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Representations of the Special Lie Superalgebra with p-Character of Height One

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Abstract. Let $\mathbf{k}$ be an algebraically closed field of prime characteristic and $S(n)$ be the special Lie superalgebra of Cartan type over $\mathbf{k}$. Define $\bar{S}(n) = S(n) \oplus \mathbf{k}\text{-}\{\xi_1 D_1\}$. So $\bar{S}(n)_0 \cong \mathfrak{gl}(n)$. Let $\mathfrak{g} = S(n)$ or $\bar{S}(n)$. We investigate in this paper the representations of $\mathfrak{g}$ when χ is restricted or $\mathrm{ht}(\chi) = 1$. The main results are listed below.

(1) When $\mathrm{ht}(\chi) = 1$, the irreducible representations of $U_\chi(\mathfrak{g})$ are considered. Precisely, the composition factors of the Kac modules are confirmed and the dimensions of simple modules are given.

(2) When $\chi = 0$ or $\mathrm{ht}(\chi) = 1$, the structures of indecomposable projective modules are studied and the Cartan invariants of $U_\chi(\mathfrak{g})$ are given.

(3) When $\chi = 0$ or $\mathrm{ht}(\chi) = 1$, we show that the representation category over $U_\chi(\mathfrak{g})$ has only one block (reckoning parities in).

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1. Introduction

Throughout this paper, let $\mathbf{k}$ be an algebraically closed field of characteristic $p > 2$. Recall that by Kac's classification theorem ([12]), finite-dimensional simple Lie superalgebras over the complex field are either of classical type or of Cartan type, with the latter consisting of infinite series of four types $W(n)$, $S(n)$, $\tilde{S}(n)$ and $H(n)$. Regretfully, the classification of Lie superalgebras over $\mathbf{k}$ has not been accomplished. Fortunately, the Lie superalgebras of Cartan type remain simple and there is no doubt that they will be one of the most important types of Lie superalgebras over $\mathbf{k}$. So it is meaningful to study their representations. The special Lie superalgebra, which is denoted by $S(n)$, is the main research object of this article.

We know that the representations of Lie superalgebras over $\mathbb{C}$ have achieved great progress since [12], see for example the books [4, 13] and the papers [1, 2, 3]. However, the modular representations of Lie superalgebras were initiated only in recent years, see [6, 15, 16, 17, 20] etc.

Recently, we got some results on the representations of $W(n)$. In paper [5], the irreducible representations of $W(n)$ were studied when the height of the character

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equals one. Based on these results, paper [7] gave a calculation of Cartan invariants for $U_\chi(W(n))$ when the height of χ is less than one. So blocks can be described clearly for $U_\chi(W(n))$ when the height of χ is less than one. Until now, there has been some work on the irreducible representations of $S(n)$, see for example [14, 18, 19]. In [14], authors studied the sufficient and necessary conditions for the Kac modules to be simple when the character is restricted. In [18], a criterion on the irreducibility of Kac module was given when the character is regular semisimple, which is a special case of this paper. In [19], results in [14] were generalized to Lie superalgebra $\bar{S}(m, n, \mathbf{1})$, i.e. sufficient and necessary conditions for the Kac modules to be simple were given when the character is restricted.

Based on the above results, it is natural to ask the following question, what is the situation for $S(n)$ when χ is not restricted. This paper will give some answers.

Now let $\mathfrak{g} = S(n)$ or $\bar{S}(n)$. The paper is organized as follows. In Section 2, we recall some basic definitions and properties on $S(n)$. In addition, we get the result that $GL(n)$ can be identified with a subgroup of $\mathrm{Aut}_p(\mathfrak{g})$. In Section 3, results on irreducible representations of $\mathfrak{sl}(n)$ and $\mathfrak{gl}(n)$ are given for later use. In Section 4, we list some important criterions for the irreducibility of Kac modules. In Sections 5, 6 and 7, we study the irreducible representations of $U_\chi(\mathfrak{g})$ when the height of χ equals to one. The sufficient and necessary conditions for the Kac modules to be simple are given. Moreover, when the Kac module is reducible, composition factors of the Kac module are confirmed. Meanwhile, dimensions of the simple $U_\chi(\mathfrak{g})$-modules are given. In Section 8, we give a description of the indecomposable projective modules. The multiplicities of $\nabla_\chi(\mu)$ in $P_\chi(\lambda)$ are given. In Section 9, the Cartan invariants of $U_\chi(\mathfrak{g})$ are calculated. Meanwhile, the blocks of the category of $U_\chi(\mathfrak{g})$-modules are confirmed.

2. Preliminaries

2.1. Lie superalgebras of Witt and special type

Let $\Lambda(n)$ be the Grassmann superalgebra over $\mathbf{k}$ with n odd generators $\xi_1, \xi_2, \cdots, \xi_n$. The *Witt Lie superalgebra* $W(n)$ is defined to be the superderivations of $\Lambda(n)$. One can check that the elements of $W(n)$ consist of the following set.

$$W(n) = \Big\{ \sum_{i=1}^{n} f_i D_i \mid f_i \in \Lambda(n) \Big\},$$

where D_i is a superderivation of $\Lambda(n)$ defined through $D_i(\xi_j) = \delta_{ij}$, $i, j \in \{1, 2, ..., n\}$. Let $\deg(\xi_i) = 1$, $1 \le i \le n$. Then $\Lambda(n)$ becomes a superalgebra with compatible $\mathbb{Z}$-grading. By the definition of D_i, we have $\deg(D_i) = -1$, $i \in \{1, 2, \cdots, n\}$. So $W(n)$ has a natural $\mathbb{Z}$-grading with

$$W(n)_i = \mathbf{k}\text{-span}\{\xi_{t_1} \cdots \xi_{t_{i+1}} D_s \mid 1 \le t_1 < \cdots < t_{i+1} \le n, 1 \le s \le n\}$$

and $W(n) = \bigoplus_{i=-1}^{n-1} W(n)_i$. The $\mathbb{Z}_2$-grading on $W(n)$ is compatible with the $\mathbb{Z}$-grading of $W(n)$. So the even subspace (resp. odd subspace) is

$$W(n)_{\bar{0}} = \sum_{\bar{i}=\bar{0}} W(n)_i \Big(\text{ resp. } W(n)_{\bar{1}} = \sum_{\bar{i}=\bar{1}} W(n)_i \Big).$$

The Lie-(super)bracket $[\ ,\ ]$ on $W(n)$ is given by

$$[fD_i, gD_j] = fD_i(g)D_j - (-1)^{\deg(fD_i)\deg(gD_j)} gD_j(f)D_i, \tag{1}$$

where fD_i, gD_j are both $\mathbb{Z}$-homogeneous elements of $W(n)$.
Let **div** be the divergence mapping from $W(n)$ to $\Lambda(n)$ defined by:

$$\mathbf{div} : W(n) \ \to\ \Lambda(n),\ \ \sum_{i=1}^{n} f_i D_i \ \mapsto\ \sum_{i=1}^{n} D_i(f_i).$$

The special Lie superalgebra $S(n)$ is defined to be the subsuperalgebra of $W(n)$, consisting of all elements $x \in W(n)$ such that $\mathbf{div}(x) = 0$.

2.2. Some structure information on $S(n)$

In order to describe the structure of $S(n)$ clearly, we introduce the mapping D_{ij}, where $1 \leq i \neq j \leq n$, in the following way:

$$D_{ij} : \Lambda(n) \ \to\ W(n),\ \ f \ \mapsto\ D_i(f)D_j + D_j(f)D_i.$$

Let $m \in \{1, 2, \cdots, n\}$. Denote by I_m the set of ordered sequence $(i_1, i_2, \cdots, i_m)$, where $1 \leq i_1 < i_2 < \cdots < i_m \leq n$. Let $I = (i_1, i_2, \cdots, i_m)$ be an element of I_m. Denote by ξ_I the element $\xi_{i_1}\xi_{i_2}\cdots\xi_{i_m}$ and by I^c the set $\{x \mid x \in \mathbb{Z},\ 1 \leq x \leq n, \text{ but } x \notin \{i_1, \cdots, i_m\}\}$.

Lemma 2.1. *Some structure information on $S(n)$ is listed in the following.*

1. *$S(n)$ is the $\mathbf{k}$-linear span of the elements belonging to $\{D_{ij}(f) \mid f \in \Lambda^k(n),\ k = 1, 2, \cdots, n\}$.*
2. *$S(n)$ is a $\mathbb{Z}$-graded subalgebra of $W(n)$. Denote by $S(n)_m$ the elements of $S(n)$ with degree m, then $S(n)_m = \mathbf{k}\text{-span}\{\, \xi_I(\xi_i D_i - \xi_j D_j) \mid I \in I_m,\ i, j \in I^c$ with $i \neq j\} \cup \mathbf{k}\text{-span}\{\xi_I D_k \mid I \in I_{m+1}, k \in I^c\}$.*
3. *$S(n)_0$ is isomorphic to $\mathfrak{sl}(n)$ as a Lie algebra.*
4. *$S(n)$ is a restricted Lie superalgebra. The $[p]$-mapping on $S(n)_{\bar{0}}$ is given by the pth power composition of derivations. Specifically, let I be an element of I_m, then the $[p]$-mapping on even part is given by*

$$\begin{cases} (\xi_I(\xi_i D_i - \xi_j D_j))^{[p]} &= \delta_{0m}(\xi_i D_i - \xi_j D_j), \\ (\xi_I D_k)^{[p]} &= 0, \quad (k \in I^c). \end{cases} \tag{2}$$

Proof: For (1),(2),(3), the reader can refer to [12, Prop. 3.3.1]. For (4), we can check it by the definition of [p]-mapping. ∎

Define $\bar{S}(n) = S(n) \oplus \mathbf{k}\text{-}\{\xi_1 D_1\}$. Let $\mathsf{d} = \xi_1 D_1 + \xi_2 D_2 + \cdots + \xi_n D_n$. We can check that when $p \nmid n$, $\bar{S}(n) = S(n) \oplus \mathbf{k}\mathsf{d}$. Obviously, $\bar{S}(n)$ is a $\mathbb{Z}$-graded subalgebra of $W(n)$.

From now on, we always denote $\mathfrak{g} = S(n)$ or $\bar{S}(n)$. Since $\mathfrak{g}$ is $\mathbb{Z}$-graded, there is a natural filtration on $\mathfrak{g}$. We have $\mathfrak{g} = \mathfrak{g}^{(-1)} \supseteq \mathfrak{g}^{(0)} \supseteq \cdots \supseteq \mathfrak{g}^{(n-2)} \supseteq 0$, where $\mathfrak{g}^{(i)} = \bigoplus_{j \geq i} \mathfrak{g}_j$, $i = -1, 0, 1, \ldots, n-2$.

Define $\mathfrak{h} = \mathbf{k}\text{-span}\{\xi_i D_i \mid 1 \le i \le n\}$, $\bar{\mathfrak{h}} = \mathbf{k}\text{-span}\{\xi_i D_i - \xi_{i+1} D_{i+1} \mid 1 \le i \le n-1\}$, $\mathfrak{n}^- = \mathbf{k}\text{-span}\{\xi_i D_j \mid 1 \le j < i \le n\}$, and $\mathfrak{n}^+ = \mathbf{k}\text{-span}\{\xi_i D_j \mid 1 \le i < j \le n\}$. Due to Lemma 2.1(3), $S(n)_0$ (resp. $\bar{S}(n)_0$) has a standard triangular decomposition: $S(n)_0 = \mathfrak{n}^- \oplus \bar{\mathfrak{h}} \oplus \mathfrak{n}^+$ (resp. $\bar{S}(n)_0 = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$). Denote by $\mathfrak{b}$ the standard Borel subalgebra of $\mathfrak{g}_0$, i.e. when $\mathfrak{g} = \bar{S}(n)$ (resp. $\mathfrak{g} = S(n)$), $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ (resp. $\mathfrak{b} = \bar{\mathfrak{h}} \oplus \mathfrak{n}^+$). Write $\mathfrak{g}^+ = \mathfrak{g}^{(0)} = \mathfrak{g}_0 \oplus \mathfrak{g}^{(1)}$, $\mathfrak{g}^- = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$. They are both Lie subalgebras of $\mathfrak{g}$. The Cartan subalgebra of $\mathfrak{g}_0$ is also considered as a Cartan subalgebra of $\mathfrak{g}$. As usual, define $\epsilon_i \in \mathfrak{h}^*$ through $\epsilon_i(\xi_j D_j) = \delta_{ij}$.

Definition 2.2. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ be a p-character of $\mathfrak{g}$. We always regard χ as an element of $\mathfrak{g}^*$ by letting $\chi(\mathfrak{g}_{\bar{1}}) = 0$. The *height* of χ, which is denoted by $\mathrm{ht}(\chi)$, is defined to be

$$\mathrm{ht}(\chi) := \min\left\{\mathrm{i} \in \mathbb{Z} \mid \chi\left(\mathfrak{g}^{(\mathrm{i})}\right) = 0\right\}.$$

Since $\chi(\mathfrak{g}_{\bar{1}}) = 0$, $\mathrm{ht}(\chi)$ is always odd. Obviously, $\mathrm{ht}(\chi) = -1$ if and only if $\chi = 0$, $\mathrm{ht}(\chi) = 1$ if and only if $\chi(\mathfrak{g}_0) \neq 0$ and $\chi(\mathfrak{g}_i) = 0$ for any $i \neq 0$. The main purpose of the paper is to study the representations of $U_\chi(\mathfrak{g})$ when $\mathrm{ht}(\chi) \le 1$. By the same argument as [5, Lemma 3.3] we have the following lemma.

Lemma 2.3. *Let χ be a p-character of $\mathfrak{g}$. When $\chi = 0$ or $\mathrm{ht}(\chi) = 1$, $U_\chi(\mathfrak{g})$ is a $\mathbb{Z}$-graded (super)algebra.*

Lemma 2.4. *$GL(n)$ can be identified with a subgroup of $\mathrm{Aut}_p(S(n))$ and $\mathrm{Aut}_p(\bar{S}(n))$, where $\mathrm{Aut}_p(\bar{S}(n))$ (resp. $\mathrm{Aut}_p(S(n))$) denotes the group of all restricted automorphisms of $\bar{S}(n)$ (resp. $S(n)$).*

Proof: $GL(n)$ can be viewed as a subgroup of $\mathrm{Aut}(\Lambda(n))$ through $\sigma(\xi_{i_1}\xi_{i_2}\cdots\xi_{i_s}) = (\sigma\xi_{i_1})(\sigma\xi_{i_2})\cdots(\sigma\xi_{i_s})$ for any $\sigma \in GL(n)$. Now using the same notation as in [12, Proposition 3.3.1] one can check that when $\omega = \theta\xi_1 \wedge \theta\xi_2 \wedge \cdots \wedge \theta\xi_n$ we obtain $\sigma(\theta\xi_1 \wedge \theta\xi_2 \wedge \cdots \wedge \theta\xi_n) = |\sigma|\theta\xi_1 \wedge \theta\xi_2 \wedge \cdots \wedge \theta\xi_n$. Therefore the condition in [12, Prop. 3.3.1(f)] is satisfied. Thus $GL(n)$ is a subgroup of $\mathrm{Aut}(S(n))$. Since $GL(n)$ acts on $S(n)$ by conjugation (i.e. for any $\sigma \in GL(n)$ and $D \in S(n)$, $(\sigma \cdot D)(f) = (\sigma D \sigma^{-1})(f)$ for $f \in \Lambda(n)$), it commutes with the $[p]$-mapping of $S(n)$. Thus $GL(n)$ acts on $S(n)$ as restricted automorphism.

For the case $\bar{S}(n)$, we notice that $\bar{S}(n)_0 \cong \mathfrak{gl}(n)$ and $\mathrm{Lie}(GL(n)) = \mathfrak{gl}(n)$. So $GL(n)$ can be naturally viewed as a subgroup of the group of all restricted automorphisms of $\bar{S}(n)_0$ and the action of $GL(n)$ on $\mathfrak{gl}(n)$ is also conjugation. Thus the lemma follows. ∎

3. Some results on the representations of $\mathfrak{gl}(n)$ and $\mathfrak{sl}(n)$

3.1. The representations of $\mathfrak{gl}(n)$

Let $\mathrm{char}(\mathbf{k}) = p$. Write $G = GL(n)$, which is a connected reductive algebraic group satisfying $\mathrm{Lie}(G) = \mathfrak{gl}(n)$. Then G satisfies the following hypotheses described in [10, Section 6.3].

(H_1) The derived group DG of G is simply connected.

(H_2) The prime p is good for $\mathfrak{gl}(n)$.

(H_3) There exists a G-invariant non-degenerate bilinear form on $\mathfrak{gl}(n)$, which is given by $(X, Y) \mapsto \mathrm{tr}(XY)$.

The hypothesis (H_3) ensures the following G-module isomorphism.

$$\varphi : \mathfrak{gl}(n) \to \mathfrak{gl}(n)^*, \quad Y \mapsto \chi : X \mapsto \mathrm{tr}(XY),$$

where $\mathfrak{gl}(n)^*$ is made into a G-module through the following coadjoint action

$$(g.\chi)(X) = (\mathrm{Ad}(g)(\chi))(X) = \chi(\mathrm{Ad}(g^{-1})(X)), \quad \forall X \in \mathfrak{gl}(n). \tag{3}$$

The isomorphism φ implies that for any $\chi \in \mathfrak{gl}(n)^*$, there exists a unique matrix $Y \in \mathfrak{gl}(n)$, such that

$$\chi(X) = \mathrm{tr}(XY), \quad \forall X \in \mathfrak{gl}(n). \tag{4}$$

Definition 3.1. Let χ be an element of $\mathfrak{gl}(n)^*$. Then the matrix Y in formula (4) is called the *corresponding matrix* of χ, which is denoted by M_χ.

Let χ be an element of $\mathfrak{gl}(n)^*$. Then χ is called semisimple (resp. nilpotent) if M_χ is semisimple (resp. nilpotent). Since M_χ has the Jordan decomposition $M_\chi = M_{\chi,s} + M_{\chi,n}$, where $M_{\chi,s}$ (resp. $M_{\chi,n}$) is the semisimple (resp. nilpotent) part of M_χ, we have a Jordan decomposition for $\chi = \varphi^{-1}(M_\chi)$, i.e.

$$\chi = \chi_s + \chi_n,$$

where $\chi_s = \varphi^{-1}(M_{\chi,s})$ and $\chi_n = \varphi^{-1}(M_{\chi,n})$. Moreover, χ_s (resp. χ_n) is called the semisimple (resp. nilpotent) part of χ.

Because $\mathrm{Lie}(GL(n)) = \mathfrak{gl}(n)$, we have an algebraic isomorphism from $U_\chi(\mathfrak{gl}(n))$ to $U_{g\cdot\chi}(\mathfrak{gl}(n))$ for any $g \in GL(n)$ and $\chi \in \mathfrak{gl}(n)^*$, where $g \cdot \chi$ is the coadjoint action described in equality (3). Consequently, in order to study the irreducible representations of $U_\chi(\mathfrak{gl}(n))$ for all $\chi \in \mathfrak{gl}(n)^*$ up to isomorphism, we just need to fix a p-character in each coadjoint orbit of $\mathfrak{gl}(n)^*$ under the coadjoint action of $GL(n)$. Let $\chi \in \mathfrak{gl}(n)^*$, then

$$(\mathrm{Ad}(g)\cdot\chi)(X) = \chi(\mathrm{Ad}(g)^{-1}\cdot X) = \chi(g^{-1}Xg) = \mathrm{tr}(g^{-1}XgM_\chi) = \mathrm{tr}(XgM_\chi g^{-1}). \tag{5}$$

The equality (5) shows that the corresponding matrix of $\mathrm{Ad}(g)\cdot\chi$ is $gM_\chi g^{-1}$. Hence in the $GL(n)$ coadjoint orbit of χ, there exists a character χ' corresponding to a Jordan matrix. Consequently, without loss of generality, we can further assume that M_χ satisfies the following hypothesis (H_4).

(H_4) M_χ is a standard Jordan matrix.

We give the definition of standard Jordan matrix below in Definition 3.2.

Definition 3.2. Let Y be a Jordan matrix with the following form

$$Y = \begin{pmatrix} J_1 & & & \mathrm{O} \\ & J_2 & & \\ & & \ddots & \\ \mathrm{O} & & & J_m \end{pmatrix}, \quad \text{where } J_i = \begin{pmatrix} j_i & 1 & & & \mathrm{O} \\ & j_i & 1 & & \\ & & \ddots & \ddots & \\ & & & j_i & 1 \\ \mathrm{O} & & & & j_i \end{pmatrix}_{t_i \times t_i}$$

is a Jordan block of size $t_i \times t_i$, $i = 1, 2, \cdots, m$. We say that Y is a *standard Jordan matrix* if the Jordan blocks $J_i, 1 \leq i \leq m$, are ordered according to the following criteria:

(1) $J_1 = J_2 = \cdots = J_r = O$, but $J_{r+1}, J_{r+2}, \cdots, J_m$ are all nonzero, $0 \le r \le m$.

(2) The Jordan blocks, which have the same eigenvalue, are adjacent.

(3) When $(J_i)_{t_i \times t_i}$ and $(J_k)_{t_k \times t_k}$ have the same eigenvalue, then J_i is placed above J_k if $t_i > t_k$.

From now on, we always assume that the corresponding matrix of χ is a standard Jordan matrix, i.e. M_χ is a standard Jordan matrix. Then through the formula (4) one can check that $\chi(\mathfrak{n}^+) = 0$. Hence we can define the corresponding baby Verma module

$$Z^0_\chi(\lambda) := U_\chi(\mathfrak{gl}(n)) \otimes_{U_\chi(\mathfrak{b}^+)} \mathbf{k}_\lambda,$$

where $\mathbf{k}_\lambda = \mathbf{k}\text{-span}\{v_\lambda\}$ is the one-dimensional $\mathfrak{b}^+$-module such that $h \cdot v_\lambda = \lambda(h) v_\lambda$ and $\mathfrak{n}^+ \cdot v_\lambda = 0$. Here λ is an element of Λ_χ, which is described in the following.

$$\begin{aligned} \Lambda_\chi &= \left\{\lambda \in \mathfrak{h}^* \mid \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \text{ for all } h \in \mathfrak{h}\right\}. \\ &= \left\{\lambda \in \mathfrak{h}^* \mid \lambda(h_i)^p - \lambda(h_i) = \chi(h_i)^p \text{ for } i = 1, 2, \cdots, n\right\}. \end{aligned} \tag{6}$$

Now $h_i = \xi_i D_i$ satisfies $h_i^{[p]} = h_i$, $1 \le i \le n$ (see formula (2)). Obviously, $1 \otimes v_\lambda$ is a generator of $Z^0_\chi(\lambda)$.

Remark 3.3. (1) Let $\{X_{-\alpha_1}, X_{-\alpha_2}, \cdots, X_{-\alpha_N}\}$ be an ordered basis of $\mathfrak{n}^-$. Then the set $\left\{X^{m_1}_{-\alpha_1} X^{m_2}_{-\alpha_2} \cdots X^{m_N}_{-\alpha_N} \otimes v_\lambda \mid 0 \le m_i \le p-1\right\}$ is a basis of $Z^0_\chi(\lambda)$ by the PBW basis theorem.

(2) By [10, Prop. 6.7] we have that every simple $U_\chi(\mathfrak{gl}(n))$-module is the homomorphic image of $Z^0_\chi(\lambda)$ for some $\lambda \in \Lambda_\chi$.

Lemma 3.4. *The baby Verma module $Z^0_\chi(\lambda)$ has a unique simple head for every $\lambda \in \Lambda_\chi$, which is denoted by $L^0_\chi(\lambda)$.*

Proof: Because M_χ is a standard Jordan matrix, the conditions of [10, Rem.10.2] are satisfied. Thus the lemma follows. ■

Proposition 3.5. *Let χ be an element of $\mathfrak{gl}(n)^*$ such that $M_\chi = \mathrm{diag}(x, x, \cdots, x, y)$, $x \ne y$. Then the following two statements hold:*

(1) $L^0_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n) = \mathbf{k}\text{-span}\left\{E^{k_{n-1}}_{n,n-1} E^{k_{n-2}}_{n,n-2} \cdots E^{k_1}_{n,1} \otimes v_\lambda \mid 0 \le k_i \le p-1\right\}$. *So* $\dim(L^0_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)) = p^{(n-1)}$.

(2) $L^0_\chi(\lambda) \cong L^0_\chi(\mu)$ *if and only if* $\lambda = \mu$.

Proof: Under this condition, $\chi = \chi_s$ is semisimple and the centralizer of χ (see [10, Sec.7] for its definition), which is denoted by $\mathfrak{l}_\chi$, is equal to the following set.

$$\mathfrak{l}_\chi = \left\{ \begin{pmatrix} A & O \\ O & z \end{pmatrix} \mid A \in M_{(n-1)\times(n-1)}, z \in \mathbf{k} \right\}.$$

Let $\{\mathcal{L}^0_\chi(\mu) \mid \mu \in T_\chi \subseteq \Lambda_\chi\}$ be the set of isomorphism classes of simple $\mathfrak{l}_\chi$-modules. Denote by $\mathfrak{u} = \mathbf{k}\text{-span}\{E_{1,n}, E_{2,n}, \cdots, E_{n-1,n}\}$ and $\mathfrak{p} = \mathfrak{l}_\chi \oplus \mathfrak{u}$. By [10, Prop.7.4] we gt $L^0_\chi(\mu) = U_\chi(\mathfrak{gl}(n)) \otimes_{U_\chi(\mathfrak{p})} \mathcal{L}^0_\chi(\mu)$. Meanwhile $\{L^0_\chi(\mu) \mid \mu \in T_\chi\}$ is the set of isomorphism classes of simple $U_\chi(\mathfrak{gl}(n))$-modules.

So in order to prove (1), we need to show that $\mathcal{L}^0_\chi(a\epsilon_1+\cdots+a\epsilon_{n-1}+b\epsilon_n) = \mathbf{k}\text{-span}\{v_\lambda\}$. Set

$$\mathfrak{l}'_\chi = \left\{ \begin{pmatrix} A & O \\ O & 0 \end{pmatrix} \mid A \in M_{(n-1)\times(n-1)} \right\}$$

and

$$\mathfrak{l}''_\chi = \left\{ \begin{pmatrix} O & O \\ O & z \end{pmatrix} \mid z \in \mathbf{k} \right\} = \mathbf{k}\text{-span}\{E_{n,n}\}.$$

Obviously, $\mathfrak{l}_\chi = \mathfrak{l}'_\chi \oplus \mathfrak{l}''_\chi$. We can check that $\mathfrak{l}''_\chi$ belongs to the center of $\mathfrak{l}_\chi$. So for any $\mathcal{L}^0_\chi(\mu)$, $\mathfrak{l}''_\chi$ acts on $\mathcal{L}^0_\chi(\mu)$ through the multiplicity $\mu(E_{nn})$. We conclude that $\mathcal{L}^0_\chi(a\epsilon_1+\cdots+a\epsilon_{n-1}+b\epsilon_n)$ is a simple $\mathfrak{l}'_\chi$-module. Because

$$\mathcal{L}^0_\chi(a\epsilon_1+\cdots+a\epsilon_{n-1}+b\epsilon_n)|_{\mathfrak{l}'_\chi} \cong \mathcal{L}^0_{\chi|_{\mathfrak{l}'_\chi}}(a\epsilon_1+\cdots+a\epsilon_{n-1})$$

and $\mathcal{L}^0_{\chi|_{\mathfrak{l}'_\chi}}(a\epsilon_1+\cdots+a\epsilon_{n-1})$ is one-dimensional, the result in (1) follows.

For proving the result (2), we assume $\lambda = \lambda_1\epsilon_1+\lambda_2\epsilon_2+\cdots+\lambda_{n-1}\epsilon_{n-1}+\lambda_n\epsilon_n$ and $\mu = \mu_1\epsilon_1+\mu_2\epsilon_2+\cdots+\mu_{n-1}\epsilon_{n-1}+\mu_n\epsilon_n$. We will show that if $\mathcal{L}^0_\chi(\lambda) \cong \mathcal{L}^0_\chi(\mu)$, then $\lambda=\mu$. First, $\lambda_n=\mu_n$. Otherwise we would have $E_{nn}\cdot\mathcal{L}^0_\chi(\lambda) = \lambda_n\mathcal{L}^0_\chi(\lambda)$ and $E_{nn}\cdot\mathcal{L}^0_\chi(\mu) = \mu_n\mathcal{L}^0_\chi(\mu)$, a contradiction. So $\mathcal{L}^0_\chi(\lambda)\cong\mathcal{L}^0_\chi(\mu)$ as irreducible $U_{\chi|_{\mathfrak{l}'_\chi}}(\mathfrak{l}'_\chi)$-modules. This is equivalent to saying that

$$\mathcal{L}^0_\chi(\lambda_1\epsilon_1+\lambda_2\epsilon_2+\cdots+\lambda_{n-1}\epsilon_{n-1}) \cong \mathcal{L}^0_\chi(\mu_1\epsilon_1+\mu_2\epsilon_2+\cdots+\mu_{n-1}\epsilon_{n-1})$$

as irreducible $\mathfrak{l}'_\chi$-modules. So we have $\lambda_i=\mu_i$ for $1\le i\le n-1$ since $\chi|_{\mathfrak{l}'_\chi}$ corresponds to the matrix $\operatorname{diag}(x,x,\cdots,x)$. Now the proposition follows. ■

Proposition 3.6. *Let χ be an element of $\mathfrak{gl}(n)^*$ and M_χ be a non-diagonalizable matrix. If $M_{\chi,s} = \operatorname{diag}(x,x,\cdots,x,y), x\neq y$, then there exist $1\le j_0<i_0\le n-1$ such that $E_{i_0j_0}\otimes v_\lambda$ is nonzero in the simple module $L^0_\chi(\lambda)$ for any $\lambda\in\Lambda_\chi$.*

Proof: We use the same notations as in Proposition 3.5 for $\mathfrak{l}_\chi$, $\mathfrak{l}'_\chi$ and $\mathcal{L}^0_\chi(\lambda)$. Denote by $\mathfrak{n}^+_{\mathfrak{l}_\chi} = \mathbf{k}\text{-span}\{E_{i,j}\mid 1\le i<j\le n-1\}$, $\mathfrak{h}_{\mathfrak{l}} = \mathbf{k}\text{-span}\{E_{i,i}\mid 1\le i\le n\}$, $\mathfrak{n}^-_{\mathfrak{l}_\chi} = \mathbf{k}\text{-span}\{E_{i,j}\mid 1\le j<i\le n-1\}$ and $\mathfrak{b}_{\mathfrak{l}_\chi} = \mathfrak{h}_{\mathfrak{l}}\oplus\mathfrak{n}^+_{\mathfrak{l}_\chi}$. Then $\mathfrak{b}_{\mathfrak{l}_\chi}$ is a subalgebra of $\mathfrak{l}_\chi$ and $\mathfrak{n}^+_{\mathfrak{l}_\chi}$ is a nilpotent ideal of $\mathfrak{b}_{\mathfrak{l}_\chi}$. Let $\lambda=\lambda_1\epsilon_1+\lambda_2\epsilon_2+\cdots+\lambda_n\epsilon_n$ be an element of Λ_χ. Since $\chi|_{\mathfrak{n}^+_{\mathfrak{l}_\chi}}=0$, every irreducible $U_\chi(\mathfrak{b}_{\mathfrak{l}_\chi})$-module is of the form $\mathbf{k}\text{-span}\{v_\lambda\}$, where $\mathfrak{n}^+_{\mathfrak{l}_\chi}\cdot v_\lambda=0$ and $h\cdot v_\lambda=\lambda(h)v_\lambda$ for any $h\in\mathfrak{h}_{\mathfrak{l}}$. So $\mathcal{L}^0_\chi(\lambda)$ is the simple head of $U_\chi(\mathfrak{l}_\chi)\otimes_{U_\chi(\mathfrak{b}_{\mathfrak{l}_\chi})}v_\lambda$. Because $E_{n,n}$ belongs to the center of $\mathfrak{l}_\chi$, $E_{n,n}$ acts on $\mathcal{L}^0_\chi(\lambda)$ through the multiplicity of λ_n. Hence $\mathcal{L}^0_\chi(\lambda)$ can be viewed as a simple $U_{\chi|_{\mathfrak{l}'_\chi}}(\mathfrak{l}'_\chi)$-module. Because M_χ is a non-diagonalizable standard Jordan matrix and $x\neq y$, we get $\chi|_{\mathfrak{l}'_\chi}\neq 0$. So by [10, Prop.7.6] we have $\dim\big(\mathcal{L}^0_\chi(\lambda)\big)\ge p$. Since $\mathcal{L}^0_\chi(\lambda)$ is the simple head of $U_\chi(\mathfrak{l}_\chi)\otimes_{U_\chi(\mathfrak{b}_{\mathfrak{l}_\chi})}v_\lambda$, there exists $1\le j_0<i_0\le n-1$ such that $E_{i_0j_0}\otimes v_\lambda$ is nonzero in $\mathcal{L}^0_\chi(\lambda)$. Thus the proposition follows since $L^0_\chi(\lambda) = U_\chi(\mathfrak{gl}(n))\otimes_{U_\chi(\mathfrak{p})}\mathcal{L}^0_\chi(\lambda)$. ■

Definition 3.7. Let W be the Weyl group of $\mathfrak{gl}(n)$. Then for any $w\in W$ and $\lambda\in\mathfrak{h}^*$, the *dot action* "$\cdot$" is defined to be

$$w\cdot\lambda := w(\lambda+\rho)-\rho,$$

where ρ is the half sum of all the positive roots.

Let $R = \{\epsilon_i - \epsilon_j \mid 1 \le i \neq j \le n\}$ be the set of root systems of $\mathfrak{gl}(n)$ and $\Pi = \{\epsilon_i - \epsilon_{i+1} \mid 1 \le i \le n-1\}$ be the set of all simple roots.

Definition 3.8. Let χ be an element of $\mathfrak{gl}(n)^*$. We say that χ has a *standard Levi form* if and only if $\chi(\mathfrak{b}^+) = 0$ and there exists a subset I of Π such that

$$\chi(\mathfrak{gl}(n)_{-\alpha}) = \begin{cases} \neq 0, & \text{if } \alpha \in I, \\ = 0, & \text{if } \alpha \in R \setminus I. \end{cases}$$

Let χ be an element of $\mathfrak{gl}(n)^*$. We can check that if M_χ is a nilpotent Jordan matrix, then χ has a standard Levi form. Denote by R_I the set $R \cap \mathbb{Z}I$ and by W_I the subgroup of the Weyl group generated by s_α with $\alpha \in I$. In fact, W_I is the Weyl group of the Levi subalgebra $\mathfrak{l}_{\mathfrak{gl}(n)} := \mathfrak{h} \oplus \bigoplus_{\alpha \in R_I} \mathfrak{gl}(n)_\alpha$.

Remark 3.9. In the $\mathfrak{sl}(n)$ case, the definition of standard Levi form and the notation W_I can also be given(see Definition 3.8).

Denote by α_i the simple root $\epsilon_i - \epsilon_{i+1}$. Then $x_{\alpha_i} = E_{i,i+1}, x_{-\alpha_i} = E_{i+1,i}$ and $h_{\alpha_i} = E_{ii} - E_{i+1,i+1}$. In addition, the set $\{x_{\alpha_i}, x_{-\alpha_i}, h_{\alpha_i}\}$ is isomorphic to $\mathfrak{sl}(2)$ as a Lie algebra. By the same argument as [10, Exam. 6.9] we have the following proposition.

Proposition 3.10. *Let χ be a character of $\mathfrak{gl}(n)^*$ or $\mathfrak{sl}(n)^*$ and λ be an element of Λ_χ. If there exists a simple root α such that $\chi(x_{-\alpha}) \neq 0$ and $\chi(h_\alpha) = 0$, then $Z^0_\chi(\lambda) \cong Z^0_\chi(s_\alpha \cdot \lambda)$. Naturally, $L^0_\chi(\lambda) \cong L^0_\chi(s_\alpha \cdot \lambda)$. In particular, if χ has a standard Levi form, then $L^0_\chi(\lambda) \cong L^0_\chi(s \cdot \lambda)$ for $s \in W_I$.*

Proof: Let λ be an element of Λ_χ. We have $\lambda(h_\alpha)^p - \lambda(h_\alpha) = \chi(h_\alpha)^p = 0$. So there exists an integer $0 \le a \le p-1$ such that $\lambda(h_\alpha) = a \cdot 1$. By the same argument as [BGG Prop 1.4] we have that $x^{a+1}_{-\alpha} \otimes v_\lambda$ is a maximal weight vector (i.e., $\mathfrak{n}^+ \cdot x^{a+1}_{-\alpha} \otimes v_\lambda = 0$). Thus the following homomorphism exists.

$$\begin{aligned} \varphi: \quad Z^0_\chi(\lambda - (a+1)\alpha) &\rightarrow Z^0_\chi(\lambda) \\ v_{\lambda-(a+1)\alpha} &\mapsto x^{a+1}_{-\alpha} \otimes v_\lambda \end{aligned}$$

In addition, the isomorphism φ is surjective because $x^{[p]}_{-\alpha} = 0$,

$$x^{p-(a+1)}_{-\alpha} \cdot (x^{a+1}_{-\alpha} \otimes v_\lambda) = x^p_{-\alpha} \otimes v_\lambda = (x^{[p]}_{-\alpha} + \chi(x_{-\alpha})^p) \otimes v_\lambda = (x_{-\alpha})^p v_\lambda,$$

and $\chi(x_{-\alpha}) \neq 0$. Furthermore, φ is an isomorphism since $\dim(Z^0_\chi(\lambda - (a+1)\alpha)) = \dim Z^0_\chi(\lambda)$. Through calculation we have $s_\alpha \cdot \lambda = \lambda - (a+1)\alpha$. Thus the proposition follows. ■

Proposition 3.11. *Let χ be an element of $\mathfrak{gl}(n)^*$ and $M_\chi = E_{n-1,n}$. Then*

$$\begin{aligned} &L^0_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n) \\ &= \mathbf{k}\text{-span}\left\{E^{k_{n-1}}_{n,n-1} E^{k_{n-2}}_{n,n-2} \cdots E^{k_1}_{n,1} \otimes v_\lambda \mid 0 \le k_i \le p-1, 1 \le i \le n-1\right\}. \end{aligned}$$

Proof: Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$, where

$$\mathfrak{l} = \left\{\begin{pmatrix} A & O \\ O & z \end{pmatrix} \mid A \in M_{(n-1)\times(n-1)}, z \in \mathbf{k}\right\}, \quad \mathfrak{u} = \mathbf{k}\text{-span}\{E_{1,n}, E_{2,n}, \cdots, E_{n-1,n}\}.$$

We can check that $\mathfrak{l}$ is a subalgebra of $\mathfrak{gl}(n)$ and $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{gl}(n)$. Because $\mathfrak{u}$ is a nilpotent subalgebra of $\mathfrak{p}$ and $\chi|_{\mathfrak{u}} = 0$, any irreducible $U_\chi(\mathfrak{p})$-module can be realized as an irreducible $U_\chi(\mathfrak{l})$-module with trivial $\mathfrak{u}$-action. Let $\mathcal{L}^0_\chi(\lambda)$ be the simple $U_\chi(\mathfrak{p})$-module corresponding to weight λ. Since $\chi|_{\mathfrak{l}} = 0$ and E_{nn} belongs to the center of $\mathfrak{l}$, we have that $\mathcal{L}^0_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n) = \mathbf{k}\text{-span}\{v_\lambda\}$ by the same argument as Proposition 3.5(1). Define

$$Z^0_{\chi,\mathfrak{p}}(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n) := U_\chi(\mathfrak{gl}(n)) \otimes_{U_\chi(\mathfrak{p})} \mathcal{L}^0_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n).$$

Then $L^0_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)$ is the simple head of $Z^0_{\chi,\mathfrak{p}}(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)$. Since $\dim Z^0_{\chi,\mathfrak{p}}(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n) = p^{n-1}$, the following inequality holds

$$\dim L^0_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n) \leq p^{n-1}. \tag{7}$$

On the other hand, one can check $\dim G \cdot \chi = 2(n-1)$. So the dimension of $L^0_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)$ is divisible by p^{n-1} by [10, Prop.7.6]. Hence

$$\dim L^0_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n) \geq p^{n-1}. \tag{8}$$

Now comparing formulas (7) and (8) we get $\dim L^0_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n) = p^{n-1}$. So $L^0_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n) = Z^0_{\chi,\mathfrak{p}}(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)$. The proposition follows. ■

3.2. The representations of $\mathfrak{sl}(n)$

In this subsection, we will determine the representations of $\mathfrak{sl}(n)$ under some conditions. Recall that $\text{char}(\mathbf{k}) = p$. When $p \nmid n$, $\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \mathbf{k}\text{-span}\{\mathbf{d}\}$, where $\mathbf{d} = \text{diag}\{1, 1, \cdots, 1\}$ belongs to the center of $\mathfrak{gl}(n)$. So every finite-dimensional simple $\mathfrak{gl}(n)$-module restricts to a simple $\mathfrak{sl}(n)$-module and every simple $\mathfrak{sl}(n)$-module can be got through this method. However, when $p \mid n$, the situation is different. Fortunately, there are some discussions on this situation in paper [11].

Lemma 3.12. [11, Prop. 1] *Let $\mathbf{k}$ be an algebraically closed field of characteristic $p > 0$ and $p \mid n$. The restriction of a simple $\mathfrak{gl}(n)$-module to $\mathfrak{sl}(n)$-module is either simple or has length p with all composition factors isomorphic to each other.*

Lemma 3.13. [11, Prop. 2] *Let χ be a character of $\mathfrak{gl}(n)^*$ such that M_χ is a Jordan matrix. If all Jordan blocks of M_χ have size divisible by p, then there is a unique (up to isomorphism) simple $\mathfrak{gl}(n)$-module E with p-character χ such that the restriction of E to $\mathfrak{sl}(n)$ is not simple. Otherwise, all simple $\mathfrak{gl}(n)$-modules with p-character χ restrict to simple $\mathfrak{sl}(n)$-modules.*

Denote by $\text{tr}(-, X)$ the element of $\mathfrak{gl}(n)^*$ with $\text{tr}(-, X)(Y) = \text{tr}(XY)$ where $Y \in \mathfrak{gl}(n)$. From Subsection 3.1 we get that the set $\{\text{tr}(-, X) \mid x \in \mathfrak{gl}(n)\}$ equals to $\mathfrak{gl}(n)^*$. Because there is a natural monomorphism from $\mathfrak{sl}(n)$ to $\mathfrak{gl}(n)$, we get an epimorphism $\phi : \mathfrak{gl}(n) \to \mathfrak{gl}(n)^* \to \mathfrak{sl}(n)^*$. So the set $\{\text{tr}(-, X) \mid X \in \mathfrak{gl}(n)\}$ can describe all characters belonging to $\mathfrak{sl}(n)^*$. By Lemma 2.4, $GL(n)$ is a subgroup of $\text{Aut}_p(S(n))$. By the same discussion as formula (5) we can see that in order to describe the representations of $U_\chi(\mathfrak{sl}(n))$ for any $\chi \in \mathfrak{sl}(n)^*$ up to isomorphism, we just need to consider the characters belonging to the following set

$$\{\text{tr}(-, X) \mid X \text{ is a standard Jordan matrix}\}.$$

From now on, we always assume that χ is of the form $\text{tr}(-, X)$, where $X \in \mathfrak{gl}(n)$ is a standard Jordan matrix.

Definition 3.14. If χ is of the form $\mathrm{tr}(-,M)$ (i.e. $\chi(Y)=\mathrm{tr}(MY)$ for any $Y\in\mathfrak{sl}(n)$), we say that χ *corresponds* to the matrix M.

Recall that we use the notation $\bar{\mathfrak{h}}=\mathbf{k}\text{-span}\{\xi_iD_i-\xi_{i+1}D_{i+1}\mid 1\le i\le n-1\}$, which is the standard Cartan subalgebra of $\mathfrak{sl}(n)$.

Remark 3.15. Let χ be an element of $\mathfrak{sl}(n)^*$ corresponding to M_χ. Then every element of Λ_χ can be written in the form $\lambda=\lambda_1\epsilon_1+\lambda_2\epsilon_2+\cdots+\lambda_n\epsilon_n$, where $(\lambda_i-\lambda_j)^p-(\lambda_i-\lambda_j)=\chi(E_{ii}-E_{jj})^p$, $1\le i\ne j\le n$. We remind the reader that $\epsilon_1+\epsilon_2+\cdots+\epsilon_n=0$ in $\bar{\mathfrak{h}}^*$.

Proposition 3.16. *Let χ be an element of $\mathfrak{sl}(n)^*$. If χ corresponds to the matrix* $\mathrm{diag}(x,x,\cdots,x,y)$, $x\ne y$, *then the following results hold:*

(1) $L_\chi^0(a\epsilon_1+\cdots+a\epsilon_{n-1}+b\epsilon_n)=\mathbf{k}\text{-span}\left\{E_{n,n-1}^{k_{n-1}}E_{n,n-2}^{k_{n-2}}\cdots E_{n,1}^{k_1}\otimes v_\lambda\mid 0\le k_i\le p-1\right\}$.

So $\dim(L_\chi^0(a\epsilon_1+\cdots+a\epsilon_{n-1}+b\epsilon_n))=p^{(n-1)}$.

(2) $L_\chi^0(\lambda)\cong L_\chi^0(\mu)$ *if and only if* $\lambda=\mu$ *in* $\bar{\mathfrak{h}}^*$.

Proof: When $p\nmid n$, the results are obvious due to Proposition 3.5.

We discuss now the case when $p|n$. We first introduce some notation. Denote by $Z_\chi^0(\lambda)^{\mathfrak{gl}(n)}=U_\chi(\mathfrak{gl}(n))\otimes_{U_\chi(\mathfrak{b})}\mathbf{k}_\lambda$, $Z_\chi^0(\lambda)^{\mathfrak{sl}(n)}=U_\chi(\mathfrak{sl}(n))\otimes_{U_\chi(\mathfrak{b})}\mathbf{k}_\lambda$, $M_\chi^0(\lambda)^{\mathfrak{gl}(n)}$ (resp. $M_\chi^0(\lambda)^{\mathfrak{sl}(n)}$) the maximal submodule of $Z_\chi^0(\lambda)^{\mathfrak{gl}(n)}$ (resp. $Z_\chi^0(\lambda)^{\mathfrak{sl}(n)}$) and of $L_\chi^0(\lambda)^{\mathfrak{gl}(n)}$ (resp. $L_\chi^0(\lambda)^{\mathfrak{sl}(n)}$) the simple head of $Z_\chi^0(\lambda)^{\mathfrak{gl}(n)}$ (resp. $Z_\chi^0(\lambda)^{\mathfrak{sl}(n)}$). By Remark 3.3 we get that $Z_\chi^0(\lambda)^{\mathfrak{sl}(n)}$ and $Z_\chi^0(\lambda)^{\mathfrak{gl}(n)}$ have the same basis. Obviously, $M_\chi^0(\lambda)^{\mathfrak{gl}(n)}$ is an $\mathfrak{sl}(n)$-module. So every simple $\mathfrak{sl}(n)$-module can be viewed as one simple quotient module of a simple $\mathfrak{gl}(n)$-module. Because $M_\chi=\mathrm{diag}(x,x,...,x,y)$, by Lemma 3.13 every simple $\mathfrak{sl}(n)$-module is meanwhile a $\mathfrak{gl}(n)$-module. So every simple $\mathfrak{sl}(n)$-module can be viewed as the restriction of a simple $\mathfrak{gl}(n)$-module. Hence every $\mathfrak{sl}(n)$-maximal weight vector in $L_\chi^0(\lambda)^{\mathfrak{sl}(n)}$ can be viewed as a $\mathfrak{gl}(n)$-maximal weight vector meanwhile.

We now prove the result in (1): By Lemma 3.13 and the above discussion, we get that $L_\chi^0(a\epsilon_1+\cdots+a\epsilon_{n-1}+b\epsilon_n)^{\mathfrak{gl}(n)}$ is a simple $\mathfrak{sl}(n)$-module. So by Proposition 3.5, the result in (1) follows.

For the result in (2): We only need to prove the necessity. Recall that we have $\bar{\mathfrak{h}}=\mathbf{k}\text{-span}\{E_{11}-E_{22},E_{22}-E_{33},\cdots,E_{n-1,n-1}-E_{nn}\}$. By the above discussion we get that if $L_\chi^0(\lambda)^{\mathfrak{sl}(n)}$ is an irreducible module, then there exists a simple $\mathfrak{gl}(n)$-module $L_\chi^0(\lambda')^{\mathfrak{gl}(n)}$ such that the restriction of $L_\chi^0(\lambda')^{\mathfrak{gl}(n)}$ to $\mathfrak{sl}(n)$-module is isomorphic to $L_\chi^0(\lambda)^{\mathfrak{sl}(n)}$ and $\lambda'\mid_{\bar{\mathfrak{h}}^*}=\lambda$. i.e.,

$$L_\chi^0(\lambda')^{\mathfrak{gl}(n)}|_{\mathfrak{sl}(n)}\cong L_\chi^0(\lambda)^{\mathfrak{sl}(n)}. \tag{9}$$

Because $L_\chi^0(\lambda)^{\mathfrak{sl}(n)}\cong L_\chi^0(\mu)^{\mathfrak{sl}(n)}$, the following isomorphism holds.

$$L_\chi^0(\lambda')^{\mathfrak{gl}(n)}|_{\mathfrak{sl}(n)}\cong L_\chi^0(\mu)^{\mathfrak{sl}(n)}. \tag{10}$$

By the formula (9) (resp. (10)) we get that there exists a $\mathfrak{gl}(n)$ maximal weight vector $v_{\lambda'}$ (resp. $v_{\mu'}$) in $L_\chi^0(\lambda')^{\mathfrak{gl}(n)}$ such that $\lambda'\mid_{\bar{\mathfrak{h}}}=\lambda$ (resp. $\mu'\mid_{\bar{\mathfrak{h}}}=\mu$). If $\lambda\ne\mu$, then $\lambda' \neq \mu'$, which means that there are two maximal weight vectors $v_{\lambda'}$ and $v_{\mu'}$ in $L^0_\chi(\lambda')^{\mathfrak{gl}(n)}$. So $L^0_\chi(\lambda')^{\mathfrak{gl}(n)} \cong L^0_\chi(\mu')^{\mathfrak{gl}(n)}$. this is a contradiction to Proposition 3.5(2). Thus $\lambda = \mu$. ■

Proposition 3.17. *Let χ be an element of $\mathfrak{sl}(n)^*$ corresponding to M. If M is non-diagonalizable and $M_s = \mathrm{diag}(x, x, \cdots, x, y)$ with $x \neq y$, then there exist $1 \leq j_0 < i_0 \leq n-1$ such that $E_{i_0 j_0} \otimes v_\lambda$ is nonzero in the simple module $L^0_\chi(\lambda)$ for any $\lambda \in \Lambda_\chi$.*

Proof: Because $M_s = \mathrm{diag}(x, x, \cdots, x, y)$ with $x \neq y$, M has one Jordan block of size one. By Lemma 3.13 and the same argument as Proposition 3.16 we get that every simple $U_\chi(\mathfrak{sl}(n))$-module can be viewed as the restriction of a simple $U_\chi(\mathfrak{gl}(n))$-module. By Proposition 3.6 the proposition follows. ■

Proposition 3.18. *Let χ be an element of $\mathfrak{sl}(n)^*$ corresponding to $E_{n-1,n}$. Then*

$$\begin{aligned} &L^0_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n) \\ &= \mathbf{k}\text{-span}\left\{E^{k_{n-1}}_{n,n-1}E^{k_{n-2}}_{n,n-2}\cdots E^{k_1}_{n,1} \otimes v_\lambda \mid 0 \leq k_i \leq p-1, 1 \leq i \leq n-1\right\}. \end{aligned}$$

Proof: Because $p \geq 3$ and the size of Jordan blocks of $E_{n-1,n}$ is either 1 or 2, which is not divisible by p, the proposition can be proved by the same argument as Proposition 3.17. ■

4. Some preliminary result

Let $\mathfrak{g} = \bar{S}(n)$ or $S(n)$. In this section, we always assume that χ is an element of $\mathfrak{g}^*$ satisfying $\mathrm{ht}(\chi) = 1$, i.e. $\chi(\mathfrak{g}_0) \neq 0$ but $\chi(\mathfrak{g}_i) = 0$ for all $i \neq 0$. By Lemma 2.4, $GL(n)$ can be regarded as a subgroup of $\mathrm{Aut}_p(\mathfrak{g})$. We can also check that $GL(n)$ keeps the grading of $\mathfrak{g}$.

Definition 4.1. Let χ be an element of $\mathfrak{g}^*$ with $\mathrm{ht}(\chi) = 1$. If $\chi|_{\mathfrak{g}_0}$ corresponds to the matrix Y, then we say that χ corresponds to the matrix Y or Y is the corresponding matrix of χ. When $\mathfrak{g} = \bar{S}(n)$, the *corresponding matrix* of χ is denoted by M_χ.

Remark 4.2. We state the following remarks.

(1) When $\mathfrak{g} = \bar{S}(n)$, the corresponding matrix of χ is unique. When $\mathfrak{g} = S(n)$, the corresponding matrix of χ is not unique.

(2) When $\mathfrak{g} = S(n)$, we have the following results:

- (a) If λ is zero in $\mathfrak{sl}(n)^*$, then the corresponding matrix of λ is of the form kE_n for some $k \in \mathbf{k}$, where $E_n = \mathrm{diag}\{1, 1, \cdots, 1\}$.
- (b) If χ corresponds to both M'_χ and M''_χ, then there exists some $k \in \mathbf{k}$ such that $M'_\chi = M''_\chi + kE_n$. So the definitions of semisimple and nilpotent characters are well defined for χ.
- (c) Let g be an element of $GL(n)$. If M is a corresponding matrix of χ, then by the same argument as formula (5) we get that the corresponding matrix of $g.\chi$ is of the form $g(M + \mathbf{k}E_n)g^{-1}, k \in \mathbf{k}$. Because $U_\chi(\mathfrak{g}) \cong U_{g.\chi}(\mathfrak{g})$, we can also assume that χ corresponds to a standard Jordan matrix up to isomorphism.

Proof: We only need to prove (2)(b). Let γ be an element of $\mathfrak{sl}(n)^*$ corresponding to the matrix $M'_\chi - M''_\chi$. Then we can check that γ is zero in $\mathfrak{sl}(n)^*$. By the result in (2)(a) we have $M'_\chi - M''_\chi = kE_n$ for some $k \in \mathbf{k}$, i.e. $M'_\chi = M''_\chi + kE_n$. ■

From now on, we always assume that χ corresponds to the matrix M_χ, which is a standard Jordan matrix.

Definition 4.3. Let N be a simple $U_\chi(\mathfrak{g}_0)$-module. The induced module

$$K_\chi(N) \triangleq U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}^+)} N$$

is called a *Kac module.* Here, N is regarded as a $U_\chi(\mathfrak{g}^+)$-module through trivial $\mathfrak{g}^{(1)}$-action.

Because M_χ is a standard Jordan matrix, by Lemma 3.4 the baby Verma module $Z^0_\chi(\lambda)$ has a unique simple quotient $L^0_\chi(\lambda)$ for any $\lambda \in \Lambda_\chi$. We will write $K_\chi(\lambda)$ for the Kac module $K_\chi(L^0_\chi(\lambda))$ for simplicity.

Lemma 4.4. [5, Prop.3.9] *Let N be a simple $U_\chi(\mathfrak{g}_0)$-module. Then the Kac module $K_\chi(N)$ has a unique maximal proper submodule and a unique simple head, which are denoted by $M_\chi(N)$ and $L_\chi(N)$ respectively. Furthermore, the set $\{L_\chi(N) \mid N \in \Gamma\}$ constitutes the complete set of isomorphism classes of irreducible $U_\chi(\mathfrak{g})$-modules, where Γ denotes the set of representatives of the isomorphism classes of irreducible $U_\chi(\mathfrak{g}_0)$-modules.*

In particular, if χ satisfies the hypothesis (H_4), then any Kac module $K_\chi(\lambda)$ has a unique maximal proper submodule and a unique simple quotient, which are denoted by $M_\chi(\lambda)$ and $L_\chi(\lambda)$ respectively. Consequently, the set $\{L_\chi(\lambda) \mid L^0_\chi(\lambda) \in \Gamma\}$ constitutes the complete set of isomorphism classes of irreducible $U_\chi(\mathfrak{g})$-modules.

Moreover, we have $L_\chi(\lambda) \cong L_\chi(\mu)$ if and only if $L^0_\chi(\lambda) \cong L^0_\chi(\mu)$.

From now on, we always assume that λ is an element of Λ_χ. By the same argument as [5, Lem.4.1] we have Lemma 4.5 below.

Lemma 4.5. *The set $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes L^0_\chi(\lambda)$ is contained in any non-zero $U_\chi(\mathfrak{g})$-submodule of $K_\chi(\lambda)$.*

Denote by $\varphi : K_\chi(\lambda) \longrightarrow L_\chi(\lambda)$ the canonical map from $K_\chi(\lambda)$ to $L_\chi(\lambda)$.

Lemma 4.6. *If there exists an element v_0 in $L^0_\chi(\lambda)$ such that the expression $\varphi(D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes v_0)$ is non-zero, then $L_\chi(\lambda) = K_\chi(\lambda)$.*

Proof: Obviously, φ is a $U_\chi(\mathfrak{g})$-module homomorphism. If there exists $v'_0 \in L^0_\chi(\lambda)$ such that $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes v'_0$ is zero in $L_\chi(\lambda)$, then by the simplicity of $L^0_\chi(\lambda)$ and the following commutation relations in $U_\chi(\mathfrak{g})$

$$(\xi_i D_j) \cdot (D_1 \wedge D_2 \wedge \cdots \wedge D_n) = \begin{cases} (D_1 \wedge D_2 \wedge \cdots \wedge D_n) \cdot (\xi_i D_j), & i \neq j; \\ (D_1 \wedge D_2 \wedge \cdots \wedge D_n) \cdot (\xi_i D_i - 1), & i = j, \end{cases} \tag{11}$$

we know that every element in $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes L^0_\chi(\lambda)$ is zero in $L_\chi(\lambda)$. So if $\varphi(D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes v_0)$ is non-zero, then every element in $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes L^0_\chi(\lambda)$ is non-zero in $L_\chi(\lambda)$.

Now assume $L_\chi(\lambda) \neq K_\chi(\lambda)$. Then by Lemma 4.4 we have that $K_\chi(\lambda)$ has a non-zero maximal submodule $M_\chi(\lambda)$ such that $L_\chi(\lambda) \cong K_\chi(\lambda)/M_\chi(\lambda)$. Because $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes L^0_\chi(\lambda) \subseteq M_\chi(\lambda)$ (Lemma 4.5), $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes v_0$ is zero in $L_\chi(\lambda)$, a contradiction. Thus the lemma follows. ■

Lemma 4.7. *Assume $\lambda = \lambda_1\epsilon_1+\lambda_2\epsilon_2+\cdots+\lambda_n\epsilon_n$ and $\varphi(D_{i_1}\wedge D_{i_2}\wedge\cdots\wedge D_{i_s}\otimes v_\lambda) \neq 0$, $1 \leq i_1 < i_2 < \cdots < i_s \leq n$. Let i, k, l be three different positive integers with $i < i_1$. Then the following results hold.*

(1) *If $k, l \in \{i_1, i_2, \cdots, i_s\}$ and $\lambda_k - \lambda_l \neq 0$, then $D_i \wedge D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$.*

(2) *If $k, l \notin \{i_1, i_2, \cdots, i_s\}$ and $\lambda_k - \lambda_l \neq 0$, then $D_i \wedge D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$.*

(3) *If $k \notin \{i_1, i_2, \cdots, i_s\}$, $l \in \{i_1, i_2, \cdots, i_s\}$ and $\lambda_k - \lambda_l \neq -1$, then $D_i \wedge D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$.*

Proof: We prove the results in (1) and (2) together. For (1), assume $k = i_m$ and $l = i_t$. Since $i < i_1$ and $1 \leq i_1 < i_2 < \cdots < i_s \leq n$, the following equalities hold:

$$\begin{aligned}
&\xi_i(\xi_k D_k - \xi_l D_l)\cdot D_i \wedge D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda \\
&\quad = [\xi_i(\xi_k D_k - \xi_l D_l), D_i \wedge D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s}] \otimes v_\lambda \\
&\quad = (\xi_k D_k - \xi_l D_l)\cdot D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda \\
&\qquad +(-1)^{m-1} D_i \wedge D_{i_1} \wedge \cdots \wedge D_{i_{m-1}} (\xi_i D_k) D_{i_{m+1}} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda \\
&\qquad -(-1)^{t-1} D_i \wedge D_{i_1} \wedge \cdots \wedge D_{i_{t-1}} (\xi_i D_l) D_{i_{t+1}} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda \\
&\quad = (\lambda_k - \lambda_l) D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda.
\end{aligned} \tag{12}$$

For (2), since $k, l \notin \{i_1, i_2, \cdots, i_s\}$, we have the following equality:

$$\xi_i(\xi_k D_k - \xi_l D_l)\cdot D_i \wedge D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda = (\lambda_k - \lambda_l) D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda. \tag{13}$$

Because $\lambda_k - \lambda_l \neq 0$ and $D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$, both formulas (12) and (13) imply that $D_i \wedge D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$.

For (3), assume $l = i_t$. Since $k \notin \{i_1, i_2, \cdots, i_s\}$, the following equalities hold.

$$\begin{aligned}
&\xi_i(\xi_k D_k - \xi_l D_l)\cdot D_i \wedge D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda \\
&\quad = [\xi_i(\xi_k D_k - \xi_l D_l), D_i \wedge D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s}] \otimes v_\lambda \\
&\quad = (\xi_k D_k - \xi_l D_l)\cdot D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda \\
&\qquad +(-1)^{t} D_i \wedge D_{i_1} \wedge \cdots \wedge D_{i_{t-1}} (\xi_i D_l) D_{i_{t+1}} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda \\
&\quad = (\lambda_k - \lambda_l + 1) D_{i_1} \wedge D_{i_2} \wedge \cdots D_{i_s} \otimes v_\lambda \in L_\chi(\lambda).
\end{aligned} \tag{14}$$

Because $\lambda_k - \lambda_l \neq -1$ and $D_{i_1} \wedge D_{i_2} \wedge \cdots D_{i_s} \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$, (14) implies that $D_i \wedge D_{i_1} \wedge D_{i_2} \wedge \cdots D_{i_s} \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$. ■

Lemma 4.8. *Assume that M is a submodule of $K_\chi(\lambda)$, $\lambda = \lambda_1\epsilon_1+\lambda_2\epsilon_2+\cdots+\lambda_n\epsilon_n$. If there exist $1 \leq i_1 < i_2 < \cdots < i_s \leq n$ such that $D_{i_1} \wedge D_{i_2} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda$ belongs to M, then we have the following results:*

(1) *If $\lambda_k - \lambda_l \neq 0$, $k, l \notin \{i_1, i_2, \cdots, i_s\}$ or $k, l \in \{i_2, i_3, \cdots, i_s\}$, then $D_{i_2} \wedge D_{i_3} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda$ belongs to M.*

(2) *If $\lambda_k - \lambda_l \neq -1$, where $k \notin \{i_1, i_2, \cdots, i_s\}$ but $l \in \{i_2, \cdots, i_s\}$, then $D_{i_2} \wedge D_{i_3} \wedge \cdots \wedge D_{i_s} \otimes v_\lambda$ belongs to M.*

Proof: For (1), by the same calculation as formula (12) and formula (13), we have the following equality.

$$\xi_{i_1}(\xi_k D_k - \xi_l D_l)\cdot D_{i_1}\wedge D_{i_2}\wedge\cdots\wedge D_{i_s}\otimes v_\lambda$$
$$= (\lambda_k - \lambda_l)D_{i_2}\wedge D_{i_3}\wedge\cdots\wedge D_{i_s}\otimes v_\lambda. \tag{15}$$

Since $\lambda_k - \lambda_l \neq 0$, eq. (15) implies that $D_{i_2}\wedge D_{i_3}\wedge\cdots\wedge D_{i_s}\otimes v_\lambda$ belongs to M.
For (2), assume $l = i_t, 2 \leq t \leq s$.

$$\begin{aligned}
&\xi_{i_1}(\xi_k D_k - \xi_l D_l)\cdot D_{i_1}\wedge D_{i_2}\wedge\cdots\wedge D_{i_s}\otimes v_\lambda\\
&= [\xi_{i_1}(\xi_k D_k - \xi_l D_l), D_{i_1}\wedge D_{i_2}\wedge\cdots\wedge D_{i_s}]\otimes v_\lambda\\
&= (\xi_k D_k - \xi_l D_l)\cdot D_{i_2}\wedge\cdots\wedge D_{i_s}\otimes v_\lambda \qquad (16)\\
&\quad + (-1)^{t-1}D_{i_1}\wedge\cdots\wedge D_{i_{t-1}}(\xi_{i_1}D_{i_t})D_{i_{t+1}}\wedge\cdots\wedge D_{i_s}\otimes v_\lambda\\
&= (\lambda_k - \lambda_l + 1)D_{i_2}\wedge\cdots\wedge D_{i_s}\otimes v_\lambda.
\end{aligned}$$

Since $\lambda_k - \lambda_l \neq -1$, $D_{i_2}\wedge\cdots\wedge D_{i_s}\otimes v_\lambda$ belongs to M. ■

Definition 4.9. Let χ be an element of $\mathfrak{g}^*$ corresponding to M_χ. When $\mathrm{ht}(\chi) = 1$ or $\chi = 0$, define l_χ to be the number of different eigenvalues of M_χ. This is equivalent to saying that if $M_{\chi,s} = \mathrm{diag}\{x_1,\cdots,x_1,x_2,\cdots,x_2,\cdots,x_s,\cdots,x_s\}$, $x_i \neq x_j$ when $i \neq j$, then $l_\chi = s$.

Remark 4.10. By Remark 4.2(3) we can see that l_χ is well defined for $S(n)$.

5. The case when $l_\chi \geq 3$

Let $\mathfrak{g} = S(n)$ or $\bar{S}(n)$ and $\chi \in \mathfrak{g}^*$. In this section we always assume $l_\chi \geq 3$, i.e.

$$M_{\chi,s} = \mathrm{diag}\{\underbrace{x_1,\cdots,x_1}_{k-1},\underbrace{x_2,\cdots,x_2}_{l},\cdots,\underbrace{x_s,\cdots,x_s}_{m}\},$$

$s \geq 3$ and $x_i \neq x_j$ for $1 \leq i \neq j \leq s$. We will study the irreducible representations of $U_\chi(\mathfrak{g})$ under the above conditions.

Theorem 5.1. *If $l_\chi \geq 3$, $K_\chi(\lambda)$ is irreducible.*

Proof: By Remark (3.15) we have $(\lambda_i - \lambda_j)^p - (\lambda_i - \lambda_j) = \chi(\xi_i D_i - \xi_j D_j)^p$ for $1 \leq i \neq j \leq n$. So the following inequalities hold:

$$\lambda_1 - \lambda_k \neq 0 \ \text{ or } \ -1; \tag{17}$$
$$\lambda_1 - \lambda_n \neq 0 \ \text{ or } \ -1; \tag{18}$$
$$\vdots$$
$$\lambda_k - \lambda_n \neq 0 \ \text{ or } \ -1. \tag{19}$$

We consider the canonical map $\varphi : K_\chi(\lambda) \longrightarrow L_\chi(\lambda)$. First the equation below

$$\xi_n(\xi_1 D_1 - \xi_k D_k)\cdot D_n \otimes v_\lambda = (\lambda_1 - \lambda_k)v_\lambda$$

implies that $D_n \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$.

Then based on the inequality (17) and the repeated use of Lemma 4.7(2) we have that $D_{n-1} \wedge D_n \otimes v_\lambda$, $D_{n-2} \wedge D_{n-1} \wedge D_n \otimes v_\lambda$, $\cdots$, $D_{k+1} \wedge D_{k+2} \wedge \cdots \wedge D_n \otimes v_\lambda$ have non-zero images in $L_\chi(\lambda)$. Now the inequality (18) and Lemma 4.7(3) imply that $D_k \wedge \cdots \wedge D_n \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$. Finally, by the inequality (19) and the repeated use of Lemma 4.7(1) we have that $D_{k-1} \wedge D_k \wedge \cdots \wedge D_n \otimes v_\lambda$, $\cdots$, $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes v_\lambda$ have non-zero images in $L_\chi(\lambda)$. Consequently, by Lemma 4.6 the theorem follows. ■

6. The case when $l_\chi = 2$

Let $\mathfrak{g} = S(n)$ or $\bar{S}(n)$. We will study the irreducible representations of $U_\chi(\mathfrak{g})$ when $l_\chi = 2$. Assume that χ corresponds to M_χ, which is a standard Jordan matrix. Then based on Remark 4.2(2)(c) and Lemma 2.4 we can write

$$M_{\chi,s} = \mathrm{diag}\{\underbrace{x, \cdots, x}_{k}, \underbrace{y, \cdots, y}_{t}\},\ x \neq y \text{ and } k \geq t.$$

Proposition 6.1. *If $t \geq 2$, $K_\chi(\lambda)$ is irreducible.*

Proof: Since $t \geq 2$, $k+1 \neq n$. We have the following inequalities.

$$\lambda_1 - \lambda_{k+1} \neq 0 \text{ or } -1; \tag{20}$$

$$\lambda_1 - \lambda_n \neq 0 \text{ or } -1; \tag{21}$$

$$\lambda_k - \lambda_n \neq 0 \text{ or } -1. \tag{22}$$

Because the formula (20) holds, by repeated use of Lemma 4.7(2) we have that $D_{k+2} \wedge \cdots \wedge D_n \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$. Since $k \geq t \geq 2$, by Lemma 4.7(3) and the formula (21) we get that $D_{k+1} \wedge \cdots \wedge D_n \otimes v_\lambda, \cdots$, $D_2 \wedge D_3 \wedge \cdots \wedge D_n \otimes v_\lambda$ have non-zero images in $L_\chi(\lambda)$. Finally, since $k \geq 2$, by Lemma 4.7 (1) and the inequality (22) we know that $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$. Now the proposition follows due to Lemma 4.6. ■

We will focus on the case when $M_{\chi,s} = \mathrm{diag}(x, \cdots, x, y), x \neq y$ (i.e.$t = 1$) in the following.

Proposition 6.2. *Let $M_{\chi,s} = \mathrm{diag}(x, ..., x, y)$, $x \neq y$. If $\lambda \neq a\epsilon_1 + ... + a\epsilon_{n-1} + b\epsilon_n$, then $K_\chi(\lambda)$ is irreducible.*

Proof: Let $\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \cdots + \lambda_n\epsilon_n$. Since $\lambda \neq a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n$, there exist $1 \leq i \neq j \leq n-1$, such that $\lambda_i \neq \lambda_j$. By the following equation we know that $D_n \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$.

$$\xi_n(\xi_i D_i - \xi_j D_j) \cdot D_n \otimes v_\lambda = (\lambda_i - \lambda_j)v_\lambda \neq 0.$$

Because $x \neq y$, $\lambda_1 - \lambda_n \notin \mathbb{F}_p$. By repeated use of Lemma 4.7(3) we have that $D_{n-1} \wedge D_n \otimes v_\lambda$, $\cdots$, $D_2 \wedge D_3 \wedge \cdots \wedge D_n \otimes v_\lambda$ have non-zero images in $L_\chi(\lambda)$. In addition, since $\lambda_{n-1} - \lambda_n \notin \mathbb{F}_p$, $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$ by Lemma 4.7(1). Hence the proposition follows by Lemma 4.6. ■

Lemma 6.3. *Let $M_{\chi,s} = \mathrm{diag}(x, \cdots, x, y), x \neq y$. If M is a non-zero $U_\chi(\mathfrak{g})$-submodule of $K_\chi(\lambda)$, then $D_n \otimes v_\lambda \in M$.*

Proof: We apply Lemma 4.5 and get that $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes v_\lambda$ belongs to M. Since $x \neq y$, $\lambda_n - \lambda_{n-1} \notin \mathbb{F}_p$. Through repeated use of Lemma 4.8(1) we get that $D_2 \wedge D_3 \wedge \cdots \wedge D_n \otimes v_\lambda$, $D_3 \wedge D_4 \wedge \cdots \wedge D_n \otimes v_\lambda, \cdots$, $D_{n-1} \wedge D_n \otimes v_\lambda$ belong to M. Since $\lambda_n - \lambda_1 \notin \mathbb{F}_p$, Lemma 4.8(2) implies that $D_n \otimes v_\lambda$ belongs to M. ■

Proposition 6.4. *Let χ be a non-semisimple character and $l_\chi = 2$. Then $K_\chi(\lambda)$ is irreducible.*

Proof: Based on Propositions 6.1 and 6.2 we only need to check that when $M_{\chi,s} = \mathrm{diag}(x, \cdots, x, y)$, $x \neq y$ and $\lambda = a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n$, $K_\chi(\lambda)$ is simple. Let N be a non-zero $U_\chi(\mathfrak{g})$-submodule of $K_\chi(\lambda)$. Then by Lemma 6.3, $D_n \otimes v_\lambda$ belongs to N. Since χ is non-semisimple, by Propositions 3.6 and 3.17, there exist $1 \leq j_0 < i_0 \leq n-1$ such that $E_{i_0 j_0} \otimes v_\lambda$ is non-zero in $L^0_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)$. Now the result holds since $\xi_n \xi_{i_0} D_{j_0} \cdot D_n \otimes v_\lambda = \xi_{i_0} D_{j_0} \otimes v_\lambda \in L^0_\chi(\lambda) \cap N$. ■

From now on, we always use the notation $N^+ = \mathfrak{n}^+ \oplus \mathfrak{g}^{(1)}$.

Lemma 6.5. *Let χ be a semisimple character with $M_\chi = \mathrm{diag}(x, \cdots, x, y)$. If $\lambda = a(\epsilon_1 + \cdots + \epsilon_{n-1}) + b\epsilon_n$, $D_n \otimes v_\lambda$ is annihilated by N^+ in the Kac module $K_\chi(\lambda)$.*

Proof: We need to show the following equalities (23), (24) and (25).

$$\mathfrak{g}_1 \cdot (D_n \otimes v_\lambda) = 0; \tag{23}$$

$$\mathfrak{n}^+ \cdot (D_n \otimes v_\lambda) = 0; \tag{24}$$

$$\mathfrak{g}^{(2)} \cdot (D_n \otimes v_\lambda) = 0. \tag{25}$$

(24) can be checked directly by calculation. (25) is obvious since in $U_\chi(\mathfrak{g})$ we have $\mathfrak{g}^{(2)} D_n \subseteq \mathfrak{g}^{(1)} + D_n \mathfrak{g}^{(2)}$, and v_λ is annihilated by any element belonging to $\mathfrak{g}^{(1)}$. We prove (23) here. Let x be an element of $\mathfrak{g}_1$. We divide it into two cases.

Case I: When $x = \xi_i \xi_j D_k$ $(1 \leq i \neq j \neq k \leq n)$. This case can be checked directly through the formula (1), Propositions 3.5 and 3.16.

Case II: When $x = \xi_i(\xi_j D_j - \xi_k D_k)$ $(1 \leq i \neq j \neq k \leq n)$.

(1) If $1 \leq i, j, k \leq n-1$, then (23) can be checked by the formula (1).

(2) If $i = n$, then there exist $j, k \in \mathbb{Z}$ such that $1 \leq j \neq k \leq n-1$ and $x = \xi_n(\xi_j D_j - \xi_k D_k)$. Now (23) holds due to the following equation:

$$x \cdot (D_n \otimes v_\lambda) = (\lambda_j - \lambda_k) v_\lambda = 0.$$

(3) If $i \neq n$, $j = n$, then $x = \xi_i(\xi_n D_n - \xi_k D_k), k \neq n$. Using the commutative relation in $U_\chi(\mathfrak{g})$, the result in (23) can be proved through the following equality:

$$x \cdot (D_n \otimes v_\lambda) = -\xi_i D_n \otimes v_\lambda = 0.$$

When $\mathrm{ht}(\chi) \leq 1$, the reduced enveloping superalgebra $U_\chi(\mathfrak{g})$ is $\mathbb{Z}$-graded. By [8, Prop.3.5], every simple $U_\chi(\mathfrak{g})$-module is $\mathbb{Z}$-graded. Actually, the $\mathbb{Z}$-gradings of $K_\chi(\lambda)$ and $L_\chi(\lambda)$ are both determined by the degree of the standard generator v_λ. From now on, we always assume that the $\mathbb{Z}$-grading of v_λ is zero. Now let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a $\mathbb{Z}$-graded $U_\chi(\mathfrak{g})$-module. Define the *$\mathbb{Z}$-grading length* of M, which is denoted by $\mathbb{Z}l(M)$, to be

$$\mathbb{Z}l(M) = \sharp\{i \mid M_i \neq \emptyset\} - 1,$$

where $\sharp A$ means the number of elements belonging to a set A.

Assume that both A and B are sets. We now introduce a notation $A\backslash B$, which is defined by

$$A\backslash B = \{a \mid a \in A \text{ but } a \notin B\}.$$

Denote by $\Lambda_\chi^{a,b}$ a subset of Λ_χ whose elements are of the form $a\epsilon_1+\cdots+a\epsilon_{n-1}+b\epsilon_n$.

Proposition 6.6. *Let χ be a semisimple character with $M_\chi = \mathrm{diag}(x, x, \cdots, x, y)$, $x \neq y$. Then the following results hold:*

(1) *$K_\chi(\lambda)$ is simple if and only if $\lambda \notin \Lambda_\chi^{a,b}$.*
For $\lambda \in \Lambda_\chi \backslash \Lambda_\chi^{a,b}$, $\dim(L_\chi(\lambda) = 2^n \dim(L_\chi^0(\lambda))$.

(2) *When $\lambda = a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n$, we have the following exact sequence.*

$$0 \to L_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + (b-1)\epsilon_n) \to K_\chi(\lambda) \to L_\chi(\lambda) \to 0.$$

(3) *In the Grothendieck group of the category of $U_\chi(\mathfrak{g})$-modules, we have*

$$[L_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)] = \frac{1}{2}\sum_{i=0}^{p-1}(-1)^i[K_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + (b-i)\epsilon_n)].$$

In addition, $\dim(L_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)) = 2^{n-1}p^{n-1}$.

Proof. For item (1), based on Proposition 6.2 we need to show that when $\lambda = a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n$, $K_\chi(\lambda)$ has a non-zero proper $U_\chi(\mathfrak{g})$-submodule. Let M be the $U_\chi(\mathfrak{g})$-submodule of $K_\chi(\lambda)$ generated by $D_n \otimes v_\lambda$. By Lemma 6.5 we have $M = U_\chi(\mathfrak{g}_{-1} \oplus \mathfrak{b}^-) \cdot D_n \otimes v_\lambda$. Since the $\mathbb{Z}$-gradings of homogeneous elements in $U_\chi(\mathfrak{g}_{-1} \oplus \mathfrak{b}^-)$ are less then zero, the gradings of the homogeneous elements belonging to M are less then -1. Hence M is a non-zero proper $U_\chi(\mathfrak{g})$-submodule of $K_\chi(\lambda)$. Now by Lemma 4.4, Propositions 3.5(2) and 3.16(2), the result in (a) follows.

For (2), we first show the following claim.

Claim: $\mathbb{Z}l(L_\chi((a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)) = n-1$.

As a vector space, $K_\chi(\lambda) \cong \wedge(\mathfrak{g}_{-1}) \otimes L_\chi^0(\lambda)$. By the definition of $\mathbb{Z}$-grading length we have

$$\mathbb{Z}l(K_\chi(\lambda)) = n. \tag{26}$$

Because $K_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)$ is reducible, the maximal proper submodule $M_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)$ of $K_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)$ is non-zero. By Lemma 4.5, the set $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes L_\chi^0(\lambda)$ is contained in $M_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)$. So $\mathbb{Z}l(L_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n) \leq n-1$.

Now we prove $\mathbb{Z}l(L_\chi((a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)) \geq n-1$. Since $\lambda_n - \lambda_1 \notin \mathbb{F}_p$, by repeated use of Lemma 4.7(2) we have that v_λ, $D_{n-1} \otimes v_\lambda$, $D_{n-2} \wedge D_{n-1} \otimes v_\lambda, \cdots$, $D_2 \wedge \cdots \wedge D_{n-1} \otimes v_\lambda$ have non-zero images in $L_\chi(\lambda)$. Since $\lambda_n - \lambda_2 \notin \mathbb{F}_p$, we obtain that $D_1 \wedge D_2 \wedge \cdots \wedge D_{n-1} \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$ by Lemma 4.7(3). Thus the claim follows.

Now let W be a non-zero $U_\chi(\mathfrak{g})$-module, then by Proposition 6.2, formula (26) and the above claim we get

$$\mathbb{Z}l(W) \geq n-1. \tag{27}$$

Let $\lambda = a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n$, we will prove that $K_\chi(\lambda)$ has only two composition factors.

Assume that N is a non-zero submodule of $K_\chi(\lambda)$. By Lemma 4.5 we know that the set $K_\chi(\lambda)_{-n} = D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes L^0_\chi(\lambda)$ is contained in N. So

$$(K_\chi(\lambda)/N)_{-n} = \emptyset, \tag{28}$$

which implies that the $\mathbb{Z}$-grading length of the quotient module $K_\chi(\lambda)/N$ is less than $n-1$. Now combined with the formula (27) we have $\mathbb{Z}l(K_\chi(\lambda)/N) = n-1$. So the image of $L^0_\chi(\lambda)$ in $K_\chi(\lambda)/N$ must be non-zero. Obviously, $v_\lambda \in K_\chi(\lambda)/N$. By the irreducibility of $L^0_\chi(\lambda)$ we get $(K_\chi(\lambda)/N)_0 = L^0_\chi(\lambda) = (L_\chi(\lambda))_0$. Consequently, $K_\chi(\lambda)/N$ is a simple module and of course $K_\chi(\lambda)/N = L_\chi(\lambda)$. Hence, N is a maximal submodule of $K_\chi(\lambda)$. Moreover, by Lemma 4.4 , we get that N is the unique maximal submodule of $K_\chi(\lambda)$. So $K_\chi(\lambda)$ has only one non-zero proper submodule. Hence N must be the submodule generated by $D_n \otimes v_\lambda$ in the process of the proof for (1). By Lemma 6.5, $D_n \otimes v_\lambda$ is a maximal weight vector of weight $a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-1} + (b-1)\epsilon_n$. So $N = M_\chi(\lambda) \cong L_\chi(a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-1} + (b-1)\epsilon_n)$.

The proof of (3) follows from (2) directly. One only needs to notice that p is odd under our assumption that $p > 2$ and $(a-p)\epsilon_n = a\epsilon_n$. For the dimension of $L_\chi(\lambda)$ we can refer to Propositions 3.5 and 3.16. ∎

Theorem 6.7. *Let χ be a character with $l_\chi = 2$. Then we have the following results.*

(I) *When χ is not semisimple, $L_\chi(\lambda) = K_\chi(\lambda)$ and $\dim(L_\chi(\lambda) = 2^n \dim(L^0_\chi(\lambda))$.*

(II) *When χ is semisimple, i.e. $M_\chi = \mathrm{diag}\{\underbrace{x, \cdots, x}_{k}, \underbrace{y, \cdots, y}_{t}\}$, $x \neq y$ and $k \geq t$, the following results hold:*

(1) *If $t \geq 2$, $L_\chi(\lambda) = K_\chi(\lambda)$. So $\dim(L_\chi(\lambda) = 2^n \dim(L^0_\chi(\lambda))$.*

(2) *If $M_\chi = \mathrm{diag}(x, x, \cdots, x, y)$, $x \neq y$, then*

(a) *$K_\chi(\lambda)$ is simple if and only if $\lambda \in \Lambda_\chi \setminus \Lambda^{a,b}_\chi$. In this case, $\dim(L_\chi(\lambda)) = 2^n \dim(L^0_\chi(\lambda))$.*

(b) *When $\lambda = a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n$, there exists an exact sequence*

$$0 \to L_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + (b-1)\epsilon_n) \to K_\chi(\lambda) \to L_\chi(\lambda) \to 0.$$

(c) *When λ is of the form $a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n$, then $\dim(L_\chi(\lambda)) = 2^{n-1}p^{n-1}$.*

Proof. The results in (I),(II)(1) and (II)(2) can be obtained from Propositions 6.4, 6.1 and 6.6 respectively. ∎

7. The case when $l_\chi = 1$

Denote by $\mathfrak{g} = S(n)$ or $\bar{S}(n)$. In this section, we always assume that M_χ is the corresponding matrix of χ with $l_\chi = 1$, i.e. $M_\chi = M_{\chi,s} + M_{\chi,n}$, where

$$M_{\chi,s} = \mathrm{diag}\{x, x, \cdots, x\}, \ x \in \mathbf{k}, \text{ and}$$

$$M_{\chi,n} = \begin{pmatrix} J_1 & & & O \\ & J_2 & & \\ & & \ddots & \\ O & & & J_t \end{pmatrix}, \qquad J_i = \begin{pmatrix} 0 & 1 & & & O \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ O & & & & 0 \end{pmatrix}.$$

Let I be the subset of Π defined as $I=\{\alpha\in\Pi \mid \chi(x_{-\alpha})\neq 0\}$. Denote by $|I|$ the number of elements belonging to I.

7.1. The nilpotent case

In this subsection, we always assume that χ is a nilpotent character, i.e. $M_\chi = M_{\chi,n}$. One can check that χ has a standard Levi form (see Definition 3.8). Recall that the definition of W_I is given between Definition 3.8 and Lemma 3.10. Denote by Ω_χ the subset of Λ_χ described in the following:

$$\Omega_\chi=\{\Lambda\in\Lambda_\chi \mid \lambda=a\epsilon_1+\cdots+a\epsilon_{i-1}+b\epsilon_i+(a+1)\epsilon_{i+1}+\cdots+(a+1)\epsilon_n, 1\leq i\leq n\}.$$

Proposition 7.1. *If $\lambda\notin\Omega_\chi$, $K_\chi(\lambda)$ is irreducible.*

Proof: Set $\lambda^{\min}=\min\{j \mid \lambda_j\neq\lambda_1\}$.
Since $\lambda\notin\Omega_\chi$, $\lambda\neq a\epsilon_1+a\epsilon_2+\cdots+a\epsilon_n$. Hence, $\lambda^{\min}$ is well defined. In addition, $\lambda\neq a\epsilon_1+\cdots+a\epsilon_{n-1}+b\epsilon_n$, so $2\leq\lambda^{\min}\leq n-1$.

Case I: When $\lambda^{\min}=n-1$, then $\lambda=a\epsilon_1+\cdots+a\epsilon_{n-2}+b\epsilon_{n-1}+c\epsilon_n$ with $c\neq a+1$ and $b\neq a$.

Because the following equality holds,

$$\xi_n(\xi_{n-1}D_{n-1}-\xi_1D_1)\cdot D_n\otimes v_\lambda=(b-a)v_\lambda,$$

$D_n\otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$. Let $k=1, l=n$. By repeated use of Lemma 4.7(3) we get that $D_{n-1}\wedge D_n\otimes v_\lambda$, $D_{n-2}\wedge D_{n-1}\wedge D_n\otimes v_\lambda,\cdots$, $D_2\wedge D_3\wedge\cdots\wedge D_n\otimes v_\lambda$ have non-zero images in $L_\chi(\lambda)$.

- If $n\geq 4$, then
$$\begin{aligned}&\xi_1(\xi_{n-1}D_{n-1}-\xi_{n-2}D_{n-2})\cdot D_1\wedge D_2\wedge\cdots\wedge D_n\otimes v_\lambda\\&=(b-a)D_2\wedge D_3\wedge\cdots\wedge D_n\otimes v_\lambda.\end{aligned}\tag{29}$$

- If $n=3$, $\lambda=a\epsilon_1+b\epsilon_2+c\epsilon_3$. Now $b\neq c$ since $\lambda\notin\Omega_\chi$. The following equality holds:
$$\begin{aligned}&\xi_1(\xi_2D_2-\xi_3D_3)\cdot D_1\wedge D_2\wedge D_3\otimes v_\lambda\\&=(b-c)D_2\wedge D_3\otimes v_\lambda.\end{aligned}\tag{30}$$

Both formulas (29) and (30) imply that $D_1\wedge D_2\wedge\cdots\wedge D_n\otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$.

Case II: When $2\leq\lambda^{\min}\leq n-2$, $\lambda=a\epsilon_1+\cdots+a\epsilon_{i-1}+b\epsilon_i+\lambda_{i+1}\epsilon_{i+1}+\cdots+\lambda_n\epsilon_n$ with $b\neq a$, i.e. $\lambda_i-\lambda_1\neq 0$.

Under this condition, the following equality holds.

$$\xi_n(\xi_iD_i-\xi_1D_1)\cdot D_n\otimes v_\lambda=(\lambda_i-\lambda_1)v_\lambda.$$

So $D_n\otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$. Let $k=i, l=1$. Then by repeated use of Lemma 4.7(2) we get that $D_{n-1}\wedge D_n\otimes v_\lambda$, $D_{n-2}\wedge D_{n-1}\wedge D_n\otimes v_\lambda,\cdots$, $D_{i+1}\wedge D_{i+2}\wedge\cdots\wedge D_n\otimes v_\lambda$ have non-zero images in $L_\chi(\lambda)$. Since $\lambda\notin\Omega_\chi$, we have the following two cases.

(1) There exist $i+1 \leq k, l \leq n$, such that $\lambda_k \neq \lambda_l$.

(2) $\lambda_{i+1} = \lambda_{i+2} = \cdots = \lambda_n \neq a+1$.

- For case (1), we use Lemma 4.7(1) repeatedly and get in consequence that $D_i \wedge D_{i+1} \wedge \cdots \wedge D_n \otimes v_\lambda$, $D_{i-1} \wedge D_i \wedge \cdots \wedge D_n \otimes v_\lambda$, $\cdots$, $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes v_\lambda$ have non-zero images in $L_\chi(\lambda)$.
- For case (2), $\lambda = a\epsilon_1 + \cdots + a\epsilon_{i-1} + b\epsilon_i + c\epsilon_{i+1} + \cdots + c\epsilon_n$ with $a \neq b$, and $c \neq a+1$. So $\lambda_1 - \lambda_n \neq -1$ Set $k=1, l=n$, by repeated use of Lemma 4.7(3) we get that $D_i \wedge D_{i+1} \wedge \cdots \wedge D_n \otimes v_\lambda$, $D_{i-1} \wedge D_i \wedge \cdots \wedge D_n \otimes v_\lambda$, $\cdots$, $D_2 \wedge D_3 \wedge \cdots \wedge D_n \otimes v_\lambda$ have non-zero images in $L_\chi(\lambda)$.
  - If $i \geq 3$, set $k=2, l=i$. Now $\lambda_k - \lambda_l \neq 0$. By Lemma 4.7(1) we know that $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$.
  - If $i=2$, then $\lambda = a\epsilon_1 + b\epsilon_2 + c(\epsilon_3 + \cdots + \epsilon_n)$, $b \neq a$. Because $\lambda \notin \Omega_\chi$, $b \neq c$. Set $k=2, l=3, \lambda_k - \lambda_l \neq 0$. By Lemma 4.7(1) we know that $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$.

Now the proposition follows from Lemma 4.6. ■

Proposition 7.2. *Let χ be a nilpotent character. If $|I| \geq 2$, $K_\chi(\lambda)$ is irreducible.*

Proof: Because M_χ is a standard Jordan matrix and $|I| \geq 2$, there exists a subgroup of W_I generated by s_{n-1} and $s_k, k=n-2$ or $k=n-3$ (when $n=3$, it must be $\langle s_2, s_1 \rangle = W_I$). Considering Proposition 7.1, we only need to check the case when $\lambda \in \Omega_\chi$.

Claim: If $\lambda \in \Omega_\chi$, there exists an element s belonging to W_I, such that $s \cdot \lambda \notin \Omega_\chi$. We will prove the claim below in Case (1) and Case (2).

Case 1. When $\lambda \neq a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_n$. Then $\lambda = a\epsilon_1 + b\epsilon_2 + b\epsilon_3 + \cdots + b\epsilon_n$, $b \neq a$ or there exists $2 \leq i \leq n$ such that $b \neq a$ and

$$\lambda = a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{i-1} + b\epsilon_i + (a+1)\epsilon_{i+1} + \cdots + (a+1)\epsilon_n.$$

If $\lambda = a\epsilon_1 + b\epsilon_2 + b\epsilon_3 + \cdots + b\epsilon_n, b \neq a$, then

$$s_{n-1} \cdot \lambda = a\epsilon_1 + b\epsilon_2 + \cdots + b\epsilon_{n-2} + (b-1)\epsilon_{n-1} + (b+1)\epsilon_n.$$

Since $b \neq a$, $s_{n-1} \cdot \lambda \notin \Omega_\chi$.

If $\lambda = a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{i-1} + b\epsilon_i + (a+1)\epsilon_{i+1} + \cdots + (a+1)\epsilon_n$, $b \neq a$, $2 \leq i \leq n$, then we divide the situationin to the following cases.

(a) If $i \leq n-2$, then
$s_{n-1} \cdot \lambda = a(\epsilon_1 + \ldots + \epsilon_{i-1}) + b\epsilon_i + (a+1)\epsilon_{i+1} + \ldots + (a+1)\epsilon_{n-2} + a\epsilon_{n-1} + (a+2)\epsilon_n \notin \Omega_\chi$.

(b) If $i = n-1$, then $\lambda = a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-2} + b\epsilon_{n-1} + (a+1)\epsilon_n$.

— When $s_{n-2} \in W_I$.

$s_{n-2} \cdot s_{n-1} \cdot (a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-2} + b\epsilon_{n-1} + (a+1)\epsilon_n)$
$= s_{n-2} \cdot (a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-3} + a\epsilon_{n-2} + a\epsilon_{n-1} + (b+1)\epsilon_n)$
$= a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-3} + (a-1)\epsilon_{n-2} + (a+1)\epsilon_{n-1} + (b+1)\epsilon_n$

Since $a \neq b, a+1 \neq b+1$. So $(s_{n-2}s_{n-1}) \cdot \lambda \notin \Omega_\chi$.

— When $s_{n-3} \in W_I$.

$s_{n-1} \cdot s_{n-3} \cdot (a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-2} + b\epsilon_{n-1} + (a+1)\epsilon_n)$
$= s_{n-1} \cdot (a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-4} + (a-1)\epsilon_{n-3} + (a+1)\epsilon_{n-2} + b\epsilon_{n-1} + (a+1)\epsilon_n)$
$= a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-4} + (a-1)\epsilon_{n-3} + (a+1)\epsilon_{n-2} + a\epsilon_{n-1} + (b+1)\epsilon_n \notin \Omega_\chi$.

So $(s_{n-1}s_{n-3}) \cdot \lambda \notin \Omega_\chi$.

(c) If $i = n$. $\lambda = a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-1} + b\epsilon_n, b \neq a$.

— When $s_{n-2} \in W_I$.

$s_{n-1} \cdot s_{n-2} \cdot (a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-1} + b\epsilon_n)$
$= s_{n-1} \cdot (a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-3} + (a-1)\epsilon_{n-2} + (a+1)\epsilon_{n-1} + b\epsilon_n)$
$= a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-3} + (a-1)\epsilon_{n-2} + (b-1)\epsilon_{n-1} + (a+2)\epsilon_n \notin \Omega_\chi$.

So $(s_{n-1}s_{n-2}) \cdot \lambda \notin \Omega_\chi$.

— When $s_{n-3} \in W_I$

$s_{n-3} \cdot (a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-1} + b\epsilon_n)$
$= a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-4} + (a-1)\epsilon_{n-3} + (a+1)\epsilon_{n-2} + a\epsilon_{n-1} + b\epsilon_n \notin \Omega_\chi$.

Case 2. When $\lambda = a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_n$.

$$s_{n-1} \cdot \lambda = a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-2} + (a-1)\epsilon_{n-1} + (a+1)\epsilon_n.$$

which is the situation in Case(1)(b). So there exists an element $s \in W_I$ such that $s \cdot \lambda \notin \Omega_\chi$.

Now combining the Claim with Lemma 4.4, Propositions 3.10 and 7.1, the proposition follows. ■

Lemma 7.3. *When $M_\chi = E_{n-1,n}$, $n \geq 4$. Let*

$$\lambda = a(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{i-1}) + b\epsilon_i + (a+1)\epsilon_{i+1} + \cdots + (a+1)\epsilon_n$$

with $b \neq a$. If $i \leq n-2$, $K_\chi(\lambda)$ is irreducible.

Proof: Because $M_\chi = E_{n-1,n}$, $W_I = \{Id, s_{n-1}\}$. Through calculation, the following equality holds:

$$s_{n-1} \cdot \lambda = a(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{i-1}) + b\epsilon_i + (a+1)\epsilon_{i+1} + \cdots + (a+1)\epsilon_{n-2} + a\epsilon_{n-1} + (a+2)\epsilon_n.$$

Since $b \neq a$, $s_{n-1} \cdot \lambda \notin \Omega_\chi$. So $K_\chi(s_{n-1} \cdot \lambda)$ is irreducible. By Lemma 4.4, Propositions 3.10 and 7.1, $K_\chi(\lambda) \cong K_\chi(s_{n-1} \cdot \lambda)$. Thus $K_\chi(\lambda)$ is irreducible. ■

Remark 7.4. Due to Lemma 7.3, Propositions 7.1 and 7.2, we only need to consider the case when $M_\chi = E_{n-1,n}$ with the weights belonging to the following two cases.

(1) $\lambda = a(\epsilon_1 + \cdots + \epsilon_{n-2}) + b\epsilon_{n-1} + (a+1)\epsilon_n, b \neq a$.

(2) $\lambda = a(\epsilon_1 + \cdots + \epsilon_{n-1}) + b\epsilon_n, b \in \mathbb{F}_p$.

Remark 7.5. (1) When $n = 3$, $\lambda \in \Omega_\chi$ if and only if λ is of the three forms

$$\lambda = a\epsilon_1 + b\epsilon_2 + b\epsilon_3. \tag{31}$$

$$\lambda = a\epsilon_1 + b\epsilon_2 + (a+1)\epsilon_3. \tag{32}$$

$$\lambda = a\epsilon_1 + a\epsilon_2 + c\epsilon_3. \tag{33}$$

For the case (31), if $b = a$, then λ is of the case in Remark 7.4(2). If $b = a+1$, then $\lambda = a\epsilon_1 + (a+1)\epsilon_2 + (a+1)\epsilon_3$, which is the case in Remark 7.4(1).

If $b \neq a$ and $b \neq a+1$, then $s_2 \cdot \lambda = a\epsilon_1 + (b-1)\epsilon_2 + (b+1)\epsilon_3 \notin \Omega_\chi$, so $K_\chi(\lambda)$ is irreducible. Cases (32) and (33) are special cases of Lemmas 7.4(1) and 7.4(2) respectively.

(2) Combining Proposition 3.10, Lemma 4.4 and the following equation

$$\begin{aligned} & s_{n-1} \cdot (a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-2} + b\epsilon_{n-1} + (a+1)\epsilon_n) \\ = \; & a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-2} + a\epsilon_{n-1} + (b+1)\epsilon_n \end{aligned}$$

we can see that we only need to consider one of the two cases in Remark 7.4.

Lemma 7.6. *Let $M_\chi = E_{n-1,n}$. If $\lambda = a(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1}) + b\epsilon_n$, $D_n \otimes v_\lambda$ can be annihilated by N^+ in the Kac module $K_\chi(\lambda)$.*

Proof: By Propositions 3.11 and 3.18 we have

$$\begin{aligned} & L_\chi^0(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n) \\ & = \mathbf{k}\text{-span}\left\{ E_{n,n-1}^{k_{n-1}} E_{n,n-2}^{k_{n-2}} \cdots E_{n,1}^{k_1} \otimes v_\lambda \mid 0 \leq k_i \leq p-1, 1 \leq i \leq n-1 \right\}. \end{aligned}$$

The same argument as Lemma 6.5 will complete this lemma. ■

Now let N be the submodule of $K_\chi(\lambda)$ generated by $D_n \otimes v_\lambda$. By Lemma 7.6 we have $N = U_\chi(\mathfrak{g}_{-1} \oplus \mathfrak{b}^-) \cdot D_n \otimes v_\lambda$. Obviously, the gradings of homogeneous elements belonging to N are less then -1. Hence N is a non-zero proper $U_\chi(\mathfrak{g})$-submodule of $K_\chi(\lambda)$. So $K_\chi(\lambda)$ is not simple. By Lemma 4.5, $D_1 \wedge D_2 \wedge \cdots \wedge D_n \otimes L_\chi^0(\lambda) \subseteq N$. So $\mathbb{Z}l(L_\chi(\lambda)) \leq n-1$.

Lemma 7.7. *Let $M_\chi = E_{n-1,n}$. If $\lambda = a(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1}) + b\epsilon_n$, then we have $\mathbb{Z}l(L_\chi(\lambda)) = n-1$.*

Proof: Based on the above discussion, we only need to show that $\mathbb{Z}l(L_\chi(\lambda)) \geq n-1$.

(1) When $b \neq a$ and $b \neq a-1$, then $\lambda_n - \lambda_1 \neq 0$. By repeated use of Lemma 4.7(2) we get that $D_{n-1} \otimes v_\lambda$, $D_{n-2} \wedge D_{n-1} \otimes v_\lambda, \cdots$, $D_2 \wedge \cdots \wedge D_{n-1} \otimes v_\lambda$ have non-zero images in $L_\chi(\lambda)$. Since $\lambda_n - \lambda_{n-1} \neq -1$, by Lemma 4.7(3) we have that $D_1 \wedge D_2 \wedge \cdots \wedge D_{n-1} \otimes v_\lambda$ has non-zero image in $L_\chi(\lambda)$. Thus the result holds.

(2) When $b = a$ or $b = a-1$. Set

$$\lambda' = s_{n-1} \cdot \lambda = a(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-2}) + (b-1)\epsilon_{n-1} + (a+1)\epsilon_n.$$

Then $\lambda'_{n-1} - \lambda'_1 = b-a-1 = -1$ or -2. By repeated use of Lemma 4.7(2) we get that $D_n \otimes v_{\lambda'}$, $D_{n-2} \wedge D_n \otimes v_{\lambda'}, \cdots$, $D_2 \wedge \cdots \wedge D_{n-2} \wedge D_n \otimes v_{\lambda'}$ have non-zero images in $L_\chi(\lambda')$. Because $a-b+1 = 1$ or 2, the calculation below implies that $D_1 \wedge D_2 \wedge \cdots \wedge D_{n-2} \wedge D_n \otimes v_{\lambda'}$ has non-zero image in $L_\chi(\lambda')$:

$$\begin{aligned} & \xi_1(\xi_n D_n - \xi_{n-1} D_{n-1}) \cdot D_1 \wedge D_2 \wedge \cdots \wedge D_{n-2} \wedge D_n \otimes v_{\lambda'} \\ & \qquad = (a-b+1) D_2 \wedge \cdots \wedge D_{n-2} \wedge D_n \otimes v_{\lambda'}. \end{aligned}$$

Now the lemma follows from Proposition 3.10 and Lemma 4.4. ■

Denote by $\Lambda_\chi^{\overline{a,b}}$ a subset of Λ_χ consisting of elements of the form $a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n$ or $a\epsilon_1 + \cdots + a\epsilon_{n-2} + b\epsilon_{n-1} + (a+1)\epsilon_n$.

Theorem 7.8. *Let χ be a nilpotent character, i.e. $M_\chi = M_{\chi,n}$. Then the irreducible representations of $U_\chi(\mathfrak{g})$ can be described below.*

(I) *When $|I| = 1$, i.e. $M_\chi = E_{n-1,n}$, the following results hold.*

(1) *$K_\chi(\lambda)$ is reducible if and only if $\lambda \in \Lambda_\chi^{\overline{a,b}}$. In addition,*

$$K_\chi(a\epsilon_1 + ... + a\epsilon_{n-1} + b\epsilon_n) \cong K_\chi(a\epsilon_1 + ... + a\epsilon_{n-2} + (b-1)\epsilon_{n-1} + (a+1)\epsilon_n).$$

(2) *If $\lambda \in \Lambda_\chi \backslash \Lambda_\chi^{\overline{a,b}}$, we have* $\dim(L_\chi(\lambda)) = 2^n \dim(L_\chi^0(\lambda))$.

(3) *If $\lambda = a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n$, then*

(a) *There exists the following exact sequence*

$$0 \to L_\chi(\lambda - \epsilon_n) \to K_\chi(\lambda) \to L_\chi(\lambda) \to 0.$$

(b) *In the Grothendieck group of the category of $U_\chi(\mathfrak{g})$-modules, we have*

$$[L_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n)] = \frac{1}{2}\sum_{i=0}^{p-1}(-1)^i[K_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + (b-i)\epsilon_n)].$$

(c) *If $\lambda \in \Lambda_\chi^{\overline{a,b}}$,* $\dim(L_\chi(\lambda)) = 2^{n-1}p^{n-1}$.

(II) *When $|I| \geq 2$, $K_\chi(\lambda)$ is irreducible. So* $\dim(L_\chi(\lambda)) = 2^n \dim(L_\chi^0(\lambda))$.

Proof: We first prove the results in (I)(1). Under this condition, $W_I = \{Id, s_{n-1}\}$. Because $s_{n-1} \cdot (a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n) = a\epsilon_1 + \cdots + a\epsilon_{n-2} + (b-1)\epsilon_{n-1} + (a+1)\epsilon_n$, we have $K_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n) \cong K_\chi(a\epsilon_1 + \cdots + a\epsilon_{n-2} + (b-1)\epsilon_{n-1} + (a+1)\epsilon_n)$ by Proposition 3.10 and Lemma 4.4. The reader can refer to Remark 7.5(2), Lemmas 7.3 and 7.7 for the remaining result in (I)(1). The result in (I)(2) follows from the fact that if $\lambda \in \Lambda_\chi \backslash \Lambda_\chi^{\overline{a,b}}$, then $K_\chi(\lambda)$ is irreducible, which is stated in (I)(1). For (I)(3), one can get it by the same argument as Proposition 6.6. The result in (II) has been proved in Proposition 7.2. ■

Theorem 7.9. *Let χ be a nilpotent character. If $|I| = 0$, then $\chi = 0$, which is a restricted character. The representations of $U_0(\mathfrak{g})$ have been studied in* [14].

7.2. The non-nilpotent case

In this subsection, we consider the case when $l_\chi = 1$ and M_χ is not nilpotent. i.e. $M_\chi = M_{\chi,s} + M_{\chi,n}$ with $M_{\chi,s} = \text{diag}\{x, x, \cdots, x\}, x \neq 0$. We will establish a Morita-equivalence between the categories of $U_\chi(\mathfrak{g})$-modules and $U_{\chi_n}(\mathfrak{g})$-modules to solve the problems.

The reader should notice that under this condition, when $\mathfrak{g} = S(n)$, $\chi|_{\bar{\mathfrak{h}}} = 0$, which is of the nilpotent case. We have studied it in subsection 7.1. So we always assume $\mathfrak{g} = \bar{S}(n)$ throughout this subsection.

Let α be a fixed element satisfying $\alpha^p - \alpha = x^p$ and $\sigma = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$. Then $\alpha\sigma$ is an element of Λ_{χ_s}. Because the characteristic p is odd, $-\alpha\sigma$ is an element of $\Lambda_{-\chi_s}$. Now we introduce two functors between the categories of $U_{\chi_n}(\mathfrak{g})$-modules and $U_\chi(\mathfrak{g})$-modules:

$$\begin{array}{rcl} \mathcal{F}_\alpha : \{U_{\chi_n}(\mathfrak{g})\text{-modules}\} & \to & \{U_\chi(\mathfrak{g})\text{-modules}\} \\ M & \mapsto & M \otimes_{\mathbf{k}} L_{\chi_s}(\alpha\sigma) \end{array}$$

$$\begin{array}{rcl} \mathcal{G}_{-\alpha} : \{U_\chi(\mathfrak{g})\text{-modules}\} & \to & \{U_{\chi_n}(\mathfrak{g})\text{-modules}\} \\ N & \mapsto & N \otimes_{\mathbf{k}} L_{-\chi_s}(-\alpha\sigma) \end{array}$$

We can check that $\mathcal{F}_\alpha \circ \mathcal{G}_{-\alpha} = \mathrm{Id}|_{U_\chi(\mathfrak{g})\text{-modules}}$ and $\mathcal{G}_{-\alpha} \circ \mathcal{F}_\alpha = \mathrm{Id}|_{U_{\chi_n}(\mathfrak{g})\text{-modules}}$. So the categories of $U_\chi(\mathfrak{g})$-modules and $U_{\chi_n}(\mathfrak{g})$-modules are Morita-equivalent.

Proposition 7.10. *Let λ be an element of Λ_χ such that*

$$\lambda = \alpha\sigma + i_1\epsilon_1 + i_2\epsilon_2 + \cdots + i_n\epsilon_n .$$

Then we have the following isomorphisms.

(1) $\mathcal{F}_\alpha(L_{\chi_n}(i_1\epsilon_1 + i_2\epsilon_2 + \cdots + i_n\epsilon_n)) \cong L_\chi(\lambda)$.

(2) $\mathcal{F}_\alpha(K_{\chi_n}(i_1\epsilon_1 + i_2\epsilon_2 + \cdots + i_n\epsilon_n)) \cong K_\chi(\lambda)$.

Proof: (1) Assume that v_0 is the highest weight vector of $L_{\chi_n}(i_1\epsilon_1+i_2\epsilon_2+\cdots+i_n\epsilon_n)$ and $v_{\alpha\sigma}$ is the highest weight vector of $L_{\chi_s}(\alpha\sigma)$. The reader can check that $v_0 \otimes v_{\alpha\sigma}$ is a highest weight vector of $\mathcal{F}_\alpha(L_{\chi_n}(i_1\epsilon_1 + i_2\epsilon_2 + \cdots + i_n\epsilon_n))$. Since $\mathcal{F}_\alpha$ and $\mathcal{G}_{-\alpha}$ are Morita-equivalent they send simple objects to simple objects.

Hence $\mathcal{F}_\alpha(L_{\chi_n}(i_1\epsilon_1 + i_2\epsilon_2 + \cdots + i_n\epsilon_n))$ is simple.

Consequently we obtain $\mathcal{F}_\alpha(L_{\chi_n}(i_1\epsilon_1 + i_2\epsilon_2 + \cdots + i_n\epsilon_n)) \cong L_\chi(\lambda)$.

The result of (2) can be proved through the following equalities. For the third isomorphism, the reader can refer to [7, Lemma 5.1].

$$\begin{aligned}
&\mathcal{F}_\alpha(K_{\chi_n}(i_1\epsilon_1 + i_2\epsilon_2 + \cdots + i_n\epsilon_n)) \\
&= K_{\chi_n}(i_1\epsilon_1 + i_2\epsilon_2 + \cdots + i_n\epsilon_n) \otimes_{\mathbf{k}} L_{\chi_s}(\alpha\sigma) \\
&= \big(U_{\chi_n}(\mathfrak{g}) \otimes_{U_{\chi_n}(\mathfrak{g}^+)} L^0_{\chi_n}(i_1\epsilon_1 + i_2\epsilon_2 + \cdots + i_n\epsilon_n)\big) \otimes_{\mathbf{k}} L_{\chi_s}(\alpha\sigma) \\
&\cong U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}^+)} \big(L^0_{\chi_n}(i_1\epsilon_1 + i_2\epsilon_2 + \cdots + i_n\epsilon_n) \otimes_{\mathbf{k}} L_{\chi_s}(\alpha\sigma)\big)\,|_{U_\chi(\mathfrak{g}^+)} \\
&\cong U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}^+)} L^0_\chi(\lambda) = K_\chi(\lambda).
\end{aligned}$$

Now the proposition follows. ■

Theorem 7.11. *Let χ be a character with the property that $M_\chi = M_{\chi,s} + M_{\chi,n}$ with $M_{\chi,s} = \mathrm{diag}\{x, x, \cdots, x\}$, $x \neq 0$. Then the conclusions on irreducible representations of $U_\chi(\mathfrak{g})$ are as follows:*

(I) *If $|I| \geq 2$, $K_\chi(\lambda)$ is irreducible.*

(II) *If $|I| = 1$, the following results hold.*

(1) *$K_\chi(\lambda)$ is reducible if and only if $\lambda \in \Lambda_\chi^{\overline{a,b}}$. In addition,* $K_\chi(a\epsilon_1 + ... + a\epsilon_{n-1} + b\epsilon_n) \cong K_\chi(a\epsilon_1 + ... + a\epsilon_{n-2} + (b-1)\epsilon_{n-1} + (a+1)\epsilon_n)$.

(2) *For $\lambda \in \Lambda_\chi \backslash \Lambda_\chi^{\overline{a,b}}$, we have* $\dim(L_\chi(\lambda)) = 2^n \dim(L^0_\chi(\lambda))$.

(3) *When $\lambda = a\epsilon_1 + \cdots + a\epsilon_{n-1} + b\epsilon_n$, then*

(a) *There exists the following exact sequence*

$$0 \to L_\chi(\lambda - \epsilon_n) \to K_\chi(\lambda) \to L_\chi(\lambda) \to 0.$$

(b) *In the Grothendieck group of the category of $U_\chi(\mathfrak{g})$-modules, we have*

$$[L_\chi(a\epsilon_1+\cdots+a\epsilon_{n-1}+b\epsilon_n)] = \frac{1}{2}\sum_{i=0}^{p-1}(-1)^i[K_\chi(a\epsilon_1+\cdots+a\epsilon_{n-1}+(b-i)\epsilon_n)].$$

(c) *If $\lambda \in \Lambda_\chi^{\overline{a,b}}$,* $\dim(L_\chi(\lambda)) = 2^{n-1}p^{n-1}$.

(III) *If* $|I|=0$*, i.e.,* $M_\chi = \mathrm{diag}\{x, x, \cdots, x\}, x \neq 0$*, then*

(1) *When* $\lambda = a\sigma$*, the following exact sequences exist:*

$$0 \to J_\chi(\lambda) \to K_\chi(\lambda) \to L_\chi(a\sigma) \to 0,$$
$$0 \to L_\chi((a-1)\sigma) \to J_\chi(\lambda) \to L_\chi(a\sigma - \epsilon_n) \to 0.$$

(2) *When* $\lambda = a\sigma + b\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n, b \neq 1$*, the cases are listed below.*

(a) *If* $b = 2$*, i.e.* $\lambda = (a+1)\sigma + \epsilon_1$*, then*

$$0 \to J_\chi(\lambda) \to K_\chi(\lambda) \to L_\chi(\lambda) \to 0,$$
$$0 \to L_\chi((a+1)\sigma - \epsilon_n) \to J_\chi(\lambda) \to L_\chi((a+1)\sigma) \to 0.$$

(b) *If* $b \in \{3, 4, \cdots, p-1\}$*, then*

$$0 \to L_\chi(a\sigma + (b-1)\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n) \to K_\chi(\lambda) \to L_\chi(\lambda) \to 0.$$

(3) *When* $\lambda = a\sigma + b\epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n, i \in \{2, 3, \cdots, n-1\}$*, we have the following exact sequences:*

(a) *If* $b = 1$*, then*

$$0 \to J_\chi(\lambda) \to K_\chi(\lambda) \to L_\chi(\lambda) \to 0,$$
$$0 \to L_\chi(a\sigma + \epsilon_{i+1} + \cdots + \epsilon_n) \to J_\chi(\lambda) \to L_\chi(a\sigma) \to 0.$$

(b) *If* $b \in \{2, 3, \cdots, p-1\}$*, then*

$$0 \to L_\chi(a\sigma + (b-1)\epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n) \to K_\chi(\lambda) \to L_\chi(\lambda) \to 0.$$

(4) *When* $\lambda = a\sigma + b\epsilon_n$*, the cases are listed below.*

(a) *If* $b = 1$*, then*

$$0 \to J_\chi(\lambda) \to K_\chi(\lambda) \to L_\chi(\lambda) \to 0,$$
$$0 \to L_\chi(a\sigma - \epsilon_{n-1} + \epsilon_n) \to J_\chi(\lambda) \to L_\chi(a\sigma) \to 0.$$

(b) *If* $b \in \{2, 3, \cdots, p-2\}$*, then*

$$0 \to L_\chi(a\sigma + (b-1)\epsilon_n) \to K_\chi(\lambda) \to L_\chi(\lambda) \to 0.$$

(c) *If* $b = p-1$*, then*

$$0 \to J_\chi(\lambda) \to K_\chi(\lambda) \to L_\chi(\lambda) \to 0,$$
$$0 \to L_\chi(a\sigma + (p-2)\epsilon_n) \to J_\chi(\lambda) \to L_\chi((a-1)\sigma) \to 0.$$

(5) *Otherwise,* $L_\chi(\lambda) = K_\chi(\lambda)$.

Proof: By Proposition 7.10, the categories of $U_\chi(\mathfrak{g})$-modules and $U_{\chi_n}(\mathfrak{g})$-modules are Morita-equivalent. So the results in (I), (II),(III) can be obtained through Theorems 7.8(II),7.8(I) and 7.9 respectively. ■

8. The indecomposable projective module

Let $\mathfrak{g} = S(n)$ or $\bar{S}(n)$. In this section we always assume that χ is an element of $\mathfrak{g}^*$ corresponding to M_χ. Recall that $\sigma = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$ and $\sigma = 0$ when $\mathfrak{g} = S(n)$. Denote by $P^0_\chi(\lambda)$ (resp. $P_\chi(\lambda)$) the projective cover of $L^0_\chi(\lambda)$ (resp. $L_\chi(\lambda)$) in the category of $U_\chi(\mathfrak{g}_0)$-modules (resp. $U_\chi(\mathfrak{g})$-modules). Denote by $\nabla_\chi(N)$ the induced module $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}^-)} N$ where N is a $U_\chi(\mathfrak{g}_0)$-module with trivial $\mathfrak{g}_{-1}$-action. By the same argument as [7, Lemma 4.10] we have the following proposition.

Proposition 8.1. (BGG-reciprocity) *The following reciprocity holds*

$$\big(P_\chi(\lambda) : \nabla_\chi(P^0_\chi(\mu))\big) = [K_\chi(\mu + \sigma) : L_\chi(\lambda)].$$

Theorem 8.2. *Let χ be a character of $\mathfrak{g}^*$ such that $\mathrm{ht}(\chi) = -1$ or 1. Then the following results hold.*

(I) *When $l_\chi \geq 3$, we have $P_\chi(\lambda) = \nabla_\chi(P^0_\chi(\lambda - \sigma))$.*

(II) *When $l_\chi = 2$, we have the following results.*

  (1) *If χ is not semisimple, $P_\chi(\lambda) = \nabla_\chi(P^0_\chi(\lambda - \sigma))$.*

  (2) *If χ is semisimple with $M_\chi = \mathrm{diag}\{\underbrace{x, \cdots, x}_{k}, \underbrace{y, \cdots, y}_{t}\}$, $k \geq t \geq 2$, then $P_\chi(\lambda) = \nabla_\chi(P^0_\chi(\lambda - \sigma))$.*

  (3) *If χ is semisimple with $M_\chi = \mathrm{diag}\{x, \cdots, x, y\}$, then*

    (a) *If $\lambda = a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-1} + b\epsilon_n$, then $[P_\chi(\lambda) : \nabla_\chi(P^0_\chi(\mu))] \neq 0$ if and only if*

$$\mu = \lambda - \sigma \text{ or } \mu = \lambda - \sigma + \epsilon_n.$$

    (b) *Otherwise, $P_\chi(\lambda) = \nabla_\chi(P^0_\chi(\lambda - \sigma))$.*

(III) *When $l_\chi = 1$, i.e. $M_\chi = kE_n + A$, where A is a nilpotent standard Jordan matrix. Denote by I_A the set of simle roots α such that $\chi((\mathfrak{g}_0)_{-\alpha}) \neq 0$. We have the following results.*

  (1) *When $|I_A| \geq 2$, $P_\chi(\lambda) = \nabla_\chi(P^0_\chi(\lambda - \sigma))$.*

  (2) *When $|I_A| = 1$,*

    (a) *If $\lambda = a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-1} + b\epsilon_n$, then $[P_\chi(\lambda) : \nabla_\chi(P^0_\chi(\mu))] \neq 0$ if and only if $\mu = \lambda - \sigma$ or $\mu = \lambda - \sigma + \epsilon_n$.*
    The reader should notice that under this condition, we have

$$P_\chi(a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-1} + b\epsilon_n) \cong P_\chi(a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-2} + (b-1)\epsilon_{n-1} + (a+1)\epsilon_n)$$

    and

$$P^0_\chi(a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-1} + b\epsilon_n) \cong P^0_\chi(a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-2} + (b-1)\epsilon_{n-1} + (a+1)\epsilon_n).$$

    (b) *Otherwise, when $\lambda \neq a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-1} + b\epsilon_n$ or $\lambda \neq a\epsilon_1 + a\epsilon_2 + \cdots + a\epsilon_{n-2} + b\epsilon_{n-1} + (a+1)\epsilon_n$, we have $P_\chi(\lambda) = \nabla_\chi(P^0_\chi(\lambda - \sigma))$.*

(3) *When* $|I_A| = 0$,

(a) *If* $\lambda = a\sigma$, *then* $[P_\chi(\lambda) : \nabla_\chi(P^0_\chi(\mu))] \neq 0$ *if and only if*

$$\mu = (a-1)\sigma, (a-1)\sigma + \epsilon_1, a\sigma + (p-1)\epsilon_n,$$
$$(a-1)\sigma + \epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n,\ 1 \leq i \leq n.$$

(b) *If* $\lambda = a\sigma - \epsilon_n$, *then* $[P_\chi(\lambda) : \nabla_\chi(P^0_\chi(\mu))] \neq 0$ *if and only if*

$$\mu = (a-1)\sigma,\ (a-1)\sigma + \epsilon_1,\ (a-1)\sigma - \epsilon_n.$$

(c) *If* $\lambda = a\sigma + b\epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_n$ *but* $\lambda \neq a\sigma, a\sigma - \epsilon_n$, *then we have* $[P_\chi(\lambda) : \nabla_\chi(P^0_\chi(\mu))] \neq 0$ *if and only if*

$$\mu = \lambda - \sigma \quad or \quad \mu = \lambda - \sigma + \epsilon_i.$$

(d) *Otherwise,* $P_\chi(\lambda) = \nabla_\chi(P^0_\chi(\lambda - \sigma))$.

(IV) *All of the above multiplicities satisfy that if* $[P_\chi(\lambda) : \nabla_\chi(P^0_\chi(\mu))] \neq 0$, *then* $[P_\chi(\lambda) : \nabla_\chi(P^0_\chi(\mu))] = 1$.

Proof: Based on Proposition 8.1, the results in (I),(II) can be proved by Theorems 5.1 and 6.7 respectively. The results in (III) are consequences of Theorems 7.8, 7.9 and 7.11. ■

9. Cartan invariants

Let $\mathfrak{g} = S(n)$ or $\bar{S}(n)$. We are now in a position to calculate the Cartan invariants of $U_\chi(\mathfrak{g})$ when $\mathrm{ht}(\chi) = -1$ or 1. We start with the calculation when χ is regular semisimple, i.e. χ is semisimple and $\chi(\xi_i D_i) \neq \chi(\xi_j D_j)$ for $1 \leq i \neq j \leq n$, (equivalent to the condition that $l_\chi = n$). We remind the reader that we always use the notation $\Lambda_0 = \sum_{i=1}^n \mathbb{F}_p \epsilon_i$ and $\Lambda_\chi = \{\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \mid \lambda_i^p - \lambda_i = \chi(\xi_i D_i)\}$ whenever $\mathfrak{g} = S(n)$ or $\mathfrak{g} = \bar{S}(n)$ in this section.

9.1. The regular semisimple case

In this subsection we always assume that χ is a regular semisimple character. By Theorem 8.2(1) we know that when χ is regular semisimple, $P_\chi(\lambda) = \nabla_\chi(P^0_\chi(\lambda - \sigma))$.

Lemma 9.1. *We have the following conclusions when* $\mathfrak{g} = S(n)$ *with* $p \mid n$.

(1) $L^0_\chi(\lambda) = Z^0_\chi(\lambda)$. *In addition,* $L^0_\chi(\lambda) \cong L^0_\chi(\mu)$ *if and only if* $\lambda = \mu$.

(2) $P^0_\chi(\lambda) = Z^0_\chi(\lambda)$ *for any* λ *in* Λ_χ.

Proof: (1) Because χ is regular semisimple, by [10, Proposition 7.3] $L^0_\chi(\lambda)^{\mathfrak{gl}(n)} = Z^0_\chi(\lambda)^{\mathfrak{gl}(n)}$. The same argument as Proposition 3.16 can be applied to get that every simple $\mathfrak{sl}(n)$-module can be viewed as the simple head of some $L^0_\chi(\lambda)^{\mathfrak{gl}(n)}$. By Lemma 3.13 we have that $L^0_\chi(\lambda)^{\mathfrak{gl}(n)}|_{\mathfrak{sl}(n)}$ is simple. So $L^0_\chi(\lambda)^{\mathfrak{sl}(n)} = Z^0_\chi(\lambda)^{\mathfrak{gl}(n)}|_{\mathfrak{sl}(n)}$. Now we prove that if $L^0_\chi(\lambda) \cong L^0_\chi(\mu)$, then $\lambda = \mu$. Since $L^0_\chi(\lambda) \cong L^0_\chi(\mu)$ implies $Z^0_\chi(\lambda) \cong Z^0_\chi(\mu)$, the same argument as [9, Lemma 3.1] can be applied to get this result.

(2) This result can be proved by the same argument as [10, Proposition 7.3]. ■

By Lemma 9.1, Theorem 8.2(1) and [10, Prop.7.3] we have $P_\chi(\lambda) = \nabla_\chi(Z^0_\chi(\lambda-\sigma))$. So we will focus on the calculation of $[\nabla_\chi(Z^0_\chi(\lambda)) : L_\chi(\mu)]$ for any $\lambda, \mu \in \Lambda_\chi$. Define

$$Q = \left\{ D_l, \xi_i D_j, \xi_{i_1}\xi_{i_2}\cdots\xi_{i_k}D_n \;\middle|\; \begin{array}{l} 1 \le l \le n,\ 1 \le i \le j \le n, \\ 1 \le i_1 < i_2 < \cdots < i_k \le n-1 \end{array} \right\}.$$

We can check that Q is a subalgebra of $\mathfrak{g}$. Recall that $\mathfrak{n}^+ = \{\xi_i D_j \mid 1 \le i < j \le n\}$, $\mathfrak{h} = \{\xi_i D_i \mid 1 \le i \le n\}$ when $\mathfrak{g} = \bar{S}(n)$ and $\mathfrak{h} = \{\xi_i D_i - \xi_{i+1}D_{i+1} \mid 1 \le i \le n-1\}$ when $\mathfrak{g} = S(n)$, $\mathfrak{b}_0 = \mathfrak{h} \oplus \mathfrak{n}^+$. Define

$$Q^+ = \{\xi_{i_1}\xi_{i_2}\cdots\xi_{i_k}D_n \mid 1 \le i_1 < i_2 < \cdots < i_k \le n-1\}.$$

We can check that $\mathfrak{g}_{-1} \oplus \mathfrak{n}^+$ (resp. $\mathfrak{n}^+ \oplus Q^+$, $\mathfrak{g}_{-1} \oplus \mathfrak{n}^+ \oplus Q^+$) is a p-nilpotent ideal of $\mathfrak{g}_{-1} \oplus \mathfrak{b}_0$ (resp. $\mathfrak{b}_0 \oplus Q^+$, Q).

Lemma 9.2. *Denote $\mathfrak{g}_{-1} \oplus \mathfrak{n}^+ \oplus Q^+$ by Q^p. Let M be a simple $U_\chi(Q)$-module. Then M can be viewed as a simple $U_\chi(\mathfrak{h})$-module with trivial Q^p-action. In particular, M is one-dimensional.*

Proof: Because Q^p is p-nilpotent, the set $N = \{m \in M \mid a \cdot m = 0, \forall a \in Q^p\}$ is non-empty. Now let $b \in Q, n \in N, a \in Q^p$. Since Q^p is a ideal in Q, we have the following equality

$$a \cdot (b \cdot n) = b \cdot (a \cdot n) + [a, b] \cdot n = 0,$$

which means that N is a submodule of M. So $M = N$ by the irreducibility of M. Hence M is annihilated by Q^p. Now by [10, Sec. 6.2] we get that M is one-dimensional. ■

Remark 9.3. (1) The same argument can be applied to $\mathfrak{g}_{-1}\oplus\mathfrak{b}_0$ (resp. $\mathfrak{b}_0\oplus Q^+$) to get the result that the set of simple $\mathfrak{g}_{-1} \oplus \mathfrak{b}_0$ (resp. $\mathfrak{b}_0 \oplus Q^+$)-modules can be parametrized by $\{\mathbf{k}_\lambda \mid \lambda \in \Lambda_\chi\}$.

(2) The simple $U_\chi(Q)$-modules can be parametrized by $\{\mathbf{k}_\lambda \mid \lambda \in \Lambda_\chi\}$ due to the regular semi-simplicity of χ.

Let $N = \left(\sum_{\lambda\in\Lambda_\chi} N_\lambda\right)_{\bar{0}} \bigoplus \left(\sum_{\mu\in\Lambda_\chi} N_\mu\right)_{\bar{1}}$ be a $\mathbb{Z}_2$-graded $U_\chi(\mathfrak{h})$-module. Define

$$\pi \cdot \mathrm{ch}N = \sum_{N_\lambda \subseteq N_{\bar{0}}} \dim N_\lambda e^\lambda + \pi\left(\sum_{N_\mu \subseteq N_{\bar{1}}} \dim N_\mu e^\mu\right).$$

Lemma 9.4. *Let $\mathfrak{g} = \bar{S}(n)$ or $S(n)$ with $n \ge 6$. Then in the category of $U_\chi(Q)$-modules, we have the following equality:*

$$[U_\chi(Q) \otimes_{U_\chi(\mathfrak{g}_{-1}\oplus\mathfrak{b}_0)} \mathbf{k}_\lambda] = 2^{m_1-n}p^{m_0}\left[\sum_{\lambda\in\Lambda_\chi} U_\chi(Q) \otimes_{U_\chi(\mathfrak{b}_0\oplus Q^+)} \mathbf{k}_\lambda\right],$$

$m_0 = \dim Q^+_{\bar{0}} - n,\ m_1 = \dim Q^+_{\bar{1}}$.

Proof: When $n \geq 6$, define

$$I = \left\{ \begin{array}{l} \xi_1\xi_2\xi_3 D_n,\ \xi_1\xi_2\xi_4 D_n,\ \xi_1\xi_2\xi_5 D_n, \\ \xi_1\xi_2\xi_3\xi_4\xi_5 D_n,\ \xi_1\xi_3\xi_4 D_5,\ \xi_2\xi_3\xi_4 D_5,\ \xi_1\xi_2\xi_i D_n \end{array} \middle| \ 6 \leq i \leq n-1 \right\},$$

which is a subset of $Q_{\bar{0}}^{+}$. The weights of the elements belonging to I are listed in the following set

$$\begin{aligned} I_R = & \{\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_n, \epsilon_1+\epsilon_2+\epsilon_4-\epsilon_n, \epsilon_1+\epsilon_2+\epsilon_5-\epsilon_n, \epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5-\epsilon_n\} \\ & \cup \{\epsilon_1+\epsilon_3+\epsilon_4-\epsilon_5, \epsilon_2+\epsilon_3+\epsilon_4-\epsilon_5, \epsilon_1+\epsilon_2+\epsilon_i-\epsilon_n \mid 6 \leq i \leq n-1\}. \end{aligned}$$

One can check that the weights belonging to I_R are linear equivalent to the set $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$. Consequently, the following equality holds:

$$\prod_{\alpha \in I_R} (1 + e^{\alpha} + (e^{\alpha})^2 + \cdots + (e^{\alpha})^{p-1}) = \sum_{\lambda \in \Lambda_0} e^{\mu}.$$

Now let $M = U_\chi(Q) \otimes_{U_\chi(\mathfrak{g}_{-1} \oplus \mathfrak{b}_0)} \mathbf{k}_\lambda$. For any $\chi' \in (\mathfrak{g}_0)^*$ and $\lambda' \in \Lambda_{\chi'}$, we have $e^{\lambda'} \sum_{\lambda \in \Lambda_0} e^{\mu} = \sum_{\delta \in \Lambda_{\chi'}} e^{\delta}$. Denote by $(Q_{\bar{0}}^{+})_R$ the weights of elements belonging to $Q_{\bar{0}}^{+}$. So the following equalities hold:

$$\begin{aligned} \pi \cdot \mathrm{ch} M &= e^{\lambda} \prod_{\alpha \in (Q_{\bar{0}}^{+})_R \setminus I_R} \left(1 + e^{\alpha} + (e^{\alpha})^2 + \cdots + (e^{\alpha})^{p-1}\right) \left(\sum_{\mu \in \Lambda_0} e^{\mu}\right) \times \prod_{\alpha \in Q_{\bar{1}}^{+}} (1 + \pi e^{\alpha}) \\ &= e^{\lambda} p^{m_0} \sum_{\mu \in \Lambda_0} e^{\mu} \times (1+\pi)^{m_1} \ (\pi^2 = 1) \\ &= p^{m_0} (1+\pi)^{m_1} \sum_{\nu \in \Lambda_\chi} e^{\nu} = p^{m_0} 2^{m_1 - 1} (1+\pi) \sum_{\nu \in \Lambda_\chi} e^{\nu}. \end{aligned}$$

Hence,

$$[U_\chi(Q) \otimes_{U_\chi(\mathfrak{g}_{-1} \oplus \mathfrak{b}_0)} \mathbf{k}_\lambda] = p^{m_0} 2^{m_1 - 1} \sum_{\lambda \in \Lambda_\chi} \left([\mathbf{k}_\lambda] + [\mathbf{k}_\lambda \langle \bar{1} \rangle]\right). \tag{34}$$

Now let $N = U_\chi(Q) \otimes_{U_\chi(\mathfrak{b}_0 \oplus Q^+)} \mathbf{k}_\lambda$. Then $N \cong \bigwedge(\mathfrak{g}_{-1}) \otimes \mathbf{k}_\lambda$ as a vector space. For $1 \leq i_1 < i_2 < \cdots < i_s \leq n$, we have

$$\sum_{\lambda \in \Lambda_\chi} D_{i_1} \wedge D_{i_2} \cdots \wedge D_{i_s} \otimes \mathbf{k}_\lambda = \sum_{\lambda \in \Lambda_\chi} \mathbf{k}_\lambda \langle \bar{s} \rangle.$$

Consequently,

$$\left[\sum_{\lambda \in \Lambda_\chi} U_\chi(Q) \otimes_{U_\chi(\mathfrak{b}_0 \oplus Q^+)} \mathbf{k}_\lambda\right] = 2^{n-1} \sum_{\lambda \in \Lambda_\chi} [\mathbf{k}_\lambda] + [\mathbf{k}_\lambda \langle \bar{1} \rangle]. \tag{35}$$

Since $n \geq 6$, we have $m_1 > n$. The proof can be completed by combining formulas (34) and (35). ■

Lemma 9.5. *Let $\mathfrak{g} = S(n)$ or $\bar{S}(n)$ with $n \geq 6$. Then in the category of $U_\chi(\mathfrak{b}_0 \oplus \mathfrak{g}_{\geq 1})$-modules, we have the following equality:*

$$\left[U_\chi(\mathfrak{b}_0 \oplus \mathfrak{g}_{\geq 1}) \otimes_{U_\chi(\mathfrak{b}_0 \oplus Q^+)} \mathbf{k}_\lambda\right] = p^{k-n} 2^{t-1} \sum_{\lambda \in \Lambda_\chi} [\mathbf{k}_\lambda] + [\mathbf{k}_\lambda \langle \bar{1} \rangle],$$

where $k = \sum_{i \geq 1} \dim \mathfrak{g}_{2i} - \dim Q_{\bar{0}}^+$, $t = \sum_{i \geq 0} \dim \mathfrak{g}_{2i+1} - \dim Q_{\bar{1}}^+$.

Proof: Define

$$\begin{aligned} I = &\{\xi_1\xi_2(\xi_3 D_3 - \xi_4 D_4),\ \xi_n\xi_1(\xi_3 D_3 - \xi_4 D_4),\ \xi_n\xi_2(\xi_3 D_3 - \xi_4 D_4)\} \\ &\cup \{\xi_n\xi_i(\xi_1 D_1 - \xi_2 D_2) \mid 3 \leq i \leq n-1\}, \end{aligned}$$

which is a subset of $\mathfrak{g}_{\geq 1} \backslash Q$ with n-elements. Moreover,the weights of the elements belonging to I are *linear equivalent* to the set $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$. Now the same discussion as Lemma 9.4 can be applied to have the Lemma. ■

Theorem 9.6. *Let $\mathfrak{g} = \bar{S}(n)$ or $S(n)$ with $n \geq 6$ and χ be a regular semisimple character. Define $a = \dim\left(\sum_{i \geq 1} \mathfrak{g}_{2i}\right)$ and $b = \dim\left(\sum_{i \geq 0} \mathfrak{g}_{2i+1}\right)$. Then for any $\lambda, \mu \in \Lambda_\chi$, we have the following results:*

(1) *When $\mathfrak{g} = \bar{S}(n)$, then* $[P_\chi(\lambda) : L_\chi(\mu)] = [P_\chi(\lambda) : L_\chi(\mu)\langle 1 \rangle] = 2^{b-1} p^{a-n}$.

(2) *When $\mathfrak{g} = S(n)$, then* $[P_\chi(\lambda) : L_\chi(\mu)] = [P_\chi(\lambda) : L_\chi(\mu)\langle 1 \rangle] = 2^{b-1} p^{a-n+1}$.

Proof: (1) By Theorem 5.1 we know that when $\mathfrak{g} = \bar{S}(n)$ with $n \geq 6$ and χ is regular semisimple, $L_\chi(\lambda) = K_\chi(\lambda)$. In addition, $L_\chi^0(\lambda) = Z_\chi^0(\lambda)$. Thus the following result holds:

$$L_\chi(\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b}_0 \oplus \mathfrak{g}_{\geq 1})} \mathbf{k}_\lambda. \tag{36}$$

Now the theorem can be checked by the following equalities.

$$\begin{aligned} &[\nabla_\chi(P_\chi^0(\lambda))] = \left[U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}^-)} Z_\chi^0(\lambda)\right] \\ &= \left[U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}^-)} U_\chi(\mathfrak{g}_0) \otimes_{U_\chi(\mathfrak{b}_0)} \mathbf{k}_\lambda\right] = \left[U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{g}_{-1} \oplus \mathfrak{b}_0)} \mathbf{k}_\lambda\right] \\ &= \left[U_\chi(\mathfrak{g}) \otimes_{U_\chi(Q)} U_\chi(Q) \otimes_{U_\chi(\mathfrak{g}_{-1} \oplus \mathfrak{b}_0)} \mathbf{k}_\lambda\right] \\ &= 2^{m_1 - n} p^{m_0} \sum_{\mu \in \Lambda_\chi} \left[U_\chi(\mathfrak{g}) \otimes_{U_\chi(Q)} U_\chi(Q) \otimes_{U_\chi(\mathfrak{b}_0 \oplus Q^+)} \mathbf{k}_\mu\right] \text{ (Lemma 9.4)} \\ &= 2^{m_1 - n} p^{m_0} \sum_{\mu \in \Lambda_\chi} \left[U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b}_0 \oplus Q^+)} \mathbf{k}_\mu\right] \\ &= 2^{m_1 - n} p^{m_0} \sum_{\mu \in \Lambda_\chi} \left[U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b}_0 \oplus \mathfrak{g}_{\geq 1})} \left[U_\chi(\mathfrak{b}_0 \oplus \mathfrak{g}_{\geq 1}) \otimes_{U_\chi(\mathfrak{b}_0 \oplus Q^+)} \mathbf{k}_\mu\right]\right] \\ &= 2^{b-1-n} p^{a-n} \sum_{\mu \in \Lambda_\chi} \left[U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b}_0 \oplus \mathfrak{g}_{\geq 1})} [\mathbf{k}_\mu] + [\mathbf{k}_\mu \langle \bar{1} \rangle]\right] \text{ (Lemma 9.5)} \\ &= 2^{b-1} p^{a-n} \sum_{\mu \in \Lambda_\chi} [L_\chi(\mu) + L_\chi(\mu)\langle \bar{1} \rangle] \text{ (36)}. \end{aligned}$$

(2) The case $\mathfrak{g} = S(n)$. We should notice that $\sigma = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n = 0$. So $\lambda = \lambda + \sigma = \lambda + 2\sigma = \cdots = \lambda + (p-1)\sigma$. Meanwhile, the discussion above also holds. So $[P_\chi(\lambda) : L_\chi(\mu)] = \Sigma_{i=0}^{p-1} [P_\chi(\lambda) : L_\chi(\mu + i\sigma)] = 2^{b-1} p^{a-n+1}$. ■

9.2. The general case

In this subsection, we will calculate the Cartan invariants of $U_\chi(\mathfrak{g})$ for $\chi \in \mathfrak{g}^*$ when $\mathrm{ht}(\chi) = 1$ or $\mathrm{ht}(\chi) = -1$. First, denote by Ω_χ the subset of Λ_χ such that $\mathrm{Irr}(U_\chi(\mathfrak{g}_0)) = \{L^0_\chi(\lambda) \mid \lambda \in \Omega_\chi\}$. Then by the same argument as [7, Theorem 4.7] we have the main theorem of this subsection.

Theorem 9.7. *Let $\mathfrak{g} = S(n)$ or $\bar{S}(n)$. For $\nu \in \Omega_\chi$, define*

$$t_\nu = \sum_{\gamma\in\Lambda_\chi} [\Delta_\chi(Z^0_\chi(\gamma)) : L_\chi(\nu)] = \sum_{\gamma\in\Lambda_\chi,\xi\in\Omega_\chi} [Z^0_\chi(\gamma) : L^0_\chi(\xi)][\Delta_\chi(L^0_\chi(\xi)) : L_\chi(\nu)].$$

The following results hold for any $\lambda, \mu \in \Omega_\chi$ and $\bar{i}, \bar{j} \in \mathbb{Z}_2$:

(1) *When $\mathfrak{g} = \bar{S}(n)$,*

$$[P_\chi(\lambda) : L_\chi(\mu)] = p^{m_0} 2^{m_1} t_\lambda t_\mu. \tag{37}$$

$$[P_\chi(\lambda)\langle\bar{i}\rangle : L_\chi(\mu)\langle\bar{j}\rangle]_s = p^{m_0} 2^{m_1-1} t_\lambda t_\mu. \tag{38}$$

(2) *When $\mathfrak{g} = S(n)$,*

$$[P_\chi(\lambda) : L_\chi(\mu)] = p^{m_0+1} 2^{m_1} t_\lambda t_\mu. \tag{39}$$

$$[P_\chi(\lambda)\langle\bar{i}\rangle : L_\chi(\mu)\langle\bar{j}\rangle]_s = p^{m_0+1} 2^{m_1-1} t_\lambda t_\mu. \tag{40}$$

The definitions of m_0 and m_1 have been given in Lemma 9.4.

Corollary 9.8. *Let $\mathfrak{g} = S(n)$ or $\bar{S}(n)$. Then both the category of $U_\chi(\mathfrak{g})$-modules and the category of $U_\chi(\mathfrak{g})$-supermodules have only one block.*

Proof: By the same discussion as [7, Corollary 4.9] we have the corollary. ∎

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