

Cohomological Integral Transform Associated to θ -Stable Parabolic Subalgebras of Holomorphic Type

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Abstract. Every elliptic adjoint orbit X of a real reductive group carries naturally a complex manifold structure. This article proves a necessary and sufficient condition on X for which the (generalized) Radon-Penrose transform on Dolbeault cohomologies on X maps into the space of holomorphic sections.

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1. Statement of results

Let G be a real reductive Lie group contained in a connected complexification $G_{\mathbb{C}}$, θ a Cartan involution, K the fixed point subgroup of θ , and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition of the Lie algebra \mathfrak{g} of G . For an element $H \in \mathfrak{k}$, we write L for the centralizer of H in G . Then the homogeneous space G/L carries a G -invariant complex manifold structure with holomorphic tangent space $\mathfrak{g}_{\mathbb{C}}/\mathfrak{q}$ at the origin (see e.g., [7]), where $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ is a θ -stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ defined as the sum of eigenspaces with eigenvalues in $\sqrt{-1}\mathbb{R}_{\geq 0}$.

Extending the work of Eastwood-Penrose-Wells [2] for $G = SU(2, 2)$, we constructed in [10, 11] for any real reductive group G a Penrose transform \mathcal{R} from Dolbeault cohomologies over G/L into the space of smooth sections for a G -equivariant vector bundle over the Riemannian symmetric space G/K , which we recall briefly now.

We note that $K/L \cap K$ is a compact complex manifold, of which the complex dimension is denoted by S . Write $\iota: K/L \cap K \hookrightarrow G/L$ for the natural embedding. Take a maximal torus T of $L \cap K$, and fix a positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ containing the weights $\Delta(\mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. For a dominant character μ of T , we denote by V_{μ} the irreducible K -module with highest weight μ , and form a G -equivariant vector bundle $\mathcal{V}_{\mu} := G \times_K V_{\mu}$ over the Riemannian symmetric space G/K .

Let \mathbb{C}_{λ} be a character of L , and $\mathcal{L}_{\lambda} := G \times_L \mathbb{C}_{\lambda}$ a homogeneous holomorphic line bundle over G/L . We write ℓ_g for the action of $g \in G$ on \mathcal{L}_{λ} . Then the natural map

$$\widetilde{\mathcal{R}}: \mathcal{E}^{0,S}(G/L, \mathcal{L}_{\lambda}) \times G \rightarrow \mathcal{E}^{0,S}(K/L \cap K, \iota^* \mathcal{L}_{\lambda}), \quad (\alpha, g) \mapsto \iota^* \ell_g^* \alpha \quad (1)$$

goes down to Dolbeault cohomologies:

$$H_{\bar{\partial}}^S(G/L, \mathcal{L}_{\lambda}) \times G \rightarrow H_{\bar{\partial}}^S(K/L \cap K, \iota^* \mathcal{L}_{\lambda}), \quad ([\alpha], g) \mapsto [\iota^* \ell_g^* \alpha].$$

The Borel-Weil-Bott theorem tells us that the target space is K -isomorphic to V_{μ_λ} if $\mu_\lambda := \mathbb{C}_\lambda \otimes \Lambda^S(\mathfrak{k}_\mathbb{C}/(\mathfrak{q} \cap \mathfrak{k}_\mathbb{C}))$ is $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ -dominant. In turn, the above map induces a G -intertwining operator ([11, Thm. 2.6]):

$$\mathcal{R}: H_{\bar{\partial}}^S(G/L, \mathcal{L}_\lambda) \rightarrow \mathcal{E}(G/K, \mathcal{V}_{\mu_\lambda}), \quad [\alpha] \mapsto (g \mapsto [\iota^* \ell_g^* \alpha]). \tag{2}$$

In this article we prove two holomorphicity theorems. One is to give a domain of holomorphy of $\mathcal{R}([\alpha])$, namely, we prove that there exists a complex neighbourhood $Y_\mathbb{C}$ of G/K in $G_\mathbb{C}/K_\mathbb{C}$ such that $\mathcal{R}([\alpha])$ extends holomorphically to $Y_\mathbb{C}$ for all $[\alpha] \in H_{\bar{\partial}}^S(G/L, \mathcal{L}_\lambda)$, see Theorem 2.2. The other is to give an algebraic necessary and sufficient condition on the θ -stable parabolic subalgebra \mathfrak{q} such that $\mathcal{R}([\alpha])$ is already holomorphic on G/K for all α , see Theorem 1.3 below. To be precise, we assume that G is a simple Lie group of Hermitian type for the second holomorphicity assertion. This means that the Lie algebra \mathfrak{k} has a one-dimensional center $\mathfrak{c}(\mathfrak{k})$, which yields a three graded structure of $\mathfrak{g}_\mathbb{C}$ by $\mathfrak{g}_\mathbb{C} = \mathfrak{p}^- + \mathfrak{k}_\mathbb{C} + \mathfrak{p}^+$.

Definition 1.1. (1) (Adams [1]) A θ -stable parabolic subalgebra \mathfrak{q} is *holomorphic* (resp. *anti-holomorphic*) if $\mathfrak{u} \cap \mathfrak{p}_\mathbb{C} \subset \mathfrak{p}^+$ (resp. if $\mathfrak{u} \cap \mathfrak{p}_\mathbb{C} \subset \mathfrak{p}^-$).

(2) We say the corresponding complex manifold $G/L \simeq \text{Ad}(G)\mathfrak{q}$ is of *holomorphic type* (resp. of *anti-holomorphic type*).

We remark that different $H \in \mathfrak{k}$ may give the same centralizer L but different θ -stable parabolic subalgebras and hence different complex structures on the same manifold G/L , some being of holomorphic type and some others not being.

Example 1.2. [3, Ex. 3.7] For $G = SU(2, 2)$, there are 18 isomorphism classes of complex homogeneous spaces G/L up to conjugacy and up to cocompact normal factors of L . Among them, 9 are of holomorphic type or of anti-holomorphic type.

See [8, Table C.1] for the complete list of defining elements H for θ -stable parabolic subalgebras \mathfrak{q} of holomorphic type or anti-holomorphic type when G is a simple Lie group.

We prove the following holomorphicity theorem:

Theorem 1.3. *Let G be a connected simple Lie group of Hermitian type, and G/L is equipped with complex structure by a θ -stable parabolic subalgebra \mathfrak{q} of $\mathfrak{g}_\mathbb{C}$.*

- (1) *The following two conditions are equivalent:*
 - (i) *\mathfrak{q} is of holomorphic type.*
 - (ii) *Image $\mathcal{R} \subset \mathcal{O}(G/K, \mathcal{V}_{\mu_\lambda})$ for any L -character \mathbb{C}_λ .*
- (2) *If μ_λ is $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ -dominant, Image \mathcal{R} contains the lowest K -type V_{μ_λ} in $\mathcal{O}(G/K, \mathcal{V}_{\mu_\lambda})$.*

Remark 1.4. Here are some remarks for the implication (i) \Rightarrow (ii) in Theorem 1.3(1).

(1) Adams proved in [1] that Zuckerman’s derived functor modules are lowest weight modules (Definition 5.3) if they are cohomologically induced from a θ -stable parabolic subalgebra \mathfrak{q} of holomorphic type. Since any irreducible lowest weight modules can be realized in the spaces of holomorphic sections for homogeneous holomorphic vector bundles over the Hermitian symmetric space G/K and vice

versa (see e.g., [6, Chap. 3]), the implication (i) \Rightarrow (ii) may be viewed as a geometric counterpart of Adams' result.

(2) Special cases of the implication (i) \Rightarrow (ii) were proved in [2, 10, 11] in the settings where G is a classical group and where $\dim V_{\mu_\lambda} = 1$, and more generally in [15]. The methods include a theorem of Godement on spherical functions and Kobayashi's theory on discretely decomposable restrictions [3, 4, 5], and the Leray spectral sequence [15].

This article provides yet another elementary proof for the implication (i) \Rightarrow (ii) by using an extended Penrose transform (Theorem 2.2). The main novelty of this article is the converse implication (ii) \Rightarrow (i). For this, we utilize an algebraic techniques by using the associated varieties of (\mathfrak{g}, K) -modules (Section 5).

2. Extended Penrose transform

In this section we consider a natural complex neighbourhood $Y_{\mathbb{C}}$ of G/K in $G_{\mathbb{C}}/K_{\mathbb{C}}$, see (3), to which the Penrose transform $\mathcal{R}([\alpha])$ can be extended as a holomorphic section of a certain holomorphic vector bundle over $Y_{\mathbb{C}}$ for any cohomology class $[\alpha]$, see Theorem 2.2 below.

Let us recall how the complex structure is defined on G/L . Given $H \in \mathfrak{k}$, we decompose $\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}^- + \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$, the sum of eigenspaces of $\text{ad}(H)$ with eigenvalues in $\sqrt{-1}\mathbb{R}_{<0}$, 0, and $\sqrt{-1}\mathbb{R}_{>0}$, respectively. Let Q be the normalizer of $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ in $G_{\mathbb{C}}$. Then the centralizer L of H in G equals $Q \cap G$. The natural morphism $G \hookrightarrow G_{\mathbb{C}}$ induces an open embedding $G/L \hookrightarrow G_{\mathbb{C}}/Q$ (a generalized Borel embedding), through which G/L carries a G -invariant complex manifold structure (see e.g., [7] for details).

Suppose \mathbb{C}_λ is a character of L . We extend it to a holomorphic character of Q by letting the unipotent radical act trivially, and form a $G_{\mathbb{C}}$ -equivariant holomorphic line bundle $\mathcal{L}_\lambda = G_{\mathbb{C}} \times_Q \mathbb{C}_\lambda$ over the flag variety $G_{\mathbb{C}}/Q$. If $\mu_\lambda = \mathbb{C}_\lambda \otimes \Lambda^S(\mathfrak{k}_{\mathbb{C}}/(\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}))$ is $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ -dominant, the $K_{\mathbb{C}}$ -module $H_{\mathfrak{g}}^S(K_{\mathbb{C}}/Q \cap K_{\mathbb{C}}, \iota^* \mathcal{L}_\lambda)$ is finite-dimensional and irreducible with highest weight μ_λ .

Definition-Lemma 2.1. (Extended Penrose transform \mathcal{R}) *Retain our setting that \mathfrak{q} is a θ -stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ (not necessarily of holomorphic type). We define an open subset $G_{\mathbb{C}}(K; Q) := \{g \in G_{\mathbb{C}} : gK \subset GQ\}$ of $G_{\mathbb{C}}$.*

(1) *The set $G_{\mathbb{C}}(K; Q)$ is left G -invariant and right $K_{\mathbb{C}}$ -invariant. We set*

$$Y_{\mathbb{C}} := G_{\mathbb{C}}(K; Q)/K_{\mathbb{C}}. \tag{3}$$

Let λ be a character of L such that μ_λ is $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ -dominant. We form a G -equivariant holomorphic vector bundle over $Y_{\mathbb{C}}$ by

$$\tilde{\mathcal{V}}_{\mu_\lambda} := G_{\mathbb{C}}(K; Q) \times_{K_{\mathbb{C}}} V_{\mu_\lambda} \rightarrow Y_{\mathbb{C}}. \tag{4}$$

The connected component of $Y_{\mathbb{C}}$ is sometimes referred to as the cycle space.

(2) *For $g \in G_{\mathbb{C}}(K; Q)$, one has $(\ell_g \circ \iota)(K/L \cap K) \subset G/L$, hence the following extension of (1) is well-defined:*

$$\tilde{\mathcal{R}}: \mathcal{E}^{0,S}(G/L, \mathcal{L}_\lambda) \times G_{\mathbb{C}}(K; Q) \rightarrow \mathcal{E}^{0,S}(K/L \cap K, \iota^* \mathcal{L}_\lambda), \quad (\alpha, g) \mapsto (\ell_g \circ \iota)^* \alpha. \tag{5}$$

Passing to Dolbeault cohomologies, one obtains a G -intertwining operator, denoted by the same letter with (2):

$$\mathcal{R}: H_{\bar{\partial}}^S(G/L, \mathcal{L}_\lambda) \rightarrow \mathcal{E}(Y_{\mathbb{C}}, \tilde{\mathcal{V}}_{\mu_\lambda}), \quad [\alpha] \mapsto \tilde{\mathcal{R}}(\alpha, \cdot). \tag{6}$$

Proof. (1) Since K is compact and GQ is open, $G_{\mathbb{C}}(K; Q)$ is an open subset in $G_{\mathbb{C}}$. Moreover, it is clear from definition that $G_{\mathbb{C}}(K; Q)$ is left G -invariant. To see that it is also right $K_{\mathbb{C}}$ -invariant, we observe that $K/L \cap K \simeq K_{\mathbb{C}}/Q \cap K_{\mathbb{C}}$. Thus one has $K(Q \cap K_{\mathbb{C}}) = K_{\mathbb{C}}$, hence $gK \subset GQ$ if and only if $gK_{\mathbb{C}} \subset GQ$, yielding the equality:

$$G_{\mathbb{C}}(K; Q) = \{g \in G_{\mathbb{C}} : gK_{\mathbb{C}} \subset GQ\}. \tag{7}$$

Hence $G_{\mathbb{C}}(K; Q)$ is right $K_{\mathbb{C}}$ -invariant.

(2) For $k \in K_{\mathbb{C}}$, the left translation $\ell_k: G_{\mathbb{C}}/Q \rightarrow G_{\mathbb{C}}/Q$ commutes with the natural embedding $\iota: K_{\mathbb{C}}/Q \cap K_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}}/Q$. Therefore one has

$$[\tilde{\mathcal{R}}(\alpha, hk)] = k^{-1}[\tilde{\mathcal{R}}(\alpha, h)] \quad \text{in } H_{\bar{\partial}}^S(K_{\mathbb{C}}/Q \cap K_{\mathbb{C}}, \iota^* \mathcal{L}_\lambda) \simeq V_{\mu_\lambda}, \tag{8}$$

for any $h \in G_{\mathbb{C}}(K; Q)$ and for any $\bar{\partial}$ -closed S -form $\alpha \in \mathcal{E}^{0,S}(G/L, \mathcal{L}_\lambda)$. This shows that the linear map (6) is well-defined. Moreover, one has

$$\tilde{\mathcal{R}}(\alpha, gh) = \tilde{\mathcal{R}}(\ell_g^* \alpha, h) \quad \text{for any } g \in G \text{ and } h \in G_{\mathbb{C}}(K; Q),$$

hence the map (6) is a G -homomorphism. ■

We now give a natural domain of holomorphy for the Penrose transform:

Theorem 2.2. *Let $\tilde{\mathcal{V}}_{\mu_\lambda} \rightarrow Y_{\mathbb{C}} = G_{\mathbb{C}}(K; Q)/K_{\mathbb{C}}$ be the holomorphic vector bundle introduced in (4), and $\mathcal{O}(Y_{\mathbb{C}}, \tilde{\mathcal{V}}_{\mu_\lambda})$ the space of holomorphic sections. The image of the extended Penrose transform (6) is contained in $\mathcal{O}(Y_{\mathbb{C}}, \tilde{\mathcal{V}}_{\mu_\lambda})$.*

For the proof, we use the following elementary but useful lemma. Denote by V^\vee the dual vector space of a complex vector space V , by \mathcal{V}^\vee the dual vector bundle of \mathcal{V} , and by $\langle \cdot, \cdot \rangle$ the canonical pairing between the fibers of \mathcal{V}^\vee and \mathcal{V} .

Lemma 2.3. *Let Z be a compact complex manifold of dimension S , D another complex manifold, and \mathcal{L} a holomorphic line bundle over $X := D \times Z$. Denote by K_Z the canonical line bundle of Z , and set a holomorphic line bundle $\mathcal{W} := (\mathcal{L}|_Z)^\vee \otimes K_Z$ over Z , and $F := \mathcal{O}(Z, \mathcal{W})$. For $\alpha \in \mathcal{E}^{0,S}(X, \mathcal{L})$, we define an element of $\mathcal{E}(D) \otimes F^\vee$, denoted by $\int_Z \alpha|_Z$, as the integration along the submanifold Z :*

$$D \times F \rightarrow \mathbb{C}, \quad (y, \gamma) \mapsto \int_Z \langle \gamma, i_y^* \alpha \rangle, \tag{9}$$

where $i_y: Z \hookrightarrow X$ is the natural embedding $z \mapsto (y, z)$. Then the integral (9) induces a linear map

$$H_{\bar{\partial}}^S(X, \mathcal{L}) \rightarrow \mathcal{O}(D) \otimes F^\vee, \quad [\alpha] \mapsto \int_Z \alpha|_Z.$$

Proof. Let us prove that the function $\langle \gamma, \int_Z \alpha|_Z \rangle$, namely, $D \ni y \mapsto \int_Z \langle \gamma, i_y^* \alpha \rangle \in \mathbb{C}$, depends only on the cohomology class $[\alpha]$ and is holomorphic on D for any $\gamma \in F$.

We denote by $\text{pr}_2: X \rightarrow Z$ the second projection. For $\alpha \in \mathcal{E}^{0,S}(X, \mathcal{L})$ and $\gamma \in F = \mathcal{O}(Z, \mathcal{W})$, we set $\beta := \langle \text{pr}_2^* \gamma, \alpha \rangle \in \mathcal{E}^{0,S}(X, \text{pr}_2^*(K_Z))$. Since $\text{pr}_2 \circ i_y = \text{id}_y$, one has

$$i_y^* \beta = \langle \gamma, i_y^* \alpha \rangle \in \mathcal{E}^{0,S}(Z, K_Z) \simeq \mathcal{E}^{S,S}(Z)$$

for any $y \in D$. If α is $\bar{\partial}$ -closed [resp. $\bar{\partial}$ -exact], then β is $\bar{\partial}$ -closed [resp. $\bar{\partial}$ -exact]. Thus the integral (9) depends only on the cohomology class of a $\bar{\partial}$ -closed form α .

Suppose α is $\bar{\partial}$ -closed. We decompose $\beta = \beta_0 + \dots + \beta_S$ such that

$$\beta_q \in (\mathcal{E}^{0,q}(D) \otimes \mathcal{E}^{0,S-q}(Z)) \otimes_{\mathcal{E}(X)} \mathcal{E}(X, \text{pr}_2^*(K_Z))$$

for $0 \leq q \leq S$. Since $X = D \times Z$ is a direct product manifold, one has $\bar{\partial} = \bar{\partial}_D + \bar{\partial}_Z$. Therefore the condition $\bar{\partial}\beta = 0$ implies $\bar{\partial}_Z \beta_{q+1} + \bar{\partial}_D \beta_q = 0$ for all $0 \leq q \leq S$.

We write $\int_Z \beta|_Z$ for the function on D given by $y \mapsto \int_Z i_y^* \beta$. We recall that $i_y^* \beta \in \mathcal{E}^{0,S}(Z, K_Z) \simeq \mathcal{E}^{S,S}(Z)$, and note $i_y^* \beta_q \in \mathcal{E}^{0,S-q}(Z, K_Z) \simeq \mathcal{E}^{S,S-q}(Z)$ for $0 \leq q \leq S$. Therefore $\int_Z \beta|_Z = \int_Z \beta_0|_Z$, and one has

$$\bar{\partial}_D \int_Z \beta|_Z = \int_Z (\bar{\partial}_D \beta_0)|_Z = - \int_Z (\bar{\partial}_Z \beta_1)|_Z = - \int_Z \bar{\partial}_Z (\beta_1|_Z) = 0.$$

Thus $D \ni y \mapsto \int_Z i_y^* \beta \in \mathbb{C}$ is a holomorphic function in D . Since $\int_Z i_y^* \beta = \int_Z \langle \gamma, i_y^* \alpha \rangle$, we have shown that $\int_Z \alpha|_Z$ is holomorphic as an F^\vee -valued function on D . ■

Proof of Theorem 2.2. We recall from Definition-Lemma 2.1 that $G_{\mathbb{C}}(K; Q)$ is an open subset of $G_{\mathbb{C}}$. Let us prove that $\tilde{\mathcal{R}}(\alpha, \cdot): G_{\mathbb{C}}(K; Q) \rightarrow V_{\mu_\lambda}$ is a holomorphic map for every $\bar{\partial}$ -closed form $\alpha \in \mathcal{E}^{0,S}(G/L, \mathcal{L}_\lambda)$.

We write $m: G_{\mathbb{C}} \times G_{\mathbb{C}}/Q \rightarrow G_{\mathbb{C}}/Q$ for the multiplication map given by $(g, x) \mapsto \ell_g x$. Let $Z := \iota(K_{\mathbb{C}}/Q \cap K_{\mathbb{C}})$ be a complex submanifold of $G_{\mathbb{C}}/Q$, and take an element $g \in G_{\mathbb{C}}(K; Q)$. Since $m(\{g\} \times Z) = \ell_g Z \subset G/L$ by (7), there exists a neighbourhood D of g in $G_{\mathbb{C}}(K; Q)$ such that $X := D \times Z$ satisfies $m(X) \subset G/L$. Consider the natural embedding $i_y: Z \hookrightarrow X$, $z \mapsto (y, z)$ for each fixed $y \in D$.

We apply Lemma 2.3 to the pull-back $m^* \mathcal{L}_\lambda \rightarrow X = D \times Z$ of the holomorphic bundle $\mathcal{L}_\lambda \rightarrow G/L$. By the Borel-Weil theory, the $K_{\mathbb{C}}$ -module

$$F := \mathcal{O}(Z, (\mathcal{L}_\lambda|_Z)^\vee \otimes K_Z) \simeq \mathcal{O}(K_{\mathbb{C}}/Q \cap K_{\mathbb{C}}, \mathcal{L}_{-\mu_\lambda})$$

is irreducible with lowest weight $-\mu_\lambda$, hence it is isomorphic to the contragredient representation $V_{\mu_\lambda}^\vee$. Hence the composition of morphisms

$$H_{\bar{\partial}}^S(G/L, \mathcal{L}_\lambda) \xrightarrow{m^*} H_{\bar{\partial}}^S(X, m^* \mathcal{L}_\lambda) \rightarrow \mathcal{O}(D) \otimes V_{\mu_\lambda}$$

gives a holomorphic function

$$D \ni y \mapsto \int_Z \langle \gamma, i_y^* m^* \alpha \rangle \in \mathbb{C}$$

for every $[\alpha] \in H_{\bar{\partial}}^S(G/L, \mathcal{L}_\lambda)$ and $\gamma \in V_{\mu_\lambda}^\vee \simeq F$. Observing that $\ell_y = m \circ i_y$, one has

$$\tilde{\mathcal{R}}(\alpha, y) = \iota^* \ell_y^* \alpha = \iota^* i_y^* m^* \alpha$$

for any $\bar{\partial}$ -closed form $\alpha \in \mathcal{E}^{0,S}(G/L, \mathcal{L}_\lambda)$, hence for every $\gamma \in F = \mathcal{O}(Z, \mathcal{L}_{-\mu_\lambda})$, one has $\langle \gamma, i_y^* m^* \alpha \rangle = \langle \gamma, (\iota^*)^{-1} \tilde{\mathcal{R}}(\alpha, y) \rangle \in \mathcal{E}^{0,S}(Z, K_Z) \simeq \mathcal{E}^{S,S}(Z)$.

Hence we have shown that $\int_Z \langle \gamma, i_y^* m^* \alpha \rangle = \int_{K/L \cap K} \langle \iota^* \gamma, \widetilde{\mathcal{R}}(\alpha, y) \rangle$ is a holomorphic function of $y \in D$, namely, $\int_Z (m^* \alpha)|_Z = \widetilde{\mathcal{R}}(\alpha, \cdot)$ is a V_{μ_λ} -valued holomorphic function in the neighbourhood D of g in $G_{\mathbb{C}}(K; Q)$. Since g is an arbitrary element of $G_{\mathbb{C}}(K; Q)$, we conclude that $\widetilde{\mathcal{R}}(\alpha, \cdot)$ is holomorphic in $G_{\mathbb{C}}(K; Q)$. ■

3. Geometry G/L of holomorphic type

Retain our setting that G/L is equipped with a G -invariant complex manifold structure by a θ -stable parabolic subalgebra \mathfrak{q} . If \mathfrak{q} is of holomorphic type, G/L has a stronger holomorphic structure, namely, G/L is a K -equivariant holomorphic fiber bundle over the compact complex manifold $K/(L \cap K)$, see [12]. In this section we prepare some other structural results on the non-compact manifold G/L . Although some of them are known, e.g., [11, 15], we provide a short proof for the sake of completeness.

Lemma 3.1. *We set $L' := L \cap K$. If \mathfrak{q} is of holomorphic type, then G/L' carries a G -invariant complex structure such that the natural projection $\pi: G/L' \rightarrow G/L$ is holomorphic, and the typical fiber is the Hermitian symmetric space $L/L' = L/(L \cap K)$.*

Proof. Let $\mathfrak{b}_{\mathfrak{k}}$ be the Borel subalgebra of $\mathfrak{k}_{\mathbb{C}}$ associated to the positive system $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ which contains $\Delta(\mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Since both \mathfrak{q} and $\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^+$ are parabolic subalgebras containing the same Borel subalgebra $\mathfrak{b}_{\mathfrak{k}} + \mathfrak{p}^+$ of $\mathfrak{g}_{\mathbb{C}}$, their intersection

$$\mathfrak{q}' := \mathfrak{q} \cap (\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^+)$$

is also a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We write Q' for the corresponding parabolic subgroup of $G_{\mathbb{C}}$. By definition, the Levi decomposition $\mathfrak{q}' = \mathfrak{l}'_{\mathbb{C}} + \mathfrak{u}'$ with $\mathfrak{l}'_{\mathbb{C}} \subset \mathfrak{l}_{\mathbb{C}}$ and $\mathfrak{u}' \supset \mathfrak{u}$ satisfies

$$\mathfrak{u}' \cap \mathfrak{k}_{\mathbb{C}} = \mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}, \quad \mathfrak{u}' \cap \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+, \quad \text{and} \quad \mathfrak{l}'_{\mathbb{C}} = \mathfrak{l}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}. \tag{10}$$

In particular, \mathfrak{q}' is a θ -stable parabolic subalgebra of holomorphic type (see Definition 1.1), and the normalizer $N_G(\mathfrak{q}')$ equals $L' = L \cap K$. We equip the homogeneous space G/L' with a complex structure via the open embedding $G/L' \hookrightarrow G_{\mathbb{C}}/Q'$. Since the natural projection $G_{\mathbb{C}}/Q' \rightarrow G_{\mathbb{C}}/Q$ is a $G_{\mathbb{C}}$ -equivariant holomorphic bundle with typical fiber $Q/Q' \simeq L_{\mathbb{C}}/(L_{\mathbb{C}} \cap Q')$, the G -equivariant bundle $\pi: G/L' \rightarrow G/L$ keeps the holomorphic structure with typical fiber $L/L' \simeq L/L \cap K$, as the diagram below shows their open embeddings.

$$\begin{array}{ccc} G/L' & \xhookrightarrow{\text{open}} & G_{\mathbb{C}}/Q' \\ \pi \downarrow & & \downarrow \\ L/L' & \xhookrightarrow{\text{open}} & Q/Q' \\ \downarrow & & \downarrow \\ G/L & \xhookrightarrow{\text{open}} & G_{\mathbb{C}}/Q \end{array} \quad \blacksquare$$

Remark 3.2. Retain the setting and notation of Lemma 3.1. In particular, $L' = L \cap K$. Applying a generalized Harish-Chandra embedding [12] to both G/L and G/L' , one sees that there exist L' -invariant bounded domains Ω and Ω' in $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}$ and \mathfrak{p}^- , respectively such that $G/L \simeq K \times_{L'} \Omega$ and $G/L' \simeq K \times_{L'} \Omega'$ are K -equivariant holomorphic bundles over K/L' . We write $\text{pr}_1: \mathfrak{p}^- \rightarrow \mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}$ for the first projection of the direct sum decomposition $\mathfrak{p}^- = (\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}}) \oplus (\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{p}^-)$.

Then one has $\text{pr}_1(\Omega') = \Omega$ and the diagram below commutes.

$$\begin{array}{ccccc}
 K/L' & \xrightarrow[\text{closed}]{\iota'} & G/L' & \simeq & K \times_{L'} \Omega' & \xrightarrow{\varpi} & K/L \cap K \\
 \parallel & & \downarrow \pi & & \downarrow 1 \times \text{pr}_1 & & \parallel \\
 K/L' & \xrightarrow[\text{closed}]{\iota} & G/L & \simeq & K \times_{L'} \Omega & \xrightarrow{\varpi} & K/L \cap K
 \end{array} \tag{11}$$

As seen in the above diagram, the projection $\pi: G/L' \rightarrow G/L$ is injective when restricted to the submanifold $\iota'(K/L \cap K)$. We consider Dolbeault cohomologies at the same degree S , which is the complex dimension of $K/L \cap K$, and compare the two Penrose transforms:

$$\begin{aligned}
 \mathcal{R}: H_{\bar{\partial}}^S(G/L, \mathcal{L}_\lambda) &\rightarrow \mathcal{E}(G/K, \mathcal{V}_{\mu_\lambda}), \\
 \mathcal{R}': H_{\bar{\partial}}^S(G/L', \pi^* \mathcal{L}_\lambda) &\rightarrow \mathcal{E}(G/K, \mathcal{V}_{\mu_\lambda}).
 \end{aligned}$$

Proposition 3.3. *(Factorization of the Penrose transform)*

Assume that \mathfrak{q} is of holomorphic type. For any G -equivariant holomorphic line bundle \mathcal{L} over G/L , the projection $\pi: G/L' \rightarrow G/L$ induces a G -intertwining operator $\pi^*: H_{\bar{\partial}}^S(G/L, \mathcal{L}) \rightarrow H_{\bar{\partial}}^S(G/L', \pi^* \mathcal{L})$ and one has $\mathcal{R} = \mathcal{R}' \circ \pi^*$.

Proof. Since π is holomorphic, the pull-back $\pi^*: \mathcal{E}^{0,S}(G/L, \mathcal{L}) \rightarrow \mathcal{E}^{0,S}(G/L', \pi^* \mathcal{L})$ sends $\bar{\partial}$ -closed forms (resp. $\bar{\partial}$ -exact forms) on G/L to those on G/L' , respectively. Thus π^* induces a G -intertwining operator, to be denoted by the same letter π^* , $H_{\bar{\partial}}^S(G/L, \mathcal{L}) \rightarrow H_{\bar{\partial}}^S(G/L', \pi^* \mathcal{L})$.

For a $\bar{\partial}$ -closed form $\alpha \in \mathcal{E}^{0,S}(G/L, \mathcal{L})$, the definition (1) shows

$$\widetilde{\mathcal{R}}'(\pi^* \alpha, g) = (\iota')^*(\ell'_g)^* \pi^* \alpha = (\pi \circ \ell'_g \circ \iota')^* \alpha = \iota^* \ell_g^* \alpha = \widetilde{\mathcal{R}}(\alpha, g).$$

Passing to cohomologies, we have shown $\mathcal{R}' \circ \pi^* = \mathcal{R}$. ■

We use a finer structure of the set $G_{\mathbb{C}}(K; Q)$ and the extended Penrose transform in Definition-Lemma 2.1 for the parabolic subalgebra \mathfrak{q}' as below.

Proposition 3.4. *Assume \mathfrak{q} is of holomorphic type such that $\mathfrak{q}' = \mathfrak{q}$, namely, $\mathfrak{q} \subset \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^+$, $L \subset K$ and $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^-$.*

(1) *There exists a bounded domain D in \mathfrak{p}^- such that*

$$GQ = GK_{\mathbb{C}}P^+ = G \exp(D)Q = G_{\mathbb{C}}(K; Q).$$

(2) *For any $h \in G_{\mathbb{C}}(K; Q)$, $k \in K_{\mathbb{C}}$, $p \in P^+$, and for any $\bar{\partial}$ -closed form $\alpha \in \mathcal{E}^{0,S}(G/L, \mathcal{L}_\lambda)$, one has*

$$\widetilde{\mathcal{R}}(\alpha, hkp) = k^{-1} \widetilde{\mathcal{R}}(\alpha, h).$$

Proof. (1) Let $o := eK$ be the origin of $G/K \subset G_{\mathbb{C}}/K_{\mathbb{C}}P^+$. We recall that

$$i: \mathfrak{p}^- \xrightarrow{\sim} \exp(\mathfrak{p}^-) \cdot o \subset G_{\mathbb{C}}/K_{\mathbb{C}}P^+, \quad Y \mapsto e^Y \cdot o$$

defines the open Bruhat cell and that there exists a bounded open domain D in \mathfrak{p}^- such that $D \rightarrow G/K$, $Y \mapsto e^Y \cdot o$ is biholomorphic (the Harish-Chandra decomposition).

By our assumption that $\mathfrak{q} \subset \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^+$ and $L \subset K$, one has $KQ = K_{\mathbb{C}}P^+$ because $K_{\mathbb{C}} = K(Q \cap K_{\mathbb{C}})$. Combining this with $G \cdot o = \exp(D) \cdot o$ in $G_{\mathbb{C}}/K_{\mathbb{C}}P^+$, one has

$$GQ = GKQ = GK_{\mathbb{C}}P^+ = G \exp(D)K_{\mathbb{C}}P^+ = G \exp(D)KQ. \tag{12}$$

The set (12) is right K -invariant, namely, $GQK = GQ$. This shows

$$G_{\mathbb{C}}(K; Q) = \{g \in G_{\mathbb{C}} : gK \subset GQK\} = GQK = GQ.$$

(2) We have already seen the $K_{\mathbb{C}}$ -covariance of $\tilde{\mathcal{R}}$ in (8). For any $p \in P^+$ and $k \in K_{\mathbb{C}}$, one has $pkQ = k(k^{-1}pk)Q = kQ$ because $\mathfrak{p}^+ = \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}} \subset \mathfrak{q}$. Hence the left translation $\ell_p: G_{\mathbb{C}}/Q \rightarrow G_{\mathbb{C}}/Q$ is the identity on the submanifold $\iota(K/L)$ in $G_{\mathbb{C}}/Q$, namely, $\ell_p \circ \iota = \iota$. Thus $\mathcal{R}(\alpha, gp) = \tilde{\mathcal{R}}(\alpha, g)$ for any $p \in P^+$. Hence the proposition is proved. ■

4. Proof of the implication (i) \Rightarrow (ii) in Theorem 1.3

We recall that G/K is a totally real submanifold of $Y_{\mathbb{C}}$ ($\subset G_{\mathbb{C}}/K_{\mathbb{C}}$) (see Definition-Lemma 2.1) for any θ -stable parabolic subalgebra \mathfrak{q} , whereas G/K itself has a complex structure as an open submanifold of $G_{\mathbb{C}}/K_{\mathbb{C}}P^+$ when G/K is of Hermitian type. We shall see that two holomorphicity assertions on $Y_{\mathbb{C}}$ (Theorem 2.2) and on G/K (Proposition 4.1 below) are compatible by using Proposition 3.4. We give a proof of the following proposition by different approaches from [2, 10, 11, 15]:

Proposition 4.1. *Suppose we are in the setting of Theorem 1.3. If \mathfrak{q} is of holomorphic type, then one has $\text{Image } \mathcal{R} \subset \mathcal{O}(G/K, \mathcal{V}_{\mu_{\lambda}})$.*

Proof of Proposition 4.1. By the factorization theorem of the Penrose transform (Proposition 3.3), the proof of Proposition 4.1 is reduced to the case $\mathfrak{q} = \mathfrak{q}'$. So we may and do assume that $\mathfrak{q} = \mathfrak{q}'$, namely, $\mathfrak{q} \subset \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^+$, $L \subset K$ and $\mathfrak{u}^- \cap \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^-$. Let $\mathcal{V}'_{\mu_{\lambda}}$ be the holomorphic vector bundle over $GK_{\mathbb{C}}P^+/K_{\mathbb{C}}P^+$ associated to the representation $V_{\mu_{\lambda}}$ of $K_{\mathbb{C}}P^+$ with trivial action by P^+ . Then one has a natural morphism between G -equivariant vector bundles:

$$\begin{array}{ccc} \tilde{\mathcal{V}}_{\mu_{\lambda}} & \longrightarrow & GK_{\mathbb{C}}P^+/K_{\mathbb{C}} & = & Y_{\mathbb{C}} \\ \downarrow & & \downarrow & & \\ \mathcal{V}'_{\mu_{\lambda}} & \longrightarrow & GK_{\mathbb{C}}P^+/K_{\mathbb{C}}P^+ & \simeq & G/K. \end{array}$$

In consequence of Proposition 3.4 we obtain that the extended Penrose transform $\mathcal{R}: H_{\bar{\partial}}^S(G/L, \mathcal{L}_{\lambda}) \rightarrow \mathcal{O}(Y_{\mathbb{C}}, \tilde{\mathcal{V}}_{\mu_{\lambda}})$ induces a G -homomorphism

$$H_{\bar{\partial}}^S(G/L, \mathcal{L}_{\lambda}) \rightarrow \mathcal{O}(Y_{\mathbb{C}}, \tilde{\mathcal{V}}_{\mu_{\lambda}})^{P^+} \simeq \mathcal{O}(G_{\mathbb{C}}(K; Q)/K_{\mathbb{C}}P^+, \mathcal{V}'_{\mu_{\lambda}}).$$

The natural embedding $G \hookrightarrow G_{\mathbb{C}}(K; Q)$ induces a G -equivariant biholomorphic map $G/K \xrightarrow{\sim} GK_{\mathbb{C}}P^+/K_{\mathbb{C}}P^+ = G_{\mathbb{C}}(K; Q)/K_{\mathbb{C}}P^+$, hence one identifies the two holomorphic bundles $\mathcal{V}_{\mu_{\lambda}} \rightarrow G/K$ and $\mathcal{V}'_{\mu_{\lambda}} \rightarrow G_{\mathbb{C}}(K; Q)/K_{\mathbb{C}}P^+$. Thus one has a natural G -isomorphism $\mathcal{O}(G_{\mathbb{C}}(K; Q)/K_{\mathbb{C}}P^+, \mathcal{V}'_{\mu_{\lambda}}) \simeq \mathcal{O}(G/K, \mathcal{V}_{\mu_{\lambda}})$. Hence the proposition is proved. ■

For the last statement of Theorem 1.3, we prepare the following:

Definition-Lemma 4.2. *Let G/K be a Hermitian symmetric space, and (σ, V) be an irreducible K -module. We write $\mathcal{V} := G \times_K V$ for the homogeneous vector bundle over G/K . Then one has $\dim_{\mathbb{C}} \text{Hom}_K(V, \mathcal{O}(G/K, \mathcal{V})) = 1$. We write \mathbb{V} for the unique subspace $\mathcal{O}(G/K, \mathcal{V})$ on which K acts as (σ, V) .*

Proof. Via the Harish-Chandra realization $\mathfrak{p}^- \supset D \xrightarrow{\sim} G/K$, the bundle $\mathcal{V} = G \times_K V$ over G/K is trivialized as $\mathcal{V} \simeq D \times V$ as K -equivariant vector bundles. This gives a K -structure of $\mathcal{O}(G/K, \mathcal{V})$ by

$$\text{Sym}(\mathfrak{p}^+) \otimes V \simeq \text{Pol}(\mathfrak{p}^-) \otimes V \simeq \mathcal{O}(G/K, \mathcal{V})_{K\text{-finite}}.$$

Since a central element of \mathfrak{k} acts on $S(\mathfrak{p}^+) = \bigoplus_{\ell=0}^{\infty} S^{\ell}(\mathfrak{p}^+)$ by different scalars depending on the homogeneous degree ℓ , one concludes

$$\text{Hom}_K(V, S(\mathfrak{p}^+) \otimes V) \simeq \text{Hom}_K(V, V) \simeq \mathbb{C}. \quad \blacksquare$$

Now the second assertion of Theorem 1.3 follows from the proposition below:

Proposition 4.3. *Assume \mathfrak{q} is of holomorphic type. Then for any L -character \mathbb{C}_{λ} such that μ_{λ} is $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ -dominant, one has $\mathbb{V}_{\mu_{\lambda}} \subset \text{Image } \mathcal{R}$.*

Remark 4.4. We do not assume that λ is “very regular” in Proposition 4.3, and allows the case where \mathcal{R} is not injective, see [11] for such an example.

Proof of Proposition 4.3. By [12, Thm. 4.1], $\text{Image } \mathcal{R}$ contains the K -type $V_{\mu_{\lambda}}$. By Theorem 2.2, $\text{Image } \mathcal{R} \subset \mathcal{O}(G/K, \mathcal{V}_{\mu_{\lambda}})$. Thus the proposition follows from Definition-Lemma 4.2. ■

5. Proof of the implication (ii) \Rightarrow (i) in Theorem 1.3

This section completes the proof of Theorem 1.3 by showing Theorem 5.1 below.

Following the terminology in [13], we say an L -character \mathbb{C}_{λ} is in the *good range* with respect to \mathfrak{q} if $\langle \lambda + \rho_{\mathfrak{l}}, \alpha \rangle > 0$ for any $\alpha \in \Delta(\mathfrak{u}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, where $\rho_{\mathfrak{l}}$ is half the sum of positive roots in $\Delta(\mathfrak{t}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. (This condition is independent of the choice of $\Delta^+(\mathfrak{t}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and there exist countably many such L -characters \mathbb{C}_{λ} .)

The implication (ii) \Rightarrow (i) in Theorem 1.3 follows from the theorem below.

Theorem 5.1. *If $\text{Image } \mathcal{R} \subset \mathcal{O}(G/K, \mathcal{V}_{\mu_{\lambda}})$ for some L -character \mathbb{C}_{λ} in the good range, then the θ -stable parabolic subalgebra \mathfrak{q} is of holomorphic type (Definition 1.1).*

The rest of this section is devoted to the proof of Theorem 5.1. We shall utilize the associated variety $\mathcal{V}(X)$ of a finitely generated \mathfrak{g} -module X , which is defined as an algebraic variety in $\mathfrak{g}_{\mathbb{C}}^{\vee}$ [14]. The following basic property is well-known (e.g., [9, Lem. 3.1] for (2)).

Lemma 5.2. (1) *If Y is a \mathfrak{g} -submodule of X , then $\mathcal{V}(Y) \subset \mathcal{V}(X)$.*
 (2) *Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Then $\mathcal{V}(X) \subset \mathfrak{h}^\perp$ if \mathfrak{h} acts locally finitely on X , where $\mathfrak{h}^\perp := \{x \in \mathfrak{g}_\mathbb{C}^\vee : x|_{\mathfrak{h}} = 0\}$.*

Since \mathfrak{g} is semisimple, we shall regard the associated variety $\mathcal{V}(X)$ as a subset of $\mathfrak{g}_\mathbb{C}$ via the isomorphism $\mathfrak{g}_\mathbb{C}^\vee \simeq \mathfrak{g}_\mathbb{C}$ induced by the Killing form.

Suppose μ_λ is $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ -dominant. Then the Penrose transform

$$\mathcal{R}: H_{\bar{\partial}}^S(G/L, \mathcal{L}_\lambda) \rightarrow C^\infty(G/K, \mathcal{V}_{\mu_\lambda})$$

is non-zero by [12, Thm. 4.1]. If λ is in the good range then μ_λ is $\Delta^+(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ -dominant and the G -module $H_{\bar{\partial}}^S(G/L, \mathcal{L}_\lambda)$ is irreducible by [13, 16]. Therefore \mathcal{R} is injective. Since the associated variety of the underlying (\mathfrak{g}, K) -module of $H_{\bar{\partial}}^S(G/L, \mathcal{L}_\lambda)$ is given by $\text{Ad}(K_\mathbb{C})(\mathfrak{u}^- \cap \mathfrak{p}_\mathbb{C})$, see e.g., [5, Lem. 2.7], that of $\text{Image } \mathcal{R}$ is also equal to $\text{Ad}(K_\mathbb{C})(\mathfrak{u}^- \cap \mathfrak{p}_\mathbb{C})$.

From now on, we assume that G is a connected simple Lie group of Hermitian type with complexified Lie algebra $\mathfrak{g}_\mathbb{C} = \mathfrak{p}^- + \mathfrak{k}_\mathbb{C} + \mathfrak{p}^+$.

Definition 5.3. An irreducible (\mathfrak{g}, K) -module X is called a *highest weight (\mathfrak{g}, K) -module* (resp. *lowest weight (\mathfrak{g}, K) -module*) if X has a non-zero vector annihilated by \mathfrak{p}^+ (resp. \mathfrak{p}^-).

The highest weight (\mathfrak{g}, K) -modules and lowest weight (\mathfrak{g}, K) -modules are characterized by their associated varieties:

Lemma 5.4. [9, Lem. 3.5] *Suppose that X is an irreducible (\mathfrak{g}, K) -module. Then X is a highest weight (\mathfrak{g}, K) -module if and only if $\mathcal{V}(X) \subset \mathfrak{p}^+$. Likewise, X is a lowest weight (\mathfrak{g}, K) -module if and only if $\mathcal{V}(X) \subset \mathfrak{p}^-$.*

Let $\mathcal{V} \rightarrow G/K$ be a G -equivariant holomorphic vector bundle where G/K is equipped with a complex structure via the open embedding $G/K \subset G_\mathbb{C}/K_\mathbb{C}P_+$ as before. Then any (\mathfrak{g}, K) -submodule Y of $\mathcal{O}(G/K, \mathcal{V})$ is a lowest weight module, hence one has $\mathcal{V}(Y) \subset \mathfrak{p}^-$ by Lemma 5.2.

Proof of Theorem 5.1. If $\text{Image } \mathcal{R} \subset \mathcal{O}(G/K, \mathcal{V}_{\mu_\lambda})$, then $\text{Ad}(K_\mathbb{C})(\mathfrak{u}^- \cap \mathfrak{p}_\mathbb{C}) \subset \mathfrak{p}^-$, hence $\mathfrak{u}^- \cap \mathfrak{p}_\mathbb{C} \subset \mathfrak{p}^-$, which is equivalent to $\mathfrak{u} \cap \mathfrak{p}_\mathbb{C} \subset \mathfrak{p}^+$. Thus \mathfrak{q} is of holomorphic type. ■

Thus the proof of Theorem 1.3 is complete.

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