

Bracket Width of the Lie Algebra of Vector Fields on a Smooth Affine Curve

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Abstract. We prove that the bracket width of the simple Lie algebra of vector fields $\text{Vec}(C)$ of a smooth irreducible affine curve C with a trivial tangent sheaf is at most three. In addition, if C is a plane curve, the bracket width of $\text{Vec}(C)$ is at most two and if moreover C has a unique place at infinity, the bracket width of $\text{Vec}(C)$ is exactly two. We also show that in case C is rational, the width of $\text{Vec}(C)$ equals one.

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1. Introduction

Given a Lie algebra L over an infinite field \mathbb{k} , we define its bracket width as the supremum of lengths $\ell(x)$, where x runs over the derived algebra $[L, L]$ and $\ell(x)$ is defined as the smallest number n of Lie brackets $[y_i, z_i]$ needed to represent x in the form

$$\sum_{i=1}^n [y_i, z_i].$$

The bracket width applies in studying different aspects of Lie algebras, see [5]. In particular, in [5] the author proves that for any $n \in \mathbb{N}$ there exists a non-simple Lie algebra with the bracket width n . There exists a similar notion of width for groups and similar results for groups, see for example [4]. However, the first example of a simple Lie algebra with the bracket width strictly bigger than one was found only very recently in [1, Theorem A] among Lie algebras of vector fields of smooth affine curves which are simple by [2] and [6, Proposition 1]. In the current note we provide an upper bound on the bracket width of a Lie algebra of vector fields on an irreducible smooth affine curve C with certain properties. Our first main result is the following statement which partially answers [1, Question 2].

Theorem 1.1. *Let C be an irreducible smooth affine curve with trivial tangent sheaf. Then the bracket width of the Lie algebra $\text{Vec}(C)$ is smaller than or equal to three. In addition, if C is a plane curve, the bracket width of $\text{Vec}(C)$ is smaller than or equal to two.*

The upper bound for the bracket width given in Theorem A is, in particular, of interest since it allows us to compute the bracket width of $\text{Vec}(C)$ for a certain family of smooth plane affine curves.

Corollary 1.2. *Let C be an irreducible non-rational smooth plane affine curve with a unique place at infinity. Then the width of the simple Lie algebra $\text{Vec}(C)$ equals two.*

There are many examples of affine curves with only one place at infinity (see Definition 2.1), and they were studied in many different contexts, see, e.g., a paper of Kollár [3] and the references therein. A simple class of examples is given by affine hyperelliptic plane curves $C \subset \mathbb{A}^2$ defined by equations $y^2 = h(x)$, where $h(x)$ is a monic polynomial of odd degree strictly greater than one which has only simple roots ([1, Example 2]).

We believe that the assumption on the curve C in Corollary 1.2 can be lightened and we have the following conjecture.

Conjecture 1.3. *Assume C is a non-rational affine smooth plane curve. Then the bracket width of $\text{Vec}(C)$ is exactly two.*

Assume f is a regular function on C . We define a principal open subset $C_f \subset C$ as $\{x \in C \mid f(x) \neq 0\} \subset C$. Note that C_f is a smooth affine curve itself.

Theorem 1.4. *Let C be an irreducible smooth affine curve and C_f be its principal open subset. Then the bracket width of $\text{Vec}(C_f)$ is smaller than or equal to the bracket width of $\text{Vec}(C)$.*

We do not know an example of a smooth affine curve C with a principal open subset $U \subset C$ such that the width of $\text{Vec}(U)$ is strictly smaller than $\text{Vec}(C)$.

As a consequence of Theorem B we have the following statement that disproves [1, Conjecture 1].

Corollary 1.5. *If \mathbb{k} is algebraically closed, the bracket width of the Lie algebra $\text{Vec}(C)$ of a rational smooth affine curve C is one.*

2. Proof of Theorem 1.1

Denote by $\mathcal{O}(C)$ the ring of regular functions on C . Since by hypothesis the tangent sheaf of C is trivial, we have $\text{Vec}(C) = \mathcal{O}(C) \cdot \tau$ for a certain nowhere vanishing global vector field $\tau \in \text{Vec}(C)$, unique up to multiplication by a nonzero constant. It is well-known that every smooth affine variety of dimension d can be embedded into \mathbb{A}^{2d+1} and that the bound $2d + 1$ is optimal ([7, Corollary 1]). In particular, a smooth affine curve can be embedded into \mathbb{A}^3 and this bound is sharp. Hence, $\mathcal{O}(C) \simeq \mathbb{k}[x, y, z]/I$, where $I \subset \mathbb{k}[x, y, z]$ is some ideal and we have the natural surjections

$$\pi: \mathbb{k}[x, y, z] \twoheadrightarrow \mathbb{k}[x, y, z]/I$$

and $\pi_*: \{\nu \in \text{Derk}[x, y, z] = \text{Vec}(\mathbb{A}^3) \mid \nu(I) \subset I\} \twoheadrightarrow \text{Vec}(C)$.

Note that $\{\nu \in \text{Derk}[x, y, z] \mid \nu(I) \subset I\} \subset \text{Derk}[x, y, z]$ is a Lie subalgebra and π_* is a homomorphism of Lie algebras. Then there exist $\tilde{P}, \tilde{Q}, \tilde{R} \in \mathbb{k}[x, y, z]$ such that τ is the image of a derivation $\tilde{\tau} = \tilde{P} \frac{\partial}{\partial x} + \tilde{Q} \frac{\partial}{\partial y} + \tilde{R} \frac{\partial}{\partial z} \in \text{Vec}(\mathbb{A}^3)$ that preserves the ideal I . Further, define $P, Q, R \in \mathcal{O}(C)$, $P = \pi(\tilde{P})$, $Q = \pi(\tilde{Q})$, $R = \pi(\tilde{R})$.

Claim 1: $\left\{ [\tilde{\tau}, \tilde{f}\tilde{\tau}] + [y\tilde{\tau}, \tilde{g}\tilde{\tau}] + [z\tilde{\tau}, \tilde{h}\tilde{\tau}] \mid \tilde{f}, \tilde{g}, \tilde{h} \in \mathbb{k}[x, y, z] \right\} = (\tilde{P}, \tilde{Q}, \tilde{R})\tau,$

where $(\tilde{P}, \tilde{Q}, \tilde{R})$ denotes the ideal of $\mathbb{k}[x, y, z]$ generated by \tilde{P}, \tilde{Q} and \tilde{R} . Indeed,

$$\begin{aligned} [\tilde{\tau}, \tilde{f}\tilde{\tau}] + [y\tilde{\tau}, \tilde{g}\tilde{\tau}] + [z\tilde{\tau}, \tilde{h}\tilde{\tau}] &= (\tilde{P}\tilde{f}'_x + \tilde{Q}\tilde{f}'_y + \tilde{R}\tilde{f}'_z + \\ &+ y(\tilde{P}\tilde{g}'_x + \tilde{Q}\tilde{g}'_y + \tilde{R}\tilde{g}'_z) - \tilde{g}\tilde{Q} + z(\tilde{P}\tilde{h}'_x + \tilde{Q}\tilde{h}'_y + \tilde{R}\tilde{h}'_z) - \tilde{h}\tilde{R})\tilde{\tau} \end{aligned}$$

which equals

$$\begin{aligned} &(\tilde{P}(\tilde{f}'_x + y\tilde{g}'_x + z\tilde{h}'_x) + \tilde{Q}(\tilde{f}'_y + y\tilde{g}'_y + z\tilde{h}'_y - \tilde{g}) + \tilde{R}(\tilde{f}'_z + y\tilde{g}'_z + z\tilde{h}'_z - \tilde{h}))\tilde{\tau} \\ &= (\tilde{P}(\tilde{f} + y\tilde{g} + z\tilde{h})'_x + \tilde{Q}((\tilde{f} + y\tilde{g} + z\tilde{h})'_y - 2\tilde{g}) + \tilde{R}((\tilde{f} + y\tilde{g} + z\tilde{h})'_z - 2\tilde{h}))\tilde{\tau}. \end{aligned} \tag{1}$$

Define $\tilde{r} = \tilde{f} + y\tilde{g} + z\tilde{h} \in \mathcal{O}(C)$. Now, the expression (1) can be written as

$$(\tilde{P}\tilde{r}'_x + \tilde{Q}(\tilde{r}'_y - 2\tilde{g}) + \tilde{R}(\tilde{r}'_z - 2\tilde{h}))\tilde{\tau}.$$

For any $F, G, H \in \mathbb{k}[x, y, z]$ there exist $\tilde{r}, \tilde{g}, \tilde{h}$ such that $F = \tilde{r}'_x$, $G = \tilde{r}'_y - 2\tilde{g}$ and $H = \tilde{r}'_z - 2\tilde{h}$. Hence,

$$\begin{aligned} &\left\{ (\tilde{P}\tilde{r}'_x + \tilde{Q}(\tilde{r}'_y - 2\tilde{g}) + \tilde{R}(\tilde{r}'_z - 2\tilde{h}))\tilde{\tau} \mid \tilde{r}, \tilde{g}, \tilde{h} \in \mathbb{k}[x, y, z] \right\} = \\ &= \left\{ (\tilde{P}F + \tilde{Q}G + \tilde{R}H)\tilde{\tau} \mid F, G, H \in \mathbb{k}[x, y, z] \right\} = (\tilde{P}, \tilde{Q}, \tilde{R})\tilde{\tau}. \end{aligned}$$

This proves Claim 1.

We claim now that any vector field from $\text{Vec}(C)$ can be written as the sum

$$[\tau, f\tau] + [\pi(y)\tau, g\tau] + [\pi(z)\tau, h\tau] \tag{2}$$

for some regular functions $f, g, h \in \mathcal{O}(C)$. Indeed, $(\tilde{P}, \tilde{Q}, \tilde{R})$ is preserved by $\tilde{\tau}$, hence $\pi((\tilde{P}, \tilde{Q}, \tilde{R}))$ is preserved by τ . Whence $\pi((\tilde{P}, \tilde{Q}, \tilde{R}))\tau$ is the ideal in $\text{Vec}(C)$. Therefore, because of simplicity of $\text{Vec}(C)$ we have

$$\pi_*((\tilde{P}, \tilde{Q}, \tilde{R})\tau) = \text{Vec}(C) \quad \text{or equivalently} \quad \pi((\tilde{P}, \tilde{Q}, \tilde{R})) = \mathcal{O}(C).$$

Finally, by Claim 1 we have

$$\begin{aligned} &\left\{ \pi_*([\tilde{\tau}, \tilde{f}\tilde{\tau}]) + \pi_*([y\tilde{\tau}, \tilde{g}\tilde{\tau}]) + \pi_*([z\tilde{\tau}, \tilde{h}\tilde{\tau}]) \mid \tilde{f}, \tilde{g}, \tilde{h} \in \mathbb{k}[x, y, z] \right\} \\ &= \pi_*((\tilde{P}, \tilde{Q}, \tilde{R})\tau) = \text{Vec}(C). \end{aligned}$$

Thus, the first statement of the theorem follows as π_* is a homomorphism of Lie algebras.

If C is a plane curve, then, similarly as above, $\text{Vec}(C) = \mathcal{O}(C) \cdot \tau$, $\mathcal{O}(C) \simeq \mathbb{k}[x, y]/I$, where $I \subset \mathbb{k}[x, y]$ is an ideal and we have the natural surjections

$$\pi: \mathbb{k}[x, y] \rightarrow \mathbb{k}[x, y]/I$$

and $\pi_*: \{ \nu \in \text{Der}\mathbb{k}[x, y] = \text{Vec}(\mathbb{A}^2) \mid \nu(I) \subset I \} \rightarrow \text{Vec}(C)$.

Then τ is the image of a derivation $\tilde{\tau} = \tilde{P} \frac{\partial}{\partial x} + \tilde{Q} \frac{\partial}{\partial y}$ that preserves the ideal I . Further, define $P, Q \in \mathcal{O}(C)$, $P = \pi(\tilde{P})$, $Q = \pi(\tilde{Q})$.

The second statement of the theorem follows if we prove that any vector field of $\text{Vec}(C)$ can be written as the sum

$$[\tau, f\tau] + [\pi(y)\tau, g\tau] \tag{3}$$

for some regular functions $f, g \in \mathcal{O}(C)$.

Similarly as above (3) follows from the next equality

$$\left\{ [\tilde{\tau}, \tilde{f}\tilde{\tau}] + [y\tilde{\tau}, \tilde{g}\tilde{\tau}] \mid \tilde{f}, \tilde{g}, \in \mathbb{k}[x, y] \right\} = (\tilde{P}, \tilde{Q})\tilde{\tau}$$

which is proved analogously to Claim 1. ■

Definition 2.1. An irreducible smooth affine curve C is said to have a *unique place at infinity* if it is equal to the complement of a single closed point in a smooth projective curve \bar{C} .

Proof of Corollary 1.2. By [1, Theorem A] the bracket width of $\text{Vec}(C)$ is strictly greater than one. Since C is a smooth irreducible plane affine curve, its tangent sheaf is trivial. Now by Theorem A the width of $\text{Vec}(C)$ is less or equal than two which proves the claim. ■

3. Proof of Theorem 1.4

Denote by $\mathcal{K}(C)$ the field of fractions of $\mathcal{O}(C)$. Then

$$\mathcal{K}(C) \otimes_{\mathcal{O}(C)} \text{Der}(\mathcal{O}(C)) \simeq \text{Der}(\mathcal{K}(C)) \tag{4}$$

and the latter is known to be a free $\mathcal{K}(C)$ -module of rank equal to $\dim C = 1$. By (4) $\text{Vec}(C) = \text{Der}\mathcal{O}(C) \subset \text{Der}\mathcal{K}(C) = \mathcal{K}(C)\tau$, where $\tau \in \text{Vec}(C)$ is a global vector field. Hence, $\text{Vec}(C) = \mathcal{R}(C) \cdot \tau$ for a certain $\mathcal{O}(C)$ -module $\mathcal{R}(C) \subset \mathcal{K}(C)$. As a consequence, $\text{Vec}(C_f) = \mathcal{R}(C)[f^{-1}]\tau$.

Now, any pair of elements in $\text{Vec}(C_f) = \mathcal{R}(C)[f^{-1}]\tau$ has the form $(\frac{a}{f^k}\tau, \frac{b}{f^k}\tau)$ for some $a, b \in \mathcal{R}(C)$ and $k \in \mathbb{N} \cup \{0\}$ and we have:

$$\begin{aligned} \left[\frac{a}{f^k}\tau, \frac{b}{f^k}\tau \right] &= \left(\frac{a}{f^k}\tau\left(\frac{b}{f^k}\right) - \frac{b}{f^k}\tau\left(\frac{a}{f^k}\right) \right) \tau \\ &= \left(\frac{a}{f^k} \frac{\tau(b)f^k - b\tau(f^k)}{f^{2k}} - \frac{b}{f^k} \frac{\tau(a)f^k - a\tau(f^k)}{f^{2k}} \right) \tau \\ &= \frac{a\tau(b)}{f^{2k}} - \frac{kab}{f^{2k+1}}\tau(f) - \frac{b\tau(a)}{f^{2k}} + \frac{kab}{f^{2k+1}}\tau(f) = \frac{1}{f^{2k}} [a, b] \tau. \end{aligned} \tag{5}$$

Further, for any $h \in \mathcal{R}(C)[f^{-1}]$ there exists $g \in \mathcal{R}(C)$ such that

$$h = \frac{g}{f^{2k}}$$

for some $k \in \mathbb{N}$. Assume that

$$g\tau = [a_1\tau, b_1\tau] + \dots + [a_n\tau, b_n\tau],$$

where $a_1, \dots, a_n, b_1, \dots, b_n \in \mathcal{R}(C)$.

Then using (5) we have

$$h\tau = \frac{g}{f^{2k}}\tau = \left[\frac{a_1}{f^k}\tau, \frac{b_1}{f^k}\tau \right] + \cdots + \left[\frac{a_n}{f^k}\tau, \frac{b_n}{f^k}\tau \right].$$

This completes the proof. ■

Proof of Corollary 1.5. Let us recall that every rational smooth affine curve C is isomorphic to $\mathbb{A}^1 \setminus \Lambda$, where Λ is a finite set of $r \geq 0$ points. In particular, it admits a closed embedding into \mathbb{A}^2 . Indeed, C can also be seen as the complement in \mathbb{A}^1 of a finite number (possibly zero) of points and we can consider the closed embedding $\mu: C \rightarrow \mathbb{A}^2$ given by $x \mapsto (x, \frac{1}{f(x)})$, where $f \in \mathbb{k}[x]$ is a polynomial whose roots are exactly the removed points. Note that the image of μ is the curve of \mathbb{A}^2 defined by the equation $f(x)y = 1$.

By [1, Proposition 1] the bracket width of $\text{Vec}(\mathbb{A}^1)$ is one. By Theorem B the bracket width of $\text{Vec}(\mathbb{A}^1 \setminus \Lambda)$ is smaller than or equal to the bracket width of $\text{Vec}(\mathbb{A}^1)$ which is one. The proof follows. ■

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