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On Semisimple Invariant *CR* Structures of Maximal Rank on the Compact Symplectic Group

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Abstract. We characterize semisimple invariant *CR* structures of maximal rank on the compact symplectic group $\mathrm{USp}_{2n}(\mathbb{C})$ for $n \neq 4$. This is equivalent to characterizing complex semisimple subalgebras of maximal dimension in $\mathrm{sp}_{2n}(\mathbb{C})$ having trivial intersection with $\mathrm{usp}_{2n}(\mathbb{C})$. We conjecture that our classification remains valid for $n = 4$. This extends previous results by Ounaïes-Khalgui and the author for the compact groups $\mathrm{SU}_n(\mathbb{C})$ and $\mathrm{SO}_n(\mathbb{R})$.

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1. Introduction

In [5, 6], Ounaïes-Khalgui and the author studied semisimple invariant *CR* structures on the compact Lie groups $\mathrm{SU}_n(\mathbb{C})$ and $\mathrm{SO}_n(\mathbb{R})$. In particular, those whose rank is maximal are determined. In this paper, we characterize semisimple *CR* structures of maximal rank on the compact symplectic group.

Let G_0 be a real compact Lie group, $\mathfrak{g}_0$ its Lie algebra and $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. A subbundle T of the complexification of the tangent bundle of G_0 is an invariant *CR* structure on G_0 if T is formally integrable, $T_x \cap \overline{T_x} = \{0\}$ for all $x \in G_0$, and $T_{g.x} = g.T_x$ for all $g, x \in G_0$ (see [1, 2] for a more detailed account on *CR* manifolds). These conditions imply that an invariant *CR* structure is completely determined by its fibre at the identity element of G_0, which is a complex Lie subalgebra $\mathfrak{h}_T$ of $\mathfrak{g}$ verifying $\mathfrak{h}_T \cap \mathfrak{g}_0 = \{0\}$, and the dimension of $\mathfrak{h}_T$ over $\mathbb{C}$ is the rank of the *CR* structure. Consequently, invariant *CR* structures on G_0 are in bijection with the set $CR(G_0)$ of complex Lie subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h} \cap \mathfrak{g}_0 = \{0\}$. So we have a purely Lie algebra setting to study invariant *CR* structures on G_0.

In [3], Charbonnel and Ounaïes-Khalgui studied invariant *CR* structures of maximal rank. They proved that $\max\{\dim \mathfrak{h}; \mathfrak{h} \in CR(G_0)\} = [\frac{\dim \mathfrak{g}}{2}]$, and if $\mathfrak{h} \in CR(G_0)$ verifies $\dim \mathfrak{h} = [\frac{\dim \mathfrak{g}}{2}]$, then $\mathfrak{h}$ is solvable.

In [5, 6], an invariant *CR* structure T on G_0 is called *semisimple* if $\mathfrak{h}_T$ is semisimple. In particular, the following description of semisimple invariant *CR* structures on G_0 of maximal rank for $G_0 = \mathrm{SU}_n(\mathbb{C})$ and $G_0 = \mathrm{SO}_n(\mathbb{R})$ is established.

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Theorem 1.1. [6] *Let* $CRSS(G_0) = \{\mathfrak{h} \in CR(G_0)\,;\mathfrak{h}$ *semisimple*$\}$.

(1) *Let* $n \in \mathbb{N}^*$. *Then*

$$\max\{\dim \mathfrak{h}; \mathfrak{h} \in CRSS(\mathrm{SU}_n(\mathbb{C}))\} = \begin{cases} 0 & \text{if } n \leqslant 2, \\ \frac{n(n-1)}{2} & \text{if } n > 2. \end{cases}$$

Moreover, if $n > 2$ *and* $\mathfrak{h} \in CRSS(\mathrm{SU}_n(\mathbb{C}))$ *is of maximal dimension, then either* $\mathfrak{h} \simeq \mathrm{so}_n(\mathbb{C})$ *or* $n = 2m+1$ *is odd and* $\mathfrak{h} \simeq \mathrm{sp}_{2m}(\mathbb{C})$.

(2) *Let* $n = 2m + \varepsilon \in \mathbb{N}^*$ *where* $m \in \mathbb{N}$ *and* $\varepsilon \in \{0,1\}$. *Then*

$$\max\{\dim \mathfrak{h}; \mathfrak{h} \in CRSS(\mathrm{SO}_n(\mathbb{R}))\} = \begin{cases} 0 & \text{if } n < 5, \\ 3 & \text{if } n = 5, \\ m^2 + (\varepsilon - 1)m & \text{if } n > 5. \end{cases}$$

Moreover, if $\mathfrak{h} \in CRSS(\mathrm{SO}_n(\mathbb{R}))$ *is of maximal dimension, then* $\mathfrak{h} \simeq \mathrm{so}_3(\mathbb{C})$ *if* $n = 5$ *and* $\mathfrak{h} \simeq \mathrm{so}_{m+\varepsilon}(\mathbb{C}) \times \mathrm{so}_m(\mathbb{C})$ *if* $n > 5$.

In this paper, we treat the case of the compact symplectic group, and our main result (Theorems 5.1 and 6.1) can be summarized as follows :

Theorem 1.2. *Let* $n \in \mathbb{N}^*$ *be such that* $n \neq 4$. *Then*

$$\max\{\dim \mathfrak{h}; \mathfrak{h} \in CRSS(\mathrm{USp}_{2n}(\mathbb{C}))\} = n^2 - 1.$$

Moreover, if $\mathfrak{h} \in CRSS(\mathrm{USp}_{2n}(\mathbb{C}))$ *is of maximal dimension, then* $\mathfrak{h} \simeq \mathrm{sl}_n(\mathbb{C})$.

The approach of the proof for the case $n \geqslant 5$ is closed to the one in [6], and the details are covered in sections 2 to 5. In Section 6, we present an approach to treat the remaining cases by computer verifications. While the case $n \leqslant 2$ is simple, work is required for the case $n = 3$ to reduce to a checkable number of cases. Unfortunately, for the case $n = 4$, we have not managed to reduce to a checkable number of cases (see Section 6 for the detailed explanations). We conjecture that the theorem remains valid for $n = 4$.

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2. A necessary condition

Let us first fix some notations which will be used throughout the paper.

Let $n \in \mathbb{N}^*$, and V be a complex vector space of dimension $2n$ endowed with the Hermitian inner product $\langle \cdot, \cdot \rangle$. Let us fix an orthonormal basis $\mathcal{B} = (e_1, \ldots, e_{2n})$ of V, so we have

$$\left\langle \sum_{i=1}^{2n} \lambda_i e_i, \sum_{i=1}^{2n} \mu_i e_i \right\rangle = \sum_{i=1}^{2n} \overline{\lambda_i} \mu_i$$

for all $\lambda_1, \ldots, \lambda_{2n}, \mu_1, \ldots, \mu_{2n} \in \mathbb{C}$.

We shall denote by Φ the non degenerate alternating form on V whose matrix with respect to the basis $\mathcal{B}$ is

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

[1] `https://romeo.univ-reims.fr`

Recall that $\mathrm{USp}(V) = \mathrm{Sp}(V) \cap \mathrm{U}(V)$ is the maximal simply-connected compact subgroup inside $\mathrm{Sp}(V)$. Its Lie algebra $\mathrm{usp}(V) = \mathrm{sp}(V) \cap \mathrm{u}(V)$ is therefore the set of anti-self-adjoint endomorphisms of V with respect to both Φ and $\langle \cdot, \cdot \rangle$. We have

$$\dim \mathrm{sp}(V) = 2n^2 + n$$

which is also the dimension of $\mathrm{usp}(V)$ as a vector space over $\mathbb{R}$.

For any complex subspace W of V, we shall denote by $W^{\perp}$ the orthogonal of W in V with respect to $\langle \cdot, \cdot \rangle$, and by W° the orthogonal of W in V with respect to Φ. We say that W is Φ-regular if $W \cap W^{\circ} = \{0\}$.

Given a complex subspace W of V, we shall identify $\mathrm{u}(W)$ with the set of elements of $\mathrm{u}(V)$ which stabilize W and are identically zero on $W^{\perp}$. In the same manner, if W is Φ-regular, then we shall identify $\mathrm{sp}(W)$ (resp. $\mathrm{usp}(W)$) with the set of elements of $\mathrm{sp}(V)$ (resp. $\mathrm{usp}(V)$) which stabilize W and are identically zero on W°.

Let W be a Φ-regular complex subspace of V. The following proposition provides a necessary condition for $\mathrm{sp}(W)$ to define an invariant semisimple *CR* structure on $\mathrm{USp}(V)$.

Proposition 2.1. *Let W be a Φ-regular complex subspace of V. If $\dim W > n$, then we have $\mathrm{sp}(W) \cap \mathrm{usp}(V) \neq \{0\}$.*

Proof. Let $\tau : V \to V$ be the $\mathbb{R}$-linear map defined by

$$\tau\left(\sum_{i=1}^{2n} \lambda_i e_i\right) = \sum_{i=1}^{n} \overline{\lambda_{n+i}} e_i - \sum_{i=1}^{n} \overline{\lambda_i} e_{n+i}.$$

Thus τ is a semilinear map with respect to complex conjugation. We check easily that $\tau^2 = -\mathrm{Id}_V$ and

$$\Phi(\tau(x), y) = \langle x, y \rangle$$

for all $x, y \in V$. Let us assume that $\dim W > n$. Set $P = W \cap \tau(W)$. Then $\dim P \geqslant 2$ and P is τ-stable. For any non zero $x \in P$, we have

$$\Phi(\tau(x), x) = \langle x, x \rangle > 0.$$

We deduce readily that P is a Φ-regular complex subspace of W, and $P^{\circ} = P^{\perp}$. Consequently,

$$\mathrm{usp}(P) \subset \mathrm{usp}(V) \ \text{ and } \ \mathrm{usp}(P) \subset \mathrm{sp}(P) \subset \mathrm{sp}(W).$$

Since $\mathrm{usp}(P)$ has real dimension greater than or equal to 3, the result follows. ■

3. An invariant semisimple *CR* structure defined by $\mathrm{sl}_n(\mathbb{C})$

In this section, we shall construct an invariant semisimple *CR* structure defined by $\mathrm{sl}_n(\mathbb{C})$. This is in fact the derived subalgebra of the Levi factor of a certain parabolic subalgebra.

Via the basis $\mathcal{B}$, we may identify the Lie algebras $\mathrm{sp}(V)$ and $\mathrm{usp}(V)$ respectively with the following Lie subalgebras of $\mathrm{gl}_{2n}(\mathbb{C})$:

$$\mathrm{sp}_{2n}(\mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} ;\ A, B, C \in \mathrm{gl}_n(\mathbb{C}) \text{ with } B \text{ and } C \text{ symmetric} \right\} \text{ and}$$

$$\mathrm{usp}_{2n}(\mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} ;\ A, B \in \mathrm{gl}_n(\mathbb{C}),\ A \text{ anti-hermitian and } B \text{ symmetric} \right\}.$$

For any symmetric matrix $S \in \mathrm{gl}_n(\mathbb{C})$, we set

$$\mathfrak{h}_S = \left\{ \begin{pmatrix} A & AS + S^tA \\ 0 & -{}^tA \end{pmatrix} ;\ A \in \mathrm{gl}_n(\mathbb{C}) \right\} \subset \mathrm{sp}_{2n}(\mathbb{C}).$$

Observe that $\mathfrak{h}_0$ is a Lie subalgebra of $\mathrm{sp}_{2n}(\mathbb{C})$ isomorphic to $\mathrm{gl}_n(\mathbb{C})$, and since

$$\mathfrak{h}_S = \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix}^{-1} \mathfrak{h}_0 \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix},$$

$\mathfrak{h}_S$ is also a Lie subalgebra of $\mathrm{sp}_{2n}(\mathbb{C})$ isomorphic to $\mathrm{gl}_n(\mathbb{C})$.

Proposition 3.1. *There exists a symmetric matrix $S \in \mathrm{gl}_n(\mathbb{R})$ which satisfies $\mathfrak{h}_S \cap \mathrm{usp}_{2n}(\mathbb{C}) = \{0\}$. Thus there exists a semisimple Lie subalgebra of $\mathrm{sp}(V)$ isomorphic to $[\mathfrak{h}_S, \mathfrak{h}_S] \simeq \mathrm{sl}_n(\mathbb{C})$ which defines an invariant semisimple CR structure on $\mathrm{USp}(V)$.*

Proof. Let $S \in \mathrm{gl}_n(\mathbb{R})$ be symmetric, and suppose that

$$\begin{pmatrix} A & AS + S^tA \\ 0 & -{}^tA \end{pmatrix} \in \mathfrak{h}_S \cap \mathrm{usp}_{2n}(\mathbb{C})$$

for some $A \in \mathrm{gl}_n(\mathbb{C})$. Then $-{}^tA = \overline{A}$ and $AS + S^tA = 0$.
Let $A_0, A_1 \in \mathrm{gl}_n(\mathbb{R})$ be such that $A = A_0 + iA_1$. Then these conditions become

$$A_0 = -{}^tA_0\ ,\ A_1 = {}^tA_1\ ,\ A_0S = SA_0\ ,\ A_1S = -SA_1.$$

Take $S = \mathrm{diag}(1, 2, \ldots, n)$, then we must have $A_0 = A_1 = 0$. Hence there exists $S \in \mathrm{gl}_n(\mathbb{R})$ such that $\mathfrak{h}_S \cap \mathrm{usp}_{2n}(\mathbb{C}) = \{0\}$. ■

4. On the dimension of a semisimple Lie subalgebra of sp(V)

For any $m \in \mathbb{N}^*$, we set $\Delta(m) = 2m^2 + m = \dim \mathrm{sp}_{2m}(\mathbb{C})$.
For any simple complex Lie algebra $\mathfrak{g}$, we set $\delta_{\mathfrak{g}}$ to be the strictly positive integer such that $2\delta_{\mathfrak{g}}$ is the minimal dimension of a symplectic representation of $\mathfrak{g}$. The values of $\delta_{\mathfrak{g}}$ given in Table 1 can be found in [6, Section 4].
In particular, we have $\dim \mathfrak{g} \leqslant \dim \mathrm{sp}_{2\delta_{\mathfrak{g}}}(\mathbb{C}) = \Delta(\delta_{\mathfrak{g}})$.

Proposition 4.1. *Let $\mathfrak{s}$ be a non trivial semisimple Lie subalgebra of $\mathrm{sp}(V)$. Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_r$ be simple Lie algebras such that $\mathfrak{s} \simeq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r$. Then there exists $\mathcal{I} \subset \{1, \ldots, r\}$ such that*

$$\dim \mathfrak{s} \leqslant \sum_{p \in \mathcal{I}} \Delta(\delta_{\mathfrak{g}_p}) \quad \text{and} \quad \sum_{p \in \mathcal{I}} \delta_{\mathfrak{g}_p} \leqslant n.$$

The rest of this section is devoted to the proof of this proposition. We shall use an induction argument similar to the one used in [6] for special orthogonal groups.

Dynkin type of $\mathfrak{g}$	$\dim \mathfrak{g}$	$\delta_{\mathfrak{g}}$
A_1	3	1
$A_\ell, \ell \geqslant 2$	$\ell^2 + 2\ell$	$\ell + 1$
B_2, C_2	10	2
$B_\ell, \ell \geqslant 3$	$2\ell^2 + \ell$	$2\ell + 1$
$C_\ell, \ell \geqslant 3$	$2\ell^2 + \ell$	ℓ
$D_\ell, \ell \geqslant 4$	$2\ell^2 - \ell$	2ℓ
E_6	78	27
E_7	133	28
E_8	248	248
F_4	52	26
G_2	14	7

Table 1: Values of $\delta_{\mathfrak{g}}$

Without loss of generality, we may assume that $\delta_{\mathfrak{g}_1} = \max\{\delta_{\mathfrak{g}_k}; 1 \leqslant k \leqslant r\}$. We shall first establish some results on the structure of V as a $\mathfrak{g}_1$-module.

Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_1$ and a system of positive roots Γ in the root system defined by $\mathfrak{h}$. Denote by P^+ the set of dominant weights with respect to Γ, and for $\lambda \in P^+$, L_λ denotes a simple $\mathfrak{g}_1$-module of highest weight λ. So L_0 is a $\mathfrak{g}_1$-module of dimension 1. Recall that the lowest weight of L_λ is $w(\lambda)$, where w is the longest element of the Weyl group with respect to Γ.

Since $\mathfrak{g}_1$ is simple, V decomposes into isotypic components

$$V = \bigoplus_{\lambda \in P^+} V(\lambda).$$

Thus $\dim V(\lambda) = d_\lambda m_\lambda$ where $d_\lambda = \dim L_\lambda$ and m_λ is the multiplicity of L_λ in V. Observe that for any simple $\mathfrak{g}_1$-submodule L in V, its Φ-orthogonal L° is $\mathfrak{g}_1$-stable, and therefore $L \cap L^\circ$ is either $\{0\}$ or L. Thus L is either Φ-regular or Φ-totally isotropic. More generally, we have the following easy but technical lemmas for decomposing $\mathfrak{g}_1$-modules. We have included their proofs for the sake of completeness.

Lemma 4.2. *Let $\lambda, \mu \in P^+$, $E \subset V(\lambda)$ and $F \subset V(\mu)$ be simple $\mathfrak{g}_1$-submodules, e_λ a highest weight vector of E and f_μ a lowest weight vector of F.*

1. *If $\Phi(e_\lambda, f_\mu) = 0$, then the restriction $\Phi|_{E \times F}$ of Φ to $E \times F$ is identically zero. In particular, if $\lambda + w(\mu) \neq 0$, then $\Phi|_{E\times F} = 0$.*
2. *If $\Phi(e_\lambda, f_\mu) \neq 0$, then $\Phi|_{E\times F}$ is non degenerate.*

Proof. (1) For any monomial $u = x_1 \cdots x_m$ in the envelopping algebra $U(\mathfrak{g}_1)$ of $\mathfrak{g}_1$ with $x_1, \ldots, x_m \in \mathfrak{g}_1$, we denote $u^* = (-1)^m x_m \cdots x_1$.

We extend this notation by linearity to any linear combination of monomials in $U(\mathfrak{g}_1)$. Thus for $(e,f,u) \in E \times F \times U(\mathfrak{g}_1)$, we have

$$\Phi\big(u(e), f\big) = \Phi\big(e, u^*(f)\big). \tag{1}$$

Let $(f_1, \dots, f_m, f_\mu)$ be a basis of F consisting of weight vectors, and $(\pi_1, \dots, \pi_m, \pi)$ its dual basis. For any root vector $x_\alpha \in \mathfrak{g}_1$ associated to a positive root α, we have by (1)

$$\Phi\big(e_\lambda, x_\alpha(f_\mu)\big) = \Phi\big(-x_\alpha(e_\lambda), f_\mu\big) = 0.$$

We deduce immediately that $\Phi(e_\lambda, f_k) = 0$ for $1 \leqslant k \leqslant m$. Hence for any $f \in F$, we have

$$\Phi(e_\lambda, f) = \pi(f)\Phi(e_\lambda, f_\mu).$$

Now if $(e,f) \in E \times F$, there exist $u, v \in U(\mathfrak{g}_1)$ such that $e = u(e_\lambda)$ and $f = v(f_\mu)$. Hence

$$\Phi(e,f) = \Phi\big(u(e_\lambda), v(f_\mu)\big) = \Phi\big(e_\lambda, (u^*v)(f_\mu)\big) = \pi\big((u^*v)(f_\mu)\big)\Phi(e_\lambda, f_\mu).$$

So if $\Phi(e_\lambda, f_\mu) = 0$, then $\Phi|_{E\times F} = 0$. In particular, if $\lambda + w(\mu) \neq 0$, then there exists $h \in \mathfrak{h}$ such that $\big(\lambda + w(\mu)\big)(h) \neq 0$. Taking $e = e_\lambda$, $f = f_\mu$ and $u = h$ in (1), we deduce that $\Phi(e_\lambda, f_\mu) = 0$, and so $\Phi|_{E\times F} = 0$.

(2) Since E and F are simple, $E^\circ \cap F$ is either $\{0\}$ or F, and $F^\circ \cap E$ is either $\{0\}$ or E. If $\Phi(e_\lambda, f_\mu) \neq 0$, then $E^\circ \cap F = \{0\} = F^\circ \cap E$. Hence $\Phi|_{E\times F}$ is non degenerate. ■

It follows directly from Lemma 4.2 that for any $\lambda, \mu \in P^+$, we have $\Phi|_{V(\lambda)\times V(\mu)} = 0$ if $\lambda + w(\mu) \neq 0$, and $\Phi|_{V(\lambda)\times V(-w(\lambda))}$ is non degenerate. In particular, $d_\lambda = d_{-w(\lambda)}$ and $m_\lambda = m_{-w(\lambda)}$.

There exists therefore a finite subset $Q \subset P^+$ such that $V(\lambda)$ is non trivial for all $\lambda \in Q$, and if we set $V_\lambda = V(\lambda) + V\big(-w(\lambda)\big)$ for $\lambda \in Q$, then

$$V = \bigoplus_{\lambda \in Q} V_\lambda$$

is a Φ-orthogonal direct sum decomposition.

Lemma 4.3. *Let $\lambda \in Q$ be such that $\lambda \neq 0$.*

(1) *Suppose that $\lambda \neq -w(\lambda)$. Then there exist simple $\mathfrak{g}_1$-modules $E_1, \dots, E_{m_\lambda}$ in $V(\lambda)$ and $F_1, \dots, F_{m_\lambda}$ in $V(-w(\lambda))$ such that*

$$V(\lambda) = \bigoplus_{k=1}^{m_\lambda} E_k\ ,\ \ V(-w(\lambda)) = \bigoplus_{k=1}^{m_\lambda} F_k \quad \textit{and}$$

   (a) *$H_k = E_k \oplus F_k$ is Φ-regular for $1 \leqslant k \leqslant m_\lambda$.*

   (b) *$V_\lambda = H_1 \oplus \cdots \oplus H_{m_\lambda}$ is a Φ-orthogonal direct sum decomposition.*

   In particular, $d_\lambda \geqslant \delta_{\mathfrak{g}_1}$.

(2) *Suppose that $\lambda = -w(\lambda)$, so $V_\lambda = V(\lambda)$.*

   (a) *If $m_\lambda > 1$, then there exist Φ-totally isotropic simple $\mathfrak{g}_1$-modules E, F in V_λ such that $H = E \oplus F$ is Φ-regular.*

(b) *If* $m_\lambda = 2\ell$ *is even, there exist* Φ*-totally isotropic simple* $\mathfrak{g}_1$*-modules* $E_1, \dots, E_\ell$*,* $F_1, \dots, F_\ell$ *in* V_λ *such that* $H_k = E_k \oplus F_k$ *is* Φ*-regular for* $1 \leqslant k \leqslant \ell$ *and*

$$V_\lambda = H_1 \oplus \cdots \oplus H_\ell$$

is a Φ*-orthogonal direct sum decomposition. In particular,* $d_\lambda \geqslant \delta_{\mathfrak{g}_1}$*.*

(c) *If* $m_\lambda = 2\ell + 1$ *is odd, then there exist* Φ*-totally isotropic simple* $\mathfrak{g}_1$*-modules* $E_1, \dots, E_\ell$*,* $F_1, \dots, F_\ell$ *and a* Φ*-regular simple* $\mathfrak{g}_1$*-module* G *in* V_λ *such that* $H_k = E_k \oplus F_k$ *is* Φ*-regular for* $1 \leqslant k \leqslant \ell$ *and*

$$V_\lambda = H_1 \oplus \cdots \oplus H_\ell \oplus G$$

is a Φ*-orthogonal direct sum decomposition. In particular,* $d_\lambda \geqslant 2\delta_{\mathfrak{g}_1}$*.*

Proof. (1) Let $E_1 \subset V(\lambda)$ be a simple submodule. Since $\lambda \neq -w(\lambda)$, E_1 is Φ-totally isotropic and its Φ-orthogonal $W(-w(\lambda)) = E_1^\circ \cap V(-w(\lambda))$ in $V(-w(\lambda))$ is a $\mathfrak{g}_1$-stable subspace. It follows that if F_1 is a $\mathfrak{g}_1$-stable complementary subspace of $W(-w(\lambda))$ in $V(-w(\lambda))$, then F_1 is simple, and $H_1 = E_1 \oplus F_1$ is Φ-regular.

Let $W(\lambda) = F_1^\circ \cap V(\lambda)$. Then $V(\lambda) = E_1 \oplus W(\lambda)$ and $V(-w(\lambda)) = F_1 \oplus W(-w(\lambda))$. The result follows by applying induction on $W_\lambda = W(\lambda) \oplus W(-w(\lambda))$.

(2) Let us prove (a). Suppose that $m_\lambda > 1$. We claim that there exists a Φ-totally isotropic simple $\mathfrak{g}_1$-submodule in V_λ.

Let us prove our claim. Let E be a simple $\mathfrak{g}_1$-submodule of V_λ. If E is Φ-totally isotropic, we are done. So let us assume that E is Φ-regular. Since $m_\lambda > 1$, $E^\circ \cap V_\lambda$ is non trivial. Let F be a simple $\mathfrak{g}_1$-submodule in $E^\circ \cap V_\lambda$. If F is Φ-totally isotropic, again we are done. If F is Φ-regular, then $H = E \oplus F$ is a Φ-orthogonal direct sum.

Let $(x_1, \dots, x_{d_\lambda})$ be a basis of E consisting of weight vectors such that x_1 is a highest weight vector and x_{d_λ} is a lowest weight vector, and $(\pi_1, \dots, \pi_{d_\lambda})$ its dual basis. There exist $u_1, \dots, u_{d_\lambda} \in U(\mathfrak{g}_1)$ such that $x_k = u_k(x_1)$ for $1 \leqslant k \leqslant d_\lambda$. By Lemma 4.2 and its proof, $\Phi(x_1, x_{d_\lambda}) \neq 0$, $\Phi(x_1, x_k) = 0$ if $1 \leqslant k \leqslant d_\lambda - 1$, and for $1 \leqslant p, q \leqslant d_\lambda$, we have

$$\Phi(x_p, x_q) = \pi_{d_\lambda}\big(u_p^* u_q(x_1)\big)\Phi(x_1, x_{d_\lambda}).$$

Let us fix an isomorphism of $\mathfrak{g}_1$-modules $\sigma : E \to F$. For $1 \leqslant k \leqslant d_\lambda$, set $y_k = \sigma(x_k)$. So $(y_1, \dots, y_{d_\lambda})$ is a basis of F consisting of weight vectors such that y_1 is a highest weight vector and y_{d_λ} is a lowest weight vector.

Again by Lemma 4.2, $\Phi(y_1, y_{d_\lambda}) \neq 0$, $\Phi(y_1, y_k) = 0$ if $1 \leqslant k \leqslant d_\lambda - 1$, and so for $1 \leqslant p, q \leqslant d_\lambda$, we have

$$\Phi(y_p, y_q) = \Phi\big(u_p(\sigma(x_1)), u_q(\sigma(x_1))\big) = \Phi\big(y_1, \sigma(u_p^* u_q(x_1))\big) = \pi_{d_\lambda}\big(u_p^* u_q(x_1)\big)\Phi(y_1, y_{d_\lambda}).$$

Let $\alpha, \beta \in \mathbb{C}^*$ be such that $\alpha^2 = -\Phi(x_1, x_{d_\lambda})$ and $\beta^2 = \Phi(y_1, y_{d_\lambda})$. The maps $\Upsilon_\pm : E \to E \oplus F$ defined by $x \mapsto \beta x \pm \alpha\sigma(x)$ are injective morphisms of $\mathfrak{g}_1$-modules. Clearly, we have $E \oplus F = \operatorname{im}\Upsilon_+ \oplus \operatorname{im}\Upsilon_-$ as a direct sum of $\mathfrak{g}_1$-modules, and by our description of $\Phi(x_p, x_q)$ and $\Phi(y_p, y_q)$ above, we check readily that for $1 \leqslant p, q \leqslant d_\lambda$, $\Phi\big(\Upsilon_+(x_p), \Upsilon_+(x_q)\big) = 0 = \Phi\big(\Upsilon_-(x_p), \Upsilon_-(x_q)\big)$. Thus $\operatorname{im}\Upsilon_+$ and $\operatorname{im}\Upsilon_-$ are Φ-totally isotropic. We have therefore proved our claim.

So there exists a Φ-totally isotropic simple $\mathfrak{g}_1$-submodule E in V_λ. Then for any $\mathfrak{g}_1$-stable complementary subspace F of $E^\circ \cap V_\lambda$ in V_λ, F is simple and the subspace $H = E \oplus F$ is Φ-regular. If F is Φ-totally isotropic, then we have a). If F is Φ-regular, then $E' = F^\circ \cap H$ is a Φ-regular $\mathfrak{g}_1$-stable subspace such that $H = E' \oplus F$ is a Φ-orthogonal direct sum decomposition. It follows from the proof of our claim above that there exist Φ-totally isotropic simple $\mathfrak{g}_1$-submodules E_1 and F_1 in H such that $H = E_1 \oplus F_1$.

We have therefore proved part (a).

Parts (b) and (c) are now immediate consequences of part a). ■

Since $\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r$ commutes with $\mathfrak{g}_1$, it follows from Schur's lemma that each isotypic component $V(\lambda)$ is $\mathfrak{s}$-stable. In particular, V_λ is also $\mathfrak{s}$-stable for each $\lambda \in Q$. We have therefore the following commutative diagram of Lie algebra morphisms:

$$\begin{array}{ccccc} \mathfrak{s} & \hookrightarrow & \prod_{\mu \in Q} \mathrm{sp}(V_\mu) & \hookrightarrow & \mathrm{sp}(V) \\ & \searrow^{\pi_\lambda} & \downarrow & & \\ & & \mathrm{sp}(V_\lambda) & & \end{array}$$

for any $\lambda \in Q$.

Let us fix a non zero $\lambda \in Q$. Then π_λ induces an embedding of $\mathfrak{g}_1$ in $\mathrm{sp}(V_\lambda)$. By renumbering, we may assume that $\ker \pi_\lambda = \mathfrak{g}_{s+1} \times \cdots \times \mathfrak{g}_r$ for some $s > 0$. Therefore

$$\mathrm{im}\, \pi_\lambda \simeq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_s,$$

and we have the embeddings

$$\mathfrak{g}_{s+1} \times \cdots \times \mathfrak{g}_r \hookrightarrow \prod_{\mu \in Q \setminus \{\lambda\}} \mathrm{sp}(V_\mu) \hookrightarrow \mathrm{sp}_{2n - \dim V_\lambda}(\mathbb{C}). \tag{2}$$

Lemma 4.4. *Suppose that $\lambda \neq -w(\lambda)$. Then we have a Lie algebra embedding*

$$\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r \hookrightarrow \mathrm{sp}_{2(n-\delta_{\mathfrak{g}_1})}(\mathbb{C}).$$

Proof. Let us use the decomposition of simple submodules in part 1) of Lemma 4.3. There exist bases $\mathcal{E}_1, \ldots, \mathcal{E}_{m_\lambda}$ of $E_1, \ldots, E_{m_\lambda}$ and $\mathcal{F}_1, \ldots, \mathcal{F}_{m_\lambda}$ of $F_1, \ldots, F_{m_\lambda}$ such that the matrix of $\Phi|_{H_k}$ in the basis $\mathcal{E}_k \cup \mathcal{F}_k$ is

$$\begin{pmatrix} 0 & I_{d_\lambda} \\ -I_{d_\lambda} & 0 \end{pmatrix}$$

and therefore the matrix of $\Phi|_{V_\lambda}$ in the basis $\mathcal{C} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{m_\lambda} \cup \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{m_\lambda}$ is

$$\begin{pmatrix} 0 & I_{d_\lambda m_\lambda} \\ -I_{d_\lambda m_\lambda} & 0 \end{pmatrix}.$$

It follows that for any $x \in \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s$, there exists $M(x) \in \mathrm{gl}_{d_\lambda m_\lambda}(\mathbb{C})$ such that the matrix of x in the basis $\mathcal{C}$ is

$$\begin{pmatrix} M(x) & 0 \\ 0 & -{}^t M(x) \end{pmatrix}.$$

Since $\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s$ commutes with $\mathfrak{g}_1$, there exists $A(x) = (a_{p,q})_{1 \leqslant p,q \leqslant m_\lambda} \in \mathrm{gl}_{m_\lambda}(\mathbb{C})$ such that

$$M(x) = \begin{pmatrix} a_{1,1} I_{d_\lambda} & \cdots & a_{1,m_\lambda} I_{d_\lambda} \\ \vdots & \ddots & \vdots \\ a_{m_\lambda,1} I_{d_\lambda} & \cdots & a_{m_\lambda,m_\lambda} I_{d_\lambda} \end{pmatrix}.$$

The map $x \mapsto A(x)$ defines an injective Lie algebra morphism from $\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s$ to $\mathrm{gl}_{m_\lambda}(\mathbb{C})$ whose image is in $\mathrm{sl}_{m_\lambda}(\mathbb{C})$ because $\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s$ is semisimple.
If $m_\lambda = 1$, then $\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s = \{0\}$, that is $s = 1$, and we have by (2) that

$$\mathfrak{g}_{s+1} \times \cdots \times \mathfrak{g}_r = \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r \hookrightarrow \mathrm{sp}_{2(n-d_\lambda)}(\mathbb{C}) \hookrightarrow \mathrm{sp}_{2(n-\delta_{\mathfrak{g}_1})}(\mathbb{C}).$$

If $m_\lambda > 1$, then $\delta_{\mathfrak{g}_1} + m_\lambda \leqslant d_\lambda + m_\lambda \leqslant d_\lambda m_\lambda$ because both d_λ and m_λ are greater than or equal to 2. Thus we have the embeddings

$$\begin{aligned} \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r &\hookrightarrow \mathrm{sl}_{m_\lambda}(\mathbb{C}) \times \mathrm{sp}_{2(n-d_\lambda m_\lambda)}(\mathbb{C}) \\ &\hookrightarrow \mathrm{sp}_{2m_\lambda}(\mathbb{C}) \times \mathrm{sp}_{2(n-\delta_{\mathfrak{g}_1}-m_\lambda)}(\mathbb{C}) \hookrightarrow \mathrm{sp}_{2(n-\delta_{\mathfrak{g}_1})}(\mathbb{C}) \end{aligned}$$

as required. ∎

Lemma 4.5. *Suppose that $\lambda = -w(\lambda)$ and m_λ is even.*

(1) *If $m_\lambda = 2$ and $\delta_{\mathfrak{g}_1} = d_\lambda$, then* $\dim(\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_s) \leqslant \delta_{\mathfrak{g}_1}^2 - 1 \leqslant \Delta(\delta_{\mathfrak{g}_1})$ *and we have a Lie algebra embedding*

$$\mathfrak{g}_{s+1} \times \cdots \times \mathfrak{g}_r \hookrightarrow \mathrm{sp}_{2(n-\delta_{\mathfrak{g}_1})}(\mathbb{C}).$$

(2) *If $m_\lambda > 2$ or $\delta_{\mathfrak{g}_1} < d_\lambda$, then we have a Lie algebra embedding*

$$\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r \hookrightarrow \mathrm{sp}_{2(n-\delta_{\mathfrak{g}_1})}(\mathbb{C}).$$

Proof. Applying the same arguments in the proof of Lemma 4.4 to the direct sum decomposition of simple modules in part 2(b) of Lemma 4.3, we obtain a Lie algebra embedding

$$\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s \hookrightarrow \mathrm{sp}_{m_\lambda}(\mathbb{C}).$$

Suppose that $m_\lambda = 2$ and $\delta_{\mathfrak{g}_1} = d_\lambda$. Then

$$\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s = \{0\} \quad \text{or} \quad \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s \simeq \mathrm{sp}_2(\mathbb{C}).$$

On the other hand, for any $x \in \mathfrak{g}_1$, there exists $M(x) \in \mathrm{gl}_{d_\lambda}(\mathbb{C})$ such that the matrix of x in the basis $\mathcal{C}$ (constructed in the same manner as in the proof of Lemma 4.4) is

$$\begin{pmatrix} M(x) & 0 \\ 0 & -{}^t M(x) \end{pmatrix}.$$

It follows that $x \mapsto M(x)$ defines an embedding of $\mathfrak{g}_1$ in $\mathrm{sl}_{d_\lambda}(\mathbb{C})$.

If $d_\lambda = 2$, then $\mathfrak{g}_1 \simeq \mathrm{sl}_2(\mathbb{C})$, which contradicts the fact that $\delta_{\mathfrak{g}_1} = d_\lambda = 2$. So $d_\lambda > 2$.

Suppose that $\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s = \{0\}$, then

$$\dim(\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_s) = \dim \mathfrak{g}_1 \leqslant d_\lambda^2 - 1 = \delta_{\mathfrak{g}_1}^2 - 1.$$

Suppose that $\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s \simeq \mathrm{sl}_2(\mathbb{C})$. Since $\mathfrak{g}_1$ commutes with $\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s$, we obtain by a simple matrix computation that $M(x)$ is antisymmetric for all $x \in \mathfrak{g}_1$.

Since $d_\lambda > 2$, we have $\dim \mathfrak{g}_1 \leqslant d_\lambda^2 - 4$, and so

$$\dim(\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_s) \leqslant d_\lambda^2 - 1 = \delta_{\mathfrak{g}_1}^2 - 1.$$

and by (2), we have $\mathfrak{g}_{s+1} \times \cdots \times \mathfrak{g}_r \hookrightarrow \mathrm{sp}_{2(n-\delta_{\mathfrak{g}_1})}(\mathbb{C})$.

Suppose now that $m_\lambda > 2$ or $\delta_{\mathfrak{g}_1} < d_\lambda$. Then

$$d_\lambda m_\lambda - 2d_\lambda - m_\lambda = (d_\lambda - 1)(m_\lambda - 2) - 2 \geqslant 0$$

because $d_\lambda \geqslant 2$, and m_λ is even. It follows that

$$2\delta_{\mathfrak{g}_1} + m_\lambda < 2d_\lambda + m_\lambda \leqslant d_\lambda m_\lambda.$$

So we have the Lie algebra embeddings

$$\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r \hookrightarrow \mathrm{sp}_{m_\lambda}(\mathbb{C}) \times \mathrm{sp}_{2n-d_\lambda m_\lambda}(\mathbb{C}) \hookrightarrow \mathrm{sp}_{2(n-\delta_{\mathfrak{g}_1})}(\mathbb{C})$$

as required. ■

Lemma 4.6. *Suppose that $\lambda = -w(\lambda)$ and m_λ is odd.*

(1) *If $d_\lambda = 2$, then $\dim(\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r) = r\Delta(1)$.*

(2) *If $d_\lambda > 2$, then we have a Lie algebra embedding*

$$\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r \hookrightarrow \mathrm{sp}_{2(n-\delta_{\mathfrak{g}_1})}(\mathbb{C}).$$

Proof. Suppose that $d_\lambda = 2$, then by part 2(c) of Lemma 4.3, we have $\delta_{\mathfrak{g}_1} = 1$. So $\delta_{\mathfrak{g}_k} = 1$ and by Table 1, $\mathfrak{g}_k \simeq \mathrm{sl}_2(\mathbb{C})$ for all $1 \leqslant k \leqslant r$. Hence

$$\dim(\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r) = r\Delta(1).$$

Suppose now $d_\lambda > 2$. Applying the same arguments in the proof of Lemma 4.4 to the direct sum decomposition of simple modules in part 2c) of Lemma 4.3, we obtain a Lie algebra embedding

$$\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s \hookrightarrow \mathrm{sl}_{m_\lambda}(\mathbb{C}).$$

If $m_\lambda = 1$, then $\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s = \{0\}$, that is $s = 1$, and we have by (2) that

$$\mathfrak{g}_{s+1} \times \cdots \times \mathfrak{g}_r = \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r \hookrightarrow \mathrm{sp}_{2n-d_\lambda}(\mathbb{C}) \hookrightarrow \mathrm{sp}_{2(n-\delta_{\mathfrak{g}_1})}(\mathbb{C}),$$

because $2\delta_{\mathfrak{g}_1} \leqslant d_\lambda$ by Lemma 4.3.

If $m_\lambda > 1$, then $d_\lambda m_\lambda - 2m_\lambda - d_\lambda = (d_\lambda - 2)(m_\lambda - 1) - 2 \geqslant 0$ because m is odd and $d_\lambda > 2$. Hence $2\delta_{\mathfrak{g}_1} + 2m_\lambda \leqslant d_\lambda + 2m_\lambda \leqslant d_\lambda m_\lambda$.

We have therefore the embeddings

$$\begin{aligned}\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r &\hookrightarrow \mathrm{sl}_{m_\lambda}(\mathbb{C}) \times \mathrm{sp}_{2(n-d_\lambda m_\lambda)}(\mathbb{C}) \\ &\hookrightarrow \mathrm{sp}_{2m_\lambda}(\mathbb{C}) \times \mathrm{sp}_{2(n-\delta_{\mathfrak{g}_1}-m_\lambda)}(\mathbb{C}) \hookrightarrow \mathrm{sp}_{2(n-\delta_{\mathfrak{g}_1})}(\mathbb{C})\end{aligned}$$

as required. ■

Proof of Proposition 4.1. Without loss of generality, we may assume that $\delta_{\mathfrak{g}_1} = \max\{\delta_{\mathfrak{g}_k}; 1 \leqslant k \leqslant r\}$. Observe also that r is at most the rank of $\mathrm{sp}(V)$, so $r \leqslant n$.

In view of Lemmas 4.4, 4.5 and 4.6, we obtain the result by induction on n. ■

5. The main result for $n \geqslant 5$

We prove in this section the following theorem. Recall that $\dim V = 2n$.

Theorem 5.1. *Assume that* $n \geqslant 5$. *Let* $\mathfrak{s}$ *be a semisimple Lie subalgebra of* $\mathrm{sp}(V)$ *such that* $\mathfrak{s} \cap \mathrm{usp}(V) = \{0\}$. *Then*

$$\dim \mathfrak{s} \leqslant n^2 - 1.$$

Moreover, the upper bound is attained, and equality holds if and only if $\mathfrak{s} \simeq \mathrm{sl}_n(\mathbb{C})$.

Proof. We may assume that $\mathfrak{s} \neq \{0\}$. As in Section 4, let $\mathfrak{g}_1, \dots, \mathfrak{g}_r$ be simple Lie algebras such that $\mathfrak{s} \simeq \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r$ and $\delta_{\mathfrak{g}_1} = \max\{\delta_{\mathfrak{g}_k}; 1 \leqslant k \leqslant r\}$. We shall use the notations of Section 4.

Case 1: $2\delta_{\mathfrak{g}_1} < n$.

By Proposition 4.1, there exists $\mathcal{I} \subset \{1, \dots, r\}$ such that

$$\dim \mathfrak{s} \leqslant \sum_{p \in \mathcal{I}} \Delta(\delta_{\mathfrak{g}_p}) \quad \text{and} \quad \sum_{p \in \mathcal{I}} \delta_{\mathfrak{g}_p} \leqslant n.$$

Observe that we have the following inequalities for $p, q \in \mathbb{N}^*$ verifying $p \geqslant q$:

$$\Delta(p) \geqslant \Delta(q) \ , \ \Delta(p+q) > \Delta(p) + \Delta(q) \ , \ \Delta(p+1) + \Delta(q-1) > \Delta(p) + \Delta(q). \quad (3)$$

Since $2\delta_{\mathfrak{g}_1} < n$ and $n \geqslant 5$, we obtain using the inequalities (3) that

$$\sum_{p \in \mathcal{I}} \Delta(\delta_{\mathfrak{g}_p}) \leqslant \Delta\left(\left[\frac{n}{2}\right]\right) + \Delta\left(\left[\frac{n-1}{2}\right]\right) + \Delta(1) < n^2 - 1.$$

Case 2: $2\delta_{\mathfrak{g}_1} \geqslant n$.

So there is only room for at most two non trivial symplectic representations of $\mathfrak{g}_1$ in V. Note that since $n \geqslant 5$, we have $d_\lambda \geqslant \delta_{\mathfrak{g}_1} \geqslant 3$.

Thus we have the following three distinct subcases :

(a) V contains a unique non trivial symplectic representation of $\mathfrak{g}_1$.

(b) V is the Φ-orthogonal direct sum of two non-isomorphic non trivial symplectic representations of $\mathfrak{g}_1$.

(c) V is the Φ-orthogonal direct sum of two isomorphic non trivial symplectic representations of $\mathfrak{g}_1$.

Let us suppose first that we are in subcase (a) or (b). Then there exist $\lambda, \mu \in P^+$ with λ non zero such that

$$V = V_\lambda \oplus V_\mu$$

and V_λ is a single non trivial symplectic representation of $\mathfrak{g}_1$. In particular, since $2\delta_{\mathfrak{g}_1} \geqslant n$, we have $m_\lambda \in \{1, 2\}$.

If $\mathfrak{g}_1$ is not of type C_ℓ, then we deduce from Table 1 that $\dim \mathfrak{g}_1 \leqslant \delta_{\mathfrak{g}_1}^2 - 1$, and we have equality if and only if $\mathfrak{g}_1$ is of type $A_{\delta_{\mathfrak{g}_1}-1}$. It follows therefore from Lemmas 4.4, 4.5(1) and 4.6(2) that

$$\dim \mathfrak{s} \leqslant \delta_{\mathfrak{g}_1}^2 - 1 + \Delta(n - \delta_{\mathfrak{g}_1}) = 3\delta_{\mathfrak{g}_1}^2 - (4n+1)\delta_{\mathfrak{g}_1} + 2n^2 + n - 1.$$

Since $n \leqslant 2\delta_{\mathfrak{g}_1} \leqslant 2n$, by studying the maximum of the function

$$t \mapsto 3t^2 - (4n+1)t + 2n^2 + n - 1$$

in the interval $[n/2, n]$, we deduce that

$$\dim \mathfrak{s} \leqslant n^2 - 1,$$

and equality holds if and only if $n = \delta_{\mathfrak{g}_1}$. We conclude that $\dim \mathfrak{s} \leqslant n^2 - 1$, and we have equality if and only if $\mathfrak{s} \simeq \mathfrak{g}_1 \simeq \mathrm{sl}_n(\mathbb{C})$.

If $\mathfrak{g}_1$ is of type C_ℓ, then $\ell = \delta_{\mathfrak{g}_1}$, and the second smallest dimension of a finite-dimensional simple $\mathfrak{g}_1$-module is $(\ell-1)(2\ell+1)$ (see for example [6, table 1]). Since $\ell \geqslant 3$, we have

$$\dim V_\lambda \leqslant \dim V = 2n \leqslant 4\ell < (\ell-1)(2\ell+1).$$

This implies $\dim V_\lambda = 2\delta_{\mathfrak{g}_1}$ and $\mathfrak{g}_1 = \mathrm{sp}(V_\lambda)$. Since $\mathfrak{g}_1 \cap \mathrm{usp}(V) \subset \mathfrak{s} \cap \mathrm{usp}(V) = \{0\}$, it follows by Proposition 2.1 that $\dim V_\lambda = 2\delta_{\mathfrak{g}_1} = n$. In particular, n is even.

Consequently, $\ker \pi_\lambda = \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r$, $\dim V_\mu = 2\delta_{\mathfrak{g}_1}$ and we have a Lie algebra embedding

$$\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r \hookrightarrow \mathrm{sp}(V_\mu).$$

Without loss of generality, we may assume that $\delta_{\mathfrak{g}_2} = \max\{\delta_{\mathfrak{g}_k}; 2 \leqslant k \leqslant r\}$. Suppose that $\delta_{\mathfrak{g}_2} < \delta_{\mathfrak{g}_1}$, then by Proposition 4.1, there exists $\mathcal{J} \subset \{2, \ldots, r\}$ such that

$$\dim \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r \leqslant \sum_{p \in \mathcal{J}} \Delta(\delta_{\mathfrak{g}_p}) \quad \text{and} \quad \sum_{p \in \mathcal{J}} \delta_{\mathfrak{g}_p} \leqslant \delta_{\mathfrak{g}_1}.$$

Using the inequalities (3) in Case 1, we obtain that

$$\sum_{p \in \mathcal{J}} \Delta(\delta_{\mathfrak{g}_p}) \leqslant \Delta(\delta_{\mathfrak{g}_1} - 1) + \Delta(1).$$

Hence $\dim \mathfrak{s} \leqslant \Delta(\delta_{\mathfrak{g}_1}) + \Delta(\delta_{\mathfrak{g}_1} - 1) + \Delta(1) < 4\delta_{\mathfrak{g}_1}^2 - 1 = n^2 - 1$ because $\delta_{\mathfrak{g}_1} \geqslant 3$.

Suppose that $\delta_{\mathfrak{g}_2} = \delta_{\mathfrak{g}_1}$. We claim that $\mathfrak{g}_2$ is not of type C_ℓ. In fact, if $\mathfrak{g}_2$ is of type C_ℓ, then $\ell = \delta_{\mathfrak{g}_2}$ and $\mathfrak{g}_2 \simeq \mathfrak{g}_1$. Consequently,

$$\dim \mathfrak{s} \geqslant 2 \dim \mathfrak{g}_1 = 2\Delta(\delta_{\mathfrak{g}_1}) = n^2 + n > \frac{\dim \mathrm{sp}(V)}{2}$$

which is absurd because $\mathfrak{s} \cap \mathrm{usp}(V) = \{0\}$.

So $\mathfrak{g}_2$ is not of type C_ℓ and by the argument above, we have $\dim \mathfrak{g}_2 \leqslant \delta_{\mathfrak{g}_2}^2 - 1$. Since $\delta_{\mathfrak{g}_2} = \delta_{\mathfrak{g}_1} = n/2 > 2$ and $\dim V_\mu = n$, we obtain by applying Lemmas 4.4, 4.5 and 4.6 to the embedding of $\mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r$ in $\mathrm{sp}(V_\mu)$, that

$$\dim \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r \leqslant \delta_{\mathfrak{g}_1}^2 - 1.$$

Hence
$$\dim \mathfrak{s} \leqslant \Delta(\delta_{\mathfrak{g}_1}) + \delta_{\mathfrak{g}_1}^2 - 1 < 4\delta_{\mathfrak{g}_1}^2 - 1 = n^2 - 1.$$

Finally, let us suppose that we are in subcase (c). Then there exists $\lambda \in P^+$ non zero such that $V = V_\lambda$. In particular, π_λ is injective and $2\delta_{\mathfrak{g}_1} = n$.

If $\lambda \neq -w(\lambda)$, then $m_\lambda = 2$ and $d_\lambda m_\lambda = n = \delta_{\mathfrak{g}_1} m_\lambda$. So by the embeddings at the end of the proof of Lemma 4.4, we obtain

$$\dim \mathfrak{s} \leqslant \Delta(\delta_{\mathfrak{g}_1}) + \dim \mathrm{sl}_{m_\lambda}(\mathbb{C}) = \Delta(\delta_{\mathfrak{g}_1}) + \Delta(1) < n^2 - 1$$

because $\delta_{\mathfrak{g}_1} \geqslant 3$.

If $\lambda = -w(\lambda)$, then $(m_\lambda, d_\lambda) = (2, 2\delta_{\mathfrak{g}_1})$ or $(m_\lambda, d_\lambda) = (4, \delta_{\mathfrak{g}_1})$. In both cases, we have $d_\lambda m_\lambda = 2n$, and so by the embeddings at the end of the proof of Lemma 4.5, we have

$$\dim \mathfrak{s} \leqslant \Delta(\delta_{\mathfrak{g}_1}) + \dim \mathrm{sp}_{m_\lambda}(\mathbb{C}) \leqslant \Delta(\delta_{\mathfrak{g}_1}) + \Delta(2) < n^2 - 1$$

because $\delta_{\mathfrak{g}_1} \geqslant 3$.

The proof is now complete since by Proposition 3.1, the upper bound is attained. ■

6. The cases $n = 1, 2, 3, 4$.

The condition $n \geqslant 5$ is essential in the proof of Theorem 5.1 for the conclusion in case 1 and to guarantee that $\delta_{\mathfrak{g}_1} \geqslant 3$ in case 2. In this section, we shall study the validity of Theorem 5.1 for $n = 1, 2, 3, 4$.

Let $\mathfrak{s}$ be a semisimple Lie algebra of $\mathrm{sp}_{2n}(\mathbb{C})$ verifying $\mathfrak{s} \cap \mathrm{usp}_{2n}(\mathbb{C}) = \{0\}$. By [3], $\dim \mathfrak{s}$ is at most $[\Delta(n)/2] - 1$. Since the rank of $\mathfrak{s}$ is at most n, we obtain by Table 1 the list of possible simple factors of $\mathfrak{s}$, and hence the possible dimensions for $\mathfrak{s}$. These are given in the following table.

n	1	2	3	4
$\left[\frac{\Delta(n)}{2}\right] - 1$	0	4	9	17
possible simple factors of $\mathfrak{s}$	–	A_1	A_1, A_2	A_1, A_2, A_3, C_2
maximal dimension possible of $\mathfrak{s}$	0	3	9	16
$n^2 - 1$	0	3	8	15

So Theorem 5.1 is valid for $n \leqslant 2$ by Proposition 3.1,

Now, any semsimple Lie algebra of dimension 8 is isomorphic to $\mathrm{sl}_3(\mathbb{C})$, and any semisimple Lie algebra of dimension 15 of rank less than or equal to 4 is isomorphic to $\mathrm{sl}_4(\mathbb{C})$. In view of Proposition 3.1, to show that Theorem 5.1 is valid for $n = 3$ and 4, we are reduced to proving that for $n = 3, 4$, any semisimple Lie subalgebra of dimension n^2 in $\mathrm{sp}_{2n}(\mathbb{C})$ has non trivial intersection with $\mathrm{usp}_{2n}(\mathbb{C})$.

Let us suppose that $n \in \{3, 4\}$ and let $\mathfrak{s}$ be a semsimple Lie subalgebras of dimension n^2 in $\mathrm{sp}_{2n}(\mathbb{C})$. Then $\mathfrak{s}$ is isomorphic to

$$\mathrm{sp}_2(\mathbb{C}) \times \mathrm{sp}_2(\mathbb{C}) \times \mathrm{sp}_2(\mathbb{C}) \text{ if } n = 3, \text{ and } \mathrm{sp}_4(\mathbb{C}) \times \mathrm{sp}_2(\mathbb{C}) \times \mathrm{sp}_2(\mathbb{C}) \text{ if } n = 4.$$

This can be obtained easily from the above table because such a semisimple Lie subalgebra must be of rank n, and so it corresponds to a root subsystem of cardinal $n^2 - n$ in the root system of type C_n. Moreover we check readily that these root subsystems are conjugate under the Weyl group. Thus there is single conjugacy class of such semisimple Lie subalgebras. In particular, this allows us to choose a semisimple Lie subalgebra $\mathfrak{s}$ in $\mathrm{sp}_{2n}(\mathbb{C})$ of dimension n^2 which is stable under complex conjugation and under conjugation by diagonal matrices in $\mathrm{Sp}_{2n}(\mathbb{C})$.

Recall from [4] that we have the Iwasawa decomposition $\mathrm{Sp}_{2n}(\mathbb{C}) = \mathbf{KAN} = \mathbf{KNA}$ where $\mathbf{K} = \mathrm{USp}_{2n}(\mathbb{C})$, $\mathbf{A} = \left\{\mathrm{diag}(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}); a_1, \ldots, a_n \in \mathbb{R}_+^*\right\}$ and

$$\mathbf{N} = \left\{ \begin{pmatrix} A & AS \\ 0 & {}^t A^{-1} \end{pmatrix}; \begin{array}{l} A, S \in \mathrm{M}_n(\mathbb{C}) \text{ with } S \text{ symmetric} \\ \text{and } A \text{ unipotent upper triangular} \end{array} \right\}$$

Elements of $\mathbf{K}$ normalize $\mathrm{usp}_{2n}(\mathbb{C})$, and by our choice of $\mathfrak{s}$, elements of $\mathbf{A}$ normalize $\mathfrak{s}$, thus to prove the validity of Theorem 5.1, it suffices to show that

$$g\mathfrak{s}g^{-1} \cap \mathrm{usp}_{2n}(\mathbb{C}) \neq \{0\}$$

for all $g \in \mathbf{N}$. Since $\mathbf{N} \simeq \mathbb{C}^{n^2} \simeq \mathbb{R}^{2n^2}$, a general element of $\mathbf{N}$ can be viewed as a matrix g_T with coefficients in $\mathbb{R}[T_1, \ldots, T_{2n^2}]$. Let $\mathcal{B} = (x_1, \ldots, x_{2n^2+n})$ be an $\mathbb{R}$-basis of $\mathrm{usp}_{2n}(\mathbb{C})$ and $\mathcal{S} = (y_1, \ldots, y_{n^2})$ be a $\mathbb{C}$-basis of $\mathfrak{s}$. There exist $L = (\lambda_{pq})_{p,q}, M = (\mu_{p,q}) \in \mathrm{M}_{2n^2+n,n^2}(\mathbb{R}[T_1, \ldots, T_{2n^2}])$ such that

$$g_T y_q g_T^{-1} = \sum_{p=1}^{2n^2+n} (\lambda_{p,q} + i\mu_{p,q})x_p$$

for $q = 1, \ldots, n^2$. Thus it boils down to showing that

$$\mathrm{rank}(g_T y_1 g_T^{-1}, \ldots, g_T y_{n^2} g_T^{-1}, ig_T y_1 g_T^{-1}, \ldots, ig_T y_{n^2} g_T^{-1}, x_1, \ldots, x_{2n^2+n}) < 4n^2 + n,$$

which is equivalent to

$$\mathrm{rank} \begin{pmatrix} L & -M & I_{2n^2+n} \\ M & L & 0 \end{pmatrix} < 4n^2 + n \text{ , or simply } \mathrm{rank}\, \Upsilon < 2n^2$$

where $\Upsilon = \begin{pmatrix} M & L \end{pmatrix}$.

- **Case** $n = 3$. We take

$$\mathfrak{s} = \left\{ \begin{pmatrix} D_1 & D_2 \\ D_3 & -D_1 \end{pmatrix} ; D_1, D_2, D_3 \in \mathrm{gl}_3(\mathbb{C}) \text{ diagonal} \right\} \simeq \mathrm{sp}_2(\mathbb{C}) \times \mathrm{sp}_2(\mathbb{C}) \times \mathrm{sp}_2(\mathbb{C}).$$

Observe that we have the decomposition $\mathbf{N} = \mathbf{N}_0\mathbf{N}_1$ where

$$\mathbf{N}_0 = \left\{ \begin{pmatrix} A & AS \\ 0 & {}^tA^{-1} \end{pmatrix} \in \mathbf{N} \text{ ; } S \in \mathrm{gl}_3(\mathbb{C}) \text{ symmetric with zeros on the diagonal} \right\}$$

and

$$\mathbf{N}_1 = \left\{ \begin{pmatrix} I_3 & D \\ 0 & I_3 \end{pmatrix} ; D \in \mathrm{gl}_3(\mathbb{C}) \text{ diagonal} \right\}.$$

Note that while $\mathbf{N}_1$ is a subgroup of $\mathbf{N}$, $\mathbf{N}_0$ is not a subgroup of $\mathbf{N}$.

Since elements of $\mathbf{N}_1$ normalize $\mathfrak{s}$, we can replace $\mathbf{N}$ by $\mathbf{N}_0$ and hence reduce the number of variables from 18 to 12. By choosing "standard" bases $\mathcal{B}$ and $\mathcal{S}$ for $\mathrm{usp}_6(\mathbb{C})$ and $\mathfrak{s}$ respectively, the coefficients of the 21 by 18 matrix Υ are elements of $\mathbb{Q}[T_1, \ldots, T_{12}]$ of degree at most 4.

Applying standard Gaussian elimination with polynomial coefficients on Υ, we are reduced to verifying that a certain 12 by 9 matrix Y has rank at most 8. This matrix Y has coefficients in $\mathbb{Q}[T_1, \ldots, T_{12}]$ of degree at most 5.

To reduce to a finite number of verifications on matrix with coefficients in $\mathbb{Q}$, we use the following obersvation.

Observation. Let $P \in \Bbbk[X_1, \ldots, X_m]$ and $A_1, \ldots, A_m \subset \Bbbk$ be non empty subsets such that for $1 \leqslant k \leqslant m$, we have $\sharp A_k > \deg_{X_k}(P)$, the degree of P in X_k. If $P(a_1, \ldots, a_m) = 0$ for all $(a_1, \ldots, a_m) \in A_1 \times \cdots \times A_m$, then $P = 0$.

We were able to compute an upper bound for the degree in each T_k for all 9 by 9 minors in Y which are respectively $10, 8, 10, 10, 8, 10, 4, 6, 6, 5, 6, 6$. By the above observation, to check that $\mathrm{rank}(Y) < 9$, it suffices to check that the matrix $Y(t_1, \ldots, t_{12})$ obtained by evaluating the coefficients of Y in $(t_1, \ldots, t_{12}) \in \mathbb{Q}^{12}$, has rank at most 8 for $-5 \leqslant t_1, t_3, t_4, t_6 \leqslant 5$, $-4 \leqslant t_2, t_5 \leqslant 4$, $-3 \leqslant t_8, t_9, t_{11}, t_{12} \leqslant 3$, $-2 \leqslant t_{10} \leqslant 3$ and $-2 \leqslant t_7 \leqslant 2$. The fact that $\mathfrak{s}$ is stable under complex conjugation allows us to assume further that $t_1, t_2, t_3, t_4, t_6 > 0$, $t_8 \geqslant 0$ and $t_3 \geqslant t_6$. There are still $10\,450\,944\,000$ cases to check. The computations were carried out using GAP4 with 112 cores in parallel by the supercomputer ROMEO[2]. It took about 10 hours and confirmed that $\mathrm{rank}(Y) < 9$.

Theorem 6.1. *Theorem* 5.1 *is valid for* $n \leqslant 3$.

- **Case** $n = 4$. We take

$$\mathfrak{s} = \left\{ \begin{pmatrix} D_1 & D_2 \\ D_3 & -D_1 \end{pmatrix} ; D_1, D_2, D_3 \in \mathbf{L} \text{ with } D_2, D_3 \text{ symmetric} \right\}$$

where
$$\mathbf{L} = \left\{ \begin{pmatrix} a & 0 & 0 & b \\ 0 & c & 0 & 0 \\ 0 & 0 & d & 0 \\ e & 0 & 0 & f \end{pmatrix} ; a, b, c, d, e, f \in \mathbb{C} \right\}.$$

Here we use the decomposition $\mathbf{N} = \mathbf{N}_0 \mathbf{N}_1$ with $\mathbf{N}_0$ consists of matrices of the form

$$\begin{pmatrix} A & AS \\ 0 & {}^t A^{-1} \end{pmatrix}$$

where
$$A = \begin{pmatrix} 1 & a & b & 0 \\ 0 & 1 & c & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & p & q & 0 \\ p & 0 & r & s \\ q & r & 0 & t \\ 0 & s & t & 0 \end{pmatrix}$$

and $\mathbf{N}_1$ consists of matrices of the form

$$\begin{pmatrix} A & AS \\ 0 & {}^t A^{-1} \end{pmatrix}$$

where $A, S \in \mathbf{L}$ with A unipotent upper triangular and S symmetric.

As in the case $n = 3$, $\mathbf{N}_0$ is not a subgroup of $\mathbf{N}$, and elements of $\mathbf{N}_1$ normalize $\mathfrak{s}$. This allows us to reduce the number of variables from 32 to 20. With Gaussian elimination, we managed to reduce to verifying that a certain 20 by 16 matrix Y has rank at most 15. The maximal degree in each T_k of all 16 by 16 minors in Y is

$$16, 16, 25, 21, 22, 16, 16, 25, 21, 22, 9, 10, 10, 14, 14, 9, 10, 10, 14, 14$$

respectively. Even with further reductions using the fact that $\mathfrak{s}$ is stable under complex conjugation, the number of cases to check is over 5×10^{22}, which is far too many to check !!!

However, we did check on many values, and the rank of $Y(t_1, \ldots, t_{20})$ is at most 15. Since we are not able to reduce to a checkable number of cases, we propose the following conjecture.

Conjecture 6.2. *Theorem* 5.1 *is valid for* $n = 4$.

[2] The implementation of the computations on the supercomputer ROMEO was done by Guillaume Dollé.

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