Journal of Lie Theory
Volume **33** (2023) 875–886

Triangular Structures on Flat Lie Algebras

Amine Bahayou

Communicated by M. Schlichenmaier

Abstract. We study a large class of exact Lie bialgebras arising from noncommutative deformations of Poisson-Lie groups endowed with a left invariant Riemannian metric. We call these structures *triangular metaflat Lie bialgebras*. We show that given the metaflatness geometrical condition, these exact bialgebra structures arise necessarily from a solution of the classical Yang-Baxter equation. Moreover, the dual Lie bialgebra is also metaflat constituting an important kind of symmetry.

Mathematics Subject Classification: 17B38, 17B62, 53D17.

Key Words: Lie bialgebra, Poisson-Lie group, Yang-Baxter equation.

1. Introduction

Hawkins [4, 5] studied and introduced necessary conditions for the existence of a deformation of the graded algebra of differential forms $\Omega^*(M)$ on a Riemannian manifold (M, g) ([5], Theorem 2.3, p. 393).

We previously studied [1], the so-called *Hawkins conditions* on a Poisson-Lie group endowed with a left invariant Riemannian metric, proved it is equivalent to ([1], Theorem 3.1, p. 446):

1. The dual Lie algebra $\mathfrak{g}^*$ is flat, i.e. decomposes into an orthogonal sum $\mathfrak{g}^* = \mathfrak{b}^* \oplus [\mathfrak{g}^*, \mathfrak{g}^*]$ (for the associated scalar product on $\mathfrak{g}^*$), where $\mathfrak{b}^*$ is an abelian subalgebra and the commutator ideal $[\mathfrak{g}^*, \mathfrak{g}^*]$ is even-dimensional and abelian.

2. The flat Lie bialgebra $(\mathfrak{g}^*, \xi^*)$ is metaflat, i.e. for all $x, y \in \mathfrak{b}^*$ $\mathrm{ad}_x^2 \xi^*(y) = 0$, where ξ^* is the 1-cocycle defining the dual bialgebra structure of $\mathfrak{g}^*$ and ad_x is the extension of the adjoint representation of $\mathfrak{g}^*$ to bivectors. (See the preliminary section for the definitions regarding flat Lie algebras and Lie bialgebras).

The main theorem of this paper is the following

Theorem 1.1. *Let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ be a flat Lie algebra and ξ the coboundary of $r \in \wedge^2\mathfrak{g}$. Then if the pair $(\mathfrak{g}, \xi)$ is metaflat, r is a solution of the classical Yang-Baxter equation. Moreover, the dual Lie bialgebra $(\mathfrak{g}^*, \xi^*)$ is flat and metaflat.*

ISSN 0949–5932 / \$2.50

The proof is a straightforward computation with a careful look to the faithful representations of $\mathfrak{s}$ into $\mathfrak{so}([\mathfrak{g},\mathfrak{g}])$ (the Lie algebra of skew symmetric matrices on $[\mathfrak{g},\mathfrak{g}]$).

The paper is organized as follows. In Section 2 we introduce some preliminary notions in Poisson geometry. In Section 3, we present the main result of this paper, Theorem 3.3, and we discuss some examples in Section 4.

2. Preliminaries

For this section, we address the readers to the book [9] where they can find all the material about Poisson geometry and Poisson-Lie groups. A Poisson structure on a smooth manifold M is a Lie bracket $\{\cdot,\cdot\}$ on the space $C^\infty(M)$ of smooth functions on M which satisfies the Leibniz rule. This bracket is called Poisson bracket and a manifold M equipped with such a bracket is called Poisson manifold. Therefore, a bivector field π on M such that the bracket

$$\{f,g\} := \pi(df,dg)$$

is a Poisson bracket is called Poisson tensor or Poisson bivector field. A Poisson tensor can be regarded as a bundle map $\pi_\sharp : T^*M \to TM$:

$$\beta(\pi_\sharp(\alpha)) = \pi(\alpha,\beta).$$

A map $\phi : (M,\pi_M) \to (N,\pi_N)$ between two Poisson manifolds is called a Poisson map if for all $f,g \in C^\infty(N)$, one has

$$\{f\circ\phi, g\circ\phi\}_M = \{f,g\}_N \circ \phi$$

2.1. Lie bialgebras and Poisson Lie groups

A *Poisson-Lie group* (G,π) is a Lie group G endowed with a Poisson tensor π for which the multiplication $m : G\times G \to G$ is a Poisson map with respect to π on G and the product Poisson structure $\pi_{G\times G} = \pi \oplus \pi$ on $G\times G$. Equivalently, π is multiplicative, i.e. for any $g,h\in G$

$$\pi(gh) = (L_g)_*\pi(h) + (R_h)_*\pi(g),$$

where $(L_g)_*$ (resp. $(R_h)_*$) denotes the tangent map of the left translation of G by g (resp. the right translation of G by h). [7], p. 503.

The associated map $\pi_r : G \to \wedge^2\mathfrak{g}$, defined by $\pi_r(g) = (R_{g^{-1}})_*\pi(g)$ satisfies

$$\pi_r(gh) = \pi_r(g) + \mathrm{Ad}_g\pi_r(h),$$

which means that π_r is a 1-cocycle of G with values in $\wedge^2\mathfrak{g}$ for the Adjoint representation

$$\mathrm{Ad}_g(X\wedge Y) := \mathrm{Ad}_gX \wedge \mathrm{Ad}_gY.$$

The linear map $\xi : \mathfrak{g} \to \wedge^2\mathfrak{g}$ defined by $\xi(X) = \mathscr{L}_X\pi_r(e)$ is a 1-cocycle with respect to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}\wedge\mathfrak{g}$

$$\mathrm{ad}_x(y\wedge z) := \mathrm{ad}_x\, y \wedge z + y \wedge \mathrm{ad}_x\, z.$$

which leads to the following definition:

A *Lie bialgebra* is a couple $(\mathfrak{g},\xi)$ where $\mathfrak{g}$ is a Lie algebra and $\xi:\mathfrak{g}\to\wedge^2\mathfrak{g}$ is a linear map such that for all x,y in $\mathfrak{g}$

$$\xi([x,y]) = \mathrm{ad}_x\,\xi(y) - \mathrm{ad}_y\,\xi(x), \tag{1}$$

and the transpose map $\xi^t = [\cdot,\cdot]_* : \mathfrak{g}^*\wedge\mathfrak{g}^*\to\mathfrak{g}^*$ is a Lie bracket, where

$$\xi^t(\alpha\wedge\beta)(x) := (\alpha\wedge\beta)\left(\xi(x)\right).$$

It is important to note that a Lie bialgebra and its dual have symmetric roles. More details on Lie bialgebras can be found in [6].

2.2. Flat Lie algebras

A *flat Lie algebra* is a couple $(\mathfrak{g},\langle\,,\rangle)$ where $\mathfrak{g}$ is a real finite dimensional Lie algebra and $\langle\,,\rangle$ is a positive definite scalar product on $\mathfrak{g}$ such that the infinitesimal Levi-Civita connection, defined for all x,y,z in $\mathfrak{g}$ by

$$2\langle\nabla_x y, z\rangle = \langle[x,y],z\rangle + \langle[z,x],y\rangle + \langle[z,y],x\rangle \tag{2}$$

has zero curvature

$$R(x,y,z) = \nabla_{[x,y]}z - (\nabla_x\nabla_y z - \nabla_y\nabla_x z) \equiv 0. \tag{3}$$

In other words, the Levi-Civita connection of the unique left invariant Riemannian metric on the associated Lie group G of $\mathfrak{g}$, which extends $\langle\,,\rangle$, is flat.

Milnor in [8] characterized these flat Lie algebras. Some refinements have been provided later in [1] and [2].

Proposition 2.1. ([1], [2], [8]) *Let $(\mathfrak{g},\langle\,,\rangle)$ be a flat Lie algebra. Then $\mathfrak{g}$ decomposes orthogonally as*

$$\mathfrak{g} = \mathfrak{s}\oplus\mathfrak{z}\oplus[\mathfrak{g},\mathfrak{g}],$$

where $\mathfrak{z}$ is the center of $\mathfrak{g}$, $\mathfrak{s}$ is an abelian Lie subalgebra, $[\mathfrak{g},\mathfrak{g}]$ is the commutator ideal satisfying the following conditions:

- $[\mathfrak{g},\mathfrak{g}]$ *is abelian and even dimensional,*
- $\mathrm{ad}_x = \nabla_x$*, for any x in $\mathfrak{z}\oplus\mathfrak{s}$.*

Remark 2.2. It follows from the above proposition that $\mathfrak{g}$ is a unimodular 2-step solvable Lie algebra, whose nilradical is given by $\mathfrak{z}\oplus[\mathfrak{g},\mathfrak{g}]$. ■

Moreover, from [1], we have:

$$\mathfrak{g} = \mathrm{span}\{s_1,\dots,s_{k_0},z_1,\dots,z_{\ell_0}\}\oplus\mathrm{span}\{d_1,\dots,d_{2m}\}$$

where $\mathrm{span}\{d_1,\dots,d_{2m}\}$ is the commutator of $\mathfrak{g}$ which is abelian, $\mathrm{span}\{z_1,\dots,z_{\ell_0}\}$ its center (possibly trivial) and $\mathrm{span}\{s_1,\dots,s_{k_0}\}$ its abelian subalgebra such that there exists real coefficients λ_{ij},

$$[s_i,d_{2j-1}] = \lambda_{ij}\,d_{2j},\ [s_i,d_{2j}] = -\lambda_{ij}\,d_{2j-1}\ \text{ for all } i=1,\dots,k_0,\ j=1,\dots,m. \tag{4}$$

Indeed, the family $\{\mathrm{ad}_s,\ s\in\mathfrak{s}\}\subseteq\mathfrak{so}(2m)$ is an abelian subalgebra, then it is conjugate by an element in $\mathrm{SO}(2m)$ to a subalgebra of the maximal abelian subalgebra $\mathfrak{t}^m$ of $\mathfrak{so}(2m)$ with respect to an orthonormal basis $\{d_1,\dots,d_{2m}\}$ of $[\mathfrak{g},\mathfrak{g}]$.

Example 2.3.

1. Any commutative Lie algebra is flat.
2. The Lie algebra $\mathfrak{g} = \text{span}\{s\} \oplus \text{span}\{d_1, d_2\}$ with the brackets

$$[s, d_1] = d_2,\ [s, d_2] = -d_1,\ [d_1, d_2] = 0,$$

is the smallest non abelian flat Lie algebra.

2.3. Nondegenerate flat Lie algebras

A flat Lie algebra is *nondegenerate* if there exists a basis of $\mathfrak{s}$ such that for all $i \neq j$ in $\{1, \ldots, m\}$,

$$\text{there exists } k \text{ in } \{1, \ldots, k_0\}, \text{ such that } \lambda_{kj}^2 \neq \lambda_{ki}^2. \tag{5}$$

Otherwise, the Lie algebra is *degenerate*.

Examples 2.4.

1. The flat Lie algebra $\mathfrak{g} = \text{span}\{s\} \oplus \text{span}\{d_1, d_2, d_3, d_4\}$ with the brackets

$$[s, d_1] = d_2,\ [s, d_2] = -d_1,\ [s, d_3] = \alpha d_4,\ [s, d_4] = -\alpha d_3,\ \alpha \neq 0, -1, 1$$

is nondegenerate.

2. The flat Lie algebra $\mathfrak{g} = \text{span}\{s_1, s_2\} \oplus \text{span}\{d_1, d_2, d_3, d_4\}$ with the brackets

$$[s_i, d_{2j-1}] = \delta_{ij}\, d_{2j},\ [s_i, d_{2j}] = -\delta_{ij}\, d_{2j-1},\ i, j = 1, 2,$$

where δ stands for the Kronecker symbol, is nondegenerate. Although for the basis $\{s_1 + s_2, s_1 - s_2\}$ of $\mathfrak{s}$ the property (5) is not satisfied.

3. The flat Lie algebra $\mathfrak{g} = \text{span}\{s\} \oplus \text{span}\{d_1, \ldots, d_4\}$ with the brackets

$$[s, d_1] = d_2,\ [s, d_2] = -d_1,\ [s, d_3] = d_4,\ [s, d_4] = -d_3,$$

is degenerate, since $\lambda_{12} = \lambda_{13}$ and (5) is not satisfied for any other vector generating $\mathfrak{s}$.

In the following, we will show that the nondegeneracy property is well defined. Let $\mathfrak{g}$ be a flat Lie algebra with a fixed basis of $[\mathfrak{g}, \mathfrak{g}]$ for which

$$[s, d_{2j-1}] = \lambda_j(s)\, d_{2j},\ [s, d_{2j}] = -\lambda_j(s)\, d_{2j-1} \text{ for all } s \in \mathfrak{s} \text{ and } j = 1, \ldots, m,$$

where $\lambda_1, \ldots, \lambda_m$ are linear forms on $\mathfrak{s}$. Let $\{s_1, \ldots, s_{k_0}\}$ be a fixed basis of $\mathfrak{s}$ and let Λ be the matrix

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{21} & \lambda_{31} & \ldots & \lambda_{k_0 1} \\ \lambda_{12} & \lambda_{22} & \lambda_{32} & \ldots & \lambda_{k_0 2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \lambda_{1m} & \lambda_{2m} & \lambda_{3m} & \ldots & \lambda_{k_0 m} \end{pmatrix}, \text{ where } \lambda_j(s_i) = \lambda_{ij}.$$

We have the following characterization

Proposition 2.5. *A flat Lie algebra $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ is degenerate if and only if there is $i \neq j$ in $\{1, \ldots, m\}$ such that for all $k \in \{1, \ldots, k_0\}$, $\lambda_{ki} = \varepsilon_{ij}^k \lambda_{kj}$, with the property*

$$\varepsilon_{ij}^k = \varepsilon_{ij} = \pm 1 \text{ independently of } k, \tag{6}$$

which means that the rows L_i and L_j of Λ are linearly dependent with $L_j = \pm L_i$.

Proof. Let $\{S_1, \ldots, S_{k_0}\}$ be an arbitrary other basis of $\mathfrak{s}$ and denote by A the invertible matrix (α_{ij}) of the change of basis. We have

$$[S_k, d_{2i-1}] = \left[\sum_{p=1}^{k_0} \alpha_{pk} s_p, d_{2i-1}\right] = \left(\sum_{p=1}^{k_0} \alpha_{pk} \lambda_{pi}\right) d_{2i} = \Lambda_{ki}\, d_{2i}.$$

So that $\Lambda_{ki}^2 = \Lambda_{kj}^2$ for all $k = 1, \ldots, k_0$ if and only if

$$\left(\sum_{p=1}^{k_0} \alpha_{pk} \lambda_{pi}\right)^2 - \left(\sum_{p=1}^{k_0} \alpha_{pk} \lambda_{pj}\right)^2 = 0 \text{ for all } k = 1, \ldots, k_0$$

$$\Leftrightarrow \sum_{p=1}^{k_0} \alpha_{pk}^2 \left(\lambda_{pi}^2 - \lambda_{pj}^2\right) + 2 \sum_{1 \le p < q \le k_0} \alpha_{pk}\alpha_{qk} \left(\lambda_{pi}\lambda_{qi} - \lambda_{pj}\lambda_{qj}\right) = 0 \text{ for all } k = 1, \ldots, k_0$$

$$\Leftrightarrow \lambda_{pi}\lambda_{qi} - \lambda_{pj}\lambda_{qj} = 0 \text{ for all } 1 \le p < q \le k_0 \text{ (pick } A \text{ such that } \alpha_{pk}, \alpha_{qk} \neq 0)$$

Since $\lambda_{pi} = \varepsilon_{ij}^p \lambda_{pj}$ and $\lambda_{qi} = \varepsilon_{ij}^q \lambda_{qj}$ ($\varepsilon_{ij}^q = \pm 1$, $\varepsilon_{ij}^q = \pm 1$), then

$$\lambda_{pi}\lambda_{qi} - \lambda_{pj}\lambda_{qj} = 0 \iff \lambda_{pi}\lambda_{qi}\left(1 - \varepsilon_{ij}^p \varepsilon_{ij}^q\right) = 0.$$

$$\iff \varepsilon_{ij}^p = \varepsilon_{ij}^q = \pm 1 \text{ or } \lambda_{pi} = \lambda_{pj} = 0, \text{ or } \lambda_{qi} = \lambda_{qj} = 0. \qquad \blacksquare$$

Remark 2.6. When $k_0 = m$ then the rows of the matrix Λ are linearly independent so that the flat Lie algebra is nondegenerate.

2.4. Metaflat Lie algebras

A *metaflat* Lie algebra is a triple $(\mathfrak{g}, \langle\,,\rangle, \xi)$ where $(\mathfrak{g}, \langle\,,\rangle)$ is a flat Lie algebra and $\xi : \mathfrak{g} \to \wedge^2 \mathfrak{g}$ is a 1-cocycle such that

$$\mathrm{ad}_x\, \mathrm{ad}_y\, \xi(z) = 0 \text{ for all } x, y, z \in \mathfrak{s}. \tag{7}$$

This is equivalent to the nullity of the metacurvature tensor (introduced in [5]). This has been proved in [1].

A commutative Lie algebra is obviously flat and metaflat. In the sequel, we assume that the Lie algebra is not commutative.

3. Main theorem

In this section we prove the main theorem (i.e., Theorem 3.3).

Let ξ be the coboundary of $r \in \wedge^2\mathfrak{g}$. We have

$$\begin{aligned} r = & \sum_{1\le i<j\le k_0} a_{ij}\, s_i \wedge s_j + \sum_{\substack{1\le i\le k_0\\ 1\le j\le \ell_0}} b_{ij}\, s_i \wedge z_j + \sum_{\substack{1\le i\le k_0\\ 1\le j\le m}} c_{ij}\, s_i \wedge d_{2j-1} + \sum_{\substack{1\le i\le k_0\\ 1\le j\le m}} e_{ij}\, s_i \wedge d_{2j} \\ & + \sum_{1\le i<j\le \ell_0} f_{ij}\, z_i \wedge z_j + \sum_{\substack{1\le i\le \ell_0\\ 1\le j\le m}} g_{ij}\, z_i \wedge d_{2j-1} + \sum_{\substack{1\le i\le \ell_0\\ 1\le j\le m}} h_{ij}\, z_i \wedge d_{2j} \\ & + \sum_{1\le i<j\le m} m_{ij}\, d_{2i-1} \wedge d_{2j-1} + \sum_{1\le i,j\le m} n_{ij}\, d_{2i-1} \wedge d_{2j} + \sum_{1\le i<j\le m} p_{ij}\, d_{2i} \wedge d_{2j}. \end{aligned}$$

For all $k = 1, \dots, k_0$, $\xi(s_k) = \mathrm{ad}_{s_k} r$, thus

$$\begin{aligned} \xi(s_k) = & \sum_{\substack{1\le i\le k_0\\ 1\le j\le m}} -\lambda_{kj} e_{ij}\, s_i \wedge d_{2j-1} + \sum_{\substack{1\le i\le k_0\\ 1\le j\le m}} \lambda_{kj} c_{ij}\, s_i \wedge d_{2j} \\ & + \sum_{\substack{1\le i\le \ell_0\\ 1\le j\le m}} -\lambda_{kj} h_{ij}\, z_i \wedge d_{2j-1} + \sum_{\substack{1\le i\le \ell_0\\ 1\le j\le m}} \lambda_{kj} g_{ij}\, z_i \wedge d_{2j} \\ & + \sum_{1\le i<j\le m} \left(-\lambda_{kj} n_{ij} + \lambda_{ki} n_{ji}\right) d_{2i-1} \wedge d_{2j-1} \\ & + \sum_{1\le i<j\le m} \left(\lambda_{kj} m_{ij} - \lambda_{ki} p_{ij}\right) d_{2i-1} \wedge d_{2j} \\ & + \sum_{1\le i<j\le m} \left(-\lambda_{ki} m_{ij} + \lambda_{kj} p_{ij}\right) d_{2j-1} \wedge d_{2i} \\ & + \sum_{1\le i<j\le m} \left(\lambda_{ki} n_{ij} - \lambda_{kj} n_{ji}\right) d_{2i} \wedge d_{2j}. \end{aligned} \tag{8}$$

For all $k = 1, \dots, \ell_0$, $\xi(z_k) = \mathrm{ad}_{z_k} r = 0$ and for all $k = 1, \dots, m$

$$\xi(d_{2k-1}) = \mathrm{ad}_{d_{2k-1}} r = \Phi_k \wedge d_{2k},\ \xi(d_{2k}) = \mathrm{ad}_{d_{2k}} r = d_{2k-1} \wedge \Phi_k$$

where

$$\begin{aligned} \Phi_k = & \left(\sum_{j=2}^{k_0} -\lambda_{jk} a_{1j}\right) s_1 + \sum_{p=2}^{k_0-1} \left(\sum_{i=1}^{p-1} \lambda_{ik} a_{ip} + \sum_{j=p+1}^{k_0} -\lambda_{jk} a_{pj}\right) s_p + \left(\sum_{i=1}^{k_0-1} \lambda_{ik} a_{ik_0}\right) s_{k_0} \\ & + \sum_{j=1}^{\ell_0} \left(\sum_{i=1}^{k_0} \lambda_{ik} b_{ij}\right) z_j + \sum_{j=1}^{m} \left(\sum_{i=1}^{k_0} \lambda_{ik} c_{ij}\right) d_{2j-1} + \sum_{j=1}^{m} \left(\sum_{i=1}^{k_0} \lambda_{ik} e_{ij}\right) d_{2j}. \end{aligned}$$

Proposition 3.1. *The* metaflatness *condition* (7) *for a coboundary* ξ *is equivalent to*

$$\xi(s_k) = 0 \text{ for all } k = 1, \dots, k_0.$$

Proof. Since ξ is a coboundary, by (8) we can simply write

$$\begin{aligned}\xi(s_k) = &\sum_{\substack{1\le i\le k_0\\1\le j\le m}} c^k_{ij}\, s_i\wedge d_{2j-1} + \sum_{\substack{1\le i\le k_0\\1\le j\le m}} e^k_{ij}\, s_i\wedge d_{2j} + \sum_{\substack{1\le i\le \ell_0\\1\le j\le m}} g^k_{ij}\, z_i\wedge d_{2j-1}\\ &+ \sum_{\substack{1\le i\le \ell_0\\1\le j\le m}} h^k_{ij}\, z_i\wedge d_{2j} + \sum_{1\le i<j\le m} m^k_{ij}\, d_{2i-1}\wedge d_{2j-1}\\ &+ \sum_{1\le i,j\le m} n^k_{ij}\, d_{2i-1}\wedge d_{2j} + \sum_{1\le i<j\le m} p^k_{ij}\, d_{2i}\wedge d_{2j},\end{aligned}$$

and

$$\begin{aligned}\mathrm{ad}^2_{s_\ell}\,\xi(s_k) = &\sum_{\substack{1\le i\le k_0\\1\le j\le m}} -\lambda^2_{\ell j} c^k_{ij}\, s_i\wedge d_{2j-1} + \sum_{\substack{1\le i\le k_0\\1\le j\le m}} -\lambda^2_{\ell j} e^k_{ij}\, s_i\wedge d_{2j}\\ &+ \sum_{\substack{1\le i\le \ell_0\\1\le j\le m}} -\lambda^2_{\ell j} g^k_{ij}\, z_i\wedge d_{2j-1} + \sum_{\substack{1\le i\le \ell_0\\1\le j\le m}} -\lambda^2_{\ell j} h^k_{ij}\, z_i\wedge d_{2j}\\ &+ \sum_{1\le i<j\le m} \left(-(\lambda^2_{\ell i}+\lambda^2_{\ell j}) m^k_{ij} + 2\lambda_{\ell i}\lambda_{\ell j} p^k_{ij}\right)\, d_{2i-1}\wedge d_{2j-1}\\ &+ \sum_{1\le i<j\le m} \left(-(\lambda^2_{\ell i}+\lambda^2_{\ell j}) n^k_{ij} + 2\lambda_{\ell i}\lambda_{\ell j} n^k_{ji}\right)\, d_{2i-1}\wedge d_{2j}\\ &+ \sum_{1\le i<j\le m} \left(2\lambda_{\ell i}\lambda_{\ell j} n^k_{ij} - (\lambda^2_{\ell i}+\lambda^2_{\ell j}) n^k_{ji}\right)\, d_{2j-1}\wedge d_{2i}\\ &+ \sum_{1\le i<j\le m} \left(2\lambda_{\ell i}\lambda_{\ell j} m^k_{ij} - (\lambda^2_{\ell i}+\lambda^2_{\ell j}) p^k_{ij}\right)\, d_{2i}\wedge d_{2j}.\end{aligned}$$

So that $\mathrm{ad}^2_{s_\ell}\,\xi(s_k)=0$ for all $k,\ell\in\{1,\dots,k_0\}$ if and only if

$$c^k_{ij} = e^k_{ij} = g^k_{ij} = h^k_{ij} = 0$$

and

$$\begin{pmatrix}(\lambda^2_{\ell i}+\lambda^2_{\ell j}) & -2\lambda_{\ell i}\lambda_{\ell j}\\ -2\lambda_{\ell i}\lambda_{\ell j} & (\lambda^2_{\ell i}+\lambda^2_{\ell j})\end{pmatrix}\begin{pmatrix}m^k_{ij}\\ p^k_{ij}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}(\lambda^2_{\ell i}+\lambda^2_{\ell j}) & -2\lambda_{\ell i}\lambda_{\ell j}\\ -2\lambda_{\ell i}\lambda_{\ell j} & (\lambda^2_{\ell i}+\lambda^2_{\ell j})\end{pmatrix}\begin{pmatrix}n^k_{ij}\\ n^k_{ji}\end{pmatrix}.$$

If $\mathfrak{g}$ is non degenerate, i.e. for all $1\le i<j\le m$ there is $\ell\in\{1,\dots,k_0\}$ such that the determinant $(\lambda^2_{\ell i}-\lambda^2_{\ell j})^2\neq 0$, then

$$m^k_{ij} = p^k_{ij} = n^k_{ij} = n^k_{ji} = 0, \text{ and from (8), } \xi(s_k)=0.$$

Suppose $\mathfrak{g}$ is degenerate, i.e. for all $\ell\in\{1,\dots,k_0\}$ $\lambda_{\ell j}=\varepsilon_{ij}\lambda_{\ell i}$, $(\varepsilon_{ij}=\pm 1)$. We can choose ℓ such that $\lambda_{\ell i}\neq 0$. Otherwise, d_{2i} would be in the center, which is impossible. Then from the characterization (6) of $\mathfrak{g}$ we get, for all $1\le i<j\le m$, $p^k_{ij}=\varepsilon_{ij}m^k_{ij}$ and $n^k_{ji}=\varepsilon_{ij}n^k_{ij}$. By (8), we get the lemma, since for a degenerate flat algebra, we have

$$\lambda_{kj}n_{ij} - \lambda_{ki}n_{ji} = (\varepsilon_{ij}\lambda_{ki})\, n_{ij} - \lambda_{ki}\,(\varepsilon_{ij}n_{ij}) = 0$$

and similarly for the other coefficients. ∎

As a direct consequence, we get the following proposition:

Proposition 3.2. *Let $\mathfrak{g}$ be a flat Lie algebra and ξ the coboundary of $r \in \wedge^2\mathfrak{g}$. If the pair $(\mathfrak{g}, \xi)$ is metaflat then*

1. *If $\mathfrak{g}$ is nondegenerate, then*

$$r = \sum_{1\leq i<j\leq k_0} a_{ij}\, s_i \wedge s_j + \sum_{\substack{1\leq i\leq k_0\\ 1\leq j\leq \ell_0}} b_{ij}\, s_i \wedge z_j + \sum_{1\leq i<j\leq \ell_0} f_{ij}\, z_i \wedge z_j + \sum_{1\leq i\leq m} n_{ii}\, d_{2i-1} \wedge d_{2i}$$

2. *If $\mathfrak{g}$ is degenerate then,*

$$\begin{aligned} r = &\sum_{1\leq i<j\leq k_0} a_{ij}\, s_i \wedge s_j + \sum_{\substack{1\leq i\leq k_0\\ 1\leq j\leq \ell_0}} b_{ij}\, s_i \wedge z_j + \sum_{1\leq i<j\leq \ell_0} f_{ij}\, z_i \wedge z_j \\ &+ \sum_{1\leq i<j\leq m} m_{ij}\, d_{2i-1} \wedge d_{2j-1} + \sum_{1\leq i,j\leq m} n_{ij}\, d_{2i-1} \wedge d_{2j} + \sum_{1\leq i<j\leq m} p_{ij}\, d_{2i} \wedge d_{2j} \end{aligned}$$

such that $p_{ij} = \varepsilon_{ij} m_{ij}$ and $n_{ji} = \varepsilon_{ij} n_{ji}$ $(\varepsilon_{ij} = \pm 1)$.

We can now state and prove the main theorem.

Theorem 3.3. *Let $\mathfrak{g}$ be a flat Lie algebra and let ξ be a coboundary of $r \in \wedge^2\mathfrak{g}$. If $(\mathfrak{g}, \xi)$ is metaflat, then r is a solution of the classical Yang-Baxter equation $[r, r] = 0$. Moreover, the dual Lie bialgebra $\mathfrak{g}^*$ is flat and metaflat for the dual positive-definite scalar product.*

Proof. From the proposition above, in both the degenerate and nondegenerate cases, we can assume

$$\begin{aligned} r = &\sum_{1\leq i<j\leq k_0} a_{ij}\, s_i \wedge s_j + \sum_{\substack{1\leq i\leq k_0\\ 1\leq j\leq \ell_0}} b_{ij}\, s_i \wedge z_j + \sum_{1\leq i<j\leq \ell_0} f_{ij}\, z_i \wedge z_j \\ &+ \sum_{1\leq i<j\leq m} m_{ij}\, d_{2i-1} \wedge d_{2j-1} + \sum_{1\leq i,j\leq m} n_{ij}\, d_{2i-1} \wedge d_{2j} + \sum_{1\leq i<j\leq m} p_{ij}\, d_{2i} \wedge d_{2j} \end{aligned}$$

Recall that the Schouten-Nijenhuis bracket is $\mathbb{R}$-bilinear with respect to its two arguments and is defined for decomposable multivectors as follows [3]

$$\begin{aligned} &[X_1 \wedge \cdots \wedge X_m, Y_1 \wedge \cdots \wedge Y_n] \\ &\quad = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_m \wedge Y_1 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_n \end{aligned}$$

where the hat sign indicates that the argument below has been omitted.
For all vector fields X_1, X_2, Y_1, Y_2 we have

$$\begin{aligned} [X_1 \wedge X_2, Y_1 \wedge Y_2] = [X_1, Y_1] \wedge X_2 \wedge Y_2 - [X_1, Y_2] \wedge X_2 \wedge Y_1 \\ - [X_2, Y_1] \wedge X_1 \wedge Y_2 + [X_2, Y_2] \wedge X_1 \wedge Y_1. \end{aligned}$$

Moreover, the bracket is symmetric when restricted to bivectors

$$[X_1 \wedge X_2, Y_1 \wedge Y_2] = [Y_1 \wedge Y_2, X_1 \wedge X_2].$$

Therefore $\frac{1}{2}[r,r]$ is given by

$$\sum_{\substack{1\le i<j\le k_0\\ 1\le k<\ell\le m}} a_{ij}m_{kl}\,[s_i\wedge s_j, d_{2k-1}\wedge d_{2\ell-1}] + \sum_{\substack{1\le i<j\le k_0\\ 1\le k<\ell\le m}} a_{ij}n_{kl}\,[s_i\wedge s_j, d_{2k-1}\wedge d_{2\ell}]$$
$$+ \sum_{\substack{1\le i<j\le k_0\\ 1\le k<\ell\le m}} a_{ij}p_{kl}\,[s_i\wedge s_j, d_{2k}\wedge d_{2\ell}] + \sum_{\substack{1\le i,j\le k_0\\ 1\le k<\ell\le m}} b_{ij}m_{kl}\,[s_i\wedge z_j, d_{2k-1}\wedge d_{2\ell-1}]$$
$$+ \sum_{\substack{1\le i,j\le k_0\\ 1\le k<\ell\le m}} b_{ij}n_{kl}\,[s_i\wedge z_j, d_{2k-1}\wedge d_{2\ell}] + \sum_{\substack{1\le i,j\le k_0\\ 1\le k<\ell\le m}} b_{ij}m_{kl}\,[s_i\wedge z_j, d_{2k}\wedge d_{2\ell}].$$

By using $\Gamma_{ij}^{\ell} = \lambda_{j\ell}s_i - \lambda_{i\ell}s_j$, we have

$$\begin{aligned}
\frac{1}{2}[r,r] = &\sum_{1\le i<j\le k_0}\sum_{1\le k<\ell\le m} a_{ij}m_{kl}\left(\Gamma_{ij}^{\ell}\wedge d_{2k-1}\wedge d_{2\ell} - \Gamma_{ij}^{k}\wedge d_{2\ell-1}\wedge d_{2k}\right)\\
&+ \sum_{1\le i<j\le k_0}\sum_{1\le k,\ell\le m} a_{ij}n_{kl}\left(-\Gamma_{ij}^{\ell}\wedge d_{2k-1}\wedge d_{2\ell-1} + \Gamma_{ij}^{k}\wedge d_{2k}\wedge d_{2\ell}\right)\\
&+ \sum_{1\le i<j\le k_0}\sum_{1\le k<\ell\le m} a_{ij}p_{kl}\left(-\Gamma_{ij}^{k}\wedge d_{2k-1}\wedge d_{2\ell} + \Gamma_{ij}^{\ell}\wedge d_{2\ell-1}\wedge d_{2k}\right)\\
&+ \sum_{\substack{1\le i\le k_0\\ 1\le j\le \ell_0}}\sum_{1\le k<\ell\le m} b_{ij}m_{kl}\,z_j\wedge\left(-\lambda_{i\ell}\,d_{2k-1}\wedge d_{2\ell} + \lambda_{ik}\,d_{2\ell-1}\wedge d_{2k}\right)\\
&+ \sum_{\substack{1\le i\le k_0\\ 1\le j\le \ell_0}}\sum_{1\le k,\ell\le m} b_{ij}n_{kl}\,z_j\wedge\left(\lambda_{i\ell}\,d_{2k-1}\wedge d_{2\ell-1} - \lambda_{ik}\,d_{2k}\wedge d_{2\ell}\right)\\
&+ \sum_{\substack{1\le i\le k_0\\ 1\le j\le \ell_0}}\sum_{1\le k<\ell\le m} b_{ij}p_{kl}\,z_j\wedge\left(\lambda_{ik}\,d_{2k-1}\wedge d_{2\ell} - \lambda_{i\ell}\,d_{2\ell-1}\wedge d_{2k}\right).
\end{aligned}$$

Now, we show that the 2nd and the 5th terms are zero and by combining the terms 1st,4th with 3rd,6th respectively, we get $[r,r]=0$. We also show that above equality always holds, whether $\mathfrak{g}$ is degenerate or not. If $\mathfrak{g}$ is nondegenerate, then

$$\begin{aligned}
[r,r] = &\sum_{1\le i<j\le k_0}\sum_{1\le k\le m} a_{ij}n_{kk}\left(-\Gamma_{ij}^{k}\wedge d_{2k-1}\wedge d_{2k-1} + \Gamma_{ij}^{k}\wedge d_{2k}\wedge d_{2k}\right)\\
&+ \sum_{\substack{1\le i\le k_0\\ 1\le j\le \ell_0}}\sum_{1\le k\le m} b_{ij}n_{kk}\,z_j\wedge\left(\lambda_{ik}\,d_{2k-1}\wedge d_{2k-1} - \lambda_{ik}\,d_{2k}\wedge d_{2k}\right) = 0.
\end{aligned}$$

If $\mathfrak{g}$ is degenerate, then for all $k<\ell$ in $\{1,\dots,m\}$ and for all i,j in $\{1,\dots,k_0\}$,

$$\lambda_{i\ell} = \varepsilon_{k\ell}\,\lambda_{ik},\ \ \lambda_{j\ell} = \varepsilon_{k\ell}\,\lambda_{jk},\ \ p_{k\ell} = \varepsilon_{k\ell}\,m_{k\ell},\ \ n_{\ell k} = \varepsilon_{k\ell}\,n_{k\ell}.$$

Therefore, $\Gamma_{ij}^{\ell} = (\varepsilon_{k\ell}\,\lambda_{jk})\,s_i - (\varepsilon_{k\ell}\,\lambda_{ik})\,s_j = \varepsilon_{k\ell}\Gamma_{ij}^{k}$, so

$$\begin{aligned}
m_{k\ell}\Gamma_{ij}^{\ell} - p_{k\ell}\Gamma_{ij}^{k} &= m_{k\ell}(\varepsilon_{k\ell}\,\Gamma_{ij}^{k}) - (\varepsilon_{k\ell}\,m_{k\ell})\Gamma_{ij}^{k} &= 0,\\
-m_{k\ell}\Gamma_{ij}^{k} + p_{k\ell}\Gamma_{ij}^{\ell} &= -m_{k\ell}\Gamma_{ij}^{k} + (\varepsilon_{k\ell}\,m_{k\ell})(\varepsilon_{k\ell}\Gamma_{ij}^{k}) &= 0.
\end{aligned}$$

Thus the sum of the 1st term with the 3rd term is zero. For the same reasons, combining the other terms gives zero.

Let $\mathfrak{g}^*$ be the dual vector space with dual basis

$$\left\{s_1^*, \dots, s_{k_0}^*, z_1^*, \dots, z_{\ell_0}, d_1^*, \dots, d_{2m}^*\right\}.$$

The subspace $\mathrm{span}\{s_1^*, \dots, s_{k_0}^*, z_1^*, \dots, z_{\ell_0}^*\}$ is a commutative subalgebra, its complementary subspace $\mathrm{span}\{d_1^*, \dots, d_{2m}^*\}$ is an abelian ideal (containing the commutator) and satisfies the following equations

$$\begin{aligned}
[s_i^*, d_{2j-1}^*] &= -s_i^*\left(\Phi_j\right) d_{2j}^*, \quad [s_i^*, d_{2j}^*] = s_i^*\left(\Phi_j\right) d_{2j-1}^*,\\
[z_i^*, d_{2j-1}^*] &= -z_i^*\left(\Phi_j\right) d_{2j}^*, \quad [z_i^*, d_{2j}^*] = z_i^*\left(\Phi_j\right) d_{2j-1}^*,
\end{aligned}$$

where

$$\begin{aligned}
\Phi_k = \left(\sum_{j=2}^{k_0} -\lambda_{jk} a_{1j}\right) s_1 + \sum_{p=2}^{k_0-1} \left(\sum_{i=1}^{p-1} \lambda_{ik} a_{ip} + \sum_{j=p+1}^{k_0} -\lambda_{jk} a_{pj}\right) s_p \\
+ \left(\sum_{i=1}^{k_0-1} \lambda_{ik} a_{ik_0}\right) s_{k_0} + \sum_{j=1}^{\ell_0} \left(\sum_{i=1}^{k_0} \lambda_{ik} b_{ij}\right) z_j.
\end{aligned}$$

In matrix form $\widehat{\Lambda} = \Lambda\, M(a|b)$, where

$$\Lambda = \left(\lambda_{ij}\right)_{\substack{1\le j\le m,\\ 1\le i\le k_0}} \quad \widehat{\Lambda} = \left(s_i^*\left(\phi_k\right) \middle| z_j^*\left(\phi_k\right)\right)_{\substack{1\le k\le m\\ 1\le i\le k_0,\ 1\le j\le \ell_0}}$$

and

$$M(a|b) = \left(\begin{array}{cccc|cccc}
0 & a_{12} & \cdots & a_{1k_0} & b_{11} & b_{12} & \dots & b_{1\ell_0}\\
-a_{12} & 0 & \cdots & a_{2k_0} & b_{21} & b_{22} & \dots & b_{2\ell_0}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
-a_{1k_0} & -a_{2k_0} & \cdots & 0 & b_{k_01} & b_{k_02} & \dots & b_{k_0\ell_0}
\end{array}\right). \tag{9}$$

We denote by $\langle\, , \rangle^*$ the positive-definite scalar product on $\mathfrak{g}^*$ associated to $\langle\, , \rangle$, via the isomorphism

$$\begin{array}{rccc}
\sharp: & \mathfrak{g} & \to & \mathfrak{g}^*\\
 & x & \mapsto & \sharp(x)(y) := \langle x, y\rangle,
\end{array}$$

by setting, $\langle \sharp(x), \sharp(y)\rangle^* := \langle x, y\rangle$. We conclude that $\mathfrak{g}^*$ decomposes orthogonally

$$\mathfrak{g}^* = \mathrm{span}\{s_1^*, \dots, s_{k_0}^*\} \oplus \{z_1^*, \dots, z_{\ell_0}^*\} \oplus \mathrm{span}\{d_1^*, \dots, d_{2m}^*\},$$

i.e. $\mathfrak{g}^*$ is flat. Since $\xi^*(x) = 0$ for all x in $\mathrm{span}\{s_1^*, \dots, s_{k_0}^*, z_1^*, \dots, z_{\ell_0}^*\}$ and for all $k = 1, \dots, m$

$$\xi^*(d_{2k-1}^*) = -\Psi_k \wedge d_{2k}^*, \ \xi^*(d_{2k}^*) = \Psi_k \wedge d_{2k-1}^* \text{ where } \Psi_k = \sum_{i=1}^{k_0} \lambda_{ik} s_i^*,$$

then the dual Lie algebra $\mathfrak{g}^*$ is metaflat. ■

Let $(\mathfrak{g}, \langle\, ,\, \rangle, \xi)$ be a triangular metaflat Lie algebra, then its dual $(\mathfrak{g}^*, \langle\, ,\, \rangle^*, \xi^*)$ is metaflat but not necessarily triangular. Indeed, if $\xi = 0$ then $\mathfrak{g}^*$ is abelian and ξ^* is not exact since it is not trivial:

$$\text{for all } k = 1, \ldots, m,\ \psi_k = \sum_{p=1}^{k_0} \lambda_{pk} s_p^* \neq 0,\ \text{thus } \xi^*(d_{2k-1}^*) = -\psi_k \wedge d_{2k}^* \neq 0.$$

However, a necessary and sufficient condition for ξ^* to be exact is given by the following proposition. We will use the notations of the main theorem.

Proposition 3.4. *Let $(\mathfrak{g}, \xi)$ be a triangular metaflat Lie algebra. The dual Lie bialgebra $(\mathfrak{g}^*, \xi^*)$ is triangular if and only if*

$$\operatorname{rank} M(a|b) = k_0. \tag{10}$$

Proof. We denote the r-matrices of $\mathfrak{g}$ and $\mathfrak{g}^*$ by

$$r = \sum_{1 \le i < j \le k_0} a_{ij}\, s_i \wedge s_j + \sum_{\substack{1 \le i \le k_0 \\ 1 \le j \le \ell_0}} b_{ij}\, s_i \wedge z_j + \sum_{1 \le i < j \le \ell_0} f_{ij}\, z_i \wedge z_j + \cdots$$

$$r^* = \sum_{1 \le i < j \le k_0} A_{ij}\, s_i^* \wedge s_j^* + \sum_{\substack{1 \le i \le k_0 \\ 1 \le j \le \ell_0}} B_{ij}\, s_i^* \wedge z_j^* + \sum_{1 \le i < j \le \ell_0} F_{ij}\, z_i^* \wedge z_j^* + \cdots$$

Then $(\mathfrak{g}^*, \xi^*)$ is triangular, if and only if,

$$\text{for all } k = 1, \ldots, m,\ -\psi_k \wedge d_{2k}^* = \xi^*(d_{2k-1}^*) = \operatorname{ad}^*_{d_{2k-1}} r^* = \phi_k \wedge d_{2k}^*.$$

By (9), this is equivalent to

$$\Lambda = \widehat{\Lambda} \left(\begin{array}{cccc|cccc} 0 & A_{12} & \cdots & A_{1k_0} & -B_{11} & -B_{12} & \ldots & -B_{1\ell_0} \\ -A_{12} & 0 & \cdots & A_{2k_0} & -B_{21} & -B_{22} & \ldots & -B_{2\ell_0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_{1k_0} & -A_{2k_0} & \cdots & 0 & -B_{k_01} & -B_{k_02} & \ldots & -B_{k_0\ell_0} \end{array} \right)^t \quad \text{and}$$

$$0 = \widehat{\Lambda} \left(\begin{array}{cccc|cccc} B_{11} & B_{21} & \ldots & B_{k_01} & 0 & F_{12} & \cdots & F_{1\ell_0} \\ B_{12} & B_{22} & \ldots & B_{k_02} & -F_{12} & 0 & \cdots & F_{2\ell_0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{1\ell_0} & B_{2\ell_0} & \ldots & B_{k_0\ell_0} & -F_{1\ell_0} & -F_{2\ell_0} & \cdots & 0 \end{array} \right)^t.$$

By the injectivity of Λ we get from the first identity above

$$M(a|b) \left(\begin{array}{cccc|cccc} 0 & A_{12} & \cdots & A_{1k_0} & -B_{11} & -B_{12} & \ldots & -B_{1\ell_0} \\ -A_{12} & 0 & \cdots & A_{2k_0} & -B_{21} & -B_{22} & \ldots & -B_{2\ell_0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_{1k_0} & -A_{2k_0} & \cdots & 0 & -B_{k_01} & -B_{k_02} & \ldots & -B_{k_0\ell_0} \end{array} \right)^t = \operatorname{Id}_{k_0},$$

i.e. $\operatorname{rank} M(a|b) = k_0$. Moreover, $\mathfrak{g}^* = \mathfrak{s}^* \oplus \mathfrak{z}^* \oplus \operatorname{span}\{d_1^*, \ldots, d_{2m}^*\}$ with $[\mathfrak{g}^*, \mathfrak{g}^*] = \operatorname{span}\{d_1^*, \ldots, d_{2m}^*\}$ (for all $k = 1, \ldots, m$, d_{2k-1}^*, d_{2k}^* can't be in the center). The center of $\mathfrak{g}^*$ is of dimension ℓ_0, since $\dim \mathfrak{z}^* = \dim \ker \widehat{\Lambda} = \dim \ker M(a|b) = \ell_0$ and $\mathfrak{s}^*$ is its orthogonal in the subspace $\operatorname{span}\{s_1^*, \ldots, s_{k_0}^*, z_1^*, \ldots, z_{\ell_0}^*\}$. From the second identity above, we can take the rows of the matrix $M(a|b)$ as a basis of $\mathfrak{s}^*$. From (6) and (9), $\mathfrak{g}^*$ is degenerate if and only if $\mathfrak{g}$ is. ∎

4. Examples

(1) The metaflatness in theorem (3.3) is not a necessary condition to a coboundary to arise from an r-matrix (i.e. a solution of the classical Yang-Baxter equation). Indeed, if ξ is the coboundary of an r-matrix of the form $r = z \wedge x$, where z is a central element and x in the commutator or

$$r = \sum_{1 \le i < j \le m} m_{ij}\, d_{2i-1} \wedge d_{2j-1} + \sum_{1 \le i < j \le m} n_{ij}\, d_{2i-1} \wedge d_{2j} + \sum_{1 \le i < j \le m} p_{ij}\, d_{2i} \wedge d_{2j},$$

(where $m_{ij} \neq \varepsilon p_{ij}$ or $n_{ji} \neq \varepsilon n_{ij}$, $\varepsilon = \pm 1$) then ξ is not metaflat.

(2) From (10), if $(\mathfrak{g}, \xi)$ is the triangular metaflat Lie bialgebra of the r-matrix

$$r = \sum_{1 \le i < j \le k_0} a_{ij}\, s_i \wedge s_j$$

then the dual Lie bialgebra $(\mathfrak{g}^*, \xi^*)$ is triangular if and only if the skew symmetric matrix $(a_{ij}) = M(a|0)$ is invertible (so k_0 is even).

Acknowledgements. The author wishes to thank the editor and the referee for their valuable comments which improved the presentation of the paper.

References

[1] A. Bahayou, M. Boucetta: *Metacurvature of Riemannian Poisson-Lie groups*, J. Lie Theory 19/3 (2009) 439–462.

[2] M. L. Barberis, I. Dotti, A. Fino: *Hyper-Kähler quotients of solvable Lie groups*, J. Geom. Phys. 56 (2006) 691–711.

[3] M. Crainic, R. L. Fernandes, I. Marcut: *Lectures on Poisson Geometry*, American Mathematical Society, Providence (2021).

[4] E. Hawkins: *Noncommutative rigidity*, Com. Math. Phys. 246 (2004) 211–235.

[5] E. Hawkins: *The structure of noncommutative deformations*, J. Diff. Geom. 77 (2007) 385–424.

[6] Y. Kosmann-Schwarzbach: *Lie bialgebras, Poisson Lie groups and dressing transformations*, in: *Integrability of Nonlinear Systems*, Y. Kosmann-Schwarzbach et al. (eds.), Proc. CIMPA School, Pondicherry Univ., India 1996, Lecture Notes in Physics 495, Springer, Berlin (1997) 104–170.

[7] J. H. Lu, A. Weinstein: *Poisson Lie groups, dressing transformations and Bruhat decompositions*, J. Differ. Geom. 31 (1990) 501–526.

[8] J. Milnor: *Curvatures of left invariant metrics on Lie groups*, Adv. Math. 21 (1976) 293–329.

[9] I. Vaisman: *Lectures on the Geometry of Poisson Manifolds*, Birkhäuser, Basel (1994).

Amine Bahayou, Dept. of Mathematics, Kasdi Merbah University, Ouargla, Algeria;
amine.bahayou@gmail.com.

Received September 25, 2022
and in final form April 17, 2023