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A Fundamental Domain for the General Linear Group by Means of Successive Minima

Masami Fujimori

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Abstract. We present a variant of the method of construction of Watanabe of a fundamental domain for the translation action by the subgroup of rational points on the whole group of adèlic points of the general linear group over an arbitrary number field. The obtained domain is a generalization of Grenier's one, not Minkowski's. For several fields, we profit our presentation to show that the fundamental domain is in a so-called Siegel set. As an example, inequalities bounding our fundamental domain are explicitly determined in the case of degree 3 over the imaginary quadratic field with discriminant -3, which will be interesting to some people.

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In celebration of the 75th birthday of Professor Yasuo Morita.

1. Introduction

In one of a series of papers studying generalizations of the Hermite constant, Watanabe [8] introduced a subset, called a Ryshkov domain, of the set of adèlic points of an arbitrarily given isotropic connected reductive group over a number field. He showed a procedure to construct a fundamental domain for the translation action by the discrete subgroup of rational points from the Ryshkov domain, which is well-suited for the search for the Hermite-like constants. In the same paper, as an example over the rational number field, he pointed out that an adequate application of the procedure to the general linear group is identical to Grenier's construction of a fundamental domain. In this sense, Watanabe gave an extension of the method of Grenier, or rather, the method of Hermite, Korkine, and Zolotareff, to the case over an arbitrary number field. Watanabe's extension depends on the induction on the degree of general linear group like Grenier's original construction.

Following Watanabe, Weng [10] made an explicit calculation of matrices which are needed for the description of the connected components of the quotient symmetric space over an arbitrary number field. Weng's result is the realization of Watanabe's construction. Unfortunately, its geometric meaning is still not very clear because of the indirect nature of induction.

So in our present paper, we try to clarify the geometric meaning of the fundamental domain of Watanabe. We make full use of Watanabe's tool itself, but discarding the

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induction, we rely on the use of a kind of successive minima. Strictly speaking, our fundamental domain is a little bit smaller than Watanabe's, but some open subset is dense in both. To explain our result, we start with recalling Grenier's fundamental domain. We describe it in terms of quadratic forms.

Let $\xi_1, \dots, \xi_{n-1}, \eta$ be variable numbers; $\mathbf{x} = (\xi_1, \dots, \xi_{n-1})$; $q(\mathbf{x}, \eta)$ a positive definite quadratic form in n variables $\xi_1, \dots, \xi_{n-1}, \eta$; and $a_n^2 := q(0, \dots, 0, \pm 1)$. After completion of the square of η, the quadratic form q is of the form

$$q(\mathbf{x}, \eta) = \tilde{q}(\mathbf{x}) + a_n^2(\xi_1 u_1 + \cdots + \xi_{n-1}u_{n-1} + \eta)^2, \tag{1}$$

where $u_1, \dots, u_{n-1}$ are some real numbers. The quadratic form $\tilde{q}$ in $n-1$ variables $\mathbf{x} = (\xi_1, \dots, \xi_{n-1})$ is uniquely defined and positive definite.

Definition 1.1. (cf. [6]) The positive definite quadratic form q is *reduced* in the sense of Hermite or in the sense of Korkine and Zolotareff (*HKZ-reduced*, for short) if and only if the following conditions are satisfied:

- $a_n^2 = \min\{q(\alpha_1, \dots, \alpha_n) \mid (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n - \{0\}\}$
- $|u_i| \le \frac{1}{2} \quad (i = 1, \dots, n-1)$ and $u_{n-1} \ge 0$
- The quadratic form $\tilde{q}$ is HKZ-reduced.

Grenier [4] proved the set of HKZ-reduced forms is a fundamental domain for the natural action of $\mathrm{GL}_n(\mathbb{Z})$ on the symmetric space of positive definite quadratic forms in n variables.
When $n = 1$, all positive definite quadratic forms are HKZ-reduced.
When $n = 2$, a positive definite quadratic $q(\xi, \eta)$ is of the form

$$q(\xi, \eta) = a_1^2\xi^2 + a_2^2(\xi u + \eta)^2 \quad (a_1, a_2, u \in \mathbb{R}).$$

The form q is HKZ-reduced if and only if $a_2^2 = \min\{q(\alpha, \beta) \mid (\alpha, \beta) \in \mathbb{Z}^2 - \{0\}\}$ and $0 \le u \le 1/2$. As an example, let

$$\tilde{f}(\xi, \eta) := 4\xi^2 + 3\eta^2 \quad \text{and} \quad \tilde{g}(\xi, \eta) := 4\xi^2 + 4\xi\eta + 4\eta^2 = 3\xi^2 + 4\Big(\frac{\xi}{2} + \eta\Big)^2.$$

The minimum value of $\tilde{f}$ at the integral points other than $(0,0)$ is 3. The form $\tilde{f}$ is obviously HKZ-reduced. The smallest value different from 0 of $\tilde{g}$ is easily seen to be 4 taken at the points $(\pm 1, 0)$, $(\pm 1, \mp 1)$, $(0, \pm 1)$, hence $\tilde{g}$ is also HKZ-reduced.
When $n = 3$, put

$$f(\xi, \eta, \zeta) := \tilde{f}(\xi, \eta) + 4\Big(\frac{\eta}{2} + \zeta\Big)^2 = 4\xi^2 + 4\eta^2 + 4\eta\zeta + 4\zeta^2$$

and

$$g(\xi, \eta, \zeta) := \tilde{g}(\xi, \eta) + 4\zeta^2 = f(\zeta, \xi, \eta).$$

Since the minimum other than 0 of $\tilde{g}$ is 4, the positive definite quadratic form g is HKZ-reduced. The relation $f(\xi, \eta, \zeta) = g(\eta, \zeta, \xi)$ tells us that the smallest value different from 0 of f is also 4. Thus f is HKZ-reduced, too.

The quadratic forms f and g above are 'equivalent'. In general, two quadratic forms q and p in n variables are said to be *equivalent* if and only if under the natural right action of $\mathrm{GL}_n(\mathbb{Z})$, there exists an element $x \in \mathrm{GL}_n(\mathbb{Z})$ such that $qx = p$.

In the sense of our present paper, the quadratic forms $\tilde{f}$, $\tilde{g}$, and f are regarded as reduced, but g is not (cf. Remark 3.6). This is because in order that a positive definite quadratic form $q(\mathbf{x}, \eta)$ is reduced, we require the minimum other than 0 of $\tilde{q}(\mathbf{x})$ in the decomposition (1) is the smallest possible value among the ones of equivalent quadratic forms.

For the purpose of stating our result, we need several symbols. We restrict ourselves in this introduction to the case over the rational number field.

We denote by $r_i(x)$ the lower $i \times n$-block of a rational integral matrix $x \in \mathrm{GL}_n(\mathbb{Z})$. For $g \in \mathrm{GL}_n(\mathbb{R})$ and $i = 1, \ldots, n-1$, let

$$\tilde{H}_i(xg) := \det\left(r_i(x) g\, {}^{\mathrm{t}}g\, {}^{\mathrm{t}}r_i(x)\right),$$

where ${}^{\mathrm{t}}\cdot$ is the matrix transposition. This $\tilde{H}_i(xg)$ is the square of the volume of i-dimensional rhombohedron spanned by the rows of $r_i(x)g$ with respect to the standard Euclidean metric on $\mathbb{R}^n$. Put

$$\tilde{H}(g) := (\tilde{H}_1(g), \ldots, \tilde{H}_{n-1}(g)) \in \mathbb{R}^{n-1}$$

and

$$\tilde{m}(g) := \min\{\tilde{H}(xg) \mid x \in \mathrm{GL}_n(\mathbb{Z})\} \in \mathbb{R}^{n-1},$$

where we consider $\mathbb{R}^{n-1}$ lexicographically ordered. The minimum always exists. In this way, we define a kind of successive minima $\tilde{m}$ of (squares of) volumes of the flags of faces of integral parallelepipeds for the Euclidean metric determined by the positive definite symmetric matrix $g\, {}^{\mathrm{t}}g$. Call B the upper triangular Borel subgroup of GL_n. Denote by K the standard orthogonal subgroup of $\mathrm{GL}_n(\mathbb{R})$. We see easily that the subset

$$\tilde{R} := \{g \in \mathrm{GL}_n(\mathbb{R}) \mid \tilde{H}(g) = \tilde{m}(g)\}$$

is left $B(\mathbb{Z})$-invariant and right K-invariant. It is closed in $\mathrm{GL}_n(\mathbb{R})$ because of the continuities of $\tilde{H}$ and $\tilde{m}$. Let Ω be any fundamental domain for the left $B(\mathbb{Z})$-action on the set $\tilde{R}$. Note that translation by elements of $B(\mathbb{Z})$ is very simple. We designate Ω° the interior of Ω in $\mathrm{GL}_n(\mathbb{R})$. Our result over the rational number field is the following:

Theorem 1.2. *The subset Ω° is an open fundamental domain for the left translation action by $\mathrm{GL}_n(\mathbb{Z})$ on $\mathrm{GL}_n(\mathbb{R})$.*

In Section 2, we prove an extension of this theorem to the cases over arbitrary number fields in terms of adèles. In Section 3 and in Section 4, we show respectively over the rational number field and over some imaginary quadratic fields that our fundamental domain is contained in a so-called Siegel set. As a corollary, we see our fundamental domain is cut out by a finite number of zero loci of effectively calculable continuous functions. The reasoning of this part is the same as Grenier's [4, Theorem 1] in essence. During the way, we also find that our fundamental domain for the rational number field is included in Grenier's one but does not exactly coincide with it. We give an explicit form of inequalities bounding our fundamental domain in a few examples of small degree. As another application of the fact that our fundamental domain is in a Siegel set, we observe that over the fields mentioned above, Mahler's compactness criterion holds, though it is known to the experts.

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2. Heights and a fundamental domain

In this section, we describe in detail a method of construction of a fundamental domain for the general linear group. Our approach will be useful for applications. The domain is originally obtained by Watanabe extending Grenier's one.

First, we fix some notation.

Let F be a finite extension field of the rational number field $\mathbb{Q}$; $\mathfrak{M}^{\mathrm{fin}}$ the set of finite places on F; $\mathfrak{M}^{\infty}$ the set of infinite ones; $\mathfrak{M} := \mathfrak{M}^{\mathrm{fin}} \cup \mathfrak{M}^{\infty}$; F_v the completion of F at $v \in \mathfrak{M}$; $\mathfrak{o}_v\,(v \in \mathfrak{M}^{\mathrm{fin}})$ the ring of integers in F_v; $\mathbb{A}$ the adèle ring of F; $\mathbb{A}^\times$ the idèle group of its invertible elements; $|\cdot|_v\,(v \in \mathfrak{M})$ the absolute value on F_v normalized so that

$$|2|_v = \begin{cases} 2 & (v\colon \text{real}) \\ 2^2 & (v\colon \text{complex}) \end{cases}$$

and the module (idèle norm) $|\cdot|_{\mathbb{A}^\times} := \prod_{v\in\mathfrak{M}} |\cdot|_v$ on $\mathbb{A}^\times$ takes a constant value 1 on F (Artin's product formula [9, Chap. IV, §4, Theorem 5]); $X_1,\ldots,X_n$ indeterminates; $V := \oplus_{i=1}^n FX_i$ the F-vector space of linear forms in $X_1,\ldots,X_n$; $\check{V}$ the dual vector space to V; and $\check{X}_1,\ldots,\check{X}_n$ its dual basis to $X_1,\ldots,X_n$.

For an element $f = \sum_{i=1}^n \alpha_i \otimes X_i$ of $F_v \otimes_F V$, we define a logarithmic local height h^v of f with respect to the basis $X_1,\ldots,X_n$ as usual:

$$h^v(f) := \begin{cases} \log\max\{|\alpha_1|_v,\ldots,|\alpha_n|_v\} & (v\colon \text{finite}) \\ \log\sqrt{|\alpha_1|_v^2+\cdots+|\alpha_n|_v^2} & (v\colon \text{real}) \\ \log(|\alpha_1|_v+\cdots+|\alpha_n|_v) & (v\colon \text{complex}) \end{cases}$$

Denote by GL_V the general linear group of linear transformations over F of V with its action on V being from the right, by N the subgroup of GL_V composed of unipotent upper triangular matrices with respect to the basis $X_1,\ldots,X_n$, and by A the diagonal subgroup of GL_V with respect to the same basis. For each place $v \in \mathfrak{M}$ on F, a maximal compact subgroup K_v of the group of F_v-valued points $\mathrm{GL}_V(F_v)$ of GL_V is given as the isotropy group of the local height function h^v. Explicitly, when $v \in \mathfrak{M}^{\mathrm{fin}}$, we see $K_v = \mathrm{GL}(\mathfrak{o}_v \otimes_{\mathbb{Z}} \sum_{i=1}^n \mathbb{Z}X_i)$ and when $v \in \mathfrak{M}^\infty$, we see K_v is the standard orthogonal group or the standard unitary group with respect to the basis $X_1,\ldots,X_n$ according as the place v is real or complex. As is well-known, we have an Iwasawa decomposition

$$\mathrm{GL}_V(F_v) = N(F_v)A(F_v)K_v$$

for all $v \in \mathfrak{M}$. Put $K := \prod_{v\in\mathfrak{M}} K_v$. Let $g^v\,(v \in \mathfrak{M})$ be the v-factor of an $\mathbb{A}$-valued point g of GL_V. Since for almost all $v \in \mathfrak{M}$, the F_v-valued point g^v of GL_V belongs to K_v, we get the global Iwasawa decomposition

$$\mathrm{GL}_V(\mathbb{A}) = N(\mathbb{A})A(\mathbb{A})K.$$

We call P_i $(i = 1,\ldots,n-1)$ the maximal parabolic subgroup of GL_V whose points are represented by the matrices with their lower-left $i \times (n-i)$-corners being zero.

The intersection $P_1 \cap \cdots \cap P_{n-1}$ is the Borel subgroup $B = NA$. Watanabe [8] has associated each maximal parabolic subgroup of any isotropic connected reductive group over a number field with an exponential height function on the set of $\mathbb{A}$-valued points of the reductive group. For a general linear group GL_V, the height function H_i attached to P_i $(i = 1, \ldots, n-1)$ is expressed up to a power as follows: If the A-part of the Iwasawa decomposition of $g \in \mathrm{GL}_V(\mathbb{A})$ is $\mathrm{diag}(a_1, \ldots, a_n)$, then

$$H_i(g) = |a_{n-i+1} \cdots a_n|_{\mathbb{A}^\times}^n \, |\det g|_{\mathbb{A}^\times}^{-i}.$$

If we denote by l the lower-right $i \times i$-corner of an F-valued point p of P_i, we get

$$H_i(pg) = |(\det l) \cdot a_{n-i+1} \cdots a_n|_{\mathbb{A}^\times}^n \, |\det p \cdot \det g|_{\mathbb{A}^\times}^{-i}.$$

The product formula $|F^\times|_{\mathbb{A}^\times} = 1$ tells us that $H_i(pg) = H_i(g)$ for any $p \in P_i(F)$. Notice that the image of a compact subgroup of $\mathrm{GL}_V(\mathbb{A})$ through the determinant character followed by the idèle norm is trivial. The function H_i is a left $P_i(F)$-invariant and right K-invariant continuous function. For an arbitrarily fixed element g of $\mathrm{GL}_V(\mathbb{A})$, restricting the domain of definition to the subset $\mathrm{GL}_V(F)g$, we obtain a usual twisted exponential height function $H_i(xg)$ $(x \in P_i(F) \backslash \mathrm{GL}_V(F))$ on the set of F-valued points of the Grassmann variety $P_i \backslash \mathrm{GL}_V$ induced by the Plücker embedding into a projective space. Note that $(P_i \backslash \mathrm{GL}_V)(F) \simeq P_i(F) \backslash \mathrm{GL}_V(F)$ and $(P_i \backslash \mathrm{GL}_V)(\mathbb{A}) \simeq P_i(\mathbb{A}) \backslash \mathrm{GL}_V(\mathbb{A})$ (e.g. [1, §7]). In particular, for a fixed $g \in \mathrm{GL}_V(\mathbb{A})$, there are only finitely many cosets $P_i(F)x$ $(x \in \mathrm{GL}_V(F))$ in the group $\mathrm{GL}_V(F)$ with its height $H_i(xg)$ bounded above.

We define a kind of successive minima function on $\mathrm{GL}_V(\mathbb{A})$, which we will use to obtain a fundamental domain on $\mathrm{GL}_V(\mathbb{A})$ for the discrete subgroup $\mathrm{GL}_V(F)$.

Let $g \in \mathrm{GL}_V(\mathbb{A})$. We consider $\mathbb{R}^{n-1}$ lexicographically ordered. Put

$$H(g) := (H_1(g), \ldots, H_{n-1}(g)) \in \mathbb{R}^{n-1},$$

$$m(g) = (m_1(g), \ldots, m_{n-1}(g)) := \min\{H(xg) \mid x \in \mathrm{GL}_V(F)\} \in \mathbb{R}^{n-1},$$

and

$$R := \{g \in \mathrm{GL}_V(\mathbb{A}) \mid H(g) = m(g)\}. \tag{2}$$

We see immediately that the functions $m_1, \ldots, m_{n-1}$ are left $\mathrm{GL}_V(F)$-invariant and right K-invariant functions.

Recall that $B = P_1 \cap \cdots \cap P_{n-1}$ is a Borel subgroup of upper triangular matrices with respect to the basis $X_1, \ldots, X_n$ of V.

Lemma 2.1. *We have $B(F)RK = R$ and $\mathrm{GL}_V(\mathbb{A}) = \mathrm{GL}_V(F)R$.*

Proof. The left $P_i(F)$-invariance and the right K-invariance of H_i and m_i for $i = 1, \cdots, n-1$ give the first equality.
There exists for any $g \in \mathrm{GL}_V(\mathbb{A})$ an element $x \in \mathrm{GL}_V(F)$ with $H(xg) = m(g)$. Since m is left $\mathrm{GL}_V(F)$-invariant, we obtain $xg \in R$. ■

For each $g \in \mathrm{GL}_V(\mathbb{A})$, let

$$M(g) := \{x \in B(F) \backslash \mathrm{GL}_V(F) \mid H(xg) = m(g)\}. \tag{3}$$

Lemma 2.2. *The set $M(g)$ of cosets modulo $B(F)$ is finite and non-empty.*

Proof. It is obvious that $M(g)$ is non-empty.

For a fixed $g \in \mathrm{GL}_V(\mathbb{A})$, the number of cosets $P_i(F)x$ with $x \in M(g)$ is finite for any $i = 1, \dots, n-1$, because $H_i(\cdot g)$ is a (twisted) height function on a Grassmann variety $P_i \backslash \mathrm{GL}_V$. Suppose that two elements $x, y \in M(g)$ satisfy

$$P_i(F)x = P_i(F)y \quad (i = 1, \dots, n-1).$$

We conclude
$$xy^{-1} \in P_1(F) \cap \cdots \cap P_{n-1}(F) = B(F).$$
■

Lemma 2.3. *For any $g \in \mathrm{GL}_V(\mathbb{A})$, there exist a neighborhood $\mathcal{U}$ of g in $\mathrm{GL}_V(\mathbb{A})$ and a finite subset X of $B(F)\backslash \mathrm{GL}_V(F)$ such that*

$$m(u) = \min_{x \in X} H(xu) \quad (u \in \mathcal{U}).$$

Thus we find that the functions $m_1, \dots, m_{n-1}$ are continuous and that the subset $R \subset \mathrm{GL}_V(\mathbb{A})$ is closed, because the vector-valued height function H is continuous.

Proof. Fix $g \in \mathrm{GL}_V(\mathbb{A})$. For an arbitrary $h \in \mathrm{GL}_V(\mathbb{A})$, let

$$h = b \cdot k \quad (b \in B(\mathbb{A}),\ k \in K)$$

be the Iwasawa decomposition. By the definition of height functions $H_1, \dots, H_{n-1}$, we see for $i = 1, \dots, n-1$ that

$$H_i(hg) = H_i(bkg) = H_i(b)H_i(kg) = H_i(h)H_i(kg).$$

An auxiliary lower bound function ℓ on $\mathrm{GL}_V(\mathbb{A})$ is defined as

$$\ell(g) = \min_{i=1,\dots,n-1} \inf \left\{ \frac{H_i(hg)}{H_i(h)} \;\middle|\; h \in \mathrm{GL}_V(\mathbb{A}) \right\} = \min_{i=1,\dots,n-1} \min_{k \in K} H_i(kg).$$

The function ℓ takes positive values and is continuous. We have by definition

$$H_i(h)\ell(g) \le H_i(hg) \quad (h, g \in \mathrm{GL}_V(\mathbb{A});\ i = 1, \dots, n-1).$$

Put
$$X_1 := \{x \in B(F)\backslash \mathrm{GL}_V(F) \mid H_1(xg) = m_1(g)\},$$

$$Y_1 := \left\{ y \in B(F)\backslash \mathrm{GL}_V(F) \;\middle|\; H_1(y) \le \frac{3m_1(g)}{\ell(g)} \right\} - X_1,$$

and
$$Z_1 := \left\{ z \in B(F)\backslash \mathrm{GL}_V(F) \;\middle|\; H_1(z) > \frac{3m_1(g)}{\ell(g)} \right\}.$$

Since
$$H_1(zg) \ge H_1(z)\ell(g) > 3m_1(g) > m_1(g) \quad (z \in Z_1),$$

the right hand side of $B(F)\backslash \mathrm{GL}_V(F) = X_1 \sqcup Y_1 \sqcup Z_1$ is a disjoint union. Continuity of the function ℓ implies that there is a neighborhood $\mathcal{V}$ of g in $\mathrm{GL}_V(\mathbb{A})$ such that

$$\ell(v) > \frac{2}{3}\,\ell(g) \quad (v \in \mathcal{V}).$$

We observe for $z \in Z_1$ and $v \in \mathcal{V}$,

$$H_1(zv) \ge H_1(z)\ell(v) > \frac{3m_1(g)}{\ell(g)} \cdot \frac{2}{3}\,\ell(g) = 2m_1(g).$$

Remember that $H_1(\cdot g)$ and $H_1(\cdot)$ are exponential height functions on the set of F-rational points on the Grassmann variety $P_1\backslash \mathrm{GL}_V$ and that the function H_1 is left $P_1(F)$-invariant. We see images of the subsets X_1 and Y_1 of $B(F)\backslash \mathrm{GL}_V(F)$ in a quotient set $P_1(F)\backslash \mathrm{GL}_V(F)$ are finite sets, hence the number of functions $H_1(x\cdot)$ $(x \in X_1)$ and $H_1(y\cdot)$ $(y \in Y_1)$ on $\mathrm{GL}_V(\mathbb{A})$ are finite. We have by definition

$$\max\{H_1(xg) \mid x \in X_1\} = m_1(g) < \min\{H_1(yg) \mid y \in Y_1\}.$$

Continuity of the function H_1 tells us that there exists a neighborhood $\mathcal{W}_1$ of g in $\mathrm{GL}_V(\mathbb{A})$ such that for $w \in \mathcal{W}_1$

$$\max\{H_1(xw) \mid x \in X_1\} < \min\left\{2m_1(g), \min\{H_1(yw) \mid y \in Y_1\}\right\}.$$

Letting $\mathcal{U}_1 := \mathcal{V} \cap \mathcal{W}_1$, we find

$$m(u) = \min\{H_1(xu) \mid x \in X_1\} \quad (u \in \mathcal{U}_1),$$

because we have adopted the lexicographic order on $\mathbb{R}^{n-1}$.

Now we choose any set $X_1^\dagger$ $(\subset X_1)$ of representatives of the finite set $P_1(F)\backslash P_1(F)X_1$. The set $X_1^\dagger$ is of course finite. Put

$$\begin{aligned}
m_2^\dagger &:= \max\{H_2(xg) \mid x \in X_1^\dagger\},\\
X_2 &:= \{x \in X_1 \mid H_2(xg) \le m_2^\dagger\},\\
Y_2 &:= \left\{y \in X_1 \;\middle|\; H_2(y) \le \frac{3m_2^\dagger}{\ell(g)}\right\} - X_2, \text{ and}\\
Z_2 &:= \left\{z \in X_1 \;\middle|\; H_2(z) > \frac{3m_2^\dagger}{\ell(g)}\right\}.
\end{aligned}$$

We know $\qquad H_2(zg) \ge H_2(z)\ell(g) > 3m_2^\dagger > m_2^\dagger \quad (z \in Z_2),$

so the subsets X_2, Y_2, Z_2 are disjoint and $X_1 = X_2 \sqcup Y_2 \sqcup Z_2$. We see also for $z \in Z_2$ and $v \in \mathcal{V}$,

$$H_2(zv) \ge H_2(z)\ell(v) > \frac{3m_2^\dagger}{\ell(g)} \cdot \frac{2}{3}\,\ell(g) = 2m_2^\dagger.$$

Since $H_2(\cdot g)$ and $H_2(\cdot)$ are height functions on the set of F-rational points on a Grassmann $P_2\backslash \mathrm{GL}_V$, the images of X_2 and Y_2 $(\subset B(F)\backslash \mathrm{GL}_V(\mathbb{A}))$ in the quotient $P_2(F)\backslash \mathrm{GL}_V(\mathbb{A})$ are finite sets. The number of functions $H_2(x\cdot)$ $(x \in X_2)$ and $H_2(y\cdot)$ $(y \in Y_2)$ on $\mathrm{GL}_V(\mathbb{A})$ are finite because of left $P_2(F)$-invariance of the function H_2. We have

$$\max\{H_2(xg) \mid x \in X_2\} = m_2^\dagger < \min\{H_2(yg) \mid y \in Y_2\}.$$

Thanks to continuity of the function H_2, we obtain a neighborhood $\mathcal{W}_2$ of g in $\mathrm{GL}_V(\mathbb{A})$ such that for $w \in \mathcal{W}_2$

$$\max\{H_2(xw) \mid x \in X_2\} < \min\left\{2m_2^\dagger, \min\{H_2(yw) \mid y \in Y_2\}\right\}.$$

Let $\mathcal{U}_2 := \mathcal{U}_1 \cap \mathcal{W}_2$. We understand

$$m(u) = \min\{H(xu) \mid x \in X_2\} \quad (u \in \mathcal{U}_2).$$

Denote by $X_2^\dagger$ ($\subset X_2$) any set of representatives of the diagonal image of X_2 ($\subset B(F)\backslash \mathrm{GL}_V(F)$) in $(P_1(F)\backslash \mathrm{GL}_V(F)) \times (P_2(F)\backslash \mathrm{GL}_V(F))$. The set $X_2^\dagger$ is finite.

Put $\quad m_3^\dagger := \max\{H_3(xg) \mid x \in X_2^\dagger\}$ and $X_3 := \{x \in X_2 \mid H_3(xg) \leq m_3^\dagger\}$.

With the same reasoning as above, we get a neighborhood $\mathcal{U}_3$ of g in $\mathrm{GL}_V(\mathbb{A})$, which depends only on g, such that

$$m(u) = \min\{H(xu) \mid x \in X_3\} \quad (u \in \mathcal{U}_3).$$

Proceeding in this way, we pick up constant numbers $m_1^\dagger = m_1(g), m_2^\dagger, m_3^\dagger, \ldots, m_{n-1}^\dagger$ and a neighborhood $\mathcal{U}$ of g in $\mathrm{GL}_V(\mathbb{A})$, all depending only on g, which satisfy

$$m(u) = \min\{H(xu) \mid x \in X\} \quad (u \in \mathcal{U}),$$

where $\quad X = \{x \in B(F)\backslash \mathrm{GL}_V(F) \mid H_i(xg) \leq m_i^\dagger \ (i = 1, \ldots, n-1)\}.$

We can show finiteness of the cardinality of X as in Lemma 2.2. ∎

Lemma 2.4. *For each $g \in \mathrm{GL}_V(\mathbb{A})$, there is a neighborhood $\mathcal{U}$ of g in $\mathrm{GL}_V(\mathbb{A})$ such that $M(g) \supset M(u)$ for all $u \in \mathcal{U}$.*

Proof. Let X be the finite set in the proof of Lemma 2.3. Then $M(u) \subset X$ ($u \in \mathcal{U}$) for $\mathcal{U}$ there. Assume that $x \in X - M(g)$. Among $1, \ldots, n-1$, we have a number q which satisfies

$$H_q(xg) \neq m_q(g).$$

Since H_q and m_q are continuous on $\mathrm{GL}_V(\mathbb{A})$, there exists a neighborhood $\mathcal{V}$ of g with

$$H_q(xu) \neq m_q(u) \quad \text{for every} \quad u \in \mathcal{V},$$

hence $x \notin M(u)$ ($u \in \mathcal{V}$). The set X being finite, if we take the neighborhood $\mathcal{V}$ small enough, then we see the intersection of $X - M(g)$ with $M(u)$ ($u \in \mathcal{V}$) is empty. Namely, $M(u) \subset M(g)$ ($u \in \mathcal{U} \cap \mathcal{V}$). ∎

In order to prove some facts which we need later, we now recall the formal definition of the height functions $H_1, \ldots, H_{n-1}$.

Let Z be the center of GL_V and, for each $i = 1, \ldots, n-1$, let L_i be the unique Levi subgroup of P_i containing the diagonal subgroup A of GL_V with respect to the basis $X_1, \ldots, X_n$ of V. The subgroup Z consists of scalar matrices and L_i is the subgroup of P_i with its upper-right $(n-i) \times i$-corner being zero. Regard the character group $X^*(L_i/Z)$ of L_i/Z as a subgroup of the character group $X^*(A/Z)$ of A/Z by restriction. The groups $X^*(L_1/Z), \ldots, X^*(L_{n-1}/Z)$ generate a subgroup of finite index in $X^*(A/Z)$. The abelian group $X^*(L_i/Z)$ is free of rank 1. Call χ_i one of the generators of $X^*(L_i/Z)$ which is, as an element of $X^*(A/Z)$, a product of non-negative powers of positive roots with respect to the Borel subgroup $B = P_1 \cap \cdots \cap P_{n-1}$ of GL_V. Explicitly, for $a = \mathrm{diag}(a_1, \ldots, a_n) \in A(\mathbb{A})$,

$$\chi_i(a) = (a_1 \cdots a_{n-i})^{i/d} (a_{n-i+1} \cdots a_n)^{-(n-i)/d},$$

where d is the greatest common divisor $\gcd(i, n-i)$ of i and $n-i$.

With this notation we obtain the expression

$$H_i(a) = |\chi_i(a)|_{\mathbb{A}^\times}^{-d} \quad \text{for} \quad a \in A(\mathbb{A}),$$

from which we see our height function H_i is the d-th power of the original Watanabe's height function H_{P_i} ([8, §1, Example 1]).

We denote by W the Weyl group of GL_V as to A, which is the normalizer of A in GL_V modulo the centralizer of A. We often identify an element of W with one of its representatives in the normalizer of A. We have an induced right action of W on $X^*(A/Z)$: For $\chi \in X^*(A/Z)$ and $w \in W$,

$$(\chi w)(a) = \chi\big(waw^{-1}\big) \quad (a \in A(R)),$$

where R is an arbitrary F-algebra. It is known to be faithful. The action of W on $X^*(A/Z)$ can be extended to an action on the $\mathbb{R}$-vector space $\mathbb{R} \otimes_{\mathbb{Z}} X^*(A/Z)$.

Lemma 2.5. *For any $w \in W - \{1\}$, there exist positive integers q, p and a cocharacter ξ of A such that for all $\lambda \in \mathbb{A}^\times$*

$$H_i(w\xi(\lambda)w^{-1}) = H_i(\xi(\lambda)) \quad (i = 1, \ldots, q-1)$$

and

$$H_q(w\xi(\lambda)w^{-1}) = |\lambda|_{\mathbb{A}^\times}^{p \gcd(q,n-q)} H_q(\xi(\lambda)).$$

Proof. Since the characters $\chi_1, \ldots, \chi_{n-1}$ span the vector space $\mathbb{R} \otimes_{\mathbb{Z}} X^*(A/Z)$ and the action of W on $\mathbb{R} \otimes_{\mathbb{Z}} X^*(A/Z)$ is faithful, there exists a positive integer q such that

$$\chi_i w = \chi_i \ (i = 1, \ldots, q-1) \quad \text{and} \quad \chi_q w \neq \chi_q.$$

We get a cocharacter ξ of A such that the composition $((\chi_q w)^{-1}\chi_q) \circ \xi$ is a non-constant automorphism of the multiplicative group $\mathbb{G}_{\mathrm{m}}$. Hence there exists an integer p other than 0 with

$$(\chi_q w)(\xi(\lambda))^{-1}\chi_q(\xi(\lambda)) = \lambda^p \quad (\lambda \in \mathbb{A}^\times).$$

We may assume that p is positive. Now, we have only to take respectively the $\gcd(i, n-i)$-th power and the $\gcd(q, n-q)$-th power of the modules $|\cdot|_{\mathbb{A}^\times}$ of both sides of the following equations for $\lambda \in \mathbb{A}^\times$

$$\chi_i(w\xi(\lambda)w^{-1})^{-1} = \chi_i(\xi(\lambda))^{-1} \quad (i = 1, \ldots, q-1)$$

and

$$\chi_q(w\xi(\lambda)w^{-1})^{-1} = \lambda^p \, \chi_q(\xi(\lambda))^{-1}.$$

■

Remember the definition (2) of R and the definition (3) of $M(\cdot)$.

Lemma 2.6. *For $g \in R$, suppose that the cardinality $\# M(g)$ of the finite set $M(g)$ is bigger than 1. Choose any $x \in M(g)$. An arbitrary neighborhood $\mathcal{U}$ of g in $\mathrm{GL}_V(\mathbb{A})$ contains a point $u \in \mathcal{U}$ such that*

$$M(u) \subset M(g) \quad \textit{and} \quad x \notin M(u).$$

Proof. Lemma 2.4 allows to assume $M(u) \subset M(g)$ for every $u \in \mathcal{U}$. Pick temporarily any $y \in M(g)$ with $y \neq B(F)$. The Bruhat decomposition [3, 5.15 Théorème]

$$\mathrm{GL}_V(F) = \bigsqcup_{w \in W} B(F)wB(F) \quad \text{(disjoint union)}$$

says that there are $w \in W - \{1\}$ and $b \in B(F)$ with $y = B(F)wb$. Due to Lemma 2.5, we have positive integers q, p and a cocharacter ξ of A such that for all $\lambda \in \mathbb{A}^\times$

$$H_i(w\xi(\lambda)w^{-1}) = H_i(\xi(\lambda)) \quad (i = 1, \ldots, q-1)$$

and
$$H_q(w\xi(\lambda)w^{-1}) = |\lambda|_{\mathbb{A}^\times}^{p\,\mathrm{gcd}(q,n-q)} H_q(\xi(\lambda)).$$

Put $g_\lambda := b^{-1}\xi(\lambda)bg$ ($\lambda \in \mathbb{A}^\times$). We see for any $i = 1, \ldots, n-1$

$$\begin{aligned} H_i(g_\lambda) &= H_i(\xi(\lambda)bg) = H_i(\xi(\lambda))H_i(bg) = H_i(\xi(\lambda))H_i(g) \\ &= H_i(\xi(\lambda))m_i(g) \end{aligned}$$

and
$$\begin{aligned} H_i(yg_\lambda) &= H_i(w\xi(\lambda)bg) = H_i(w\xi(\lambda)w^{-1})H_i(wbg) \\ &= H_i(w\xi(\lambda)w^{-1})H_i(yg) = H_i(w\xi(\lambda)w^{-1})m_i(g). \end{aligned}$$

When $x = B(F)$, take $\lambda \in \mathbb{A}^\times$ sufficiently close to 1 with $|\lambda|_{\mathbb{A}^\times} < 1$ so that $g_\lambda \in \mathcal{U}$. If for some positive integer r with $r < q$

$$H_i(g_\lambda) = m_i(g_\lambda)\ (i = 1, \ldots, r-1) \quad \text{and} \quad H_r(g_\lambda) > m_r(g_\lambda),$$

then $x \notin M(g_\lambda)$. If $H_i(g_\lambda) = m_i(g_\lambda)$ $(i = 1, \ldots, q-1)$, then for $i = 1, \ldots, q-1$

$$H_i(yg_\lambda) = H_i(w\xi(\lambda)w^{-1})m_i(g) = H_i(\xi(\lambda))m_i(g) = H_i(g_\lambda) = m_i(g_\lambda)$$

and
$$H_q(yg_\lambda) = H_q(w\xi(\lambda)w^{-1})m_q(g) < H_q(\xi(\lambda))m_q(g) = H_q(g_\lambda),$$

thus $x \notin M(g_\lambda)$.

When $x \neq B(F)$, let $y = x$ from the beginning of the present proof and take $\lambda \in \mathbb{A}^\times$ sufficiently close to 1 so that $g_\lambda \in \mathcal{U}$, keeping $|\lambda|_{\mathbb{A}^\times} > 1$ at this time. If for some positive integer r with $r < q$

$$H_i(xg_\lambda) = m_i(g_\lambda)\ (i = 1, \ldots, r-1) \quad \text{and} \quad H_r(xg_\lambda) > m_r(g_\lambda),$$

then $x \notin M(g_\lambda)$. If $H_i(xg_\lambda) = m_i(g_\lambda)$ $(i = 1, \ldots, q-1)$, then for $i = 1, \ldots, q-1$

$$H_i(g_\lambda) = H_i(\xi(\lambda))m_i(g) = H_i(w\xi(\lambda)w^{-1})m_i(g) = H_i(xg_\lambda) = m_i(g_\lambda)$$

and
$$H_q(g_\lambda) = H_q(\xi(\lambda))m_q(g) < H_q(w\xi(\lambda)w^{-1})m_q(g) = H_q(xg_\lambda),$$

hence eventually $x \notin M(g_\lambda)$. ∎

Let
$$R^\circ := \{g \in R \mid \#M(g) = 1\} = \{g \in \mathrm{GL}_V(\mathbb{A}) \mid M(g) = \{B(F)\}\}.$$

We can prove the next lemma and a theorem completely in the same manner as in Watanabe's paper [8, Lemmas 9, 10, 11, 12, 14 and Theorem 15].

Lemma 2.7. *The following assertions hold:*

- *The set* R° *coincides with the interior of* R *in* $\mathrm{GL}_V(\mathbb{A})$.
- *The set* $\mathrm{GL}_V(F)R^\circ$ *is open and dense in* $\mathrm{GL}_V(\mathbb{A})$.
- *For* $\gamma \in \mathrm{GL}_V(F)$, *the set* $R^\circ \cap \gamma R$ *is non-empty if and only if* $\gamma \in B(F)$.
- *Denote by* $\overline{R^\circ}$ *the closure of* R° *in* $\mathrm{GL}_V(\mathbb{A})$. *We have*

$$\mathrm{GL}_V(\mathbb{A}) = \bigcup_\gamma \gamma \overline{R^\circ},$$

where γ *runs through the representatives of* $\mathrm{GL}_V(F)/B(F)$.

Theorem 2.8. *Let* Ω *be an open fundamental domain for the action of* $B(F)$ *on* $\overline{R^\circ}$. *We denote respectively by* Ω°, $\overline{\Omega}$, *and* $\overline{\Omega^\circ}$ *the interior of* Ω, *the closure of* Ω, *and the closure of* Ω° *in* $\mathrm{GL}_V(\mathbb{A})$. *We have*

$$\Omega^\circ = \Omega \cap R^\circ \quad \text{and} \quad \overline{\Omega} = \overline{\Omega^\circ}.$$

The subset Ω° *of* $\mathrm{GL}_V(\mathbb{A})$ *is an open fundamental domain for the action of* $\mathrm{GL}_V(F)$ *on* $\mathrm{GL}_V(\mathbb{A})$.

Remark 2.9. The left action of the discrete subgroup $B(F)$ is very clear in terms of the Iwasawa decomposition $\mathrm{GL}_V(\mathbb{A}) = B(\mathbb{A})K = N(\mathbb{A})A(\mathbb{A})K$.

For the use in Section 3 and Section 4, we shall look more closely at the height functions $H_1, \dots, H_{n-1}$.

Let $\bigwedge^i V$ be the i-th exterior product of the vector space V over F of linear forms in $X_1, \dots, X_n$. Like the function h^v on $F_v \otimes_F V$ for $v \in \mathfrak{M}$, we can define a logarithmic local height function h_i^v on $F_v \otimes_F \bigwedge^i V$ with respect to the basis $X_{q(1)} \wedge \dots \wedge X_{q(i)}$ $(1 \le q(1) < \dots < q(i) \le n)$. On the subset $\left(\bigwedge^i V\right)\mathrm{GL}_V(\mathbb{A})$ of the adèle space $\mathbb{A} \otimes_F \bigwedge^i V$, we obtain a logarithmic global height function h_i as the sum of the local height functions h_i^v $(v \in \mathfrak{M})$:

$$h_i(fg) := \sum_{v \in \mathfrak{M}} h_i^v(fg^v) \quad \Big(f \in \bigwedge^i V,\ g = (g^v)_{v \in \mathfrak{M}} \in \mathrm{GL}_V(\mathbb{A})\Big)$$

The next lemma follows at once:

Lemma 2.10. (cf. [8, Example 1]) *We have for* $g \in \mathrm{GL}_V(\mathbb{A})$

$$\log H_i(g) = n\, h_i((X_{n-i+1} \wedge \dots \wedge X_n)g) - i \log |\det g|_{\mathbb{A}^\times}.$$

3. The case of the rational number field

Below, we shall show for several fields that the fundamental domain $\overline{\Omega}$ constructed in Section 2 is included in a so-called Siegel set. As a result, the boundary of the fundamental domain $\overline{\Omega}$ is observed to be the union of a finite number of zero sets of continuous functions. We calculate it explicitly in 4 examples of low degree. Mahler's compactness criterion is also derived as another consequence. For the specific case of the rational number field, we observe that our fundamental domain $\overline{\Omega}$ is smaller than Grenier's a little.

The notation is the same as in Section 2. In addition, we define $F_\infty := \mathbb{R} \otimes_{\mathbb{Q}} F$, $K^{\mathrm{fin}} := \prod_{v \in \mathfrak{M}^{\mathrm{fin}}} K_v$ the finite part of the compact subgroup K of $\mathrm{GL}_V(\mathbb{A})$, $\mathfrak{o}_F$ the ring of integers in F, and $\mathrm{GL}_V(\mathfrak{o}_F) := \mathrm{GL}(\mathfrak{o}_F \otimes_{\mathbb{Z}} \sum_{i=1}^n \mathbb{Z}X_i)$.

In this section, the base field F is the rational number field $\mathbb{Q}$.

Since the class number of $F = \mathbb{Q}$ equals 1, we have a single double coset decomposition [1, 2.2 Proposition]

$$\mathrm{GL}_V(\mathbb{A}) = \mathrm{GL}_V(\mathbb{Q}) \left(\mathrm{GL}_V(\mathbb{Q}_\infty) \times K^{\mathrm{fin}}\right).$$

On the other hand, we also have a 'rational Iwasawa' decomposition (e.g. [7, Theorem 3])

$$\mathrm{GL}_V(\mathbb{Q}) = B(\mathbb{Q})\ \mathrm{GL}_V(\mathbb{Z}), \tag{4}$$

where B is the Borel subgroup consisting of upper triangular matrices with respect to the basis $X_1, \ldots, X_n$ of V. Thus we see in any way

$$\mathrm{GL}_V(\mathbb{A}) = B(\mathbb{Q}) \left(\mathrm{GL}_V(\mathbb{Q}_\infty) \times K^{\mathrm{fin}}\right).$$

For $g \in \mathrm{GL}_V(\mathbb{A})$, call g^∞ any element of $\mathrm{GL}_V(\mathbb{Q}_\infty)$ such that there exist elements $b \in B(\mathbb{Q})$ and $k \in K^{\mathrm{fin}}$ satisfying $g = bg^\infty k$. Here we regard respective elements of $\mathrm{GL}_V(\mathbb{Q}_\infty)$ or K^{fin} as elements of $\mathrm{GL}_V(\mathbb{A})$ under the canonical identification of $\mathrm{GL}_V(\mathbb{Q}_\infty)$ or K^{fin} with a subgroup of $\mathrm{GL}_V(\mathbb{A})$. The left $B(\mathbb{Q})$-invariance and the right K-invariance of the height functions $H_1, \ldots, H_{n-1}$ and the successive minimum functions $m_1, \ldots, m_{n-1}$ imply

$$H_i(g) = H_i(g^\infty) \quad \text{and} \quad m_i(g) = m_i(g^\infty) \qquad (i = 1, \ldots, n-1).$$

Note that a continuous homomorphism $|\det(\cdot)|_{\mathbb{A}^\times}$ takes the constant value $1 \in \mathbb{R}^\times_{>0}$ on any compact subgroup of $\mathrm{GL}_V(\mathbb{A})$ and on the group $\mathrm{GL}_V(F)$ of rational points. The latter fact is due to the product formula. In particular, we have

$$|\det g|_{\mathbb{A}^\times} = |\det g^\infty|_{\mathbb{A}^\times} = |\det g^\infty|_\infty.$$

Denote respectively by

$$a = \mathrm{diag}(a_1, \ldots, a_n) \quad \text{and} \quad a^\infty = \mathrm{diag}(a_1^\infty, \ldots, a_n^\infty)$$

the A-parts of the Iwasawa decompositions of $g \in \mathrm{GL}_V(\mathbb{A})$ and of $g^\infty \in \mathrm{GL}_V(\mathbb{Q}_\infty)$. By definition, we know

$$H_1(g) = |a_n|^n_{\mathbb{A}^\times}\, |\det g|^{-1}_{\mathbb{A}^\times},$$

$$\frac{H_{i+1}(g)}{H_i(g)} = \frac{|a_{n-i} \cdots a_n|^n_{\mathbb{A}^\times}\, |\det g|^{-i-1}_{\mathbb{A}^\times}}{|a_{n-i+1} \cdots a_n|^n_{\mathbb{A}^\times}\, |\det g|^{-i}_{\mathbb{A}^\times}} = |a_{n-i}|^n_{\mathbb{A}^\times}\, |\det g|^{-1}_{\mathbb{A}^\times} \quad (i = 1, \ldots, n-2),$$

and

$$H_{n-1}(g) = |a_2 \cdots a_n|^n_{\mathbb{A}^\times}\, |\det g|^{-n+1}_{\mathbb{A}^\times} = |a_1|^{-n}_{\mathbb{A}^\times}\, |\det g|_{\mathbb{A}^\times}.$$

Analogous equations are valid for g^∞, too.

Lemma 3.1. *When $F = \mathbb{Q}$, we have*

$$|a_i|_{\mathbb{A}^\times} = |a_i^\infty|_\infty \quad (i = 1, \ldots, n)$$

and if $g \in \mathrm{GL}_V(\mathbb{A})$ is in the subset R of (2), then g^∞ is also in R.

Proof. From what we have seen just above, the equalities

$$|a_1|_{\mathbb{A}^\times}^n = |\det g|_{\mathbb{A}^\times}\, H_{n-1}(g)^{-1} = |\det g^\infty|_\infty\, H_{n-1}(g^\infty)^{-1} = |a_1^\infty|_\infty^n,$$

$$|a_i|_{\mathbb{A}^\times}^n = \frac{H_{n-i+1}(g)}{H_{n-i}(g)}\, |\det g|_{\mathbb{A}^\times} = \frac{H_{n-i+1}(g^\infty)}{H_{n-i}(g^\infty)}\, |\det g^\infty|_\infty = |a_i^\infty|_\infty^n$$

for $i = 2, \dots, n-1$, and

$$|a_n|_{\mathbb{A}^\times}^n = H_1(g)\, |\det g|_{\mathbb{A}^\times} = H_1(g^\infty)\, |\det g^\infty|_\infty = |a_n^\infty|_\infty^n$$

hold. Moreover, if $g \in R$, then by the definition of R, we get

$$H_i(g^\infty) = H_i(g) = m_i(g) = m_i(g^\infty) \quad (i = 1, \dots, n-1),$$

hence $g^\infty \in R$. ■

Now consider for each $i = 1, \dots, n-1$, the logarithmic height function h_i on the i-th exterior product $\left(\bigwedge^i V\right) \mathrm{GL}_V(\mathbb{A})$. For an integral matrix $x \in \mathrm{GL}_V(\mathbb{Z})$, let $r_i(x)$ be the lower $i \times n$-block of x. An elementary calculation shows that for $g \in \mathrm{GL}_V(\mathbb{A})$

$$h_i((X_{n-i+1} \wedge \cdots \wedge X_n) x g^\infty) = \log \sqrt{\det(r_i(x) g^\infty\, {}^{\mathrm{t}}g^\infty\, {}^{\mathrm{t}}r_i(x))}. \tag{5}$$

If $x \in P_{i-1}(\mathbb{Z})$, then $r_i(x)$ is of the form

$$r_i(x) = \begin{pmatrix} \mathbf{x} & \mathbf{y} \\ 0 & y \end{pmatrix},$$

where $\mathbf{x} \in \mathbb{Z}^{n-i+1}$, $\mathbf{y} \in \mathbb{Z}^{i-1}$, and y is an integral matrix of degree $i-1$. Let $g^\infty = n^\infty a^\infty k^\infty$ ($n^\infty \in N(\mathbb{Q}_\infty)$, $k^\infty \in K_\infty$) be the Iwasawa decomposition, where $a^\infty = \mathrm{diag}(a_i^\infty) \in A(\mathbb{Q}_\infty)$ as before. We denote respectively by b_1, b_2, and b_3 the upper-left $(n-i+1) \times (n-i+1)$-corner, the upper-right $(n-i+1) \times (i-1)$-corner, and the lower-right $(i-1) \times (i-1)$-corner of the matrix $n^\infty a^\infty$:

$$n^\infty a^\infty = \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix}$$

We have $r_i(x) g^\infty = \begin{pmatrix} \mathbf{x} & \mathbf{y} \\ 0 & y \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} k^\infty = \begin{pmatrix} \mathbf{x} b_1 & \mathbf{x} b_2 + \mathbf{y} b_3 \\ 0 & y b_3 \end{pmatrix} k^\infty.$

Using (5), we observe

$$\begin{aligned} &\exp 2h_i((X_{n-i+1} \wedge \cdots \wedge X_n) x g^\infty) = \det r_i(x) g^\infty\, {}^{\mathrm{t}}g^\infty\, {}^{\mathrm{t}}r_i(x) \\ &= \det \begin{pmatrix} \mathbf{x} b_1\, {}^{\mathrm{t}}b_1\, {}^{\mathrm{t}}\mathbf{x} + (\mathbf{x} b_2 + \mathbf{y} b_3)\, {}^{\mathrm{t}}(\mathbf{x} b_2 + \mathbf{y} b_3) & (\mathbf{x} b_2 + \mathbf{y} b_3)\, {}^{\mathrm{t}}b_3\, {}^{\mathrm{t}}y \\ y b_3\, {}^{\mathrm{t}}(\mathbf{x} b_2 + \mathbf{y} b_3) & y b_3\, {}^{\mathrm{t}}b_3\, {}^{\mathrm{t}}y \end{pmatrix} \\ &= (\mathbf{x} b_1\, {}^{\mathrm{t}}b_1\, {}^{\mathrm{t}}\mathbf{x}) \det(y b_3\, {}^{\mathrm{t}}b_3\, {}^{\mathrm{t}}y) = (\mathbf{x} b_1\, {}^{\mathrm{t}}b_1\, {}^{\mathrm{t}}\mathbf{x})(a_{n-i+2}^\infty \cdots a_n^\infty)^2, \end{aligned}$$

because $\det y = \pm 1$ and $\det b_3 = a_{n-i+2}^\infty \cdots a_n^\infty$.

Theorem 3.2. *When $F = \mathbb{Q}$, we have for any $g \in R$*

$$|a_i|_{\mathbb{A}^\times} \geq \frac{\sqrt{3}}{2}\, |a_{i+1}|_{\mathbb{A}^\times} \quad (i = 1, \dots, n-1).$$

Proof. Call $n^\infty_{i,j}$ the (i,j)-entry of n^∞ and denote by $s_{i,j}$ the identity matrix of degree n with the i-th and the j-th rows exchanged. In the above calculation, we respectively substitute $n-i$ for i and $s_{i,i+1}$ for x. We get

$$\mathbf{x}b_1 = (0,\, \ldots,\, 0,\, a^\infty_i,\, n^\infty_{i,i+1}a^\infty_{i+1}),$$

hence

$$\begin{aligned}&\exp 2h_{n-i}((X_{i+1}\wedge\cdots\wedge X_n)s_{i,i+1}g^\infty)\\ &= \left((a^\infty_i)^2 + (n^\infty_{i,i+1}a^\infty_{i+1})^2\right)(a^\infty_{i+2}\cdots a^\infty_n)^2\\ &= (a^\infty_i a^\infty_{i+2}\cdots a^\infty_n)^2 + (n^\infty_{i,i+1}a^\infty_{i+1}a^\infty_{i+2}\cdots a^\infty_n)^2.\end{aligned}$$

On the assumption $g\in R$, we know $g^\infty\in R$ from Lemma 3.1. We see from the fact that $s_{i,i+1}\in(P_1\cap\cdots\cap P_{n-i-1})(\mathbb{Q})$

$$H_j(s_{i,i+1}g^\infty) = H_j(g^\infty) = m_j(g^\infty) \quad (j=1,\ldots,n-i-1)$$

and

$$H_{n-i}(s_{i,i+1}g^\infty) \geq m_{n-i}(g^\infty) = H_{n-i}(g^\infty).$$

By virtue of Lemma 2.10, we obtain

$$\begin{aligned}&\log\left((a^\infty_i a^\infty_{i+2}\cdots a^\infty_n)^2 + (n^\infty_{i,i+1}a^\infty_{i+1}a^\infty_{i+2}\cdots a^\infty_n)^2\right)\\ &\geq 2h_{n-i}((X_{i+1}\wedge\cdots\wedge X_n)g^\infty) = \log(a^\infty_{i+1}a^\infty_{i+2}\cdots a^\infty_n)^2.\end{aligned}$$

Notice that we can select g^∞ so that $|n^\infty_{i,i+1}|_\infty \leq 1/2$, because we may alter the representative g^∞ to bg^∞ for an arbitrary $b\in B(\mathbb{Z})$. Thus we get

$$\log\left(\left(\frac{a^\infty_i}{a^\infty_{i+1}}\right)^2 + \frac{1}{4}\right) \geq \log\left(\left(\frac{a^\infty_i}{a^\infty_{i+1}}\right)^2 + (n^\infty_{i,i+1})^2\right) \geq 0.$$

Thanks to Lemma 3.1, we arrive at the conclusion. ∎

We know $B(\mathbb{Q}_\infty)\cap K_\infty = A(\mathbb{Z}) = \{\mathrm{diag}(\pm1,\ldots,\pm1)\}$ and

$$B(\mathbb{Q})\cap\left(\mathrm{GL}_V(\mathbb{Q}_\infty)\times K^{\mathrm{fin}}\right) = B(\mathbb{Z}) = N(\mathbb{Z})A(\mathbb{Z}).$$

We denote by K^-_∞ any open fundamental domain for the translation by scalar matrices $\{\pm1\}$ on K_∞. A concrete example of an open fundamental domain is $\Omega^- = \Omega^-_\infty\times K^{\mathrm{fin}}$ with a subset

$$\Omega^-_\infty := \left\{g^\infty \;\middle|\; g^\infty\in R^\circ,\ |n^\infty_{i,j}|_\infty < \frac{1}{2},\ n^\infty_{i,i+1}>0,\ a^\infty_i>0 \text{ for all } i\neq j,\ k^\infty\in K^-_\infty\right\} \quad (6)$$

of $\mathrm{GL}_V(\mathbb{Q}_\infty)$. For later use, put $\Omega^\circ_\infty := \Omega^-_\infty K_\infty$. This is about the twice of Ω^-_∞. Let

$$\mathfrak{S}_\infty := \left\{g^\infty \;\middle|\; |n^\infty_{i,j}|_\infty \leq \frac{1}{2},\ n^\infty_{i,i+1}\geq 0,\ a^\infty_i \geq \frac{\sqrt{3}}{2}a^\infty_{i+1} > 0 \text{ for all } i\neq j\right\},$$

a subset of $\mathrm{GL}_V(\mathbb{Q}_\infty)$. This is an example of Siegel set or Siegel domain [2]. A Siegel set in $\mathrm{GL}_V(\mathbb{Q}_\infty)$ is in general a subset of the form $\nu A_t K_\infty$ with a positive parameter t, where ν is a compact subset of $N(\mathbb{Q}_\infty)$ and

$$A_t := \left\{a^\infty\in A(\mathbb{Q}_\infty) \;\middle|\; \frac{a^\infty_i}{a^\infty_{i+1}} \geq t\ (i=1,\cdots,n-1),\ a^\infty_n>0\right\}$$

with our notation. For $\mathrm{GL}_n(\mathbb{C})$, Siegel sets are defined in a similar way, the orthogonal group being replaced by the unitary group, or, with the use of embedding of $\mathrm{GL}_n(\mathbb{C})$ into $\mathrm{GL}_{2n}(\mathbb{R})$ via Weil restriction. Both result in the same thing.

Corollary 3.3. $\Omega^- \subset \mathfrak{S} := \mathfrak{S}_\infty \times K^{\mathrm{fin}}$.

Proof. An immediate conclusion of Lemma 3.1 and Theorem 3.2. ∎

As an application of Theorem 3.2, we shall see the open fundamental domain Ω^- is in fact almost determined by a finite number of inequalities which yield Grenier's fundamental domain. Before doing that, we will see a relation of the set Ω^-_∞ to Grenier's fundamental domain.

Lemma 3.4. *When $F = \mathbb{Q}$, if $x \in P_{i-1}(\mathbb{Z})$, then*

$$\log H_i(xg^\infty) = \frac{n}{2}\log(\mathbf{x}b_1\,{}^{\mathrm{t}}b_1\,{}^{\mathrm{t}}\mathbf{x}) + n\log|a^\infty_{n-i+2}\cdots a^\infty_n|_\infty - i\log|\det g^\infty|_\infty,$$

where $\mathbf{x} \in \mathbb{Z}^{n-i+1}$ is the $(n-i+1)$-dimensional left component of the $(n-i+1)$-th row of the matrix x and b_1 is the upper-left square corner of degree $n-i+1$ of the NA-part $n^\infty a^\infty$ of the Iwasawa decomposition $g^\infty = n^\infty a^\infty k^\infty$.

Proof. Apply Lemma 2.10 to the calculation just before Theorem 3.2. ∎

Assume temporarily that $\xi_1, \ldots, \xi_n$ are variable real numbers and let $\mathbf{x}_i := (\xi_1, \ldots, \xi_i)$ $(i = 1, \ldots, n)$. We denote by $q_i(\mathbf{x}_{n-i+1})$ the quadratic form in the coefficients of $\mathbf{x}_{n-i+1}$ attached to a positive definite symmetric matrix $b_1{}^{\mathrm{t}}b_1$ in Lemma 3.4:

$$q_i(\mathbf{x}_{n-i+1}) := \mathbf{x}_{n-i+1}b_1\,{}^{\mathrm{t}}b_1\,{}^{\mathrm{t}}\mathbf{x}_{n-i+1} \quad (i = 1, \ldots, n)$$

Note that we have

$$q_1(\mathbf{x}_n) = \mathbf{x}_n(n^\infty a^\infty)\,{}^{\mathrm{t}}(n^\infty a^\infty)\,{}^{\mathrm{t}}\mathbf{x}_n = \mathbf{x}_n g^\infty\,{}^{\mathrm{t}}g^\infty\,{}^{\mathrm{t}}\mathbf{x}_n$$

and

$$q_i(\mathbf{x}_{n-i+1}) = \tag{7}$$

$$q_{i+1}(\mathbf{x}_{n-i}) + \left(a^\infty_{n-i+1}\right)^2\left(\xi_1 n^\infty_{1,n-i+1} + \cdots + \xi_{n-i}n^\infty_{n-i,n-i+1} + \xi_{n-i+1}\right)^2$$

for $i = 1, \ldots, n-1$. Put

$$\tilde{\Omega}_\infty := \{g^\infty \in \mathrm{GL}_V(\mathbb{Q}_\infty) \mid g^\infty \in R,\ |n^\infty_{i,j}|_\infty \le \tfrac{1}{2},\ n^\infty_{i,i+1} \ge 0 \text{ for all } i \ne j\}.$$

Proposition 3.5. *If $g^\infty \in \tilde{\Omega}_\infty$, then the quadratic form q_1 attached to $g^\infty\,{}^{\mathrm{t}}g^\infty$ is HKZ-reduced (Definition 1.1).*

Proof. Suppose $g^\infty \in R$. By the definition (2) of R, Lemma 2.10, and the equation (5) we have

$$\begin{aligned}\frac{(a^\infty_n)^n}{|\det g^\infty|_\infty} &= H_1(g^\infty) = m_1(g^\infty) = \min\{H_1(xg^\infty) \mid x \in \mathrm{GL}_V(\mathbb{Z})\}\\ &= \min\left\{\frac{q_1(\mathbf{x}_n)^{n/2}}{|\det g^\infty|_\infty}\,\middle|\, x \in \mathrm{GL}_V(\mathbb{Z})\right\},\end{aligned}$$

where $\mathbf{x}_n$ is the last row of the integral matrix $x \in \mathrm{GL}_V(\mathbb{Z})$. In particular, elements of $P_1(\mathbb{Z})$ take the minimum value $m_1(g^\infty)$ of $H_1(xg^\infty)$ $(x \in \mathrm{GL}_V(\mathbb{Z}))$. We obtain

$$(a^\infty_n)^2 = \min\{q_1(\alpha_1, \ldots, \alpha_n) \mid (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n - \{0\}\}.$$

Since the stabilizer of P_1 under the (left) translation action on GL_V on itself is of course P_1, we see from Lemma 3.4

$$\begin{aligned}\frac{(a_{n-1}^{\infty}a_n^{\infty})^n}{|\det g^{\infty}|_{\infty}^2} &= H_2(g^{\infty}) = m_2(g^{\infty}) \leq \min\{H_2(xg^{\infty}) \mid x \in P_1(\mathbb{Z})\} \\ &= \min\left\{\frac{q_2(\mathbf{x}_{n-1})^{n/2}(a_n^{\infty})^n}{|\det g^{\infty}|_{\infty}^2} \,\middle|\, x \in P_1(\mathbb{Z})\right\},\end{aligned}$$

where $\mathbf{x}_{n-1}$ is the last row of the upper-left square corner of degree $n-1$ of $x \in P_1(\mathbb{Z})$. Elements of $(P_1 \cap P_2)(\mathbb{Z})$ give the minimum $m_2(g^{\infty})$. We get

$$\left(a_{n-1}^{\infty}\right)^2 = \min\{q_2(\alpha_1,\ldots,\alpha_{n-1}) \mid (\alpha_1,\ldots,\alpha_{n-1}) \in \mathbb{Z}^{n-1} - \{0\}\}.$$

The definition of m_3 by means of the lexicographic order on $\mathbb{R}^{n-1}$ tells us that

$$\begin{aligned}\frac{(a_{n-2}^{\infty}a_{n-1}^{\infty}a_n^{\infty})^n}{|\det g^{\infty}|_{\infty}^3} &= H_3(g^{\infty}) = m_3(g^{\infty}) \leq \min\{H_3(xg^{\infty}) \mid x \in (P_1 \cap P_2)(\mathbb{Z})\} \\ &= \min\left\{\frac{q_3(\mathbf{x}_{n-2})^{n/2}(a_{n-1}^{\infty}a_n^{\infty})^n}{|\det g^{\infty}|_{\infty}^3} \,\middle|\, x \in (P_1 \cap P_2)(\mathbb{Z})\right\},\end{aligned}$$

where $\mathbf{x}_{n-2}$ is the last row of the upper-left square corner of degree $n-2$ of $x \in (P_1 \cap P_2)(\mathbb{Z})$. Hence

$$\left(a_{n-2}^{\infty}\right)^2 = \min\{q_3(\alpha_1,\ldots,\alpha_{n-2}) \mid (\alpha_1,\ldots,\alpha_{n-2}) \in \mathbb{Z}^{n-2} - \{0\}\}.$$

In this way, we find for any $i = 1, 2, 3, \ldots, n$

$$(a_i^{\infty})^2 = \min\{q_{n-i+1}(\alpha_1,\ldots,\alpha_i) \mid (\alpha_1,\ldots,\alpha_i) \in \mathbb{Z}^i - \{0\}\}.$$

Thanks to the equation (7) and the condition $|n_{i,j}^{\infty}| \leq 1/2$, $n_{i,i+1}^{\infty} \geq 0$, we understand that q_1 is HKZ-reduced. ■

Remark 3.6. The converse of Proposition 3.5 is not valid. Actually, let

$$g^{\infty} = \begin{pmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The quadratic form $q_1(\xi,\eta,\zeta)$ attached to g^{∞} is

$$\begin{aligned}q_1(\xi,\eta,\zeta) &= (\sqrt{3}\xi, \xi + 2\eta, 2\zeta)\,{}^{\mathrm{t}}(\sqrt{3}\xi, \xi + 2\eta, 2\zeta) \\ &= 4\xi^2 + 4\xi\eta + 4\eta^2 + 4\zeta^2,\end{aligned}$$

which is nothing but $g(\xi,\eta,\zeta)$ in the Introduction. As we saw there, the quadratic form $q_1 = g$ is HKZ-reduced.

We denote by $x \in \mathrm{GL}_3(\mathbb{Z})$ a matrix whose lower 2×3-block $r_2(x)$ has the form

$$r_2(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

From Lemma 2.10 and the equality (5), we observe

$$H_1(g^\infty) = \exp\left(\frac{3}{2}\log q_1(0,0,1) - \log|\det g^\infty|_\infty\right)$$
$$= \exp(3\log 2 - \log 4\sqrt{3}) = \frac{8}{4\sqrt{3}} = \frac{2}{\sqrt{3}}$$

and
$$H_1(xg^\infty) = \exp\left(\frac{3}{2}\log q_1(0,1,0) - \log|\det g^\infty|_\infty\right) = \frac{2}{\sqrt{3}}.$$

Since the minimum value other than 0 of q_1 at the rational integral points is 4, we have

$$m_1(g^\infty) = H_1(g^\infty) = H_1(xg^\infty) = \frac{2}{\sqrt{3}}.$$

On the other hand, the equation (5) shows us that

$$\exp 2h_2((X_2 \wedge X_3)g^\infty) = \det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g^\infty \, {}^{\mathrm{t}}g^\infty \, {}^{\mathrm{t}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} {}^{\mathrm{t}}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4^2 = 16, \text{ and}$$

$$\exp 2h_2((X_2 \wedge X_3)xg^\infty) = \det r_2(x) g^\infty \, {}^{\mathrm{t}}g^\infty \, {}^{\mathrm{t}}r_2(x)$$
$$= \det \begin{pmatrix} \sqrt{3} & 1 \\ 0 & 2 \end{pmatrix} {}^{\mathrm{t}}\begin{pmatrix} \sqrt{3} & 1 \\ 0 & 2 \end{pmatrix} = (2\sqrt{3})^2 = 12.$$

By Lemma 2.10, we know that

$$H_2(g^\infty) = \exp\left(3h_2((X_2 \wedge X_3)g^\infty) - 2\log|\det g^\infty|_\infty\right) = 16^{3/2}(4\sqrt{3})^{-2} = \frac{4}{3}$$

and
$$H_2(xg^\infty) = 12^{3/2}(4\sqrt{3})^{-2} = \frac{\sqrt{3}}{2}.$$

We gain $H_2(g^\infty) > H_2(xg^\infty)$, hence $g^\infty \notin R$.

This example implies that HKZ-reduced quadratic forms do not necessarily come from $\tilde{\Omega}_\infty$.

We refine the notion of HKZ-reducedness as follows:

Definition 3.7. Using the notation from Definition 1.1 we call a quadratic form q is *strictly reduced* if and only if the next conditions are satisfied:

- $a_n^2 < q(\alpha_1, \dots, \alpha_n)$ for $(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n - \{0, (0, \dots, 0, \pm 1)\}$
- $|u_i| < \frac{1}{2}$ $(i = 1, \dots, n-1)$ and $u_{n-1} > 0$
- $\tilde{q}$ is strictly reduced.

Remark 3.8. A strictly reduced quadratic form is HKZ-reduced.

Lemma 3.9. *Returning to the notation in Section* 3*, the quadratic form* q_1 *is strictly reduced if and only if*

- $(a_i^\infty)^2 < q_{n-i+1}(\alpha_1, \dots, \alpha_i)$ *for* $(\alpha_1, \dots, \alpha_i) \in \mathbb{Z}^i - \{0, (0, \dots, 0, \pm 1)\}$ *and* $i = 1, \dots, n$
- $|n_{i,j}^\infty| < \frac{1}{2}$ *and* $n_{i,i+1}^\infty > 0$ *for any* $i < j$

Proof. Repeated application of Definition 3.7 to the equality (7). ∎

Lemma 3.10. *Let $P_0 := \mathrm{GL}_V$ and fix $i \in \{1, \dots, n-1\}$. When $F = \mathbb{Q}$, except a finite number of cosets $x \in (P_{i-1} \cap P_i)(\mathbb{Z}) \backslash P_{i-1}(\mathbb{Z})$, we have*

$$H_i(xg) > H_i(g) \quad \textit{for any } g \in \mathfrak{S},$$

where $\mathfrak{S}$ is the subset of $\mathrm{GL}_V(\mathbb{A})$ in Corollary 3.3.

Proof. The symbols $\mathbf{x}$ and b_1 have the same meaning as in Lemma 3.4. Put $\mathbf{x} =: (\xi_1, \dots, \xi_{n-i+1})$. We see

$$\begin{aligned}\mathbf{x} b_1 {}^{\mathrm{t}} b_1 {}^{\mathrm{t}}\mathbf{x} = (a_1^\infty)^2 \xi_1^2 + (a_2^\infty)^2 (\xi_1 n_{1,2}^\infty + \xi_2)^2 + \cdots \\ + (a_{n-i+1}^\infty)^2 (\xi_1 n_{1,n-i+1}^\infty + \cdots + \xi_{n-i} n_{n-i,n-i+1}^\infty + \xi_{n-i+1})^2.\end{aligned}$$

Since $g \in \mathfrak{S}$, we observe

$$\begin{aligned}\mathbf{x} b_1 {}^{\mathrm{t}} b_1 {}^{\mathrm{t}}\mathbf{x} \geq \biggl(\left(\frac{3}{4}\right)^{n-i} \xi_1^2 + \left(\frac{3}{4}\right)^{n-i-1} (\xi_1 n_{1,2}^\infty + \xi_2)^2 + \cdots \\ + (\xi_1 n_{1,n-i+1}^\infty + \cdots + \xi_{n-i} n_{n-i,n-i+1}^\infty + \xi_{n-i+1})^2 \biggr) (a_{n-i+1}^\infty)^2,\end{aligned}$$

hence, leaving a finite number of $\mathbf{x} \in \mathbb{Z}^{n-i+1}$ aside, the inequality

$$\mathbf{x} b_1 {}^{\mathrm{t}} b_1 {}^{\mathrm{t}}\mathbf{x} > (a_{n-i+1}^\infty)^2$$

holds. Note that the map

$$(P_{i-1} \cap P_i)(\mathbb{Z}) \backslash P_{i-1}(\mathbb{Z}) \ni x \mapsto \mathbf{x} \in \left(\mathbb{Z}^{n-i+1} - \{0\}\right) \big/ \{\pm 1\}$$

is injective. From Lemma 3.4 and the right K-invariance of H_i, we know

$$\log H_i(xg) - \log H_i(g) = \frac{n}{2} \log \frac{\mathbf{x} b_1 {}^{\mathrm{t}} b_1 {}^{\mathrm{t}}\mathbf{x}}{(a_{n-i+1}^\infty)^2}.$$

Thus the assertion is true. ∎

Denote by F_i $(i = 1, \dots, n-1)$ the set of cosets

$$x \in (P_0 \cap \cdots \cap P_i)(\mathbb{Z}) \backslash (P_0 \cap \cdots \cap P_{i-1})(\mathbb{Z})$$

satisfying $x \neq (P_0 \cap \cdots \cap P_i)(\mathbb{Z})$ and

$$H_i(xg) \leq H_i(g) \quad \text{for some} \quad g \in \mathfrak{S}.$$

Due to Lemma 3.10, the cardinality of F_i is finite for any $i \in \{1, \dots, n-1\}$. Let

$$E := \{ g \in \mathrm{GL}_V(\mathbb{A}) \mid H_i(xg) = H_i(g) \text{ for some } i \text{ and } x \in F_i \}.$$

The set E is a finite union of zero loci of continuous functions on $\mathrm{GL}_V(\mathbb{A})$.

Theorem 3.11. *The attached quadratic form q_1 is strictly reduced if and only if $g^\infty \in \Omega_\infty^\circ \times \{1\} - E$. Here $\Omega_\infty^\circ = \Omega_\infty^- K_\infty$ and the set $\Omega_\infty^- \times \{1\}$ is the infinite part of our fundamental domain $\Omega^- = \Omega_\infty^- \times K^{\mathrm{fin}}$.*

Proof. Suppose $g^\infty \in \Omega_\infty^\circ \times \{1\} - E$. Since $g^\infty \in R^\circ$, we have at the first place

$$H_1(g^\infty) = m_1(g^\infty) \le H_1(xg^\infty) \quad (x \in \mathrm{GL}_V(\mathbb{Z})).$$

The fact that $g^\infty \in \mathfrak{S} - E$ forces

$$H_1(g^\infty) < H_1(xg^\infty) \quad \text{for all} \quad x \in \mathrm{GL}_V(\mathbb{Z}) - P_1(\mathbb{Z}),$$

which is the same as saying (cf. the proof of Proposition 3.5)

$$(a_n^\infty)^2 < q_1(\alpha_1, \dots, \alpha_n) \text{ for } (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n - \{0, (0, \dots, 0, \pm 1)\}.$$

As a consequence, we get

$$H_2(g^\infty) = m_2(g^\infty) = \min\{H_2(xg^\infty) \mid x \in P_1(\mathbb{Z})\}.$$

Again, the fact $g^\infty \in \mathfrak{S} - E$ implies

$$H_2(g^\infty) < H_2(xg^\infty) \quad \text{for any} \quad x \in P_1(\mathbb{Z}) - P_2(\mathbb{Z}),$$

which is equivalent to

$$(a_{n-1}^\infty)^2 < q_2(\alpha_1, \dots, \alpha_{n-1}) \text{ for } (\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{Z}^{n-1} - \{0, (0, \dots, 0, \pm 1)\}.$$

As a further consequence, we obtain

$$H_3(g^\infty) = m_3(g^\infty) = \min\{H_3(xg^\infty) \mid x \in (P_1 \cap P_2)(\mathbb{Z})\}.$$

Once again, from $g^\infty \in \mathfrak{S} - E$, we see

$$H_3(g^\infty) < H_3(xg^\infty) \quad \text{for every} \quad x \in (P_1 \cap P_2)(\mathbb{Z}) - P_3(\mathbb{Z}),$$

i.e.,

$$(a_{n-2}^\infty)^2 < q_3(\alpha_1, \dots, \alpha_{n-2}) \text{ for } (\alpha_1, \dots, \alpha_{n-2}) \in \mathbb{Z}^{n-2} - \{0, (0, \dots, 0, \pm 1)\}.$$

In such a way, after this process is terminated, we observe

$$(a_i^\infty)^2 < q_{n-i+1}(\alpha_1, \dots, \alpha_i)$$

for arbitrary pairs of $i = 1, \dots, n$ and $(\alpha_1, \dots, \alpha_i) \in \mathbb{Z}^i - \{0, (0, \dots, 0, \pm 1)\}$. Lemma 3.9 implies q_1 is strictly reduced.

Suppose this time the quadratic form q_1 is strictly reduced. Due to Lemma 3.9, we have

$$(a_i^\infty)^2 < q_{n-i+1}(\alpha_1, \dots, \alpha_i)$$

for any combination of $i = 1, \dots, n$ and $(\alpha_1, \dots, \alpha_i) \in \mathbb{Z}^i - \{0, (0, \dots, 0, \pm 1)\}$. From what we have seen just above, these inequalities are gathered into

$$H_i(g^\infty) < H_i(xg^\infty) \quad \text{for} \quad x \in (P_0 \cap \dots \cap P_{i-1})(\mathbb{Z}) - P_i(\mathbb{Z})$$

$(i = 1, \dots, n-1)$, where $P_0 = \mathrm{GL}_V$ as before. By the lexicographic definition of the minimum function $m(g^\infty)$, the set $M(g^\infty)$ of (3) in Section 2 consists of a trivial element only. This means $g^\infty \in R^\circ$. The condition $g^\infty \notin E$ is apparent. The other conditions for g^∞ to be a member of Ω_∞° are trivial. ∎

Corollary 3.12. (cf. Grenier [4, Theorem 1]) *The open fundamental domain* Ω^- *is almost determined by a finite number of inequalities in the coordinates of the Iwasawa decomposition. More precisely, a dense subset* $\Omega^-_\infty \times \{1\} - E$ *of the infinite part* $\Omega^-_\infty \times \{1\}$ *of* $\Omega^- = \Omega^-_\infty \times K^{\mathrm{fin}}$ *is defined as the set of elements* g^∞ *of* $\mathrm{GL}_V(\mathbb{Q}_\infty)$ *which satisfy* $k^\infty \in K^-_\infty$ *and*

$$|n^\infty_{i,j}|_\infty < \frac{1}{2}, \quad n^\infty_{i,i+1} > 0, \quad a^\infty_i \geq \frac{\sqrt{3}}{2} a^\infty_{i+1} > 0, \quad H_i(xg^\infty) > H_i(g^\infty) \tag{8}$$

for all i, j *with* $i \neq j$, $x \in F_i$.

Proof. Assume $g^\infty \in \Omega^-_\infty \times \{1\} - E$. Completely in the same manner as in the first half of the proof of Theorem 3.11, we find the inequalities

$$H_i(g^\infty) < H_i(xg^\infty) \quad (x \in F_i;\ i = 1, \dots, n-1)$$

are fulfilled. The other inequalities in (8) follow from the definition of Ω^-_∞ and Theorem 3.2.

To show the reverse implication, assume $g^\infty \in \mathrm{GL}_V(\mathbb{Q}_\infty)$ meets the inequalities in (8). We have $g^\infty \in \mathfrak{S}$. By the definition of F_1,

$$H_1(xg^\infty) > H_1(g^\infty) \quad \text{for all} \quad x \in \mathrm{GL}_V(\mathbb{Z}) - P_1(\mathbb{Z}).$$

As a consequence, we get $H_1(g^\infty) = m_1(g^\infty)$ and

$$m_2(g^\infty) = \min\{H_2(xg^\infty) \mid x \in P_1(\mathbb{Z})\}.$$

Since we know from the definition of F_2 that

$$H_2(xg^\infty) > H_2(g^\infty) \quad \text{for any} \quad x \in P_1(\mathbb{Z}) - P_2(\mathbb{Z}),$$

we obtain $H_2(g^\infty) = m_2(g^\infty)$ and

$$m_3(g^\infty) = \min\{H_3(xg^\infty) \mid x \in (P_1 \cap P_2)(\mathbb{Z})\}.$$

In this fashion, we finally see that for every $i = 1, \dots, n-1$

$$H_i(xg^\infty) > H_i(g^\infty) \quad (x \in (P_1 \cap \dots \cap P_{i-1})(\mathbb{Z}) - P_i(\mathbb{Z}))$$

and $H_i(g^\infty) = m_i(g^\infty)$, which implies $g^\infty \in R^\circ$, hence $g^\infty \in \Omega^-_\infty \times \{1\}$. The condition $g^\infty \notin E$ is obvious. ∎

We shall see a few examples of non-trivial defining inequalities $H_i(xg^\infty) > H_i(g^\infty)$ $(x \in F_i;\ i = 1, \dots, n-1)$, which will be observed identical to Grenier's ones. Here, for simplicity of notation, we will omit the super- or sub-script ∞.

For $x \in F_i$, let $\mathbf{x} = (\xi_1, \dots, \xi_{n-i+1})$ be the last row of the upper-left $(n-i+1) \times (n-i+1)$-corner of x as before. The proof of Lemma 3.10 shows that there exist real numbers $n_{i,j}$ $(1 \leq i < j \leq n)$ with $|n_{i,j}| \leq 1/2$ and $n_{i,i+1} \geq 0$ such that

$$\begin{aligned} &\left(\frac{3}{4}\right)^{n-i} \xi_1^2 + \left(\frac{3}{4}\right)^{n-i-1} (\xi_1 n_{1,2} + \xi_2)^2 + \cdots \\ &+ (\xi_1 n_{1,n-i+1} + \cdots + \xi_{n-i} n_{n-i,n-i+1} + \xi_{n-i+1})^2 \leq 1. \end{aligned} \tag{9}$$

Example 3.13. ($n = 2$) The index i must be 1.

If $|\xi_1| \geq 2$, then $(\frac{3}{4})^{n-i}\xi_1^2 = \frac{3}{4}\xi_1^2 \geq 3$. The inequality (9) does not stand.

If $\xi_1 = 0$, then x would belong to $P_1(\mathbb{Z})$, hence x could not be in F_1.

Thus we have $\xi_1 = \pm 1$. When this is the case, the inequality (9) becomes

$$(\pm n_{1,2} + \xi_2)^2 \leq \frac{1}{4}, \quad \text{i.e.,} \quad |\pm n_{1,2} + \xi_2| \leq \frac{1}{2}.$$

Possible is $|\xi_2| = 1$ only when $n_{1,2} = 1/2$. This means if $|\xi_1| = |\xi_2| = 1$, then $H_1(xg) > H_1(g)$ for $g \in \Omega_\infty^\circ$ because $0 < n_{1,2} < 1/2$ for $g \in \Omega_\infty^\circ$. So we can ignore the case $|\xi_1| = |\xi_2| = 1$. There remains the case $|\xi_1| = 1$ and $\xi_2 = 0$. When $\mathbf{x} = (\xi_1, \xi_2) = (\pm 1, 0)$, the defining inequality $H_1(xg) > H_1(g)$ is expressed as

$$\left(\frac{a_1}{a_2}\right)^2 + (n_{1,2})^2 > 1.$$

Let
$$A_+^1 := \{\mathrm{diag}(a_1, a_2) \mid a_1 a_2 = 1,\ a_2 > 0\}$$
and
$$\mathcal{H} := \{z \in \mathbb{C} \mid z = x + y\sqrt{-1},\ x \in \mathbb{R},\ y \in \mathbb{R}_{>0}\}.$$

Identify $N(\mathbb{R})A_+^1$ with $\mathcal{H}$ via the map

$$N(\mathbb{R})A_+^1 \ni \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \mapsto x + y\sqrt{-1} \in \mathcal{H}.$$

Over the complex upper half plane $\mathcal{H}$, the above defining inequality takes the form

$$y^2 + x^2 > 1 \qquad (x + y\sqrt{-1} \in \mathcal{H}),$$

one of the well-known inequalities bounding an open fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$.

Example 3.14. ($n = 3$) When the index i equals 2, the inequality (9) is completely the same as in the example $n = 2$. We get a defining inequality

$$\left(\frac{a_1}{a_2}\right)^2 + (n_{1,2})^2 > 1. \tag{10}$$

Assume $i = 1$. If $|\xi_1| \geq 2$, then $(\frac{3}{4})^{n-i}\xi_1^2 = (\frac{3}{4})^2\xi_1^2 \geq \frac{9}{4}$. The inequality (9) does not hold. In the case $\xi_1 = 0$, the inequality (9) becomes

$$\frac{3}{4}\xi_2^2 + (\xi_2 n_{2,3} + \xi_3)^2 \leq 1.$$

Again, this is of the same form as the one in the previous example. We obtain the defining inequality

$$\left(\frac{a_2}{a_3}\right)^2 + (n_{2,3})^2 > 1. \tag{11}$$

In addition to the assumption $i = 1$, suppose that $\xi_1 = \pm 1$. The inequality (9) transforms into

$$\frac{3}{4}\,(\pm n_{1,2} + \xi_2)^2 + (\pm n_{1,3} + \xi_2 n_{2,3} + \xi_3)^2 \leq \frac{7}{16}.$$

If $|\xi_2| \geq 2$, then $\frac{3}{4}\,(\pm n_{1,2}+\xi_2)^2 \geq \frac{3}{4}\left(\frac{3}{2}\right)^2 = \frac{27}{16}$, impossible.

In the case $|\xi_2| \leq 1$, if $|\xi_3| \geq 2$, then $(\pm n_{1,3}+\xi_2 n_{2,3}+\xi_3)^2 \geq 1$. It has to be $|\xi_3| \leq 1$.

When $\mathbf{x} = (\xi_1, \xi_2, \xi_3) = (\pm 1, 0, 0)$, the defining inequality $H_1(xg) > H_1(g)$ is described as

$$\left(\frac{a_1}{a_2}\right)^2\left(\frac{a_2}{a_3}\right)^2 + (n_{1,2})^2\left(\frac{a_2}{a_3}\right)^2 + (n_{1,3})^2 > 1. \tag{12}$$

If $\mathbf{x} = (\pm 1, 0, \pm 1)$ or $(\pm 1, 0, \mp 1)$, then we would have

$$\left(\frac{a_1}{a_2}\right)^2\left(\frac{a_2}{a_3}\right)^2 + (n_{1,2})^2\left(\frac{a_2}{a_3}\right)^2 + (n_{1,3}\pm 1)^2 > 1,$$

which is not necessary, because when the inequality (12) is fulfilled and $|n_{1,3}| \leq 1/2$, the unnecessary inequality stands automatically.

If $\mathbf{x} = (\pm 1, \pm 1, 0)$, then we would obtain

$$\left(\frac{a_1}{a_2}\right)^2\left(\frac{a_2}{a_3}\right)^2 + (n_{1,2}+1)^2\left(\frac{a_2}{a_3}\right)^2 + (n_{1,3}+n_{2,3})^2 > 1,$$

which is again not necessary. This is because we have $a_1/a_2,\ a_2/a_3 \geq \sqrt{3}/2$ and $n_{1,2} \geq 0$ on the Siegel set $\mathfrak{S}_\infty$, the sum of the first two terms of the left side is already greater than 1. When $\mathbf{x} = (\pm 1, \mp 1, 0)$, the condition $H_1(xg) > H_1(g)$ says

$$\left(\frac{a_1}{a_2}\right)^2\left(\frac{a_2}{a_3}\right)^2 + (n_{1,2}-1)^2\left(\frac{a_2}{a_3}\right)^2 + (n_{1,3}-n_{2,3})^2 > 1. \tag{13}$$

If $\mathbf{x} = (\pm 1, \pm 1, \pm 1)$ or $(\pm 1, \pm 1, \mp 1)$, then we would get

$$\left(\frac{a_1}{a_2}\right)^2\left(\frac{a_2}{a_3}\right)^2 + (n_{1,2}+1)^2\left(\frac{a_2}{a_3}\right)^2 + (n_{1,3}+n_{2,3}\pm 1)^2 > 1,$$

which is always satisfied on the set $\mathfrak{S}_\infty$ as before. When $\mathbf{x} = (\pm 1, \mp 1, \pm 1)$, the defining inequality $H_1(xg) > H_1(g)$ is

$$\left(\frac{a_1}{a_2}\right)^2\left(\frac{a_2}{a_3}\right)^2 + (n_{1,2}-1)^2\left(\frac{a_2}{a_3}\right)^2 + (n_{1,3}-n_{2,3}+1)^2 > 1. \tag{14}$$

If $\mathbf{x} = (\pm 1, \mp 1, \mp 1)$, then we would gain

$$\left(\frac{a_1}{a_2}\right)^2\left(\frac{a_2}{a_3}\right)^2 + (n_{1,2}-1)^2\left(\frac{a_2}{a_3}\right)^2 + (n_{1,3}-n_{2,3}-1)^2 > 1.$$

This is redundant. On $\mathfrak{S}_\infty$, the first term of left side of the inequality is at least $9/16$, the second term is at least $3/16$, and the third is at least $1/4 = 4/16$ because of $n_{2,3} \geq 0$, hence their sum is at least 1.

In summary, we have found that the essential defining inequalities in the case $n = 3$ are 5 inequalities (10), (11), (12), (13), and (14). The evident remaining inequalities, the ones for the K-part of Iwasawa decomposition being left aside, are

$$0 < n_{1,2},\ n_{2,3} < \frac{1}{2}; \quad |n_{1,3}| < \frac{1}{2}; \quad a_1,\ a_2,\ a_3 > 0.$$

Note that under the conditions (10), (11), and $0 < n_{1,2},\ n_{2,3} < 1/2$, we always have $|a_1/a_2|,\ |a_2/a_3| \geq \sqrt{3}/2$. Our inequalities are the same as Grenier's (see [4, § 6] or [5, § 2]).

Remark 3.15. When $F = \mathbb{Q}$, the fundamental domain $\overline{\Omega}$ does *not* lead to Minkowski's fundamental domain for the space of positive definite symmetric matrices if $n = \dim_F V > 2$, because it is known that Minkowski-reduced quadratic forms are different from HKZ-reduced quadratic forms when $n > 2$.

Let $\varpi\colon \mathrm{GL}_V(\mathbb{A}) \to \mathrm{GL}_V(F)\backslash \mathrm{GL}_V(\mathbb{A})$ be the natural projection. Another well-known application of the property like the one in Theorem 3.2 is the following:

Corollary 3.16. (Mahler's compactness criterion) *When $F = \mathbb{Q}$, for a subset S of $\mathrm{GL}_V(\mathbb{A})$, the image $\varpi(S)$ in the quotient space $\mathrm{GL}_V(F)\backslash \mathrm{GL}_V(\mathbb{A})$ is relatively compact if and only if the additive character $\log|\det(\cdot)|_{\mathbb{A}^\times}$ is bounded on S and the logarithm of the 'Hermite' function m_1 is bounded from below on S.*

For convenience of the reader and in order to remember how we can deduce this kind of criterion from Theorem 3.2, we shall add a proof.

Proof. Since the functions $|\det(\cdot)|_{\mathbb{A}^\times}$ and m_1 are continuous and left $\mathrm{GL}_V(F)$-invariant, if $\varpi(S)$ is relatively compact, then the set of values of $\log|\det(\cdot)|_{\mathbb{A}^\times}$ and $\log m_1$ on S are relatively compact, hence bounded.

Conversely, suppose there exists a positive constant number δ such that

$$\delta < |\det(g)|_{\mathbb{A}^\times} < \frac{1}{\delta} \quad \text{and} \quad \delta < m_1(g) \qquad (g \in S).$$

Due to the left $\mathrm{GL}_V(F)$-invariance, we may assume $S \subset R$. For any $g \in S \subset R$, let g^∞ be as before. It suffices to prove that g^∞ belongs to a compact subset of $\mathrm{GL}_V(\mathbb{Q}_\infty)$. We denote by $\mathrm{diag}(a_1^\infty, \ldots, a_n^\infty)$ the A-part of the Iwasawa decomposition of g^∞.

By the definitions of the Iwasawa decomposition and g^∞, we have

$$|a_1^\infty \cdots a_n^\infty|_\infty = |\det(g^\infty)|_\infty = |\det(g)|_{\mathbb{A}^\times}.$$

From Lemma 3.1, we see

$$|a_n^\infty|_\infty^n = |a_n|_{\mathbb{A}^\times}^n = |\det(g)|_{\mathbb{A}^\times} H_1(g) = |\det(g)|_{\mathbb{A}^\times} m_1(g).$$

By virtue of Theorem 3.2, we also observe

$$|a_i^\infty|_\infty \geq \frac{\sqrt{3}}{2}|a_{i+1}^\infty|_\infty \geq \cdots \geq \left(\frac{\sqrt{3}}{2}\right)^{n-i} |a_n^\infty|_\infty \quad (i = 1, \ldots, n-1).$$

Hence we get $$|a_i^\infty|_\infty > \left(\frac{\sqrt{3}}{2}\right)^{n-i} \delta^{2/n} \quad (i = 1, \ldots, n).$$

Using these inequalities in turn, we further obtain at least

$$\frac{1}{\delta} > |a_1^\infty \cdots a_n^\infty|_\infty > |a_i^\infty|_\infty \left(\frac{\sqrt{3}}{2}\right)^{n(n-1)/2} \delta^2,$$

that is, $$|a_i^\infty|_\infty < \left(\frac{\sqrt{3}}{2}\right)^{-n(n-1)/2} \frac{1}{\delta^3} \quad (i = 1, \ldots, n).$$

Translating by an appropriate element of $N(\mathfrak{o}_F) = N(\mathbb{Z})$, we know the unipotent N-part of the Iwasawa decomposition of g^∞ can always be chosen from a fixed compact set. In this way, we understand each part of the Iwasawa decomposition of g^∞ is respectively contained in some compact subset of $\mathrm{GL}_V(\mathbb{Q}_\infty)$. ∎

4. The case of some imaginary quadratic fields

In this section, the base field F is an imaginary quadratic field whose discriminant D is one of the integers -3, -4, -7, -8, or -11.

The class number of F is also known to be 1. Every F-rational invertible matrix can be transformed into an upper triangular matrix by $\mathfrak{o}_F$-integral elementary column operations. Thus we obtain decompositions

$$\mathrm{GL}_V(\mathbb{A}) = \mathrm{GL}_V(F)\left(\mathrm{GL}_V(F_\infty) \times K^{\mathrm{fin}}\right)$$

and

$$\mathrm{GL}_V(F) = B(F)\ \mathrm{GL}_V(\mathfrak{o}_F).$$

Since there exists only one place ∞ at infinity, all the discussion for $\mathbb{Q}$ is valid when F is one of the above five fields, except that the absolute value at infinity is squared and that quadratic forms are replaced with hermitian forms, thanks to the next:

Lemma 4.1. *Let $F = \mathbb{Q}(\sqrt{D})$ with $D = -3, -4, -7, -8$, or -11. Embed F into the complex number field $\mathbb{C}$ as a subfield. For any $z \in \mathbb{C}$, there is an integer $\alpha \in \mathfrak{o}_F$ such that*

$$|z - \alpha|_\infty \le u,$$

where

$$u = \frac{1}{3},\ \frac{1}{2},\ \frac{4}{7},\ \frac{3}{4},\ \text{or}\ \frac{9}{11}$$

according respectively as $D = -3, -4, -7, -8$, or -11.

Note that the absolute value $|\cdot|_\infty$ is by definition the square of the ordinary norm on $\mathbb{C}$.

Proof. If $D = -4$ or -8, then the complex plane is paved with rectangles of respective diameter $\sqrt{2}$ or $\sqrt{3}$ whose centers are placed on the integer points.

If $D = -3, -7$, or -11, then the complex plane is covered with a honeycomb whose hexagons are respectively of diameter $2/\sqrt{3}$, $4/\sqrt{7}$, or $6/\sqrt{11}$ with their centers placed at the integral points ∎

The fruits are the following:

Theorem 4.2. *When $F = \mathbb{Q}(\sqrt{D})$ with $D = -3, -4, -7, -8$, or -11, we have for any element g in the set R of (2)*

$$|a_i|_{\mathbb{A}^\times} \ge (1-u)\,|a_{i+1}|_{\mathbb{A}^\times} \quad (i = 1, \ldots, n-1),$$

where $u = 1/3$, $1/2$, $4/7$, $3/4$, or $9/11$ according respectively as $D = -3, -4, -7, -8$, or -11.

Corollary 4.3. *When F is as above, an open fundamental domain Ω° in Section 2 is almost determined by a finite number of inequalities in the coordinates of the Iwasawa decomposition.*

Corollary 4.4. *When F is as above, for a subset S of $\mathrm{GL}_V(\mathbb{A})$, the image $\varpi(S)$ in the quotient space $\mathrm{GL}_V(F)\backslash\mathrm{GL}_V(\mathbb{A})$ is relatively compact if and only if the additive character $\log|\det(\cdot)|_{\mathbb{A}^\times}$ is bounded on S and the function $\log m_1$ is bounded from below on S.*

We will illustrate our fundamental domain for the mentioned imaginary quadratic fields with examples of discriminant $D=-3$.

Now put $\rho := (1+\sqrt{-3})/2$, a generator of the unit group of integer ring $\mathfrak{o}_F$ ($F=\mathbb{Q}(\sqrt{-3})$), a sixth root of unity. Let U be the interior of a regular hexagon in $F_\infty \simeq \mathbb{C}$ whose vertices are $(3\pm\sqrt{-3})/6$, $\pm\sqrt{-3}/3$, $(-3\pm\sqrt{-3})/6$; T the subset of U whose elements belong to the interior of a regular triangle spanned by $(3\pm\sqrt{-3})/6$, 0; K_∞^ρ any open fundamental domain for the scalar translation by the unit group of $\mathfrak{o}_F$ on K_∞;

$$\Omega_\infty^\rho := \{g^\infty \in \mathrm{GL}_V(F_\infty) \mid g^\infty \in R^\circ,\, n_{i,j}^\infty \in U,\, n_{i,i+1}^\infty \in T,\, a_i^\infty > 0, k^\infty \in K_\infty^\rho\};$$

and $\Omega^\rho := \Omega_\infty^\rho \times K^{\mathrm{fin}}$. The set Ω^ρ is an open fundamental domain for $\mathrm{GL}_V(F)$ in $\mathrm{GL}_V(\mathbb{A})$ when $F=\mathbb{Q}(\sqrt{-3})$.

Denote by $\bar{U}$ and by $\bar{T}$ the respective closures in $\mathbb{C}$ of U and of T and let

$$\mathfrak{S}_\infty := \{g^\infty \mid n_{i,j}^\infty \in \bar{U},\, n_{i,i+1}^\infty \in \bar{T},\, a_i^\infty \geq \sqrt{\frac{2}{3}}\, a_{i+1}^\infty > 0\}.$$

This is a Siegel set in $\mathrm{GL}_V(\mathbb{C})$. Theorem 4.2 gives the next:

Corollary 4.5. *The set Ω_∞^ρ is contained in the Siegel set $\mathfrak{S}_\infty$.*

For simplicity of notation, we drop the superscript ∞ from here.

For $x \in P_{i-1}(\mathfrak{o}_F)$, let $\mathbf{x} := (\xi_1, \ldots, \xi_{n-i+1})$ be the last row of the upper-left corner of degree $n-i+1$ of x. If $H_i(xg) = H_i(g)$ becomes a boundary of Ω_∞^ρ, then there exist elements $n_{i,j} \in U$ with $n_{i,i+1} \in T$ ($1 \leq i < j \leq n$) such that

$$\begin{aligned}&\left(\frac{2}{3}\right)^{n-i}|\xi_1|_\infty + \left(\frac{2}{3}\right)^{n-i-1}|\xi_1 n_{1,2}+\xi_2|_\infty + \cdots \\ &+ |\xi_1 n_{1,n-i+1} + \cdots + \xi_{n-i}n_{n-i,n-i+1} + \xi_{n-i+1}|_\infty \leq 1. \end{aligned} \tag{15}$$

We have for $\xi, \eta \in \mathbb{R}$

$$|\xi+\eta\rho|_\infty = \left|\xi + \eta\frac{1+\sqrt{-3}}{2}\right|_\infty = \xi^2 + \xi\eta + \eta^2.$$

Small values of $|\cdot|_\infty$ on $\mathfrak{o}_F$ are $0, 1, 3, 4, 7, 9, 12, 13, \ldots$

Example 4.6. ($D=-3$, $n=2$) The index $i=1$.

If $|\xi_1|_\infty \geq 3$, then $\left(\frac{2}{3}\right)^{n-i}|\xi_1|_\infty \geq 2$. The inequality (15) does not stand.

If $\xi_1 = 0$, then x would belong to $P_1(\mathfrak{o}_F)$. We should always have $H_1(xg) = H_1(g)$. If $\xi_1 = \rho^r$ ($r \in \mathbb{Z}$), then the inequality (15) becomes

$$|\rho^r n_{1,2} + \xi_2|_\infty \leq \frac{1}{3}.$$

Since $\rho^r n_{1,2}$ is in a regular triangle $\rho^r T$, this is only possible if $\xi_2 = 0$ or $\xi_2 = -\rho^r$. When $\mathbf{x} = (\xi_1, \xi_2) = (\rho^r, 0)$ ($r \in \mathbb{Z}$), the defining inequality $H_1(xg) > H_1(g)$ is expressed as

$$\left|\frac{a_1}{a_2}\right|_\infty + |n_{1,2}|_\infty > 1. \tag{16}$$

When $\mathbf{x} = (\xi_1, \xi_2) = (\rho^r, -\rho^r)$ ($r \in \mathbb{Z}$), the defining inequality $H_1(xg) > H_1(g)$ is paraphrased to

$$\left|\frac{a_1}{a_2}\right|_\infty + |n_{1,2} - 1|_\infty > 1,$$

which is not necessary. This is because we have $|n_{1,2} - 1|_\infty > |n_{1,2}|_\infty$ for $n_{1,2} \in T$. The unnecessary inequality is already fulfilled when the inequality (16) is valid.

When $D = -3$ and $n = 2$, a unique essential defining inequality of Ω^ρ_∞ is given by (16). Trivial inequalities are

$$n_{1,2} \in T; \quad a_1,\, a_2 > 0; \quad k^\infty \in K^\rho_\infty.$$

We do not need $|a_1/a_2|_\infty \geq 2/3$, which is implied by (16) and $n_{1,2} \in T$.

Example 4.7. ($D = -3$, $n = 3$) When the index $i = 2$, the inequality (15) is completely the same as in the example $n = 2$. We get one of defining inequalities

$$\left|\frac{a_1}{a_2}\right|_\infty + |n_{1,2}|_\infty > 1. \tag{17}$$

Assume i equals 1. If $|\xi_1|_\infty \geq 3$, then $\left(\frac{2}{3}\right)^{n-i} |\xi_1|_\infty \geq \frac{4}{9} \times 3 = \frac{4}{3}$. The inequality (15) does not hold. In the case $\xi_1 = 0$, the inequality (15) becomes

$$\frac{2}{3}|\xi_2|_\infty + |\xi_2 n_{2,3} + \xi_3|_\infty \leq 1.$$

Again, this is of the same form as the one in the previous example. We obtain another one of defining inequalities

$$\left|\frac{a_2}{a_3}\right|_\infty + |n_{2,3}|_\infty > 1. \tag{18}$$

In addition to the assumption $i = 1$, suppose that $\xi_1 = \rho^r$ ($r \in \mathbb{Z}$). The inequality (15) transforms into

$$\frac{2}{3}\,|\rho^r n_{1,2} + \xi_2|_\infty + |\rho^r n_{1,3} + \xi_2 n_{2,3} + \xi_3|_\infty \leq \frac{5}{9}.$$

Keep the facts $\rho^r n_{1,2} \in \rho^r T \subset U$ and $\rho^r n_{1,3} \in U$ in mind. If $|\xi_2|_\infty \geq 3$, then $\frac{2}{3}\,|\rho^r n_{1,2} + \xi_2|_\infty \geq \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$, impossible. In the case $\xi_2 = 0$, if $|\xi_3|_\infty \geq 3$, then $|\rho^r n_{1,3} + \xi_2 n_{2,3} + \xi_3|_\infty \geq \frac{4}{3}$. Hence $|\xi_3|_\infty \leq 1$. When $\mathbf{x} = (\xi_1, \xi_2, \xi_3) = (\rho^r, 0, 0)$ ($r \in \mathbb{Z}$), the defining inequality $H_1(xg) > H_1(g)$ is described as

$$\left|\frac{a_1}{a_2}\right|_\infty \left|\frac{a_2}{a_3}\right|_\infty + |n_{1,2}|_\infty \left|\frac{a_2}{a_3}\right|_\infty + |n_{1,3}|_\infty > 1. \tag{19}$$

If $\mathbf{x} = (\rho^r, 0, \rho^{r+s})$ $(r, s \in \mathbb{Z})$, then we would have

$$\left|\frac{a_1}{a_2}\right|_\infty \left|\frac{a_2}{a_3}\right|_\infty + |n_{1,2}|_\infty \left|\frac{a_2}{a_3}\right|_\infty + |n_{1,3} + \rho^s|_\infty > 1,$$

which is not necessary. For it is always true when the inequality (19) holds and $n_{1,3} \in U$ because of $|n_{1,3} + \rho^s|_\infty > |n_{1,3}|_\infty$ $(n_{1,3} \in U)$.

In the case $|\xi_2|_\infty = 1$, we have $\frac{2}{3}|\rho^r n_{1,2} + \xi_2|_\infty \geq \frac{2}{3} \times \frac{1}{4} = \frac{1}{6}$. It needs to be

$$|\rho^r n_{1,3} + \xi_2 n_{2,3} + \xi_3|_\infty \leq \frac{7}{18}.$$

The complex number $\xi_2^{-1}\rho^r n_{1,3} + n_{2,3}$ $(n_{1,3} \in U,\ n_{2,3} \in T)$ runs through the interior of a hexagon whose vertices are $(-3 \pm \sqrt{-3})/6$, $\pm\rho$, $(3 \pm \sqrt{-3})/3$. We see $|\xi_3|_\infty \leq 1$ or ξ_3 is one of certain two integers, depending on the value of ξ_2, among six integers ξ with $|\xi|_\infty = 3$.

Suppose the condition $|a_i/a_{i+1}|_\infty \geq 2/3$ $(i = 1, 2)$ is already satisfied. When $\mathbf{x} = (\rho^r, \rho^{r+s}, 0)$ $(r, s \in \mathbb{Z})$, the requirement $H_1(xg) > H_1(g)$ says

$$\left|\frac{a_1}{a_2}\right|_\infty \left|\frac{a_2}{a_3}\right|_\infty + |n_{1,2} + \rho^s|_\infty \left|\frac{a_2}{a_3}\right|_\infty + \left|\rho^{-s} n_{1,3} + n_{2,3}\right|_\infty > 1. \tag{20}$$

If $s \equiv 0, 1, 5 \mod 6$, then $|n_{1,2} + \rho^s|_\infty > 1$ $(n_{1,2} \in T)$. Since we are in the situation $|a_i/a_{i+1}|_\infty \geq 2/3$ $(i = 1, 2)$, the sum of the first two terms of the left side would be at least $(2/3)^2 + 1 \times 2/3 = 10/9$. Thus, for (20), three values $s = 2, 3, 4$ will suffice.

When $\mathbf{x} = (\rho^r, \rho^{r+s}, \rho^{r+s+t})$ $(r, s, t \in \mathbb{Z})$, the defining inequality $H_1(xg) > H_1(g)$ is

$$\left|\frac{a_1}{a_2}\right|_\infty \left|\frac{a_2}{a_3}\right|_\infty + |n_{1,2} + \rho^s|_\infty \left|\frac{a_2}{a_3}\right|_\infty + \left|\rho^{-s} n_{1,3} + n_{2,3} + \rho^t\right|_\infty > 1. \tag{21}$$

We may assume $s = 2, 3$, or 4 as in the previous inequality. Furthermore, the values of t can also be assumed $2, 3$, or 4 by the fact that

$$\left|\rho^{-s} n_{1,3} + n_{2,3} + \rho^t\right|_\infty > \left|\rho^{-s} n_{1,3} + n_{2,3}\right|_\infty$$

for $n_{1,3} \in U,\ n_{2,3} \in T$ when $t \equiv 0, 1, 5 \mod 6$.

When $\mathbf{x} = (\rho^r, \rho^{r+s}, \rho^{r+s}\xi)$ $(r, s \in \mathbb{Z};\ \xi \in \mathfrak{o}_F,\ |\xi|_\infty = 3)$, we would obtain

$$\left|\frac{a_1}{a_2}\right|_\infty \left|\frac{a_2}{a_3}\right|_\infty + |n_{1,2} + \rho^s|_\infty \left|\frac{a_2}{a_3}\right|_\infty + \left|\rho^{-s} n_{1,3} + n_{2,3} + \xi\right|_\infty > 1.$$

If $\xi \neq (-3 \pm \sqrt{-3})/2$, then $|\rho^{-s} n_{1,3} + n_{2,3} + \xi|_\infty > 1$ $(n_{1,3} \in U,\ n_{2,3} \in T)$.

The inequality above in this case is redundant. If $\xi = (-3 \pm \sqrt{-3})/2$, then

$$\left|\rho^{-s} n_{1,3} + n_{2,3} + \xi\right|_\infty > \left|\rho^{-s} n_{1,3} + n_{2,3} + \rho^t\right|_\infty$$

for $n_{1,3} \in U$, $n_{2,3} \in T$, and the pairs $(\xi, t) = ((-3 \pm \sqrt{-3})/2, \pm 2)$. Thus the above inequality is not needed in fact at all.

To summarize, put $\omega := \rho^2$. The number ω is a third root of unity. We have found that the essential defining inequalities in the case $D = -3$ and $n = 3$ are 15 in number: (17); (18); (19); (20) with $\rho^s = -1, \omega, \omega^2$; and (21) with $\rho^s, \rho^t = -1, \omega, \omega^2$.

We cannot reduce these 15 inequalities anymore. This is because after substitution of the following values of $(|a_1/a_2|_\infty; |a_2/a_3|_\infty; n_{1,2}; n_{1,3}; n_{2,3})$, we see all except one of the values of the left side of the 15 inequalities are at least 1:

$$\begin{aligned}
&\left(\left|\frac{a_1}{a_2}\right|_\infty; \left|\frac{a_2}{a_3}\right|_\infty; n_{1,2}; n_{1,3}; n_{2,3}\right)\\
&= \left(\frac{2}{3}; \frac{3}{2}; 0; 0; 0\right),\ \left(\frac{3}{2}; \frac{2}{3}; 0; 0; 0\right),\ \left(\frac{2}{3}; \frac{2}{3}; \frac{1}{2} + \frac{\sqrt{-3}}{6}; 0; \frac{1}{2} + \frac{\sqrt{-3}}{6}\right),\\
&\left(\frac{2}{3}; \frac{2}{3}; \frac{\sqrt{-3}}{3} - \rho^2; -\rho^2\left(\frac{1}{2} + \frac{\sqrt{-3}}{6}\right); \frac{1}{2} + \frac{\sqrt{-3}}{6}\right),\\
&\left(\frac{2}{3}; \frac{2}{3}; -\frac{1}{2} + \frac{\sqrt{-3}}{6} - \rho^3; -\rho^3\left(\frac{1}{2} + \frac{\sqrt{-3}}{6}\right); \frac{1}{2} + \frac{\sqrt{-3}}{6}\right),\\
&\left(\frac{2}{3}; \frac{2}{3}; -\frac{\sqrt{-3}}{3} - \rho^4; -\rho^4\left(\frac{1}{2} + \frac{\sqrt{-3}}{6}\right); \frac{1}{2} + \frac{\sqrt{-3}}{6}\right),\\
&\left(\frac{2}{3}; \frac{2}{3}; \frac{\sqrt{-3}}{3} - \rho^2; \rho^2\left(-\frac{\sqrt{-3}}{3}\right); \frac{1}{2} - \frac{\sqrt{-3}}{6}\right),\\
&\left(\frac{2}{3}; \frac{2}{3}; \frac{\sqrt{-3}}{3} - \rho^2; \rho^2\left(\frac{1}{2} + \frac{\sqrt{-3}}{6}\right); \frac{1}{2} - \frac{\sqrt{-3}}{6}\right),\\
&\left(\frac{2}{3}; \frac{2}{3}; \frac{\sqrt{-3}}{3} - \rho^2; \rho^2\left(\frac{\sqrt{-3}}{3}\right); \frac{1}{2} + \frac{\sqrt{-3}}{6}\right),\\
&\left(\frac{2}{3}; \frac{2}{3}; -\frac{1}{2} + \frac{\sqrt{-3}}{6} - \rho^3; \rho^3\left(-\frac{\sqrt{-3}}{3}\right); \frac{1}{2} - \frac{\sqrt{-3}}{6}\right),\\
&\left(\frac{2}{3}; \frac{2}{3}; -\frac{1}{2} + \frac{\sqrt{-3}}{6} - \rho^3; \rho^3\left(\frac{1}{2} + \frac{\sqrt{-3}}{6}\right); \frac{1}{2} - \frac{\sqrt{-3}}{6}\right),\\
&\left(\frac{2}{3}; \frac{2}{3}; -\frac{1}{2} + \frac{\sqrt{-3}}{6} - \rho^3; \rho^3\left(\frac{\sqrt{-3}}{3}\right); \frac{1}{2} + \frac{\sqrt{-3}}{6}\right),\\
&\left(\frac{2}{3}; \frac{2}{3}; -\frac{\sqrt{-3}}{3} - \rho^4; \rho^4\left(-\frac{\sqrt{-3}}{3}\right); \frac{1}{2} - \frac{\sqrt{-3}}{6}\right),\\
&\left(\frac{2}{3}; \frac{2}{3}; -\frac{\sqrt{-3}}{3} - \rho^4; \rho^4\left(\frac{1}{2} + \frac{\sqrt{-3}}{6}\right); \frac{1}{2} - \frac{\sqrt{-3}}{6}\right),\\
&\left(\frac{2}{3}; \frac{2}{3}; -\frac{\sqrt{-3}}{3} - \rho^4; \rho^4\left(\frac{\sqrt{-3}}{3}\right); \frac{1}{2} + \frac{\sqrt{-3}}{6}\right)
\end{aligned}$$

The evident inequalities are

$$n_{1,2},\ n_{2,3} \in T; \quad n_{1,3} \in U; \quad a_1,\ a_2,\ a_3 > 0; \quad k^\infty \in K_\infty^\rho.$$

We do not have to require $|a_1/a_2|_\infty$, $|a_2/a_3|_\infty \geq 2/3$ which is a consequence of the condition (17), (18), and $n_{1,2}$, $n_{2,3} \in T$.

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Masami Fujimori, Kanagawa Institute of Technology, Atsugi, Kanagawa, Japan;
fujimori@gen.kanagawa-it.ac.jp.

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