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On the Universal L_∞-Algebroid of Linear Foliations

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Abstract. We compute an L_∞-algebroid structure on a projective resolution of some classes of singular foliations on a vector space V induced by the linear action of some Lie subalgebras of $\mathfrak{gl}(V)$. This L_∞-algebroid provides invariants of the singular foliations, and also provides a constant-rank replacement of the singular foliation. The computation consists of first constructing a projective resolution of the foliation induced by the linear action of the Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$, and then computing the L_∞-algebroid structure. We then generalize these constructions to a vector bundle E, where the role of the origin is now taken by the zero section L.
We then show that the fibers over a singular point of a projective resolution of any singular foliation can be computed directly from the foliation, without needing the projective resolution. For linear foliations, we also provide a way to compute the action of the isotropy Lie algebra in the origin on these fibers directly from the foliation.

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1. Introduction

Let M be a smooth manifold, equipped with a singular foliation $\mathcal{F}$. By *singular foliation*, we mean a subsheaf $\mathcal{F}$ of the sheaf of vector fields $\mathfrak{X}$ on M such that

(a) for all $U \subset M$ open, $\mathcal{F}(U)$ is a $C^\infty(U)$-submodule of $\mathfrak{X}(U)$,

(b) for all $U \subset M$ open and $X, Y \in \mathcal{F}(U)$ we have $[X,Y] \in \mathcal{F}(U)$,

(c) for all $x \in M$, there exists an open subset U_x of M containing x, such that $\mathcal{F}(U_x)$ is a finitely generated $C^\infty(U_x)$-module.

This definition of singular foliations was used in [9]. An equivalent definition, using compactly supported vector fields, appeared in [11, 1] among other places. This equivalence was shown in [13, Proposition 2.1.9], and the construction of the sheaf out of compactly supported vector fields appeared in [3].

In [9], it was shown that under certain conditions on $\mathcal{F}$ one can associate an L_∞-algebroid over M to $(M, \mathcal{F})$. Here an L_∞-algebroid is a non-positively graded vector bundle $E = \bigoplus_{i \in \mathbb{Z}_{\leq 0}} E_i$, with a collection of multibrackets

$$\{\ell_k : \Gamma(\wedge^k E) \to \Gamma(E)\}_{k \geq 1},$$

where ℓ_k has degree $2-k$, and a vector bundle map $\rho : E_0 \to TM$ intertwining ℓ_2 with the Lie bracket of vector fields, called the anchor, satisfying some quadratic

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identities. L_∞-algebroids were first defined in [12] as higher analogues of Lie algebroids. When $M = \{*\}$ is a single point, the definition reduces to that of a non-positively graded L_∞-algebra, which appeared in [8] as strongly homotopy Lie algebra. For the definition and important properties, we refer to [10, Section 2.1].

The construction of [9] can be broken into two parts:

(i) Choosing a resolution of $\mathcal{F}$ in the category of C^∞_M-modules by finitely generated projective modules[1],

(ii) Constructing an L_∞-algebroid structure on the complex given by the resolution.

In step (i) the conditions posed on $\mathcal{F}$ are used. Neither of the steps is constructive, but plenty of examples are given. Because of (i), the L_∞-algebroid constructed in (ii) satisfies a universality property, which implies uniqueness up to a notion of homotopy ([9, Corollary 2.9]). It will therefore be referred to as a *universal* L_∞-algebroid of $\mathcal{F}$. Because of the uniqueness up to homotopy, this L_∞-algebroid captures invariants of the singular foliation $\mathcal{F}$. Moreover, it allows to replace the singular foliation by a collection of constant-rank objects, which provides a framework to extend some results from the theory of Lie algebroids to singular foliations. Further, knowing a universal L_∞-algebroid of a singular foliation allows to compute the modular class of a singular foliation as in [10].

In this article, we generalize the example of the foliation $\mathcal{F}_0$ consisting of vector fields vanishing in the origin given in [9]. The foliation $\mathcal{F}_0$ is induced by the canonical linear $\mathfrak{gl}(V)$-action on V. The universal L_∞-algebroid given in [9, Example 3.99] only has two nonzero operations, turning it into a differential graded Lie algebroid (dg-Lie algebroid): $\ell_k = 0$ for $k \geq 3$. This raises several questions:

(1) Can we construct the universal L_∞-algebroids for linear actions of other Lie algebras explicitly?

(2) Can this approach be generalized to higher-dimensional leaves, with the corresponding isotropy Lie algebra?

(3) Does the universal L_∞-algebroid for such a foliation always admit a dg-Lie algebroid structure (i.e. an L_∞-algebroid structure for which only the unary and binary brackets are non-zero)?

Main results. We address the questions above in the following examples:

- The Lie subalgebra $\mathfrak{gl}(V,W) \subset \mathfrak{gl}(V)$ for a given subspace $W \subset V$,
- the Lie subalgebra $\mathfrak{sl}(V) \subset \mathfrak{gl}(V)$ of traceless endomorphisms,
- the Lie subalgebra $\mathfrak{sp}(V,\omega) \subset \mathfrak{gl}(V)$ of endomorphisms preserving a non-degenerate skew-symmetric 2-form $\omega \in \wedge^2 V^*$.

All three questions have a positive answer in the cases $\mathfrak{gl}(V,W)$ and $\mathfrak{sl}(V)$. We answer questions (1) and (2) partially in the case of $\mathfrak{sp}(V,\omega)$, and we do not know the answer to question (3) in this case.

[1] As we work in the smooth category, this is equivalent to choosing a resolution of $\mathcal{F}(M)$ in the category of $C^\infty(M)$-modules by sections of vector bundles. We therefore do not distinguish between the sheaf and its global sections.

The resolutions of the module $\mathcal{F}$ we construct are *minimal* at the origin, which means that all differentials, being vector bundle maps, vanish at the origin. An advantage of this is that two L_∞-algebroid structures constructed on minimal resolutions are not only homotopy equivalent, but actually L_∞-isomorphic in a neighborhood of the origin, as explained at the end of section 2.1.

In section 2 we address questions (1) and (3).

- In section 2.1 we recall the construction for $\mathfrak{gl}(V)$, which induces the foliation given by all vector fields vanishing in $0 \in V$, as given in Example 3.99 of [9].
- In section 2.2 we consider the case of $\mathfrak{gl}(V, W)$, which induces the foliation generated by the linear vector fields tangent to the subspace W. We give a geometric resolution and describe an L_∞-algebroid structure with only a unary and binary bracket in Proposition 2.4 yielding a positive answer to question (3).
- In section 2.3 we consider the case of $\mathfrak{sl}(V)$, which induces the foliation generated by linear vector fields preserving a constant volume form on V. We compute a geometric resolution in Proposition 2.6, and describe an L_∞-algebroid structure with only a unary and binary bracket in Proposition 2.8 yielding a positive answer to question (3).
- In section 2.4 we fix a non-degenerate element $\omega \in \wedge^2 V^*$ and consider the case of $\mathfrak{sp}(V, \omega)$. We compute the geometric resolution in Proposition 2.12, and give a binary bracket in Proposition 2.15 depending on a map r^ω we chose. We show that this bracket does not satisfy the Jacobi identity, and give an expression for the ternary bracket. In the appendix A we investigate if the binary brackets can be simplified by picking r^ω to be a cochain map in some degrees, and show that this cannot be done when V is 4-dimensional. The answer to question 3) remains inconclusive in this case.

In section 3 we turn our attention to the higher-dimensional analogues of the above-mentioned cases and address the corresponding questions 2) and 3). In each of the cases the results of the earlier sections generalize.

- In section 3.1 we consider the foliation of vector fields on a vector bundle E which are tangent to the zero section. We compute the geometric resolution in Proposition 3.5, and describe an L_∞-algebroid structure in Proposition 3.7.
- In section 3.2 we consider the foliation of vector fields on a vector bundle which are tangent to a vector subbundle, of which the zero section is a special case. The geometric resolution and L_∞-algebroid structure are given in Proposition 3.8.
- In section 3.3 we consider the foliation on an orientable vector bundle $E \to L$, with non-vanishing section $\mu \in \Gamma(\wedge^n E)$, where $n = \mathrm{rk}(E)$, generated (as C^∞_E-module) by the linear vector fields which preserve μ. We give the geometric resolution in Proposition 3.10, and the L_∞-algebroid structure in Proposition 3.11.
- In section 3.4 we consider the foliation on a vector bundle $E \to L$ generated by the linear vector fields which preserve a non-degenerate $\omega \in \Gamma(\wedge^2 E)$. The projective resolution is given in Proposition 3.12, and a binary bracket of the L_∞-algebroid structure in Proposition 3.13.

Finally, in section 4 we consider a general foliation $\mathcal{F}$ on a vector space V, for which the origin p is a singular point. We show that the fibers over p of any geometric resolution which is minimal at the origin can be computed directly from $\mathcal{F}$, *without needing to find a geometric resolution* (Proposition 4.2). In the case that $\mathcal{F}$ is linear, we additionally show that part of the structure of the isotropy L_∞-algebra (see [9, Section 4.2]), which is an invariant of the foliation $\mathcal{F}$, can be recovered from the foliation directly (Proposition 4.3).

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2. Zero-dimensional leaves

Convention. Throughout this article, for a finite-dimensional real vector space W, we will consider the trivial vector bundle $W \times V$ over a finite-dimensional real vector space V. Its global sections will be denoted by $\Gamma(W)$.

Unless stated otherwise, repeated indices will be summed over.

In this section, we compute a universal L_∞-algebroid for some classes of singular foliations generated by some Lie subalgebra of the Lie algebra of linear vector fields on a vector space V, addressing questions (1) and (3) from Section 1.

2.1. Vector fields vanishing at the origin

In this section we recall Example 3.99 of [9]. Let V be a real vector space of dimension $n \geq 0$, and let

$$\mathcal{F}_0(V) = \{X \in \mathfrak{X}(V) \mid X(0) = 0\} \tag{1}$$

be the submodule of vector fields on V vanishing in the origin. It is easy to see that it is a singular foliation. A resolution of $\mathcal{F}_0$ can be constructed using the following lemma.

Lemma 2.1. *The complexes*

$$0 \longrightarrow \Gamma(\wedge^n V^*) \xrightarrow{d_n} \dots \xrightarrow{d_2} \Gamma(V^*) \xrightarrow{\rho} I_q \longrightarrow 0\,, \tag{2}$$

$$0 \longrightarrow \Gamma(\wedge^n V^*) \xrightarrow{d_n} \dots \xrightarrow{d_2} \Gamma(V^*) \xrightarrow{d_1} C^\infty(V) \xrightarrow{ev_q} \mathbb{R} \longrightarrow 0 \tag{3}$$

are exact. Here for $k = 1, \dots, n$, $d_k : \Gamma(\wedge^k V^*) \to \Gamma(\wedge^{k-1} V^*)$ and $\rho : \Gamma(V^*) \to I_q$ are the contraction with the Euler vector field $x^i \partial_{x^i}$, I_q is the ideal of functions vanishing at the origin $q \in V$ and ev_q is the evaluation of a function at $q = 0$. In particular, the complexes remain exact when applying the functor $- \otimes_{C^\infty(V)} \Gamma(W)$ for some vector bundle $W \times V \to V$.

Taking $W = V$ and tensoring (2) with $\Gamma(V) \cong \mathfrak{X}(V)$ we obtain the exact sequence

$$0 \longrightarrow \Gamma(\wedge^n V^* \otimes V) \xrightarrow{d_n} \dots \xrightarrow{d_2} \Gamma(V^* \otimes V) \xrightarrow{\rho} I_p \mathfrak{X}(V) = \mathcal{F}_0(V) \longrightarrow 0. \tag{4}$$

Here, and in the rest of this article we use the convention that $\Gamma(V^* \otimes V)$ sits in degree 0, and the differential $d_\bullet$ has degree 1.

An L_∞-algebroid structure on $\Gamma(\wedge^\bullet V^* \otimes V)$ can be given as follows: for the unary bracket, we take $d_\bullet$, as in (4). For the binary bracket we take the Nijenhuis-Richardson bracket: For $0 \le k_1, k_2 \le n-1$ define

$$[-,-] : \Gamma\left(\wedge^{k_1+1}V^* \otimes V\right) \times \Gamma\left(\wedge^{k_2+1}V^* \otimes V\right) \to \Gamma\left(\wedge^{k_1+k_2+1}V^* \otimes V\right) \quad \text{by}$$

$$[f_1 \cdot (\phi_1 \otimes w_1), f_2 \cdot (\phi_2 \otimes w_2)] := f_1 f_2 \cdot (\phi_1 \iota_{w_1}(\phi_2) \otimes w_2 - (-1)^{k_1 k_2} \phi_2 \iota_{w_2}(\phi_1) \otimes w_1)$$
$$+ (f_1 \rho(\phi_1 \otimes w_1)(f_2) \cdot (\phi_2 \otimes w_2) - f_2 \rho(\phi_2 \otimes w_2)(f_1) \cdot (\phi_1 \otimes w_1)) \qquad (5)$$

for $f_i \in C^\infty(V)$, $\phi_i \in \wedge^{k_i}V^*$, $w_i \in V$ $(i = 1, 2)$. Here, for $v \in V, \alpha \in \wedge^k V^*$, $\iota_v(\alpha) \in \wedge^{k-1}V^*$ is the insertion of v into the first slot of α, and $\rho(\phi_i \otimes w_i)$ is understood to vanish if $k_i \neq 0$.

One can check that this defines a dg-Lie algebroid over V for which the image of ρ is exactly $\mathcal{F}_0$. We denote it by $L_\infty(\mathcal{F}_0)$.

Note that the differentials d_p vanish at the origin for $p = 2, \dots, n$. This implies that any L_∞-algebroid structure with the same property is L_∞-isomorphic to the one above in a neighborhood of the origin: by [9, Corollary 2.9], any two L_∞-algebroid structures are homotopy equivalent by an L_∞-morphism Φ. By minimality and [9, Lemma 4.13(iii)], this implies that the homotopy equivalence is an isomorphism in the origin. As invertibility is an open condition it follows that it is an isomorphism in a neighborhood of the origin.

2.2. Linear vector fields preserving a subspace

Let V be a real vector space of dimension n, and $W \subset V$ a linear subspace. Let

$$\mathcal{F}_W(V) := \{X \in \mathcal{F}_0(V) \mid X(I_W) \subset I_W\}$$

be the $C^\infty(V)$-submodule of linear vector fields tangent to the subspace W. This is a singular foliation, and is induced by the action of the Lie subalgebra $\mathfrak{gl}(V, W)$ of $\mathfrak{gl}(V)$ given by

$$\mathfrak{gl}(V, W) = \{A \in \mathfrak{gl}(V) \mid A(W) \subset W\},$$

the endomorphisms of V preserving W. The leaves of this foliation consist of the origin, the connected components of $W \setminus \{0\}$, and the connected components of $V \setminus W$.

Example 2.2. Let $V = \mathbb{R}^2, W = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$. Then $\mathcal{F}_W(V)$ is generated by the vector fields $x\partial_x, y\partial_x, y\partial_y$, and the leaves are the positive x-axis, the origin, the negative x-axis, the upper half plane and the lower half plane. In this case $\mathfrak{gl}(V, W)$ consists of all upper triangular matrices.

We can describe a minimal universal L_∞-algebroid of $\mathcal{F}_W$ as an L_∞ -subalgebroid of $L_\infty(\mathcal{F}_0)$. In particular, it will again be a dg-Lie algebroid.

Definition 2.3. Let $j \in \{1, \dots, n\}$. Define $K_j \subset \wedge^j V^* \otimes V = \mathrm{Hom}(\wedge^j V, V)$ by

$$K_j := \{\phi \in \wedge^j V^* \otimes V \mid \forall w \in W, \forall v_1, \dots, v_{j-1} \in V : \phi(w, v_1, \dots, v_{j-1}) \in W\}.$$

Proposition 2.4. (i) *The differential* $d_j : \Gamma(\wedge^j V^* \otimes V) \to \Gamma(\wedge^{j-1} V^* \otimes V)$ *as in* (4) *restricts to a map* $d_j : \Gamma(K_j) \to \Gamma(K_{j-1})$.

(ii) *The bracket* (5) *restricts to the subspaces* $\Gamma(K_j)$.

(iii) *The subcomplex*

$$0 \longrightarrow \Gamma(K_n) \xrightarrow{d_n} \Gamma(K_{n-1}) \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} \Gamma(K_1) \xrightarrow{\rho_W} \mathcal{F}_W(V) \longrightarrow 0 \qquad (6)$$

is exact, where $\rho_W = \rho|_{\Gamma(K_1)}$.

Consequently, $\Gamma(K_\bullet)$ *with the restrictions of* $d_\bullet$ *and* $[-,-]$ *is a minimal universal* L_∞*-algebroid of the foliation* $\mathcal{F}_W$.

Proof. Items (i) and (ii) are straightforward computations. For item (iii), fix a complement C of W in V. Then K_i can be identified with $\wedge^i V^* \otimes W \oplus \wedge^i C^* \otimes C$, and the complex (6) decomposes as

$$(\Gamma(K_\bullet), \partial) = (\Gamma(\wedge^\bullet V^* \otimes W) \oplus \Gamma(\wedge^\bullet C^* \otimes C), \partial_W + \partial_C),$$

where $\partial_W(\phi) = x^i \iota_{e_i}(\phi)$ for $\phi \in \Gamma(\wedge^i V^* \otimes W)$, and $\{e_i\}_{i=1}^n$ is a basis for V, with linear coordinates $\{x^i\}_{i=1}^n$. For $\psi \in \Gamma(\wedge^i C^* \otimes C)$, $\partial_C(\psi) = y^i \iota_{f_i}(\psi)$, where $\{f_i\}_{i=1}^r$ is a basis for C, and $\{y^i\}_{i=1}^r$ are the corresponding linear coordinates. By Lemma 2.1, both are exact, concluding the proof. ■

2.3. Vector fields preserving a volume form

The next choice for a Lie algebra $\mathfrak{g}$ acting linearly on a vector space V we consider is $\mathfrak{g} = \mathfrak{sl}(V)$, the Lie algebra of traceless endomorphisms. Observe that the partition of V is identical to the case of $\mathfrak{gl}(V)$, but that the underlying submodules of $\mathfrak{X}_V$ are different. Let $\mu \in \wedge^n V^*$ be a non-zero element, and denote the foliation given by the action of $\mathfrak{sl}(V)$ by $\mathcal{F}_\mu$.

2.3.1. The projective resolution. As it is in general not possible to restrict a projective resolution of a module to a submodule, one cannot directly get a projective resolution of the module of vector fields generated by the action of $\mathfrak{sl}(V)$, by restricting all modules to live over $\mathfrak{sl}(V)$. But for the most part, the resolution we will construct is related to the one given in (4). Consider the following diagram of $C^\infty(V)$-modules:

$$\begin{array}{ccc} \Gamma(\wedge^2 V^* \otimes V) & \xrightarrow{d_2} & \Gamma(V^* \otimes V) \\ & & \downarrow \mathrm{Tr} \\ \Gamma(V^*) & \xrightarrow{\partial_1} & \Gamma(\mathbb{R}) \end{array} \qquad (7)$$

where $\partial_1 : \Gamma(V^*) \to \Gamma(\mathbb{R})$ is the contraction with the negative of the Euler vector field $x^i \partial_{x^i}$ and Tr is the trace of endomorphisms.

Now there is a linear map $\phi_2 : \wedge^2 V^* \otimes V \to V^*$ by taking partial traces: for $\psi \in \wedge^2 V^*, v \in V$, we set

$$\phi_2(\psi \otimes v) = -\iota_v(\psi).$$

We now claim that (the constant extension of) ϕ_2 completes (7) to an anti-commutative square.

Indeed: let $\{e_i\}_{i=1}^n$ be a basis of V, with corresponding coordinates $\{x^i\}_{i=1}^n$, and let $\psi \otimes v \in \Gamma(\wedge^2 V^* \otimes V)$. Then

$$\begin{aligned}\partial_1(\phi_2(\psi \otimes v)) &= -\partial_1(\iota_v(\psi)) = x^i \iota_{e_i}\iota_v(\psi) = -x^i \iota_v \iota_{e_i}(\psi) \\ &= -\mathrm{Tr}(x^i \iota_{e_i}(\psi) \otimes v) = -\mathrm{Tr}(d_2(\psi \otimes v)).\end{aligned}$$

More generally, for $1 \leq k \leq n$, we can define the anti-symmetrized partial trace map $\phi_k : \wedge^k V^* \otimes V \to \wedge^{k-1} V^*$. For $\alpha \in \wedge^k V^*, v \in V$, we set

$$\phi_k(\alpha \otimes v) = (-1)^{k-1} \iota_v(\alpha).$$

Observe that ϕ_1 is the usual trace.

Note that the map $\partial_1 : \Gamma(V^*) \to \Gamma(\mathbb{R})$ as in (7) of free $C^\infty(V)$-modules can be extended to obtain a cochain complex

$$0 \longrightarrow \Gamma(\wedge^n V^*) \xrightarrow{\partial_n} \Gamma(\wedge^{n-1} V^*) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} \Gamma(V^*) \xrightarrow{\partial_1} C^\infty(V) \longrightarrow 0\,. \quad (8)$$

The cochain complex is concentrated in negative degrees, with $C^\infty(V)$ being in degree -1. Note that by Lemma 2.1, the complex is exact in degrees $-2, \ldots, -n-1$, as it is the truncation of (3). The following lemma describes the compatibility of ϕ with the respective differentials:

Lemma 2.5. *The map $\phi : (\Gamma(\wedge^\bullet V^* \otimes V), d_\bullet) \to (\Gamma(\wedge^{\bullet-1} V^*), \partial_\bullet)$ is a cochain map of degree -1, which is surjective in degrees $0, \ldots, -(n-2)$, and an isomorphism in degree $-n+1$, where $d_\bullet$ is as in (4), and $\partial_\bullet$ is as in (8).*

Proof. Let $\alpha \otimes v \in \Gamma(\wedge^{k+1} V^* \otimes V)$. We first show that ϕ anti-commutes with the respective differential:

$$\begin{aligned}\partial_k(\phi_{k+1}(\alpha \otimes v)) &= \partial_k((-1)^k \iota_v(\alpha)) = (-1)^{k+1} x^i \iota_{e_i}(\iota_v(\alpha)) = (-1)^k x^i \iota_v(\iota_{e_i}(\alpha)) \\ &= -\phi_k(x^i \iota_{e_i}(\alpha) \otimes v) = -\phi_k(d_{k+1}(\alpha \otimes v)).\end{aligned}$$

To see the surjectivity, pick a basis $\{e_i\}_{i=1}^n$ of V, and a dual basis $\{e^i\}_{i=1}^n$ of V^* such that $\mu = e^1 \wedge \cdots \wedge e^n$. For $k \in \{1, \ldots, n\}$, a basis for $\wedge^{k-1} V^*$ is given by $\{e^{i_1} \wedge \cdots \wedge e^{i_{k-1}} \mid 1 \leq i_1 < \cdots < i_{k-1} \leq n\}$. Given $e^{i_1} \wedge \cdots \wedge e^{i_{k-1}}$, let $q \in \{1, \ldots, n\} - \{i_1, \ldots, i_{k-1}\}$. Then

$$\phi_k(e^{i_1} \wedge \cdots \wedge e^{i_{k-1}} \wedge e^q \otimes e_q) = e^{i_1} \wedge \cdots \wedge e^{i_{k-1}},$$

where q is *not* summed over. Further, under the identification

$$V \to \wedge^n V^* \otimes V, \quad v \mapsto e^1 \wedge \cdots \wedge e^n \otimes v,$$

ϕ_n is the map $V \to \wedge^{n-1} V^*$ given by contraction with the volume form $e^1 \wedge \cdots \wedge e^n$, which is an isomorphism. ∎

We use the properties of ϕ to construct a projective resolution for $\mathcal{F}_\mu$.

Proposition 2.6. *For $i=1,...,n$, let $K_i \subset \wedge^i V^* \otimes V$ be defined by $K_i := \ker(\phi_i)$. The sequence*

$$0 \longrightarrow \Gamma(\wedge^n V^*) \xrightarrow{d_n \phi_n^{-1} \partial_n} \Gamma(K_{n-1}) \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} \Gamma(K_1) \xrightarrow{\rho_\mu} \mathcal{F}_\mu(V) \longrightarrow 0 \tag{9}$$

is exact, where $\rho_\mu = \rho|_{\Gamma(K_1)}$.

Proof. Note that by definition of ϕ_1, $K_1 = \mathfrak{sl}(V)$, so ρ_μ is surjective by definition of $\mathcal{F}_\mu(V)$. Let $i \in \{1, \dots, n-2\}$. Consider the following diagram with (anti)-commuting squares, where the middle and bottom rows are exact by Lemma 2.1:

$$\begin{array}{ccccccc}
\Gamma(K_{i+2}) & \xrightarrow{d_{i+2}} & \Gamma(K_{i+1}) & \xrightarrow{d_{i+1}} & \Gamma(K_i) & \xrightarrow{d_i} & \Gamma(K_{i-1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma(\wedge^{i+2} V^* \otimes V) & \xrightarrow{d_{i+2}} & \Gamma(\wedge^{i+1} V^* \otimes V) & \xrightarrow{d_{i+1}} & \Gamma(\wedge^{i} V^* \otimes V) & \xrightarrow{d_i} & \Gamma(\wedge^{i-1} V^* \otimes V) \\
\downarrow{\scriptstyle \phi_{i+2}} & & \downarrow{\scriptstyle \phi_{i+1}} & & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i-1}} \\
\Gamma(\wedge^{i+1} V^*) & \xrightarrow{\partial_{i+1}} & \Gamma(\wedge^{i} V^*) & \xrightarrow{\partial_i} & \Gamma(\wedge^{i-1} V^*) & \xrightarrow{\partial_{i-1}} & \Gamma(\wedge^{i-2} V^*)
\end{array}$$

For exactness at $\Gamma(K_i)$, take $\chi \in \Gamma(K_i)$ such that $d_i(\chi) = 0$, where d_1 is understood to be ρ_μ. Then by exactness of the middle row, there exists $\psi \in \Gamma(\wedge^{i+1} V^* \otimes V)$ such that $d_{i+1}(\psi) = \chi$. Now ψ may not be an element of $\Gamma(K_{i+1})$, so we consider $\phi_{i+1}(\psi)$. Note that

$$\partial_i \phi_{i+1}(\psi) = -\phi_i(\partial_{i+1}(\psi)) = -\phi_i(\chi) = 0,$$

so by exactness of (8) there exists $\tau \in \Gamma(\wedge^{i+1} V^*)$ such that

$$\phi_{i+1}(\psi) = \partial_{i+1}(\tau).$$

Using surjectivity of ϕ_{i+2}, lift τ to an element $\tilde{\tau} \in \Gamma(\wedge^{i+2} V^* \otimes V)$. Then

$$\phi_{i+1}(\psi + \partial_{i+2}(\tilde{\tau})) = \phi_{i+1}(\psi) - \partial_{i+1}(\tau) = 0,$$

so $\psi + \partial_{i+2}(\tilde{\tau}) \in K_{i+1}$, and $\partial_{i+1}(\psi + \partial_{i+2}(\tilde{\tau})) = \chi$.

For exactness at $\Gamma(K_{n-1})$, consider

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & \Gamma(\wedge^n V^*) & \xrightarrow{d_n (\phi_n)^{-1} \partial_n} & \Gamma(K_{n-1}) & \xrightarrow{d_{n-1}} & \Gamma(K_{n-2}) \\
 & & & & & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & \Gamma(\wedge^n V^* \otimes V) & \xrightarrow{d_n} & \Gamma(\wedge^{n-1} V^* \otimes V) & \xrightarrow{d_{n-1}} & \Gamma(\wedge^{n-2} V^* \otimes V) \\
 & & & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_{n-1}} & & \downarrow{\scriptstyle \phi_{n-2}} \\
\Gamma(\wedge^n V^*) & \xrightarrow{\partial_n} & \Gamma(\wedge^{n-1} V^*) & \xrightarrow{\partial_{n-1}} & \Gamma(\wedge^{n-2} V^*) & \xrightarrow{\partial_{n-2}} & \Gamma(\wedge^{n-3} V^*) & &
\end{array} .$$

Let $\xi \in \Gamma(K_{n-1})$ such that $d_{n-1}(\xi) = 0$.

Then there exists $\eta \in \Gamma(\wedge^n V^* \otimes V)$ such that $d_n(\eta) = \xi$.

As ϕ_n is an isomorphism, we have $\eta = \phi_n^{-1}(\phi_n(\eta))$.

Moreover, we know that $\partial_{n-1}(\phi_n(\eta)) = -\phi_{n-1}(d_n(\eta)) = -\phi_{n-1}(\xi) = 0$, so we have $\phi_n(\eta) = \partial_n(\pi)$ for some $\pi \in \Gamma(\wedge^n V^*)$. Consequently,

$$\xi = d_n(\phi_n^{-1}(\partial_n(\pi))).$$

Finally, exactness at $\Gamma(\wedge^n V^*)$ is clear. ∎

2.3.2. The L_∞-algebroid structure. In this section, we will construct the L_∞-algebroid structure on the resolution (9) of $\mathcal{F}_\mu(V)$. As in most degrees the spaces involved in the resolution of $\mathcal{F}_\mu$ are contained in the spaces involved in the resolution of $\mathcal{F}_0$, we try to use the restriction of (5).
The following lemma shows that this can be done:

Lemma 2.7. *The bracket* (5) *restricts to the subspaces* $\Gamma(K_i)$*.*

This gives us a hint on how to extend the bracket to (9): on the subcomplex given by the part up until degree $n-1$, it is given by (5). Note that there is no issue when $k_1+k_2-1=n$: since the bracket should land in $\Gamma(K_n)=0$, we can unambiguously extend this definition when we replace $\Gamma(K_n)$ by $\Gamma(\wedge^n V^*)$.

For degree reasons and the Leibniz identity in a L_∞-algebroid, we only have to specify what happens when we pair the constant extension of $X \in K_1 = \mathfrak{sl}(V)$ with the constant extension of $\mu \in \wedge^n V^*$. Due to the requirement that the differential is a derivation of the binary bracket, there is only one choice for this: We set

$$[X, \mu] := 0 \in \Gamma(\wedge^n V^*). \tag{10}$$

We then obtain:

Proposition 2.8. *The binary operation defined by the restriction of* (5) *on the spaces* $\Gamma(K_i)$*, together with the extension of* (10) *defines a dg-Lie algebroid structure on the resolution* (9) *of* $\mathcal{F}_\mu(V)$*. This is a universal* L_∞*-algebroid of* $\mathcal{F}_\mu$*, which is minimal at the origin.*

2.4. Vector fields preserving the linear symplectic form

Next, we consider the symplectic Lie algebra. Given a vector space V of even dimension n, and a non-degenerate skew-symmetric bilinear map $\omega : V \times V \to \mathbb{R}$, we consider the Lie subalgebra of $\mathfrak{gl}(V)$ preserving ω:

Definition 2.9. Let (V, ω) be a symplectic vector space. The *symplectic Lie algebra* is the Lie subalgebra of $\mathfrak{gl}(V)$ given by

$$\mathfrak{sp}(V, \omega) := \{A \in \mathfrak{gl}(V) \mid \omega(Ax, y) + \omega(x, Ay) = 0 \ \ \forall x, y \in V\}, \tag{11}$$

By restricting the anchor ρ to $\Gamma(\mathfrak{sp}(V, \omega))$, we obtain a singular foliation

$$\mathcal{F}_\omega(V) := \rho(\Gamma(\mathfrak{sp}(V, \omega))).$$

In section 2.4.1 we construct a projective resolution of $\mathcal{F}_\omega(V)$ (Proposition 2.12). In section 2.4.2 we construct a part of the L_∞-algebroid structure. We give an expression for the binary bracket (Proposition 2.15), depending on a choice of left inverse r^ω of an injective cochain map. This bracket does not satisfy the

Jacobi identity, so we give an expression for the ternary bracket, which serves as a contracting homotopy for the Jacobiator. In appendix A we investigate whether r^ω can be chosen to be a cochain map in some degrees, which would simplify the binary bracket. We show that when $\dim(V) = 4$, r^ω can not be chosen as a cochain map in any degree (Proposition A.5).

2.4.1. The projective resolution. As before, we first construct the projective resolution of the foliation $\mathcal{F}_\omega = \rho(\Gamma(\mathfrak{sp}(V, \omega)))$ on V. The starting point is the same as for $\mathfrak{sl}(V)$, but the rest of the approach will be quite different, as the analogue of the map ϕ is not surjective in negative degrees. First consider the map

$$\phi_1^\omega : \mathfrak{gl}(V) \to \wedge^2 V^* \text{ given by } \phi_1^\omega(A) = A \cdot \omega \text{ for } A \in \mathfrak{gl}(V),$$

where for $x, y \in V$, $(A \cdot \omega)(x, y) = \omega(Ax, y) + \omega(x, Ay)$.

The next step is to extend this map to the entire dg-Lie algebra $\Gamma(\wedge^\bullet V^* \otimes V)$, as in (4) and (5). This immediately raises the question what the codomain should be. We define for $p = 1, \ldots, n$

$$\phi_p^\omega : \wedge^p V^* \otimes V \to \wedge^{p-1} V^* \otimes \wedge^2 V^*$$

by $$\phi_p^\omega(\alpha \otimes X) := (-1)^{p-1} \iota_{e_i}(\alpha) \otimes e^i \wedge \iota_X \omega,$$

where $\{e_i\}_{i=1}^n$ and $\{e^i\}_{i=1}^n$ are dual bases of V and V^* respectively. When viewing the domain and codomain of ϕ_p^ω as $\mathrm{Hom}(\wedge^p V, V)$ and $\mathrm{Hom}(\wedge^{p-1} V, \wedge^2 V^*)$ respectively, the map ϕ_p^ω can equivalently be described as

$$\phi_p^\omega(\psi)(v_1, \ldots, v_{p-1}) = \psi(v_1, \ldots, v_{p-1}, -) \cdot \omega$$

for $\psi \in \mathrm{Hom}(\wedge^p V, V)$, $v_1, \ldots, v_{p-1} \in V$.

We now equip the graded $C^\infty(V)$-module $\Gamma(\wedge^\bullet V^* \otimes \wedge^2 V^*)$ with the differential

$$\partial_p : \Gamma(\wedge^p V^* \otimes \wedge^2 V^*) \to \Gamma(\wedge^{p-1} V^* \otimes \wedge^2 V^*)$$

given by $$\partial_p(\alpha \otimes \tau) := -x^i \iota_{e_i}(\alpha) \otimes \tau.$$

Here the grading is chosen as

$$\Gamma(\wedge^\bullet V^* \otimes \wedge^2 V^*)^p = \Gamma(\wedge^{p+1} V^* \otimes \wedge^2 V^*).$$

Finally, there is an action of the dg-Lie algebra $\Gamma(\wedge^\bullet V^* \otimes V)$ on $\Gamma(\wedge^\bullet V^* \otimes \wedge^2 V^*)$:

for $$\alpha \otimes X \in \wedge^p V^* \otimes V, \ \beta \otimes \tau \in \wedge^q V^* \otimes \wedge^2 V^*,$$

we set $$(\alpha \otimes X) \cdot (\beta \otimes \tau) := (-1)^{p-1} \alpha \iota_X(\beta) \otimes \tau + \iota_{e_i}(\alpha) \beta \otimes e^i \wedge \iota_X(\tau) \tag{12}$$

for constant sections, and extend it to non constant sections using the Leibniz rule with respect to the anchor of $\Gamma(\mathfrak{gl}(V))$. Note that by multiplying by $-(-1)^{(p-1)(q-1)}$, we can turn this into a right action. We have the following lemma summarizing the properties of the data above, of which the proof is a direct computation.

Lemma 2.10. (i) *ϕ_1^ω is surjective, and* $\ker(\phi_1^\omega) = \mathfrak{sp}(V,\omega)$.

(ii) *For $k \geq 2$, ϕ_k^ω is injective. In particular, ϕ_2^ω is an isomorphism.*

(iii) *ϕ^ω is a cochain map of degree -1, i.e. we have $\phi_p^\omega d_{p+1} + \partial_p \phi_{p+1}^\omega = 0$.*

(iv) *The operation defined in* (12) *is a dg-Lie algebra action. Consequently, there is a dg-Lie algebra structure on $\Gamma(\wedge^\bullet V^* \otimes V) \oplus \Gamma(\wedge^\bullet V^* \otimes \wedge^2 V^*)$ encoding this action.*
For $p \geq 0$, the degree p-part is given by $\Gamma(\wedge^{p+1} V^* \otimes V) \oplus \Gamma(\wedge^{p+1} V^* \otimes \wedge^2 V^*)$, and the differential is given by $d + \partial$.

(v) *ϕ^ω is a derivation of the bracket on $\Gamma(\wedge^\bullet V^* \otimes V) \oplus \Gamma(\wedge^\bullet V^* \otimes \wedge^2 V^*)$: for $\alpha \otimes X \in \Gamma(\wedge^p V^\vee \otimes V), \beta \otimes Y \in \Gamma(\wedge^q V^\vee \otimes V)$, we have the equality*

$$\phi_{p+q-1}^\omega([\alpha \otimes X, \beta \otimes Y]) = [\phi_p^\omega(\alpha \otimes X), \beta \otimes Y] + (-1)^{p-1}[\alpha \otimes X, \phi_q^\omega(\beta \otimes Y)]. \quad (13)$$

Corollary 2.11. *By property* (iii)*, the differentials $d_\bullet$ and $\partial_\bullet$ restrict and descend to the kernel and cokernel of ϕ^ω respectively. We denote the differential induced by $\partial_\bullet$ on* $\Gamma(\operatorname{coker}(\phi^\omega))$ *by $\bar{\partial}_\bullet$.*

Using these properties, we can construct a projective resolution:

Proposition 2.12. *For $i = 3, \dots, n+1$, let C_i be defined as $C_i := \operatorname{coker}(\phi_i^\omega)$. The sequence*

$$0 \longrightarrow \Gamma(C_{n+1}) \xrightarrow{\bar{\partial}_n} \dots \xrightarrow{\bar{\partial}_3} \Gamma(C_3) \xrightarrow{d_2(\phi_2^\omega)^{-1}\partial_2} \Gamma(\mathfrak{sp}(V,\omega)) \longrightarrow \mathcal{F}_\omega(V) \longrightarrow 0\,, \quad (14)$$

is exact. Here $\phi_{n+1}^\omega : 0 \to \Gamma(\wedge^n V^* \otimes \wedge^2 V^*)$ is understood to be the zero map.

Proof. We start by proving exactness at $\Gamma(C_p)$ for $p = 4, \dots, n+1$. Consider the following diagram in which we use the notations $A_n := \Gamma(\wedge^n V^* \otimes V)$ and $B_n := \Gamma(\wedge^n V^* \otimes \wedge^2 V^*)$ merely for the purpose to get the diagram into the textframe:

$$\begin{array}{ccccccc}
A_{p+1} & \xrightarrow{d_{p+1}} & A_p & \xrightarrow{d_p} & A_{p-1} & \xrightarrow{d_{p-1}} & A_{p-2} \\
\downarrow{\scriptstyle \phi_{p+1}^\omega} & & \downarrow{\scriptstyle \phi_p^\omega} & & \downarrow{\scriptstyle \phi_{p-1}^\omega} & & \downarrow{\scriptstyle \phi_{p-2}^\omega} \\
B_p & \xrightarrow{\partial_p} & B_{p-1} & \xrightarrow{\partial_{p-1}} & B_{p-2} & \xrightarrow{\partial_{p-2}} & B_{p-3} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma(C_{p+1}) & \xrightarrow{\overline{\partial_p}} & \Gamma(C_p) & \xrightarrow{\overline{\partial_{p-1}}} & \Gamma(C_{p-1}) & \xrightarrow{\overline{\partial_{p-2}}} & \Gamma(C_{p-2})
\end{array}$$

For $\tau \in \Gamma(\wedge^{p-1} V^* \otimes \wedge^2 V^*)$, assume that there exists $X \in \Gamma(\wedge^{p-1} V^* \otimes V)$ such that

$$\partial_{p-1}(\tau) = \phi_{p-1}^\omega(X).$$

Then $\phi_{p-2}^\omega(d_{p-1}(X)) = -\partial_{p-2}(\phi_{p-1}^\omega(X)) = 0$, and by injectivity of ϕ_{p-1}^ω, it follows that $d_{p-1}(X) = 0$. By exactness of (4) we find $X = d_p(Y)$. Now

$$\partial_{p-1}(\tau + \phi_p^\omega(Y)) = \phi_{p-1}^\omega(X) - \phi_{p-1}^\omega(X) = 0,$$

so by exactness of (3), with $W = \wedge^2 V^*$, we find that there exists $\mu \in \Gamma(\wedge^p V^* \otimes \wedge^2 V^*)$ such that $\tau = \partial_p(\mu) - \phi_p^\omega(Y)$, showing that the class of τ modulo the image of ϕ^ω is a coboundary.

For exactness at $\Gamma(C_3)$, assume that for $\tau \in \Gamma(\wedge^2 V^* \otimes \wedge^2 V^*)$ such that

$$d_2(\phi_2^\omega)^{-1}\partial_2(\tau) = 0.$$

Then there exists $X \in \Gamma(\wedge^3 V^* \otimes V)$ with

$$(\phi_2^\omega)^{-1}\partial_2(\tau) = d_3(X), \quad \text{or} \quad \partial_2(\tau + \phi_2^\omega(X)) = 0,$$

which implies that there exists $\mu \in \Gamma(\wedge^3 V^* \otimes \wedge^2 V^*)$ such that

$$\partial_3(\mu) = \tau + \phi_2^\omega(X),$$

so the class of τ module the image of ϕ^ω is a coboundary.

Finally, by definition of $\mathcal{F}_\omega$ the anchor restricted to $\mathfrak{sp}(V,\omega)$ is surjective, so it suffices to show that the kernel is precisely $(d_2(\phi_2^\omega)^{-1}\partial_2)(\Gamma(\wedge^2 V^* \wedge^2 V^*)$. Take some $A \in \Gamma(\mathfrak{sp}(V,\omega))$ such that $\rho_\omega(A) = 0$. Then $A = d_2(X)$ for some $X \in \Gamma(\wedge^2 V^* \otimes V)$. As ϕ_2^ω is an isomorphism, we have $A = d_2(\phi_2^\omega)^{-1}\phi_2^\omega(X)$.

As $\partial_1\phi_2^\omega(X) = -\phi_1^\omega(d_2(X)) = \phi_1^\omega(A) = 0$, it follows that $\phi_2^\omega(X) = \partial(\mu)$ for some $\mu \in \Gamma(\wedge^2 V^* \otimes \wedge^2 V^*)$, and $A = d_2(\phi_2^\omega)^{-1}\partial_2(\mu)$, concluding the proof. ■

2.4.2. The (partial) L_∞-algebroid structure

In this section we construct part of an L_∞-algebroid structure on the resolution (14). Due to the algebraic structures present, there is a canonical Lie bracket $\{-,-\}$ on the graded $C^\infty(V)$-module underlying the resolution (14). It is given by

$$\{A, B\} = [A, B]$$

for $A, B \in \Gamma(\mathfrak{sp}(V,\omega))$, where $[-,-]$ is the usual bracket, and

$$\{A, \omega_k + \mathrm{im}(\phi_{k+1}^\omega)\} = [A, \omega_k] + \mathrm{im}(\phi_{k+1}^\omega)$$

for $A \in \Gamma(\mathfrak{sp}(V,\omega)), \omega_k \in \Gamma(C_{k+1})$. Here the bracket $[-,-]$ on the right hand side is the semi-direct product bracket as described in Lemma 2.10iv).

Remark 2.13. One way to see that $\{-,-\}$ is well-defined, is by forgetting the differentials $d_\bullet$ and $\partial_\bullet$, and viewing $(\Gamma(\wedge^\bullet V^*) \oplus \Gamma(\wedge^\bullet V^* \otimes \wedge^2 V^*), \phi^\omega)$ as a dg-Lie algebra. The resolution (14) of $\mathcal{F}_\omega(V)$ is then precisely the cohomology of this dg-Lie algebra. Consequently, the bracket equips the graded module with a graded Lie algebra structure. ■

When at least one of the entries of $\{-,-\}$ has degree 0, the differential in (14) is a derivation of $\{-,-\}$. However, for elements $\omega_1, \omega_2 \in \Gamma(C_3)$, for the differential in (14) to be a derivation of the bracket, the equation

$$[d_2(\phi_2^\omega)^{-1}\partial_2\omega_1, \omega_2] - [\omega_1, d_2(\phi_2^\omega)^{-1}\partial_2\omega_2] = \overline{\partial_3}[\omega_1, \omega_2] = 0 \in \Gamma(C_4) \tag{15}$$

must hold. This means that the expression (15) must lie in the image of ϕ_4^ω, but one can check that this is not the case: the binary operation $\{-.-\}$ therefore does not equip the resolution (14) with a L_∞-algebroid structure, as the differential is not a derivation of the binary bracket.

To rectify this, we modify $\{-.-\}$ to obtain a new binary operation $[\![-,-]\!]$ on the resolution (14), for which the differential is a derivation.

Before we define this binary operation, we make a choice of left inverse of the map

$$\phi_p^\omega : \Gamma(\wedge^p V^* \otimes V) \to \Gamma(\wedge^{p-1}V^* \otimes \wedge^2 V^*)$$

for $p \geq 2$. Define $r_p^\omega : \wedge^p V^* \otimes \wedge^2 V^* \to \wedge^{p+1}V^* \otimes V$ by

$$r_p^\omega(\omega_p \otimes \tau) = \left(\frac{1}{p+1}\omega_p \wedge \iota_{e_i}(\tau) - \frac{(-1)^p}{p(p+1)}\iota_{e_i}(\omega_p) \wedge \tau \right) \otimes \omega^{-1}(e^i) \in \wedge^{p+1}V^* \otimes V$$

for $\omega_p \in \wedge^p V^*, \tau \in \wedge^2 V^*$. Further, $\{e_i\}_{i=1}^n, \{e^i\}_{i=1}^n$ are dual bases for V and V^* respectively, and $\omega^{-1} : V^* \to V$ is the inverse of the contraction map $\omega : V \to V^*$. The proof of the following lemma is a straightforward computation:

Lemma 2.14.

(i) *For $k \geq 2$, r_k^ω intertwines the $\mathfrak{sp}(V,\omega)$-action on $\Gamma(\wedge^k V^* \otimes \wedge^2 V^*)$ and $\Gamma(\wedge^{k+1}V^* \otimes V)$.*

(ii) *For $k \geq 2$, we have $r_k^\omega \circ \phi_{k+1}^\omega = id_{\Gamma(\wedge^{k+1}V^* \otimes V)}$.*

We can now give an expression for the binary operation $[\![-,-]\!]$ for which the differential of the resolution (14) is a derivation, providing the binary bracket for an L_∞-algebroid structure on the resolution.

Proposition 2.15. *When at least one entry of $[\![-,-]\!]$ has degree 0, we set*

$$[\![-,-]\!] = \{-,-\}.$$

Now let $p, q \geq 2$. For $\omega_p \in \Gamma(\wedge^p V^* \otimes \wedge^2 V^*)$, $\omega_q \in \Gamma(\wedge^q V^* \otimes \wedge^2 V^*)$, set

$$\begin{aligned}[\![\omega_p, \omega_q]\!] := [r_{p-1}^\omega \partial_p P_p(\omega_p), P_q(\omega_q)] + [P_p(\omega_p), r_{q-1}^\omega \partial_q P_q(\omega_q)] \\ + im(\phi_{p+q}^\omega) \in \Gamma(C_{p+q}).\end{aligned} \tag{16}$$

Here $P_p : \Gamma(\wedge^p V^* \otimes \wedge^2 V^*) \to \Gamma(\ker(r_p^\omega))$ is the projection

$$P_p = id - \phi_{p+1}^\omega \circ r_p^\omega,$$

and $[-,-]$ on the right hand side is the semi-direct product bracket as described in Lemma 2.10(iv). Then the differential of (14) is a derivation of $[\![-,-]\!]$.

Proof. The proof is a direct computation using Lemma 2.10(iv) and Lemma 2.14(ii). ■

The natural question is now: Does $[\![-,-]\!]$ satisfy the Jacobi identity? To address this, we distinguish two cases. We first compute the Jacobiator when at least one of the entries has degree 0, and then when all the entries have negative degree.

- For $A, B, C \in \Gamma(\mathfrak{sp}(V,\omega))$, the Jacobiator of $[\![-,-]\!]$ is the Jacobiator of the Lie algebroid $\Gamma(\mathfrak{sp}(V,\omega))$, which vanishes.
- For $A, B \in \Gamma(\mathfrak{sp}(V,\omega))$, $\omega_k \in \Gamma(\text{coker}(\phi_{k+1}^\omega))$, the Jacobiator being zero is equivalent to $\text{coker}(\phi_{k+1}^\omega)$ being an $\mathfrak{sp}(V,\omega)$-representation.

- For $A \in \Gamma(\mathfrak{sp}(V,\omega))$, $\omega_k \in \Gamma(\mathrm{coker}(\phi^\omega_{k+1}), \omega_l \in \Gamma(\mathrm{coker}(\phi^\omega_{l+1})$, the Jacobiator being zero is equivalent to the $\mathfrak{sp}(V,\omega)$-action on $\Gamma(\mathrm{coker}(\phi^\omega_j))$ being a derivation of $[\![-,-]\!]$ restricted to negative degrees. This is the case because r_k intertwines the $\mathfrak{sp}(V,\omega)$-actions on $\Gamma(\wedge^k V^* \otimes \wedge^2 V^*)$ and $\Gamma(\wedge^{k+1} V^* \otimes \wedge^2 V^*)$.

Consequently, the Jacobiator vanishes when at least one entry has degree 0.

Now let $k, l, m \geq 2$, and

$$\omega_k \in \Gamma(\mathrm{coker}(\phi^\omega_{k+1})), \quad \omega_l \in \Gamma(\mathrm{coker}(\phi^\omega_{l+1})), \ \omega_m \in \Gamma(\mathrm{coker}(\phi^\omega_{m+1})).$$

A lengthy computation shows that the Jacobiator

$$[\![[\![\omega_k,\omega_l]\!],\omega_m]\!] + (-1)^{(k-1)(l+m)}[\![[\![\omega_l,\omega_m]\!],\omega_k]\!] + (-1)^{(m-1)(k+l)}[\![[\![\omega_m,\omega_k]\!],\omega_l]\!]$$

does not vanish, but is equal to

$$\overline{\partial}([\![-,-,-]\!])(\omega_k,\omega_l,\omega_m) = \overline{\partial}[\![\omega_k,\omega_l,\omega_m]\!] + [\![\overline{\partial}(\omega_k),\omega_l,\omega_m]\!] + (-1)^{k-1}[\![\omega_k,\overline{\partial}(\omega_l),\omega_m]\!] + (-1)^{k+l}[\![\omega_k,\omega_l,\overline{\partial}(\omega_m)]\!], \tag{17}$$

where $[\![\omega_k,\omega_l,\omega_m]\!]$ is given by the class of

$$[r_{k+l-1}\widehat{[\![\omega_k,\omega_l]\!]}, P_m(\omega_m)] + (-1)^{(k-1)(l+m)}[r_{k+l-1}\widehat{[\![\omega_l,\omega_m]\!]}, P_k(\omega_k)] + (-1)^{(m-1)(k+l)}[r_{k+l-1}\widehat{[\![\omega_m,\omega_k]\!]}, P_l(\omega_l)] \tag{18}$$

modulo the image of $\phi^\omega_{k+l+m-1}$, and

$$\widehat{[\![\omega_k,\omega_l]\!]} = [r_{k-1}\partial_k P_k(\omega_k), P_l(\omega_l)] + [P_k(\omega_k), r_{l-1}\partial_l P_l(\omega_l)] \in \Gamma(\wedge^{k+l-1}V^* \otimes \wedge^2 V^*).$$

We recognize equation (17) as a contracting homotopy for the Jacobiator: consequently, $-[\![-,-,-]\!]$ is a ternary operation satisfying the higher Jacobi identity an L_∞-algebroid must satisfy.

In particular, this does not equip the complex (14) with the structure of a dg-Lie algebroid as in the case of $\mathfrak{gl}(V)$, $\mathfrak{gl}(V,W)$, and $\mathfrak{sl}(V)$. As this structure is only unique up to L_∞-algebroid homotopy, this does of course not exclude the possibility that there exists a dg-Lie algebroid structure inducing the foliation $\mathcal{F}_\omega$.

In Appendix A, we investigate to what extent r^ω can be chosen to (anti)-commute with the differentials, as this would simplify both the binary and ternary bracket.

Remark 2.16. (1) For degree reasons, the operation $[\![-,-,-]\!]$ vanishes when $\dim V \leq 4$. This means that when $\dim V = 2$ or $\dim V = 4$, the foliation $\mathcal{F}_\omega$ does admit a universal L_∞-algebroid with only a unary and binary bracket. Of course, for $\dim V = 2$, $\mathfrak{sp}(V,\omega) = \mathfrak{sl}(V)$, for which it was already known that a dg-Lie algebroid structure structure exists.

(2) When $\dim V = 6$, the unary, binary and ternary bracket determine the full L_∞-algebroid structure. ■

3. Higher-dimensional leaves

In this section we address question (2) of section 1. Instead of considering foliations on a vector space with linear generators, we consider foliations $\mathcal{F}$ on vector *bundles* $\pi : E \to L$ which are generated by *fiberwise* linear vector fields such that the zero section is a leaf.

As for $x \in L$ the fibers $E_x = \pi^{-1}(\{x\})$ of $\pi : E \to L$ are transverse to L, the foliation $\mathcal{F}$ restricts to the fibers ([1, Proposition 1.10]). We consider foliations for which the restriction $\mathcal{F}|_{E_x}$ coincides with one of the examples in section 2. All results given in section 2 will carry over, although in order to define the analogue of $\mathcal{F}_\mu$ and $\mathcal{F}_\omega$ we need the existence of non-vanishing sections of $\wedge^{\mathrm{rk}(E)} E^*$ and $\wedge^2 E^*$ respectively.

In section 3.1, we consider the foliation of all vector fields on the vector bundle E, for which the restriction to the zero section is tangent to the zero section. This foliation consists of *all* fiberwise linear vector fields and is the analogue of $\mathcal{F}_0$ in section 2.1.

In section 3.2, we consider the foliation of all fiberwise linear vector fields on E tangent to a subbundle D, which is the analogue of $\mathcal{F}_W$ in section 2.2.

In section 3.3, we assume that E is orientable, and consider the foliation of all fiberwise linear vector fields on E which preserve a non-vanishing (on L) section $\mu \in \Gamma(\wedge^{\mathrm{rk}(E)} E^*)$, which is the analogue of $\mathcal{F}_\mu$ in section 2.3.

In section 3.4, we assume that $E \to L$ is a *symplectic* vector bundle with non-degenerate $\omega \in \Gamma(\wedge^2 E^*)$, and consider the foliation of all fiberwise linear vector fields on E which preserve ω, which is the analogue of $\mathcal{F}_\omega$ in section 2.4.

3.1. Vector fields tangent to the zero section

To generalize section 2.1, we need a generalization of the Lie algebra $\mathfrak{gl}(V)$ for a vector space V to a vector bundle. Let L be a smooth manifold, and let $\pi : E \to L$ be a (real) vector bundle of rank n. One way to generalize $\mathfrak{gl}(V)$ would be to use the Lie algebra bundle $\mathrm{End}(E)$. This is however not the right thing to consider: although it acts infinitesimally on E, for $x \in L$ the leaves of the foliation are given by $0_x \in E_x$ and the connected components of $E_x \setminus 0_x$. We are however interested in the situation where the zero section is a leaf, and the transverse foliation at a point of the zero section is given by (1). This is the case when dealing with a linearizable foliation around an embedded codimension n leaf with isotropy Lie algebra bundle $\mathrm{End}(E)$ acting on the n-dimensional fiber of the normal bundle by all vector bundle maps $E \to E$.

First recall that there are two distinguished classes of smooth functions on a vector bundle E. The fiberwise constant maps, given by the image of $\pi^* : C^\infty(L) \to C^\infty(E)$, and the fiberwise linear ones, given by $\Gamma(E^*)$. Now let

$$\mathfrak{X}_{lin}(E) = \{X \in \mathfrak{X}(E) \mid X(\pi^*(C^\infty(L))) \subset \pi^*(C^\infty(L)),\, X(\Gamma(E^*)) \subset \Gamma(E^*)\}$$

be the set of vector fields preserving the fiberwise constant functions and fiberwise linear functions. First off, we note that $\mathfrak{X}_{lin}(E)$ is isomorphic to the sections of a transitive Lie algebroid over L (see [7, Theorem 1.4]).

Lemma 3.1. (i) *There is a short exact sequence of $C^\infty(L)$-modules*

$$0 \longrightarrow \Gamma(End(E)) \xrightarrow{a} \mathfrak{X}_{lin}(E) \xrightarrow{\rho} \mathfrak{X}(L) \longrightarrow 0.$$

Here ρ is the restriction of a vector field to the subalgebra $\pi^*(C^\infty(L))$ and

$$a(A)(f) = \left.\frac{d}{dt}\right|_{t=0} f \circ exp(-tA),$$

which is a fiberwise extension of the identification of linear vector fields on a vector space with the endomorphisms on the vector space.

(ii) $\mathfrak{X}_{lin}(E)$ *is a finitely generated projective* $C^\infty(L)$*-module. So there exists a vector bundle* $\mathfrak{gl}(E)$ *such that* $\mathfrak{X}_{lin}(E) = \Gamma(\mathfrak{gl}(E))$.

(iii) $\mathfrak{X}_{lin}(E)$ *is closed under the Lie bracket of vector fields.*

(iv) *The triple* $(\mathfrak{gl}(E), \rho, [-,-])$ *is a Lie algebroid.*

Proof. (i) is a local computation, (ii) follows from (i), and (iii) and (iv) are immediate. ∎

We can now construct a singular foliation on E for which L is a leaf: let

$$\mathcal{F}_L(E) = \{X \in \mathfrak{X}(E) \mid X|_L \in \mathfrak{X}(L)\}.$$

Lemma 3.2. $\mathcal{F}_L(E) = Im(C^\infty(E) \otimes_{C^\infty(L)} \mathfrak{X}_{lin}(E) \xrightarrow{m} \mathfrak{X}(E))$, *where* m *is the natural multiplication map.*

Proof. It follows from a local computation. ∎

Remark 3.3. Note that $C^\infty(E) \otimes_{C^\infty(L)} \mathfrak{X}_{lin}(E) = \Gamma(\pi^*(\mathfrak{gl}(E)))$, which are the sections of the action Lie algebroid corresponding to the natural action of $\mathfrak{X}_{lin}(E)$ on E. The anchor of this action Lie algebroid $\pi^*(\mathfrak{gl}(E))$ is m, and the bracket is given for $f, g \in C^\infty(E)$, $X, Y \in \mathfrak{X}_{lin}(E)$ by

$$[f \otimes X, g \otimes Y] = fg \otimes [X,Y] + fX(g) \otimes Y - gY(f) \otimes X.$$

3.1.1. The projective resolution. Lemma 3.2 now gives a first step in the resolution of $\mathcal{F}_L(E)$:

$$\Gamma(\pi^*(\mathfrak{gl}(E))) \xrightarrow{m} \mathcal{F}_L(E) \longrightarrow 0.$$

Observe that m is not injective! However, the kernel can be explicitly described. As a first step, we show that the kernel only affects the direction transverse to the leaf.

Lemma 3.4. $\ker(m) \subset \Gamma(\pi^*(End(E))) \subset \Gamma(\pi^*(\mathfrak{gl}(E)))$.

The following argument is thanks to Marco Zambon.

Proof. There is a commutative diagram of $C^\infty(E)$-modules given by

$$\begin{array}{ccc} \Gamma(\pi^*(\mathfrak{gl}(E))) & \xrightarrow{m} & \mathfrak{X}(E) \\ & \searrow^{\mathrm{Id}\otimes\rho} & \downarrow{\scriptstyle d\pi} \\ & & \Gamma(\pi^*(TL)) \end{array} .$$

The statement now follows from Lemma 3.1(i). ∎

Viewing $End(E)$ as $E^* \otimes E$, we can write down a complex analogous to (4).

$$0 \longrightarrow \Gamma(\pi^*(\wedge^n E^* \otimes E)) \xrightarrow{d_n} \dots \xrightarrow{d_2} \Gamma(\pi^*(\mathfrak{gl}(E))) \longrightarrow \mathcal{F}_L(E) \longrightarrow 0, \qquad (19)$$

where $$d_k : \Gamma(\pi^*(\wedge^k E^* \otimes E)) \to \Gamma(\pi^*(\wedge^{k-1} E^* \otimes E)) \tag{20}$$

is given by $$d_k(\alpha \otimes e) = y^i \iota_{e_i}(\alpha) \otimes e, \tag{21}$$

for $\alpha \otimes e \in \Gamma(\pi^*(\wedge^k(E^* \otimes E)))$, $\{e_i\}_{i=1}^n$ a local frame of E, and $\{y^i\}_{i=1}^n$ the corresponding linear coordinates (note that this does not depend on the choice of frame and defines a global section $\epsilon \in \Gamma(\pi^* E)$). We then have:

Proposition 3.5. *The complex* (19) *is exact.*

Proof. Proving exactness is completely analogous to the case considered in section 2.1: it suffices to pick an open cover of L over which E trivializes (so $\pi^*(E)$ trivializes over the preimages of this cover), and show exactness over this open cover. But for trivial bundles the result is equivalent to the exactness of (4). ∎

3.1.2. The L_∞-algebroid structure. We claim that we can again find a dg-Lie algebroid structure on the resolution (19). Note that for degrees $-1, \ldots, -n+1$, the involved spaces are simply the fiberwise extensions of (4), so we take the fiberwise extension of (5). To incorporate $\mathfrak{X}_{lin}(E)$ inside into this, we recall the following:

Lemma 3.6. *The action of $\mathfrak{X}_{lin}(E)$ on $\Gamma(E^*)$ extends to $\Gamma(E)$, all tensor, wedge and symmetric products and their pullbacks to E.*

Proof. Recall that an action of $\mathfrak{X}_{lin}(E)$ on a vector bundle F is a flat $\mathfrak{gl}(E)$-connection on the vector bundle F. As the action on $\Gamma(E^*)$ is equivalent to a $\mathfrak{gl}(E)$-connection on E^*, one can dualize this connection, and extend it via the Leibniz rule to tensor powers.

Finally, to extend the action to the pullback, we recall that

$$\Gamma(\pi^*(E^*)) = C^\infty(E) \otimes_{C^\infty(L)} \Gamma(E^*),$$

and that both factors have a natural action of $\mathfrak{X}_{lin}(E)$.

For $g \otimes X \in C^\infty(E) \otimes_{C^\infty(L)} \mathfrak{X}_{lin}(E)$, $f \otimes \alpha \in \Gamma(\pi^*(E^*))$, the action is given by

$$(g \otimes X) \cdot (f \otimes \alpha) = gX(f) \otimes \alpha + gf \otimes X(\alpha).$$

Since duals and tensors commute with pullbacks, the result follows. ∎

Using these actions, we can describe a dg-Lie algebroid structure on the resolution (19):

Proposition 3.7. *The complex* (19) *carries a dg-Lie algebroid structure, where the binary bracket is given by the analogue of equation* (5) *on elements of degree* -1 *and lower, and the bracket involving an element $f \otimes X \in \Gamma(\pi^*(\mathfrak{gl}(E)))$ and an element $g \otimes \alpha \otimes e \in \Gamma(\pi^*(\wedge^k E^* \otimes E))$ for $f, g \in C^\infty(E)$, $X \in \mathfrak{X}_{lin}(E)$, $\alpha \in \Gamma(\wedge^k E^*)$, $e \in \Gamma(E)$ is given by*

$$[f \otimes X, g \otimes \alpha \otimes e] = fX(g) \otimes \alpha \otimes e + fg \otimes X \cdot (\alpha \otimes e).$$

The bracket involving two elements of $\Gamma(\pi^*(\mathfrak{gl}(E)))$ is given by the action Lie algebroid bracket.

3.2. Linear vector fields preserving a subbundle

Let $\pi : E \to L$ be a vector bundle and $D \subset E$ a vector subbundle. In this section we combine sections 2.2 and 3.1 to give a projective resolution of the subfoliation $\mathcal{F}_D \subset \mathcal{F}_L$ given by

$$\mathcal{F}_D(E) = \{X \in \mathcal{F}_L(E) \mid X(I_D) \subset I_D\},$$

where I_D is the vanishing ideal of $D \subset E$. In other words, $\mathcal{F}_D(E)$ consists of all vector fields which are tangent to the subbundle D. Note that when $D = 0$, we are in the situation of section 3.1.

This can be approached in a similar way as $\mathcal{F}_L$: define

$$\mathfrak{X}_{lin}(E, D) := \{X \in \mathfrak{X}_{lin}(E) \mid X(\Gamma(\mathrm{Ann}(D))) \subset \Gamma(\mathrm{Ann}(D))\},$$

where $\Gamma(\mathrm{Ann}(D)) \subset \Gamma(E^*)$ is viewed as a subset of $C^\infty(E)$.

For $i \geq 2$, let $K_i \subset \wedge^i E^* \otimes E$ be the subbundle given by

$$K_i := \{\phi \in \wedge^i E^* \otimes E \mid \forall d \in D, \forall e_1, \dots, e_{i-1} \in E : \phi(d, e_1, \dots, e_{i-1}) \in D\},$$

Here the condition should be read fiberwise. Further, define $\mathfrak{gl}(E, D) \subset \mathfrak{gl}(E)$ as the subbundle whose sections are precisely $\mathfrak{X}_{lin}(E, D)$. Then the analogue of Proposition 2.4 holds, and we find:

Proposition 3.8. (i) *For $j \geq 2$, the differential*

$$d_j : \Gamma(\pi^*(\wedge^j E^* \otimes E)) \to \Gamma(\pi^*(\wedge^{j-1} E^* \otimes E))$$

as in (19) *restricts to a map $d_j : \Gamma(\pi^*(K_j)) \to \Gamma(\pi^*(K_{j-1}))$, and we have $d_2(\Gamma(\pi^*(K_2))) \subset \Gamma(\pi^*(\mathfrak{gl}(E, D)))$.*

(ii) *The complex*

$$0 \longrightarrow \Gamma(\pi^*(K_n)) \xrightarrow{d_n} \dots \xrightarrow{d_2} \Gamma(\pi^*(\mathfrak{gl}(E, D))) \xrightarrow{\rho_D} \mathcal{F}_D(E) \longrightarrow 0 \qquad (22)$$

is exact.

(iii) *The bracket as described in Proposition* 3.7 *restricts to* (22)*.*

Consequently, (22) *with the restrictions of the differential and bracket is a universal L_∞-algebroid of the foliation $\mathcal{F}_D$, which is minimal at points of L.*

3.3. Vector fields preserving a volume form

In section 2.3 we constructed the universal L_∞-algebroid for the foliation given by the action of $\mathfrak{sl}(V)$ for an n-dimensional vector space V. Although we made the choice of a volume form, this was not strictly necessary in this case.

Now if we want to generalize this example to the case of higher dimensional leaves, i.e. a linear foliation on a vector bundle $\pi : E \to L$ of rank n, such that the zero section is a leaf and the transverse foliation on the fibers E_x for $x \in L$ is isomorphic to the one given by the action of $\mathfrak{sl}(E_x)$, one approach would be to take an appropriate Lie subalgebroid $\mathfrak{sl}(E, \mu)$ of $\mathfrak{gl}(E)$ and to look at the induced foliation on E.

To generalize the special linear subalgebra, the sections of the Lie subalgebroid $\mathfrak{sl}(E,\mu)$ should sit in a short exact sequence

$$0 \longrightarrow \Gamma(End_0(E)) \xrightarrow{a} \Gamma(\mathfrak{sl}(E,\mu)) \xrightarrow{\rho} \mathfrak{X}(L) \longrightarrow 0, \tag{23}$$

where $End_0(E)$ is the kernel of the vector bundle map $\mathrm{Tr} : End(E) \to \mathbb{R}$ given by the trace.

3.3.1. The projective resolution. Assume that E is orientable and pick a volume form $\mu \in \Gamma(\wedge^n E^*)$. Then we can identify the Lie algebroid $\mathfrak{sl}(E,\mu)$ as follows: Let

$$\mathfrak{X}^{\mu}_{lin}(E) = \{X \in \mathfrak{X}_{lin}(E) \mid X \cdot \mu = 0\},$$

where $X \cdot \mu$ is defined as in Lemma 3.6. A local computation shows that there exists a vector bundle $\mathfrak{sl}(E,\mu)$ satisfying (23) such that

$$\Gamma(\mathfrak{sl}(E,\mu)) = \mathfrak{X}^{\mu}_{lin}(E)$$

analogous to Lemma 3.1. To construct the projective resolution, we adopt a similar approach as in the case where L was a point. Define

$$\widehat{\mathrm{Tr}} : \mathfrak{X}_{lin}(E) \to \Gamma(\wedge^n E^*) \quad \text{by} \quad \widehat{\mathrm{Tr}}(X) = -X \cdot \mu.$$

Clearly, $\mathfrak{X}^{\mu}_{lin}(E) = \ker(\widehat{\mathrm{Tr}})$. Moreover, this really extends the trace:

Lemma 3.9. *For $A \in \Gamma(End(E))$, $\widehat{\mathrm{Tr}} \circ a(A) = \mathrm{Tr}(A)\mu$.*

Proof. Pick a local frame $\{e_i\}_{i=1}^n$ for E and a dual frame $\{e^i\}_{i=1}^n$ for E^*, such that $\mu = e^1 \wedge \cdots \wedge e^n$. Then

$$\begin{aligned}\widehat{\mathrm{Tr}}(a(A)) &= -\sum_{i=1}^{n} e^1 \wedge \cdots \wedge \left.\frac{d}{dt}\right|_{t=0} e^i \circ exp(-tA) \wedge \cdots \wedge e^n \\ &= -\sum_{i=1}^{n} e^1 \wedge \cdots \wedge e^i \circ \left.\frac{d}{dt}\right|_{t=0} exp(-tA) \wedge \cdots \wedge e^n \\ &= \sum_{i=1}^{n} e^1 \wedge \cdots \wedge e^i \circ A \wedge \cdots \wedge e^n = \mathrm{Tr}(A)\mu. \end{aligned}$$ ■

We now apply the same ideas as in the case where L is a point. As in Lemma 3.4, the kernel of $\rho_\mu : \Gamma(\pi^*(\mathfrak{sl}(E,\mu))) \to \mathfrak{X}(E)$ is contained in $\Gamma(\pi^*(\mathrm{End}_0(E)))$, so we proceed in a similar way as in section 2.3. Define for $i = 2, \ldots, n$ the vector bundle map $\phi_i : \wedge^i E^* \otimes E \to \wedge^{i-1} E^*$ over L by $\phi_i(\alpha \otimes e) = (-1)^{i-1}\iota_e(\alpha)$ for $\alpha \in \wedge^i E^*, e \in E$. Setting $K_i = \ker(\phi_i)$, we obtain the following analogue of Proposition 2.6:

Proposition 3.10.

$$0 \longrightarrow \Gamma(\pi^*(\wedge^n E^*)) \xrightarrow{d_n \phi_n^{-1} \partial_n} \Gamma(\pi^*(K_{n-1})) \xrightarrow{d_{n-1}} \cdots$$
$$\cdots \xrightarrow{d_2} \Gamma(\pi^*(\mathfrak{sl}(E,\mu))) \xrightarrow{\rho_\mu} \mathcal{F}^{\mu}_L(E) \longrightarrow 0 \tag{24}$$

is exact.

3.3.2. The L_∞-algebroid structure As in section 2.3.2, to define the L_∞-algebroid structure on the resolution (24), we would like to restrict the bracket as described in Proposition 3.7 to the kernel of the morphisms ϕ_k for $k \in \{1, \dots, n-1\}$. For elements of degrees -1 and lower and for two elements from $\Gamma(\pi^*(\mathfrak{sl}(E,\mu)))$ this is clear, as this is just the fiberwise extension of Lemma 2.7.

For the bracket between $\Gamma(\pi^*(\mathfrak{sl}(E,\mu)))$ and $\Gamma(\pi^*(K_q))$, take an element

$$f \otimes \alpha \otimes e \in \Gamma(K_q),$$

where $f \in C^\infty(E), \alpha \in \Gamma(\wedge^q E^*), e \in \Gamma(E)$, and an element $X \in \Gamma(\pi^*(\mathfrak{sl}(E,\mu)))$, we compute

$$\begin{aligned}(-1)^{q-1}\phi_q([X, f \otimes \alpha \otimes e]) &= \rho_\mu(X)(f) \otimes \iota_e(\alpha) + f \otimes \iota_e(X \cdot \alpha) + f \otimes \iota_{X\cdot e}(\alpha)\\ &= \rho_\mu(X)(f)\iota_e(\alpha) + f \otimes X \cdot (\iota_e(\alpha)) = 0,\end{aligned}$$

as $f \otimes \alpha \otimes e \in \Gamma(K_q)$ means that $\iota_e(\alpha) = 0$. Hence, the bracket restricts to the subspaces given by the kernels of the ϕ_k. Finally, as in section 2.3.2, we use the natural action of $\Gamma(\pi^*(\mathfrak{sl}(E,\mu)))$ on $\Gamma(\pi^*(\wedge^n E^*)) \cong C^\infty(E)\mu$ to define the bracket between degree 0 and $-n+1$.

Therefore, we again obtain a dg-Lie algebroid structure:

Proposition 3.11. *The resolution* (24) *of $\mathcal{F}_L^\mu(E)$ carries a dg-Lie algebroid structure, where the binary bracket is the restriction of the one described in Proposition 3.7 when both entries have degrees $0, \dots, -n+2$, and the bracket*

$$[X, \tau]$$

of $X \in \Gamma(\pi^*\mathfrak{sl}(E,\mu))$ with $\tau \in \Gamma(\pi^*(\wedge^n E^*))$ is given by the natural action of X on τ as in Lemma 3.6.

3.4. Linear vector fields preserving a fiberwise symplectic form

We now turn to the symplectic case: let $\pi : E \to L$ be a symplectic vector bundle with $\omega \in \Gamma(\wedge^2 E^*)$ a non-degenerate skew-symmetric bilinear form. By now we know how to construct a Lie subalgebroid of $\mathfrak{gl}(E)$ of linear vector fields preserving ω:

$$\text{consider} \qquad \mathfrak{X}_{lin}^\omega(E) := \{X \in \mathfrak{X}_{lin}(E) \mid X \cdot \omega = 0\},$$

where $X \cdot \omega$ is defined as in Lemma 3.6. Note that $\mathfrak{X}_{lin}^\omega(E)$ is closed under the Lie bracket of $\mathfrak{X}_{lin}(E)$.

As in the previous section, there exists a vector bundle $\mathfrak{sp}(E,\omega)$ over L, such that

$$\Gamma(\mathfrak{sp}(E,\omega)) = \mathfrak{X}_{lin}^\omega(E).$$

We therefore obtain a Lie subalgebroid $\mathfrak{sp}(E,\omega) \subset \mathfrak{gl}(E)$ over L, which generates a linear foliation $\mathcal{F}_L^\omega$ on E. The zero section is a leaf, and the transverse foliation on E_x for $x \in L$ is given by the standard $\mathfrak{sp}(E_x, \omega_x)$-action on E_x.

3.4.1. The projective resolution. We proceed as in section 2.4. Define for $p = 1, \dots, n+1$ the vector bundle map

$$\phi_p^\omega : \wedge^p E^* \otimes E \to \wedge^{p-1} E^* \otimes \wedge^2 E^*$$

over L, given by $\phi_p^\omega(\alpha\otimes e) = (-1)^{p-1}\iota_{e_i}(\alpha)\otimes e^i\wedge\iota_e(\omega)$, where $\{e_i\}_{i=1}^n$ and $\{e^i\}_{i=1}^n$ are dual local frames of E and E^* respectively. Note that ϕ_p^ω is independent of the choice of basis. Define the differentials

$$d_p : \Gamma(\pi^*(\wedge^p E^*\otimes E)) \to \Gamma(\pi^*(\wedge^{p-1}E^*\otimes E))$$

as in equation (21), and

$$\partial_p : \Gamma(\pi^*(\wedge^p E^*\otimes\wedge^2 E^*)) \to \Gamma(\pi^*(\wedge^{p-1}E^*\otimes\wedge^2 E^*)),$$

given by $\partial_p(\alpha\otimes\tau) = -\iota_\epsilon(\alpha)\otimes\tau$ for $\alpha\in\Gamma(\pi^*(\wedge^p E^*))$, $\tau\in\Gamma(\pi^*(\wedge^2 E^*))$. Then ϕ^ω is a cochain map of degree -1, and setting $C_i := \operatorname{coker}(\phi_i^\omega)$, with induced differentials $\overline{\partial_\bullet} : \Gamma(\pi^*(C_p)) \to \Gamma(\pi^*(C_{p-1}))$ we can describe a projective resolution as in section 2.4:

Proposition 3.12. *The sequence*

$$0 \longrightarrow \Gamma(\pi^*(C_{n+1})) \xrightarrow{\overline{\partial_n}} \dots$$
$$\dots \xrightarrow{\overline{\partial_3}} \Gamma(\pi^*(C_3)) \xrightarrow{d_2\phi_2^{-1}\partial_2} \Gamma(\pi^*(\mathfrak{sp}(E,\omega))) \xrightarrow{\rho_\omega} \mathcal{F}_L^\omega(E) \longrightarrow 0 \tag{25}$$

is exact.

3.4.2. The (partial) L_∞-algebroid structure. To obtain the binary brackets, we follow the same approach as in section 2.4. Let

$$r_p^\omega : \Gamma(\wedge^p E^*\otimes\wedge^2 E^*) \to \Gamma(\wedge^{p+1}E^*\otimes E)$$

be defined by

$$r_p^\omega(\alpha_p\otimes\tau) = \left(\frac{1}{p+1}\alpha_p\wedge\iota_{e_i}(\tau) - \frac{(-1)^p}{p(p+1)}\iota_{e_i}(\alpha_p)\wedge\tau\right)\otimes\omega^{-1}(e^i) \in \Gamma(\pi^*(\wedge^{p+1}E^*\otimes E)),$$

for $\alpha\in\Gamma(\wedge^p E^*), \tau\in\Gamma(\wedge^2 E^*)$. Then:

Proposition 3.13. *Using the notation from section* 2.4.2*, the binary operation* $[\![-,-]\!]$ *on* (25) *defined by*

- *The standard* $\pi^*(\mathfrak{sp}(E,\omega))$*-action on itself and* $\pi^*(C_i)$ *for* $i=3,\dots,n$ *when one of the entries lies in* $\Gamma(\pi^*(\mathfrak{sp}(E,\omega)))$,
- $[\![\omega_p,\omega_q]\!] = [r_{p-1}^\omega\partial_p P_p(\omega_p), P_q(\omega_q)] + [P_p\omega_p, r_{q-1}\partial_q P_q(\omega_q)]$,
  for $\omega_p\in\Gamma(\pi^*(\wedge^p E^*\otimes\wedge^2 E^*))$, $\omega_q\in\Gamma(\pi^*(\wedge^q E^*\otimes\wedge^2 E^*))$ *where*

  $$P_p : \Gamma(\pi^*(\wedge^p E^*\otimes\wedge^2 E^*)) \to \Gamma(\pi^*(\ker(r_p^\omega)))$$

  is the projection $id - \phi_{p+1}^\omega\circ r_p^\omega$.

equips (25) *with a differential graded almost Lie algebroid structure, as in* [9, Definition 3.68].

The same remarks as in section 2.4 can be made:

Remark 3.14. (1) The operation defined in Proposition 3.13 satisfies the Jacobi identity if at least one entry has degree 0, but not when all of the entries have degree ≤ -1. The analogue of (18) defines a ternary operation which serves as a contracting homotopy for the Jacobiator.

(2) When the rank of E is at most 4, the Jacobi identity is trivially satisfied.

(3) When the rank of E is equal to 6, the full L_∞-algebroid structure is determined by the differential, the binary bracket as in Proposition 3.13 and the analogue of (18).

4. The isotropy L_∞-algebra in a singular point

Let $\mathcal{F}$ be a foliation on the vector space V. Assume that the origin $p \in V$ be a leaf of $\mathcal{F}$. In [9, Section 4.2] the authors define an L_∞-algebra with trivial differential associated to a leaf of a foliation. Given a minimal resolution

$$0 \longrightarrow \Gamma(E_n) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} \Gamma(E_0) \xrightarrow{\rho} \mathcal{F} \longrightarrow 0 \tag{26}$$

of $\mathcal{F}$ at p, and an L_∞-algebroid structure $\{\ell_k, \rho\}_{k\in\mathbb{N}}$ on $\Gamma(E_\bullet)$, it is defined by restricting the multibrackets ℓ_k to the fibers $(E_i)_p$, which is well-defined because $\rho_p = 0$. This L_∞-algebra is an invariant of $\mathcal{F}$, extending the isotropy Lie algebra $\mathcal{F}/I_p\mathcal{F}$, which is canonically isomorphic to $(E_0)_p$ with the restriction of ℓ_2 ([9, Proposition 4.14]). In particular, the binary bracket ℓ_2 turns $(E_i)_p$ into a $(E_0)_p$-representation. In this section we show that the spaces $(E_i)_p$ can be recovered directly from $\mathcal{F}$, without needing to find a projective resolution of $\mathcal{F}$. Moreover, we show that if $\mathcal{F}$ is linear, the $(E_0)_p$-representations on the $(E_i)_p$ can be determined explicitly. Note that the $(E_0)_p$-representation on $(E_0)_p$ is just the adjoint representation. This construction builds on [9, Remark 4.9] which states the following:

Lemma 4.1. $$(E_i)_p \cong Tor_i^{C^\infty(V)}(\mathcal{F}, \mathbb{R}),$$

where the $C^\infty(V)$-module structure on $\mathbb{R}$ is defined by evaluation in the origin.

Proof. One way to construct $\mathrm{Tor}_i^{C^\infty(V)}(\mathcal{F}, \mathbb{R})$ is to take a projective resolution

$$0 \longrightarrow \Gamma(E_n) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_1} \Gamma(E_0) \xrightarrow{\rho} \mathcal{F} \longrightarrow 0$$

of $\mathcal{F}$, then take the tensor product with $\mathbb{R}$ over $C^\infty(V)$ to obtain

$$0 \longrightarrow \Gamma(E_n) \otimes_{C^\infty(V)} \mathbb{R} \xrightarrow{\partial_n \otimes \mathrm{id}} \dots \xrightarrow{\partial_1 \otimes \mathrm{id}} \Gamma(E_0) \otimes_{C^\infty(V)} \mathbb{R} \longrightarrow 0$$

and compute the cohomology. As $\Gamma(E_i) \otimes_{C^\infty(V)} \mathbb{R} \cong (E_i)_p$ and the differentials become trivial, the result follows. ■

It is however a well-known fact (see [14, Theorem 2.7.2] for instance) that instead of first taking a projective resolution of $\mathcal{F}$ and then taking the tensor product with $\mathbb{R}$, we can equivalently first take a projective resolution of $\mathbb{R}$, and then take the tensor product with $\mathcal{F}$ and compute cohomology. A major advantage here is that we know an explicit resolution of $\mathbb{R}$: it is given by the complex (3). We therefore obtain:

Proposition 4.2. *For $i = 0, \dots, n$, we have*

$$(E_i)_p \cong H^i(\Gamma(\wedge^\bullet V^*) \otimes_{C^\infty(V)} \mathcal{F}(V), d_\bullet \otimes id) \tag{27}$$

where $\wedge^{-1}(V^*)$ is understood to be 0, and $d = \iota_{x^i \partial_{x^i}}$.

Next, we consider the action σ_i of the isotropy Lie algebra $\mathcal{F}/I_p\mathcal{F} \cong (E_0)_p$ on $(E_i)_p$ by the binary bracket ℓ_2. It turns out that when $\mathcal{F}$ is a *linear* foliation, we can define a canonical action on the right hand side of (27) which under the isomorphism of Proposition 4.2 corresponds to σ_i.

Proposition 4.3. *Let $\mathcal{F}$ be a linear foliation on V. Then:*

(i) *The map $lin : \mathcal{F} \to \mathcal{F}$ given by $X \mapsto X^{(1)} \in \mathcal{F}$ descends to an injective Lie algebra homomorphism $\overline{lin} : \mathcal{F}/I_p\mathcal{F} \to \mathcal{F}$. Here $X^{(1)}$ denotes the linear part of the vector field X.*

(ii) *Let $i = 0, \dots, n$. For $X \in \mathcal{F}(V)$, $\alpha \otimes Y \in \Gamma(\wedge^i V^*) \otimes_{C^\infty(V)} \mathcal{F}(V)$, the assignment*

$$(X + I_p\mathcal{F}) \cdot (\alpha \otimes \mathcal{F}) := [X^{(1)}, \alpha \otimes Y]_{FN} = \mathcal{L}_{X^{(1)}}(\alpha) \otimes Y + \alpha \otimes [X^{(1)}, Y] \tag{28}$$

defines a representation of $\mathcal{F}/I_p\mathcal{F}$ on $\Gamma(\wedge^i V^*) \otimes_{C^\infty(V)} \mathcal{F}(V)$, compatible with the differential $d_\bullet$, where $[-,-]_{FN}$ is the Frölicher-Nijenhuis bracket. Consequently, there is a well-defined action on the cohomology groups.

(iii) *The $\mathcal{F}/I_p\mathcal{F}$-action on $H^i(\Gamma(\wedge^\bullet V^*) \otimes_{C^\infty(V)} \mathcal{F}(V), d_\bullet \otimes id)$ induced by the $\mathcal{F}/I_p\mathcal{F}$-action (28) is equivalent to the $\mathcal{F}/I_p\mathcal{F}$-action on $(E_i)_p$.*

Proof. (i) The fact that $\overline{lin}$ is a well-defined Lie algebra homomorphism was shown in [2, Section 4]. For the injectivity, we need to show that if a vector field $Y \in \mathcal{F}(V)$ vanishes quadratically, it can be written as a linear combination

$$Y = \sum_{i=1}^r f^i X_i,$$

where $X_i \in \mathcal{F}(V)$, and $f^i(p) = 0$ for $i = 1, \dots, r$. As $\mathcal{F}$ is a linear foliation, we can take the X_i to be linear vector fields which are linearly independent over $\mathbb{R}$. Then

$$0 = Y^{(1)} = \sum_{i=1}^r f^i(0) X_i,$$

which implies that $f^i(0) = 0$.

(ii) This follows directly from the fact that $\overline{lin}$ is a Lie algebra homomorphism, and that the Frölicher-Nijenhuis bracket satisfies the Jacobi identity. The compatibility with the differentials follows from the fact that

$$[\mathcal{L}_{X^{(1)}}, \iota_{x^i \partial_{x^i}}] = \iota_{[X^{(1)}, x^i \partial_{x^i}]} = 0,$$

as $X^{(1)}$ is linear.

(iii) For this, we recall the isomorphism (27) as described in [14]. Given the projective resolutions (3) and (26), we can take the tensor product to obtain a double complex

$$(\Gamma(\wedge^\bullet V^*) \otimes_{C^\infty(V)} \Gamma(E_\bullet), d_\bullet \otimes \mathrm{id}, \mathrm{id} \otimes \partial_\bullet).$$

From the double complex, we can construct the total complex

$$(\mathrm{Tot}(\Gamma(\wedge^\bullet V^*) \otimes_{C^\infty(V)} \Gamma(E_\bullet), d_\bullet \otimes \mathrm{id} + \mathrm{id} \otimes \partial_\bullet).$$

Then the maps

$$\mathrm{id} \otimes \rho : \mathrm{Tot}(\Gamma(\wedge^\bullet V^*) \otimes_{C^\infty(V)} \Gamma(E_\bullet)) \to \Gamma(\wedge^\bullet V^*) \otimes_{C^\infty(V)} \mathcal{F}(V)$$

and

$$\mathrm{ev}_p \otimes \mathrm{id} : \mathrm{Tot}(\Gamma(\wedge^\bullet V^*) \otimes_{C^\infty(V)} \Gamma(E_\bullet)) \to \mathbb{R} \otimes_{C^\infty(V)} \Gamma(E_\bullet)$$

induce isomorphisms in cohomology. As both maps are compatible with the $\mathcal{F}/I_p\mathcal{F}$-actions, the isomorphisms in cohomology respect the $\mathcal{F}/I_p\mathcal{F}$-action as well. ■

The proposition now allows us to compute invariants of the foliation $\mathcal{F}(V)$ without needing an explicit resolution of $\mathcal{F}$, as we do in the following example.

Example 4.4. Consider on $V = \mathbb{R}^2$ the foliation $\mathcal{F}_1(V) = \langle x\partial_x, y\partial_x \rangle_{C^\infty(V)}$. Then by Lemma 4.1,

$$(E_0)_p := \mathrm{Tor}_0^{C^\infty(V)}(\mathcal{F}(V), \mathbb{R}) = \mathcal{F}_1/I_p\mathcal{F}_1.$$

For $\mathrm{Tor}_1^{C^\infty(V)}(\mathcal{F}(V), \mathbb{R})$, a straightforward computation shows that the middle cohomology of

$$\Gamma(\wedge^2 V^*) \otimes_{C^\infty(V)} \mathcal{F}(V) \xrightarrow{d_2} \Gamma(V^*) \otimes_{C^\infty(V)} \mathcal{F}(V) \xrightarrow{d_1} \mathcal{F}(V) \qquad (29)$$

is one-dimensional, generated by the class of $\gamma := dx \otimes y\partial_x - dy \otimes x\partial_x$. Observe that this element is not exact in (29): although it can be written as

$$dx \otimes y\partial_x - dy \otimes x\partial_x = d_2(dx \wedge dy \otimes \partial_x)$$

in (4), $\partial_x \notin \mathcal{F}(V)$. Moreover, any exact element in (29) must vanish at least quadratically in the origin, which is not the case for γ.

Finally, it is easy to see that d_2 is injective, so we now know that for any minimal resolution (26), the space $(E_0)_p$ is two-dimensional, the space $(E_1)_p$ is one-dimensional, and the spaces $(E_i)_p$ for $i \geq 2$ are trivial. The Lie algebra structure on $(E_0)_p$ is the non-abelian two-dimensional Lie algebra, while the action of $(E_0)_p$ on $(E_1)_p$ is trivial.

Example 4.5. We can modify the previous example to obtain a foliation which is not linear: consider $\mathcal{F}_2(V) = \langle (x + xy)\partial_x + y^2\partial_y, y\partial_x \rangle_{C^\infty(V)}$. It is not difficult to see that $\mathcal{F}_2(V)$ is a projective $C^\infty(V)$-module. Consequently, for any minimal resolution (26), $(E_0)_p$ is two-dimensional, and $(E_i)_p = 0$ for $i \geq 1$. Although it was already known that there exists no analytic diffeomorphism of V taking the generators of $\mathcal{F}_2(V)$ to the generators of $\mathcal{F}_1(V)$ of the previous example (see [5, Proposition 1.2]), the above argument shows that there does not even exist a smooth diffeomorphism of V taking the $C^\infty(V)$-*module* $\mathcal{F}_2(V)$ to $\mathcal{F}_1(V)$, showing that not even the germs of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent, even though the modules generated by the first order approximations of the generators around $p \in V$ are equal. Of course, in this case the difference between $\mathcal{F}_1(V)$ and $\mathcal{F}_2(V)$ can be seen by considering the dimension of the regular leaves: for $\mathcal{F}_1$ they are 1-dimensional, while for $\mathcal{F}_2$ they are 2-dimensional.

Appendix

A. Compatibility of r^ω with the differentials

In this section we use the notation from section 2.4, and investigate whether the left inverse r^ω of ϕ^ω can be chosen to be a cochain map in some degrees, which would simplify the brackets of the L_∞-algebroid structure.

As the choice of r^ω in (16) is not unique, we investigate whether the left inverse r^ω can be chosen to be compatible with the differentials, as this would force $[\![-,-]\!]$ to be equal to $\{-,-\}$. However, it is clear that this is not possible in all degrees: first of all, as r_1^ω is not only a left inverse, but the unique inverse, as ϕ_2^ω is an isomorphism. Hence, there is no choice there. Then, the existence of $\widetilde{r_2^\omega} : \Gamma(\wedge^2 V^* \otimes \wedge^2 V^*) \to \Gamma(\wedge^3 V^* \otimes V)$ such that

$$r_1^\omega \partial_2 + d_3 \widetilde{r_2^\omega} = 0$$

implies that $d_2 r_1^\omega \partial_2 = 0$, which is not the case.

Nevertheless, we consider the other degrees, as compatibility with the differentials would simplify the binary and ternary brackets.

We start with the lowest degree: Let $n = \dim V$. In degrees $-n$ and $-n+1$, we get the following square

$$\begin{array}{ccc} 0 & \longrightarrow & \Gamma(\wedge^n V^* \otimes V) \\ \downarrow & & \downarrow \phi_n^\omega \\ \Gamma(\wedge^n V^* \otimes \wedge^2 V^*) & \xrightarrow{\partial_n} & \Gamma(\wedge^{n-1} V^* \otimes \wedge^2 V^*) \end{array} .$$

Given a left inverse $\widetilde{r_{n-1}^\omega} : \Gamma(\wedge^{n-1} V^* \otimes \wedge^2 V^*) \to \Gamma(\wedge^n V^* \otimes V)$ of ϕ_n^ω such that

$$\widetilde{r_{n-1}^\omega} \partial_n = 0,$$

we note that the constant extension of the value at the origin $\widetilde{r_{n-1}^\omega}(0)$ is also a left inverse of ϕ_n^ω which satisfies

$$\widetilde{r_{n-1}^\omega}(0) \partial_n = 0.$$

It therefore suffices to show that there exists no constant (in V) left inverse $\widetilde{r_{n-1}^\omega}$ of ϕ_n^ω such that

$$\widetilde{r_{n-1}^\omega} \partial_n = 0.$$

Let $\mu \in \wedge^n V^*, \tau \in \wedge^2 V^*$. Then we can view ∂_n as an injective $\mathbb{R}$-linear map

$$\partial_n : \wedge^n V^* \otimes \wedge^2 V^* \to V^* \otimes \wedge^{n-1} V^* \otimes \wedge^2 V^*,$$

as ∂_n has linear coefficient functions, and $\widetilde{r_{n-1}^\omega}$ extends by $\mathrm{id}_{V^*} \otimes \widetilde{r_{n-1}^\omega}$ to a map

$$\mathrm{id}_{V^*} \otimes \widetilde{r_{n-1}^\omega} : V^* \otimes \wedge^{n-1} V^* \otimes \wedge^2 V^* \to V^* \otimes \wedge^n V^* \otimes V$$

by $C^\infty(V)$-linearity.

Now $\mathrm{id}_{V^*} \otimes \widetilde{r^\omega_{n-1}} \partial_n(\mu \otimes \tau) = 0$ implies that

$$e^i \otimes \iota_{e_i}(\mu) \otimes \tau \in \ker(\mathrm{id}_{V^*} \otimes \widetilde{r^\omega_{n-1}}).$$

However, as $\ker(\mathrm{id}_{V^*} \otimes \widetilde{r^\omega_{n-1}}) = V^* \otimes \ker(\widetilde{r^\omega_{n-1}})$, it follows that $\iota_{e_i}(\mu) \otimes \tau \in \ker(\widetilde{r^\omega_{n-1}})$ for each $i = 1, \ldots, n$.

These elements actually generate the entirety of $\wedge^{n-1}V^* \otimes \wedge^2 V^*$, forcing $\widetilde{r^\omega_{n-1}} = 0$, contradicting the assumption that $\widetilde{r^\omega_{n-1}} \phi^\omega_n = \mathrm{id}$.

Now fix $\dim V = 4$. The general case discussed above shows that there exists no left inverse $\widetilde{r^\omega_3}$ of ϕ^ω_4 such that $d_4 \widetilde{r^\omega_3} \partial_4 = 0$. We will show that there exists no $\mathfrak{sp}(V, \omega)$-*equivariant* left inverse $\widetilde{r^\omega_2}$ of ϕ^ω_3 satisfying $d_3 \widetilde{r^\omega_2} \partial_3 = 0$.

The requirement that $\widetilde{r^\omega_2}$ is $\mathfrak{sp}(V,\omega)$-equivariant is natural, as ϕ^ω_3 is. We follow [4, Chapter 16] to determine the space of all $\mathfrak{sp}(V,\omega)$-equivariant maps $\wedge^2 V^* \otimes \wedge^2 V^* \to \wedge^3 V^* \otimes V$, and then restrict to those which are left inverses of ϕ^ω_3. For this, we decompose the respective spaces into irreducible $\mathfrak{sp}(V,\omega)$-representations:

Lemma A.1.
$$R_1 := \wedge^3 V^* \otimes V \cong \mathbb{R} \oplus W \oplus S^2(V)$$
$$R_2 := \wedge^2 V^* \otimes \wedge^2 V^* \cong \mathbb{R}^{\oplus 2} \oplus W^{\oplus 2} \oplus S^2(V) \oplus C,$$
where $W = Ann(\mathbb{R}\omega) \subset \wedge^2 V$, and C is an irreducible representation not isomorphic to $\mathbb{R}$, W or $S^2(V)$.

Now we would like to apply a variation of Schur's lemma (see for instance [6]) to compute the space of $\mathfrak{sp}(V,\omega)$-equivariant maps $\wedge^2 V^* \otimes \wedge^2 V^* \to \wedge^3 V^* \otimes V$. We first obtain:

Lemma A.2.

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{sp}(V,\omega)}(R_2, R_1) &\cong \mathrm{End}_{\mathfrak{sp}(V,\omega)}(\mathbb{R})^{\oplus 2} \oplus \mathrm{End}_{\mathfrak{sp}(V,\omega)}(W)^{\oplus 2} \oplus \mathrm{End}_{\mathfrak{sp}(V,\omega)}(S^2(V)) \\ &\cong \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}.\end{aligned}$$

Proof. By Schur's lemma, the restriction of a map of representations to irreducible factors is either 0, or an isomorphism, which proves the first isomorphism in the statement. For the second isomorphism, we observe that when complexifying, the representations

$$\mathbb{C}, W \otimes_{\mathbb{R}} \mathbb{C}, S^2_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C})$$

are irreducible $\mathfrak{sp}(V \otimes_{\mathbb{R}} \mathbb{C}, \omega)$-representations, where ω is now extended to a $\mathbb{C}$-bilinear skew-symmetric map

$$\omega : V \otimes_{\mathbb{R}} \mathbb{C} \times V \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}.$$

Moreover, it is easy to see that for any representation T, the natural map

$$\mathrm{End}_{\mathfrak{sp}(V,\omega)}(T) \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{End}_{\mathfrak{sp}(V \otimes_{\mathbb{R}} \mathbb{C}, \omega)}(T \otimes_{\mathbb{R}} \mathbb{C})$$

is an isomorphism. As the endomorphism ring of a complex irreducible representation is $\mathbb{C}$ by Schur's Lemma, it follows that the endomorphism ring of the real representations $\mathbb{R}, W, S^2(V)$ is $\mathbb{R}$, concluding the proof of the lemma. ∎

We explicitly construct the generators of $\mathrm{Hom}_{\mathfrak{sp}(V,\omega)}(R_2, R_1)$: pick a basis $\{e_i\}_{i=1}^4$ of V such that $\omega = e^1 \wedge e^3 + e^2 \wedge e^4$, and let

$$\pi_\omega = \frac{1}{2}(e_1 \otimes e_3 + e_2 \otimes e_4 - e_3 \otimes e_1 - e_4 \otimes e_2) \in V \otimes V.$$

Lemma A.3. *Let $\tau \in \wedge^2 V^*$. Define*

$$\overline{\tau} := \tau - \frac{1}{2}(\tau(e_1, e_3) + \tau(e_2, e_4))\omega.$$

Let $\tau_1, \tau_2 \in \wedge^2 V^*$. Then $\mathrm{Hom}_{\mathfrak{sp}(V,\omega)}(R_2, R_1)$ is generated by the maps

$$\begin{aligned} p_1(\tau_1 \otimes \tau_2) &= \frac{1}{4}(\tau_1(e_1, e_3) + \tau_1(e_2, e_4))(\tau_2(e_1, e_3) + \tau_2(e_2, e_4))\pi_\omega, \\ p_2(\tau_1 \otimes \tau_2) &= (\overline{\tau_1} \wedge \overline{\tau_2})(e_1, e_3, e_2, e_4)\pi_\omega, \\ q_1(\tau_1 \otimes \tau_2) &= ((\omega^\flat)^{-1} \wedge (\omega^\flat)^{-1})(\overline{\tau_1})\frac{1}{2}(\tau_2(e_1, e_3) + \tau_2(e_2, e_4)), \\ q_2(\tau_1 \otimes \tau_2) &= \frac{1}{2}(\tau_1(e_1, e_3) + \tau_1(e_2, e_4))((\omega^\flat)^{-1} \wedge (\omega^\flat)^{-1})(\overline{\tau_2}), \\ s(\tau_1 \otimes \tau_2) &= \overline{\tau_1}((\omega^\flat)^{-1}(e^k), e_j)\overline{\tau_2}(e_k, e_l)\omega^{-1}(e^j) \cdot \omega^{-1}(e^l), \end{aligned}$$

where p_1, p_2 correspond to the trivial representation, q_1, q_2 to W, and s to $S^2(V)$. Here $\cdot$ denotes the symmetric product in $S^2(V)$, $\wedge^3 V^* \otimes V$ is identified with $V \otimes V$ via the volume form $\frac{1}{2}\omega \wedge \omega$, and $\wedge^2 V$ and $S^2(V)$ sit inside $V \otimes V$ as

$$v_1 \wedge v_2 \mapsto \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1),$$

$$v_1 \cdot v_2 \mapsto \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1).$$

The lemma above allows us to formulate a condition under which

$$\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 p_1 + \mu_2 p_2 + \nu s \tag{30}$$

is a left inverse of ϕ_3^ω:

Lemma A.4. (30) *is a left inverse of ϕ_3^ω if and only if*

$$\lambda_1 = 2 - 10\lambda_2, \quad \mu_1 = \mu_2 - 2, \quad \textit{and} \quad \nu = -2.$$

It is now straightforward to show that there is no value of $\lambda_2, \mu_2 \in \mathbb{R}$ such that the corresponding map $\widetilde{r_2^\omega} = (2 - 10\lambda_2)p_1 + \lambda_2 p_2 + (\mu_2 - 2)q_1 + \mu_2 q_2 - 2s$ satisfies

$$d_3 \widetilde{r_2^\omega} \partial_3 = 0.$$

Consequently:

Proposition A.5. *When $\dim V = 4$, there exist no $\mathfrak{sp}(V,\omega)$-equivariant*

$$\widetilde{r_2^\omega} : \Gamma(\wedge^2 V^* \otimes \wedge^2 V^*) \to \Gamma(\wedge^3 V^* \otimes V), \widetilde{r_3^\omega} : \Gamma(\wedge^3 V^* \otimes \wedge^2 V^*) \to \Gamma(\wedge^4 V^* \otimes V)$$

satisfying

$$\widetilde{r_2^\omega} \partial_3 + d_4 \widetilde{r_3^\omega} = 0.$$

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