Unipotent Representations of
Exceptional Richardson Orbits

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Abstract.   We study special unipotent representations attached to complex exceptional Richardson orbits. As a consequence, we verify a conjecture of Achar and Sommers for these orbits.

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1.   Introduction

In [11], Barbasch and Vogan studied *special unipotent representations* for complex simple Lie groups G. These representations are of interest in various areas of representation theory. For instance, they are conjectured to be the 'building blocks' of the unitary dual of G. This idea is verified by Vogan [32] for $G = GL(n, \mathbb{F})$ where $\mathbb{F} = \mathbb{C}$, $\mathbb{R}$ and $\mathbb{H}$, and by Barbasch [5] for all complex classical groups. Moreover, these representations are conjectured to be quantization models of special nilpotent orbits (Conjecture 2.5). The conjecture was shown to be true by Barbasch [8] for classical nilpotent orbits, and by Losev, Mason-Brown and Matvieievskyi [19], [24] in general (Remark 2.7).

A main goal of this manuscript is to study a map defined by Lusztig and Vogan on nilpotent orbits of complex reductive Lie algebras. More explicitly, let $\mathcal{N}$ be the set of nilpotent elements of the Lie algebra $\mathfrak{g}$ of G, and $\mathcal{O} \subset \mathcal{N}$ be a nilpotent orbit. For any $e \in \mathcal{O}$, let G_e be the stabilizer subgroup of e under conjugation, and $(G_e)^\vee$ be the isomorphism class of irreducible, algebraic representations of G_e. They conjectured that there is a bijection between the sets

$$\Phi : \{(\mathcal{O}, \sigma) \mid \mathcal{O} \subset \mathcal{N}, \sigma \in (G_e)^\vee \text{ for any } e \in \mathcal{O}\} \longrightarrow \Lambda^+(G), \qquad (1)$$

where $\Lambda^+(G) \subset \mathfrak{t}^*$ be the collection of highest dominant weights of finite dimensional representations of G. The conjecture is later proved by Bezrukavnikov, and the image of Φ is computed explicitly in some cases – namely, Achar [1] computed the bijection $G = GL(n, \mathbb{C})$, and the calculations are later simplified by Rush [27]. On the other hand, Liang-Zhang [18] computed the bijection for minimal nilpotent orbits, and Zhang [38] computed the case of $G = G_2(\mathbb{C})$.

In [37], the author defined a map analogous to (1):

$$\Psi : \left\{(\mathcal{O}, \pi) \mid \mathcal{O} \subset \mathcal{N} \text{ special orbit}, \pi \in (\overline{A}(\mathcal{O}))^\vee\right\} \longrightarrow \Lambda^+(G),$$

where $\mathcal{O}$ is a special orbit in the sense of Lusztig, and $\overline{A}(\mathcal{O})$ is Lusztig's quotient of the component group $A(\mathcal{O}) := G_e/(G_e)^0$ of G_e (c.f. [20, Chapters 4, 13]). To relate Ψ to special unipotent representations and quantization, note that the domain of Ψ is in one-to-one correspondence with the special unipotent representations of G. Indeed, given the validity of the quantization conjecture mentioned above, then Ψ is equal to Φ for $\pi \in (G_e)^\vee$ that factors through $G_e \to G_e/(G_e)^0 = A(\mathcal{O}) \to \overline{A}(\mathcal{O})$. In such cases, one can effectively compute (part of) Φ by understanding Ψ. This idea is pursued in [37] for classical nilpotent orbits, so that one can prove a conjecture by Achar and Sommers [2] on the image of Φ for classical Lie groups.

In this manuscript, we compute the image of Ψ explicitly for all complex exceptional Richardson orbits. As a consequence, we prove that the Achar-Sommers conjecture also holds for these orbits (Theorem 2.10).

2.   Preliminaries

2.1. Special unipotent representations

Let G be a complex simple Lie group, with maximal compact subgroup K. We recall the construction of irreducible, admissible $(\mathfrak{g}_\mathbb{C}, K_\mathbb{C})$-modules, where $K_\mathbb{C} \cong G$ is the complexification of K.

Let $H = TA$ be the Cartan decomposition of the Cartan subgroup H of G, with $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$. We make the following identifications:

$$\mathfrak{h}_\mathbb{C} \cong \mathfrak{h} \times \mathfrak{h}, \quad \mathfrak{t}_\mathbb{C} = \{(x, -x) : x \in \mathfrak{h}\} \cong \mathfrak{h}, \quad \mathfrak{a}_\mathbb{C} \cong \{(x, x) : x \in \mathfrak{h}\} \cong \mathfrak{h}.$$

Let $(\lambda_L, \lambda_R) \in \mathfrak{h}^* \times \mathfrak{h}^*$ such that $\eta := \lambda_L - \lambda_R$ is integral, and write

$$\{\eta\} := w'\eta \tag{2}$$

as the unique dominant weight to which η is conjugate under the action of the Weyl group $w' \in W := W(\mathfrak{g}, \mathfrak{h})$. Write $\nu := \lambda_L + \lambda_R$. We can view η as a weight of T and ν a character of A. Put

$$X(\lambda_L, \lambda_R) := \mathrm{Ind}_B^G(\mathbb{C}_\eta \otimes \mathbb{C}_\nu \otimes \mathrm{triv})_{K-\mathrm{finite}},$$

where $B \supset H$ is the Borel subgroup of G determined by a choice of positive roots Δ_G^+. Then we define $J(\lambda_L, \lambda_R)$ be the unique irreducible subquotient of $X(\lambda_L, \lambda_R)$ containing the $K_\mathbb{C} \cong G$-type $V_{\{\eta\}}^G$. By [39], every irreducible admissible $(\mathfrak{g}, K)$-module has the form $J(\lambda_L, \lambda_R)$. We will refer to the pair (λ_L, λ_R) as the *Zhelobenko parameter* for the module $J(\lambda_L, \lambda_R)$.

Among all $J(\lambda_L, \lambda_R)$'s, we focus on the following collection of irreducible modules:

Definition 2.1.   Let G be a complex Lie group, and ${}^L\mathcal{O} \subset {}^L\mathfrak{g}$ be a special nilpotent orbit in the Langlands dual of $\mathfrak{g}$. Writing $\mathcal{O}$ as the Lusztig-Spaltenstein dual of ${}^L\mathcal{O}$, and $\lambda_\mathcal{O} := \frac{{}^L H}{2}$ as one half of the semisimple element in a Jacobson-Morozov triple attached to ${}^L\mathcal{O}$. Then the *special unipotent representations* attached to ${}^L\mathcal{O}$ are given by the set

$$\Pi({}^L\mathcal{O}) := \{J(\lambda_\mathcal{O}, w\lambda_\mathcal{O}) \mid AV(J(\lambda_\mathcal{O}, w\lambda_\mathcal{O})) \subseteq \overline{\mathcal{O}}\},$$

where $AV(X)$ is the associated variety of any $(\mathfrak{g}_\mathbb{C}, K_\mathbb{C})$-module X (cf. [34, Sec. 2]).

By Corollary 5.18 of [11], $J(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}})$ have associated variety greater than or equal to $\overline{\mathcal{O}}$ for all $w \in W$. Therefore $\Pi(^L\mathcal{O})$ consists of irreducible representations $J(\lambda_{\mathcal{O}}, w\lambda_{\mathcal{O}})$'s whose associated varieties are *precisely* equal to $\overline{\mathcal{O}}$. Note that by Theorem 1.5 of [11], $J(\lambda, w\mu) \cong J(x\lambda, xw\mu)$ for all $x \in W$. So we can begin with any W-conjugate of λ.

Here is the main theorem of [11]:

Theorem 2.2. ([11], Theorem III) *The cardinality of $\Pi(^L\mathcal{O})$ is equal to $|\overline{A}(\mathcal{O})^\vee|$, i.e. the number of irreducible, non-isomorphic representations of $\overline{A}(\mathcal{O})$. Moreover, by writing*

$$\Pi(^L\mathcal{O}) = \{X_{\mathcal{O},\pi} \mid \pi \in \overline{A}(\mathcal{O})^\vee\}, \tag{3}$$

then the character formula for each $X_{\mathcal{O},\pi}$ can be obtained explicitly.

A detailed proof of Theorem 2.2 is given by [11] in the case when $\lambda_{\mathcal{O}}$ is integral, i.e. $^L\mathcal{O}$ is an even orbit. In the Appendix, we complete the proof of the theorem for non-integral $\lambda_{\mathcal{O}}$.

2.2. Richardson orbits

Although one can get the character formula for all $X_{\mathcal{O},\pi}$ by Theorem 2.2, its expression can be quite complicated. Therefore, we focus on the case when $\mathcal{O}$ is Richardson, i.e, there exists a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ such that $\mathcal{O}$ is the dense orbit in the G-saturation $G \cdot \mathfrak{n}$ of $\mathfrak{n}$. In particular, their closures can be obtained by the image of the moment map

$$\mu : T^*(G/P) \cong G \times_P \mathfrak{n} \to \overline{\mathcal{O}} \tag{4}$$

given by $\mu(g, X) := g \cdot X$ for some parabolic group $P = LN$. We would like to investigate when the above map is birational:

Proposition 2.3. *Let $\mathcal{O}$ be an exceptional Richardson orbit other than*

$$A_4 + A_1, \ D_5(a_1) \ in \ E_7, \quad E_6(a_1) + A_1, \ E_7(a_3) \ in \ E_8. \tag{5}$$

Then there exists a parabolic subgroup P such that the moment map μ in (4) is birational onto $\overline{\mathcal{O}}$.

Proof. Our choices of P for all Richardson orbits other than the four orbits in (5) are listed in Sections 4–8. It is known that if $[G^e : P^e] = 1$, then μ is birational. This holds when $G_e/(G_e)^0 = A(\mathcal{O}) = 1$. Also, by the results in [25], the map is birational when $\mathcal{O}$ is even (so that the Levi subgroup L is given by the nodes marked with 0's). By checking the tables in [14], we are only left with

$$D_4(a_1) + A_1 \ in \ E_7; \quad D_6(a_1), D_7(a_2) \ in \ E_8 \tag{6}$$

along with the four orbits mentioned in the proposition.

Indeed, we can apply Lemma 5.5 in [16] to check birationality of μ, which says that for any $e \in \mathcal{O}$, the number of components in $\mu^{-1}(e)$ is equal to

$$\sum_{\psi \in \widehat{A(e)}} \dim V_\psi \cdot [\sigma_{\mathcal{O},\psi} : \operatorname{Ind}_{W(L)}^W(\operatorname{sgn})],$$

where $\sigma_{\mathcal{O},\psi}$ is the Springer representation $H^{top}(\mathcal{B}_e)^\psi$.

One can use the table in [3] to check that the above value is equal to 1 for the orbits in (6). Therefore μ is birational for all the orbits stated in the proposition. ∎

For all exceptional Richardson $\mathcal{O}$ other than the four orbits in (5), Theorem 9.11 of [11] provides a nice character formula for all $\Pi(^L\mathcal{O})$.

Theorem 2.4. *Let $\mathcal{O}$ be an exceptional Richardson orbit not equal to four orbits in (5), and $\mathfrak{l}$ be the Levi subalgebra of the parabolic subgroup corresponding to $\mathcal{O}$ given by Proposition 2.3, then the following statements hold:*

(a) *The number of elements in*

$$P(\mathcal{O}) := \{w\lambda_{\mathcal{O}} \mid w\lambda_{\mathcal{O}}|_{\mathfrak{l}} \text{ is dominant and regular}\}$$

is equal to the number of conjugacy classes of $\overline{A}(\mathcal{O})$. In particular, there exists a unique λ_1 such that $(\lambda_1, \alpha^\vee) = 1$ for all simple roots in $\mathfrak{l}$, and $\lambda_1 - \lambda$ is a sum of positive roots for all $\lambda \in P(\mathcal{O})$.

(b) *The elements in $P(\mathcal{O}) = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$ can be arranged such that we have $|\lambda_i - \lambda_1| < |\lambda_{i+1} - \lambda_1|$ for all i.*

(c) *The representations $J(\lambda_1, \lambda_i)$ exhaust all special unipotent representations in $\Pi(^L\mathcal{O})$. More precisely, suppose $\overline{A}(\mathcal{O}) \cong S_k$ for $2 \leq k \leq 5$, and $C_1 = \langle k \rangle > \cdots > C_r = \langle 1^k \rangle$ is the total ordering of partitions of k (such ordering exists since $k \leq 5$). Then $\pi_i \in \overline{A}(\mathcal{O})^\vee$ are parametrized by the partition C_i, with*

$$X_{\mathcal{O}, \pi_i} := J(\lambda_1, \lambda_i) \cong \mathrm{Ind}_L^G(V_i)$$

for some finite-dimensional representations V_i of L with the highest weight $(\lambda_1 - \lambda_i)|_{\mathfrak{l}}$. In particular, when $i = 1$, then π_1 is the trivial representation of $\overline{A}(\mathcal{O})^\vee$ and V_1 is the trivial representation of L.

Proof. The values of λ_i in (a) and (b) are computed in Sections 4–8. Assuming the results of (a) and (b) hold, then for each λ_i, let I_i be as defined in Section 9 of [11] such that the character formula of I_i is given by

$$\sum_{w \in W(L)} \mathrm{sgn}(w) X(\lambda_1, w\lambda_i), \tag{7}$$

so that $I_i \cong \mathrm{Ind}_L^G(V_{\lambda_1 - \lambda_i}^L)$ as $K_{\mathbb{C}} \cong G$-modules. Obviously, we have $AV(I_i) = \overline{\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(0)} = \overline{\mathcal{O}}$, and its composition factors must consist of special unipotent representations attached to $\mathcal{O}$. Using the arguments in 9.11 - 9.21 of [11], one can see $I_i = J(\lambda_1, \lambda_i)$ and (c) follows. ∎

2.3. Vogan's conjecture on quantization

As mentioned in the introduction, the special unipotent representations $\Pi(^L\mathcal{O})$ are conjectured to be the 'quantizations' of (local systems of) $\mathcal{O}$. More precisely, we have the following conjecture by Vogan in the 1980s:

Conjecture 2.5. [[34], Conjecture 12.1] *Let $\mathcal{O}$ be a complex special nilpotent orbit, and π be an irreducible representation of $\overline{A}(\mathcal{O})$. Then there exists an irreducible representation W_ψ of G_e that factors through $G_e/(G_e)_0 = A(\mathcal{O})$ such that:*
$$X_{\mathcal{O},\pi}|_{K_{\mathbb{C}}} \cong R(\mathcal{O}, \psi) \cong \mathrm{Ind}_{G_e}^G(W_\psi),$$

where $R(\mathcal{O}, \psi)$ is the global section of the vector bundle $G \times_{G_e} W_\psi \to G/G_e \cong \mathcal{O}$.

To provide some evidences on the validity of Conjecture 2.5, the results in [30] imply that the irreducible L-modules V_i in Theorem 2.4(c) are all 'lifts' of some $W_{\psi_i} \in A(\mathcal{O})^\vee$ in Conjecture 2.5. Namely, the restricted representation $V_i|_{L_e}$ factors through $L_e/(L_e)^0 \cong G_e/(G_e)^0$, and is isomorphic to W_{ψ_i}. Indeed, for triv $\in \overline{A}(\mathcal{O})^\vee$, one can easily verify the conjecture for exceptional Richardson orbits:

Theorem 2.6. *Let $\mathcal{O}$ be an exceptional Richardson orbit. Then $X_{\mathcal{O},\mathrm{triv}} \cong R(\mathcal{O})$.*

Proof. When $\mathcal{O}$ is not equal to the four orbits in (5), the map $\mu : T^*(G/P) \to \overline{\mathcal{O}}$ is the normalization of the orbit closure $\overline{\mathcal{O}}$. By standard arguments in algebraic geometry (see [17] for example), $R(\mathcal{O}) \cong \mathbb{C}[T^*(G/P)]$. By a result of [25], the latter is isomorphic to $\mathrm{Ind}_L^G(\mathrm{triv})$ as G-modules. This means $R(\mathcal{O}) \cong \mathrm{Ind}_L^G(\mathrm{triv}) \cong X_{\mathcal{O},\mathrm{triv}}$ and therefore Theorem 2.6 holds for these orbits. We will get the same conclusion for the four orbits in (5) in Section 9. ∎

Remark 2.7. As mentioned in the introduction, Conjecture 2.5 is proved in full generality in [19] and [24]. More explicitly, a definition of *unipotent representations* for G-equivariant covers of nilpotent orbits is given in [19, Definition 1.4.1]. It is shown in [19] (for classical groups) and [24] (for exceptional and spin groups) that all special unipotent representations $X_{\mathcal{O},\pi}$ in Definition 2.1 are unipotent representations attached to some covers of $\mathcal{O}$. Then by [19, Section 6.7], Conjecture 2.5 holds for all special unipotent representations.

2.4. The Lusztig-Vogan map

We now give a precise description of the map (1) given by Theorem 8.2 of [35]:

Definition 2.8. For each nilpotent element $\mathcal{O} \subset \mathfrak{g}$, and $W_\sigma \in (G_e)^\vee$ for any $e \in \mathcal{O}$, write

$$R(\mathcal{O},\sigma) \cong \mathrm{Ind}_{G^e}^G(W_\sigma) \cong \sum_{\lambda \in \Lambda^+(G)} m_{\mathcal{O},\sigma}(\lambda)\mathrm{Ind}_T^G(e^\lambda), \tag{8}$$

where all but finitely many $m_{\mathcal{O},\sigma}(\lambda) \in \mathbb{Z}$ are zero. Then the *Lusztig-Vogan map*

$$\Phi : \{(\mathcal{O},\sigma) \mid \mathcal{O} \subset \mathcal{N}, \sigma \in (G_e)^\vee \text{ for any } e \in \mathcal{O}\} \to \Lambda^+(G)$$

is defined by $\Phi(\mathcal{O},\sigma) = \lambda_{max}$, where $\lambda_{\max}$ is the largest dominant element in Equation (8) such that $m_{\mathcal{O},\sigma}(\lambda_{\max}) \neq 0$. ∎

The paper [37] studies Φ for all classical nilpotent orbits $\mathcal{O}$, and all $(G_e)^\vee$ that factors through $(G_e/(G_e)^0)^\vee = A(\mathcal{O})^\vee$, and consequently proved the following conjecture of Achar and Sommers for classical groups:

Conjecture 2.9. [2] *Let $\mathcal{O}$ be a special nilpotent orbit. Consider the special piece $\mathcal{S}_{{}^L\mathcal{O}}$ ([21]) of the Lusztig-Spaltenstein dual ${}^L\mathcal{O}$ of $\mathcal{O}$:*

$$S_{{}^L\mathcal{O}} := \{{}^L\mathcal{P} \subseteq \overline{{}^L\mathcal{O}} | {}^L\mathcal{P} \nsubseteq \overline{{}^L\mathcal{O}_{sp}} \text{ for other special } {}^L\mathcal{O}_{sp} \subsetneq \overline{{}^L\mathcal{O}}\}.$$

For each ${}^L\mathcal{P} \in \mathcal{S}_{{}^L\mathcal{O}}$, let $H_{{}^L\mathcal{P}}$ be the semi-simple element of a Jacobson-Morozov triple corresponding to ${}^L\mathcal{P}$. Then there exists $W_\psi \in A(\mathcal{O})^\vee$ (which can be seen as a representation of G_e that factors through $A(\mathcal{O})$) such that $\Phi(\mathcal{O},\psi) = H_{{}^L\mathcal{P}}$.

Let us describe how the above conjecture is proved in [37] for classical groups. In [37, Definition 1.5], the author defined the map

$$\Psi : \{(\mathcal{O}, \pi) \mid \mathcal{O} \text{ special nilpotent orbit}, \pi \in \overline{A}(\mathcal{O})^\vee\} \longrightarrow \Lambda^+(G), \qquad (9)$$

as follows: Suppose the character formula $X_{\mathcal{O},\pi}$ is given by

$$\sum_{w \in W} a_{\mathcal{O},\pi}(w) X(\lambda_\mathcal{O}, w\lambda_\mathcal{O}).$$

On the level of Grothendieck groups, the restriction of $X_{\mathcal{O},\pi}$ into $K_\mathbb{C}$-modules is of the form

$$X_{\mathcal{O},\pi}|_{K_\mathbb{C}} = \sum_{w \in W} a_{\mathcal{O},\pi}(w)\mathrm{Ind}_T^G(\{\lambda_\mathcal{O} - w\lambda_\mathcal{O}\}) = \sum_{\lambda \in \Lambda^+(G)} b_{\mathcal{O},\pi}(\lambda)\mathrm{Ind}_T^G(e^\lambda), \qquad (10)$$

where $\{\lambda_\mathcal{O} - w\lambda_\mathcal{O}\}$ is as defined in (2), and $b_{\mathcal{O},\pi}(\lambda) \in \mathbb{Z}$. We define $\Psi(\mathcal{O}, \pi) := \lambda_{\max}$, where $\lambda_{\max}$ is the largest dominant element in (10) such that $b_{\mathcal{O},\pi}(\lambda_{\max}) \neq 0$.

Since Conjecture 2.5 holds for all nilpotent orbits, one can relate the two maps Ψ and Φ by Equations (8) and (10). As a consequence, one can verify Conjecture 2.9 by computing Ψ. This is done for all classical special nilpotent orbits $\mathcal{O}$ and all $\pi \in \overline{A}(\mathcal{O})^\vee$ in [37]. In this manuscript, we apply the same strategy for exceptional Richardson orbits and obtain the following:

Theorem 2.10. *Let G be a complex exceptional Lie group of adjoint type, and $\mathcal{O}$ be a Richardson orbit. For every semisimple element $H_{^L\mathcal{P}}$ in a Jacobson-Morozov triple corresponding to $^L\mathcal{P} \in \mathcal{S}_{L\mathcal{O}}$, there exists $\pi \in \overline{A}(\mathcal{O})^\vee$ such that $\Psi(\mathcal{O}, \pi) = H_{^L\mathcal{P}}$. Consequently, Conjecture 2.9 holds for all exceptional Richardson orbits.*

3.   Proof of Theorem 2.10 - general case

In Sections 4–8, we write down the details for Theorem 2.4 and Theorem 2.10 for all Richardson orbits other than the four orbits in (5). Namely, for each exceptional Richardson orbit $\mathcal{O}$, we list

- the Levi subalgebra $\mathfrak{l}$ where $\mathcal{O}$ is induced from, by specifying a sub-diagram of the Dynkin diagram of $\mathfrak{g}$;

- all irreducible representations π_i of $\overline{A}(\mathcal{O}) \cong S_k$ $(k = 2, 3, 4, 5)$ in terms of partitions of k;

- the values of λ_i appearing in Theorem 2.4(a)–(b).

This verifies Theorem 2.4.

We now turn our attention to Theorem 2.10. By Equation (7) and (10), the image of Ψ can be easily computed as $\Psi(\mathcal{O}, \pi_i) = \{\lambda_1 - w_L(\lambda_i)\}$ for all $\pi_i \in \overline{A}(\mathcal{O})^\vee$, where w_L is the longest element in the Weyl group $W(L)$. The values of $\Psi(\mathcal{O}, \pi_i)$ are recorded on the second last column of the tables in terms of the weighted Dynkin diagram of $^L\mathfrak{g}$. And the last column records the orbit $^L\mathcal{P}$ whose Dynkin element $H_{^L\mathcal{P}}$ is given by the previous column, which verifies Theorem 2.10 for all but four exceptional Richardson orbits given in (5).

4. G_2 orbits

The exceptional group G_2 has three Richardson orbits.

Fix the Dynkin diagram of $\mathfrak{g}$ by:
$$\overset{(-2,1,1)\;(1,-1,0)}{\underset{3}{\bullet\!-\!\!-\!\!\bullet}}$$

$\mathcal{O}$	$\mathfrak{l}$	π_i	λ_i	$\Psi(\mathcal{O},\pi_i)$	$^L\mathcal{P}$
G_2	ϕ	$\langle 1\rangle$	$(0,0,0)$	$\overset{0\quad\ 0}{\underset{3}{\bullet\!-\!\!\bullet}}$	0
$G_2(a_1)$	$\underset{(1,\text{-}1,0)}{\bullet}$	$\langle 3\rangle$	$(0,-1,1)$	$\overset{0\quad\ 1}{\underset{3}{\bullet\!-\!\!\bullet}}$	A_1
		$\langle 2,1\rangle$	$(1,-1,0)$	$\overset{1\quad\ 0}{\underset{3}{\bullet\!-\!\!\bullet}}$	$\widetilde{A}_1$
		$\langle 1,1,1\rangle$	$(1,0,-1)$	$\overset{0\quad\ 2}{\underset{3}{\bullet\!-\!\!\bullet}}$	$G_2(a_1)$
0	$\overset{(\text{-}2,1,1)\ \ (1,\text{-}1,0)}{\underset{3}{\bullet\!-\!\!\bullet}}$	$\langle 1\rangle$	$(-1,-2,3)$	$\overset{2\quad\ 2}{\underset{3}{\bullet\!-\!\!\bullet}}$	G_2

5. F_4 orbits

The exceptional group F_4 has nine Richardson orbits. Fix the Dynkin diagram of $\mathfrak{g}$ by

$\overset{1\ \ 2\ \ 3\ \ 4}{\underset{2}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}}$ with simple roots: $\underset{(0,1-1,0)}{\bullet}\!-\!\!\!\overset{(0,0,1,-1)\,(0,0,0,1)}{\underset{2}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}}\underset{(\frac{1}{2},\frac{-1}{2}\frac{-1}{2},\frac{-1}{2})}{}$

$\mathcal{O}$	$\mathfrak{l}$	π_i	λ_i	$\Psi(\mathcal{O},\pi_i)$	$^L\mathcal{P}$
F_4	ϕ	$\langle 1\rangle$	$(0,0,0,0)$	$\overset{0\ \ 0\ \ 0\ \ 0}{\underset{2}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}}$	0
$F_4(a_1)$	$\overset{3}{\bullet}$	$\langle 2\rangle$	$(\tfrac{1}{2},0,0,\tfrac{1}{2})$	$\overset{0\ \ 0\ \ 0\ \ 1}{\underset{2}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}}$	A_1
		$\langle 1,1\rangle$	$(\tfrac{-1}{2},0,0,\tfrac{1}{2})$	$\overset{1\ \ 0\ \ 0\ \ 0}{\underset{2}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}}$	$\widetilde{A}_1$
$F_4(a_2)$	$\overset{1}{\bullet}\ \ \overset{3}{\bullet}$	$\langle 1\rangle$	$(0,\tfrac{1}{2},\tfrac{-1}{2},\tfrac{1}{2})$	$\overset{0\ \ 0\ \ 1\ \ 0}{\underset{2}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}}$	$A_1+\widetilde{A}_1$
C_3	$\overset{3\ \ 4}{\bullet\!\!-\!\!\bullet}$	$\langle 1\rangle$	$(\tfrac{1}{2},\tfrac{-1}{2},\tfrac{-1}{2},\tfrac{1}{2})$	$\overset{0\ \ 0\ \ 0\ \ 2}{\underset{2}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}}$	A_2
B_3	$\overset{1\ \ 2}{\bullet\!\!-\!\!\bullet}$	$\langle 1\rangle$	$(0,1,0,-1)$	$\overset{2\ \ 0\ \ 0\ \ 0}{\underset{2}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet}}$	$\widetilde{A}_2$

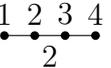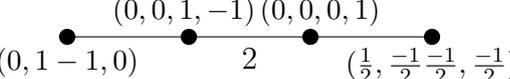| $\mathcal{O}$	$\mathfrak{l}$	π_i	λ_i	$\Psi(\mathcal{O},\pi_i)$	$^L\mathcal{P}$
$F_4(a_3)$ $\overset{1}{\bullet}$ $\overset{3}{\bullet}\!-\!\overset{4}{\bullet}$		$\langle 4\rangle$	$(\frac{3}{2},\frac{1}{2},\frac{-1}{2},\frac{1}{2})$	$\overset{0\ \ 1\ \ 0\ \ 0}{\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet}$ 2	$A_2+\widetilde{A}_1$
		$\langle 3,1\rangle$	$(1,0,-1,1)$	$\overset{1\ \ 0\ \ 1\ \ 0}{\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet}$ 2	$\widetilde{A}_2+A_1$
		$\langle 2,2\rangle$	$(\frac{1}{2},\frac{1}{2}\frac{-3}{2},\frac{1}{2})$	$\overset{1\ \ 0\ \ 0\ \ 2}{\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet}$ 2	B_2
		$\langle 2,1,1\rangle$	$(\frac{1}{2},\frac{-1}{2}\frac{-3}{2},\frac{1}{2})$	$\overset{0\ \ 1\ \ 0\ \ 1}{\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet}$ 2	$C_3(a_1)$
		$\langle 1,1,1,1\rangle$	$(\frac{-1}{2},\frac{-1}{2}\frac{-3}{2},\frac{1}{2})$	$\overset{0\ \ 0\ \ 2\ \ 0}{\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet}$ 2	$F_4(a_3)$
$\tilde{A}_2$	$\overset{2\ \ 3\ \ 4}{\bullet\!-\!\bullet\!-\!\bullet}\ 2$	$\langle 1\rangle$	$(\frac{3}{2},\frac{-3}{2},\frac{3}{2},\frac{1}{2})$	$\overset{2\ \ 2\ \ 0\ \ 0}{\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet}$ 2	B_3
A_2	$\overset{1\ \ 2\ \ 3}{\bullet\!-\!\bullet\!-\!\bullet}\ 2$	$\langle 1\rangle$	$(0,\frac{5}{2},\frac{3}{2},\frac{1}{2})$	$\overset{2\ \ 1\ \ 0\ \ 1}{\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet}$ 2	C_3
0	$\overset{1\ \ 2\ \ 3\ \ 4}{\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet}\ 2$	$\langle 1\rangle$	$(\frac{11}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2})$	$\overset{2\ \ 2\ \ 2\ \ 2}{\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet}$ 2	F_4

6. E_6 orbits

The adjoint exceptional group E_6 has fifteen Richardson orbits.

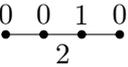Fix the Dynkin diagram $\overset{\ \ \ \ 2}{\underset{6\ 5\ 4\ 3\ 1}{\bullet\ \bullet\ \overset{\bullet}{\bullet}\ \bullet\ \bullet}}$ with simple roots:

$(0,0,0,1,1,0)$ top node;
$(0,1,-1,0,0,0)$, $(0,0,0,1,-1,0)$;
$(1,-1,0,0,0,0)$, $(0,0,1,-1,0,0)$, $(\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{1}{2},\frac{-\sqrt{3}}{2})$.

$\mathcal{O}$	$\mathfrak{l}$	π_i	λ_i	$\Psi(\mathcal{O},\pi_i)$	$^L\mathcal{P}$
E_6	ϕ	$\langle 1\rangle$	$(0,0,0,0,0,0)$	$\overset{0}{\underset{0\ 0\ 0\ 0\ 0}{\bullet\ \bullet\ \overset{\bullet}{\bullet}\ \bullet\ \bullet}}$	0
$E_6(a_1)$	$\overset{\bullet}{5}$	$\langle 1\rangle$	$(0,\frac{1}{2},\frac{-1}{2},0,0,0)$	$\overset{1}{\underset{0\ 0\ 0\ 0\ 0}{\bullet\ \bullet\ \overset{\bullet}{\bullet}\ \bullet\ \bullet}}$	A_1
D_5	$\overset{\overset{2}{\bullet}}{\underset{\overset{\bullet}{5}}{}}$	$\langle 1\rangle$	$(0,\frac{1}{2},\frac{-1}{2},\frac{1}{2},\frac{1}{2},0)$	$\overset{0}{\underset{1\ 0\ 0\ 0\ 1}{\bullet\ \bullet\ \overset{\bullet}{\bullet}\ \bullet\ \bullet}}$	$2A_1$
$E_6(a_3)$	$\overset{2}{\bullet}$	$\langle 2\rangle$	$(0,1,0,1,0,0)$	$\overset{0}{\underset{0\ 0\ 1\ 0\ 0}{\bullet\ \bullet\ \overset{\bullet}{\bullet}\ \bullet\ \bullet}}$	$3A_1$
	$\overset{\bullet}{5}\quad\overset{\bullet}{3}$	$\langle 1^2\rangle$	$(0,0,-1,1,0,0)$	$\overset{2}{\underset{0\ 0\ 0\ 0\ 0}{\bullet\ \bullet\ \overset{\bullet}{\bullet}\ \bullet\ \bullet}}$	A_2

$\mathcal{O}$	$\mathfrak{l}$	π_i	λ_i	$\Psi(\mathcal{O},\pi_i)$	$^L\mathcal{P}$
$D_5(a_1)$	(diagram: 6—5, 2 above)	$\langle 1 \rangle$	$(1,0,-1,\frac{1}{2},\frac{1}{2},0)$	(diagram: 1 0 0 0 1, 1 above)	$A_2 + A_1$
$A_4 + A_1$	(diagram: 6—5, 2 above, 3)	$\langle 1 \rangle$	$(1,0,-1,1,0,0)$	(diagram: 1 0 0 1 0, 0 above)	$A_2 + 2A_1$
D_4	(diagram: 6—5, 3—1)	$\langle 1 \rangle$	$(\frac{1}{2},\frac{-1}{2},\frac{-3}{2},\frac{1}{2},\frac{-1}{2},\frac{-\sqrt{3}}{2})$	(diagram: 2 0 0 0 2, 0 above)	$2A_2$
A_4	(diagram: 6—5—4)	$\langle 1 \rangle$	$(\frac{3}{2},\frac{1}{2},\frac{-1}{2},\frac{-3}{2},0,0)$	(diagram: 1 0 0 0 1, 2 above)	A_3
$D_4(a_1)$	(diagram: 6—5, 5—1, 3 above)	$\langle 3 \rangle$	$(1,0,-1,1,0,-\sqrt{3})$	(diagram: 0 1 0 1 0, 0 above)	$2A_2 + A_1$
		$\langle 21 \rangle$	$(\frac{1}{2},\frac{-1}{2},\frac{-3}{2},\frac{3}{2},\frac{1}{2},\frac{-\sqrt{3}}{2})$	(diagram: 1 0 0 0 1, 2 above)	$A_3 + A_1$
		$\langle 1^3 \rangle$	$(0,-1,-2,1,0,0)$	(diagram: 0 0 2 0 0, 0 above)	$D_4(a_1)$
A_3	(diagram: 6—5—4, 2 above)	$\langle 1 \rangle$	$(2,1,0,-1,2,0)$	(diagram: 2 0 0 0 2, 2 above)	A_4
$A_2 + 2A_1$	(diagram: 6—5—4, 2 above, 1)	$\langle 1 \rangle$	$(\frac{7}{4},\frac{3}{4},\frac{-1}{4},\frac{-5}{4},\frac{9}{4},\frac{-\sqrt{3}}{4})$	(diagram: 1 1 0 1 1, 1 above)	$A_4 + A_1$
$2A_2$	(diagram: 5—4—3, 2 above)	$\langle 1 \rangle$	$(0,3,2,1,0,0)$	(diagram: 0 0 2 0 0, 2 above)	D_4
A_2	(diagram: 6—5—4—3—1)	$\langle 2 \rangle$	$(\frac{3}{2},\frac{1}{2},\frac{-1}{2},\frac{-3}{2},\frac{-5}{2},\frac{-3\sqrt{3}}{2})$	(diagram: 1 1 0 1 1, 2 above)	A_5
		$\langle 1^2 \rangle$	$(1,0,-1,-2,-3,-\sqrt{3})$	(diagram: 2 0 2 0 2, 0 above)	$E_6(a_3)$
$2A_1$	(diagram: 6—5—4—3, 2 above)	$\langle 1 \rangle$	$(4,3,2,1,0,0)$	(diagram: 2 0 2 0 2, 2 above)	D_5
0	(diagram: 6—5—4—3—1, 2 above)	$\langle 1 \rangle$	$(4,3,2,1,0,-4\sqrt{3})$	(diagram: 2 2 2 2 2, 2 above)	E_6

7. E_7 orbits

The adjoint exceptional group E_7 has twenty seven Richardson orbits excluding $A_4 + A_1$ and $D_5(a_1)$. Fix the Dynkin diagram

with simple roots:

$$(\tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})$$

$(1,-1,0,0,0,0,0,0)$ $(0,0,1,-1,0,0,0,0)$ $(0,0,0,0,1,-1,0,0)$

$(0,1,-1,0,0,0,0,0)$ $(0,0,0,1,-1,0,0,0)$ $(0,0,0,0,0,1,-1,0)$.

$\mathcal{O}$	$\mathfrak{l}$	π_i	λ_i	$\Psi(\mathcal{O},\pi_i)$	$^L\mathcal{P}$
E_7	ϕ	$\langle 1 \rangle$	$(0,0,0,0,0,0,0,0)$	$\begin{smallmatrix}&&0&&\\0&0&0&0&0&0\end{smallmatrix}$	0
$E_7(a_1)$	4	$\langle 1 \rangle$	$(0,0,0,\tfrac{1}{2},\tfrac{-1}{2},0,0,0)$	$\begin{smallmatrix}&&0&&\\0&0&0&0&0&1\end{smallmatrix}$	A_1
$E_7(a_2)$	$6\ \ 4$	$\langle 1 \rangle$	$(0,\tfrac{1}{2},\tfrac{-1}{2},\tfrac{1}{2},\tfrac{-1}{2},0,0,0)$	$\begin{smallmatrix}&&0&&\\0&1&0&0&0&0\end{smallmatrix}$	$2A_1$
$E_7(a_3)$	$\begin{smallmatrix}&2&\\7&&1\end{smallmatrix}$	$\langle 2 \rangle$	$(\tfrac{1}{2},\tfrac{-1}{2},\tfrac{-1}{2},\tfrac{-1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{-1}{2},\tfrac{1}{2})$	$\begin{smallmatrix}&&0&&\\0&0&0&0&1&0\end{smallmatrix}$	$(3A_1)'$
		$\langle 1^2 \rangle$	$(0,-1,0,0,0,1,0,0)$	$\begin{smallmatrix}&&0&&\\0&0&0&0&0&2\end{smallmatrix}$	A_2
E_6	$\begin{smallmatrix}&2&\\7&5\end{smallmatrix}$	$\langle 1 \rangle$	$(\tfrac{1}{4},\tfrac{-3}{4},\tfrac{1}{4},\tfrac{-3}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4})$	$\begin{smallmatrix}&&0&&\\2&0&0&0&0&0\end{smallmatrix}$	$(3A_1)''$
$E_6(a_1)$	$\begin{smallmatrix}&2&\\7&5&3\end{smallmatrix}$	$\langle 2 \rangle$	$(\tfrac{1}{2},\tfrac{-1}{2},0,-1,\tfrac{1}{2},\tfrac{-1}{2},\tfrac{1}{2},\tfrac{1}{2})$	$\begin{smallmatrix}&&1&&\\1&0&0&0&0&0\end{smallmatrix}$	$4A_1$
		$\langle 1^2 \rangle$	$(0,-1,\tfrac{1}{2},\tfrac{-1}{2},1,0,0,0)$	$\begin{smallmatrix}&&0&&\\0&1&0&0&0&1\end{smallmatrix}$	$A_2 + A_1$
$E_7(a_4)$	$\begin{smallmatrix}&2&\\7\,6&&1\end{smallmatrix}$	$\langle 1 \rangle$	$(\tfrac{3}{4},\tfrac{-1}{4},\tfrac{-5}{4},\tfrac{-1}{4},\tfrac{1}{4},\tfrac{3}{4},\tfrac{-1}{4},\tfrac{1}{4})$	$\begin{smallmatrix}&&0&&\\0&0&0&1&0&0\end{smallmatrix}$	$A_2 + 2A_1$
$D_6(a_1)$	$7\,6\,5$	$\langle 1 \rangle$	$(\tfrac{3}{2},\tfrac{1}{2},\tfrac{-1}{2},\tfrac{-3}{2},0,0,0,0)$	$\begin{smallmatrix}&&0&&\\0&1&0&0&0&2\end{smallmatrix}$	A_3
$D_5 + A_1$	$\begin{smallmatrix}7\,6&&3\,1\end{smallmatrix}$	$\langle 1 \rangle$	$(1,0,-1,0,1,0,-1,0)$	$\begin{smallmatrix}&&0&&\\0&2&0&0&0&0\end{smallmatrix}$	$2A_2$
A_6	$\begin{smallmatrix}&2&\\7&5&3\,1\end{smallmatrix}$	$\langle 1 \rangle$	$(\tfrac{1}{4},\tfrac{-3}{4},\tfrac{1}{4},\tfrac{-3}{4},\tfrac{5}{4},\tfrac{1}{4},\tfrac{-3}{4},\tfrac{1}{4})$	$\begin{smallmatrix}&&2&&\\0&0&0&0&0&0\end{smallmatrix}$	$A_2 + 3A_1$
$E_7(a_5)$	$\begin{smallmatrix}&2&\\7\,6&&3\,1\end{smallmatrix}$	$\langle 3 \rangle$	$(1,0,-1,-1,1,0,-1,1)$	$\begin{smallmatrix}&&0&&\\0&1&0&0&1&0\end{smallmatrix}$	$2A_2 + A_1$
		$\langle 21 \rangle$	$(\tfrac{1}{2},\tfrac{-1}{2},\tfrac{-3}{2},\tfrac{-1}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{-1}{2},\tfrac{1}{2})$	$\begin{smallmatrix}&&0&&\\0&0&0&1&0&1\end{smallmatrix}$	$(A_3 + A_1)'$
		$\langle 1^3 \rangle$	$(\tfrac{1}{2},\tfrac{-1}{2},\tfrac{-3}{2},\tfrac{1}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{-1}{2},\tfrac{-1}{2})$	$\begin{smallmatrix}&&0&&\\0&0&0&0&2&0\end{smallmatrix}$	$D_4(a_1)$
D_5	$\begin{smallmatrix}&2&\\7&5\,4\end{smallmatrix}$	$\langle 1 \rangle$	$(\tfrac{-1}{4},\tfrac{-5}{4},\tfrac{3}{4},\tfrac{-1}{4},\tfrac{-5}{4},\tfrac{3}{4},\tfrac{3}{4},\tfrac{3}{4})$	$\begin{smallmatrix}&&0&&\\2&0&0&0&0&2\end{smallmatrix}$	$(A_3 + A_1)''$

$\mathcal{O}$	$\mathfrak{l}$	π_i	λ_i	$\Psi(\mathcal{O},\pi_i)$	$^L\mathcal{P}$
$E_6(a_3)$	(diagram: top 2; bottom $7\ 6\ 5\ \ 3$)	$\langle 2\rangle$	$(\frac54,\frac14,\frac{-3}4,\frac{-7}4,\frac34,\frac{-1}4,\frac{-1}4,\frac34)$	(top 0; $1\ 0\ 1\ 0\ 0\ 1$)	A_3+2A_1
		$\langle 1^2\rangle$	$(\frac54,\frac14,\frac{-3}4,\frac{-7}4,\frac34,\frac{-1}4,\frac34,\frac{-1}4)$	(top 1; $1\ 0\ 0\ 0\ 1\ 0$)	$D_4(a_1)+A_1$
$D_5(a_1)+A_1$	(diagram: $7\ 6\ \ 4\ 3\ 1$)	$\langle 1\rangle$	$(1,0,-1,\frac32,\frac12,\frac{-1}2,\frac{-3}2,0)$	(top 0; $0\ 1\ 0\ 1\ 0\ 0$)	A_3+A_2
A_4+A_2	(diagram: top 2; bottom $7\ 6\ 5\ \ 3\ 1$)	$\langle 1\rangle$	$(\frac54,\frac14,\frac{-3}4,\frac{-7}4,\frac54,\frac14,\frac{-3}4,\frac14)$	(top 0; $0\ 0\ 2\ 0\ 0\ 0$)	$A_3+A_2+A_1$
$(A_5)''$	(diagram: top 2; bottom $5\ 4\ 3$)	$\langle 1\rangle$	$(\frac{-3}2,\frac{-3}2,\frac32,\frac12,\frac{-1}2,\frac{-3}2,\frac32,\frac32)$	(top 0; $0\ 0\ 0\ 0\ 2\ 2$)	D_4
$A_3+A_2+A_1$	(diagram: top 2; bottom $7\ 6\ \ 4\ 3\ 1$)	$\langle 1\rangle$	$(0,-1,-2,2,1,0,-1,1)$	(top 0; $0\ 0\ 0\ 2\ 0\ 0$)	A_4+A_2
A_4	(diagram: top 2; bottom $7\ \ 5\ 4\ 3$)	$\langle 2\rangle$	$(-1,-2,\frac32,\frac12,\frac{-1}2,\frac{-3}2,1,2)$	(top 1; $1\ 0\ 0\ 0\ 1\ 2$)	D_4+A_1
		$\langle 1^2\rangle$	$(-1,-2,\frac32,\frac12,\frac{-1}2,\frac{-3}2,2,1)$	(top 0; $0\ 1\ 0\ 1\ 0\ 2$)	$D_5(a_1)$
$D_4(a_1)+A_1$	(diagram: $7\ 6\ 5\ 4\ 3$)	$\langle 2\rangle$	$(\frac52,\frac32,\frac12,\frac{-1}2,\frac{-3}2,\frac{-5}2,\frac{-1}2,\frac12)$	(top 0; $0\ 2\ 0\ 1\ 0\ 1$)	$(A_5)'$
		$\langle 1^2\rangle$	$(\frac52,\frac32,\frac12,\frac{-1}2,\frac{-3}2,\frac{-5}2,\frac12,\frac{-1}2)$	(top 0; $0\ 2\ 0\ 0\ 2\ 0$)	$E_6(a_3)$
D_4	(diagram: top 2; bottom $7\ 6\ 5\ 4$)	$\langle 1\rangle$	$(\frac54,\frac14,\frac{-3}4,\frac{-7}4,\frac{-11}4,\frac54,\frac54,\frac54)$	(top 0; $2\ 2\ 0\ 0\ 0\ 2$)	$(A_5)''$
$D_4(a_1)$	(diagram: top 2; bottom $7\ 6\ 5\ 4\ \ 1$)	$\langle 3\rangle$	$(\frac54,\frac14,\frac{-3}4,\frac{-7}4,\frac{-11}4,\frac54,\frac14,\frac94)$	(top 0; $2\ 1\ 0\ 1\ 0\ 1$)	A_5+A_1
		$\langle 21\rangle$	$(\frac54,\frac14,\frac{-3}4,\frac{-7}4,\frac{-11}4,\frac94,\frac14,\frac54)$	(top 1; $2\ 0\ 1\ 0\ 1\ 0$)	$D_6(a_2)$
		$\langle 1^3\rangle$	$(\frac54,\frac14,\frac{-3}4,\frac{-7}4,\frac{-11}4,\frac94,\frac54,\frac14)$	(top 0; $2\ 0\ 0\ 2\ 0\ 0$)	$E_7(a_5)$
$(A_3+A_1)''$	(diagram: top 2; bottom $5\ 4\ 3\ 1$)	$\langle 1\rangle$	$(-\frac52,-\frac52,\frac52,\frac32,\frac12,\frac{-1}2,\frac{-3}2,\frac52)$	(top 0; $0\ 2\ 0\ 0\ 2\ 2$)	D_5
A_2+3A_1	(diagram: $7\ 6\ 5\ 4\ 3\ 1$)	$\langle 1\rangle$	$(3,2,1,0,-1,-2,-3,0)$	(top 0; $0\ 2\ 0\ 2\ 0\ 0$)	A_6
$2A_2$	(diagram: $7\ \ 5\ 4\ 3\ 1$)	$\langle 1\rangle$	$(-3,-2,\frac52,\frac32,\frac12,\frac{-1}2,\frac{-3}2,\frac52)$	(top 1; $0\ 1\ 1\ 0\ 1\ 2$)	D_5+A_1
A_2	(diagram: top 2; bottom $7\ 6\ 5\ 4\ 3$)	$\langle 2\rangle$	$(\frac54,\frac14,\frac{-3}4,\frac{-7}4,\frac{-11}4,\frac{-15}4,\frac{13}4,\frac{17}4)$	(top 1; $2\ 2\ 1\ 0\ 1\ 2$)	D_6
		$\langle 1^2\rangle$	$(\frac54,\frac14,\frac{-3}4,\frac{-7}4,\frac{-11}4,\frac{-15}4,\frac{17}4,\frac{13}4)$	(top 0; $2\ 2\ 0\ 2\ 0\ 2$)	$E_7(a_3)$
$(3A_1)''$	(diagram: top 2; bottom $6\ 5\ 4\ 3\ 1$)	$\langle 1\rangle$	$(\frac{-11}2,\frac52,\frac32,\frac12,\frac{-1}2,\frac{-3}2,\frac{-5}2,\frac{11}2)$	(top 0; $0\ 2\ 0\ 2\ 2\ 2$)	E_6
0	(diagram: top 2; bottom $7\ 6\ 5\ 4\ 3\ 1$)	$\langle 1\rangle$	$(\frac54,\frac14,\frac{-3}4,\frac{-7}4,\frac{-11}4,\frac{-15}4,\frac{-19}4,\frac{49}4)$	(top 2; $2\ 2\ 2\ 2\ 2\ 2$)	E_7

8. E_8 orbits

The exceptional group E_8 has thirty two Richardson orbits excluding $E_6(a_1) + A_1$ and $E_7(a_3)$. Fix the Dynkin diagram with simple roots:

$$(0,0,0,0,0,1,-1,0)$$

$$(0,1,-1,0,0,0,0,0) \qquad (0,0,0,1,-1,0,0,0) \qquad (0,0,0,0,0,1,1,0)$$

$$(1,-1,0,0,0,0,0,0) \quad (0,0,1,-1,0,0,0,0) \quad (0,0,0,0,1,-1,0,0) \quad \left(\tfrac{-1}{2},\tfrac{-1}{2},\tfrac{-1}{2},\tfrac{-1}{2},\tfrac{-1}{2},\tfrac{-1}{2},\tfrac{-1}{2},\tfrac{-1}{2}\right).$$

$\mathcal{O}$	$\mathfrak{l}$	π_i	λ_i	$\Psi(\mathcal{O},\pi_i)$	$^L\mathcal{P}$
E_8	ϕ	$\langle 1 \rangle$	$(0,0,0,0,0,0,0,0)$	$\genfrac{}{}{0pt}{}{0}{0\,0\,0\,0\,0\,0\,0}$	0
$E_8(a_1)$	8	$\langle 1 \rangle$	$(\tfrac{1}{2},\tfrac{-1}{2},0,0,0,0,0,0)$	$\genfrac{}{}{0pt}{}{0}{1\,0\,0\,0\,0\,0\,0}$	A_1
$E_8(a_2)$	$8 \quad 6$	$\langle 1 \rangle$	$(\tfrac{1}{2},\tfrac{-1}{2},\tfrac{1}{2},\tfrac{-1}{2},0,0,0,0)$	$\genfrac{}{}{0pt}{}{0}{0\,0\,0\,0\,0\,0\,1}$	$2A_1$
$E_8(a_3)$	$\genfrac{}{}{0pt}{}{2}{8 \quad 3}$	$\langle 2 \rangle$	$(0,0,1,0,0,1,0,0)$	$\genfrac{}{}{0pt}{}{0}{0\,1\,0\,0\,0\,0\,0}$	$3A_1$
		$\langle 1^2 \rangle$	$(0,0,0,-1,0,1,0,0)$	$\genfrac{}{}{0pt}{}{0}{2\,0\,0\,0\,0\,0\,0}$	A_2
$E_8(a_4)$	$\genfrac{}{}{0pt}{}{2}{8 \; 6 \; 3}$	$\langle 2 \rangle$	$(\tfrac{1}{2},\tfrac{-1}{2},1,0,0,1,0,0)$	$\genfrac{}{}{0pt}{}{1}{0\,0\,0\,0\,0\,0\,0}$	$4A_1$
		$\langle 1^2 \rangle$	$(\tfrac{1}{2},\tfrac{-1}{2},0,-1,0,1,0,0)$	$\genfrac{}{}{0pt}{}{0}{1\,0\,0\,0\,0\,0\,1}$	$A_2 + A_1$
$E_8(b_4)$	$\genfrac{}{}{0pt}{}{2}{8\,7\;5}$	$\langle 1 \rangle$	$(1,0,-1,\tfrac{1}{2},\tfrac{-1}{2},\tfrac{1}{2},\tfrac{-1}{2},0)$	$\genfrac{}{}{0pt}{}{0}{0\,0\,1\,0\,0\,0\,0}$	$A_2 + 2A_1$
$E_8(a_5)$	$\genfrac{}{}{0pt}{}{2}{8\,7\;5\;3}$	$\langle 2 \rangle$	$(1,0,-1,1,0,1,0,0)$	$\genfrac{}{}{0pt}{}{0}{0\,0\,0\,0\,0\,1\,0}$	$A_2 + 3A_1$
		$\langle 1^2 \rangle$	$(1,0,-1,0,-1,1,0,0)$	$\genfrac{}{}{0pt}{}{0}{0\,0\,0\,0\,0\,0\,2}$	$2A_2$
$E_7(a_1)$	$8\,7\,6$	$\langle 1 \rangle$	$(\tfrac{3}{2},\tfrac{1}{2},\tfrac{-1}{2},\tfrac{-3}{2},0,0,0,0)$	$\genfrac{}{}{0pt}{}{0}{2\,0\,0\,0\,0\,0\,1}$	A_3
$E_8(b_5)$	$8\,7\;5\,4\;1$	$\langle 3 \rangle$	$(1,0,-1,1,0,-1,-1,-1)$	$\genfrac{}{}{0pt}{}{0}{0\,1\,0\,0\,0\,0\,1}$	$2A_2 + A_1$
		$\langle 21 \rangle$	$(\tfrac{1}{2},\tfrac{-1}{2},\tfrac{-3}{2},\tfrac{1}{2},\tfrac{-1}{2},\tfrac{-3}{2},\tfrac{-1}{2},\tfrac{-1}{2})$	$\genfrac{}{}{0pt}{}{0}{1\,0\,1\,0\,0\,0\,0}$	$A_3 + A_1$
		$\langle 1^3 \rangle$	$(\tfrac{1}{2},\tfrac{-1}{2},\tfrac{-3}{2},\tfrac{1}{2},\tfrac{-1}{2},\tfrac{-3}{2},\tfrac{1}{2},\tfrac{1}{2})$	$\genfrac{}{}{0pt}{}{0}{0\,2\,0\,0\,0\,0\,0}$	$D_4(a_1)$
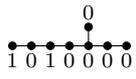
$\mathcal{O}$	$\mathfrak{l}$	π_i	λ_i	$\Psi(\mathcal{O}, \pi_i)$	${}^L\mathcal{P}$
$E_8(a_6)$	(diagram: 8 7 · 5 · 3 1, top 2)	$\langle 3 \rangle$	$(1, 0, -1, \frac{-1}{2}, \frac{-3}{2}, 1, 0, -1)$	(diagram, top 0) $0\,0\,0\,1\,0\,0\,0$	$2A_2 + 2A_1$
		$\langle 21 \rangle$	$(\frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}, 0, -1, \frac{3}{2}, \frac{-1}{2}, \frac{-1}{2})$	(diagram, top 0) $1\,0\,0\,0\,0\,1\,0$	$A_3 + 2A_1$
		$\langle 1^3 \rangle$	$(0, -1, -2, \frac{1}{2}, \frac{-1}{2}, 1, 0, 0)$	(diagram, top 1) $0\,1\,0\,0\,0\,0\,0$	$D_4(a_1) + A_1$
$D_7(a_1)$	(diagram: 8 7 6 · 4, top 2)	$\langle 1 \rangle$	$(\frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}, 1, 0, -1, 0)$	(diagram, top 0) $0\,0\,1\,0\,0\,0\,1$	$A_3 + A_2$
$E_8(b_6)$	(diagram: 8 7 6 · 3 1, top 2)	$\langle 2 \rangle$	$(1, 0, -1, -2, 0, 1, 0, -1)$	(diagram, top 0) $0\,0\,0\,0\,1\,0\,0$	$A_3 + A_2 + A_1$
		$\langle 1^2 \rangle$	$(1, 0, -1, -2, -1, 1, 0, 0)$	(diagram, top 2) $0\,0\,0\,0\,0\,0\,0$	$D_4(a_1) + A_2$
$D_7(a_2)$	(diagram: 8 7 6 · 4 3, top 2)	$\langle 2 \rangle$	$(\frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}, 2, 1, 0, -1)$	(diagram, top 0) $0\,0\,0\,1\,0\,0\,1$	$2A_3$
		$\langle 1^2 \rangle$	$(\frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}, 2, 1, 0, 1)$	(diagram, top 0) $1\,0\,0\,0\,1\,0\,0$	$A_4 + 2A_1$
E_6	(diagram: 5 4 3, top 2)	$\langle 1 \rangle$	$(0, 0, 0, 3, 2, 1, 0, 0)$	(diagram, top 0) $2\,2\,0\,0\,0\,0\,0$	D_4
$D_5 + A_2$	(diagram: 8 7 · 5 4 3 1)	$\langle 1 \rangle$	$(0, -1, -2, 1, 0, -1, 2, -1)$	(diagram, top 0) $0\,0\,2\,0\,0\,0\,0$	$A_4 + A_2$
$E_6(a_1)$	(diagram: 7 · 5 4 3, top 2)	$\langle 2 \rangle$	$(\frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}, 3, 2, 1, 0, \frac{-1}{2})$	(diagram, top 1) $2\,1\,0\,0\,0\,0\,0$	$D_4 + A_1$
		$\langle 1^2 \rangle$	$(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, 3, 2, 1, 0, \frac{1}{2})$	(diagram, top 0) $2\,0\,1\,0\,0\,0\,1$	$D_5(a_1)$
$A_6 + A_1$	(diagram: 8 7 6 5 · 3 1, top 2)	$\langle 1 \rangle$	$(\frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}, \frac{-5}{2}, 1, 0, \frac{-1}{2})$	(diagram, top 0) $0\,0\,1\,0\,0\,1\,0$	$A_4 + A_2 + A_1$
$D_6(a_1)$	(diagram: 8 7 6 5 4)	$\langle 2 \rangle$	$(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}, \frac{-5}{2}, \frac{1}{2}, \frac{-1}{2})$	(diagram, top 0) $1\,0\,1\,0\,0\,0\,2$	A_5
		$\langle 1^2 \rangle$	$(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}, \frac{-5}{2}, \frac{-1}{2}, \frac{1}{2})$	(diagram, top 0) $0\,2\,0\,0\,0\,0\,2$	$E_6(a_3)$
A_6	(diagram: 8 7 · 5 4 3, top 2)	$\langle 1 \rangle$	$(1, 0, -1, 3, 2, 1, 0, 0)$	(diagram, top 2) $2\,0\,0\,0\,0\,0\,0$	$D_4 + A_2$
$E_8(a_7)$	(diagram: 8 7 6 · 4 3 1, top 2)	$\langle 5 \rangle$	$(1, 0, -1, -2, 2, 1, 0, -3)$	(diagram, top 0) $0\,1\,0\,0\,1\,0\,0$	$A_4 + A_3$
		$\langle 41 \rangle$	$(1, 0, -1, -3, 2, 1, 0, -2)$	(diagram, top 0) $1\,0\,1\,0\,0\,1\,0$	$D_5(a_1) + A_2$
		$\langle 32 \rangle$	$(\frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}, \frac{-5}{2}, \frac{5}{2}, \frac{3}{2}, \frac{-1}{2}, \frac{-3}{2})$	(diagram, top 0) $1\,0\,0\,0\,1\,0\,1$	$A_5 + A_1$

$\mathcal{O}$	$\mathfrak{l}$	π_i	λ_i	$\Psi(\mathcal{O},\pi_i)$	${}^L\mathcal{P}$
		$\langle 31^2\rangle$	$(1,0,-2,-3,2,1,0,-1)$	top 0; 0 1 0 1 0 0 1	$E_6(a_3)+A_1$
		$\langle 2^21\rangle$	$(0,-1,-2,-3,2,1,0,-1)$	top 1; 0 1 0 0 0 1 0	$D_6(a_2)$
		$\langle 21^3\rangle$	$(1,-1,-2,-3,2,1,0,0)$	top 0; 0 0 1 0 1 0 0	$E_7(a_5)$
		$\langle 1^5\rangle$	$(0,-1,-2,-3,2,1,0,1)$	top 0; 0 0 0 2 0 0 0	$E_8(a_7)$
D_5	2 over 6 5 4 3	$\langle 1\rangle$	$(0,0,4,3,2,1,0,0)$	top 0; 2 2 0 0 0 0 2	D_5
$E_6(a_3)$	2 over 7 5 4 3 1	$\langle 2\rangle$	$(\frac{-3}{2},\frac{-3}{2},\frac{-5}{2},3,2,1,0,\frac{-5}{2})$	top 0; 2 1 0 1 0 0 1	D_5+A_1
		$\langle 1^2\rangle$	$(\frac{-5}{2},\frac{-3}{2},\frac{-5}{2},3,2,1,0,\frac{-3}{2})$	top 1; 2 1 0 0 0 1 0	$D_6(a_1)$
D_4+A_2	2 over 8 7 6 5 4	$\langle 1\rangle$	$(3,2,1,0,-1,-2,-3,0)$	top 0; 0 0 2 0 0 0 2	A_6
$A_4+A_2+A_1$	2 over 8 7 6 5 4 1	$\langle 1\rangle$	$(\frac{11}{4},\frac{7}{4},\frac{3}{4},\frac{-1}{4},\frac{-5}{4},\frac{-9}{4},\frac{-13}{4},\frac{-1}{4})$	top 0; 0 0 1 0 1 0 1	A_6+A_1
A_4+A_2	2 over 8 7 5 4 3 1	$\langle 1\rangle$	$(-1,-2,-3,3,2,1,0,-2)$	top 0; 2 0 0 2 0 0 0	D_5+A_2
$D_4(a_1)+A_2$	8 7 6 5 4 3 1	$\langle 2\rangle$	$(2,1,0,-1,-2,-3,4,-3)$	top 0; 0 1 1 0 1 0 1	A_7
		$\langle 1^2\rangle$	$(\frac{3}{2},\frac{1}{2},\frac{-1}{2},\frac{-3}{2},\frac{-5}{2},\frac{-7}{2},\frac{9}{2},\frac{-1}{2})$	top 0; 2 0 0 0 2 0 0	$E_8(b_6)$
A_4	2 over 7 6 5 4 3	$\langle 2\rangle$	$(\frac{1}{2},5,4,3,2,1,0,\frac{-1}{2})$	top 1; 2 1 0 0 0 1 2	D_6
		$\langle 1^2\rangle$	$(\frac{-1}{2},5,4,3,2,1,0,\frac{1}{2})$	top 0; 2 0 1 0 1 0 2	$E_7(a_3)$
D_4	2 over 6 5 4 3 1	$\langle 1\rangle$	$(-4,-4,4,3,2,1,0,-4)$	top 0; 2 2 2 0 0 0 2	E_6
$D_4(a_1)$	2 over 8 6 5 4 3 1	$\langle 3\rangle$	$(-3,-4,4,3,2,1,0,-5)$	top 0; 2 2 1 0 1 0 1	E_6+A_1
		$\langle 21\rangle$	$(-3,-5,4,3,2,1,0,-4)$	top 1; 2 2 0 1 0 1 0	$E_7(a_2)$
		$\langle 1^3\rangle$	$(-4,-5,4,3,2,1,0,-3)$	top 0; 2 2 0 0 0 2 0	$E_8(b_5)$
$2A_2$	2 over 8 7 6 5 4	$\langle 2\rangle$	$(6,5,4,3,2,1,0,-1)$	top 1; 1 0 1 1 0 1 2	D_7
		$\langle 1^2\rangle$	$(6,5,4,3,2,1,0,1)$	top 0; 0 2 0 0 2 0 2	$E_8(a_5)$

$\mathcal{O}$	$\mathfrak{l}$	π_i	λ_i	$\Psi(\mathcal{O},\pi_i)$	$^L\mathcal{P}$
A_2	$\overset{2}{\underset{7\ 6\ 5\ 4\ 3\ 1}{\bullet\bullet\bullet\bullet\bullet\bullet}}$	$\langle 2\rangle$	$(-8,5,4,3,2,1,0,-9)$	$\overset{1}{\underset{2\ 2\ 2\ 1\ 0\ 1\ 2}{\bullet\bullet\bullet\bullet\bullet\bullet\bullet}}$	E_7
		$\langle 1^2\rangle$	$(-9,5,4,3,2,1,0,-8)$	$\overset{0}{\underset{2\ 2\ 2\ 0\ 2\ 0\ 2}{\bullet\bullet\bullet\bullet\bullet\bullet\bullet}}$	$E_8(a_3)$
0	$\overset{2}{\underset{8\ 7\ 6\ 5\ 4\ 3\ 1}{\bullet\bullet\bullet\bullet\bullet\bullet\bullet}}$	$\langle 1\rangle$	$(6,5,4,3,2,1,0,-23)$	$\overset{2}{\underset{2\ 2\ 2\ 2\ 2\ 2\ 2}{\bullet\bullet\bullet\bullet\bullet\bullet\bullet}}$	E_8

9. Proof of Theorem 2.10 - the four exceptional cases

We now finish the proof of Theorem 2.10 for $\mathcal{O}$ equal to the four orbits in (5). Note that all these orbits have Lusztig's quotient $\overline{A}(\mathcal{O}) \cong S_2$, and the special piece $\mathcal{S}_{L\mathcal{O}} = \{^L\mathcal{O}\}$ only contains one element. Therefore, one only needs to find one $\pi \in \overline{A}(\mathcal{O})^\vee$ such that $\Psi(\mathcal{O},\pi) = H_{L\mathcal{O}}$ in order to verify Theorem 2.10.

Note that the orbits $A_4 + A_1$ in E_7 and $E_6(a_1) + A_1$ in E_8 are called *exceptional* in Section 4 of [11].

9.1. $\mathcal{O} = A_4 + A_1$ in E_7

By checking the tables of [4] directly (the calculations for other orbits can be found in the Appendix), the two left cell representations attached to $\mathcal{O}$ are equal to $J_{D_6+A_1}^{E_7}(\sigma_1)$, $J_{D_6+A_1}^{E_7}(\sigma_2)$, where σ_1, σ_2 are the two left cell representations attached to $\mathcal{O}' = [332211] + [11]$ in $D_6 + A_1$. Using Proposition 6.6 of [11], the character formulas for $X_{\mathcal{O},\pi}$ can be derived from that of $\mathcal{O}'$ in

$$D_6 + A_1 = \quad \text{(Dynkin diagram with labels } (1,-1,0,0,0,0,0,0),\ (0,0,1,-1,0,0,0,0),\ (0,0,0,0,1,-1,0,0),\ (0,1,-1,0,0,0,0,0),\ (0,0,0,1,-1,0,0,0),\ (\tfrac{-1}{2},\tfrac{-1}{2},\tfrac{-1}{2},\tfrac{-1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}),\ (0,0,0,0,0,0,1,-1)\text{)}.$$

Using Theorem 3.4 in [37],

$$\Psi(\mathcal{O}',\langle 2\rangle) = \overset{0}{\underset{0\ 2\ 0\ 2\ 0}{\bullet\bullet\bullet\bullet\bullet}}\ \underset{2}{\bullet\bullet}\ ; \quad \Psi(\mathcal{O}',\langle 1^2\rangle) = \overset{1}{\underset{0\ 2\ 1\ 0\ 1}{\bullet\bullet\bullet\bullet\bullet}}\ \underset{2}{\bullet\bullet}.$$

Therefore, one can compute that

$$\Psi(A_4+A_1,\langle 2\rangle) = \{(1,1,-1,-1,-3,-3,4,2)\} = \left(\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-3}{2},\frac{-3}{2},\frac{-5}{2},\frac{11}{2}\right),$$

$$\Psi(A_4+A_1,\langle 1^2\rangle) = \{(1,1,-1,-2,-2,-3,4,2)\} = \left(\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-3}{2},\frac{-5}{2},\frac{11}{2}\right).$$

In terms of Dynkin diagram, we have

$$\Psi(A_4+A_1,\langle 2\rangle) = \overset{0}{\underset{0\ 1\ 0\ 1\ 0\ 1}{\bullet\bullet\bullet\bullet\bullet\bullet}}\ ; \quad \Psi(A_4+A_1,\langle 1^2\rangle) = \overset{1}{\underset{1\ 0\ 0\ 0\ 1\ 1}{\bullet\bullet\bullet\bullet\bullet\bullet}}$$

Note that we have $\Psi(A_4+A_1,\langle 2\rangle) = H_{A_4+A_1} = H_{L(A_4+A_1)}$ and Theorem 2.10 holds.

9.2. $\mathcal{O} = D_5(a_1)$ in E_7

As in the above subsection, one can derive the character formulas of $X_{\mathcal{O},\pi}$ from that of $\mathcal{O}' = [332211]$ in D_6. More precisely, we have

$$\Psi(\mathcal{O}', \langle 2\rangle) = \underset{0\,2\,0\,2\,0\ \ 0}{\overset{0}{\bullet\,\bullet\,\bullet\,\bullet\,\bullet}}\ \ \bullet\ ;\quad \Psi(\mathcal{O}', \langle 1^2\rangle) = \underset{0\,2\,1\,0\,1\ \ 0}{\overset{1}{\bullet\,\bullet\,\bullet\,\bullet\,\bullet}}\ \ \bullet$$

Therefore, one can compute that

$$\Psi(D_5(a_1), \langle 2\rangle) = \{(1, 1, -1, -1, -3, -3, 3, 3)\} = (1, 1, -1, -1, -1, -1, -3, 5),$$

$$\Psi(D_5(a_1), \langle 1^2\rangle) = \{(1, 1, -1, -2, -2, -3, 3, 3)\} = (1, 0, 0, -1, -1, -1, -3, 5).$$

In terms of Dynkin diagram, we have

$$\Psi(D_5(a_1), \langle 2\rangle) = \underset{0\,2\,0\,0\,0\,2}{\overset{0}{\bullet\,\bullet\,\bullet\,\bullet\,\bullet\,\bullet}}\ ;\quad \Psi(D_5(a_1), \langle 1^2\rangle) = \underset{1\,0\,1\,0\,0\,2}{\overset{0}{\bullet\,\bullet\,\bullet\,\bullet\,\bullet\,\bullet}}$$

Note that $\Psi(D_5(a_1), \langle 2\rangle) = H_{A_4} = H_{L(D_5(a_1))}$, and Theorem 2.10 holds.

9.3. $\mathcal{O} = E_6(a_1) + A_1$ in E_8

The character formulas for $X_{\mathcal{O},\pi}$ can be derived from that of $\mathcal{O}' = D_5(a_1) + [11]$ in $E_7 + A_1$, as before, which, by the last subsection, can be derived from that of $[332211] + [11]$ in $D_6 + A_1$. Using the same argument as above, we have

$$\Psi(E_6(a_1) + A_1, \langle 2\rangle) = \underset{1\,0\,1\,0\,0\,0\,1}{\overset{0}{\bullet\,\bullet\,\bullet\,\bullet\,\bullet\,\bullet\,\bullet}}\ ;\quad \Psi(E_6(a_1) + A_1, \langle 1^2\rangle) = \underset{1\,1\,0\,0\,0\,0\,0}{\overset{1}{\bullet\,\bullet\,\bullet\,\bullet\,\bullet\,\bullet\,\bullet}}$$

We have $\Psi(E_6(a_1) + A_1, \langle 2\rangle) = H_{A_4+A_1} = H_{L(E_6(a_1)+A_1)}$ and Theorem 2.10 follows.

9.4. $\mathcal{O} = E_7(a_3)$ in E_8

Finally, we can derive the character formula of $X_{\mathcal{O},\pi}$ from that of $\mathcal{O}' = [332211]$ in D_6. As in the previous subsections, we have

$$\Psi(E_7(a_3), \langle 2\rangle) = \underset{2\,0\,0\,0\,0\,0\,2}{\overset{0}{\bullet\,\bullet\,\bullet\,\bullet\,\bullet\,\bullet\,\bullet}}\ ;\quad \Psi(E_7(a_3), \langle 1^2\rangle) = \underset{2\,0\,0\,0\,0\,1\,0}{\overset{0}{\bullet\,\bullet\,\bullet\,\bullet\,\bullet\,\bullet\,\bullet}}$$

We have $\Psi(E_7(a_3), \langle 2\rangle) = H_{A_4} = H_{L(E_7(A_3))}$, and Theorem 2.10 holds. This finishes the proof of Theorem 2.10. ∎

Remark 9.1. For the four orbits above, the sum of the two special unipotent representations attached to $A_4 + A_1$, $D_5(a_1)$ in $G = E_7$, and $E_6(a_1) + A_1$, $E_7(a_3)$ in $G = E_8$ are isomorphic to $\operatorname{Ind}_L^G(\mathrm{triv})$, where the semisimple part of L is of type $A_4 + A_1$ for the orbits $A_4 + A_1$, $E_6(a_1) + A_1$, and of type A_4 for $D_5(a_1)$, $E_7(a_3)$. In fact, the orbit $[332211]$ in D_6 that shows up in all the above examples is also a Richardson orbit, induced from $GL(5) \times SO(2)$. The sum of the two special unipotent representations attached to $[332211]$ is isomorphic to $\operatorname{Ind}_{GL(5)\times SO(2)}^{SO(12)}(\mathrm{triv})$.

There are two consequences of this observation: (a) Non-birationality of μ: We give an alternative proof of Proposition 2.3 for these four orbits – for example, suppose

on the contrary that $\mu : T^*(G/P) \to \overline{D_5(a_1)}$ is birational with the semisimple part of L is of Type A_4, then

$$R(D_5(a_1)) \cong \mathbb{C}[T^*(G/P)] \cong \mathrm{Ind}_L^{E_7}(\mathrm{triv}) \cong X_{D_5(a_1),\mathrm{triv}} \oplus X_{D_5(a_1),\mathrm{sgn}}.$$

However, the multiplicity of $\overline{D_5(a_1)}$ in $R(D_5(a_1))$ is one, while the multiplicity in $X_{D_5(a_1),\mathrm{triv}} \oplus X_{D_5(a_1),\mathrm{sgn}}$ is greater than one.

(b)   An alternative proof of Conjecture 2.5 for these orbits goes as follows: By [34], we have $X_{\mathcal{O},\mathrm{triv}} = R(\mathcal{O}) - Y_1$ and $X_{\mathcal{O},\mathrm{sgn}} = R(\mathcal{O}, \mathrm{sgn}) - Y_2$ for some genuine $K_{\mathbb{C}}$-modules Y_1, Y_2. Therefore,

$$\mathrm{Ind}_L^G(\mathrm{triv}) = X_{\mathcal{O},\mathrm{triv}} \oplus X_{\mathcal{O},\mathrm{sgn}} = R(\mathcal{O}) + R(\mathcal{O}, \mathrm{sgn}) - Y_1 - Y_2.$$

By Proposition 4.6.1 of [8], the left hand side is equal to $R(\mathcal{O}) + R(\mathcal{O}, \mathrm{sgn})$. So $Y_1 = Y_2 = 0$ and hence we have $X_{\mathcal{O},\mathrm{triv}} \cong R(\mathcal{O})$ and $X_{\mathcal{O},\mathrm{sgn}} \cong R(\mathcal{O}, \mathrm{sgn})$.   ∎

The method we used in (b) in the above remark can also be applied to verify Conjecture 2.5 for other Richardson orbits with non-birational moment maps. For example, let $\mathcal{O} = F_4(a_3)$ in F_4. If we take the the parabolic subgroup whose Levi is of type $A_2 + \widetilde{A_1}$, then one can show that $X_{\mathcal{O},\langle 31 \rangle} \cong R(\mathcal{O}, \langle 31 \rangle)$. On the other hand, if we take the the parabolic subgroup whose Levi is of type B_2, then one can show that $X_{\mathcal{O},\langle 22 \rangle} \cong R(\mathcal{O}, \langle 22 \rangle)$.

A.   Special unipotent representations for non-even $^L\mathcal{O}$

Based on the results of [11] on the structure of $\Pi(^L\mathcal{O})$ for even $^L\mathcal{O}$, we give a sketch proof of Theorem 2.2 when the special orbit $^L\mathcal{O}$ is not even (or equivalently $\lambda_{\mathcal{O}}$ is not integral). Although it is known by the experts that the results in [11] carry over to all $\lambda_{\mathcal{O}}$ regardless of its integrality, we do not find a proof of such result in the literature.

Sketch Proof of Theorem 2.2 *for non-even* $^L\mathcal{O}$:

(I) Let $\lambda := \lambda_{\mathcal{O}}$. Consider the Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ with roots α satisfying $\langle \alpha^\vee, \lambda \rangle \in \mathbb{Z}$. It turns out that there is a choice of simple roots of $\mathfrak{g}'$ such that their inner products with λ is equal to either 0 or 1.

(II) Upon restricting to $\mathfrak{g}'$, the result in [11] applies, and we can obtain *integral* special unipotent representations with infinitesimal character $\lambda = \frac{^L H'}{2}$ for some *even* orbit $^L\mathcal{O}' \in {}^L\mathfrak{g}'$.

(III) More explicitly, let $\mathcal{O}' \in \mathfrak{g}'$ be the Lusztig-Spaltenstein dual of $^L\mathcal{O}'$, and $V^L(\mathcal{O}')$ be the left cell representation of W' denoted by $V^L(w_0 w_{\mathcal{O}'})$ in [11]. By Chapter 4 of [20], the irreducible representations in $V^L(\mathcal{O}')$ are parametrized by $\{\sigma'_x \mid x \in [\overline{A}(\mathcal{O}')]\}$ (i.e. partitions $[n_1^{m_1} n_2^{m_2} \dots]$ of k if $\overline{A}(\mathcal{O}') \cong S_k$).

By Theorem III of [11], the unipotent representations $\Pi({}^L\mathcal{O}')$ are parametrized by $\pi \in \overline{A}(\mathcal{O}')^\vee$, and their character formulas are given by

$$J_{G'}(\lambda, w_\pi \lambda) = \frac{1}{|\overline{A}(\mathcal{O}')|} \sum_{x \in [\overline{A}(\mathcal{O}')]} \mathrm{tr}_\pi(x)|x| \sum_{w' \in W'} \mathrm{tr}_{\sigma'_x}(w') X_{G'}(\lambda, w'\lambda).$$

Under the above identification of elements in $V^L(\mathcal{O}')$, the representation $\sigma'_{sp} := \sigma'_{[1^k]}$ is special, and is the Springer representation of $\mathcal{O}'$. All other irreducible representations $J_{G'}(\lambda, w\lambda)$ with our choice of λ have character formulas involving special representations $\tau'_{sp} \in (W')^\vee$, where $\deg(\tau'_{sp}) < \deg(\sigma'_{sp})$.

(IV) By a version of the Kazhdan-Lusztig conjecture (c.f. [5, Theorem 4.2]),

$$J(\lambda, w_\pi \lambda) = \frac{1}{|\overline{A}(\mathcal{O}')|} \sum_{x \in [\overline{A}(\mathcal{O}')]} \mathrm{tr}_\pi(x)|x| \sum_{w' \in W'} \mathrm{tr}_{\sigma'_x}(w') X(\lambda, w'\lambda) \qquad (11)$$

are character formulas of irreducible representations of G for all $\pi \in \overline{A}(\mathcal{O}')^\vee$. The same result goes for character formulas of other irreducible representations $J_{G'}(\lambda, w\lambda)$ that involves $\tau'_{sp} \in (W')^\vee$.

(V) Let $J^W_{W'}(\sigma'_{sp})$ is the J-truncated induction defined in [20, Section 4.3]. It contains a unique special representation σ_{sp} with $\deg(\sigma_{sp}) = \deg(\sigma'_{sp})$ (and similarly we have $\tau_{sp} \in J^W_{W'}(\tau'_{sp})$ for all special representations $\tau'_{sp} \in (W')^\vee$). By studying the $W \times W$-module $\mathbb{C}[W]$ (c.f. [11, Proposition 6.6]), these character formulas in G contain expressions involving σ_{sp} (or τ_{sp}), and their associated varieties can be determined by the Springer correspondence of σ_{sp} (or τ_{sp}). Since $\deg(\tau_{sp}) < \deg(\sigma_{sp})$, the irreducible representations $J(\lambda, w_\pi \lambda)$ in (11) are the ones with smallest possible associated variety for our choice of λ.

(VI) We verify the following statements in the next couple of sections:

- For all orbits we are studying, $\overline{A}(\mathcal{O}) \cong \overline{A}(\mathcal{O}')$, and $\sigma_{sp} = J^W_{W'}(\sigma'_{sp})$ is non-zero and irreducible. More precisely, σ_{sp} is the Springer representation attached to $\mathcal{O}$, the Lusztig-Spaltenstein dual of ${}^L\mathcal{O}$.

- If $\mathcal{O}$ is not equal to the three exceptional orbits ($A_4 + A_1$ in E_7, $A_4 + A_1$ and $E_6(a_1) + A_1$ in E_8) listed in [11, Definition 4.5], $J^W_{W'}(\sigma'_x)$ is non-zero and irreducible for all $\sigma'_x \in V^L(\mathcal{O}')$. Since $V^L(\mathcal{O}) = J^W_{W'}(V^L(\mathcal{O}'))$ by [10, Proposition 3.18], the elements of $V^L(\mathcal{O})$ can be parametrized by

$$x \in [A(\mathcal{O})] \cong [A(\mathcal{O}')] \longleftrightarrow \sigma_x := J^W_{W'}(\sigma'_x) \in V^L(\mathcal{O}).$$

This identification of elements in $V^L(\mathcal{O})$ matches with that of [20].

The first point guarantees that all $J(\lambda, w_\pi \lambda)$ have associated variety equal to $\overline{\mathcal{O}}$ (see also Theorem 5.2 of [26]), and they have the same cardinality as the number of irreducible representations of $\overline{A}(\mathcal{O})$, i.e. all $J(\lambda, w_\pi \lambda) \in \Pi({}^L\mathcal{O})$ and $|\Pi({}^L\mathcal{O})| = |\overline{A}(\mathcal{O})^\vee|$. The second point and [11, Proposition 6.6] reformulate Equation (11) into

$$J(\lambda, w_\pi \lambda) = \frac{1}{|\overline{A}(\mathcal{O})|} \sum_{x \in [\overline{A}(\mathcal{O})]} \mathrm{tr}_\pi(x)|x| \sum_{w \in W} \mathrm{tr}_{\sigma_x}(w) X(\lambda, w\lambda).$$

In other words, the character formula of $J(\lambda, w_\pi \lambda)$ is given precisely by Theorem III of [11], and the result follows.

A.1. Classical Lie algebras

We apply the strategy in the previous section to classify all classical special unipotent representations. Recall from [14] that each classical nilpotent orbit $^L\mathcal{O} \subset {}^L\mathfrak{g}$ can be uniquely characterized by a partition, except in Type D that there are two orbits corresponding to a single *very even* partition of the form

$$[2a \geq 2a \geq 2b \geq 2b \geq \cdots \geq 2c \geq 2c]$$

(the *very even orbits*). Since all very even orbits are even, the results in [11] can be applied directly. For convenience, we denote all non-very even $^L\mathcal{O}$ by their corresponding partition.

Proposition A.1 ([14], Proposition 6.3.7)**.**

- $^L\mathfrak{g}$ **of Type** B**:** *Let* $^L\mathcal{O} = [r_{2k} \geq r_{2k-1} \geq \cdots \geq r_0]$ *be a nilpotent orbit of Type* B*, i.e.* $|\{i \mid r_i = \alpha\}|$ *is even for all* **even** *integers* α*. Then* $^L\mathcal{O}$ *is a special orbit if and only if its dual partition defines a nilpotent orbit of Type* B *or, equivalently,* $r_{2l+1} + r_{2l}$ *is even for all* $l \leq k - 1$*.*

- $^L\mathfrak{g}$ **of Type** C**:** *Let* $^L\mathcal{O} = [r_{2k+1} \geq r_{2k} \geq \cdots \geq r_1]$ *be a nilpotent orbit of Type* C*, i.e.* $|\{i \mid r_i = \alpha\}|$ *is even for all* **odd** *integers* α*. Then* $^L\mathcal{O}$ *is a special orbit if and only if its dual partition defines a nilpotent orbit of Type* C *or, equivalently,* $r_{2l+1} + r_{2l}$ *is even for all* $l \leq k$*.*

- $^L\mathfrak{g}$ **of Type** D**:** *Let* $^L\mathcal{O} = [r_{2k+1} \geq r_{2k} \geq \cdots \geq r_0]$ *be a non-very even orbit of Type* D*, i.e.* $|\{i \mid r_i = \alpha\}|$ *is even for all* **even** *integers* α*. Then* $^L\mathcal{O}$ *is a special orbit if and only if its dual partition defines a nilpotent orbit of Type* C *or, equivalently,* $r_{2l+1} + r_{2l}$ *is even for all* $l \leq k$*.*

For any special orbit $^L\mathcal{O}$, the Lusztig's quotient $\overline{A}(^L\mathcal{O})$ is given by:

Proposition A.2 ([37], Proposition 2.2)**.**

- $^L\mathfrak{g}$ **of Type** B**:** *Let* $^L\mathcal{O} = [r_{2k} \geq r_{2k-1} \geq \cdots \geq r_0]$ *be a special orbit. Separate all* **even** *rows (which must be of the form* $r_{2l+1} = r_{2l} = \alpha$*), along with odd row pairs of the form* $r_{2l} = r_{2l-1} = \beta$ *and get*

$$^L\mathcal{O} = [r''_{2q} > r''_{2q-1} \geq r''_{2q-2} > \cdots \geq r''_2 > r''_1 \geq r''_0] \cup \bigcup_i [\alpha_i, \alpha_i] \cup \bigcup_j [\beta_j, \beta_j],$$

- $^L\mathfrak{g}$ **of Type** C**:** *Let* $^L\mathcal{O} = [r_{2k+1} \geq r_{2k} \geq \cdots \geq r_1]$ *be a special orbit. Separate all* **odd** *rows (which must be of the form* $r_{2l+1} = r_{2l} = \alpha$*), and even row pairs of the form* $r_{2l} = r_{2l-1} = \beta$ *and get*

$$^L\mathcal{O} = [r''_{2q+1} \geq r''_{2q} > r''_{2q-1} \geq \cdots > r''_3 \geq r''_2 > r''_1] \cup \bigcup_i [\alpha_i, \alpha_i] \cup \bigcup_j [\beta_j, \beta_j],$$

- $^L\mathfrak{g}$ **of Type** D**:** *Let* $^L\mathcal{O} = [r_{2k+1} \geq r_{2k} \geq \cdots \geq r_0]$ *be a special, non-very even orbit. Separate all* **even** *rows (which must be of the form* $r_{2l+1} = r_{2l} = \alpha$*), and all odd row pairs* $r_{2l} = r_{2l-1} = \beta$ *and get*

$$^L\mathcal{O} = [r''_{2q+1} \geq r''_{2q} > r''_{2q-1} \geq \cdots \geq r''_2 > r''_1 \geq r''_0] \cup \bigcup_i [\alpha_i, \alpha_i] \cup \bigcup_j [\beta_j, \beta_j],$$

Then $\overline{A}(^L\mathcal{O}) = (\mathbb{Z}/2\mathbb{Z})^q$*, regardless of the number of* α_i*'s and* β_j*'s.*

For any special orbit $^L\mathcal{O}$, consider $^L\mathcal{O} = {^L\mathcal{P}} \cup {^L\mathcal{Q}}$, where $^L\mathcal{P}$ is the orbit of the same type as $^L\mathcal{O}$ without the α_i's, and $^L\mathcal{Q} := \bigcup_{i=1}^{x}[\alpha_i, \alpha_i]$. Note that $^L\mathcal{P}$ is even with $\overline{A}(^L\mathcal{P}) \cong \overline{A}(^L\mathcal{O})$ and the results in [11] hold for $^L\mathcal{P}$. More precisely, the coordinates of $\lambda_\mathcal{O}$ consists of:

- $^L\mathfrak{g}$ of **Type** B: integers coming from $^L\mathcal{P}$, half-integers coming from $^L\mathcal{Q}$.

- $^L\mathfrak{g}$ of **Type** C: half-integers coming from $^L\mathcal{P}$, integers coming from $^L\mathcal{Q}$.

- $^L\mathfrak{g}$ of **Type** D: integers coming from $^L\mathcal{P}$, half-integers coming from $^L\mathcal{Q}$.

Let $^L\mathfrak{g}'$ be the Lie subalgebra of $^L\mathfrak{g}$ whose roots are given by the roots $\alpha^\vee$ in $^L\mathfrak{g}$ satisfying $\langle \alpha^\vee, \lambda_\mathcal{O} \rangle \in \mathbb{Z}$. We will study integral special unipotent representations $\Pi(^L\mathcal{O}')$, where $^L\mathcal{O}' \subset {^L\mathfrak{g}'} = {^L\mathfrak{g}'_1} + {^L\mathfrak{g}'_2}$ is an even orbit given by $^L\mathcal{O}' = {^L\mathcal{P}} + {^L\mathcal{Q}}$, with

- $^L\mathfrak{g}$ of **Type** B: $^L\mathcal{P} \subset {^L\mathfrak{g}'_1}$ is of Type B; $^L\mathcal{Q} \subset {^L\mathfrak{g}'_2}$ is of Type D.

- $^L\mathfrak{g}$ of **Type** C: $^L\mathcal{P} \subset {^L\mathfrak{g}'_1}$ is of Type C; $^L\mathcal{Q} \subset {^L\mathfrak{g}'_2}$ is of Type C.

- $^L\mathfrak{g}$ of **Type** D: $^L\mathcal{P} \subset {^L\mathfrak{g}'_1}$ is of Type D; $^L\mathcal{Q} \subset {^L\mathfrak{g}'_2}$ is of Type D.

Note that $\overline{A}(\mathcal{O}) = \overline{A}(^L\mathcal{O}) = \overline{A}(^L\mathcal{P}) = \overline{A}(^L\mathcal{O}') = \overline{A}(\mathcal{O}')$.

We now study the unipotent representations $\Pi(^L\mathcal{P})$ and $\Pi(^L\mathcal{Q})$ individually. For $^L\mathcal{Q}$, it has trivial Lusztig quotient, and *the* special unipotent representation attached to it is $\Pi(^L\mathcal{Q}) = \{\mathrm{Ind}_{GL(\alpha_1) \times \cdots \times GL(\alpha_x)}^{G'_2}(\mathrm{triv} \boxtimes \cdots \boxtimes \mathrm{triv})\}$. Moreover, the (unique) left cell representation is $V^L(\mathcal{Q}) = J_{A_{\alpha_1-1} \times \cdots \times A_{\alpha_x-1}}^{W(G'_2)}(\mathrm{sgn} \boxtimes \cdots \boxtimes \mathrm{sgn})$.

On the other hand, suppose the left cell representation of $\mathcal{P}$ is $V^L(\mathcal{P}) = \bigoplus_{x \in \overline{A}(\mathcal{P})} \sigma'_x$.

Then
$$V^L(\mathcal{O}') = \bigoplus_{x \in \overline{A}(\mathcal{O}')} \sigma'_x \boxtimes J_{A_{\alpha_1-1} \times \cdots \times A_{\alpha_x-1}}^{W(G'_2)}(\mathrm{sgn} \boxtimes \cdots \boxtimes \mathrm{sgn}).$$

For $x \in \overline{A}(\mathcal{O}') \cong \overline{A}(\mathcal{O})$, let

$$\sigma_x := J_{W(G')}^{W(G)}(\sigma'_x \boxtimes J_{A_{\alpha_1-1} \times \cdots \times A_{\alpha_x-1}}^{W(G'_2)}(\mathrm{sgn} \boxtimes \cdots \boxtimes \mathrm{sgn}))$$

$$= J_{W(G'_1) \times A_{\alpha_1-1} \times \cdots \times A_{\alpha_x-1}}^{W(G)}(\sigma'_x \boxtimes \mathrm{sgn} \boxtimes \cdots \boxtimes \mathrm{sgn}).$$

One can use Equation (4.6.5) of [20] to check that the right hand side is non-zero and irreducible, and hence Step (VI) holds for all special $\mathcal{O}$. More explicitly, $\Pi(^L\mathcal{O})$ is given by

$$\Pi(^L\mathcal{O}) = \{\mathrm{Ind}_{G'_1 \times GL(\alpha_1) \times \cdots \times GL(\alpha_x)}^{G}(J(\lambda', w_\pi\lambda') \boxtimes \mathrm{triv} \boxtimes \cdots \boxtimes \mathrm{triv}) \mid J(\lambda', w_\pi\lambda') \in \Pi(^L\mathcal{P})\}.$$

A.2. Exceptional Lie algebras

For exceptional special nilpotent orbits, we only focus on $^L\mathcal{O} \subset {^L\mathfrak{g}}$ such that $^L\mathcal{O}$ is not even. Since there are no such orbits in G_2, we will only study exceptional Lie algebras of Type E and F.

In the following tables, we list all non-even special orbits $^L\mathcal{O}$ and their Lusztig-Spaltenstein dual $\mathcal{O}$. In the third column, we get the subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ determined by $\lambda_\mathcal{O}$ (Step (I)). Then $\lambda_\mathcal{O}$ determines an even orbit $^L\mathcal{O}' \subset {}^L\mathfrak{g}'$ (Step (II)), whose Lusztig-Spaltenstein dual $\mathcal{O}'$ is recorded in the fourth column. Afterwards, we can use [21] to compute $V^L(\mathcal{O}')$ (see Example A.3 below), and then we have $V^L(\mathcal{O}) = \bigoplus_{\sigma' \in V^L(\mathcal{O}')} J_{W(G')}^{W(G)}(\sigma')$ (Step(VI)), which is given in the second last column of the table, with $\sigma_{sp} \in V^L(\mathcal{O})$ always appears first in the list. And the last column records the degree $\deg(\sigma_{sp}) = \deg(\sigma'_{sp})$.

The computations below are carried out by LiE [31] and MATLAB.

A.2.1. Type F_4. The results for F_4 are as follows:

$^L\mathcal{O}$	$\mathcal{O}$	$\mathfrak{g}'$	$\mathcal{O}'$	$V^L(\mathcal{O})$	$\deg(\sigma_{sp})$
$\widetilde{A}_1$	$F_4(a_1)$	B_4	$[711]$	$4_2, 2_1$	1
$A_1 + \widetilde{A}_1$	$F_4(a_2)$	$C_3 + A_1$	$[3^2] + [2]$	9_1	2
C_3	A_2	$C_3 + A_1$	$[1^6] + [2]$	8_2	9

A.2.2. Type E_6. The results for E_6 are as follows:

$^L\mathcal{O}$	$\mathcal{O}$	$\mathfrak{g}'$	$\mathcal{O}'$	$V^L(\mathcal{O})$	$\deg(\sigma_{sp})$
A_1	$E_6(a_1)$	$A_5 + A_1$	$[6] + [1^2]$	6_p	1
$2A_1$	D_5	D_5	$[71^3]$	20_p	2
$A_2 + A_1$	$D_5(a_1)$	$A_5 + A_1$	$[41^2] + [1^2]$	64_p	4
$A_2 + 2A_1$	$A_4 + A_1$	D_5	$[3^31]$	60_p	5
A_3	A_4	D_5	$[51^5]$	81_p	6
$A_4 + A_1$	$A_2 + 2A_1$	$A_5 + A_1$	$[21^4] + [1^2]$	$60'_p$	11
$D_5(a_1)$	$A_2 + A_1$	D_5	$[2^21^6]$	$64'_p$	13

A.2.3. Type E_7. The results for E_7 are as follows:

$^L\mathcal{O}$	$\mathcal{O}$	$\mathfrak{g}'$	$\mathcal{O}'$	$V^L(\mathcal{O})$	$\deg(\sigma_{sp})$
A_1	$E_7(a_1)$	$D_6 + A_1$	$[11,1] + [1^2]$	$7'_a$	1
$2A_1$	$E_7(a_2)$	$D_6 + A_1$	$[91^3] + [2]$	27_a	2
$A_2 + A_1$	$E_6(a_1)$	$D_6 + A_1$	$[731^2] + [1^2]$	$120_a, 15'_a$	4
$A_2 + 2A_1$	$E_7(a_4)$	$D_6 + A_1$	$[53^21] + [2]$	$189'_b$	5
A_3	$D_6(a_1)$	$D_6 + A_1$	$[71^5] + [2]$	210_a	6
$D_4(a_1) + A_1$	$E_6(a_3)$	$D_6 + A_1$	$[531^4] + [1^2]$	$405_a, 216'_a$	8
$A_3 + A_2$	$D_5(a_1) + A_1$	$D_6 + A_1$	$[3^31^3] + [2]$	$378'_a$	9
$A_4 + A_1$	$A_4 + A_1$	$D_6 + A_1$	$[3^22^21^2] + [1^2]$	$512'_a$	11
$D_5(a_1)$	A_4	$D_6 + A_1$	$[3^21^6] + [2]$	$420'_a, 84'_a$	13
$D_5 + A_1$	$2A_2$	$D_6 + A_1$	$[31^9] + [1^2]$	$168'_a$	21
$D_6(a_1)$	A_3	$D_6 + A_1$	$[2^21^8] + [2]$	$210'_a$	21

A.2.4. Type E_8**.** The results for E_8 are as follows:

$^L\mathcal{O}$	$\mathcal{O}$	$\mathfrak{g}'$	$\mathcal{O}'$	$V^L(\mathcal{O})$	$\deg(\sigma_{sp})$
A_1	$E_8(a_1)$	$E_7 + A_1$	$E_7 + [1^2]$	8_z	1
$2A_1$	$E_8(a_2)$	D_8	$[13, 1^3]$	35_x	2
$A_2 + A_1$	$E_8(a_4)$	$E_7 + A_1$	$E_7(a_3) + [1^2]$	$210_x, 50_x$	4
$A_2 + 2A_1$	$E_8(b_4)$	D_8	$[93^21]$	560_z	5
A_3	$E_7(a_1)$	D_8	$[11, 1^5]$	567_x	6
$D_4(a_1)+A_1$	$E_8(a_6)$	$E_7 + A_1$	$E_7(a_5) + [1^2]$	$1400_x, 1050_x, 175_x$	8
$A_3 + A_2$	$D_7(a_1)$	D_8	$[73^21^3]$	3240_z	9
$A_4 + A_1$	$E_6(a_1)+A_1$	$E_7 + A_1$	$D_5(a_1) + [1^2]$	4096_z	11
$A_4 + 2A_1$	$D_7(a_2)$	D_8	$[53^31^2]$	$4200_x, 840_x$	12
$D_5(a_1)$	$E_6(a_1)$	D_8	$[731^6]$	$2800_z, 700_{xx}$	13
$D_5(a_1)+A_1$	$E_7(a_4)$	$E_7 + A_1$	$(A_3 + A_2) + [2]$	6075_x	14
$A_4+A_2+A_1$	$A_6 + A_1$	$E_7 + A_1$	$(A_3+A_2+A_1)+[1^2]$	2835_x	14
$D_6(a_1)$	$E_6(a_3)$	D_8	$[531^8]$	$5600'_z, 3200'_x$	21
$A_6 + A_1$	$A_4+A_2+A_1$	$E_7 + A_1$	$(A_2 + 3A_1) + [1^2]$	$2835'_x$	22
$E_7(a_4)$	$D_5(a_1)+A_1$	$E_7 + A_1$	$(A_2 + 2A_1) + [2]$	$6075'_x$	22
$D_7(a_2)$	$A_4 + 2A_1$	D_8	$[3^22^21^6]$	$4200'_x, 840'_x$	24
$E_6(a_1)+A_1$	$A_4 + A_1$	$E_7 + A_1$	$(A_2 + A_1) + [1^2]$	$4096'_z$	26
$E_7(a_3)$	A_4	$E_7 + A_1$	$A_2 + [2]$	$2268'_x, 972'_x$	30
$E_7(a_1)$	A_3	$E_7 + A_1$	$A_1 + [2]$	$567'_x$	46

Here is an example on how the above results are obtained:

Example A.3. Let $^L\mathcal{O} = D_4(a_1) + A_1$ be a nilpotent orbit of Type E_8. By calculating $\frac{1}{2}{}^LH$ explicitly, one can check that $^L\mathfrak{g}'$ is of Type $E_7 + A_1$, and the coroots $\alpha^\vee$ such that $\langle \alpha^\vee, \frac{1}{2}{}^LH \rangle = 0$ forms a Lie subalgebra $^L\mathfrak{l}'$ of Type $(A_5+A_1)+0$. Then $^L\mathcal{O}' \subset {}^L\mathfrak{g}'$ is the even orbit

$$^L\mathcal{O}' = \operatorname{Ind}_{^L\mathfrak{l}'}^{^L\mathfrak{g}'}(0) = D_4(a_1) + [2].$$

By looking at the tables of [12], one can check that Lusztig-Spaltenstein dual of $^L\mathcal{O}$ and $^L\mathcal{O}'$ are $\mathcal{O} = E_8(a_6)$ and $\mathcal{O}' = E_7(a_5) + [1^2]$ respectively. Also, one can check from [14] that $\overline{A}(\mathcal{O}) = \overline{A}(\mathcal{O}') = S_3$.

We now study the left cell $V^L(\mathcal{O}')$: Firstly, note that $E_7(a_5) = \operatorname{Ind}_{E_6}^{E_7}(D_4(a_1))$. By [21], the special piece attached to $D_4(a_1)$ is equal to $\{D_4(a_1), A_3 + A_1, 2A_2 + A_1\}$, and their Springer representation constitute the left cell

$$V^L(D_4(a_1)) = 80_s \oplus 60_s \oplus 10_s$$

(in fact, $V^L(D_4(a_1))$ can also be obtained directly from (4.11.2) of [20], but this perspective is useful in determining left cells for classical orbits with large Lusztig quotient).

Let $[1^3]$, $[21]$, $[3]$ be the conjugacy classes of S_3, then by Proposition 4.14 of [11] (which is valid since $\mathcal{O}'$ is even),

$$V^L(\mathcal{O}') = J^{E_7+A_1}_{E_6+A_1}(80_s \boxtimes \mathrm{sgn}) \oplus J^{E_7+A_1}_{E_6+A_1}(60_s \boxtimes \mathrm{sgn}) \oplus J^{E_7+A_1}_{E_6+A_1}(10_s \boxtimes \mathrm{sgn}).$$

Using the notations of Step (III) in the first section, we obtain

$$\sigma'_{[1^3]} = J^{E_7+A_1}_{E_6+A_1}(80_s \boxtimes \mathrm{sgn}),$$
$$\sigma'_{[21]} = J^{E_7+A_1}_{E_6+A_1}(60_s \boxtimes \mathrm{sgn})$$
and
$$\sigma'_{[3]} = J^{E_7+A_1}_{E_6+A_1}(10_s \boxtimes \mathrm{sgn}).$$

By Proposition 3.18 of [10] and (4.13.3) of [20], we have

$$\begin{aligned}
V^L(\mathcal{O}) &= J^{E_8}_{E_7+A_1}(\sigma'_{[1^3]}) \oplus J^{E_8}_{E_7+A_1}(\sigma'_{[21]}) \oplus J^{E_8}_{E_7+A_1}(\sigma'_{[3]}) \\
&= J^{E_8}_{E_6+A_1}(80_s \boxtimes \mathrm{sgn}) \oplus J^{E_8}_{E_6+A_1}(60_s \boxtimes \mathrm{sgn}) \oplus J^{E_8}_{E_6+A_1}(10_s \boxtimes \mathrm{sgn}) \\
&= 1400_x \oplus 1050_x \oplus 175_x.
\end{aligned}$$

Moreover, $\sigma_{sp} = 1400_x$ is the special representation corresponding to the $\mathcal{O}$ under the Springer correspondence. Therefore, this verifies Step (VI) and Table A.2.4 for this orbit. ∎

We end the Appendix by mentioning the character formulas of the special unipotent representations $\Pi(^L\mathcal{O})$ when $\mathcal{O}$ is equal to the three exceptional orbits, i.e. A_4+A_1 in E_7, A_4+A_1 and $E_6(a_1)+A_1$ in E_8.

From the Tables A.2.3 and A.2.4 above, one can check that for the $\mathcal{O}' \subset \mathfrak{g}'$ corresponding to these $\mathcal{O}$'s, $\overline{A}(\mathcal{O}) \cong \overline{A}(\mathcal{O}') \cong S_2$ and

$$V^L(\mathcal{O}') = \sigma_{[1^2]} \oplus \sigma_{[2]}.$$

However, their truncated inductions are given by

$$J^W_{W'}(\sigma'_{[1^2]}) = \sigma_{sp}, \quad J^W_{W'}(\sigma'_{[2]}) = 0,$$

so $V^L(\mathcal{O}) = \sigma_{sp}$ contains one irreducible representation only. In other words, the second bullet point of Step (VI) does not hold for these orbits, and one cannot express their character formulas in the form of Theorem III of [11].

Nevertheless, the character formulas of $\Pi(^L\mathcal{O})$ can still be obtained by using Step (IV), and the number of representations is equal to the number of irreducible representations of $\overline{A}(\mathcal{O}) \cong \overline{A}(\mathcal{O}')$.

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