

On Odd Parameters in Geometry

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Abstract. (1) In 1976, looking at the list of simple finite-dimensional complex Lie superalgebras, J. Bernstein and I, and independently M. Duflo, observed that some divergence-free vectorial Lie superalgebras have deformations with odd parameters and conjectured that no other simple Lie superalgebras have such deformations. Here, I prove this conjecture and overview the known classification of simple finite-dimensional complex Lie superalgebras, their presentations, realizations, and relations with simple Lie (super)algebras over fields of positive characteristic.

(2) Any ringed space of the form (a manifold M , the sheaf of sections of the exterior algebra of a vector bundle over M) is called split supermanifold. Gawędzki (1977) and Batchelor (1979) proved that every smooth supermanifold is split. In 1982, P. Green and Palamodov showed that a complex-analytic supermanifold can be non-split. So far, researchers considered only even obstructions to splitness. This led them to the conclusion that any supermanifold of superdimension $m|1$ is split. I'll show there are non-split supermanifolds of superdimension $m|1$; e.g., certain superstrings, the obstructions to their splitness depend on odd parameters.

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1. Introduction

In this paper, I mainly consider two types of occurrences of odd parameters:

- (i) in deformations of several simple Lie superalgebras;
- (ii) in obstructions to splitness of complex-analytic supermanifolds.

The Lie superalgebras considered are mainly finite-dimensional over \mathbb{C} , unless otherwise specified; the characteristic p of the ground field \mathbb{K} can not be confused with parity also denoted by p .

I overview many related results and offer several intriguing open problems.

1.1. Prologue: Four types of main characters, two questions

In the late 1960s, many mathematicians (first, in Moscow, and later in the whole Universe, until the vogue changed) were discussing two topics:

- (1) **Kac's classification of simple \mathbb{Z} -graded Lie algebras of polynomial growth over \mathbb{C}** (under a technical assumption later dismissed by O. Mathieu in a difficult paper [119]) consisting of finite-dimensional examples (classified by Cartan and Killing), vectorial Lie algebras with polynomial coefficients (classified by Cartan) and two new ingredients: loop algebras with values in simple finite-dimensional algebras and twisted versions of these loops, and the complexified Lie algebra of vector fields on the circle, see [78].

(2) **The Kostrikin-Shafarevich conjecture** on classification of simple finite-dimensional *modular* (i.e., over \mathbb{K} of characteristic $p > 0$) Lie algebras over algebraically closed fields of characteristic $p > 7$ (initially formulated only for restricted algebras); for a review and update, see [20].

In both classifications, the \mathbb{Z} -graded infinite-dimensional simple Lie algebras of vector fields over \mathbb{C} play a prominent role.

It became more and more obvious with time that two digressions from the main goal (classification of *simple and \mathbb{Z} -graded* objects) are important: classification of (a) deformations of \mathbb{Z} -graded Lie algebras and (b) Lie algebras of derivations and central extensions of simple Lie algebras.

The deformation of the Lie algebra of Hamiltonian vector fields induced by quantization of the Poisson Lie algebra leads out of the set of \mathbb{Z} -graded Lie algebras considered in topic (1), and was ignored by mathematicians interested in classifications (1) and (2), whereas all other examples of simple \mathbb{Z} -graded Lie algebras are rigid. (This claim on rigidity is correct from a *certain point of view* only; I will not digress into this mysterious area here; let me just mention an *ostensible* contradiction between, on one hand, the correct, under the definitions considered, proof of the fact that $H^2(\mathfrak{g}; \mathfrak{g}) = 0$ for the loop algebras \mathfrak{g} , see [96], and, on the other hand, the explicit examples of such deformations in [61, 76] later evolved into a rich theory of Krichever-Novikov algebras, see a review [147]. There is no contradiction, actually, because $H^2(\mathfrak{g}; \mathfrak{g}) = 0$ does not guarantee *rigidity*, which is lack of deformations. Observe also that $H^2(\mathfrak{g}; \mathfrak{g}) \neq 0$ does not imply that the deformed algebra is not isomorphic to the initial algebra; this phenomenon is called *semi-trivial deformation*, see [22, 25] with many examples.)

Although in topic (2), i.e., in the non-restricted KSh-conjecture, the deformations come to the foreground, they still did not draw as much attention as I think they should have, except for the filtered deformations. (A plausible, very probable, reason for this negligence: of the two types of simple Lie algebras, the non-filtered *deforms* – results of deformations – of the vectorial Lie algebras are isomorphic to the known simple Lie algebras, whereas the Lie algebras of the other type – the ones with Cartan matrix and their simple relatives – are rigid, at least for $p > 3$. The situation is totally different if $p = 2$, see Shchepochkina’s exceptional example – the deform of the divergence-free algebra – in [20].)

Observe two more types of characters that came to the foreground in the mid-1970s:

(3) **Lie superalgebras** were known in topology and deformation theory under the self-contradictory¹ name “graded Lie algebras” since at least 1940, mainly over finite fields or \mathbb{Z} (in which case they are super Lie rings). From the mid-1970s, Lie superalgebras started to draw attention of researchers: first, in characteristic 0, thanks to applications of Lie superalgebras to high energy physics and solid body physics.

Some physicists, more clairvoyant than others, began to perceive the usefulness – in physics – of what is nowadays called “Lie superalgebras” even before the notion “Lie superalgebra” was explicitly distinguished and separated from “ $\mathbb{Z}/2$ -graded Lie algebra”, see prefaces in [50]. Encouraged by one such physicist, Kac gave examples

¹ The old name is self-contradictory because Lie superalgebras and Lie algebras (graded or not) satisfy different identities.

of three series of simple vectorial Lie superalgebras, see [79] and more accessible Kac's preface in [50].

(4) For decades the mathematicians interested in simple Lie (super)algebras mostly ignored the examples of *filtered simple Lie algebras which can not be obtained as a result of deformation of a \mathbb{Z} -graded simple Lie algebra*. In physics, such Lie (super)algebras appeared in higher spin theories, see [153, 154]; in mathematics, they were used to interpret known orthogonal polynomials, see [113, 141]. Among such simple Lie (super)algebras, the simplest to describe is $\mathfrak{sl}(\lambda)$, the Lie algebra of traceless matrices of complex size $\lambda \in \mathbb{C}\mathbb{P}^1$. Recall its construction.

Let $\mathfrak{g} = \mathfrak{sl}(2)$, and $c_1 = \lambda^2 - 1$, where $\lambda = \mu + 1 \in \mathbb{C}$ for the highest weight μ of an irreducible $\mathfrak{sl}(2)$ -module. Let C_1 be the quadratic Casimir operator. The associative algebra $U_\lambda := U(\mathfrak{g})/(C_1 - c_1)$, where $U(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} and $(C_1 - c_1)$ is a two-sided ideal generated by $C_1 - c_1$, has an ideal

$$I_\lambda = \begin{cases} 0 & \text{if } \lambda \notin \mathbb{Z} \setminus \{0\} \\ \text{of finite codimension} & \text{otherwise.} \end{cases}$$

Let A be any associative algebra, let A^L be the Lie algebra whose space is A but multiplication is given by the commutator $[a, b] := ab - ba$ for any $a, b \in A$. If $\lambda = n \in \mathbb{Z} \setminus \{0\}$, then $U_n/I_n \simeq \text{Mat}(|n|)$, hence the name $\mathfrak{gl}(\lambda) := (U_\lambda/I_\lambda)^L$ for any $\lambda \in \mathbb{C}$, see [53].

While the trace can be normalized at will, J. Bernstein suggested a definition of the trace on $\mathfrak{gl}(\lambda)$ such that $\text{tr}(1) = \lambda$ for $\lambda \in (\mathbb{C} \setminus \mathbb{Z})$, see [90], resembling $\text{tr}(1_{|n|}) = |n|$ for $n \in \mathbb{Z} \setminus \{0\}$. Let

$$\mathfrak{sl}(\lambda) := \{X \in \mathfrak{gl}(\lambda) \mid \text{tr}X = 0\}. \quad (1)$$

If $\lambda \notin \mathbb{Z} \setminus \{0\}$, then U_λ and its limit as $\lambda \rightarrow \infty \in \mathbb{C}\mathbb{P}^1$, see [64], are central simple as well as U_n/I_n for $n \in \mathbb{Z} \setminus \{0\}$. According to a Herstein–Montgomery theorem, for any central simple associative algebra A with center Z , the Lie algebra $L(A) := (A^L)' / ((A^L)' \cap Z)$, where \mathfrak{g}' designates the first derived (commutant) of the Lie (super)algebra \mathfrak{g} , is simple, and the same is true for Lie superalgebras under a certain condition, see [121, Th.3.8].

The Lie algebras $(U_\lambda)^L$ are “quantized” versions of the Lie algebras of functions on the orbits of the co-adjoint representation of simple finite-dimensional Lie algebras over \mathbb{C} , see [93], considered with the Poisson bracket. The natural filtration of $U(\mathfrak{sl}(2))$ induces a filtration in $(U_\lambda)^L$; the associated graded Lie algebra is isomorphic for $\lambda \neq 0$ to the Lie algebra $\mathfrak{po}(2)_{\bar{0}}$ realized on the space of even degree polynomials in 2 indeterminates with respect to the Poisson bracket, see [74]; the elements of non-negative degree in the Lie grading of $\mathfrak{po}(2)_{\bar{0}}$ form a Lie algebra with codimension-3 radical complementing $\mathfrak{sl}(2)$.

Consider the following generalizations of $\mathfrak{sl}(\lambda)$, see eq. (1). Let \mathfrak{g} be any simple finite-dimensional Lie algebra over \mathbb{C} , let ρ be the half-sum of positive roots of \mathfrak{g} , let μ be the highest weight of an irreducible \mathfrak{g} -module with highest weight vector v^μ , let the C_i be the Casimirs, i.e., the generators of the center of $U(\mathfrak{g})$. Set (the shift by ρ is need to get the c_i symmetric relative the Weyl group)

$$U_c := U(\mathfrak{g}) / (C_i - c_i)_{i=1}^{\text{rk}\mathfrak{g}}, \quad \text{where } c_i = C_i|_{v^{\mu+\rho}} \in \mathbb{C}. \quad (2)$$

For almost all values of $c = (c_1, \dots, c_{\text{rk}\mathfrak{g}})$, the algebra U_c is central simple, so $L(U_c)$ is a simple Lie algebra, see [106], where Lie superalgebras \mathfrak{g} in this construction are also considered.

Question 1: What are the deformations of the simple \mathbb{Z} -graded Lie superalgebras (finite-dimensional and of polynomial growth) that can not be captured by classification of equivalence classes of the tensors the algebra preserves, cf. [20, Subsection 24.2]? In § 7, I will show that no simple *finite-dimensional* vectorial simple Lie superalgebra over \mathbb{C} has such deformations.

Contrariwise, the deforms of $\mathfrak{osp}(4|2)$ (see Subsection 6.1.2) provide us with examples of Lie superalgebras that can not be described by equivalence classes of the bilinear forms preserved by certain members of the parametric family of these algebras.

Question 2: Over \mathbb{C} , is there a simple finite-dimensional Lie superalgebra whose associated \mathbb{Z} -graded algebra has no negative part, like $\mathfrak{sl}(\lambda)$? For the answer, see [83] and the gist of the ideas of the proof in § 8: no such Lie superalgebras. The answer for the simple finite-dimensional Lie algebras over fields of characteristic > 3 is the same. The same question for Lie superalgebras over fields of positive characteristic and Lie algebras in characteristics 3 and 2 is an open problem.

1.2. Disclaimer

It was clear from the very beginning that certain “relatives” of some of the simple Lie (super)algebras \mathfrak{g} , such as Lie (super)algebras of derivations and central extensions of \mathfrak{g} , are more (at least, no less) interesting in applications than \mathfrak{g} itself: e.g., affine Kac–Moody (super)algebras vs. loop (super)algebras and their twisted versions, Poisson Lie (super)algebras vs. Hamiltonian Lie (super)algebras. (Observe that the affine Kac–Moody algebras are most known examples of recently distinguished notion of *double extensions*, this notion simultaneously requires a derivation, a central extension, and a third ingredient – a non-degenerate \mathfrak{g} -invariant symmetric bilinear form on \mathfrak{g} , see [28] and references therein.)

Until now, the vectorial Lie superalgebras (same as vectorial Lie algebras over fields of positive characteristic) and several other types of Lie (super)algebras are often described as if the author of the description is obsessed with *simple* Lie (super)algebras. This approach leads to the notation with a wrong emphasis that manifestly has to be improved: the *natural* and *basic* object is the *Cartan prolong* (the result of the Cartan prolongation) rediscovered by Tanaka and generalized by Shchepochkina (see [20, 146]), whereas the simple object is the derived algebra or a quotient of the prolong. Similar arguments identify $\mathfrak{g}(A)$, $\mathfrak{pe}(n)$, $\mathfrak{q}(n)$, $\mathfrak{po}(0|n)$, defined in what follows, as the protagonists, more natural than their simple subquotients.

However, the *simple* Lie (super)algebras are the first to be classified, of course.

1.3. Simple Lie superalgebras. First definitions and examples

First examples of Lie superalgebras, actually Lie super **rings** over \mathbb{Z} , appeared in 1941, in topology as the sets of homotopy groups with the Whitehead product, see [157]. Associated with these examples were modular Lie superalgebras over finite fields. Similarity of Lie superalgebras and modular Lie algebras is so striking that sometimes one hears and reads that “when $p = 2$, there is no difference between Lie algebras and Lie superalgebras”, which is not true as is lucidly explained in [20].

Weibel writes: “Homological algebra had its origins in the 19-th century, via the work of Riemann (1857) and Betti (1871) on “homology numbers”, and the rigorous development of the notion of homology numbers by Poincaré in 1895. An observation of Emmy Noether in 1925 shifted the attention to the “homology groups” of a space, and algebraic techniques were developed for computational purposes in the 1930’s. Yet, homology remained a part of the realm of topology until about 1945.”, see [156, p. 797–836].

The Lie superalgebras or super rings that thus appeared in topology over \mathbb{Z} or finite fields were nilpotent (hence, “not very interesting” and nobody tried to find out any simple (without ideals) examples), and did not draw much attention until it was discovered that the representations of the quantum groups $U_q(\mathfrak{g})$ when q is the p th root of unit are related with the representation of the characteristic p version of the simple complex Lie algebra \mathfrak{g} and a certain Lie superalgebra, see, e.g., [1, 38, 155] and references therein. By that time simple finite-dimensional Lie superalgebras over \mathbb{C} were already widely known.

However, even now, the study of modular Lie (super)algebras, even simple ones, is not a “trendy” topic, despite very interesting recent advances, see, e.g., [87] and references therein.

The steps of classification of *simple* finite-dimensional Lie superalgebras considered naively are as follows.

Over \mathbb{C} . For the first works with examples, starting with [79], see the classification in particular cases [88, 89], [135] and [44–46, 48, 75]. For a classification of simple \mathbb{Z} -graded finite-dimensional Lie superalgebras together with rounding up of all (in particular, vectorial) finite-dimensional cases, see [80–82] culminated in Part 1 of [83]. In these first papers, in the book [134], and in many works published later, Lie superalgebras \mathfrak{g} were, and often still are, considered as pairs $(\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}})$ consisting of the Lie algebra $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{0}}$ -module $\mathfrak{g}_{\bar{1}}$ such that the squaring $\text{sq} : S^2(\mathfrak{g}_{\bar{1}}) \rightarrow \mathfrak{g}_{\bar{0}}$ satisfies the super Jacobi identity. Interpretations of each Lie superalgebra as preserving “something” (realizations) are more insightful; they help to understand the structure of the superalgebra, its representations, and this “something”. (Observe that such an interpretation of several important Lie (super)algebras – e.g., of affine Kac-Moody (super)algebras and of non-trivial central extensions of simple stringy Lie (super)algebras – is lacking to this day.)

Over \mathbb{K} of characteristic $p > 0$. For a conjectural classification of simple finite-dimensional Lie superalgebras over algebraically close fields for $p > 5$, see the most recent [20]. Clearly, the smaller p the more difficult the task is. It is therefore incredible that in the most difficult case $p = 2$ the answer is obtained, although with a catch: modulo classification of simple finite-dimensional Lie algebras which is a (very tough) open problem, see [20, 41] and [25].

1.4. Realizations

Several of simple Lie superalgebras are realized as linear algebras preserving a tensor, see [19, 21], or a distribution. That is precisely how – as preserving a distribution – Cartan and Killing used to describe not only all 5 exceptional simple Lie algebras, but also $\mathfrak{sl}(n)$, $\mathfrak{sp}(2n)$, and $\mathfrak{o}(n)$, at the time when description in terms of roots and Dynkin diagrams was not yet discovered. For usefulness of such a realization of certain simple Lie algebras in characteristic 3, see [67], where examples of Skryabin’s

and several other simple Lie algebras are interpreted and described more precisely than previously.

Each of the serial finite-dimensional simple Lie algebras has two super analogs: a direct (simple-minded) one and one of a purely super nature. None of the exceptional simple Lie algebras over \mathbb{C} has a super analog, at least over \mathbb{C} . For various interesting interpretations of the exceptional complex simple Lie superalgebras, see [149]; for a way to understand them by considering their versions in characteristic 3, see [51, 52]. I particularly like Elduque's Super Magic Super Square (rectangle, actually): Elduque's approach indicates a road not fully explored yet (cf. [87]) and shows the important role the "incarnations" of exceptional simple Lie superalgebras over fields of positive characteristic play in interpretation of their namesakes over \mathbb{C} . This Super Square also describes new examples indigenous to positive characteristic, see [52], where several examples from [21] not directly entering the Super Square are further interpreted, whereas [21] contains classification of finite-dimensional modular Lie superalgebras with indecomposable Cartan matrix.

The infinite-dimensional Lie algebras of vector fields with polynomial (or formal power series) coefficients have finite-dimensional analogs both over fields of positive characteristic and among Lie superalgebras (over any fields), see [20]. For the steps of classification of the infinite-dimensional simple vectorial Lie superalgebras over \mathbb{C} , see Part 2 of [83] with numerous corrections, the latest few being [32–34, 40]; for another approach, see [114–116, 144, 145] and references therein.

Part 2 of [83] contains also an overview of representation theory, in particular, of solvable Lie superalgebras, corrected in [140], of real forms of simple finite-dimensional complex Lie superalgebras, corrected in [136], classification of finite-dimensional complex Lie superalgebras with Cartan matrix, corrected in [21, 27, 36], and a highest weight theorem for irreducible representations, corrected in [12, 13, 30, 139].

In view of all these corrections and [81], I had to be sure the list of simple Lie superalgebras to be deformed is complete. So §8 contains summary of the key ideas of the correct and interesting proof given in Part 1 of [83]; I hope this summary is more helpful as a guide to the proof of the classification of simple finite-dimensional complex Lie superalgebras in [83] than [134] which was supposed to clarify some vague places in Part 1 of [83], but is too lengthy, I think, and does not distinguish the main ideas.

1.5. Cartan matrices

Kac observed, see [82, 83], that several of simple Lie superalgebras \mathfrak{g} can be realized as $\mathfrak{g} = \mathfrak{g}(A)$, where A is an analog of Cartan matrix he defined. Kac mistakenly claimed that $\mathfrak{psl}(n|n)$ also had Cartan matrix, whereas it is its *double extension* $\mathfrak{gl}(n|n)$ that has Cartan matrix. For correct definitions of Cartan matrices of a Lie superalgebra, analogs of the Dynkin diagrams, and roots, see [21, 27, 36]; for a review of double extensions, see [28], especially its two last sections.

1.6. Presentations

For defining relations (replacing Serre relations) between analogs of the Chevalley generators of $\mathfrak{g}(A)$, see [66] and – for comparison – [159] where, same as in [66], *non*-Serre relations are also given, despite the title.

There are also other presentations of $\mathfrak{g}(A)$, e.g., in terms of a pair (triple, actually) of Jacobson generators, see [64], where all simple finite-dimensional Lie algebras are considered. Not every simple finite-dimensional Lie superalgebra with Cartan matrix can be similarly described, see [65, 109].

For presentations of simple vectorial Lie superalgebras needed in this paper, see [70]. Sixteen years after [70], presentations of 2 of the 4 series of simple *finite-dimensional* vectorial Lie superalgebras were given in [35]; the form of these latter presentations, though interesting, is not useful for this paper.

1.7. Deformations with odd parameters

Some of simple Lie superalgebras – over fields of any characteristic – have deformations thus resembling simple modular Lie algebras.

Here, I give a complete classification of deformations of simple finite-dimensional Lie superalgebras over \mathbb{C} , with proofs for the first time. In order to do this, I need to introduce the adequate definition of Lie superalgebra and deformations with odd parameters. Help of A. Lebedev in clarifying these notions was invaluable.

1.8. Several other appearances of odd parameters

Two instances of odd parameters were observed earlier, see [100]. More accessible sources in English where these instances are described are the following ones:

- A. A. Kirillov observed a supersymmetry between solutions of the Korteweg-de Vries and Schrödinger equations, but missed the odd parameters; for these parameters, a generalization of Kirillov's construction to some of the N -extended stringy superalgebras and the restriction on N , see [103].
- In [117], Manin defined the connection in terms of the differential forms, which is dual to the classical definition. This definition of connections on supermanifolds naturally leads to odd parameters, see [104].
- One more instance of odd parameters was discovered by Shander who expounded few pages on integration in [11, 100] to 86 pages in [101, no. 31/1988-14, 45–131], see his Chapter “Integration” [14, Ch. 6]. In particular, in 1986, Shander defined cochains dual to chains and hence bigraded by pairs of numbers; here the odd parameters are needed. Shander also hinted at a way to define an analog of the Stokes' formula with “over-supermanifold” of codimension $(0| - 1)$, see [14, Ch. 6]. A similar formula (with the $(0|1)$ -dimensional superline as the boundary of the point) was suggested by Palamodov, see [128].
- Let \mathcal{C} be a supercommutative superalgebra. One more instance of odd parameter is given for the group $\text{GQ}(n; \mathcal{C})$ of invertible matrices of the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$, where $A \in \text{GL}(n; \mathcal{C}_0)$ and $B \in \text{Mat}(n; \mathcal{C}_1)$, by the *queer determinant* – a group homomorphism $\text{GQ}(n; \mathcal{C}) \rightarrow \text{GL}(1; \mathcal{C}[\tau])$, defined by the formula (cf. with a different, non-functorial, definition in [13])

$$\text{qet}(A, B) := 1 + \tau \sum_{i \geq 1} \frac{\text{tr}((A^{-1}B)^{2i-1})}{2i-1} \in \mathcal{C}[\tau], \text{ where } p(\tau) = \bar{1} \text{ and } \tau^2 = 0.$$

In [143], Shander described, using odd parameters, invariant functions on supermatrices. Functions of this type were never considered before (and, regrettably, never after – to this day).

• The papers of Bettadapura, see [15, 16], also address odd parameters of deformations. The claims of his texts are, however, misleading, as I’ll show in Subsection 5.3.

2. What “Lie superalgebra” is, naively

The elements of $\mathbb{Z}/2$ will be denoted by $\bar{0}$ and $\bar{1}$ to distinguish them from elements of \mathbb{Z} . The $\mathbb{Z}/2$ -graded space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is called a *superspace*. The *parity* of any non-zero element $v \in V_i$ is denoted by $p(v) = i \in \mathbb{Z}/2$. Superizing Milnor’s K_0 -functor, we say that the *superdimension* of V is $\text{sdim } V = \dim V_{\bar{0}} + \varepsilon \dim V_{\bar{1}}$, where $\varepsilon^2 = 1$. This is briefly denoted $a|b$, where $a = \dim V_{\bar{0}}$ and $b = \dim V_{\bar{1}}$.

A *superalgebra* is any $\mathbb{Z}/2$ -graded algebra. In the conventional theory, the *morphisms* of superalgebras must preserve parity, i.e., only even homomorphisms are morphisms of superalgebras; for a dissident definition, see Subsection 3.2. (NB: according to this definition any $\mathbb{Z}/2$ -graded *Lie algebra* is a superalgebra, but it is not a “Lie” superalgebra because it satisfies the anti-commutativity and Jacobi identities NOT amended by the Sign Rule.)

2.1. The Sign Rule, skew and anti

The definition of *Lie superalgebra* for any characteristic $p \neq 2$ or 3 is obtained from the definition of the Lie algebra using the Sign Rule “*if something of parity a is moved past something of parity b , the sign $(-1)^{ab}$ accrues; formulas defined on homogeneous elements are extended to all elements via linearity*”.

In this note, all supercommutative superalgebras are supposed to be associative with 1 ; their morphisms should send 1 to 1 ; the morphisms of all superalgebras (supercommutative and Lie) must preserve parity.

(Observe that the requirement to preserve parity, currently universally accepted, is not required if the superalgebras are considered as “abstract”, i.e., if given by generators and relations that do not involve parity, cf. Subsection 3.2.)

Since commutative algebra and supercommutative superalgebra can appear in one sentence, I never drop the prefix “super” as is often done in futile attempts to save space, but successfully causing confusion.

Observe that sometimes applying the Sign Rule requires some dexterity. For example, we have to distinguish between two versions both of which turn in the nonsuper case into one, called either skew- or anti-commutativity, see [63]:

$$\begin{aligned} ba &= (-1)^{p(b)p(a)} ab && \text{(super commutativity)} \\ ba &= -(-1)^{p(b)p(a)} ab && \text{(super anti-commutativity)} \\ ba &= (-1)^{(p(b)+1)(p(a)+1)} ab && \text{(super skew-commutativity)} \\ ba &= -(-1)^{(p(b)+1)(p(a)+1)} ab && \text{(super antiskew-commutativity)} \end{aligned}$$

The *skew* formulas are those that can be “straightened” by the reversal of parity of the space on which the structure is considered, whereas the prefix *anti* requires an overall minus sign regardless of parity. The super commutator of two elements is denoted by the same symbol as the commutator; the first derived superalgebra of the Lie algebra or Lie superalgebra \mathfrak{g} is denoted by \mathfrak{g}' .

A *Lie superalgebra* understood naively is any superalgebra satisfying the anti-commutativity and Jacobi identities amended by the Sign Rule.

The supercommutative superalgebra of “functions” on the $0|n$ -dimensional supermanifold (which is just a superpoint in this situation) is the Grassmann algebra $\Lambda^*(n)$ on n anticommuting generators, call them $\xi = (\xi_1, \dots, \xi_n)$. We consider $\Lambda^*(n) := \mathbb{C}[\xi]$ as a superalgebra by setting $p(\xi_i) = \bar{1}$ for all i ; clearly, $\Lambda^*(n)$ is supercommutative.

An example not immediately seen as a result of application of the Sign Rule is the explicit formula of the *trace*, called *supertrace* for emphasis, which is any linear functional on \mathfrak{g} that vanishes on \mathfrak{g}' . For its correct formula on odd supermatrices in $\mathfrak{gl}(m|n; \mathcal{C})$ with elements in the supercommutative superalgebra \mathcal{C} , see eqs. (12), (13).

3. What “Lie superalgebra” is from categorical point of view

3.1. Morphisms of supervarieties

Over \mathbb{R} . Recall that the smooth supermanifolds are *ringed spaces*,² i.e., pairs $\mathcal{M} := (M, \mathcal{O}_{\mathcal{M}})$, where M is an m -dimensional manifold called the *underlying manifold* of \mathcal{M} , and $\mathcal{O}_{\mathcal{M}}$ is the structure sheaf of \mathcal{M} . In the same way as the manifold M is glued from coordinate patches locally diffeomorphic to a ball in \mathbb{R}^m , the space of sections of $\mathcal{O}_{\mathcal{M}}$ over the domain U in M is isomorphic to $C^\infty(U) \otimes \Lambda^*(n)$. Pairs $\mathcal{U} := (U, \mathcal{O}_{\mathcal{U}} \otimes \Lambda^*(n))$, where $\mathcal{O}_{\mathcal{U}}(U) := C^\infty(U)$, are called smooth *superdomains*, see, e.g., [42, p.65].

If the ground field is \mathbb{C} and $\mathcal{O}_{\mathcal{U}}(U) := \mathcal{A}(U)$ is the algebra of complex-analytic functions, then pairs $\mathcal{U} := (U, \mathcal{O}_{\mathcal{U}} \otimes \Lambda^*(n))$ are called *analytic superdomains* that glue into an *almost complex supermanifold*, see Subsection 5.1.

We say that the *superdimension* of \mathcal{M} is $\text{sdim } \mathcal{M} = m|n$. A *morphism* of supermanifolds $\mathcal{M} \rightarrow \mathcal{N}$ is any pair (φ, φ^*) , where $\varphi : M \rightarrow N$ is a diffeomorphism and $\varphi^* : \mathcal{O}_{\mathcal{N}} \rightarrow \mathcal{O}_{\mathcal{M}}$ is a morphism of sheaves of superalgebras (conventionally, preserving the natural parity of the structure sheaves, cf. Subsection 3.2). Observe that, unlike the case of manifolds, the morphism φ^* is not recovered from φ and this is precisely what makes the supermanifold theory rich.

Over any ground field \mathbb{K} . Consider a superdomain \mathcal{U} of superdimension $0|n$. Unlike superdomains of superdimension $a|b$ with $a \neq 0$, we can consider \mathcal{U} over any ground field \mathbb{K} and call it *superpoint*. The underlying domain of \mathcal{U} is a single point $*$. Since $\mathcal{O}_{\mathcal{U}}(*) = \Lambda^*(n)$, the superpoint \mathcal{U} has a lot of nontrivial automorphisms, namely the group $\text{Aut}^0 \Lambda^*(n)$ of parity preserving automorphisms of $\Lambda^*(n)$ (or – in the category to be studied in future – the even larger group $\text{Aut} \Lambda^*(n)$ of all automorphisms, cf. Subsection 3.2). Let the ξ_j for $1 \leq j \leq n$ be generators of $\Lambda^*(n)$. Any automorphism in $\text{Aut}^0 \Lambda^*(n)$ is of the form

$$\xi_j \mapsto \sum_r \varphi_j^r \xi_r + \sum_{s \geq 1} \sum_{j_1 < \dots < j_{2s+1}} \varphi_j^{j_1 \dots j_{2s+1}} \xi_{j_1} \dots \xi_{j_{2s+1}}, \quad (3)$$

where the matrix (φ_j^r) with elements in \mathbb{K} is invertible. We see that such automorphisms constitute an algebraic group (or a Lie group if $\text{char } \mathbb{K} = 0$) whose Lie algebra consists of the even elements of the Lie superalgebra $\mathbf{vect}(0|n) := \mathbf{der} \Lambda^*(n)$.

² I recommend [118] as the most clear and shortest source from which to understand the definition of ringed spaces.

What corresponds to the odd vector fields from $\text{der } \Lambda^*(n)$? Let me give the answer in the following more general setting, the one involving both even and odd indeterminates, but only for $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . In the absence of even indeterminates, i.e., for superpoints, the formulas (4) and (5) are meaningful for any \mathbb{K} .

Let E be the trivial vector bundle over a domain U of dimension m with fiber V of dimension n ; let $\Lambda^*(E)$ be the exterior algebra of E . To the bundle E , we assign the superdomain $\mathcal{U} = (U, C^\infty(\mathcal{U}))$, where $C^\infty(\mathcal{U})$ is the superalgebra of smooth sections of $\Lambda^*(E)$. Clearly, each automorphism of the pair $(U, \Lambda^*(E))$, i.e., of the vector bundle $\Lambda^*(E)$, induces an automorphism of the superdomain \mathcal{U} .

However, *not all* automorphisms of the superdomain \mathcal{U} are obtained in this way. By definition we conclude that every *morphism of superdomains* $(\varphi, \varphi^*): \mathcal{U} \rightarrow \mathcal{V}$ is in one-to-one correspondence with a homomorphism of the superalgebras of functions $\varphi^*: C^\infty(\mathcal{V}) \rightarrow C^\infty(\mathcal{U})$ since φ is determined by φ^* .

Every homomorphism φ^* is defined on the (topological³) generators of the superalgebra, in other words: on “coordinates” (the even ones $u := (u_1, \dots, u_m)$ and the odd ones $\xi := (\xi_1, \dots, \xi_n)$). Consider the corresponding formulas

$$\begin{cases} \varphi^*(u_i) = \varphi_i^0(u) + \boxed{\sum_{r \geq 1} \sum_{i_1 < \dots < i_{2r}} \varphi_i^{i_1 \dots i_{2r}}(u) \xi_{i_1} \dots \xi_{i_{2r}}} & \text{for all } i, \\ \varphi^*(\xi_j) = \sum_{r \geq 0} \sum_{j_1 < \dots < j_{2r+1}} \psi_j^{j_1 \dots j_{2r+1}}(u) \xi_{j_1} \dots \xi_{j_{2r+1}} & \text{for all } j. \end{cases} \tag{4}$$

(A) The terms $\varphi_i^0(u)$ for all i determine an endomorphism of the underlying domain U .

(B) The linear terms $\sum_i \psi_j^i(u) \xi_i$ for all j determine endomorphisms of the fiber V (over each point its own fiber, as the dependence on u shows).

(C) The terms of higher degree in ξ in the right-hand side of the expression of $\varphi^*(\xi_j)$ in eq. (4) determine an endomorphism of the larger fiber – the Grassmann superalgebra $\Lambda^*(V)$.

The endomorphisms (A)–(C) existed in Differential Geometry, and no need to introduce a flashy prefix “super” was felt.

The difference between the vector bundle $\Lambda^*(E)$ and the superdomain \mathcal{U} is most easily understood when the reader looks at the boxed terms in (4). These terms, meaningless in the conventional Differential Geometry (on manifolds), make sense in the new paradigm:

In the category of superdomains there are more morphisms than in the category of vector bundles: morphisms with non-vanishing boxed terms in (4) are exactly the additional ones.

However, even the boxed terms in eq. (4) is not all we get in the new setting: we still did not describe any of odd parameters of endomorphisms. *To account for the odd*

³A *topological algebra* A over a topological field \mathbb{K} is a topological vector space together with a bilinear multiplication $A \times A \rightarrow A$, continuous in a certain sense, and such that A is an algebra over \mathbb{K} . Usually the continuity of the multiplication means that the multiplication is continuous as a map between topological spaces $A \times A \rightarrow A$. A set S is a *generating set* of a topological algebra A if the smallest closed subalgebra of A containing S is A .

parameters, we have to consider the functor from the category of supercommutative superalgebras to the category of groups $C \mapsto \text{Aut}_C^{\bar{0}}(C^\infty(\mathcal{U}) \otimes C)$, i.e., the parity preserving C -linear automorphisms of the form

$$\left\{ \begin{array}{l} \varphi^*(u_i) = \varphi_i(u, \xi) := \varphi_i^0(u) + \boxed{\sum_{r \geq 1} \sum_{i_1 < \dots < i_r} \varphi_i^{i_1 \dots i_r}(u) \xi_{i_1} \dots \xi_{i_r}} \text{ for all } i, \\ \varphi^*(\xi_j) = \psi_j(u, \xi) := \sum_{r \geq 0} \sum_{j_1 < \dots < j_{2r+1}} \psi_j^{j_1 \dots j_{2r+1}}(u) \xi_{j_1} \dots \xi_{j_{2r+1}} \\ \quad + \boxed{\psi_j^0(u) + \sum_{r \geq 1} \sum_{j_1 < \dots < j_{2r}} \psi_j^{j_1 \dots j_{2r}}(u) \xi_{j_1} \dots \xi_{j_{2r}}} \text{ for all } j, \end{array} \right. \quad (5)$$

where for all r the even and the odd parameters are, respectively,

$$\begin{aligned} \varphi_i^0(u), \varphi_i^{i_1 \dots i_{2r}}(u), \psi_j^{j_1 \dots j_{2r+1}}(u) &\in C_{\bar{0}}, \\ \psi_j^0(u), \psi_j^{j_1 \dots j_{2r}}(u), \varphi_i^{i_1 \dots i_{2r+1}}(u) &\in C_{\bar{1}}. \end{aligned}$$

These are parameters of the infinite-dimensional supergroup of automorphisms of $C^\infty(\mathcal{U})$ or, equivalently, of diffeomorphisms of \mathcal{U} ; infinitesimally: of Lie superalgebra $\mathbf{vect}(m|n)$.

3.2. On automorphisms that do not preserve parity

It took a while to acknowledge the fact that there are automorphisms of the Grassmann algebra $\mathbb{C}[\xi]$, considered as an associative algebra (and more generally, of $\mathbb{C}[x, \xi]$, where $x = (x_1, \dots, x_m)$ are even commuting indeterminates and $\xi = (\xi_1, \dots, \xi_n)$ are anti-commuting odd ones) not preserving parity, compare [7, 126] with [6, 47, 111, 112]. The meaning of such more general than (5) automorphisms is unclear at the moment; most of researchers ignore them as “lacking physical meaning”. The situation with arbitrary automorphisms and inhomogeneous subalgebras reminds me the first cautious greetings of supersymmetries (through 1970s till the end of 1980s, some researchers, more outspoken than other, even branded them “useless to physics”, e.g., see a reference in [10]). The inhomogeneous with respect to parity subalgebras of supercommutative superalgebras are *metabelean*, i.e., satisfying the identity

$$[a, [b, c]] = 0 \text{ for any its elements } a, b, c, \text{ where } [a, b] := ab - ba.$$

A theorem by Volichenko proves that any metabelean algebra has a universal enveloping supercommutative superalgebra. Recently, U. Iyer proved that *Volichenko algebras*, defined as the inhomogeneous (relative parity) subalgebras of Lie superalgebras, play the role of Lie algebras for the groups of C -points of such general automorphisms, see [77]. Both metabelean and Volichenko algebras are not $\mathbb{Z}/2$ -graded, but 2-step filtered which is a very natural generalization of $\mathbb{Z}/2$ -gradedness. Observe that Volichenko’s attempt to describe what we now call Volichenko algebras by a set of polynomial identities failed; to find the full set of such identities is an Open problem to this day, see [29].

3.3. Supervarieties and superschemes (after [97])

Over the ground field \mathbb{R} or \mathbb{C} , let E be a vector bundle over M with fiber V . Let $\Lambda^*(V)$ be the Grassmann superalgebra on V , let $U \subset M$ be an open domain.

A *supermanifold* is a ringed space $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$, where M is a manifold, and the sheaf $\mathcal{O}_{\mathcal{M}}$ is locally isomorphic to $\mathcal{O}_U \otimes \Lambda^*(V)$. A supermanifold isomorphic to the ringed space whose structure sheaf is the sheaf of sections of the vector bundle $\Lambda^*(E)$ with fiber V over M is called *split*.

A *supervariety* is a ringed space $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$, where M is a variety (i.e., not necessarily smooth), and the sheaf $\mathcal{O}_{\mathcal{M}}$ is locally isomorphic to a quotient of $\mathcal{O}_U \otimes \Lambda^*(V)$.

Observe that every object in the category of smooth supermanifolds is split (for a clear proof, more instructive, I think, than the first publications [5, 60], see [117, Subsection 4.1.3]), so there is a one-to-one correspondence between the set of objects in the category of vector bundles over supermanifolds and the set of objects in the category of smooth supermanifolds. The latter category has, however, many more morphisms than the former, see eq. (5), to say nothing about even more general automorphisms in the category considered in Subsection 3.2.

A purely algebraic version of the supermanifold over any field (or any commutative ring) of any characteristic is an *affine superscheme* $\text{Spec } C$, where C is a supercommutative superalgebra or a superring. The affine superscheme is defined literally as the affine scheme: its points are prime ideals defined literally as in the commutative case, i.e., $\mathfrak{p} \subsetneq C$ is *prime*⁴

$$\text{if } a, b \in C \text{ and } ab \in \mathfrak{p}, \text{ then either } a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}. \quad (6)$$

The space of any affine scheme is endowed with Zariski topology and the structure sheaf, defined as in the commutative case, see [118], whose 1968 edition was the source of inspiration for [97].

There is just one subtlety: the localization of the superalgebra (or superring) C at the prime ideal \mathfrak{p} should be performed with respect to the multiplicative system $S_{\mathfrak{p}} := C \setminus \mathfrak{p}$ and it is not obvious what should we consider – in order to have well-defined fractions with non-homogeneous denominators b – only left fractions $b^{-1}a$ or only right fractions ab^{-1} or *the equivalence (equality) of fractions does not depend on the choice (left or right)*. Mathematically, a 1-line-proof of the italicized statement is the only non-trivial place in the superization of Grothendieck’s schemes (succinctly described in [118]) to superschemes.

3.4. The functor of points (co)represented by a supervariety or a superscheme

Smooth manifolds can be described as sets of points with a topology. For *manifolds-with-boundary* (which, strictly speaking, are not manifolds, hence the suggestive notation “in-one-word”) or over fields of characteristic $p > 0$, the set of points does not define the variety or scheme; the same is true for supervarieties and

⁴ K. Coulembier pointed out to me that the so far conventional definitions in the non-commutative case are at variance with the commutative case and common sense: at the moment, if (6) holds, \mathfrak{p} is called (say, in Wikipedia) *completely prime* while it would be natural to retain the term *prime*, as is done in [97] and by J. Bernstein, P. Deligne et al. in [42], since the definition is the same as in the commutative case despite the fact that supercommutative rings are not commutative, whereas the term *prime* ideal of the non-commutative ring R is (currently) applied to any ideal $P \subsetneq R$ which satisfies the following version of condition (6) for any two ideals A and B in R : if $AB \subset P$, then either $A \subset P$ or $B \subset P$.

superschemes. To determine one of such objects \mathcal{M} , we consider it parametrized by a superscheme $\text{Spec } C$ for various C . In other words, we consider collections $\text{Hom}(\text{Spec } C, \mathcal{M})$ of $(\text{Spec } C)$ -points of \mathcal{M} , usually called *C-points of \mathcal{M}* . If \mathcal{M} can be recovered from its algebra of functions \mathcal{F} , as is the case, e.g., with affine (super)schemes \mathcal{M} , we can consider $\text{Hom}(\mathcal{F}, C)$ instead of $\text{Hom}(\text{Spec } C, \mathcal{M})$.

3.4.1. Linear supervariety \longleftrightarrow linear superspace. The set \mathbb{K}^n can be considered as a linear space and as a manifold. Linear superspaces are just $\mathbb{Z}/2$ -graded linear spaces, they are not supermanifolds. So, first of all, let me introduce *linear supervariety* or *linear supermanifold*

$$\mathcal{V} = (V_{\bar{0}}, \mathcal{O}_{V_{\bar{0}}} \otimes \Lambda^*(V_{\bar{1}}^*)).$$

corresponding to the linear superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$.

The *morphisms* of linear superspaces constitute the space $\text{Hom}_{\bar{0}}(V, W)$, whereas the *supervariety of linear homomorphisms* $V \rightarrow W$ is the linear supervariety corresponding to the superspace $\underline{\text{Hom}}(V, W) := \text{Hom}(V, W)$.

In various instances, e.g., dealing with actions of supergroups, it is more convenient in computations⁵ to consider, instead of the vector superspace V or the linear supermanifold \mathcal{V} , the functor $\text{ScommSalgs}_{\mathbb{K}} \rightsquigarrow \text{Mod}_{\mathbb{K}}$ from the category of supercommutative \mathbb{K} -superalgebras C to the category of C -modules represented by V or, equivalently, \mathcal{V} :

$$C \longmapsto \mathcal{V}(C) = V(C) := V \otimes C \text{ for any } C,$$

where “any” is understood inside a suitable category (e.g., of superalgebras finitely generated over \mathbb{K}).

3.4.2. Lie superalgebras ([94, 95]). In the above terms, a *Lie superalgebra in the category of supervarieties* is a vector superspace \mathfrak{g} , or a linear supervariety (supermanifold) \mathcal{G} corresponding to it, corepresenting the functor from the category of supercommutative \mathbb{K} -superalgebras C to the category of Lie superalgebras understood “naively”. In other words, considering *corepresenting* functor instead of a *representing* one, we replace \mathfrak{g} , or the linear supervariety corresponding to it, by the algebra $P(\mathfrak{g})$ of polynomial functions on \mathfrak{g} . (Even over \mathbb{R} we have to replace the spaces by the algebras of polynomial functions on these spaces.)

Clearly, $P(\mathfrak{g})$ is a free supercommutative superalgebra generated by \mathfrak{g}^* , i.e., there is a natural isomorphism of functors

$$C \mapsto \text{Hom}_{\text{ScommSalgs}_{\mathbb{K}}}(P(\mathfrak{g}), C) \text{ and } C \mapsto \text{Hom}_{\mathbb{K}\text{-Vect}}(\mathfrak{g}^*, C),$$

whereas the second functor is naturally isomorphic to the forgetful functor

$$\mathbb{K}\text{-Vect} \rightarrow \text{Sets} : C \mapsto \mathfrak{g} \otimes C.$$

To the Lie superalgebra homomorphisms (in particular, to representations) a morphism of the respective functors should correspond.

⁵ This convenience is not just a matter of taste or experience or habit. More precisely, one has to work with either commuting diagrams, like in [118, Section 1.15.4], or with matrix realizations, as one does when working with Lie group actions. The language of points allows one to *actually compute* something, like when turning from the invariant language of operators to matrices.

Clearly, if \mathfrak{g} is a Lie superalgebra, then $\mathfrak{g}(C) := \mathfrak{g} \otimes C$ is also a Lie superalgebra for any C functorially in C . The last three words mean that

for any morphism of supercommutative superalgebras $C \rightarrow C'$, there exists a morphism of Lie superalgebras $\mathfrak{g} \otimes C \rightarrow \mathfrak{g} \otimes C'$ so that a composition of morphisms of supercommutative superalgebras

$$C \rightarrow C' \rightarrow C'' \tag{7}$$

goes into the composition of Lie superalgebra morphisms

$$\mathfrak{g}(C) \rightarrow \mathfrak{g}(C') \rightarrow \mathfrak{g}(C'');$$

the identity map goes into the identity map, etc.

An ideal $\mathfrak{h} \subset \mathfrak{g}$ represents the collection of ideals $\mathfrak{h}(C) \subset \mathfrak{g}(C)$ for every C .

A Lie or algebraic (over any field) *super*group \mathcal{G} is a group in the category of supervarieties. The *action* of \mathcal{G} in the superspace V is the collection of actions of $\mathcal{G}(C)$ in $V(C)$ for every C such that these actions are compatible with morphisms of supercommutative superalgebras $C \rightarrow C'$ in the same sense as in (7).

However, we can consider any category \mathcal{C} ; then any object $\mathfrak{g} \in \text{Ob } \mathcal{C}$ of this category representing the functor $C \mapsto \mathfrak{g}(C) := \text{Hom}_{\mathcal{C}}(\text{Spec } C, \mathfrak{g})$, i.e., satisfying conditions (7), is said to be a *Lie superalgebra in the category \mathcal{C}* .

4. Deformations

4.1. Deformations and deforms. Odd parameters

Which of the infinitesimal deformations can be extended to a global one is a separate much tougher question, usually solved *ad hoc*. Let me give two graphic examples.

(1) **Deformations of representations.** The tangent space of the moduli superspace of deformations of a representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is isomorphic to the space $H^1(\mathfrak{g}; V \otimes V^*)$. For example, if \mathfrak{g} is the $0|n$ -dimensional (i.e., purely odd) Lie superalgebra (with the only bracket possible: identically equal to zero), its only irreducible representations are the 1-dimensional trivial one, $\mathbb{1}$, and $\Pi(\mathbb{1})$, where Π is the *reversal of parity functor*, i.e. $\Pi(V)_i := V_{i+1}$ for any superspace V . Clearly,

$$\mathbb{1} \otimes \mathbb{1}^* \simeq \Pi(\mathbb{1}) \otimes \Pi(\mathbb{1})^* \simeq \mathbb{1},$$

and, because the Lie superalgebra \mathfrak{g} is *commutative* (the bracket of any two elements is identically equal to zero), the differential in the cochain complex is zero. Therefore

$$H^1(\mathfrak{g}; \mathbb{1}) \simeq \mathfrak{g}^*,$$

so there is a $(0| \dim \mathfrak{g})$ -dimensional space of odd parameters of deformations of the trivial representation. If we consider \mathfrak{g} “naively”, all of these odd parameters will be lost.

(2) **Deformations of the brackets.** Let C be a supercommutative superalgebra. After [133], where the non-super case is considered, we say that a *deformation* of a Lie superalgebra \mathfrak{g} over $\text{Spec } C$, is a Lie algebra \mathfrak{G} such that $\mathfrak{G} \simeq \mathfrak{g} \otimes C$, as spaces.

The deformation is *trivial* if $\mathfrak{G} \simeq \mathfrak{g} \otimes C$, as Lie superalgebras over C , not just as C -modules, and *non-trivial* otherwise.

Generally, the *deforms* – the results of deformations – of a given Lie superalgebra \mathfrak{g} over \mathbb{K} are Lie superalgebras $\mathfrak{G} \otimes_I \mathbb{K}$, where I is any closed point in $\text{Spec } C$.

In particular, consider a deformation with an odd parameter τ . This is a Lie superalgebra \mathfrak{G} isomorphic to $\mathfrak{g} \otimes \mathbb{K}[\tau]$ as a *superspace*. If, moreover, $\mathfrak{G} \simeq \mathfrak{g} \otimes \mathbb{K}[\tau]$ as a *Lie superalgebra*, i.e.,

$$[a \otimes f, b \otimes g] = (-1)^{p(f)p(b)}[a, b] \otimes fg \text{ for any } a, b \in \mathfrak{g} \text{ and } f, g \in \mathbb{K}[\tau],$$

then the deformation is considered *trivial* (and *non-trivial* otherwise). Observe that $\mathfrak{g} \otimes \tau$ is not an ideal of \mathfrak{G} : the ideal should be a free $\mathbb{K}[\tau]$ -module.

Comment. In a sense, the people who ignore odd parameters of deformations have a point: we (rather they) consider classification of simple Lie superalgebras (or whatever other problem) over the ground field \mathbb{K} , not over $\mathbb{K}[\tau]$, right? However, the odd parameters of deformations are no less natural, actually, than the odd part of the Lie superalgebra itself. In order to see these parameters, we have to consider whatever we are deforming not over \mathbb{K} , but over $\mathbb{K}[\tau]$.

We do the same, actually, when τ is even and we consider formal deformations over $\mathbb{K}[[\tau]]$. If the formal series in τ converges in a domain U , we can evaluate τ for any $\tau \in U$ and consider copies \mathfrak{g}_τ , where $\tau \in U$ (of the same dimension as that of \mathfrak{g} if $\dim \mathfrak{g} < \infty$). If the parameter is formal or odd, such an evaluation is only possible trivially: $\tau \mapsto 0$.

5. Deformations of complex supermanifolds. Odd parameters

Objects in the category of smooth supermanifolds are in one-to-one correspondence with vector bundles; this is proved in [5, 60], [117, Subsection 4.1.3]. P. Green, see [62], was the first to observe that in the category of complex-analytic supermanifolds there are more objects than there are vector bundles; see also [127] and Palamodov's chapter in [8, Ch. 4, §4, Sections 6–9] reproduced in [9, Ch. 3, Th.2, p.126]. These Palamodov's chapters expound a short note [127] submitted for publication 2 months after [62] was submitted, but published much later than [62]. This topic attracted a new interest with the discovery by Donagi and Witten of non-splitness of the moduli spaces of super Riemann surfaces, see [49]. For an overview of results on non-split supermanifolds obtained by Onishchik and his students, see [105]; see also [117, Ch. 4, §2, Prop. 8 (a), p. 190] and [152, Example 3.3.1 (1)]. All papers on non-split supermanifolds published so far ignored odd parameters of deformations. Let me consider the odd parameters.

5.1. The (almost) complex supermanifolds

Let the ground field be \mathbb{C} , and V an m -dimensional space. The sheaf $\mathcal{F}_{n|m}$ of germs of $\Lambda^\bullet(V)$ -valued holomorphic functions on \mathbb{C}^n defines the supermanifold $\mathcal{C}^{n|m}$. We consider the elements of $\mathcal{F}_{n|0}$ even, the elements ξ_j of a basis of V odd. A *superdomain* in $\mathcal{C}^{n|m}$ is a ringed space of the form $(U, \mathcal{F}_{n|m}|_U)$, where U is an open domain in \mathbb{C}^n . An almost *complex-analytic supermanifold* of dimension $n|m$ is a ringed space locally isomorphic to a superdomain in $\mathcal{C}^{n|m}$. Thus, if $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ is a supermanifold, then for any point $p_0 \in M$ there exist a neighborhood U of p_0 in M

and an isomorphism of the ringed space $(U, \mathcal{O}|_U)$ onto a superdomain $(\tilde{U}, \mathcal{F}_{n|m}|_{\tilde{U}})$, where \tilde{U} is an open domain in \mathbb{C}^n , called a *chart* in $\mathcal{C}^{n|m}$. Let x_1, \dots, x_n be coordinates in \mathbb{C}^n . Identifying $(U, \mathcal{O}|_U)$ with the superdomain by means of a chart, we get the sections x_i for $i = 1, \dots, n$, and ξ_j for $j = 1, \dots, m$ of $\mathcal{O}|_U$ called *local coordinates* on $(U, \mathcal{O}|_U)$.

For *complex supermanifolds* and recently distinguished notion of *real-complex supermanifolds*, such as Minkowski supermanifolds, see [23].

5.2. The retract of the supermanifold

Let (M, \mathcal{O}_M) be a supermanifold, $\mathcal{I} \subset \mathcal{O}_M$ a subsheaf of ideals; let $\mathcal{O}_M \rightarrow \mathcal{O}_M/\mathcal{I}$ be the natural projection. Setting

$$N = \{x \in M \mid \varphi(x) = 0 \text{ for all } \varphi \in \mathcal{I}\}, \quad \mathcal{O}_N = (\mathcal{O}_M/\mathcal{I})|_N,$$

we get the ringed space (N, \mathcal{O}_N) called a *subsupermanifold* of (M, \mathcal{O}_M) . If the sheaf \mathcal{I} is generated, over an open set $U \subset M$, by its homogeneous sections $\varphi_1, \dots, \varphi_s$, then it is usual to say that the subsupermanifold is determined in U by the system of equations $\varphi_i = 0$, where $i = 1, \dots, s$.

To any supermanifold $\mathcal{M} = (M, \mathcal{O}_M)$ the following split supermanifold corresponds. Consider the filtration

$$\mathcal{O}_M := \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \supset \dots \tag{8}$$

of \mathcal{O}_M by the powers of the subsheaf of ideals $\mathcal{J} = \mathcal{J}^1$ generated by the odd elements. The associated graded sheaf

$$\text{gr } \mathcal{O}_M = \bigoplus_{k \geq 0} \text{gr}^k \mathcal{O}_M,$$

where $\text{gr}^k \mathcal{O}_M = \mathcal{J}^k/\mathcal{J}^{k+1}$, gives rise to the split supermanifold $(M, \text{gr } \mathcal{O}_M)$ called the *retract* of the supermanifold \mathcal{M} .

The quotient modulo the subsheaf of ideals $(\mathcal{O}_M)_{\text{nil}}$ generated by all nilpotents of \mathcal{O}_M determines the *reduction* whose result is conventionally denoted \mathcal{M}_{red} . The quotient modulo the subsheaf \mathcal{J}^1 generated by **only odd** nilpotents determines the *odd reduction*, whose result Manin denoted \mathcal{M}_{rd} , of (M, \mathcal{O}_M) ; thus, \mathcal{M}_{rd} is a submanifold of (M, \mathcal{O}_M) .

Thus, we see that with any (almost) complex supermanifold \mathcal{M} two objects of the classical complex analytic geometry are associated: the (almost) complex manifold (M, \mathcal{O}_M) and the holomorphic vector bundle \mathbf{E} over (M, \mathcal{O}_M) with fiber V corresponding to the sheaf $\mathcal{E} = \mathcal{J}^1/\mathcal{J}^2$. It turns out that the complex supermanifold \mathcal{M} is not, in general, determined by these two objects up to an isomorphism: there exist non-split supermanifolds.

The obstructions to splitness were considered and calculated, so far, for transition functions of the form (4); e.g., see [4, 31] and references therein. In this – restricted – approach (considering only transition functions of the form (4)), the first obstructions to splitness are given by the terms of degree > 1 in the ξ_i in the expansion of $\varphi^*(x_i)$ and $\varphi^*(\xi_j)$. If the x_i and ξ_j are arbitrary local coordinates of \mathcal{M} in a neighborhood $U \subset M$, then $X_i = x_i + \mathcal{J}^2$ and $\Xi_j = \xi_j + \mathcal{J}^3$ are splitting local coordinates of $(M, \text{gr } \mathcal{O}_M)$ in U , and one gets the transition functions between these splitting coordinates, if one takes the terms of degree 0 (respectively 1) in ξ_j in the transition functions φ_i (respectively ψ_j) for \mathcal{M} , see eq. (4).

In the restricted approach, the *first* obstructions to splitness are represented by cocycles in the Čech cohomology $H^1(M; \mathcal{T} \otimes \mathcal{E}^2(\mathbf{E}))$, where $\mathcal{E}^i(\mathbf{E})$ is the sheaf corresponding to the i th exterior power $E^i(\mathbf{E})$ of the bundle \mathbf{E} , and \mathcal{T} is the tangent bundle, see [117, Ch. 4, §2, Prop. 9, p. 191] and Palamodov's chapter in [8, Ch. 4, §4, Sections 6–9] reproduced in [9, Ch. 3, Th.2, p.126].

The *higher* obstructions to splitness in the restricted approach are represented by cocycles in the Čech cohomology $H^1(M; \mathcal{T} \otimes \mathcal{E}^{2i}(\mathbf{E}))$ for $i > 1$, see [9, Ch. 3, §7, Th.2, p.135]; these obstructions describe the even obstructions to splitness in the non-restricted approach as well, but miss the odd obstructions.

In particular, this restricted approach (ignoring odd parameters, as in eq. (4)) immediately leads one to the conclusion that “any complex supermanifold of superdimension $m|1$ is split” since $E^2(\mathbf{E}) = 0$, see [117, Ch. 4, §2, Prop. 8 (a), p. 190] and [152, Example 3.3.1 (1)]. In [117, 151, 152], various deformation problems were considered over a supervariety of parameters, but – inexplicably – not in such generality in this particular case: in the study of non-splitness.

To incorporate odd parameters in a simplest case, recall the following well-known facts, see, e.g., [124, § 1.1]. Let us cover $\mathbb{C}\mathbb{P}^1$ by two affine charts U_0 and U_1 with local coordinates x and $y = \frac{1}{x}$, respectively. Let ξ and η be basis sections of line bundle \mathbf{L}_k over U_0 and U_1 , respectively, such that in $U_0 \cap U_1$ we have (up to a non-zero constant factor)

$$\eta = x^k \xi.$$

The bundle \mathbf{L}_k is said to be of *degree* $k \in \mathbb{Z}$. The sheaf of sections of this bundle is designated $\mathcal{O}(k)$. In particular, since $dy = -x^{-2}dx$, then the cotangent sheaf of 1-forms Ω^1 is $\mathcal{O}(-2)$, and hence the dual to Ω^1 tangent sheaf \mathcal{T} is $\mathcal{O}(2)$; clearly, $\mathcal{O} := \mathcal{O}(0)$ is the structure sheaf (of functions). We also have the following formulas, see “Bott's formulas” in [124, p. 4]:

$$\begin{aligned} \mathcal{O}(a) \otimes \mathcal{O}(b) &\simeq \mathcal{O}(a+b); \\ \dim H^0(\mathbb{C}\mathbb{P}^1; \mathcal{O}(a)) &= \begin{cases} a+1 & \text{if } a \geq 0 \\ 0 & \text{otherwise;} \end{cases} \end{aligned} \quad (9)$$

$\xi, x\xi, \dots, x^a\xi$ is a basis of the space of sections over U_0 ;

$$\dim H^1(\mathbb{C}\mathbb{P}^1; \mathcal{O}(a)) = \begin{cases} |a| - 1 & \text{if } a \leq -2 \\ 0 & \text{otherwise;} \end{cases}$$

for a basis of the space of sections over U_0 we can take

$$\begin{cases} \xi, x^{-1}\xi, \dots, x^{2-|a|}\xi & \text{if } a < -2 \\ \xi & \text{if } a = -2. \end{cases} \quad (10)$$

Now, if we consider transition functions taking odd parameters into account, then eq. (5) shows that there is a linear in ξ term in the new coordinates $\varphi(x_i)$. The *odd* obstructions to splitness are represented by cocycles in $H^1(M; \mathcal{T} \otimes \mathcal{E}^{2i+1}(\mathbf{E}))$ for $i \geq 0$.

In the case of the linear bundle $\mathbf{E} = \mathbf{L}_k$ over $M = \mathbb{C}\mathbb{P}^1$, the corresponding obstructions belong to (recall that $E^1(V) := \Pi(V)$)

$$H^1(M; \mathcal{T} \otimes \mathcal{E}^1(\mathbf{E})) = H^1(M; \mathcal{O}(2) \otimes \Pi(\mathcal{O}(k))) \simeq H^1(M; \Pi(\mathcal{O}(k+2))). \quad (11)$$

Eqs. (9), (10) and (11) imply the following theorem, where $\mathrm{GL}(V) = \mathrm{GL}(1)$.

Theorem 5.1. (Odd parameters of the non-split 1|1-dimensional superstring whose underlying space is \mathbb{CP}^1) *The non-split supermanifolds \mathcal{M} of complex superdimension 1|1, a.k.a. superstrings, with underlying manifold \mathbb{CP}^1 , whose retract corresponds to the line bundle $\mathbf{E} = \mathbf{L}_k$ with fiber V are described by $\mathrm{GL}(V)$ -orbits in the space $H^1(M; \Pi(\mathcal{O}(k+2)))$ – its projectivization, the number of orbits being equal to $|k+2| - 1$ for $k \leq -4$, whereas \mathcal{M} is split for $k > -4$.*

5.3. On the claims of Bettadapura, see [15–17]

Although Bettadapura sometimes uses standard notation in nonstandard ways (e.g., his ΠE is an *ad hoc* notation, which in the context of super geometry causes confusion), writes that something “will” whereas it already “is”, Palamodov’s technique published in the chapter written by Palamodov ([8, Ch.4, §4]) is misattributed to Berezin, some symbols are left undefined (e.g., E^v), etc., etc., and claims are formulated assuming the change of coordinates is performed via formulas of the form (4), not (5), it is possible to decipher that some statements are wrong if we take into account odd parameters (e.g., [15, Lemma 2.4] – a copy of the corresponding claims in [117] and [152], although these references are not given). These claims are at variance with Theorem 5.1; the other statements *do* take the odd parameters into account and contain ostensibly new examples of non-split super Riemann surfaces \mathcal{S} , but in reality they do not describe odd parameters of deformations of \mathcal{S} , but rather an incomplete study of deformations of $\mathcal{S} \times \mathcal{C}^{0|n}$.

Observe that, like many, but fortunately not everybody, Bettadapura applies the term “super Riemann surface” to 1|1-dimensional supermanifolds with a contact structure, whereas it seems natural to consider *several* types of “super Riemann surfaces” of dimension 1| N either with or without contact structure but, perhaps, preserving a volume element, as I intend to show in a paper in preparation in which the upper bounds on $N > 1$ are justified (explained) from a certain point of view.

6. Finite-dimensional simple Lie superalgebras considered naively

Certain simple Lie superalgebras (or their relatives, like $\mathfrak{gl}(n|n)$, $\mathfrak{pgl}(n|n)$, and $\mathfrak{sl}(n|n)$) were easy to describe meaningfully, by some properties of these algebras. Even before a matrix realization of $\mathfrak{osp}(m|2n)$ and its interpretation as a Lie superalgebra preserving a non-degenerate even bilinear form were discovered it was natural to call it “ortho-symplectic”, just by looking at its even part. Two series of simple Lie superalgebras were considered “strange” until the interpretations of their non-simple relatives as preserving a tensor was discovered, hence their first notation and certain awkward names (e.g., “di-spin algebra of Mitchell—Gell-Mann—Radicati”) were soon abandoned; certain confusing names – $F(4)$ and $G(3)$ instead of *ad hoc* but not confusing (with the Lie algebra \mathfrak{f}_4 and some object on the line of exceptional Lie algebras on the Vogel plane, see [120], respectively) names given by their discoverer, Kaplansky still remain in some papers. The notation “ \mathfrak{psl} ” was also being avoided for a long time, although it was already usual in the modular setting.

6.1. Linear (matrix) Lie superalgebras

Let $\mathrm{Size} := (p_1, \dots, p_{|\mathrm{Size}|})$, where $|\mathrm{Size}| := \dim V$, be an ordered collection of parities of the basis vectors of a superspace V ; for the basis vectors we take only

vectors *homogeneous with respect to parity*. The *general linear* Lie superalgebra of all supermatrices of size Size corresponding to linear operators in the superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ over the ground field \mathbb{K} is denoted by $\mathfrak{gl}(\text{Size})$. Usually, for the *standard* (a simplest from a certain point of view) format, $\mathfrak{gl}(\bar{0}, \dots, \bar{0}, \bar{1}, \dots, \bar{1})$ is abbreviated to $\mathfrak{gl}(\dim V_{\bar{0}} | \dim V_{\bar{1}})$. Any non-zero supermatrix in $\mathfrak{gl}(\text{Size})$ can be uniquely expressed as the sum of its even and odd parts; in the standard format, this is the following block expression; on non-zero summands the parity is defined:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad p\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right) = \bar{0}, \quad p\left(\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right) = \bar{1}.$$

The *trace* on the Lie superalgebra \mathfrak{g} (or *supertrace* for emphasis) is any map $\text{tr} : \mathfrak{g} \rightarrow \mathbb{K}$ that vanishes on the first derived Lie (super)algebra \mathfrak{g}' . In particular, on $\mathfrak{gl}(\text{Size})$, there is just one (up to a non-zero scalar) such map; it is given by the formula

$$\text{str}(X) := \sum (-1)^{p_i} X_{ii}. \quad (12)$$

Thus, in the standard format,

$$\text{str} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{tr}A - \text{tr}D.$$

Observe that for the Lie superalgebra $\mathfrak{gl}_{\mathcal{C}}(a|b)$ over a supercommutative superalgebra \mathcal{C} , i.e., for supermatrices with homogeneous (with respect to parity) elements in \mathcal{C} , we have

$$\text{str} X = \text{tr}A - (-1)^{p(X)} \text{tr}D \quad \text{for any } X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (13)$$

$$\text{where } p(X) = p(A_{ij}) = p(D_{kl}) = p(B_{il}) + \bar{1} = p(C_{kj}) + \bar{1},$$

so on odd supermatrices with entries in \mathcal{C} such that $\mathcal{C}_{\bar{1}} \neq 0$, the expression of supertrace coincides with the conventional trace of a matrix.

Since $\text{str}[x, y] = 0$, the supertraceless supermatrices span a Lie subsuperalgebra called *special linear* and denoted $\mathfrak{sl}(\text{Size})$.

There are, however, at least two super versions of $\mathfrak{gl}(n)$, not one; for reasons, see [14, Ch. 1, Ch. 7], where the super version of the Schur lemma and classification of central simple (finite-dimensional) associative superalgebras are considered. The natural (from the Schur lemma's point of view) other version – $\mathfrak{q}(n)$ – is called the *queer* Lie superalgebra and is defined as the one that preserves – over an algebraically closed field \mathbb{K} of characteristic $p \neq 2$ – the complex structure given by an *odd* operator J , i.e., $\mathfrak{q}(n)$ is the supercentralizer $C(J)$ of J :

$$\mathfrak{q}(n) := C(J) = \{X \in \mathfrak{gl}(n|n) \mid [X, J] = 0\}, \quad \text{where } J^2 = -\text{id}.$$

It is clear that over an algebraically closed field, we can by a change of basis reduce J to the normal form (shape) J_{2n} in the standard format, see eq. (14), and then the elements of $\mathfrak{q}(n)$ take the form

$$\mathfrak{q}(n) := \left\{ (A, B) := \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \quad \text{where } A, B \in \mathfrak{gl}(n) \text{ and } J_{2n} := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\}. \quad (14)$$

(Over any field \mathbb{K} , instead of J we can take any odd operator K such that $K^2 = a \text{id}_{n|n}$, the multiple of the identity operator, where $a \in \mathbb{K}^\times$. If \mathbb{K} is algebraically closed, the Lie superalgebras $\mathcal{C}(K)$ – supercentralizers of K – are isomorphic for distinct a ; if $p = 2$, it is natural to select $K^2 = \text{id}$.)

On $\mathfrak{q}(n)$, the odd trace, nowadays called *queertrace*, is defined:

$$\text{qtr}: (A, B) \mapsto \text{tr}B.$$

Denote by $\mathfrak{sq}(n)$ the Lie superalgebra of *queertraceless* matrices, first described in [13] together with the odd (queer) determinant of the C -points of the corresponding supergroup, see a more accessible [14, Ch. 1].

Clearly, \mathfrak{gl} and \mathfrak{q} correspond to the two cases of the super version of Schur’s lemma over an algebraically closed field:

the \mathfrak{gl} case: an irreducible module M over a collection S of homogeneous operators can be *absolutely irreducible*, i.e., have no proper invariant subspaces, then the only operator commuting with S is a scalar, and

the \mathfrak{q} case: M can have an invariant subspace, which is not a sub *super* space, then the superdimension of the module is of the form $n|n$ and an odd operator K interchanges the homogeneous components of the module.

6.1.1. Supermatrices of operators. To a linear map of superspaces $F : V \rightarrow W$ there corresponds the dual map $F^* : W^* \rightarrow V^*$ between the dual superspaces. In bases consisting of homogeneous vectors $v_i \in V$ of parity $p(v_i)$, and $w_j \in W$ of parity $p(w_j)$, the formula $F(v_j) = \sum_i w_i X_{ij}$ assigns to F the supermatrix X . In the dual bases, the *supertransposed* matrix X^{st} corresponds to F^* :

$$(X^{st})_{ij} := (-1)^{(p(v_i)+p(w_j))p(w_j)} X_{ji}.$$

In the standard supermatrix format, over the supercommutative superalgebra \mathcal{C} , we have

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto X^{st} := \begin{pmatrix} A^t & (-1)^{p(X)} C^t \\ -(-1)^{p(X)} B^t & D^t \end{pmatrix} = \begin{cases} \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix} & \text{if } p(X) = \bar{0}, \\ \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix} & \text{if } p(X) = \bar{1}. \end{cases}$$

6.1.2. Supermatrices of bilinear forms. Having selected a basis (by definition consisting of vectors homogeneous with respect to parity) of the superspace V , we define the Gram matrix $B = (B_{ij})$ of the bilinear form B^f on V by the formula

$$B_{ij} := (-1)^{p(B)p(v_i)} B^f(v_i, v_j) \quad \text{for the basis vectors } v_i \in V. \tag{15}$$

This formula for the Gram matrix of B^f allows us to identify any bilinear form $B \in B(V, W)$, where V and W are superspaces, with an operator, an element of $\text{Hom}(V, W^*)$, see [14, Ch. 1].

Recall that the *upsetting* of bilinear forms $u : \text{Bil}(V, W) \rightarrow \text{Bil}(W, V)$ is given by the formula

$$u(B^f)(w, v) := (-1)^{p(v)p(w)} B^f(v, w) \quad \text{for any } v \in V \text{ and } w \in W. \tag{16}$$

Let now $W = V$, and $\text{Bil}(V) := \text{Bil}(V, V)$. The shape of the Gram matrix B^u of a homogeneous form $u(B^f)$ in the standard format of V is as follows

$$B^u := \begin{pmatrix} R^t & (-1)^{p(B)}T^t \\ (-1)^{p(B)}S^t & -U^t \end{pmatrix} \quad \text{for } B = \begin{pmatrix} R & S \\ T & U \end{pmatrix}, \quad (17)$$

The form B^f is said to be *symmetric* if $B^u = B$, and *anti-symmetric* if $B^u = -B$. In particular, the form on $\mathfrak{gl}(V)$ (resp. $\mathfrak{q}(V)$) given by the respective trace is symmetric:

$$\begin{aligned} (X, Y) &:= \text{str}(XY) \quad \text{for any } X, Y \in \mathfrak{gl}(V) \\ (\text{resp. } (X, Y) &:= \text{qtr}(XY) \quad \text{for any } X, Y \in \mathfrak{q}(V)). \end{aligned}$$

Clearly, the *upsetting* $B \mapsto B^u$ of Gram matrices of bilinear forms is *not* supertransposition.

Observe that *the passage from V to $\Pi(V)$ turns every symmetric form \mathcal{B} on V into an anti-symmetric one \mathcal{B}^Π on $\Pi(V)$ and every anti-symmetric \mathcal{B} into a symmetric \mathcal{B}^Π by setting*

$$\mathcal{B}^\Pi(\Pi(x), \Pi(y)) := (-1)^{p(B)+p(x)+p(x)p(y)} \mathcal{B}(x, y) \quad \text{for any } x, y \in V.$$

Observe that there are no “supersymmetric” bilinear forms; the property

$$B^f(w, v) := (-1)^{p(v)p(w)} B^f(v, w) \quad \text{for any } v, w \in V \quad (18)$$

which is sometimes given as their definition just reflects braiding in the category of superspaces and has nothing to do with the symmetry of bilinear forms. (Therefore, the title of [24] should be corrected. The same confusing term is used in [3], where inhomogeneous with respect to parity bilinear forms are classified on superspaces of dimension ≤ 7 in an attempt to define a generalized mixture of the Heisenberg superalgebra and what in [20] was called the *antibracket superalgebra*. The mainstream supersymmetry theory ignores the inhomogeneous bilinear forms since they are not preserved by Lie superalgebras; but it is interesting, nevertheless, to explore their hidden depths, cf. Subsection 3.2; e.g., what is the “something” that preserves them? What linear transformations of the superspace V (only even or any) is reasonable in some (which?) sense to consider in the classification of inhomogeneous bilinear forms on V ?)

Most popular normal shapes of the (Gram matrices of the) even non-degenerate symmetric form are the ones which in the standard format are as follows:

$$\begin{aligned} B_{ev}(m|2n) &= \text{diag}(1_m, J_{2n}) := \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix} \quad \text{or} \quad \text{diag}(A_m, J_{2n}) := \begin{pmatrix} A_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \\ \text{where } J_{2n} &= \text{antidiag}(1_n, -1_n) := \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \quad \text{and } A_m = \text{antidiag}(1, \dots, 1). \end{aligned}$$

The Lie superalgebra $\mathbf{aut}(B) \subset \mathfrak{gl}(\text{Size})$ that preserves the Gram matrix B of the form $B^f \in \text{Bil}(V)$ consists of the supermatrices $X \in \mathfrak{gl}(\text{Size})$ such that

$$X^{st}B + (-1)^{p(X)p(B)}BX = 0 \quad \text{for an homogeneous matrix } B \in \mathfrak{gl}(\text{Size}).$$

The usual notation for $\mathbf{aut}(B_{ev}(m|2n))$ is $\mathbf{osp}(m|2n)$; sometimes one writes more explicitly, $\mathbf{osp}^{sy}(m|2n)$. Observe that the antisymmetric non-degenerate bilinear

form is preserved by the “symplectic-orthogonal” Lie superalgebra $\mathfrak{osp}^a(m|2n)$ isomorphic to $\mathfrak{osp}^{sy}(m|2n)$.

A non-degenerate *symmetric* odd bilinear form $B_{\text{odd}}(n|n)$ can be reduced to a normal shape whose matrix in the standard format is J_{2n} , see formula (17), *not* $\Pi_{2n} := \text{antidiag}(1_n, 1_n)$ which is *antisymmetric*, see [14], contrary to a hasty conclusion induced by the “even” experience. The usual notation for $\mathfrak{aut}(B_{\text{odd}}(\text{Size}))$ is $\mathfrak{pe}(\text{Size})$. The passage from V to $\Pi(V)$ establishes a corresponding isomorphism $\mathfrak{pe}^{sy}(\text{Size}) \cong \mathfrak{pe}^a(\text{Size})$. These isomorphic Lie superalgebras are nowadays called, as A. Weil suggested, *periplectic*.

The traceless (special) subalgebra of $\mathfrak{pe}(n)$ is denoted $\mathfrak{spe}(n)$; it is simple for $n > 2$. For a large class of Lie superalgebras either simple, or relatives of simple, Kac introduced the notion of a Cartan matrix, see [83] with improvements of the definition in [21, 27, 28, 36]. If $p \neq 2$, then neither periplectic superalgebras nor their simple relatives have Cartan matrices; this is not so for $p = 2$, see the paper [21] containing the classification of all finite-dimensional modular Lie superalgebras with indecomposable Cartan matrix over any algebraically closed field of positive characteristic. If \mathfrak{c} is a Lie algebra of scalar matrices, and $\mathfrak{g} \subset \mathfrak{gl}(n|n)$ is a Lie subsuperalgebra containing \mathfrak{c} , then the *projective* Lie superalgebra of type \mathfrak{g} is $\mathfrak{pg} := \mathfrak{g}/\mathfrak{c}$, e.g., $\mathfrak{psl}(n|n)$ and $\mathfrak{psq}(n)$.

6.2. Exceptional finite-dimensional simple Lie superalgebras

There are 3 of them: $\mathfrak{osp}_a(4|2)$, $\mathfrak{ag}(2)$ and $\mathfrak{ab}(3)$. I denote them following meaningful Serganova’s notation from the paper [138], where she proved that the notion of roots generalized to non-degenerate *pseudo*-euclidean space instead of euclidean, leads – unexpectedly – to Lie superalgebras $\mathfrak{g}(A)$ with indecomposable matrix A . A tempting open problem: superise the result of [131] by considering non-degenerate but not sign-definite sesquilinear form – an analog of Serganova’s result for complex reflections; see [138].

Kaplansky was the first to discover all three exceptional simple finite-dimensional Lie superalgebras over \mathbb{C} ([88]). They are of the form $\mathfrak{g}(A)$, i.e., have a Cartan matrix A , and thus are described in [21] over fields of any characteristic p (non-existing for some values of p), where the description as pairs $(\mathfrak{g}_0, \mathfrak{g}_1)$, although snubbed at in Subsection 1.3 above as “clumsy”, is also given because it is useful in certain calculations, see also [36]. For *interpretations* of the exceptional simple Lie algebras, see [51, 52] and [149], see also Subsection 6.4.1.

6.3. Finite-dimensional simple vectorial Lie superalgebras

Define partial derivatives in $\Lambda^\bullet(n)$ by setting $\partial_{\xi_i}(\xi_j) := \delta_{ij}$ and considering only left action of the derivations on functions. Let $\partial_i := \partial_{\xi_i}$. Then, setting $\deg \xi_i = 1$ for all i , we get the *standard* \mathbb{Z} -grading of

- the *general vectorial Lie superalgebra*

$$\mathfrak{vect}(0|n) := \left\{ \sum f_i \partial_i \mid f_i \in \Lambda^\bullet(n) \text{ for all } i \right\} = \bigoplus_{-1 \leq i \leq n-1} \mathfrak{g}_i.$$

- The *divergence* (with respect to the volume element $\text{vol}(\xi)$ in “coordinates” ξ which means that the Lie derivative L_D along the vector field D multiplies $\text{vol}(\xi)$)

by $\operatorname{div}(D)$) defined by the formula (clearly, $p(\partial_i) = \bar{1}$ for all i)

$$\operatorname{div}\left(\sum f_i \partial_i\right) := \sum (-1)^{p(f_i)} \partial_i f_i.$$

The *special* or *divergence-free* vectorial Lie superalgebra is

$$\mathbf{svect}(0|n) := \{D \in \mathbf{vect}(0|n) \mid \operatorname{div}(D) = 0\} = \{D \in \mathbf{vect}(0|n) \mid L_D(\operatorname{vol}(\xi)) = 0\}.$$

Kac was the first to observe (in analogy with characteristic > 0 , cf. [158]) that for $n = 2k$ any *volume form* $f \operatorname{vol}(\xi)$, where $f \in \Lambda^*(\xi)$, can be reduced by a change of indeterminates (an automorphism of $\mathbb{C}[\xi]$) to one of the two *normal shapes*: with either $f = 1$ or with $f = (1 + \prod \xi_i)$, see [82], [83, Section 3.3.1], where the volume forms (a particular case of *integrable forms*, see [11]) are erroneously considered as a particular case of differential forms. Kac formulated this reduction even before the notion of the volume form was correctly defined. The same argument works for functions in $n = 2k + 1$ indeterminates with values in $\mathbb{C}[\tau]$, where $p(\tau) = \bar{1}$, the normal shapes being with either $f = 1$ or with $f = (1 + \tau \prod \xi_i)$. Accordingly, the *deformed special* or *divergence-free* vectorial Lie superalgebra is

$$\widetilde{\mathbf{svect}}(0|2k) := \{D \in \mathbf{vect}(0|2k) \mid L_D((1 + t \prod \xi_i) \operatorname{vol}(\xi)) = 0\}.$$

Any two such Lie superalgebras for $t, t' \in \mathbb{C}^\times$ are clearly isomorphic, so we assume $t = 1$.

Considering the *supergroup of automorphisms* of $\mathbb{C}[\xi_1, \dots, \xi_n]$ in terms of the functor

$$\mathcal{C} \mapsto \operatorname{Aut}_{\mathcal{C}}^{\bar{0}}(\mathcal{C} \otimes \mathbb{C}[\xi]),$$

where \mathcal{C} is a supercommutative superalgebra (cf. with the description of a group scheme in [118]), we arrive at the following normal shapes of the volume forms $f \operatorname{vol}(\xi)$:

$$f = \begin{cases} 1; \\ (1 + t \prod \xi_i), & \text{where } t \in \mathbb{C}_0^\times \quad \text{for } n \text{ even,} \\ (1 + \tau \prod \xi_i), & \text{where } \tau \in \mathbb{C}_{\bar{1}} \quad \text{for } n \text{ odd,} \end{cases}$$

and hence to an odd parameter τ and the *deformed special* or *divergence-free* vectorial Lie superalgebra

$$\widetilde{\mathbf{svect}}(0|2k+1) := \{D \in \mathbb{C}[\tau] \otimes \mathbf{vect}(0|2k+1) \mid L_D((1 + \tau \prod \xi_i) \operatorname{vol}(\xi)) = 0\}.$$

Any two such Lie superalgebras for odd parameters τ, τ' are, clearly, isomorphic.

• Define the space of *differential* (also known as *exterior*) forms $\mathbb{C}[\xi, d\xi]$ by setting $p(d\xi_i) = \bar{0}$ for all i , and letting the $d\xi_i$ commute with the ξ_j and $d\xi_j$; define the *exterior differential* by the formula (together with the Leibniz and Sign Rules)

$$d(f) := \sum d\xi_i \frac{\partial f}{\partial \xi_i}.$$

Define the action of *Lie derivative* along the vector field D on the space of differential forms $\mathbb{C}[\xi, d\xi]$ by setting

$$L_D(df) := (-1)^{p(D)} d(D(f)).$$

For a normal shape of the *symplectic*, i.e., a non-degenerate and closed, 2-form on $\mathcal{C}^{0|n}$ we can take

$$\omega := \sum (d\xi_i)^2.$$

This was first claimed in [98], for the first proof, see [142]; for a conceptual proof, see [23, 69]; observe that in characteristic $p > 0$ there are many normal shapes of the symplectic forms, see [148] where this classification is performed on manifolds; to superize Skryabin’s result is an important open problem.

Let the *Lie superalgebra of Hamiltonian vector fields* on $\mathcal{C}^{0|n}$ be

$$\mathfrak{h}(0|n) := \{D \in \mathbf{vect}(0|n) \mid L_D(\omega) = 0\}.$$

Let the space of the *Poisson Lie superalgebra* $\mathfrak{po}(0|n)$ on $\mathcal{C}^{0|n}$ be $\Lambda^\bullet(n)$ with the *Poisson bracket*

$$\{f, g\}_{P.b.} := (-1)^{p(f)} \sum \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_i}.$$

Let $H_- : f \mapsto H_f$ be the projection $\mathfrak{po}(0|n) \rightarrow \mathfrak{h}(0|n)$ given by the formula

$$H_f := (-1)^{p(f)} \sum \frac{\partial f}{\partial \xi_i} \partial_{\xi_i}.$$

Clearly, the following sequence is exact

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathfrak{po}(0|n) \xrightarrow{H_-} \mathfrak{h}(0|n) \longrightarrow 0. \tag{19}$$

6.4. The list of finite-dimensional simple Lie superalgebras considered naively

Recall that the Lie superalgebra \mathfrak{g} without proper ideals and of dimension > 1 is said to be *simple*. Eq. (20) lists all non-isomorphic (minding isomorphisms (22), (23)) simple finite-dimensional Lie superalgebras over \mathbb{C} ; for first examples and subcases of classification, see [44–46, 48, 75, 79–81, 88, 135]; for a summary (ignoring odd parameters), see [83, 134] with improvements cited in eq. (20):

- $\mathfrak{sl}(m|n)$ for $m > n \geq 1$, $\mathfrak{psl}(n|n)$ for $n > 1$, $\mathfrak{osp}(m|2n)$ for $mn \neq 0$
 - $\mathfrak{psq}(n)$ and $\mathfrak{spe}(n)$ for $n > 2$;
 - $\mathfrak{osp}_\alpha(4|2)$ for $\alpha \neq 0, -1$ is given by any of its four Cartan matrices ([21])
- $$\begin{pmatrix} 0 & 1 & -1-\alpha \\ -1 & 0 & -\alpha \\ -1-\alpha & \alpha & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ \alpha & 0 & -1-\alpha \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & -\alpha \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1-\alpha & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \tag{20}$$
- $\mathfrak{ag}(2)$ and $\mathfrak{ab}(3)$ are also given by their Cartan matrices (see [21]);
 - for interpretations, see [43, 149] and very interesting papers [51, 52];
 - $\mathbf{vect}(0|n)$ for $n > 1$,
 - $\mathbf{svect}(0|n)$ for $n > 2$ and $\widetilde{\mathbf{svect}}(0|2n)$ for $n > 1$,
 - $\mathfrak{h}'(0|n) := [\mathfrak{h}(0|n), \mathfrak{h}(0|n)]$ for $n > 3$.

Occasional isomorphisms:

$$\begin{aligned} \mathfrak{sl}(m|n) &\simeq \mathfrak{sl}(n|m), & \mathbf{vect}(0|2) &\simeq \mathfrak{sl}(1|2) \simeq \mathfrak{osp}(2|2), \\ \mathfrak{spe}(3) &\simeq \mathbf{svect}(0|3), & \mathfrak{psl}(2|2) &\simeq \mathfrak{h}'(0|4), \\ \mathfrak{osp}_1(4|2) &\simeq \mathfrak{osp}_{-2}(4|2) \simeq \mathfrak{osp}_{-1/2}(4|2) \simeq \mathfrak{osp}(4|2). \end{aligned} \tag{21}$$

The isomorphisms $\mathfrak{osp}_\alpha(4|2) \simeq \mathfrak{osp}_{\alpha'}(4|2)$ are generated by the transformations (see [21]):

$$\alpha \mapsto \alpha' := -1 - \alpha, \quad \alpha \mapsto \alpha' := \frac{1}{\alpha}, \quad (22)$$

so the other isomorphisms are

$$\alpha \mapsto \alpha' := -\frac{1+\alpha}{\alpha}, \quad \alpha \mapsto \alpha' := -\frac{1}{\alpha+1}, \quad \alpha \mapsto \alpha' := -\frac{\alpha}{\alpha+1}. \quad (23)$$

In what follows, I'll list all deformation of simple finite-dimensional Lie superalgebras (see Theorems 7.2 and 7.1), in particular, I'll prove the old conjecture:

The Lie superalgebras $\widetilde{\mathfrak{svect}}(0|2n+1)$ for $n \geq 1$ are the only (bar $\mathfrak{spe}(3)$, see eqs. (21)) deformations of simple finite-dimensional Lie superalgebras with an odd parameter.

6.4.1. Exceptional simple Lie superalgebras: realizations. The irreducible modules of least dimension over $\mathfrak{ag}(2)$, $\mathfrak{ab}(3)$, and $\mathfrak{osp}_a(4|2)$ for a generic are the adjoint ones. Therefore, for the best interpretation of these algebras, see [52].

Consider $\mathfrak{osp}_a(4|2)$ for the exceptional values of a . Kac was, probably, the first to establish their irreducibles of least dimension in an unpublished preprint [84].

For $a = 1$ (and hence for $a = -2$ and $a = -\frac{1}{2}$), the Lie superalgebra $\mathfrak{osp}_a(4|2)$ is isomorphic to $\mathfrak{osp}(4|2)$ and its irreducible module of least dimension is (up to the reversal of parity) the $(4|2)$ -dimensional tautological module in which the non-degenerate even symmetric bilinear form is preserved.

For $\mathfrak{osp}_2(4|2)$ and $\mathfrak{osp}_3(4|2)$ taken in realization with respective Cartan matrices

$$\begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ -3 & 2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$$

consider the weights of irreducibles over them relative their Chevalley generators $H_i := [X_i^+, X_i^-]$.

For $a = 2$, as well as for $a = -3, \frac{1}{2}, -\frac{3}{2}, -\frac{1}{3}$ and $-\frac{2}{3}$, the irreducible module W of least dimension is (up to a reversal of parity) the $(6|4)$ -dimensional $\mathfrak{osp}_2(4|2)$ -module with an even highest weight vector of weight $(0, 0, 2)$ with respect to the Cartan subalgebra of \mathfrak{g}_0 .

For $a = 3$, as well as for $a = -4, \frac{1}{3}, -\frac{4}{3}, -\frac{1}{4}$ and $-\frac{3}{4}$, the irreducible module W of least dimension is (up to a reversal of parity) the $(6|8)$ -dimensional $\mathfrak{osp}_3(4|2)$ -module with an even highest weight vector of weight $(2, 3, 0)$ with respect to the Cartan subalgebra of \mathfrak{g}_0 .

Interestingly, the two exceptional algebras are somewhat similar in structure, see [68], being represented as a direct sum \bigoplus – as superspaces – of a subsuperalgebra \mathfrak{h} and an \mathfrak{h} -module:

$$\begin{aligned} \mathfrak{ag}(2) &= \mathfrak{osp}_3(4|2) \bigoplus \Pi(W); \\ \mathfrak{ab}(3) &= (\mathfrak{osp}_2(4|2) \bigoplus \mathfrak{sl}(2)) \bigoplus (V \boxtimes R(\omega_1)), \end{aligned}$$

where $R(\omega_1)$ is the tautological $\mathfrak{sl}(2)$ -module and $A \boxtimes B$ is the modern notation of the tensor product of modules over the sum $\mathfrak{g} \oplus \mathfrak{h}$ of algebras, e.g., of the \mathfrak{g} -module A by the \mathfrak{h} -module B .

7. Description of deformations

Theorem 7.1. (Classification of deformations with odd parameters) *The only (observe that $\mathfrak{svect}(0|3) \simeq \mathfrak{spe}(3)$, see eq. (21)) simple finite-dimensional Lie superalgebras over \mathbb{C} having deformations with odd parameters are $\mathfrak{g} = \mathfrak{svect}(0|2n+1)$ for $n \geq 1$ and $\text{sdim } H^2(\mathfrak{g}; \mathfrak{g}) = 0|1$. The deformed superalgebra is $\widetilde{\mathfrak{svect}}(0|2n+1)$.*

This theorem was conjectured in 1976 by Bernstein and me, and M. Dufflo as he told me in 1986; this conjecture was never published. In this section, I'll prove it. For the vectorial Lie superalgebras, I know only one method to prove this conjecture; it requires ingredients published in the paper [70]. This section contains also the details of the proof and description of deformations with even parameters; almost all cases were partly described earlier by other researchers.

Theorem 7.2. (Classification of deformations with even parameters) *The only (minding isomorphisms (21)) simple finite-dimensional Lie superalgebras over \mathbb{C} having deformations with even parameters are $\mathfrak{g} = \mathfrak{svect}(0|2n)$ for $n \geq 2$, $\mathfrak{h}'(0|m)$ for $m \geq 5$, and $\mathfrak{osp}(4|2)$. In all these cases, $\dim H^2(\mathfrak{g}; \mathfrak{g}) = 1$. The respective deforms are $\mathfrak{g} = \widetilde{\mathfrak{svect}}(0|2n)$, $\mathfrak{psl}(2^{k-1}|2^{k-1})$ for $m = 2k$ and $\mathfrak{psq}(2^{k-1})$ for $m = 2k - 1$, and $\mathfrak{osp}_\alpha(4|2)$. The deforms of $\mathfrak{svect}(0|2n)$, as well as the deforms of $\mathfrak{h}'(0|m)$, are isomorphic for any two non-zero values of parameter; for isomorphisms of $\mathfrak{osp}_\alpha(4|2)$ for distinct α , see eq. (22).*

7.1. General facts on cohomology of Lie algebras and Lie superalgebras

See [91, pp.288–289] and [59, Ch. 1]. For correct descriptions of $H^*(\mathfrak{g})$ for simple finite-dimensional Lie superalgebras \mathfrak{g} over \mathbb{C} , see [18], except for $\mathfrak{svect}(0|n)$, $\widetilde{\mathfrak{svect}}(0|n)$ and $\mathfrak{h}'(0|n)$ in which case to describe $H^*(\mathfrak{g})$ is an open problem, cf. [2]. For the peculiarities of description of deformations in characteristic $p > 0$, see [22, 25]; for general definitions, see a detailed discussion in [22].

7.2. Lie superalgebras $\mathfrak{g}(A)$ with indecomposable Cartan matrix A

Kac was the first to observe that some of simple Lie superalgebras have Cartan matrices, and that one Lie superalgebra can have several Cartan matrices, see [83]. For a correct definition of Cartan matrices, and how to list all Cartan matrices of a given Lie superalgebra, see [27, 28, 36]. For the Lie superalgebra $\mathfrak{g} = \mathfrak{g}(A)$, if A is invertible, the proof of the fact “ $H^2(\mathfrak{g}; \mathfrak{g}) = 0$ ” is the same as for simple Lie algebras (see [133]) with the help of quadratic Casimir element corresponding to the even non-degenerate invariant symmetric bilinear form (even NIS) on $\mathfrak{g}(A)$; for the description of such forms on Lie superalgebras $\mathfrak{g}(A)$, see [24].

However, $\mathfrak{psl}(n|n)$ has no Cartan matrix, while none of several Cartan matrices of $\mathfrak{gl}(n|n)$ is invertible.

If A is not invertible, we have to consider the simple subquotient $\mathfrak{g}'(A)/\mathfrak{c}$ of $\mathfrak{g}(A)$, and the proof is still the same with the help of the quadratic Casimir element corresponding to the NIS induced from the NIS on $\mathfrak{g}(A)$.

7.3. Lie superalgebras $\mathfrak{g} = \mathfrak{psq}(n)$ for $n > 2$

To prove that $H^2(\mathfrak{g}; \mathfrak{g}) = 0$, we use the Hochschild-Serre spectral sequence (see [59, Ch. 1, no. 5]) in which $E_2^{i,j} = H^j(\mathfrak{g}_0; (S^i(\mathfrak{g}_1^*) \otimes \mathfrak{g}))$ for $i, j \geq 0$ and $i + j = 2$. Since

$H^j(\mathfrak{sl}(n); M) = 0$ for any j and any non-trivial (finite-dimensional) $\mathfrak{sl}(n)$ -module M , we deduce with the help of [124, Table 5] that $\mathfrak{psq}(n)$ is rigid for $n > 2$.

7.4. Lie superalgebras $\mathfrak{g} = \mathfrak{spe}(n)$ for $n > 3$

Observe, that the title of the paper [55] is misleading: the paper considers q -quantized algebra $U_q(\mathfrak{pe}(n))$, not deformations of $\mathfrak{pe}(n)$ or $\mathfrak{spe}(n)$. I'll consider the case $\mathfrak{spe}(3) \simeq \mathfrak{svect}(0|3)$ while studying $\mathfrak{svect}(0|2n-1)$.

Similarly to the proof for $\mathfrak{psq}(n)$, using [124, Table 5] we deduce that $\mathfrak{spe}(n)$ is rigid for $n > 3$, whereas $\mathfrak{spe}(3)$ has a deformation. However, the proof is more complicated than in the $\mathfrak{psq}(n)$ case: we have to consider $E_2^{3,0} = H^0(\mathfrak{g}_0; S^3(\mathfrak{g}_1^*) \otimes \mathfrak{g})$ and calculate the differential in the spectral sequence.

7.4.1. Induced and coinduced modules. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie superalgebra and its subsuperalgebra, M an \mathfrak{h} -module. The induced and coinduced \mathfrak{g} -modules are constructed from M by setting

$$\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(M) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} M, \quad \mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(M) := \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), M). \quad (24)$$

Clearly, $U(\mathfrak{g})$ is both left and right \mathfrak{h} -module; hence, $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(M)$ and $\mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(M)$ are \mathfrak{g} -modules.

For any \mathfrak{h} -module M , we have ([59, Th. 1.5.4])

$$H^q(\mathfrak{g}; \mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(M)) \simeq H^q(\mathfrak{h}; M); \quad H_q(\mathfrak{g}; \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(M)) \simeq H_q(\mathfrak{h}; M). \quad (25)$$

Note that for the Lie superalgebra $\mathfrak{g} = \mathfrak{vect}(0|a)$, where $a \geq 2$, or $\mathfrak{svect}(0|b)$, where $b \geq 3$, or $\mathfrak{h}(0|c)$, where $c \geq 4$, the subalgebra $\mathfrak{g}_{>0} := \bigoplus_{i \geq 0} \mathfrak{g}_i$ is generated by \mathfrak{g}_1 , except for $\mathfrak{h}(0|c)$; it is $(\mathfrak{h}'(0|c))_{>0}$ which is generated by \mathfrak{g}_1 .

For any \mathbb{Z} -graded simple vectorial Lie superalgebras \mathfrak{g} on $\mathcal{C}^{0|n}$, let the \mathfrak{g}_0 -module M be such that $\mathfrak{g}_{>0}M = 0$, so M can be considered as $\mathfrak{g}_{\geq 0}$ -module, where $\mathfrak{g}_{\geq 0} := \bigoplus_{i \geq 0} \mathfrak{g}_i$. The \mathfrak{g} -module $T(M) := \mathrm{Coind}_{\mathfrak{g}_{\geq 0}}^{\mathfrak{g}}(M)$ is interpreted as the space of tensor fields on $\mathcal{C}^{0|n}$ – sections of the vector bundle with fiber M . For any \mathfrak{g} -module V , let $V^{\mathfrak{g}}$ denote the submodule of \mathfrak{g} -invariants. We have

$$H^*(\mathfrak{g}; T(M)) \stackrel{(25)}{\simeq} H^*(\mathfrak{g}_{\geq 0}; M) \stackrel{[\mathrm{Fu}]}{\simeq} H^*(\mathfrak{g}_0) \otimes (H^*(\mathfrak{g}_{>0}) \otimes M)^{\mathfrak{g}_0}. \quad (26)$$

For the proof of the second isomorphism in (26), see Th. 2.2.8 (due to Losik) in Fuchs' book [59]. It is given there for the Lie algebra $\mathfrak{g} = \mathfrak{vect}(n|0) = \mathfrak{der} \mathbb{C}[x]$ only, where $x = (x_1, \dots, x_n)$ are even indeterminates, but the proof can be directly translated to any Cartan prolongs $\mathfrak{g} := (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ and conjecturally – we do not need this in this paper – to the generalized Cartan prolongs – Lie (super)algebras $\mathfrak{g} := \bigoplus_{i \geq -d} \mathfrak{g}_i$ of depth $d > 1$; for the most clear definition of prolong, see [20, 146]. The proof of (26) in [59] uses the Hochschild-Serre spectral sequence for the algebra $\mathfrak{g}_{\geq 0}$ and its ideal $\mathfrak{g}_{>0}$; it is proved that $E_{\infty} = E_2$.

7.4.2. The long exact sequence. Let \mathfrak{g} be a Lie superalgebra and let

$$0 \longrightarrow A \xrightarrow{\partial_0} C \xrightarrow{\partial_1} B \longrightarrow 0, \quad (27)$$

where $p(\partial_0) = \bar{0}$ and ∂_1 is either even or odd, be a short exact sequence of \mathfrak{g} -modules. Let d be the differential in the standard cochain complex of \mathfrak{g} , cf. [59, Subsection 1.3.6].

Consider the long cohomology sequence:

$$\dots \xrightarrow{\partial} H^i(\mathfrak{g}; A) \xrightarrow{\partial_0} H^i(\mathfrak{g}; C) \xrightarrow{\partial_1} H^i(\mathfrak{g}; B) \xrightarrow{\partial} H^{i+1}(\mathfrak{g}; A) \xrightarrow{\partial_0} \dots \quad (28)$$

where the ∂_i are the differentials induced by their namesakes in (27), and $\partial = d \circ \partial_1^{-1}$. Since ∂_0 and ∂_1 commute with d , the sequence (27) is well-defined and the same arguments as for Lie algebras ([59]) demonstrate that the long cohomology sequence (28) induced by (27) is exact, see [108].

7.4.3. Vectorial Lie superalgebras as coinduced modules. Let \mathfrak{g} be a vectorial Lie superalgebra with the standard \mathbb{Z} -grading, $\mathbb{1}$ be the 1-dimensional trivial \mathfrak{g}_0 -module and $\mathbb{1}[k]$ the \mathfrak{g}_0 -module trivial with respect to the semi-simple part of \mathfrak{g}_0 and with value k on the distinguished central element ($1_n \in \mathfrak{gl}(n)$). Let the space of functions – a module over any vectorial Lie superalgebra (subalgebra of $\mathbf{vect}(0|n)$) – be denoted by

$$\mathcal{F} := T(\mathbb{1}) = \Lambda^*(n).$$

Let $\text{id}_{\mathfrak{h}} := V$ be the restriction of the tautological $\mathfrak{gl}(V)$ -module to $\mathfrak{h} \subset \mathfrak{gl}(V)$. Then:

$$\begin{aligned} \mathbf{vect}(0|n) &= \begin{cases} T(\text{id}_{\mathfrak{gl}(n)}) & \text{as } \mathbf{vect}(0|n)\text{-module;} \\ T(\text{id}_{\mathfrak{sl}(n)}) & \text{as } \mathbf{svect}(0|n)\text{-module;} \end{cases} \quad (29) \\ \mathbf{po}(0|n) &= T(\mathbb{1}) = \mathcal{F} \text{ as an } \mathfrak{h}(0|n)\text{- and } \mathfrak{h}'(0|n)\text{-modules.} \end{aligned}$$

Looking at the explicit relations in $\mathfrak{g}_{>}$, see [70], we get $H^2(\mathfrak{g}_{>})$. Clearly, $H^1(\mathfrak{g}_{>}) = \mathfrak{g}_1$.

- For $\mathfrak{v} := \mathbf{vect}(0|n)$. We have

$$H^2(\mathfrak{v}; \mathfrak{v}) \simeq \oplus_{a+b=2} H^a(\mathfrak{gl}(n)) \otimes (H^b(\mathfrak{v}_{>}) \otimes \text{id}_{\mathfrak{gl}(n)})^{\mathfrak{gl}(n)}. \quad (30)$$

Since $H^a(\mathfrak{gl}(n)) \neq 0$ for $a = 0, 1$, we have to consider $(H^b(\mathfrak{v}_{>}) \otimes \text{id}_{\mathfrak{gl}(n)})^{\mathfrak{gl}(n)}$ for $b = 2$ and 1 , respectively. Looking at the explicit weights of generators ($b = 1$) and relations ($b = 2$) in $\mathfrak{v}_{>}$ given in [70], we conclude that $(H^b(\mathfrak{v}_{>}) \otimes \text{id}_{\mathfrak{gl}(n)})^{\mathfrak{gl}(n)} = 0$, so $\mathbf{vect}(0|n)$ is rigid.

- For $\mathfrak{s} := \mathbf{svect}(0|n)$. We have an exact sequence of \mathfrak{s} -modules:

$$0 \longrightarrow \mathfrak{s} \longrightarrow \mathfrak{v} \xrightarrow{\text{div}} \text{Vol}^0 \longrightarrow 0. \quad (31)$$

In this case, the divergence is not a surjective mapping onto \mathcal{F} , its image is the space Vol^0 of functions (coefficients of vol) with integral 0. In the associated long exact sequence

$$\dots \longrightarrow H^1(\mathfrak{s}; \mathfrak{s}) \longrightarrow H^1(\mathfrak{s}; \mathfrak{v}) \longrightarrow H^1(\mathfrak{s}; \text{Vol}^0) \longrightarrow H^2(\mathfrak{s}; \mathfrak{s}) \longrightarrow H^2(\mathfrak{s}; \mathfrak{v}) \longrightarrow \dots \quad (32)$$

let us compute

$$H^i(\mathfrak{s}; \mathfrak{v}) \simeq \sum_{a+b=i} H^a(\mathfrak{sl}(n)) \otimes (H^b(\mathfrak{s}_{>}) \otimes \text{id}_{\mathfrak{sl}(n)})^{\mathfrak{sl}(n)} \text{ for } i = 1, 2 \text{ and } n > 1; \quad (33)$$

Since $H^1(\mathfrak{s}; \mathfrak{v}) = 0$ and $H^2(\mathfrak{s}; \mathfrak{v}) = 0$ by (25), (29) and also by plugging the weights of generators of $\mathbf{svect}(0|n)_{>0}$ computed in [70] in (33) for $b = 1$, we see that $H^2(\mathfrak{s}; \mathfrak{s}) \cong H^1(\mathfrak{s}; \text{Vol}^0)$. To compute it, consider the short exact sequence

$$0 \longrightarrow \text{Vol}^0 \longrightarrow \mathcal{F} \xrightarrow{f} \mathbb{C} \longrightarrow 0. \quad (34)$$

It gives rise to the long exact sequence

$$0 \rightarrow H^0(\mathfrak{s}; \text{Vol}^0) \rightarrow H^0(\mathfrak{s}; \mathcal{F}) \rightarrow H^0(\mathfrak{s}) \rightarrow H^1(\mathfrak{s}; \text{Vol}^0) \rightarrow H^1(\mathfrak{s}; \mathcal{F}) \rightarrow \dots \quad (35)$$

and since

$$H^0(\mathfrak{s}; \text{Vol}^0) \simeq \mathbb{C}, \quad H^0(\mathfrak{s}; \mathcal{F}) \simeq H^0(\mathfrak{sl}) \simeq \mathbb{C}; \quad H^1(\mathfrak{s}; \mathcal{F}) = H^0(\mathfrak{sl}) = \mathbb{C},$$

it follows that $H^1(\mathfrak{s}; \text{Vol}^0) \simeq H^0(\mathfrak{s}) \simeq \Pi^n \mathbb{C}$. Since we already know one global deformation, there are no more.

7.5. What was known about deformations of simple vectorial Lie superalgebras

For $\mathfrak{h}'(0|n)$, and for the Lie superalgebra $\mathfrak{svect}(0|n)$ of divergence-free vector fields, the deformations were computed for n small by N. V. D. Hijligenberg, Yu. Kotchetkov, and G. Post, see [74].

The result for $\mathfrak{h}'(0|n)$ is double-checked being a corollary of the description of deformations of the Poisson Lie superalgebras (physicists call it *quantization*) obtained by I. Tyutin [150]. Observe that the coincidence of the number of non-trivial deformations of $\mathfrak{po}(0|n)$ and $\mathfrak{h}(0|n)$ is not automatic: e.g., cf. the answers for $\mathfrak{po}(2a|b)$ and $\mathfrak{h}(2a|b)$ for $(2a, b) = (2, 2)$, see [115]. I was unable to understand the idea of the proof in [150], so I have to use another method, the one described in Subsection 7.4.3, see Section 7.6.

7.5.1. Filtered deformations. Let $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ be a \mathbb{Z} -graded Lie superalgebra and \mathfrak{G} a filtered Lie superalgebra with decreasing filtration such that $\text{gr } \mathfrak{G} \simeq \mathfrak{g}$. Then, the elements of $(H^2(\mathfrak{g}_-, \mathfrak{g}))^{\mathfrak{g}_0}$ of \mathfrak{g}_0 -invariants in $H^2(\mathfrak{g}_-, \mathfrak{g})$ describe the filtered deformations of \mathfrak{g} to \mathfrak{G} , see [39, 40]. The *filtered* deformations of $\mathfrak{svect}(0|2n)$ and $\mathfrak{h}'(0|n)$ are described in [130]. Observe that the filtered deformation describes only the deformation corresponding to one \mathbb{Z} -grading and only if the negative part in this grading is non-zero. To describe all deformations remained an open problem, closed in this paper.

7.6. Deformations of $\mathfrak{h}'(0|n)$ ([115, 150])

The answer for $\mathfrak{h}'(0|n)$ can be double-checked as a corollary of the description of the deformations of the Poisson Lie superalgebra $\mathfrak{po}(0|n)$, see [150], taking into account the exact sequences (19) and

$$0 \rightarrow \mathfrak{h}'(0|n) \rightarrow \mathfrak{h}(0|n) \rightarrow \mathbb{C}H_{\theta_1 \dots \theta_n} \rightarrow 0. \quad (36)$$

Theorem 7.3. (Deformations of $\mathfrak{h}'(0|n)$ ([115]) *The quantizations*

$$\mathfrak{po}(0|2n) \rightarrow \mathfrak{gl}(\Lambda^\bullet(n)) \text{ and } \mathfrak{po}(0|2n-1) \rightarrow \mathfrak{q}(\Lambda^\bullet(n))$$

induce deformations

$$\mathfrak{h}'(0|2n) \rightarrow \mathfrak{psl}(\Lambda^\bullet(n)) \text{ and } \mathfrak{h}'(0|2n-1) \rightarrow \mathfrak{psq}(\Lambda^\bullet(n)).$$

8. Summary of ideas in Kac's proof ([83])

Notation below are same as in [83] with translation into notation of §2 and information on discoverers and first partial classifications from [88, Newsletters], [122]:

For example, the simple Lie superalgebras \mathfrak{g} with simple $\mathfrak{g}_{\bar{0}}$ were classified in [45, 48]; there are 4 series of which members of the 3rd and 4th series are actually isomorphic. In modern notation, they are: $\mathfrak{osp}(1|2n)$, $\mathfrak{psq}(n)$, $\mathfrak{spe}^a(n) \simeq \mathfrak{spe}^{sy}(n)$. The Lie superalgebras of series \mathfrak{sl} , \mathfrak{psl} and \mathfrak{osp} were discovered more or less simultaneously by several groups of researchers, see [88, Newsletters], [122].

Theorem 8.1. (Kac’s Main Theorem) ([83, Theorem 5 on p. 74]) *Any simple finite-dimensional Lie superalgebra \mathfrak{g} over an algebraically closed field \mathbb{K} of characteristic 0 is either isomorphic to a simple Lie algebra, or to one of the following Lie superalgebras:*

- $A(m, n)$ ($=\mathfrak{sl}(m+1|n+1)$, where $1 \leq m < n$, or $\mathfrak{psl}(n+1|n+1)$, where $n \geq 1$),
- $B(m, n)$ ($=\mathfrak{osp}(2m+1|2n)$, where $m \geq 0, n > 0$),
- $C(n)$ ($=\mathfrak{osp}(2|2n-2)$, where $n \geq 2$),
- $D(m, n)$ ($=\mathfrak{osp}(2m|2n)$, where $m \geq 2, n > 0$),
- $D(2, 1; \alpha)$ ($=\mathfrak{osp}_{\alpha}(4|2)$; discovered as $\Gamma(A, B, C)$, see [88]),
- $F(4)$ ($=\mathfrak{ab}(3)$, see [138]; discovered as Γ_3 , see [88]),
- $G(3)$ ($=\mathfrak{ag}(2)$, see [138]; discovered as Γ_2 , see [88]),
- $P(n)$ ($=\mathfrak{spe}(n+1)$, where $n \geq 2$; discovered in [45, 48]),
- $Q(n)$ ($=\mathfrak{psq}(n+1)$, where $n \geq 2$; discovered in [45, 48]),
- $W(n)$ ($=\mathfrak{vect}(0|n)$, where $n \geq 2$; discovered in [79]),
- $S(n)$ ($=\mathfrak{svect}(0|n)$, where $n \geq 3$; discovered in [79]),
- $\widetilde{S}(2n)$ ($=\widetilde{\mathfrak{svect}}(0|2n)$, where $n \geq 2$; see [80, 82]),
- $H(n)$ ($=\mathfrak{h}'(0|n)$; discovered in [79]).

Proof. (Ideas of the proof of Kac’s Main Theorem) Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$.

First, let us look at the $L_{\bar{0}}$ -action on $L_{\bar{1}}$. If this action is irreducible, then L is what Kac calls “classical”; it is isomorphic to one of the following: $B(m, n)$, $D(m, n)$, $F(4)$, $G(3)$, $Q(n)$ or $D(2, 1; \alpha)$. This is proved in Theorem 2 on p. 44 of [83]. The proof is performed by means of the representation theory of semisimple Lie algebras.

If the $L_{\bar{0}}$ -action on $L_{\bar{1}}$ is reducible, we can take a maximal proper subalgebra $L_0 \subset L$ containing $L_{\bar{0}}$ (there can be several such subalgebras, take any of them). The subalgebra L_0 induces a filtration of L :

$$L = L_{-1} \supset L_0 \supset L_1 \supset \dots$$

Set $\text{gr } L := \bigoplus_{i \geq -1} \text{gr}_i L$. Then, (Proposition 1.3.2, p. 24) $\text{gr } L$ has the following properties:

- (1) $\text{gr } L$ is transitive (this is a direct corollary of simplicity of L);
- (2) \mathbb{Z} -grading of $\text{gr } L$ is compatible with $\mathbb{Z}/2$ -grading a.k.a. parity (this is a corollary of the fact $\text{gr}_{-1} L \subset \text{gr } L_{\bar{1}}$ because $L_0 \supset L_{\bar{0}}$).
- (3) $\text{gr } L$ is irreducible, i.e., $\text{gr}_0 L$ irreducibly acts on $\text{gr}_{-1} L$ (since L_0 is maximal).
- (4) $\text{gr}_1 L \neq 0$ (since $L_{\bar{0}}$ -action on $L_{\bar{1}}$ is reducible).

Theorem 4, p. 71, claims that any \mathbb{Z} -graded Lie superalgebra satisfying the above conditions 1–4 is one of the following:

- (I) $A(m, n), C(n), P(n)$;
- (II) $W(n), S(n), \tilde{H}(n) := \mathfrak{h}(0|n), H(n)$;
- (III) $H^\xi = H \otimes \Lambda(\xi) + \mathbb{K} \cdot \partial_\xi$, where H is a simple Lie algebra, with the grading given by setting $\deg \xi = -1$.
- (IV) $L + \mathbb{K} \cdot d$, where L is of type I, II, III, the center of $L_{\bar{0}}$ is trivial and d is the grading operator.

Let the $L_{\bar{0}}$ -action on $L_{\bar{1}}$ be reducible. Then, Theorem 4 implies Theorem 5 by case-by-case checking of these four possibilities for $\text{gr } L$.

(III) If $\text{gr } L = H^\xi$, then L is not simple. To prove this Kac uses Proposition 2.2.2, p. 35. In this particular case, one can directly see why: $L_0 = H \oplus \mathbb{K} \cdot \partial_\xi$ (as a linear space). Hence, $L_{\bar{0}} = H$. Since H is a simple Lie algebra, it acts on the 1-dimensional space $\mathbb{K} \cdot \partial_\xi$ by zero, so $L' \neq L$.

(IV) This case, and the case where $\text{gr } L = W(n)$ or $A(m|n)$ or $C(n)$, are dealt with by Proposition 1.3.1, p. 24, which claims

If for a finite-dimensional transitive filtered Lie superalgebra L the representation of $\text{gr}_0 L$ in $\text{gr}_{-1} L$ is irreducible, and the even part of the center of $\text{gr}_0 L$ is non-zero, then $L \simeq \text{gr } L$.

This claim is practically obvious. Indeed, let us normalize the element z of the center of $\text{gr}_0 L$ so as to act as a grading operator on $\text{gr } L$. Then, its pre-image in L is diagonalizable and defines a grading on L itself.

Thanks to this Proposition, the case IV does not contribute to the classification: the Lie superalgebra L is not simple, whereas if $\text{gr } L = W(n)$ or $A(m|n)$ or $C(n)$, then $L = W(n)$ or $A(m|n)$ or $C(n)$, respectively.

The case $\text{gr } L = P(n)$ (as well as with many other cases) is dealt with by Proposition 2.1.4, p. 34, which claims

If $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a simple finite-dimensional Lie superalgebra such that the $L_{\bar{0}}$ action on $L_{\bar{1}}$ is the same as that in one of the Lie superalgebras $A(m, n), B(m, n), C(n), D(m, n)$ where $(m, n) \neq (2, 1), F(4), G(3), P(n), Q(n)$, then L is isomorphic to the corresponding Lie superalgebra.

The hypothesis of this Proposition is obvious if $\text{gr } L = P(n)$ or $A(m|n)$ or $C(n)$: indeed, $L_0 = (L_0)_{\bar{0}} \oplus (L_0)_{\bar{1}}$, where $(L_0)_{\bar{0}} = \text{gr}_0 L = L_{\bar{0}}$ and $(L_0)_{\bar{1}} = \text{gr}_1 L$. Since $\text{gr}_0 L = L_{\bar{0}}$ is reductive, any its finite-dimensional representation is completely reducible, in particular, the $L_{\bar{0}}$ -action on $L_{\bar{1}}$.

The key point in the proof of Proposition 2.1.4 is Proposition 2.1.3, which claims that for the Lie superalgebras in question, the $L_{\bar{0}}$ -module $S^2(L_{\bar{1}})$ contains $L_{\bar{0}}$ with multiplicity 1.

The cases where $\text{gr } L = \tilde{H}(n) := \mathfrak{h}(0|n)$, or $H(n)$ are considered in Proposition 3.3.7, p. 66. Since $\mathfrak{h}(0|n)$ is not simple, there remains only $H(n)$.

Finally, the case of $\text{gr } L = S(n)$ is dealt with by Theorem 3.3.5, p. 64. In both these cases, the following two arguments are key ones.

The first is Proposition 3.1.3, p. 59, which claims that if $L = L_{-1} \supset L_0 \supset \dots$ is a transitive filtered Lie superalgebra such that $L_0 \supset L_{\bar{0}}$ and $\dim L_{-1}/L_0 = (0|n)$, then L can be embedded into $W(n)$.

The second is the Levi-Malcev theorem for Lie algebras claiming that any semisimple subalgebra of the Lie algebra L can, by means of conjugations, be embedded into a fixed Levi subalgebra (i.e., the complement to the radical) of L . Hence, in the case where $\text{gr } L = \widetilde{H}(n)$ or $H(n)$, the Lie superalgebra L contains $\mathfrak{o}(n)$, whereas in the case where $\text{gr } L = S(n)$, the Lie superalgebra L contains $\mathfrak{sl}(n)$. The rest is a routine application of the representation theory of simple Lie algebras. ■

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