -6r$, $p_{5,8} = -4s$, $p_{5,9} = 1$,
$p_{6,1} = -2r(63r^4+70r^2s^2+15s^4)$, $p_{6,2} = -2s(35r^4+30r^2s^2+3s^4)$,
$p_{6,3} = 10r^(2)(7r^2+3s^2)$, $p_{6,4} = 10sr(7r^2+3s^2)$, $p_{6,5} = 5r(-7r^2+s^2)$,
$p_{6,6} = -5s(9r^2+s^2)$, $p_{6,7} = 5(3r^2-s^2)$, $p_{6,8} = 20sr$, $p_{6,9} = -5r$, $p_{6,10} = -5s$,
$p_{7,1} = 4(r^2+s^2)(21r^4+14r^2s^2+s^4)$, $p_{7,2} = 8sr(7r^2+3s^2)(r^2+s^2)$,
$p_{7,3} = -8r^3(7r^2+5s^2)$, $p_{7,4} = -2s(35r^4+30r^2s^2+3s^4)$, $p_{7,5} = 5(7r^4-2r^2s^2-s^4)$, $p_{7,6} = 20sr(3r^2+s^2)$, $p_{7,7} = -20r(r^2-s^2)$, $p_{7,8} = -40sr^2$, $p_{7,9} = 10(r^(2)-s^(2))$, $p_{7,10} = 20sr$,
$p_{8,1} = -12r(3r^2+s^2)(r^2+s^2)^2$, $p_{8,2} = -4(7r^2+s^2)(r^2+s^2)^2s$,
$p_{8,3} = 2(r^2+s^2)(14r^4+r^2s^2-s^4)$, $p_{8,4} = 6sr(7r^2+3s^2)(r^2+s^2)$,
$p_{8,5} = -r(21r^4-10r^2s^2-15s^4)$, $p_{8,8} = 40r^3s$, $p_{8,9} = -10r(r^2-3s^2)$, $p_{8,10} = -10s(3r^2-s^2)$,
$p_{9,1} = (9r^2+s^2)(r^2+s^2)^3$, $p_{9,2} = 8sr(r^2+s^2)^3$, $p_{9,3} = -4r(2r^2-s^2)(r^2+s^2)^2$, $p_{9,4} = -2(7r^2+s^2)(r^2+s^2)^2s$, $p_{9,5} = (r^2+s^2)(7r^4-12r^2s^2-3s^4)$,
$p_{9,6} = 2sr(9r^2+s^2)(r^2+s^2)$, $p_{9,7} = -2r(3r^4-10r^2s^2-5s^4)$, $p_{9,8} = -4s(5r^4-s^4)$, $p_{9,9} = 5(r^2-2sr-s^2)(r^2+2sr-s^2)$, $p_{9,10} = 20sr(r-s)(r+s)$,
$p_{10,1} = -r(r^2+s^2)^4$, $p_{10,2} = -s(r^2+s^2)^4$, $p_{10,3} = (r^2-s^2)(r^2+s^2)^3$,
$p_{10,4} = 2sr(r^2+s^2)^3$, $p_{10,5} = -r(r^2-3s^2)(r^2+s^2)^2$, $p_{10,6} = -(3r^2-s^2)(r^2+s^2)^2s$, $p_{10,7} = (r^2+s^2)(r^2-2sr-s^2)(r^2+2sr-s^2)$, $p_{10,8} = 4sr(r-s)(r+s)(r^2+s^2)$, $p_{10,9} = -r(r^4-10r^2s^2+5s^4)$, $p_{10,10} = -s(5r^4-10r^2s^2+s^4)$
and $\det(P) = -1048576s^{25}$.

References

[1] M. A. Alvarez, M. C. Rodríguez-Vallarte, G. Salgado: *Contact and Frobenius solvable Lie algebras with abelian nilradical*, Commun. Algebra 46/10 (2018) 4344–4354.

[2] N. Blazić, S. Vukmirović: *Four-dimensional Lie algebras with a para-hypercomplex structure*, Rocky Mountain J. Math. 40/5 (2010) 1391–1439.

[3] M. Bordemann, A. Medina, A. Ouadfel: *Le groupe affine comme variété symplectique*, Tohoku Math. J. (2) 45/3 (1993) 423–436.

[4] W. A. De Graaf: *Classification of nilpotent associative algebras of small dimension*, Int. J. Algebra Comput. 28/1 (2018) 133–161.

[5] M. A. de Olmo, M. A. Rodriguez, P. Wintemitz, H. Zassenhaus: *Maximal Abelian subalgebras of pseudounitary Lie algebras*, Linear Algebra Appl. 135 (1990) 79–151.

[6] A. Diatta: *Left invariant contact structures on Lie groups*, Diff. Geom. Appl. 26/5 (2008) 544–552.

[7] A. Diatta, B. Manga: *On properties of principal elements of Frobenius Lie algebras*, J. Lie Theory 24/3 (2014) 849–864.

[8] A. Diatta, B. Manga, A. Mbaye: *On systems of commuting matrices, Frobenius Lie algebras and Gerstenhaber's theorem*, arXiv: 2002.08737 (2020).

[9] A. Diatta, A. Medina: *Classical Yang-Baxter equation and left invariant affine geometry on Lie groups*, Manuscripta Math. 114/4 (2004) 477–486.

[10] J. Dixmier: *Sous-anneaux abéliens maximaux dans les facteurs de type fini*, Ann. of Math. (2) 59 (1954) 279–286.

[11] M. Fernández, A. Fino, V. Manero: *G2-structures on Einstein solvmanifolds*, Asian J. Math. 19/2 (2015) 321–342.

[12] Harish-Chandra: *The characters of semisimple Lie groups*, Trans. Amer. Math. Soc. 83 (1956) 98–163.

[13] N. Jacobson: *Schur's theorems on commutative matrices*, Bull. Amer. Math. Soc. 50/6 (1944) 431–436.

[14] B. Kostant: *On the conjugacy of real Cartan subalgebras I*, Proc. Nat. Acad. Sci. U.S.A. 41 (1955) 967–970.

[15] A. Lichnerowicz, A. Medina: *On Lie groups with left-invariant symplectic or Kählerian structures*, Lett. Math. Phys. 16/3 (1988) 225–235.

[16] A. Mbaye: *2-Solvable Frobenius Lie Algebras and Applications*, Ph.D. Thesis, in preparation (2023).

[17] J. C. Ndogmo, P. Winternitz: *Solvable Lie algebras with abelian nilradicals*, J. Phys. A 27/2 (1994) 405–423.

[18] A. I. Ooms: *On Lie algebras having a primitive universal enveloping algebra*, J. Algebra 32 (1974) 488–500.

[19] A. I. Ooms: *On Frobenius Lie algebras*, Commun. Algebra 8/1 (1980) 13–52.

[20] I. Schur: *Zur Theorie der vertauschbaren Matrizen*, J. Reine Angew. Math. 130 (1905) 66–76.

[21] J. P. Serre: *Lie Algebras and Lie Groups*, Lectures Notes in Mathematics 1500, Springer, Berlin (2006).

[22] J. E. Snow: *Invariant complex structures on four dimensional solvable real Lie groups*, Manuscripta Math. 66 (1990) 397–412.

[23] M. Sugiura: *Conjugate classes of Cartan subalgebras in real semi-simple Lie algebras*, J. Math. Soc. Japan 11 (1959) 374–434.

[24] D. A. Suprunenko, R. I. Tyshkevich: *Commutative Matrices*, Academic Press, New York (1968).

[25] P. Winternitz: *Subalgebras of Lie algebras. Example of sl(3,R)*, in: *Symmetry in Physics*, CRM Proc. Lecture Notes 34, American Mathematical Society, Providence (2004) 215–227.

[26] P. Winternitz, H. Zassenhaus: *The structure of maximal Abelian subalgebras of classical Lie and Jordan algebras*, in: *XIIIth Int. Colloquium on Group Theoretical Methods in Physics*, College Park 1984, World Scientific Publishing, Singapore (1984) 115–118.

André Diatta, Aix-Marseille Université, Institut Fresnel, Marseille, France; andre.diatta@fresnel.fr.
Bakary Manga, Ameth Mbaye, Dép. de Mathématiques et Informatique, Université Cheikh Anta Diop de Dakar, Sénégal; bakary.manga@ucad.edu.sn, ameth3.mbaye@ucad.edu.sn.

Received August 26, 2022
and in final form March 10, 2023